A supplement to part 2C above.

What follows is an ad-hoc supplemented to part 2C (it will be added to the final version of it to be submitted for publication).

A. The restriction in the Main Theorem 3 can be entirely removed. This goes in two steps as follows. We refer to the notations of the proof of 6.2.3.1, with $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ of $D_n^{\mathbb{H}}$ type. Using the same ideas of principal symmetric quasi-polarizations of 4.2 T), we get immediately (standard arguments) that the Frobenius endomorphism π of a Shimura-ordinary abelian variety over a finite field k lifts to a G_1-endomorphism of its Shimura-canonical lift (here G_1-endomorphism is in the sense of [Va2, 4.4.3]). In other words, if $(M, F^1, \phi, G_{1W(k)}, (t_\alpha)_{\alpha\in\mathcal{J}'})$ is the Shimura filtered σ-crystal of such a Shimura-canonical lift, then π, as an automorphism of $M\left[\frac{1}{p}\right]$, fixes t_α, $\forall\alpha\in\mathcal{J}'$. This is Step one.

In Step 2, we need to remark that the center C of the centralizer of π in $G_{1B(k)}$ contains the generic fibre of the center $Z(R_1)$ of the reductive subgroup R_1 of $G_{1W(k)}$ whose Lie algebra corresponds to slope zero for $(\mathrm{Lie}(G_{1W(k)}), \phi)$. As the canonical lift $\mu : \mathbb{G}_m \to G_{1W(k)}$ of $(M, F^1, \phi, G_{1W(k)}, (t_\alpha)_{\alpha\in\mathcal{J}'})$ factors through $Z(R_1)$, and so through C, and as C has a logical $\mathbb{Q}$-model (cf. Step 1), we deduce (to be compared with [Va2, 2.2.9 3)]) that this Shimura-canonical lift is a CM lift.

B. Entirely in the same manner, in the general context of 6.2.3.2 1), we get that the generic fibre of a U-canonical lift of a U-ordinary point with values in $\mathbb{F}$, is a special point provided all simple factors of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ are of A, C or $D^{\mathbb{H}}$ type. In many cases it can be checked easily (i.e. without referring to the Langlands–Rapoport conjecture) that this remains true when we do have factors of B or $D^{\mathbb{R}}$ type.

C. Also from A above we deduce that, whenever the generic fibre of a U-canonical lift of a U-ordinary point with values in $\mathbb{F}$ is a special point, this lft is the only lift to DVR's of mixed characteristic $(0, p)$ to which the Frobenius endomorphism lifts (the adjoint version of it can be defined in an abstract way for any Shimura quadruple of preabelian type, by just following the pattern of trick three of 2.3). This is the full completion of the generalized Serre–Tate theory.

D. It is not made explicit, but the results for $D_n^{\mathbb{H}}$ types in mixed characteristic $(0, 2)$ refer just to the abelian type picture (cf. the limitations of [Va1, 6.8.1]). In other words, in 6.4.1.1 we only have the abelian type in mind.