

A variant of the Jacobian conjecture in positive characteristic

Let p be a prime. Let k be an algebraically closed field of characteristic p . Let $n, d \in \mathbb{N}$. Let $f_1, \dots, f_n \in k[x_1, \dots, x_n]$ be polynomials in n variables such that the maximum of the total degrees of $f_1, \dots, f_n$ is d . We consider the following system of n equations in n variables:

$$(1) \quad x_i = f_i(x_1^p, \dots, x_n^p) \quad i \in \{1, \dots, n\}.$$

It is clear that such a system defines an étale k -scheme X .

Question 1. Is it true that X is non-empty?.

Question 2. How the number of points of X changes if each f_i is replaced by $c_i + f_i$ with $c_i \in k$ a constant? Under which conditions the number of points of X does not change if each L_i is replaced by such a $c_i + L_i$?

Known cases. The answers are known (to us) only in the cases when:

(a) $d = 1$ (i.e., L_i are linear) in which case $X = \text{Spec } k^{p^m}$, with $m \in \{0, 1, \dots, n\}$;

(b) $n = 1$ (this is trivial);

(c) some cases which could be reduced naturally to (a) (like when each f_i is up to a constant additive, i.e., is a linear combination of monomials in the set $\{1\} \cup \{x_i^{p^s} \mid 1 \leq i \leq n, s \geq 0\}$);

(d) generically X is non-empty (i.e., considering the coefficients of f_i 's as variables, X will be non-empty for an open, dense subscheme of the affine space described by these coefficients; see Example 2 below for the non-empty part);

(e) cases to which the tricks below apply.

In these cases, the number of points of X does not change. However, it does change in general (see Example 1 below).

Example 1 (due to Deligne). If $n = 2$, then the system $y = y^p$ and $x = 1 + y^p x^p$ has $p^2 - p + 1$ solutions while the system $y = y^p + 1$ and $x = 1 + y^p x^p$ has p^2 solutions.

Example 2. Assume that k is an algebraic closure of $\kappa((t))$ with κ a field. If all the coefficients of f_i 's belong to $\kappa[[t]]$ and if $f_1(0, \dots, 0), \dots, f_n(0, \dots, 0) \in$

$t\kappa[[t]]$, then X is non-empty and a solution of the above system is obtained as the t -adic limit of the sequence $(y_s)_{s \geq 1}$ of n -tuples of elements of $t\kappa[[t]]$ defined by $y_1 = (f_1(0, \dots, 0), \dots, f_n(0, \dots, 0))$ and by the inductive rule $y_{s+1} := (f_1(y_s^p), \dots, f_n(y_s^p))$.

Motivations. There are mainly two (besides the simplicity of such systems of equations):

(i) one hopes to get an insight into the Jacobian conjecture in characteristic 0;

(ii) one hopes to use the result to prove that some quotients G/U are affine, where G is a connected, affine, smooth group over k and U is a connected, smooth, unipotent subgroup of G (at least that is the way we came across such systems of equations with $d \geq 2$).

Basic tricks.

(a) Adding repeatedly extra variables and equations of the form $x_{n+1} = x_i^p$ with $i \in \{1, \dots, n\}$, one can decrease the total degrees of the f_i 's to the extent that each partial degree $\deg_{x_j}(f_i)$ is at most $p-1$ (here also $j \in \{1, \dots, n\}$). Thus one can assume that $d \leq (p-1)n$.

(b) By adding extra variables via the rule $x_i = \sum u_{i,j}$, one can assume that each f_i is moreover *homogeneous*.

(b) One can *compactify* the system by adding one extra variable x_0 . Like if $f_1(x_1, \dots, x_n) = \sum a_{l_1, \dots, l_n} x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$ and if $d_1 \in \{0, 1, \dots, d\}$ is the greatest number such that there exists a nono-zero coefficient $a_{l_1, \dots, l_n}$ with $\sum_{i=1}^n l_i = d_1$, then one can replace the equation $x_1 = f_1(x_1^p, \dots, x_n^p)$ with the equation

$$x_1 x_0^{pd_1-1} = \sum a_{l_1, \dots, l_n} x_0^{pd_1-p \sum_{i=1}^n l_i} x_1^{pl_1} x_2^{pl_2} \cdots x_n^{pl_n}.$$

For $i \in \{1, \dots, n\}$ let d_i be defined similarly; one can assume that each $d_i \geq 1$. In this way one gets a projective scheme $\mathcal{X}$ in $\mathbb{P}^n$ which is the intersection of n hypersurfaces of degrees $pd_1, \dots, pd_n$. Intersection theory tells us that $\mathcal{X}$ is non-empty. The scheme X is an open subscheme of $\mathcal{X}$ corresponding to $x_0 \neq 0$.

Let g_i be a homogeneous polynomial of total degree d_i such that the total degree of $h_i := f_i - g_i$ is at most $d_i - 1$ (thus g_i is the leading form of f_i).

Criterion. *If the system of homogeneous equations $g_1(x_1^p, \dots, x_n^p) = \cdots = g_n(x_1^p, \dots, x_n^p) = 0$ has no solution in $\mathbb{P}^{n-1}$, then X is non-empty.*

Proof: The hypothesis means that $\mathcal{X}$ intersected with the hypersurface $x_0 = 0$ is empty and thus $X = \mathcal{X}$ is non-empty. $\square$

Example 3. The system $x_1 = h_1(x_1^p, x_2^p) + x_1^{p^d}$ and $x_2 = h_2(x_1^p, x_2^p) + x_2^{p^d} + x_1^{p^d}$ has solutions (the total degrees of h_1 and h_2 are less than d).

(d) The *recursive process* of Example 2.

(e) Let c be the number of i 's such that $f_i(0, \dots, 0)$ is non-zero. One can try to use *induction* on $c + n$. Using a variant of the trick (b) (involving substitutions $x_i = u_{i1} + u_{i2}$ for those i such that $f_i(0, \dots, 0) \neq 0$ and then eliminating $c - 1$ of the variables that show up in an equation of the form $x_j = c_j$ via identities $x_{j1} = c_{j1}c_{j2}^{-1}x_{j2}$), one achieves the reduction to the case when $c = 1$ (i.e., for $c \geq 2$, the pair (c, n) can be replaced by the pair $(1, n + 1)$). If $c = 1$ and let's say that $f_1(0, \dots, 0) = \alpha \neq 0$, then one can use as a parameter space $k[t]$ (with α replaced by t) and then get solutions first over either a strict henselization of $k[t]_{(t)}$ or $k[[t]]$ and one could try to see if by chance there are no poles involved at $t = \alpha$.

(f) One can assume that k is an algebraic closure of $\mathbb{F}_p$ and one could try to *perturb* the system (1) by replacing each equation $x_i = f_i(x_1^p, \dots, x_n^p)$ by an equation of the form $x_i = f_i(x_1^p, \dots, x_n^p) - x_i + x_i^{p^q}$ with q big enough so that the coefficients are in $\mathbb{F}_{p^q}$ and the Criterion applies. Each solution in $\mathbb{F}_{p^q}^n$ (there are p^{qn} solutions in k) of the perturbed system, is also a solution of (1).

(g) *Combine* the above six tricks (and maybe few others) in some smart way.