Points of integral canonical models of Shimura varieties of preabelian type, p-divisible groups, and applications

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These are abridged, essentialized notes of our talk in Berkeley on the Langlands-Rapoport conjecture, given on April 27, 1999. Wherever preprints are already availbale, quotes are made.

By (G, X, H, v) we denote a Shimura quadruple of preabelian type. Let p be the rational prime divided by the prime v of $E(G, X)$. We assume that $p \geq 3$.

Step 0. Theorem. *The integral canonical model* $\mathcal{N}$ *of* (G, X, H, v) *exists.*

References. For the unicity part see [Va1, 3.2.4]. For the existence part for $p \geq 5$ see [Va1] and [Va2, 4.6.11]. The case $p = 3$ will be made available soon.

Notation. We denote by $LR(\mathcal{N})$ the statement of the LR-conj (see [Mi1, 4.4] and [Pf]) for $\mathcal{N}$. Warning: we do not assume axiom SV4 of [Mi1]; so [Mi4, 4.4] has to be supplemented by [Pf].

Step 1 (Milne). Let $(G_1, X_1, H_1, v_1) \to (G, X, H, v)$ be a cover (in the sense of [Va1, 2.4 and 3.2.6]). Then $LR(\mathcal{N}_1)$ implies $LR(\mathcal{N})$.

References. [Mi1, 4.9].

Step 2. We consider a situation

$$(G_1, X_1, H_1, v_1) \to (G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}, H_1^{\mathrm{ad}}, v_1^{\mathrm{ad}}) = (G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}}, H_2^{\mathrm{ad}}, v_2^{\mathrm{ad}}) \leftarrow (G_2, X_2, H_2, v_2),$$

with $G_1^{\mathrm{der}} = G_2^{\mathrm{der}}$ and $k(v_1) \subset k(v_2)$. Then $LR(\mathcal{N}_1)$ implies $LR(\mathcal{N}_2)$.

References. This is easy; see [Va1, 6.2.3].

Step 3. Let $Q_0 := (G_0, X_0, H_0, v_0)$ be an adjoint Shimura quadruple of preabelian type. Then there is an injective map

$$f_1 \colon (G_1, X_1, H_1, v_1) \hookrightarrow (GSp(W, \psi), S, K_p, p)$$

such that:

a) we have $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}, H_1^{\mathrm{ad}}, v_1^{\mathrm{ad}}) = Q_0$;
b) $k(v_1) = k(v)$;
c) G_1^{der} is the maximal étale group cover of G_1^{ad} allowed for the abelian type;
d) there is a standard Hodge situation $(f_1, L_{(p)}, v_1)$ attached to f_1.

References. This is just a restatement of [Va1, 6.4.2], to which b) above is added. The fact that b) can be added is a consequence of the proof of [Va1, 6.5.1.1]. For a standard Hodge situation (to be abbreviated as SHS), see [Va2, 2.3].

Step 4 (Milne). We refer to the context of Step 3 involving the injective map f_1. Then $LR(\mathcal{N}_1)$ is true provided:

a) G_1^{der} is simply connected;
b) Milne's conjecture [Mi1, 0.1] is checked in this context;
c) we have good reduction of the tensors involved.

References. [Mi2, 6.12].

Step 5. In the context of the SHS of Step 3, Milne's conjecture is checked.

References. See [Va4].

Main Theorem. $LR(\mathcal{N})$ *is true provided one of the following two conditions is satisfied:*

a) $k(v) = \mathbb{F}_p$*;*
b) $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ *has all the factors of* A_n*, or* B_n*, or* $D_n^{\mathbb{R}}$ *type.*

Step 6. For the proof of the above theorem, we can assume that $k(v) = \mathbb{F}_p$.

References. The above Steps 1-5 take care of the A_n type case (cf. [Va2, 2.3.5]). For the types B_n and $D_n^{\mathbb{R}}$, this is an easy consequence of [Va1, 6.6.2] (see i) and d) of loc. cit.).

Step 7. This step is stated in [Va2, 1.23.1]. It can be checked directly either case by case or by a density argument (based on [Va2, 4.2.1] and on [Va2, 1.23] and its applications).

Remark. This step for G_1-ordinary points is nothing else but the statement of [Va2, 4.4.6]. Its proof is essentially contained in [Va3, 3.1.1].

Step 8 (Zink). In this step we refer to the SHS $(f_1, L_{(p)}, v_1)$ of Step 3. Over $\mathcal{N}_1$ there is a universal principally polarized abelian scheme $(\mathcal{A}_1, \mathcal{P}_{\mathcal{A}_1})$ endowed naturally with a family of Hodge cycles (cf. [Va2, 2.3]). If $H_{01} \subset G_1(\mathbb{A}_f)$ is as in loc. cit., then $\mathcal{A}_1$ descends to an abelian scheme $\mathcal{A}_{1H_{01}}$ over $\mathcal{N}_1/H_{01}$. So in what follows we work with points of $\mathcal{N}_1/H_{01}$ with values in finite fields, and with the abelian varieties obtained over such fields by pulling back $\mathcal{A}_{1H_{01}}$. In this step we show that up to isogeny we can lift the Frobenius endomorphisms and the crystalline components of the Hodge cycles involved to characteristc zero.

References. The main result of [Zi]. The essence of it is captured in the conjecture expressed in [Va2, rm.3 of 3.13.3.1].

Remarks. 1) The assumption $k(v) = \mathbb{F}_p$ gives us simplifications from the following points of view:

a) the G_1-ordinary type we get (see [Va2, 4.1]) is the ordinary type (cf. [Va2, P1 of 4.6]);

b) there are no non-isolated "compact factors" over $\mathbb{Z}_p$ (i.e. $G^{1\mathrm{ad}}_{\mathbb{Z}_p}$ is a product of factors such that the natural cocharacter $\mu\colon \mathbb{G}_m \to G^1_{W(k(v_1))}$ has a natural projection on such a factor which is either trivial or it is such that over $W(\mathbb{F})$ is non-trivial on each simple factor).

[Zi] is worked out for some $D^{\mathbb{H}}_n$ and C_n cases; it can be easily extended (in the context of $k(v_1) = \mathbb{F}_p$) to the cases of B_n and $D^{\mathbb{R}}_n$ types.

2) This is the only place where we need in the Main Theorem $k(v) = \mathbb{F}_p$ for the C_n and $D^{\mathbb{H}}_n$ factors. This condition should be easily removed (for instance, in [Zi] there are also treated cases of A_n type with non-isolated compact factors over $\mathbb{Z}_p$).

Step 9. Using Fontaine's categories we show that the lifts to characteristic zero of Step 8 are part of the initial family (and not of the family of abelian varieties imerging from some other Shimura variety of Hodge type). Of course first the isogeny property has to be checked (see [Va1, 1.7.1]).

References. For the background where the Fontaine's categories are useful see [Va1, 5.5.1]. For how one gets the isogeny property see [Va2, 1.22].

Step 10 (Conclusions). The case of a simple connected semisimple group G^{der}_1 can be deduced easily from [Mi1, 4.6] and Step 9. The case $D^{\mathbb{H}}_n$ is deduced from the $C^{\mathbb{H}}_n$ type via the following trick:

Proposition. *We assume that $(G^{\mathrm{ad}}_1, X^{\mathrm{ad}}_1)$ is simple of $D^{\mathbb{H}}_n$ type. Then we can assume that f_1 factors as follows*

$$(G_1, X_1, H_1, v_1) \overset{f_2}{\hookrightarrow} (G_2, X_2, H_2, v_2) \overset{f_3}{\hookrightarrow} (G_3, X_3, H_3, v_3) \hookrightarrow (GSp(W, \psi), S, K_p, p),$$

where:

a) f_2 is obtained using the Weil restriction (as in [Va1, 6.6.2 e)]);
b) If G^{ad}_2 is of D_m Lie type, then G^{ad}_3 is of C_{2m} Lie type;
c) f_3 is a relative PEL type situation in the sense of [Va1, 4.3.16].

References. This is just a simpler variant of the twisting process of [Va3, 3.1].

References

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