Points of integral canonical models of Shimura varieties of preabelian type, p-divisible groups, and applications

Adrian Vasiu, UC at Berkeley

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(The first part out of three)

Contents of the first part

Contents of the second part

Contents of the third part

§1 *Introduction*

1.0. Recently we managed to prove the existence of integral canonical models of Shimura varieties of preabelian type with respect to primes $p \geq 3$ (cf. [Va2, 6.4.1 and the definitions of 3.2.6]; see [Va2, 6.8], 4.6.11 and [Va3] for the case of Shimura varieties of preabelian type, and see §6 and [Va5] for the case $p = 3$). For the general theory of such integral models we refer to [Va2, §3]. With this paper, which is a natural continuation of [Va2], we start an extensive programme of proving that many well-known results for integral canonical models of Siegel modular varieties remain true (under proper formulation) for integral canonical models of Shimura varieties of preabelian type. Here we are concerned with points of these models with values in regular formally smooth schemes over rings of Witt vectors of a given length (finite or infinite) of a perfect field. Special attention is paid to points with values in (the Witt ring of) a perfect field.

The results mentioned in [Va2, 1.6-7] are proved in this paper. For a faster publication of this paper, we divide it in three parts, to be published separately. The first part contains §1–4, the second part will contain §5–10, and the third part will contain §11–14. In this first part we concentrate (this was the initial goal) in reobtaining in a new manner:

- the theory (due to Serre–Tate; see [Me]) of ordinary abelian varieties (over perfect fields) and of their canonical lifts,
- the theory (due to Dwork; see [Ka1]) of canonical coordinates of the formal moduli scheme of deformations of an ordinary abelian variety over an algebraically closed field,

and in extending these theories to the context of integral canonical models of Shimura varieties of preabelian type, and even more generally to the context (see 1.1 below) of Shimura σ-crystals. On the way we obtain many other results pertaining to p-divisible groups, to abelian varieties, and to the above mentioned integral canonical models. In particular we obtain versal global deformations, i.e. versal deformations over the p-adic completion of pro-étale schemes over smooth affine schemes over (Witt rings of) perfect fields (of characteristic $p > 0$), of p-divisible groups over such fields, cf. 3.15. In 4.12.12 (cf. also Theorem 12 of 1.14 below), we show that in the case when we are over an algebraically closed field we can drop the words "p-adic completion" and "pro-étale".

1.1. In what follows k is a perfect field of positive characteristic p, $W(k)$ is the ring of (infinite) Witt vectors of k, and σ is the Frobenius automorphism of $W(k)$. We denote by $\bar{E}$ the algebraic closure of a field E. The proof of the basic result 5.1 of [Va2] leads to the study of σ-crystals endowed with an extra structure (cf. [Va2, 5.6.5]). These are σ-crystals (M, φ) over k (so M is a free $W(k)$-module of finite rank, and φ is a σ-linear endomorphism of M producing a σ-linear automorphism of $M[\frac{1}{p}]$) such that:

i) There is a family $(v_\alpha)_{\alpha\in\mathcal{J}}$ of tensors in spaces of the form $M^{\otimes n}\otimes M^{*\otimes m}[\frac{1}{p}]$ (with M^* the dual of M, and with $n,m\in\mathbb{N}\cup\{0\}$), fixed by φ, and such that the Zariski closure G of the subgroup of $GL(M[\frac{1}{p}])$ fixing v_α, $\forall\alpha\in\mathcal{J}$, in $GL(M)$, is a quasi-split reductive group over $W(k)$;

ii) There is a cocharacter $\mu:\mathbb{G}_m\to G$ producing a direct sum decomposition $M=F^1\oplus F^0$ such that $\beta\in\mathbb{G}_m(W(k))$ acts through μ on F^i as the multiplication with β^{-i}, $i=\overline{0,1}$. Moreover F^1 is a proper direct summand of M (this condition is inserted just to avoid trivial cases, cf. 2.2.9 11));

iii) The triple (M,F^1,φ) is a filtered σ-crystal.

The resulting triple (M,φ,G) (resp. quadruple $(M,F^1,\varphi,G,)$) is called a Shimura σ-crystal (resp. a Shimura filtered σ-crystal), while the pair $(\mathrm{Lie}(G),\varphi)$ is called the Shimura Lie σ-crystal attached to the Shimura σ-crystal (M,φ,G) (we still denote by φ the resulting σ-linear automorphism of $\mathrm{Lie}(G)[\frac{1}{p}]$; here $\mathrm{Lie}(G)$ is the Lie algebra of G).

Fixing the group G, the family of tensors $(v_\alpha)_{\alpha\in\mathcal{J}}$, and the $G(W(k))$-conjugacy class of the cocharacter μ, a σ-linear endomorphism φ_1 of M corresponding to any other σ-crystal (M,φ_1) satisfying the above properties i) to iii), up to isomorphism, can be put in the form $\varphi_1=g\varphi$, with $g\in G(W(k))$.

1.2. The case $p=2$ is treated as asides. So, in all that follows, without a special reference we assume that $p>2$. Our basic results are grouped in 3.1 and 3.4. In 3.1 we present the first group of basic results:

Theorem 1. *a) Among all Shimura σ-crystals $(M,g\varphi,G)$ (with $g\in G(W(k))$), the ones which have the smallest Newton polygon (in the sense that all others have a Newton polygon strictly above it) are precisely the ones $(M,g_1\varphi,G)$ (called G-ordinary σ-crystals) for which there is a cocharacter $\mu:\mathbb{G}_m\to G$ as in 1.1 ii), such that the parabolic Lie subalgebra $\mathfrak{p}_1$ of* $\mathrm{Lie}(G)$ *corresponding to non-negative slopes of the Shimura Lie σ-crystal* $\big(\mathrm{Lie}(G),g_1\varphi\big)$ *is contained in the parabolic Lie subalgebra of* $\mathrm{Lie}(G)$ *leaving invariant the F^1-filtration of M defined (as in 1.1 ii)) by μ.*

b) If $(M,g_1\varphi,G)$ is a G-ordinary σ-crystal, then there is a unique Shimura filtered σ-crystal $(M,F^1,g_1\varphi,G)$ (called the G-canonical lift of the G-ordinary σ-crystal $(M,g_1\varphi,G)$) such that $\mathfrak{p}_1$ leaves invariant F^1.

c) Among all Shimura σ-crystals $(M,g\varphi,G)$ (with $g\in G(W(k))$), the ones which have the smallest Newton polygon of their attached Shimura Lie σ-crystals $(\mathrm{Lie}(G),g\varphi)$, *are precisely the G-ordinary σ-crystals.*

d) Any Shimura σ-crystal $(M,g\varphi,G)$ is the specialization of a $G_{W(k_1)}$-ordinary σ_1-crystal (here σ_1 is the Frobenius automorphism of the Witt ring of a perfect field k_1 containing k; we can take k_1 to be the field of fractions of the perfection of $k[[X]]$, where X is an independent variable).

1.2.0. The proof of Theorem 1 is achieved by first (an idea suggested by G. Faltings) proving it for Shimura Lie σ-crystals $(\mathrm{Lie}(G),g\varphi)$. To achieve this and to complete the proof of Theorem 1 (cf. 3.4.7) we construct global deformations of such (as in 1.1) Shimura filtered σ-crystals; their goal is: to connect any Shimura σ-crystal $(M,g\varphi,G)$ (with $g\in$

$G(W(k)))$ with (the extension to $\bar{k}$ of) a G-ordinary σ-crystal $(M, g_1\varphi, G)$. Here by global we mean: deformations over the p-adic completion of pro-étale schemes over smooth affine schemes over $W(k)$, whose special fibres are connected and have a Zariski dense set of $\bar{k}$-valued points. This takes care of d) of Theorem 1, as it can be checked easily starting from the parts of 3.5.11-12 recalled in 1.2.1 2) and 3) below.

1.2.1. The proof of Theorem 1 for Shimura Lie σ-crystals $(\mathrm{Lie}(G), g\varphi)$ (with $g \in G(W(k))$), means three things:

1) there is an element $g_1 \in G(W(k))$ such that the parabolic Lie subalgebra $\mathfrak{p}_1$ of $\mathrm{Lie}(G)$ (defined in a) of Theorem 1) leaves invariant F^1 (cf. 3.2.3);
2) for any $g \in G(W(k))$, the Newton polygon of the Lie σ-crystal $(\mathrm{Lie}(G), g\varphi)$ is strictly above or equal to the Newton polygon of the Lie σ-crystal $(\mathrm{Lie}(G), g_1\varphi)$ (cf. 3.5.12);
3) if in 2) above we have equality of Newton polygons, then an explicit expression of g can be determined, and the σ-crystal $(M, g\varphi)$ has the same Newton polygon as the σ-crystal $(M, g_1\varphi)$ (cf. 3.5.11).

Part 1) is easy (as G is a quasi-split reductive group), while 3) is obtained on the way of proving 2) (cf. 3.5.8-11 and 3.7.5). From these three facts and 1.2.0 we get easily a), b) and c) of Theorem 1.

For the proof of 2) we first show that for any $g \in G(W(k))$, the Newton polygon of $(\mathrm{Lie}(G), g\varphi)$ can not be strictly below the Newton polygon of $(\mathrm{Lie}(G), g_1\varphi)$ (cf. 3.5.12). This is achieved (cf. the proof of 3.5.6 needed to get 3.5.8, and so needed for getting 3.5.12) by making use of the Lie stable p-rank of a Shimura σ-crystal (M, φ, G) (it is defined in 3.9.1; see also 3.5.5), via Theorem 2 below. 3.5.6 is a statement on such Lie stable p-ranks capturing the essence behind 2) and 3) and Theorem 2 below. Its proof is carried out in 3.6; it involves standard techniques pertaining to parabolic subgroups of $G_{\bar{k}}$. To conclude the proof of 2) we use the global deformations mentioned above.

1.2.1.1. For a simpler way of proving Theorem 1, see 3.5.14.

1.2.2. We have (cf. 3.9.2) the following important supplement of the first group of basic results:

Theorem 2. *A Shimura σ-crystal (M, φ, G) is a G-ordinary σ-crystal iff for any $g \in G(W(k))$, the Lie stable p-rank of $(M, g\varphi, G)$ is smaller than or equal to the Lie stable p-rank of (M, φ, G).*

In particular any Shimura σ-crystal $(M, g\varphi, G)$ having the same Lie stable p-rank as a G-ordinary σ-crystal $(M, g_1\varphi, G)$, is a G-ordinary σ-crystal. Theorem 2 represents the generalization of the characterization of ordinary σ-crystals in terms of the Hasse–Witt matrices: we get Shimura–Hasse–Witt matrices (cf. 4.3.7 and 4.3.10). The essence of its proof is 3.6.

1.2.3. In 2.2 we present the basic facts pertaining to Shimura (adjoint) (filtered) (Lie) (σ- or F-) crystals. The notion of Shimura p-divisible groups (these are p-divisible groups endowed with crystalline (de Rham) cycles and satisfying some axioms inspired from the ones satisfied by Shimura crystals) is introduced in 2.2.20. The first group of basic results (and their proofs) is presented in 3.1-7, while 3.8 shows that Theorem 1 remains true if G is just a reductive group (i.e. it is not necessarily quasi-split). The construction of the global deformations mentioned in 1.2.0, is carried out in 3.4, being part of the second group of basic results.

In section 3.4 we start the presentation of ten fundamental principles of the crystalline theory. These principles are:

- the unicity principle (cf. 3.4.1.2, 3.4.18.8 and 3.4.18.8.1);
- the ∇ principle (cf. 3.4.1.3, 3.4.18.4 and 3.4.18.6 a));
- the deformation principle (to be described in 1.3.2) (cf. 3.4.14 and 3.4.14.1);
- the (very) (weak) gluing principle (cf. 3.4.2.2 and 3.4.19);
- the surjectivity principle (cf. 3.4.8.1.2 a), 3.4.8.2, 3.4.1.3 5) and 3.4.18.6 b));
- the integrability principle (cf. 3.4.18.4.1);
- the moduli principle (cf. 3.4.18.4.2 and 3.4.1.3);
- the invariance principle (cf. 4.13, §9-10 and Theorem 11 below);
- the slice principle (cf. §7);
- the integral Manin problem (cf. 1.14 and 4.12).

Seven other useful properties (the constancy property, the constructibility property, the inducing property, the liftability property, the touching property, the purity property and the rigidity property) are presented respectively in 3.4.8.9, 3.4.18.4.5, 3.4.18.5 (see also its abstract form 3.4.8.1.2 b)), 3.4.18.5.2, 3.4.18.4.3, 3.4.8.1.4 and §7 (see also 3.13.5.4). 3.4.18.4-6 can be extended to a relative situation (i.e. to a situation where a smooth group is involved, as in the context of 1.1): see 3.4.18.7. It would be too long to fully describe these principles (or properties) here. So we concentrate in outlying the main features.

1.2.4. For the definition of Fontaine's categories $\mathcal{MF}_{[0,1]}(X)$ and $\mathcal{MF}^{\nabla}_{[0,1]}(X)$, of their p-divisible objects, and how such objects are pulled back under pro-étale (affine) morphisms see 2.1, 2.2.1 c), and respectively 3.4.1.1.1. Here X stands for a regular formally smooth affine scheme over $W(k)$ such that either itself or its p-adic completion is endowed with an arbitrary (but fixed) Frobenius lift compatible with σ; sometimes X can be non-affine, and then we do not need to assume that it is endowed with a Frobenius.

Let $U = \mathrm{Spec}(R)$ be an open subscheme of G such that the identity element a_0 of $G(W(k))$ belongs to $U(W(k))$. We assume the existence of an étale morphism $U \to \mathrm{Spec}(W(k)[x_1, ..., x_{d(G)}])$, where $d(G)$ is the relative dimension of G, such that a_0 factors through the closed subscheme of U defined by $x_i = 0$, $i = \overline{1, d(G)}$. Let Φ_R be the Frobenius of the p-adic completion $U^\wedge := \mathrm{Spec}(R^\wedge)$ of U which at the level of rings takes x_i into x_i^p, $\forall i \in \{1, ..., d(G)\}$, and is compatible with the Frobenius σ of $W(k)$ (for the case of a general type of Frobenius Φ_R see 3.4.18.8.3). Let M_R be the p-divisible object of $\mathcal{MF}_{[0,1]}(U)$ defined by the triple

$$(M \otimes_{W(k)} R^\wedge, F^1 \otimes_{W(k)} R^\wedge, \Phi_{M_U}).$$

Here Φ_{M_U} is the Φ_R-linear (Frobenius) endomorphism of $M \otimes_{W(k)} R^\wedge$ which takes an element $m \in M$ into $u(\varphi(m))$, with $u \in G(R^\wedge)$ as the universal element naturally defined by the inclusion $U \hookrightarrow G$. So $\Phi_{M_U}(v_\alpha) = v_\alpha$, $\forall \alpha \in \mathcal{J}$. In their simplified form, the ∇, the moduli, and the surjectivity principles for Shimura σ-crystals can be formulated (cf. 3.4.1.3):

Theorem 3. *There is a pro-étale affine morphism $\ell : U_1 = \mathrm{Spec}(R_1) \to U$, with the special fibre of U_1 a geometrically connected scheme over k, such that:*

a) *The pull back M_{U_1} of M_U to (the p-adic completion of) U_1, is a Shimura p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(U_1)$. In other words there is a unique connection*

$$\nabla : M \otimes_{W(k)} R_1^\wedge \to M \otimes_{W(k)} \Omega_{R/W(k)} \otimes_R R_1^\wedge$$

for which the Frobenius $\Phi_{M_{U_1}}$ of $M \otimes_{W(k)} R_1^\wedge$ (defined by Φ_{M_U}; the Frobenius of $R_1^\wedge$ being the one naturally induced from Φ_R through ℓ) is ∇-parallel (in the usual sense; see [Fa2, ch. 7]). ∇ is integrable and nilpotent mod p. Moreover it respects the G-action;

b) *There is a closed subscheme $\mathcal{B}$ of the special fibre U_k of U not containing the identity element of G_k, with the property that the fibres of ℓ above points of $U_k \setminus \mathcal{B}$ are non-empty.*

From Theorem 3 and [Fa1, 7.1] we deduce the existence of a p-divisible group D_1 over $\mathrm{Spec}(R_1^\wedge)$ corresponding to this p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(U_1)$. As ∇ annihilates v_α, $\forall \alpha \in \mathcal{J}$, D_1 gets a natural structure of a Shimura p-divisible group.

In their full form, the above three principles for Shimura σ-crystals describe how we can choose ℓ to get an universal property and what extra properties it enjoys (subject to this special choice).

1.2.5. Let X be as in 1.2.4. The ∇ principle for an arbitrary object $\mathcal{C}$ of $\mathcal{MF}_{[0,1]}(X)$ shows the existence, in the étale topology of X (i.e. when we pass to a surjective étale affine morphism $X' \to X$), of connections satisfying the equations (see 3.4.1.1.1) needed for $\mathcal{C}$ to be an object of $\mathcal{MF}^\nabla_{[0,1]}(X')$. It also computes the number $\nabla(\mathcal{C})$ of such connections (in the case when the special fibre of X is a local strictly henselian scheme), cf. 3.4.18.4.4, and expresses upper bounds for this number (in the general case), cf. 3.4.18.9. The ∇ principle for a p-divisible object $\mathcal{C}$ of $\mathcal{MF}_{[0,1]}(X)$ shows the existence of such connections in the pro-étale topology of X (i.e. when we pass to a surjective pro-étale affine morphism $X' \to X$), and asserts the existence of a p-divisible group over the p-adic completion of X' whose filtered F-crystal, viewed just as a p-divisible object of $\mathcal{MF}_{[0,1]}(X')$, is the pull back of $\mathcal{C}$ to X'. It also describes how such an X-scheme X' can be obtained: the special fibre of X' is obtained from the special fibre of X by using quasi Artin–Schreier systems, while X' is obtained by lifting what we get mod p. These systems (cf. terminology of 3.4.18.4.6 A) are systems of n equations in n variables x_1,..., x_n over an algebra $R(p)$ over the field $\mathbb{F}_p$ with p elements, of the form

$$x_i = L_i(x_1^p, ..., x_n^p),$$

$i \in \{1, ..., n\}$, where L_i are linear forms (not necessarily homogeneous) in n variables with coefficients in $R(p)$. Here $n \in \mathbb{N}$.

The unicity principle tell us when the above number $\nabla(\mathcal{C})$ is 1. There are two forms of it: one in terms only of the Frobenius of X (see 3.4.18.8), and another one in terms only of the Frobenius of (the module underlying) $\mathcal{C}$ (see 3.4.18.8.1 a)). The integrability principle asserts that all the connections, whose existence is asserted by the ∇ principle, are integrable. The ideas involved in obtaining it can be easily grasped from 3.4.18.4.1. These two principles together with the inducing property of 3.4.18.5 form a natural extension of [Fa2, th. 10].

The moduli principle shows the existence of moduli schemes of connections (satisfying the equations mentioned above) for an arbitrary (p-divisible) object $\mathcal{C}$ of $\mathcal{MF}_{[0,1]}(X)$. The surjectivity principle asserts that, in the case of a p-divisible object, any connected component of the special fibre of a resulting moduli scheme, maps into a dense constructible subset of a connected component of the special fibre of X. The gluing principle has many variants (cf. 3.4.19). One such variant shows that in many cases two (Shimura) filtered F-crystals over the special fibre of a smooth scheme Y over $W(k)$, under some conditions which make them to look similar, are isomorphic over the special fibre of Y_1, where $Y_1 \to Y$ is a pro-étale affine morphism, whose imagine contains an open dense subscheme of the special fibre of Y.

1.2.5.1. The guiding main new idea for obtaining these principles: we work inductively mod p^n, $n \in \mathbb{N}$ (even in the case of a regular ring of formal power series over $W(k)$). What we gain by doing this: the systems of equations we get are of a very simple and practical form (see 3.4.8; in particular see the simple independent lemma 3.4.8.1, whose proof is based on intersection theory). We use these systems to define as well Artin–Schreier systems with coefficients in $\mathbb{F}_p$-algebras and Artin–Schreier fundamental groups of $\mathbb{F}_p$-algebras (see 3.4.18.4.6 B). See 3.4.18.4.6 C and D for the fundamental lemma of the π_1-theory in positive characteristic. The idea behind the surjectivity principle is the constancy property: the quasi Artin–Schreier systems we get working with different powers of p are very close (in shape) one to another, i.e. they differ, cf. 3.4.8.9, only by adding some constants to the linear forms involved (so the abstract surjectivity principle, cf. 3.4.1.8.2 a), applies). For an application to difference varieties see 3.4.8.1.1.

1.2.6. In 3.9.4-5 we present the notion of the refined Lie stable p-rank of a Shimura σ-crystal, and a theorem for it similar to the Theorem 2 above. In 3.10 we introduce the terminology (and notion) of cyclic Shimura Lie σ-crystals to be used in the future, as well as a formulae for them. In 3.12 we show that for many Shimura filtered F-crystals $(M, F^1, \varphi, G, H, \tilde{f})$ (defined in 2.2.10), the Shimura σ_ν-crystal over the algebraic closure ν of the field of fractions of the completion of the special fibre of H in its origin, is a $G_{W(\nu)}$-ordinary σ_ν-crystal (here σ_ν is the Frobenius automorphism of $W(\nu)$). Our hopes for a good (understandable and useful) theory of lifts of Shimura σ-crystals (M, φ, G) which are not G-ordinary (or cyclic diagonalizable; see def. b) of 2.2.1 and 2.2.22) are gathered in 3.13: we define different (too many to be itemized) deviations of Shimura (filtered) (Lie) σ-crystals. The case $p = 2$ is treated in 3.14. Some conclusions, obtained by combining the basic results of 3.1 and 3.4, are included in 3.15 (see also 3.1.8). These conclusions are refined in the case of an algebraically closed field in 4.12.12 and 4.12.12.2 (cf. Theorem 12 of 1.14 below).

1.2.7. The special features of the case $k = \bar{k}$ are presented in 3.11. Using the definitions 2.2.1 b), 2.2.22 and 2.2.12, we have (cf. rm. 5-6) of 2.2.9):

Theorem 4. *If $k = \bar{k}$, then all G-canonical lifts $(M, F^1, g\varphi, G)$ ($g \in G(W(k))$) are all isomorphic and cyclic diagonalizable, and their Shimura filtered Lie σ-crystals are cyclic diagonalizable and of Borel and parabolic type.*

In particular there is a basis $\{e_i | i \in B(M)\}$ of M, and a permutation π of the set $B(M)$, such that $g\varphi(e_i) = p^{\varepsilon_i} e_{\pi(i)}$, where ε_i is 1 or 0 depending on the fact that e_i is or is not in F^1. The proof of Theorem 4 is a simple consequence of 1.2.1 3).

1.2.8. References to Fontaine's categories $\mathcal{MF}_{[a,b]}(*)$, with $a, b \in \mathbb{Z}$, $b \geq a + 2$, are made in 3.4.8.9, 3.4.18.5.5, 3.4.18.7.1, 3.4.18.8, 3.4.18.8.1 b) and 4.14.4. Here we will just point out that most of the properties and principles we get for $\mathcal{MF}_{[0,1]}(*)$, are either not true (in general) or are true in a much weaker form for $\mathcal{MF}_{[a,b]}(*)$.

1.3. For a better presentation of the applications (see §4) of the basic results, we introduce in 2.3 the standard Hodge situation.

We start with an injective map $f : \mathrm{Sh}(G, X) \hookrightarrow \mathrm{Sh}\big(\mathrm{GSp}(W, \psi), S\big)$ of Shimura pairs. Here the pair $(\mathrm{GSp}(W, \psi), S)$ defines a Siegel modular variety, cf. [Va1, 2.5]. Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ with respect to the rational prime $p \geq 3$. We assume the existence of a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W which is crystalline well positioned for the map f (cf. def. 2.3.4). Let $K_p := \{g \in \mathrm{GSp}(W, \psi)(\mathbb{Q}_p) \mid g(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p) = L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p\}$. The hypothesis on $L_{(p)}$ implies that $L_{(p)}$ is a good $\mathbb{Z}_{(p)}$-lattice with respect to f. This means that ψ induces a perfect form $L_{(p)} \otimes_{\mathbb{Z}_{(p)}} L_{(p)} \to \mathbb{Z}_{(p)}$ (i.e. the induced map from $L_{(p)}$ into its dual $L^*_{(p)}$ is an isomorphism), and that the Zariski closure $G_{\mathbb{Z}_{(p)}}$ of G in $\mathrm{GSp}(L_{(p)}, \psi)$ is a reductive group over $\mathbb{Z}_{(p)}$. So K_p is a hyperspecial subgroup of $\mathrm{GSp}(W, \psi)(\mathbb{Q}_p)$ and the intersection $H := \mathrm{GSp}(W, \psi)(\mathbb{Q}_p) \cap K_p$ is a hyperspecial subgroup of $G(\mathbb{Q}_p)$.

We choose a $\mathbb{Z}$-lattice $L_{\mathbb{Z}}$ of W such that ψ induces a perfect form $\psi : L_{\mathbb{Z}} \otimes_{\mathbb{Z}} L_{\mathbb{Z}} \to \mathbb{Z}$ and $L_{(p)} = L_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Let $(v_\alpha)_{\alpha \in \mathcal{J}}$ be a family of tensors in spaces of the form $(W \oplus W^*)^n$, $n \in \mathbb{N}$, such that G is the subgroup of $\mathrm{GSp}(W, \psi)$ fixing the tensors of this family. The choice of the lattice $L_{\mathbb{Z}}$ and of the family $(v_\alpha)_{\alpha \in \mathcal{J}}$ allows the interpretation of $\mathrm{Sh}(G, X)(\mathbb{C})$ as the set of isomorphism classes of principally polarized abelian varieties over $\mathbb{C}$ of dimension e (with $2e = \dim_{\mathbb{Q}}(W)$), carrying a family of Hodge cycles $(w_\alpha)_{\alpha \in \mathcal{J}}$, and satisfying some additional conditions and level structures (cf. [Va2, 4.1]). Let v be a prime of the reflex field $E(G, X)$ dividing p and let $O_{(v)}$ be the localization of the ring of integers of $E(G, X)$ with respect to v.

The triple $\big(f, L_{(p)}, v\big)$ is called a standard Hodge situation.

It is known that the scheme $\mathcal{M}$ over $\mathbb{Z}_{(p)}$ parametrizing isomorphism classes of principally polarized abelian schemes of dimension e over $\mathbb{Z}_{(p)}$-schemes, having level-N symplectic similitude structure for any $N \in \mathbb{N}$ relatively prime to p, together with the canonical action of $\mathrm{GSp}(W, \psi)(\mathbb{A}_f^p)$ on it (via L), is an integral canonical model of $\mathrm{Sh}\big(\mathrm{GSp}(W, \psi), S\big)/K_p$ (see [Va2, 3.2.9]).

The normalization $\mathcal{N}$ of the Zariski closure of $\mathrm{Sh}(G,X)/H$ in $\mathcal{M}_{O_{(v)}}$ is (cf. 2.3.3) an integral canonical model of $\mathrm{Sh}(G,X)/H$ (or of the Shimura quadruple (G,X,H,v); see [Va2, 3.2.6] for the definition of these quadruples). Let $k(v)$ be the residue field of v and let $\mathcal{N}_{k(v)}$ be the special fibre of $\mathcal{N}$. The universal principally polarized abelian scheme over $\mathcal{M}$ gives birth, by pull back, to a principally polarized abelian scheme $(\mathcal{A},\mathcal{P}_{\mathcal{A}})$ over $\mathcal{N}$, which we call special.

1.3.1. The crystalline well positioned property of $L_{(p)}$ for the map f also says: any point $z \in \mathcal{N}(W(k))$ gives birth to a principally polarized abelian scheme (A,p_A) over $W(k)$ (obtained from $(\mathcal{A},\mathcal{P}_{\mathcal{A}})$ by pull back) having a family $(w_\alpha)_{\alpha\in\mathcal{J}}$ of Hodge cycles such that the filtered σ-crystal (M,F^1,φ) (with $M := H^1_{\mathrm{crys}}(A/W(k))$, with φ as the σ-linear endomorphism of M, and with F^1 as the Hodge filtration of M defined by A) gets a natural structure of a Shimura filtered σ-crystal $(M,F^1,\varphi,\tilde{G}_{W(k)})$. In other words:

i) The subgroup of $GL(M)$ obtained by taking the Zariski closure of the subgroup of $GL\big(M[\frac{1}{p}]\big)$ fixing the perfect alternating form $p_A : M\otimes_{W(k)}M \to W(k)(1)$ (induced by the polarization p_A of the abelian variety A), and the family $(t_\alpha)_{\alpha\in\mathcal{J}}$ of (crystalline) de Rham components of $(w_\alpha)_{\alpha\in\mathcal{J}}$ (these are tensors in the tensor algebra of $(M\oplus M^*)[\frac{1}{p}]$), is a reductive group $\tilde{G}_{W(k)}$ over $W(k)$.

ii) There is a cocharacter $\mu : \mathbb{G}_m \to \tilde{G}_{W(k)}$ producing a direct sum decomposition $M = F^1 \oplus F^0$, with $\beta \in \mathbb{G}_m(W(k))$ acting through μ on F^i as the multiplication with β^{-i}, $i=\overline{0,1}$. Moreover F^1 is a proper direct summand of M.

There is a finite nilpotent Galois cover $\mathrm{Spec}(W(k_1))$ of $\mathrm{Spec}(W(k))$ of index of nilpotency at most two and of a very particular type (see 2.3.9 b) and d)) such that $\tilde{G}_{W(k_1)}$ is isomorphic to $G_{W(k_1)}$. Moreover, for many standard Hodge situations we can show (see 2.3.9 c)) that it is in fact isomorphic to $G_{W(k)}$. So, just for the sake of convenience, we make the convention (see 2.3.9.2) that, without any extra reference, we consider from now on only points $\mathrm{Spec}(k) \to \mathcal{N}$ with the property that for any lift of it $\mathrm{Spec}(W(k)) \to \mathcal{N}$, the group we get, as in i) above, is $G_{W(k)}$.

Using this convention, the triple $(M,\varphi,G_{W(k)})$ (resp. the quadruple $(M,F^1,\varphi,G_{W(k)})$) is called the Shimura σ-crystal (resp. the Shimura filtered σ-crystal) attached to the point $y \in \mathcal{N}(k)$ defined by z (resp. attached to z). When we want to emphasize the principal polarization we speak also about the principally quasi-polarized Shimura (resp. Shimura filtered) σ-crystal attached to y (resp. to z). Notation: $(M,\varphi,G_{W(k)},p_A)$ (resp. $(M,F^1,\varphi,G_{W(k)},p_A)$).

From this we get easily that for any regular formally smooth scheme $\tilde{U}$ over $W(k)$, and for every morphism $\tilde{U} \to \mathcal{N}$, we get naturally (by pull back) a principally quasi-polarized Shimura p-divisible group over $\tilde{U}$, cf. 3.2.11.

1.3.2. The deformation principle in its essence says:

Theorem 5. *We assume that a starting assumption (see SA1-2 of 3.4.14) is satisfied. For any point $y : \mathrm{Spec}(k) \to \mathcal{N}$, there is a smooth morphism $\mathcal{N}(y) \to \mathcal{N}$ to which y lifts, and there is a pro-étale morphism $\mathcal{N}(y)^1 \to \mathcal{N}(y)$, surjective mod p, with the special fibre* *of $\mathcal{N}(y)^1$ a geometrically connected scheme over k, and such that the principally quasi-polarized Shimura p-divisible group over the p-adic completion of $\mathcal{N}(y)^1$, obtained by pull back, has a very nice (down to earth) expression. What we mean by this: it is obtained by pulling back (under the p-adic completion of a pro-étale morphism) the principally quasi-polarized Shimura p-divisible group over $U_1^\wedge$ defined by (cf. [Fa1, 7.1]) a principally quasi-polarized Shimura p-divisible object M_{U_1}, produced as in Theorem 3, but starting from the principally quasi-polarized Shimura σ-crystal attached to y (so here U is an open subscheme of the maximal connected subgroup of $G_{W(k)}$ fixing the bilinear form p_A). Moreover y lifts to a morphism* $\mathrm{Spec}(k) \to \mathcal{N}(y)^1$.

We get this principle by combining standard arguments with our strategy of working inductively modulo powers of p. The starting assumption refers to the fact that, working mod p, the variant of Theorem 5, obtained by working with the henselization of the local ring of the k-valued point of $\mathcal{N}_k$ defined by y, is a priori satisfied. In practice it is hard to check if a starting condition is satisfied or not. However the proof of Theorem 5 involves some very important techniques, and its variants modulo powers of p (see 3.4.14.4), involving no assumption, are also very useful. Moreover, the fact that a starting condition is satisfied or not, is at the root of many extremely important problems (see §5-8).

Of course there are variants of this principle: we work with a point $y_{H_0} : \mathrm{Spec}(k) \to \mathcal{N}/H_0$ (here H_0 is a compact open subgroup of $G(\mathbb{A}_f^p)$ such that the quotient scheme $\mathcal{N}/H_0$ is smooth over $O_{(v)}$ and $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ descends to $\mathcal{N}/H_0$), or with an étale morphism $\mathcal{N}(y) \to \mathcal{N}$, cf. 3.4.14 and 3.4.14.1-2.

1.4. As a straightforward consequence of Theorem 1, we are able to extend and reobtain the well known results (pertaining to integral canonical models of Siegel modular varieties and universal principally polarized abelian schemes over them) concerning the existence of an ordinary type and the existence of the canonical lift of a (principally polarized) abelian variety of ordinary type, to any integral canonical model $\mathcal{N}$ of a Shimura variety $\mathrm{Sh}(G, X)$ of Hodge type and to any special principally polarized abelian scheme $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ over $\mathcal{N}$, arising (as in 1.3) from a standard Hodge situation $(f, L_{(p)}, v)$. With the notations of 1.3, we get three things.

a) First we get a G-ordinary type (cf. 4.1). It depends only on f and v, but not on the choice of $L_{(p)}$. It is the formal isogeny type of abelian varieties (obtained from $\mathcal{A}$ by pull back) over geometric points of a Zariski dense open subscheme of $\mathcal{N}_{k(v)}$. It can be described as follows.

The isomorphism $L_{(p)} \tilde{\to} L^*_{(p)}$ induced by ψ allows us to identify $G_{\mathbb{Z}_{(p)}}$ with a subgroup of $GL(L^*_{(p)})$. Let $L^*_p := L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$; so $G_{\mathbb{Z}_p}$ is naturally a subgroup of $GL(L^*_p)$. Let T be a maximal torus of $G_{\mathbb{Z}_p}$ such that:

1) there is a cocharacter $\mu : \mathbb{G}_m \to T_{W(k(v))}$ which over an embedding of $W(k(v))$ in $\mathbb{C}$ (extending the composite inclusion $O_{(v)} \subset E(G, X) \subset \mathbb{C}$) is $G(\mathbb{C})$-conjugate to the cocharacters $\mu^*_x : \mathbb{G}_m \to G_{\mathbb{C}}$, $x \in X$, which are the cocharacters μ_x mentioned in [Va2, 2.2], but in the dual context (so they are the cocharacters of $G_{\mathbb{C}}$ defining the Hodge $\mathbb{Q}$–structures on W^* associated to points $x \in X$);

2) there is a Borel subgroup B of $G_{\mathbb{Z}_p}$ whose Lie algebra leaves invariant the F^1-filtration of $L_p^* \otimes_{\mathbb{Z}_p} W(k(v))$ defined by μ, i.e. $\mathrm{Lie}(B) \subset F^0(\mathrm{Lie}(G_{W(k(v))}))$.

We recall that μ gives birth to a direct sum decomposition $L_p^* \otimes_{\mathbb{Z}_p} W(k(v)) = F^1 \oplus F^0$, and that $\beta \in \mathbb{G}_m(W(k(v)))$ acts through μ on F^i as the multiplication with β^{-i}, $i = \overline{0,1}$.

The G-ordinary type we get is the formal isogeny type associated to the σ-crystal

$$(L_p^* \otimes_{\mathbb{Z}_p} W(k(v)), \varphi)$$

defined by $\varphi := \sigma \circ \mu(\frac{1}{p})$. Here σ acts as identity on L_p^* and as Frobenius on $W((k(v))$, and $\frac{1}{p} \in \mathbb{G}_m\left(W\left(k(v)\left[\frac{1}{p}\right]\right)\right)$.

b) Second we get G-ordinary points of $\mathcal{N}_{k(v)}$ (cf. 4.2). These are the points, with values in a field, with the property that the abelian varieties over them, obtained from $\mathcal{A}$ by pull back, have the above G-ordinary type as their formal isogeny type.

c) Third we get G-canonical lifts of G-ordinary points of $\mathcal{N}_{k(v)}$ with values in perfect fields (cf. 4.4). These G-canonical lifts are points of $\mathcal{N}$ with values in Witt rings over perfect fields.

The G-ordinary type we obtain is a usual ordinary type (i.e. it is the formal isogeny type $e(1,0) + e(0,1)$) if and only if $k(v)$ is $\mathbb{F}_p$ (cf. the property P1 of 4.6). If $k(v) = \mathbb{F}_p$ then the abelian variety over $W(k)$ obtained from $\mathcal{A}$ by pull back through a G-canonical lift of a G-ordinary point (with values in the perfect field k) of $\mathcal{N}_{k(v)}$, is the canonical lift of an ordinary abelian variety (cf. the property P2 of 4.6).

1.4.1. A natural question arises: for which other points of $\mathcal{N}_{k(v)}$ with values in k we can (identify and) define naturally a lift of them to $W(k)$-valued points of $\mathcal{N}$, which are unique subject to some conditions? In 4.4.13, as a digression, we answer partially this question: we define U-ordinary points and U-canonical lifts. The U-ordinary points of $\mathcal{N}_{k(v)}$ with values in algebraically closed fields are those points whose attached Shimura σ-crystals have a unique lift of quasi CM type in the sense of 2.2.17, and these unique lifts define the U-canonical lifts. Of course any G-ordinary point is an U-ordinary point; but there are tremendously many examples of SHS's for which we have (plenty of) U-ordinary points which are not G-ordinary. These examples do not show up in the classical setting of Siegel modular varieties: they typically show up when $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ has certain simple factors which are not absolutely simple.

The passage from G-ordinary points to U-ordinary points is achieved via: we move from keeping track of parabolic Lie subalgebras of $\mathrm{Lie}(G)$ corresponding to non-negative slopes to (reductive) Lie subalgebras of $\mathrm{Lie}(G)$ corresponding only to the slope zero. When these Lie subalgebras are abelian, we get a subclass of U-ordinary points which we call T-ordinary points. The G-ordinary points are just sporadically T-ordinary points (see P7 of 4.6).

Our hopes for good (i.e. unique in some sense) lifts for other points of $\mathcal{N}_{k(v)}$ are expressed in Q3 of 4.5.15.

1.5. To any point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ we attach (cf. 2.3.10) a Shimura Lie σ-crystal $(\mathfrak{g}, \varphi)$, with $\mathfrak{g}$ the Lie algebra of $G_{W(k)}$ (cf. 1.3.1). Any lift $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$ of y, makes $(\mathfrak{g}, \varphi)$ to be a filtered Lie σ-crystal, i.e. it makes it to be a p-divisible object of $\mathcal{MF}_{[-1,1]}(W(k))$. This means that we get a filtration $0 = F^2(\mathfrak{g}) \subset F^1(\mathfrak{g}) \subset F^0(\mathfrak{g}) \subset F^{-1}(\mathfrak{g}) = \mathfrak{g}$ such that $\varphi\big(\frac{1}{p}F^1(\mathfrak{g}) + F^0(\mathfrak{g}) + p\mathfrak{g}\big) = \mathfrak{g}$. $F^0(\mathfrak{g})$ is a parabolic Lie subalgebra of $\mathfrak{g}$ and $F^1(\mathfrak{g})$ is its nilpotent radical. b) of Theorem 1 implies that the point y is a G-ordinary point iff there is a lift z of it to $W(k)$ which makes the Lie σ-subcrystal of $(\mathfrak{g}, \varphi)$ corresponding to non-negative slopes to be a p-divisible object of $\mathcal{MF}_{[0,1]}(W(k))$. Such a lift z, when exists, is unique and defines (via 2.3.17) the G-canonical lift of y.

The G-ordinary points of $\mathcal{N}_{k(v)}$ and the G-canonical lifts of the G-ordinary points have all the usual properties of ordinary points (and of their canonical lifts) of an integral canonical model of a Siegel modular variety with respect to p, which can be read out from the Shimura Lie σ-crystals attached to points (with values in perfect fields). As examples of this philosophy see 1.7-9 and 4.4.4. Of course these properties have to be reformulated accordingly.

1.6. These attached Shimura Lie σ-crystals allow us (cf. 4.5) to achieve a stratification (called the canonical Lie stratification) of $\mathcal{N}_{k(v)}$ into $G(\mathbb{A}_f^p)$-invariant locally closed subschemes, indexed by Newton polygons of the attached Shimura Lie σ-crystals $(\mathfrak{g}, \varphi)$. It is similar (in nature and properties) to the well-known stratification of the special fibres of integral canonical models of Siegel modular varieties.

For any representation $\rho : G_{\mathbb{Z}_p} \to GL(N)$, with N a free $\mathbb{Z}_p$-module of finite rank, we define (cf. 4.5.4) a ρ-stratification of $\mathcal{N}_{k(v)}$ in $G(\mathbb{A}_f^p)$-invariant locally closed subschemes. The intersection of all these ρ-stratifications defines the absolute stratification. This absolute stratification is the most refined stratification which can be obtained naturally (i.e. in an adequate Shimura context, involving representations of $G_{\mathbb{Z}_p}$) with the help of Newton polygons.

We also introduce (4.5.2) the refined canonical Lie stratification and show (see 4.5.6.1) that it coincides with the absolute stratification.

The G-ordinary points of $\mathcal{N}_{k(v)}$ are points of the generic, Zariski dense open stratum (it does not matter which stratification we use, cf. 4.5.6 8)). In the case of the special fibre of an integral canonical model of a Siegel modular variety, all non-trivial stratifications (i.e. stratifications which do not consist just from one stratum) introduced above coincide with the usual Newton polygon stratification; but this is not true for an arbitrary Shimura variety of Hodge type (see 4.5.4-6).

A long list of other types of stratifications is presented in 4.5.10 and 4.5.15-17.

1.6.1. In 4.5.12 we enlarge the class of U-ordinary points: we introduce toric points in the same manner of having lifts of quasi CM type, but we do not require the unicity of such lifts.

1.7. In 4.3 we restate 1.2.2 in the case of a standard Hodge situation. For a refined restatement we use the refined Lie stable p-rank introduced in 3.9.4; in other words, each simple factor of $G^{\mathrm{ad}}_{\mathbb{Z}_p}$ can be used to define (independent) Lie stable p-ranks of its corresponding cyclic factors (see 3.10.1 for definition) of the Shimura σ-crystals attached to points of $\mathcal{N}_{k(v)}$.

1.8. In 4.6 we include some examples to illustrate the general theory presented in 4.1-5 and to point out the complexities which are arising. We also list the main properties of the general theory. As a simple application of 1.4 b) we present (cf. 4.6.9) a general principle concerning the non-ordinary reduction of an abelian variety A over a number field E, with respect to primes of E:

Theorem 6 (the non-ordinary reduction criterion). *If the reflex field of the Shimura variety defined (see [Va2, 2.12 3)]) by the Mumford–Tate group of A, is not $\mathbb{Q}$, then A has a non-ordinary reduction with respect to an infinite number of primes of E.*

4.6.9.4 says very precisely how "to cook" such primes.

1.9. We consider a standard Hodge situation $\big(f, L_{(p)}, v\big)$ as in 1.3. Let v^{ad} be the prime of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ divided by v. In 1.9-10 we assume that $k=\bar{k}$. For any point $y:\mathrm{Spec}(k)\hookrightarrow \mathcal{N}\times_{O_{(v)}} W(k)$ defined by a G-ordinary point $y_0\colon\mathrm{Spec}(k)\to\mathcal{N}_{k(v)}$, the principally quasi-polarized filtered F-crystal (defined by $(\mathcal{A},\mathcal{P}_{\mathcal{A}})$) over the completion $k[[X_1,\ldots,X_d]]$ of the local ring of y in $\mathcal{N}_k$ (with $d:=\dim\mathrm{Sh}(G,X)$), can be put in a very special form (cf. 4.7.2). Using this we get (see 4.7.30 6)):

Theorem 7. *The formal moduli scheme of G-deformations (i.e. deformations factoring through $\mathcal{N}$) of the principal polarized abelian variety (A_y, p_{A_y}) over k (obtained from $(\mathcal{A},\mathcal{P}_{\mathcal{A}})$ through the point y_0) over Artinian local $W(k)$-algebras, having k as their residue field, is naturally isomorphic to the formal multiplicative group $\mathbb{G}_m^{d\wedge}/W(k)$. Under such an isomorphism, the G-canonical lift becomes the origin.*

This is part of the general theory of crystalline coordinates for p-divisible groups over k, developed in 4.7.30. This is a generalization of the main result of [No1] (in the less general case of a standard Hodge situation) and of [De2, 2.1.3]. We obtain G-multiplicative (crystalline) coordinates. When $k(v^{\mathrm{ad}})=\mathbb{F}_p$, the situation is very close to the one in [De2, 1.4] (cf. 4.7.5-29; see also its generalization in 4.7.30 4)): we obtain G-canonical (crystalline) coordinates.

As a further generalization we have (cf. 4.7.33.1):

Proposition. *If A_k is an abelian variety over k of dimension d, and if it has a lift A (over $W(k)$) of quasi CM type (cf. def. 2.2.17), then the formal moduli scheme of deformations of A_k is naturally isomorphic to the formal torus $\mathbb{G}_m^{d^2}{}^\wedge/W(k)$, with the abelian variety A corresponding to its origin.*

This is the most general context in which we can (presently) prove the existence of good (useful) crystalline (natural) coordinates. For variants of this Proposition, including the additive case, see 4.7.33.2 and 4.7.30 5).

For a list of the new types of equations (differential or not) showing up in the theory of crystalline coordinates when we move from an ordinary context to a cyclic diagonalizable one, see end of 4.7.30 1). For the case of K3 surfaces (including the case $p = 2$) see end of 4.7.30 2).

1.10. In 4.8 we present the following Galois property of G-ordinary points of $\mathcal{N}_{k(v)}$; it answers a question of C. -L. Chai. We consider a standard Hodge situation $(f, L_{(p)}, v)$. With the notations of 1.3, we consider a G-ordinary point $\mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ and a lift of it $z : \mathrm{Spec}(V) \to \mathcal{N}$, with V a discrete valuation ring which is a finite flat extension of $W(k)$. Let $K := V\left[\frac{1}{p}\right]$. Let A_V be the abelian variety over V, obtained from $\mathcal{A}$ by pull back through the point z. We have:

Theorem 8. *There is a finite field extension K_1 of K such that the Galois representation $\rho : \mathrm{Gal}(\bar{K}/K_1) \to GL(H^1_{\acute{e}t}(A_{\bar{K}}, \mathbb{Z}_p))(\mathbb{Z}_p) = GL(L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p)(\mathbb{Z}_p)$ factors through the group of $\mathbb{Z}_p$-valued points of a solvable subgroup of $G_{\mathbb{Z}_p}$.*

The identification $H^1_{\acute{e}t}(A_{\bar{K}}, \mathbb{Z}_p) = L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ used here is not canonical, cf. [Va2, 5.1.2]. The proof of Theorem 8, as R. Pink has pointed out to us, is a direct consequence of 4.4.7 and of 4.7.2 (and so of Theorem 7). For variants of the above Galois property (including the case of a finite field) see 4.8.3 d) and e).

1.11. The phenomena of 1.4-6 are an intrinsic property of $\mathcal{N}$ (and of $\mathcal{N}_{k(v)}$), not depending on the special principally polarized abelian scheme $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ we got over $\mathcal{N}$ (only the notion of a G-ordinary type does depend on it), and so they can be extended to special fibres of integral canonical models of Shimura varieties of preabelian type. We have (cf. 4.9.8):

Theorem 9. *Let $\mathcal{N}^1_{k(v^1)}$ be the special fibre of an integral canonical model $\mathcal{N}^1$ of a Shimura variety $\mathrm{Sh}(G^1, X^1)$ of preabelian type with respect to a prime v^1 of $E(G^1, X^1)$ dividing a rational prime $p > 2$. Then $\mathcal{N}^1_{k(v^1)}$ has a refined canonical Lie stratification in $G^1(\mathbb{A}^p_f)$-invariant locally closed subschemes. It is functorial with respect to finite maps between Shimura quadruples; moreover in the case of a standard Hodge situation we regain the refined canonical Lie stratification mentioned in 1.6. The points (with values in fields) of the Zariski dense open stratum are called G^1-ordinary points of $\mathcal{N}^1_{k(v^1)}$, and any such point with values in k, has a uniquely defined G^1-canonical lift $\mathrm{Spec}(W(k)) \to \mathcal{N}^1$.*

In 4.9 we present a proof of this for primes $p \geq 5$ (for the case $p = 3$ see §6). The proof relies on the fact that for any Shimura quadruple (G_0, X_0, H_0, v_0) (with (G_0, X_0) defining a Shimura variety of adjoint abelian type, with v_0 a prime of $E(G_0, X_0)$ dividing a rational prime $p \geq 3$ such that G_0 is unramified over $\mathbb{Q}_p$, and with H_0 a hyperspecial subgroup of $G_0(\mathbb{Q}_p)$), there is a standard Hodge situation $(f^0, L^0_{(p)}, v^0)$, with f^0: $\mathrm{Sh}(G^0, X^0) \hookrightarrow \mathrm{Sh}(\mathrm{GSp}(W, \psi), S)$ an injective map such that $(G^{0\mathrm{ad}}, X^{0\mathrm{ad}}) = (G_0, X_0)$ and v^0 divides v_0 (for $p \geq 5$ cf. [Va2, 6.4.2] and 2.3.6; for $p = 3$ cf. §6).

1.11.1. Using Theorem 9 and its proof, we define Shimura adjoint (filtered) Lie σ-crystals attached to points of $\mathcal{N}^1$ (see 4.9.17), and so we define toric points (in particular U-ordinary and T-ordinary points) of $\mathcal{N}^1_{k(v^1)}$ (see 4.9.17.2 and 4.9.17.5), as well as U-canonical lifts of U-ordinary points with values in perfect fields.

1.12. We do expect (cf. 4.10.1) that the open stratum of $\mathcal{N}^1_{k(v^1)}$ is the smallest $G^1(\mathbb{A}_f^p)$-invariant open subscheme of $\mathcal{N}^1_{k(v^1)}$ (cf. also [Ch, p. 441]). This is proved for Siegel modular varieties in [Ch]. Using Theorem 9, we prove it for Shimura curves (cf. 4.10.3). For a more substantial approach to the general case see 4.10.5.

1.13. In 4.11 we prove (this is standard) the functorial behaviour of G-ordinary points and of G-canonical lifts of them. Let $f_0 : (G, X, H, v) \hookrightarrow (G_1, X_1, H_1, v_1)$ be an injective map between two Shimura quadruples of Hodge type, having integral canonical models $\mathcal{N}$, and respectively $\mathcal{N}_1$. We assume that v is relatively prime to 2. So we have a morphism (still denoted by) $f_0 : \mathcal{N} \to \mathcal{N}_1$ (cf. [Va2, rm. 4) of 3.2.7]). We also assume the existence of an injective map $f_1 : (G_1, X_1, H_1, v_1) \hookrightarrow \big(\mathrm{GSp}(W, \psi), S, \mathrm{GSp}(W, \psi)(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p), p\big)$ (with $L_{(p)}$ a $\mathbb{Z}_{(p)}$-lattice of W for which $\psi : L_{(p)} \otimes_{\mathbb{Z}_{(p)}} L_{(p)} \to \mathbb{Z}_{(p)}$ is a perfect form) such that $(f_1, L_{(p)}, v_1)$ and $(f_1 \circ f_0, L_{(p)}, v)$ are standard Hodge situations. So the Zariski closure of G (resp. G_1) in $\mathrm{GSp}(L_{(p)}, \psi)$ is a reductive groups $G_{\mathbb{Z}_{(p)}}$ (resp. $G_{1\mathbb{Z}_{(p)}}$). Let $k(v)$ and $k(v_1)$ be the residue field of v and respectively of v_1. We have:

Theorem 10. *A G-ordinary point of $\mathcal{N}_{k(v)}$ is mapped by f_0 into a G_1-ordinary point of $\mathcal{N}_{1k(v_1)}$, and a G-canonical lift of $\mathcal{N}$ is mapped by f_0 into a G_1-canonical lift of $\mathcal{N}_1$ iff $k(v) = k(v_1)$ and there is a torus T of $G_{\mathbb{Z}_p}$ such that:*

1) there is a cocharacter $\mu : \mathbb{G}_m \hookrightarrow T_{W(k(v))}$, which over an embedding of $W(k(v))$ in $\mathbb{C}$, extending the composite inclusion $O_{(v)} \subset E(G, X) \subset \mathbb{C}$, is $G(\mathbb{C})$-conjugate to the cocharacters μ_x^*, $x \in X$, defined as in 1.4 (see also 2.3.1);

2) there is a Borel subgroup B_1 of $G_{1\mathbb{Z}_p}$ containing a Borel subgroup of $G_{\mathbb{Z}_p}$ and whose Lie algebra leaves invariant the F^1-filtration of $L^*_{(p)} \otimes_{\mathbb{Z}_p} W(k(v))$ defined by μ (as in 1.4 a)).

1.14. In 4.12-13 we present two important applications inspired from the deformation principle of 3.4.14. First, in 4.12 we formulate the integral Manin problem for a standard Hodge situation, as well as for integral canonical models of Shimura varieties of preabelian type with respect to primes relatively prime to 2. For a standard Hodge situation $(f, L_{(p)}, v)$, with the notations of 1.3, the problem is to determine all the principally quasi-polarized Shimura σ-crystals attached to k-valued points of (a connected component of) $\mathcal{N}_k$; here $k = \bar{k}$. A solution (cf. §11; see 4.12 for some particular cases) to this problem is provided by the Milne's conjecture (see d) of 4.4.1 3) and 1.16) and by the completion property. The completion property is stated in 3.4.15 A as an expectation, and is proved in §11, starting from the isogeny property referred below in 1.22; see the proof of 4.12.12 for a sample.

For different local (resp. global) integral Manin problems see 4.12.14-19 (resp. 4.12.21). For a new and simple solution to the original Manin problem we refer to 4.12.12.4.

Second, in 4.13 we present (partially; to be continued in §9-10, cf. 1.21) the invariance principle. Let $p \geq 3$ be a rational prime. Let (G_i, X_i, H_i, v_i), $i = \overline{1, 2}$, be two Shimura quadruples of preabelian type, with v_i dividing p. Let $\mathcal{N}_i$ be the integral canonical model of (G_i, X_i, H_i, v_i), $i = \overline{1, 2}$. Let $H_{0i} \subset G_i(\mathbb{A}_f^p)$ be a subgroup such that the subgroup $H_{0i} \times H_i$ of $G_i(\mathbb{A}_f)$ is smooth for the pair (G_i, X_i) (see def. [Va2, 2.11]), $i = \overline{1, 2}$. Let

k be a perfect field containing $k(v_1)$ and $k(v_2)$. Let $y_i : \mathrm{Spec}(k) \to \mathcal{N}_{ik}/H_{0i}$, $i = \overline{1,2}$, be points such that the non-trivial part of the Shimura adjoint Lie σ-crystal attached to y_1 is isomorphic to the non-trivial part of the Shimura adjoint Lie σ-crystal attached to y_2 (for definitions cf. 2.3.10 and 3.10.1). Let O_{y_i} be the local ring of y_i (in $\mathcal{N}_{iW(k)}$), and let $O_{y_i}^{h\wedge}$ be the p-adic completion of its henselization, $i = \overline{1,2}$. The refined canonical Lie stratification of $\mathcal{N}_{ik(v_i)}$ makes $\mathrm{Spec}(O(p)_{y_i}^h) := \mathrm{Spec}(O_{y_i}^{h\wedge}/pO_{y_i}^{h\wedge})$ to be a stratified scheme. One variant of the local form of the invariance principle is:

Theorem 11. *We assume $k = \bar{k}$. Let $n \in \mathbb{N}$. There is an isomorphism*

$$\rho : \mathrm{Spec}(O_{y_1}^{h\wedge}) \tilde{\to} \mathrm{Spec}(O_{y_2}^{h\wedge})$$

inducing an isomorphism

$$\rho(p) : \mathrm{Spec}(O(p)_{y_1}^h) \tilde{\to} \mathrm{Spec}(O(p)_{y_2}^h)$$

of stratified schemes, and such that the pull back under ρ of the non-trivial part of the Shimura adjoint filtered Lie F-crystal over $\mathrm{Spec}(O(p)_{y_2}^h)$ *taken mod p^n, defined by the natural morphism* $\mathrm{Spec}(O_{y_2}^{h\wedge}) \to \mathcal{N}_2$*, is isomorphic to the non-trivial part of the Shimura adjoint filtered Lie F-crystal over* $\mathrm{Spec}(O(p)_{y_1}^h)$ *taken mod p^n, defined by the natural morphism* $\mathrm{Spec}(O_{y_1}^{h\wedge}) \to \mathcal{N}_1$ *(see 4.9.20 and 4.13 for the definition of these Lie F-crystals).*

As an application of what the completion property says in some particular cases in which G^{ad} is of A_n type, for future references, we state here the combined version of 4.12.12, 4.12.12.2 and 4.12.13 2). We have:

Theorem 12. *We assume $p \geq 2$ and that $k = \bar{k}$. Then for any p-divisible group D over k, there is a connected projective smooth scheme $Z(D)$ over $W(k)$ and a p-divisible group $D_{Z(D)}$ over $Z(D)$ such that:*

- *$D_{Z(D)}$ is versal in each k-valued point of $Z(D)$;*
- *in a k-valued point of $Z(D)$, $D_{Z(D)}$ becomes isomorphic to D.*

If $Z(D)$ is not $\mathrm{Spec}(W(k))$ *itself (i.e. if D and its dual are not étale), then we can assume that $Z(D)$ is of general type.*

1.15. Some afterthoughts are presented in 4.14 and in the appendix. For the case of a $p = 2$ standard Hodge situation introduced in 2.3.18, see 4.12.12.1 and 4.14.3. In the appendix we make the connexion and comparison with [RR]. Two proofs of Theorem 1 are presented. The first one starts in 3.1 and ends in 3.5.14 (it is less than 12 pages), via 3.6; it completely avoids the use of 3.4. The second one is just a more elaborated form of the first one, in the sense that it leads to a much more stronger form of d) of Theorem 1 (cf. 3.15). It starts in 3.1 and ends in 3.7.5; it uses from 3.4 just the part till 3.4.11 inclusive. As 3.4 grew much more than expected initially, we recommend that after the reading of 3.3, the reader should skip to 3.5-6, and only after that to proceed with 3.4, 3.7, 3.8, etc. Also for 4.1-8 and 4.11 which form a sequence, continuing 2.3, we need from 3.4 just 3.4.6: 3.4.6 is independent of the other part of 3.4 except of the first paragraph of 3.4.0. The parts 3.4.13-16 are needed just in 3.4.19, in 4.3.6 (as an exemplification) and in 4.12-13. They are independent of what follows after them in 3.5-15; so we do not hesitate to use for their proofs results stated in 3.5-12, and so their reading should procede only after the reading of 3.5-12; we felt that it is appropriate to include them as part of 3.4, for having the deformation part of the paper presented fluently. The sequence 3.4.17-20 is unrelated to what follows after it; so it can be read at any time after 3.4.1-9, as well as it can be skipped. For the present work we were motivated by very concrete problems, imerging from the theory of Shimura varieties. Whenever we were aware from the very beginning that the things work out in a more general context, we worked in such general context; when not, we stated (or presented) the things in the more general context as remarks following the "simpler" context. We tried to present the ideas in the most natural order. But we did not hesitate to point out different connexions between them, which somehow were going against this order.

There are two facts stated (and referred) here whose proofs are postponed to future papers: 4.2.8.1 and d) of 4.4.1 3); they are labeled with a $*$. Whenever we use them we present a second proof entirely independent of them: the only exception is the starred answer of 4.12.5, as it is easy to restate it in a way which does not rely on d) of 4.4.1 3)) (see the paragraph following it). Also we would like to point out that for the majority of cases, 4.4.7 is equally good as d) of 4.4.1 3). So the paper is self contained modulo the existence of integral canonical models of Shimura quadruples (G, X, H, v) of preabelian type with v dividing $p = 3$; warning: the limitations of [Va2, 6.8.6] are removed in 4.6.11. In other words, in 4.6.11 we prove that the criterion [Va2, 6.8.2 b)] holds; the proof relies on the simple fact that the G-canonical lifts of G-ordinary points with values in $\mathbb{F}$, in the context of a standard PEL situation (see def. 2.3.5), are having complex multiplication (cf. 4.4.5). As a conclusion we get (cf. 4.6.11 B):

Fact. *Everywhere in [Va2] the stars can be removed.*

So the reader, if desires, can everywhere we are dealing with:

– a standard Hodge situation $(f, L_{(p)}, v)$ such that v divides 3 and not all the simple factors of G^{ad} are of some A_n Lie type (cf. 2.3.5.1), can add the assumption that $\mathcal{N}$ a priori exists;

– a Shimura quadruple (G, X, H, v) such that v divides 3 and not all the simple factors of G^{ad} are of some A_n Lie type, can add the assumption that there is a standard Hodge situation $(f^1, L^1_{(p)}, v^1)$ such that $(G^{1\mathrm{ad}}, X^{1\mathrm{ad}}, H^{1\mathrm{ad}}, v^{1\mathrm{ad}}) = (G^{\mathrm{ad}}, X^{\mathrm{ad}}, H^{\mathrm{ad}}, v^{\mathrm{ad}})$ and $G^{1\mathrm{der}}$ is maximal among the isogeny covers of G^{ad} allowed by the abelian type (i.e. the analogue of [Va2, 6.4.2] holds).

Due to the fact that the results presented in this first part, have been largely detailed in this first chapter, we do no stop at the beginning of the other chapters §2-4, to outline their contents.

1.16. We present now in a very brief manner the contents of the other two parts. We recall that without stated the things otherwise we assume $p \geq 3$.

In §5 we elaborate on the notion of Shimura p-divisible groups introduced in 2.2.20, and prove the conjecture of Milne stated in [Va2, 5.6.6] (i.e. we prove d) of 4.4.1 3)), in the wider context of Shimura p-divisible groups. We also present a general theory of representation well behaved family of tensors (to be compared with [Va2, 4.3]).

As a corollary of Milne's conjecture we get the integrality theorem of Shimura p-divisible groups, and so also the integrality theorem of Shimura varieties of Hodge type. Here we formulate it only for a $W(k)$-valued point (it can be stated also for a V-valued point, with V a discrete valuation ring which is a finite flat extension of $W(k)$, or for the rings $Re(V)(k)$ and $\tilde{R}e(V)(k)$ defined similarly as in [Va2, 5.2.1], but for k instead of $\mathbb{F} := \overline{\mathbb{F}_p}$; here $e(V) := [V : W(k)]$) of an integral canonical model of a Shimura variety of Hodge type.

If $(f, L_{(p)}, v)$ is a standard Hodge situation, then, with the notation of 1.3, the p-component of the étale component of w_α is integral (i.e. an element of the tensor algebra of $H^1_{\acute{e}t}(A_{\bar{K}_0}, \mathbb{Z}_p) \oplus (H^1_{\acute{e}t}(A_{\bar{K}_0}, \mathbb{Z}_p))^*$ (with $\bar{K}_0$ the algebraic closure of $W(k)[\frac{1}{p}]$) iff the de Rham component t_α of w_α is integral (i.e. it is an element of the tensor algebra of $M \oplus M^*$). What is worth remarking: there is no restriction on the degree of w_α, $\alpha \in \mathcal{J}$, and it also holds for $p = 3$.

We apply the conjecture of Milne to show how, for a standard Hodge situation $(f, L_{(p)}, v)$, and for a representation $\rho : G_{\mathbb{Z}_p} \to GL(N)$ (with N a finitely generated $\mathbb{Z}_p$-module, not necessarily free) there is naturally associated and object (if N as a set is finite) or a quasi p-divisible object (otherwise) $\mathfrak{C}_\rho$ of $\mathcal{MF}^\nabla(\mathcal{N})$ (with $\mathcal{MF}^\nabla(\mathcal{N})$ as in [Fa1, §2]; of course the necessary precautions are taken in the case we do not get something to which [Fa1, 2.3] applies); here by quasi p-divisible object we mean the extension of a non-trivial p-divisible object by an object. If N is a free $\mathbb{Z}_p$-module of finite rank then $\mathfrak{C}_\rho$ is a filtered F-crystal on $\mathcal{N}_{k(v)}$ (cf. the conjecturally expected result mentioned in [Ch, end of $(Q1)_D$]). The association $\rho \to \mathfrak{C}_\rho$ is tannakian and functorial. It is uniquely determined by requesting two other natural properties to be satisfied.

We introduce the notion of automorphic vector bundles on an integral canonical model of a Shimura variety. We use the existence of $\mathfrak{C}_\rho$ to prove the existence and unicity of automorphic vector bundles on $\mathcal{N}$ (associated to representations of $G_{\mathbb{Z}_{(p)}}$ on $\mathbb{Z}_{(p)}$-free modules of finite rank, or to representations of $G_{O_{(v)}}$ on $O_{(v)}$-free modules of finite rank) or on $\mathcal{N} \times O_{(v_1)}$ (with $O_{(v_1)}$ a local ring which is an étale $O_{(v)}$-algebra) (associated to representations of $G_{O_{(v_1)}}$ on $O_{(v_1)}$-free modules of finite rank). We also extend this to Shimura varieties of preabelian type, under slight conditions (as in [Mi4, ch. III]) on the representations involved.

1.17. In §6 we prove that for an injective map $f : (G, X) \hookrightarrow \big(\mathrm{GSp}(W, \psi), S\big)$, any $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ (of W) good with respect to f, is automatically crystalline well positioned for the map f. We apply this for proving the existence of integral canonical models of Shimura varieties of preabelian type with respect to primes dividing 3, and the existence of integral canonical models of Shimura varieties of Hodge type with respect to primes dividing $p \geq 3$ and (plenty of) non-hyperspecial subgroups.

1.18. In §7 we introduce a method by which, from particular Shimura σ-crystals, we construct connected normal schemes over local rings which are formally pro-étale over $\mathbb{Z}_{(p)}$, with $p \geq 2$, which either have an open (dense) set of formally smooth points, or have the extension property. By this method we recover all the connected components of integral canonical models of Shimura varieties of Hodge type. We (conjecturally) expect to get some other interesting integral models, even smooth ones. The motivation of this expectation is based on the existence of Shimura σ-crystals (M, φ, G) for which the derived group G^{der} of G (cf. the notations of 1.1) is of a special type (of E_7 Lie type, etc.). We call these normal schemes twisted connected Shimura varieties. These varieties are supposed to lead to the construction of integral canonical models of Shimura varieties of special type (see Definitions 2 of [Va2, 2.5]). They either lead to very beautiful mathematics or provide a negative answer to Oort conjecture [Oo1, p. 6].

1.19. In §8, we apply some of the ideas of §3, §5 and §7 to prove the Mumford–Tate conjecture for certain abelian varieties over number fields. We also present a simple strategy meant to prove the following general criterion (it is known that it implies the Mumford–Tate conjecture):

Expectation (the split criterion). *Let A be an abelian variety over a number field E. There is $N(A) \in \mathbb{N}$ (very easily expressable in terms of A) such that if ℓ is a prime greater or equal to $N(A)$, and if the reductive group $G_{\mathbb{Q}_\ell}$ over $\mathbb{Q}_\ell$, defined as the connected algebraic envelope of the Galois representation $\rho : \mathrm{Gal}(\bar{E}/E) \to GL(H^1_{\acute{e}t}(A_{\bar{E}}, \mathbb{Q}_\ell))(\mathbb{Q}_\ell)$, is a split reductive group, then the Mumford–Tate conjecture is true for A with respect to ℓ, i.e. we have $G_{\mathbb{Q}_\ell} = H_{A\mathbb{Q}_\ell}$. Here H_A is the Mumford–Tate group of A.*

1.20. In §9 we introduce the notion of the null stratum of the special fibre of an integral canonical model of a Shimura variety of preabelian type. This is the stratum of the refined canonical Lie stratification, which is "opposite"to the dense open stratum, and which is the natural generalization of the supersingular stratum of the special fibre of an integral canonical model of a Siegel modular variety. With the notations of 1.3-5, a point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ is a point of the null stratum, iff the Shimura Lie σ-crystal attached to it has all the slopes zero. We prove that the null stratum is always non void. Of course it is closed. We formulate our expectations about the properties enjoyed by the null strata, inspired by the properties (see [LO, 0.2]) of supersingular strata of the special fibres of integral canonical models of Siegel modular varieties, and prove (partially) some of these expectations. In particular we obtain a non-supersingular criterion similar to the non-ordinary one of Theorem 6.

1.21. The goal of §10 is to show that the number $N(G, X, v)$ of strata of the refined canonical Lie stratification of the special fibre of an integral canonical model of a Shimura variety $\mathrm{Sh}(G, X)$ of preabelian type with respect to a prime v of $E(G, X)$ dividing $p > 2$, and the dimensions of the geometrically connected components of these strata are easily effectively (algorithmically) computable. The philosophy is:

Ph. *The number $N(G, X, v)$, the above mentioned dimensions, as well as the local geometry of strata (like the type of singularities in the étale topology of a given stratum,* *like which stratum specializes to which stratum, etc.) depend only on the pair* $(G^{\mathrm{adnc}}_{\mathbb{Q}_p}, \mu^{\mathrm{ad}}_g)$, *where* $G^{\mathrm{adnc}}_{\mathbb{Q}_p}$ *is the smallest direct product factor of* $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ *with the property that the generic fibre* μ^{ad}_g *of the cocharacter* $\mu^{\mathrm{ad}} : \mathbb{G}_m \to G^{\mathrm{ad}}_{W(k(v^{\mathrm{ad}}))}$ *(with* $k(v^{\mathrm{ad}})$ *the residue field of the prime* v^{ad} *of* $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ *divided by* v*) factors through the extension of* $G^{\mathrm{adnc}}_{\mathbb{Q}_p}$ *to* $W(k(v^{\mathrm{ad}}))\left[\frac{1}{p}\right]$*. Here* μ^{ad} *is a cocharacter such that it becomes (under an embedding of* $W(k(v^{\mathrm{ad}}))\left[\frac{1}{p}\right]$ *in* $\mathbb{C}$ *extending the inclusion* $E(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \subset \mathbb{C}$*)* $G^{\mathrm{ad}}(\mathbb{C})$*-conjugate to the cocharacters* μ_x*,* $x \in X^{\mathrm{ad}}$ *of [Va2, 2.2].*

This forms the global form of the invariance principle.

1.22. The main purpose of §11 is to prove the isogeny property mentioned in [Va2, 1.7]. The idea is: we follow entirely the pattern of [FC, 4.3, p. 264], but in the context of crystalline cohomology provided by Fontaine's categories (as the isogeny property is trivial for G-ordinary points). The hard part is when we deal with points of the null stratum (defined in 1.20); for such points we need to combine loc. cit. with ideas of [Mi1].

1.23. In §12 we prove the following result: Let (G, X, H, v) be a Shimura quadruple with (G, X) a Shimura pair of adjoint abelian type, and with $(v, 2) = 1$. Then there is an injective map $f : (G_0, X_0, H_0, v_0) \hookrightarrow (G_1, X_1, H_1, v_1)$ such that:

1) $(G_0^{\mathrm{ad}}, X_0^{\mathrm{ad}}) = (G, X)$ and (under the natural inclusion $E(G, X) \subset E(G_0, X_0)$) v_0 divides v;

2) (G_1, X_1, H_1, v_1) is the quadruple of a Shimura variety of PEL-type, with all the factors of G^{ad} of some A_ℓ type, $\ell \in \mathbb{N}$;

3) the residue field $k(v_0)$ of v_0 is the same as the residue field $k(v_1)$ of v_1 or $[k(v_0) : k(v_1)]$ is a power of 2, depending on the fact that (G, X) has not or has simple factors of $D_n^{\mathbb{H}}$ type with (non-trivial) involution ($n \in \mathbb{N}$, $n > 3$);

4) there is an injective map $f_1 : (G_1, X_1) \hookrightarrow (\mathrm{GSp}(W, \psi), S)$ and a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W (with p the rational prime divided by v), with the property that the triples $(f_1, L_{(p)}, v_1)$ and $(f_1 \circ f, L_{(p)}, v_0)$ are standard Hodge situations;

5) if we have $k(v_0) = k(v_1)$ or if some other simple to check condition holds, then the G-ordinary type of the standard Hodge situation $(f_1 \circ f, L_{(p)}, v_0)$ is the same as the G_1-ordinary type of the standard Hodge situation $(f_1, L_{(p)}, v)$, and a G-canonical lift of the integral model $\mathcal{N}_0$ of the Shimura quadruple (G_0, X_0, H_0, v_0) is mapped under the natural morphism $\mathcal{N}_0 \to \mathcal{N}_1$ (cf. [Va2, rm. 4) of 3.2.7]) into a G_1-canonical lift of the integral canonical model $\mathcal{N}_1$ of the Shimura quadruple (G_1, X_1, H_1, v_1)).

As a consequence of this (cf. 1.13) we prove the result concerning special points mentioned in [Va2, 1.6.1] as well as the following result.

1.23.1. Let A be an abelian scheme over the ring $W(k)$ of Witt vectors of a finite field k of characteristic $p \geq 3$. Let $B(k) := W(k)[\frac{1}{p}]$. Let $(s_\beta)_{\beta \in I}$ be a family of Hodge cycles of A, including a polarization of degree relatively prime to p. We assume that the Zariski closure of the subgroup G_p of $GL(L_p[\frac{1}{p}])$ fixing the p-components of the étale components of the Hodge cycles of the family $(s_\beta)_{\beta \in I}$ in $GL(L_p)$, with $L_p := H^1_{\acute{e}t}(A_{\overline{B(k)}}, \mathbb{Z}_p)$, is a reductive group over $\mathbb{Z}_p$. Let π be the Frobenius endomorphism of A_k. Then a power of π, as an endomorphism of $H^1_{\mathrm{crys}}(A_k/W(k))$, fixes the de Rham components of the Hodge cycles of the family $(s_\beta)_{\beta\in I}$.

1.24. In §13-14, we prove the results pertaining to the Langlands–Rapoport conjecture stated in [Va2, 1.7].

1.25. A small part of the results presented in this first part (essentially Theorems 1 and 2 in the context of a standard Hodge situation and their applications, presented in a) and b) of 1.4), is a revised and improved version of the second part of our thesis [Va1]. Strictly speaking b) of Theorem 1 was not stated in [Va1], but all the ingredients needed for getting it can be found in loc. cit. Besides the results mentioned in 1.2-22, this paper contains many other ideas, exercises and examples. The ideas we did not have time to deal with, are presented in the form of problems and questions, or as remarks. 40 instructive and simple exercises, almost all with full hints, are also included.

1.26. We would like to thank Prof. Gerd Faltings for his encouragement to approach the topics of 1.2, 1.4 and 1.7, for his deep insights and for his suggestion mentioned in 1.2. Here, as well as in [Va2], we are very much obliged to [Fa1, §2 and §7] and [Fa2, §7]. We would like to thank Prof. Richard Pink for pointing out the thing mentioned in 1.10, and Prof. Arthur Ogus for some comments, for the example of 3.4.1.4 2), and for explaining us things in connection to the case $p = 2$ for ordinary elliptic curves; 2.3.18.1 (and so implicitely part of 4.14.3) is based on these explanations. We would like to thank Rahel Boller for the typing of the first (incomplete) version of this manuscript. We would like to thank the FIM ETH Zürich, and UC at Berkeley for providing us with excellent conditions for the writing of this manuscript. This work was partially supported by the NSF grant DMF 97-05376.

§2 *Preliminaries*

2.1. Notations and conventions. Reductive groups over fields are always assumed to be connected. So the fibres of a reductive group over a scheme are connected. For a reductive group G over a scheme we denote by G^{der}, $Z(G)$, G^{ab} and G^{ad}, respectively, the derived group of G, the center of G, the maximal abelian quotient of G and the adjoint group of G. We denote by $\mathrm{Lie}(G)$ or $\mathfrak{g}$ the Lie algebra of a reductive group scheme G over an affine scheme $X = \mathrm{Spec}(R)$. For any cocharacter $\mu : \mathbb{G}_m \to G$ and for any affine X-scheme $X^1 = \mathrm{Spec}(R^1)$ we denote by $[\mu_{R^1}]$ or $[\mu_{X^1}]$ the $G(R_1)$-conjugacy class of cocharacters defined by the extension of μ to X^1. We say that a reductive group G over $\mathbb{Q}$ is unramified over $\mathbb{Q}_p$ (p being a rational prime) if $G_{\mathbb{Q}_p}$ is unramified over $\mathbb{Q}_p$, i.e. if $G_{\mathbb{Q}_p}$ is quasi-split and splits over an unramified finite field extension of $\mathbb{Q}_p$.

If X is a set endowed with an equivalence relation $R \subset X \times X$, then for any element $x \in X$, we denote by $[x] \in X/R$ the equivalence class of x. We denote by $|B|$ the number of elements of a finite set B.

The expression (G, X) always denotes a pair defining a Shimura variety, $E(G, X)$ denotes its attached reflex field, and $\mathrm{Sh}(G, X)$ denotes the Shimura variety defined by it, identified with the canonical model of the complex variety. For an arbitrary compact subgroup K of $G(\mathbb{A}_f)$, we denote by $\mathrm{Sh}_K(G, X)$ or $\mathrm{Sh}(G, X)/K$, the quotient of $\mathrm{Sh}(G, X)$ by K. Any $x \in X$ and any $\alpha \in G(\mathbb{A}_f)$ define a complex point $[x, \alpha]$ of $\mathrm{Sh}_K(G, X)$. As in this paper we encounter several quadruples, the quadruples introduced in [Va2, 3.2.6] will be called Shimura quadruples. We are using freely the notations and terminology of [Va2, 2.4 and 3.2.6].

If k is a field we denote by $\bar{k}$ its algebraic closure. For a perfect field k, $W(k)$ is the ring of Witt vectors of k, while $B(k)$ is the field of fractions of $W(k)$. For any $n \in \mathbb{N}$ $W_n(k) := W(k)/p^n W(k)$. We denote by σ (resp. $\overline{\sigma}$) the Frobenius automorphism of $W(k)$ and $B(k)$ (resp. of $W(\bar{k})$ and $B(\bar{k})$). By a σ-crystal (over k) we mean a pair (M, φ), with M a free $W(k)$-module of finite rank, and with φ a σ-linear endomorphism of M which is a σ-linear automorphism of $M \otimes_{W(k)} B(k)$. Warning: by a filtered σ-crystal (over k) we mean a triple (M, F^1, φ), with (M, φ) a σ-crystal, and with F^1 a $W(k)$-submodule of M which is a direct summand of M such that $\varphi(\frac{1}{p}F^1 + M) = M$. By a principal quasi-polarization of a σ-crystal (M, φ) we mean a symplectic perfect form $\psi : M \otimes_{W(k)} M \to W(k)$ such that $\psi(\varphi(x), \varphi(y)) = p\sigma(\psi(x, y))$, $\forall x, y \in M$. We often write $\psi : M \otimes_{W(k)} M \to W(k)(1)$. Let $n \in \mathbb{N}$; by a σ^n-isocrystal (over k) we mean a pair (M, φ), with M a finite dimensional $B(k)$-vector space, and with φ a σ^n-linear automorphism of M. When $n = 1$ we speak simply about isocrystals (over k). We still denote by φ the Frobenius of any free $W(k)$-submodule or $B(k)$-vector subspace of different tensor products (taken in any order) of a given number of copies of M with a given number of copies of its dual M^*, which is invariant under φ.

We denote by $\mathcal{MF}_{[a,b]}(R)$, $\mathcal{MF}(R)$, $\mathcal{MF}^{\nabla}_{\mathrm{big}}(R)$, $\mathcal{MF}^{\nabla}_{[a,b]}(R)$ and $\mathcal{MF}_{[a,b]}(R)$ the categories introduced in [Fa2]. We allow R to be (the p-adic completion of) a formally étale algebra (resp. affine scheme; when allowed, just scheme) over a smooth $W(k)$-algebra (resp. $W(k)$-scheme). If R and S are two $W(k)$-algebras having Frobenius lifts Φ_R and respectively Φ_S, if M is a free R-module endowed with a Φ_R-linear map $\varphi : M \to M$, and if $b : R \to S$ is a ring homomorphism compatible with the mentioned Frobenius lifts, we denote by $\varphi \otimes 1$ the Φ_S-linear map on $M \otimes_R S$ induced from φ through b.

Always V_0 denotes a Witt ring over an algebraically closed field of positive characteristic, and then K_0 automatically denotes its field of fractions. If v is a prime of a number field E, we denote by $k(v)$ the residue field of v and by $O_{(v)}$ the localization of the ring of integers of E with respect to the prime v. If O is a local ring, we denote by O^h, O^{sh} and $\widehat{O}$ respectively its henselization, its strict henselization and its completion.

Let p be a rational prime. We usually write $\mathbb{Z}_{(p)}$ instead of $O_{(p)}$. The ring of finite adèles $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is denoted by $\mathbb{A}_f$, and the ring of finite adèles with the p-component omitted is denoted by $\mathbb{A}_f^p$. We use freely different Tate-twists: $\mathbb{Q}(1), \mathbb{Q}_p(1), \mathbb{Z}_p(1), \mathbb{A}_f(1)$ etc. We denote by $\mathbb{F}_{p^d}$ the field with p^d elements ($d \in \mathbb{N}$), and by $\mathbb{F}$ its algebraic closure. If R is a faithfully flat $\mathbb{Z}_{(p)}$-algebra, we denote by $R^{\wedge}$ its p-adic completion. Similarly, if $X = \mathrm{Spec}(R)$, with R as before, we denote $X^{\wedge} := \mathrm{Spec}(R^{\wedge})$.

If A is a set and $f : A \to A$ is an endomorphism of it, we denote by f^m the m-times composite of f with itself ($m \in \mathbb{N}$). If $f : A \to B$ and $g : B \to C$ are morphisms in some category we refer to $g \circ f$ as the composite of f with g.

For every free module M of finite rank over a commutative ring R, we denote by M^* its dual. We denote by $M^{\otimes n} \otimes_R M^{*\otimes m}$, with $m, n \in \mathbb{N} \cup \{0\}$, the tensor product of n-copies of M with m-copies of M^*. If $v_\alpha \in M^{\otimes n} \otimes_R M^{*\otimes m}$, we denote by $\deg(v_\alpha) := n + m$ its degree, and we call it a homogeneous tensor of degree $n + m$. By the essential tensor algebra of $M \oplus M^*$ we mean

$$\mathcal{T}(M) := \oplus_{n,m \in \mathbb{N} \cup \{0\}} M^{\otimes n} \otimes_R M^{*\otimes m}.$$

We do not make use of the natural structure of $\mathcal{T}(M)$ as an R-algebra (it is non-commutative if the rank of M is at least 2). Similarly we define the essential tensor algebra $\mathcal{T}(\mathcal{F})$ of $\mathcal{F} \oplus \mathcal{F}^*$, for a locally free $\mathcal{O}_X$-sheaf $\mathcal{F}$ of finite rank on a ringed space $(X, \mathcal{O}_X)$. A bilinear form on M is called perfect if it induces an isomorphism from M into its dual M^*. A pair (M, ψ) with M as above, and with $\psi : M \otimes_R M \to R$ a perfect alternating form on it, is called a symplectic space over R. We usually use the same notation for two perfect alternating forms if they are obtained one from another by extension of scalars. If (W, ψ) is a symplectic space over $\mathbb{Q}$, we denote by $\mathrm{Sh}\big(\mathrm{GSp}(W, \psi), S\big)$ the Siegel modular variety associated to it.

For a finite field extension $k_0 \hookrightarrow k$ and for a reductive group G over k, $\mathrm{Res}_{k/k_0} G$ denotes the group over k_0 obtained from G by Weil restriction of scalars. If $f : G \to G_1$ is a homomorphism of group schemes which is a closed embedding, then we identify G with its image (through f) in G_1.

We use freely the terminology of Hodge cycles of abelian varieties over a field of characteristic zero used in [De3], and by natural extension, this terminology is used also for abelian schemes over a reduced $\mathbb{Q}$–scheme S. Even more: if S is a reduced flat $\mathbb{Z}_{(p)}$-scheme, and if A is an abelian scheme over S, we speak about Hodge cycles of A, though strictly speaking, we should speak of the Hodge cycles of the generic fibre of A. Any polarization of an arbitrary abelian scheme A is denoted by p_A (sometimes $\mathcal{P}_A$), and by abuse of notation we also denote by p_A the different maps on the cohomologies (homologies) of A induced by it. Pairs of the form (A, p_A) always denote a polarized abelian scheme.

We consider an abelian variety or a p-divisible group A over a field k of positive characteristic p. We denote by A^t its dual. If k is perfect, then by the σ-crystal of (or attached to) A (resp. by the isocrystal of A) we mean the σ-crystal defined by the first crystalline cohomology group of A with coefficients in $W(k)$ (resp. we mean the isocrystal over k defined by such a σ-crystal). If k is arbitrary, by the formal isogeny type of A, we mean the formal isogeny type (see [Og1]) defined by the isocrystal of $A_{\bar{k}}$.

For Artin's approximation theorem we refer to [BLR, p. 91]. All the limits and colimits are assumed to be filtered. The projective limits between schemes have affine transition morphisms. If a projective limit is indexed in the standard manner by elements of $\mathbb{N}$, then we refer to it as an $\mathbb{N}$-projective limit; similarly we speak about an $\mathbb{N}$-pro-étale scheme over another scheme.

2.2. Shimura crystals. Let k be an arbitrary perfect field of characteristic $p > 0$.

2.2.1. Definitions. a) Let $r \in \mathbb{Q}$. An isocrystal (M, φ) over k is said to be r-symmetric, if $\forall \alpha \in \mathbb{Q}$ such that $r - \alpha$ is a slope of it, $r + \alpha$ is also a slope of it of the same multiplicity.

b) A filtered σ-crystal (M, F^1, φ) is said to be cyclic diagonalizable if there is a $W(k)$-basis $\{e_i | i \in A\}$ of M (with $A = \{1, 2, \ldots, \dim_{W(k)}(M)\}$) such that $\{e_i | i \in B\}$ (with B a subset of A) is a $W(k)$-basis of F^1, and $\varphi(e_i) = p^{\varepsilon_i} e_{\pi(i)}$, with $\varepsilon_i = 1$ if $i \in B$ and zero otherwise, and with π a permutation of A. When $\pi = 1_A$ we drop the word cyclic. If π is an $|A|$-cycle, then we replace the word cyclic by circular.

c) Let R be a formally étale algebra over a smooth $W(k)$-algebra, and let Φ_R be a σ-linear endomorphism of R (resp. of $R^\wedge$) which lifts the Frobenius of R/pR. We have in mind mainly the case when the only open subscheme of $\mathrm{Spec}(R)$ containing all the $\bar{k}$-valued points of $\mathrm{Spec}(R)$, is $\mathrm{Spec}(R)$ itself; but for the sake of generality we do not assume this. A finitely generated projective R-module (resp. $R^\wedge$-module) M, together with a decreasing filtration $(F^i)_{i \in \mathbb{Z}}$ (with F^i a direct summand of M) and a Φ_R-linear endomorphism $\varphi : M \to M$, is said to be a p-divisible object of $\mathcal{MF}(R)$ if for any $i \in \mathbb{Z}$, φ restricted to F^i is of the form $p^i \varphi_i$, with $\varphi_i : F^i \to M$, and the R-linear (resp. $R^\wedge$-linear) map $\oplus_{i \in \mathbb{Z}} F^i \otimes_{R\ \Phi_R} R \to M$ (resp. $\oplus_{i \in \mathbb{Z}} F^i \otimes_{R^\wedge\ \Phi_R} R^\wedge \to M$), whose restriction to the summand $F^i \otimes_{R\, \Phi_R} R$ (resp. $F^i \otimes_{R^\wedge\, \Phi_R} R^\wedge$) is $\varphi_i \otimes 1$ ($i \in \mathbb{Z}$), is an isomorphism. We denote such a p-divisible object by a triple

$$\mathcal{C} := (M, (F^i)_{i \in \mathbb{Z}}, \varphi).$$

So for any $n \in \mathbb{N}$, $M/p^n M$, together with $(F^i/p^n F^i)_{i \in \mathbb{Z}}$ and the induced $(\varphi_i)_{i \in \mathbb{Z}}$ is an object of $\mathcal{MF}(R)$. We denote it by $\mathcal{C}/p^n\mathcal{C}$ and call it the truncation mod p^n of $\mathcal{C}$. If M is a projective R-module, we identify this p-divisible object with the one obtained from it by passing from R to $R^\wedge$ (but we do not want to lose the information of being "algebraizable"). If $\mathcal{C}_1 = (M_1, (F_1^i)_{i\in\mathbb{Z}}, \varphi_1)$ is another such p-divisible object, then by a morphism from $\mathcal{C}$ to $\mathcal{C}_1$, we mean an R-linear map $M \to M_1$ (resp., when M and M_1 are $R^\wedge$ modules, we mean an $R^\wedge$-linear map $M \to M_1$) respecting the filtrations and the Frobenius. We get the category $p - \mathcal{MF}(R)$ of p-divisible objects of $\mathcal{MF}(R)$. If moreover we fix a family of tensors $(v_\alpha)_{\alpha\in\mathcal{J}}$ of the F^0-filtration of $\mathcal{T}(M[\frac{1}{p}])$ fixed by φ, then we speak about a p-divisible object with cycles of $\mathcal{MF}(R)$.

Similarly we speak about p-divisible objects (with cycles) of $\mathcal{MF}_{[a,b]}(R)$, $\mathcal{MF}_{\mathrm{big}}(R)$, $\mathcal{MF}^\nabla_{[a,b]}(R)$, or of $\mathcal{MF}^\nabla_{\mathrm{big}}(R)$, or of a p-divisible object (with cycles) of $\mathcal{MF}_{[a,b]}(X)$, etc., with X a regular formally smooth $W(k)$-scheme (in this last case the cycles are sections in locally free sheaves of finite rank over X or over the formal scheme defined by the p-adic completion of X). We always assume that either X is affine and itself or its p-adic completion is equipped with a Frobenius lift, or that we are in a situation where [Fa1, 2.3] applies. We also speak about the categories defined by such p-divisible objects: $p - \mathcal{MF}_{[a,b]}(R)$, etc. When we speak about a p-divisible object with cycles of $\mathcal{MF}^\nabla(R)$, we require that the cycles are horizontal, i.e. are annihilated by the connection involved. Warning: as we allowed R to be regular formally smooth (not only smooth) over $W(k)$, some precautions have to be taken. In the case of a p-divisible object of $\mathcal{MF}^\nabla(R)$, we work with the p-adic completion of the module of relative differentials of R over $W(k)$. For instance if $R = W(k)[[z_1, ..., z_m]]$, $m \in \mathbb{N}$, and if M is a free R-module of finite rank, then all the connections

$$\nabla : M \to M \otimes_{W(k)} \Omega_{R/W(k)}$$

to be considered, factor through $M \otimes_R \bar{\Omega}_{R/W(k)}$, where $\bar{\Omega}_{R/W(k)}$ is the free R-submodule of $\Omega_{R/W(k)}$ generated by dz_1,...,dz_m.

We also speak about a principal quasi-polarization of a p-divisible object of $\mathcal{MF}_{[0,1]}(R)$ or of $\mathcal{MF}^\nabla_{[0,1]}(R)$. For instance, if $\mathfrak{C}$ is a p-divisible object of $\mathcal{MF}_{[0,1]}(R)$, the dual $\mathfrak{C}^*$ of $\mathfrak{C}$ is a p-divisible object of $\mathcal{MF}_{[-1,0]}(R)$, and so $\mathfrak{C}^*(1)$ is a p-divisible object of $\mathcal{MF}_{[0,1]}(R)$; so by a principal quasi-polarization of $\mathfrak{C}$, we mean an isomorphism

$$\mathfrak{C} \tilde{\to} \mathfrak{C}^*(1)$$

producing a perfect alternating form

$$\mathfrak{C} \otimes_R \mathfrak{C} \to R(1);$$

this form is a morphism in the category $p - \mathcal{MF}_{[0,2]}(R)$. Here

$$R(1) := R \otimes_{W(k)} W(k)(1).$$

Similarly we speak about principally quasi-polarized truncation mod p^n (this is an object) of a p-divisible object of $\mathcal{MF}_{[0,1]}(R)$ or of $\mathcal{MF}^\nabla_{[0,1]}(R)$; here $n \in \mathbb{N}$.

2.2.1.1. Remarks. 1) We might wonder if the terminology p-divisible objects is the right one. Our justification is: the philosophies of [Fa2, 7.1] and 3.4.18.5.1. Moreover the terminology formal is used in this paper in a totally different context (cf. 4.7).

2) We have a contravariant functor from the category of p-divisible groups over $R^\wedge$ (or R) and the category p-$\mathcal{MF}^\nabla_{[0,1]}(R)$ (this is a particular case of the Grothendieck-Messing theory; see [Me] and [BBM]). In particular to any p-divisible group over $R^\wedge$ (or R) it corresponds (or it is associated) a p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(R)$.

3) The above categories of objects or p-divisible objects depend on the choice of Φ_R. However, if $p > 2$ then the categories $\mathcal{MF}^\nabla_{[0,1]}(X)$ and $\mathcal{MF}^\nabla_{[0,1]}(X)$ do not depend on Φ_R, cf. [Fa2, 2.3].

4) Any circular diagonalizable σ-crystal has a unique slope.

2.2.2. Examples. The isocrystal of an abelian variety over k is 1/2-symmetric. If $k = \bar{k}$ then the filtered σ-crystal of the canonical lift of an ordinary abelian variety over k is diagonalizable and a p-divisible object of $\mathcal{MF}_{[0,1]}(W(k))$.

2.2.3. Definitions. 1) A Lie σ-crystal (resp. a Lie isocrystal over k) is a pair $(\mathfrak{g}, \varphi)$, with $\mathfrak{g}$ a Lie algebra over $W(k)$ (resp. over $B(k)$), which as a $W(k)$-module (resp. as a $B(k)$-vector space) is free of finite rank (resp. of finite dimension), and with φ a σ-linear Lie automorphism of $\mathfrak{g} \otimes_{W(k)} B(k)$ such that $\varphi(p\mathfrak{g}) \subset \mathfrak{g}$ (resp. with φ a σ-linear Lie automorphism of $\mathfrak{g}$).

2) A Lie σ-crystal is said to be ordinary or of ordinary type, if the slopes of the isocrystal $(\mathfrak{g} \otimes_{W(k)} B(k), \varphi)$ are (warning!) precisely -1, 0 and 1, with the multiplicity of the slope 1 equal to the multiplicity of the slope -1.

3) A filtered Lie σ-crystal is a quadruple $(\mathfrak{g}, \varphi, F^0(\mathfrak{g}), F^1(\mathfrak{g}))$ such that:

– the pair $(\mathfrak{g}, \varphi)$ is a Lie σ-crystal;

– $F^0(\mathfrak{g})$ and $F^1(\mathfrak{g})$ are Lie subalgebras of $\mathfrak{g}$ such that $F^1(\mathfrak{g})$ is an ideal of $F^0(\mathfrak{g})$;

– it defines a p-divisible object of the category $\mathcal{MF}_{[-1,1]}(W(k))$, i.e. $\varphi\bigl(\frac{1}{p}F^1(\mathfrak{g}) + F^0(\mathfrak{g}) + p\mathfrak{g}\bigr) = \mathfrak{g}$ and, as $W(k)$-submodules, $F^0(\mathfrak{g})$ and $F^1(\mathfrak{g})$ are direct summands of $\mathfrak{g}$ (we have $\mathfrak{g} = F^{-1}(\mathfrak{g})$ and $0 = F^2(\mathfrak{g})$).

2.2.4. Remarks. 1) The slopes of a filtered Lie σ-crystal are rational numbers of the interval [-1,1]. Most common, for Lie σ-crystals $(\mathfrak{g}, \varphi)$, $\mathfrak{g}$ is the Lie algebra of a reductive group and for filtered Lie σ-crystals $(\mathfrak{g}, \varphi, F^0(\mathfrak{g}), F^1(\mathfrak{g}))$, $\mathfrak{g}$ is the Lie algebra of a reductive group, $F^0(\mathfrak{g})$ is a parabolic Lie subalgebra of $\mathfrak{g}$ having $F^1(\mathfrak{g})$ as its nilpotent radical, and so $\bigl([\mathfrak{g}, \mathfrak{g}] \otimes_{W(k)} B(k), \varphi\bigr)$ is a 0-symmetric isocrystal (i.e. the multiplicity of a slope α is the same as the multiplicity of the slope $-\alpha$). We sometimes need to use Lie σ-crystals $(\mathfrak{g}, \varphi)$, with $\mathfrak{g}$ as the Lie subalgebra of the Lie algebra of a reductive group.

2) We can introduce the notion of an a-Lie σ-crystal $(\mathfrak{g}, \varphi)$ (or of an $[a, b]$-filtered Lie σ-crystal $\bigl(\mathfrak{g}, \varphi, F^{a+1}(\mathfrak{g}), \ldots, F^b(\mathfrak{g})\bigr)$) (with $a, b \in \mathbb{Z}$). For instance, for an a-Lie σ-crystal we require that $\mathfrak{g}$ is a Lie algebra over $W(k)$, that $\mathfrak{g}$ as a $W(k)$-module is free of finite rank, that φ is a σ-linear Lie automorphism of $\mathfrak{g} \otimes_{W(k)} B(k)$, and that $\varphi(p^a\mathfrak{g}) \subset \mathfrak{g}$. But we do not need this for the present paper, except for a short intermezzo in 3.4.1.5-6.

3) Let $(\mathfrak{g}, \varphi)$ be a Lie isocrystal over k. Then there is a unique Lie subalgebra $\mathfrak{p}_{\geq 0}$ of $\mathfrak{g}$ left invariant by φ and such that all the slopes of the isocrystal $(\mathfrak{p}_{\geq 0}, \varphi)$ (resp. of $(\mathfrak{g}/\mathfrak{p}_{\geq 0}, \varphi)$) are non-negative (resp. are negative). If $\mathfrak{g}$ is a semisimple Lie algebra over

$B(k)$, then $\mathfrak{p}_{\geq 0}$ is a parabolic Lie subalgebra of $\mathfrak{g}$. We call the isocrystal $(\mathfrak{p}_{\geq 0}, \varphi)$ (resp. $\mathfrak{p}_{\geq 0}$) the (Lie) subisocrystal of $(\mathfrak{g}, \varphi)$ (resp. the Lie subalgebra of $\mathfrak{g}$) corresponding to non-negative slopes of $(\mathfrak{g}, \varphi)$. Similarly we define things for non-positive (or positive, or negative) slopes. Similarly we define things for a Lie σ-crystal.

4) One may ask: why Lie crystals? There are two reasons. First, this is the best way to keep track of the relative situation which involves cycles. Second, we have natural equivalences between different categories $\mathcal{MF}_{[a,b]}(*)$ and $\mathcal{MF}_{[a+c,b+c]}(*)$, with $a, b, c \in \mathbb{Z}$, $b \geq a$. When we take End of these objects, c goes automatically away; so we have a better way to express our results (as samples see 3.4.18.7.0, 3.4.18.8.1 b), 4.7.30 4), etc.).

In some sense, this first part grew out of a systematic exploitation of Lie crystals. Accordingly, to always have a good track of when we are dealing with Lie crystals instead of just some crystals, in the filtered context, we most of the time we use the ordering $(\mathfrak{g}, \varphi, F^0(\mathfrak{g}), F^1(\mathfrak{g}))$ instead of the standard one $(\mathfrak{g}, F^0(\mathfrak{g}), F^1(\mathfrak{g}), \varphi)$.

2.2.5. Definitions. Let X be a connected scheme.

1) A Shimura group pair over X is a pair $(G, [\mu])$, with G a reductive group over X, and with $[\mu]$ a $G(X')$-conjugacy class of cocharacters $\mu : \mathbb{G}_m \to G_{X'}$ defined over a connected étale cover X' of X, such that the induced action of $\mathbb{G}_m$ on $\mathrm{Lie}(G_{X'})$ is defined (warning!) precisely by the characters: the null, the identical and the inverse of the identical character of $\mathbb{G}_m$.

2) Two such Shimura group pairs over X $(G, [\mu_i])$, $i = \overline{1,2}$, (with $\mu_i : \mathbb{G}_m \to G_{X_i}$ defined over a connected étale cover X_i of X) are said to be identical if there is a connected étale cover X' of X, dominating X_1 and X_2, and over which μ_1 and μ_2 define the same $G(X')$-conjugacy class of cocharacters of $G_{X'}$.

3) For any such Shimura group pair $(G, [\mu])$ over X, we denote by $[\mathrm{Lie}(\mu)]$ the $G^{\mathrm{ad}}(X')$-conjugacy class of the (rank one) direct summand $\mu(\mathrm{Lie}(\mathbb{G}_m))$ of $\mathrm{Lie}(G_{X'})$. We call the pair $(\mathrm{Lie}(G), [\mathrm{Lie}(\mu)])$ the Shimura Lie pair (over X) attached to the Shimura group pair $(G, [\mu])$.

4) Two such Shimura Lie pairs over X $(\mathrm{Lie}(G_i), [\mathrm{Lie}(\mu_i)])$, $i = \overline{1,2}$, (with μ_i a cocharacter of G_{X_i}; X_i being an étale connected cover of X), are said to be isomorphic if:

a) there is an isomorphism of Lie algebras $\rho : \mathrm{Lie}(G_1) \tilde{\to} \mathrm{Lie}(G_2)$ producing an isomorphism $\mathrm{Lie}(G_1^{\mathrm{ad}}) \tilde{\to} \mathrm{Lie}(G_2^{\mathrm{ad}})$ induced by an isomorphism $G_1^{\mathrm{ad}} \tilde{\to} G_2^{\mathrm{ad}}$;

b) over an étale connected cover X' of X, dominating X_1 and X_2, ρ takes $[\mathrm{Lie}(\mu_{1X'})]$ onto $[\mathrm{Lie}(\mu_{2X'})]$.

2.2.6. Remarks. **1)** Most common, in 2.2.5 1), either G is split and X is local henselian, and then we can take $X' = X$, or $X = \mathrm{Spec}(\mathbb{Q}_p)$ (or $X = \mathrm{Spec}(\mathbb{Z}_p)$) and G is unramified over $\mathbb{Q}_p$, and then there is a smallest unramified cover X' of X, over which such a cocharacter $\mu: \mathbb{G}_m \to G_{X'}$ is defined.

2) We can work 2.2.5 in terms of the faithfully flat (or étale) topology of X, but for this paper, the present form of 2.2.5 is all that we need.

3) 2.2.5 1-2) allows us to introduce the notion of morphisms (maps) between two Shimura group pairs $(G_1, [\mu_1])$ and $(G_2, [\mu_2])$ over X: These are (group) homomorphisms

$f : G_1 \to G_2$ such that for a simply connected pro-étale cover X' of X, $f_{X'}$ takes $[\mu_{1X'}]$ into $[\mu_{2X'}]$. We denote such a map by $f : (G_1, [\mu_1]) \to (G_2, [\mu_2])$. It is called an injective map (resp. an isomorphism), if as a group homomorphism is a closed embedding (resp. an isomorphism). We get the category $\mathrm{ShGp}(X)$ of Shimura group pairs over X.

4) A study of Shimura group pairs over $\mathbb{Z}_p$ together with their symplectic embeddings (i.e. injective maps into $\big(\mathrm{GSp}(W_{\mathbb{Z}_p}, \psi), [\mu]\big)$; here $(W_{\mathbb{Z}_p}, \psi)$ is a symplectic space over $\mathbb{Z}_p$, while $[\mu]$ is uniquely determined), is started in 3.10 and is fully developped in §7.

5) Any Shimura variety $\mathrm{Sh}(G, X)$ of dimension ≥ 1 gives birth to a Shimura group pair $(G, [\mu])$ over $\mathrm{Spec}(\mathbb{Q})$. Here $\mu : \mathbb{G}_m \to G_{E_1(G,X)}$ is a cocharacter, with $E_1(G, X)$ a finite field extension of $E(G, X)$, which over $\mathbb{C}$ under an embedding $E_1(G, X) \subset \mathbb{C}$ extending the inclusion $E(G, X) \subset \mathbb{C}$, is $G(\mathbb{C})$-conjugate to anyone of the cocharacters μ_x, $x \in X$, def. in [Va2, 2.2]. The association: $(G, [\mu])$ to $\mathrm{Sh}(G, X)$, is functorial.

2.2.7. Remark. If X is local henselian and if G is an X-simple adjoint split group, then the Shimura group pairs $(G, [\mu])$ are classified by [De2, 1.3.9]. We have $\left[\frac{\ell+1}{2}\right]$ isomorphism classes if G is of A_ℓ Lie type, 2 isomorphism classes if G is of D_ℓ Lie type ($\ell \geq 5$), and one such class if G is of D_4, E_6, E_7, B_ℓ or C_ℓ Lie type. There are no Shimura group pairs $(\tilde{G}, [\tilde{\mu}])$ with $\tilde{G}$ an X-simple adjoint group of E_8, F_4 or G_2 Lie type.

Moreover any Shimura group pair $(G_1, [\mu_1])$ over such an X, with G_1 an adjoint group, can be written uniquely (up to isomorphism) as a product of pairs $(G_2, [\mu_2])$ which are either having μ_2 as the trivial cocharacter, or are Shimura pairs with G_2 as an X-simple adjoint group.

2.2.8. Definitions. **1)** A Shimura filtered σ-crystal over k is a quadruple (M, F^1, φ, G), with (M, F^1, φ) a filtered σ-crystal such that F^1 is a proper direct summand of M, and with G a quasi-split reductive subgroup of $GL(M)$ such that:

a) There is a family $(v_\alpha)_{\alpha \in \mathcal{J}}$ of tensors of the F^0-filtration (defined naturally by the filtration F^1 of M) of $\mathcal{T}(M[\frac{1}{p}])$, fixed by G and φ, and such that $G_{B(k)}$ is the subgroup of $GL\big(M[\frac{1}{p}]\big)$ fixing v_α, $\forall \alpha \in \mathcal{J}$;

b) $\Big(\mathrm{Lie}(G), \varphi, F^0\big(\mathrm{Lie}(G)\big), F^1\big(\mathrm{Lie}(G)\big)\Big)$ is a filtered Lie σ-crystal; here

$$F^0\big(\mathrm{Lie}(G)\big) := \big\{x \in \mathrm{Lie}(G) \mid x(F^1) \subset F^1\big\}$$

and

$$F^1\big(\mathrm{Lie}(G)\big) := \big\{x \in \mathrm{Lie}(G) \mid x(F^1) = 0\big\};$$

c) There is an injective cocharacter $\mu : \mathbb{G}_m \hookrightarrow G$ producing a direct sum decomposition $M = F^1 \oplus F^0$, with $\beta \in \mathbb{G}_m(W(k))$ acting through μ on F^i as the multiplication with β^{-i}, $i = \overline{0,1}$.

2) A Shimura σ-crystal over k is a triple (M, φ, G) which can be extended to a quadruple (M, F^1, φ, G) defining a Shimura filtered σ-crystal over k.

3) A generalized Shimura p-divisible object of $\mathcal{MF}_{[a,b]}(W(k))$ is a quadruple

$$(M, (F^i)_{a \leq i \leq b}, \varphi, G),$$

where the triple $(M,(F^i)_{a\le i\le b},\varphi)$ is a p-divisible object of $\mathcal{MF}_{[a,b]}(W(k))$ (so $a,b\in\mathbb{Z}$, with $a\le b$) and G is a quasi-split reductive subgroup of $GL(M)$, for which the conditions a) and b) are true, but for which condition c) is replaced by:

d) There is an injective cocharacter $\mu:\mathbb{G}_m\hookrightarrow G$ producing a direct sum decomposition $M=\oplus_{a\le i\le b}\tilde F^i$, with $\beta\in\mathbb{G}_m(W(k))$ acting through μ on $\tilde F^i$ as the multiplication with β^{-i}, $i=\overline{a,b}$, with $F^i=\oplus_{i\le j\le b}\tilde F^j$, and with at least two of the $W(k)$-modules $\tilde F^i$, $i=\overline{a,b}$ being non-zero, and such that the induced cocharacter $\mu^{\rm ad}:\mathbb{G}_m\to G^{\rm ad}$ is either trivial, or gives birth to a Shimura group pair $(G^{\rm ad},[\mu^{\rm ad}])$.

4) A generalized Shimura p-divisible object over k is a triple (M,φ,G) which can be extended to a quadruple $(M,(F^i)_{a\le i\le b},\varphi,G)$ defining a generalized Shimura p-divisible object of $\mathcal{MF}_{[a,b]}(W(k))$.

5) A Shimura isocrystal over k is a triple $(M_{B(k)},\varphi,G_{B(k)})$, where $(M_{B(k)},\varphi)$ is an isocrystal over k, and where $G_{B(k)}$ is a reductive subgroup of $GL(M_{B(k)})$, such that there is a family of tensors $(v_\alpha)_{\alpha\in\mathcal{J}}$ of $\mathcal{T}(M_{B(k)})$ fixed by φ, and having the property that $G_{B(k)}$ is the subgroup of $GL(M_{B(k)})$ fixing its members.

6) A Shimura filtered isocrystal over k is a quadruple $(M_{B(k)},(F^i)_{i\in\mathbb{Z}},\varphi,G_{B(k)})$, where the triple $(M_{B(k)},\varphi,G_{B(k)})$ is a Shimura isocrystal over k, and where $(F^i)_{i\in\mathbb{Z}}$ is a decreasing filtration of $M_{B(k)}$, such that, with the notations of 5), we can choose the family of tensors $(v_\alpha)_{\alpha\in\mathcal{J}}$ to be in the F^0-filtration (defined by $(F^i)_{i\in\mathbb{Z}}$) of $\mathcal{T}(M_{B(k)})$.

2.2.9. Remarks and variants. 0) We often drop out mentioning "over k".

1) We refer to 2.2.8 1). The condition a) implies that $\varphi\big(\mathrm{Lie}(G)[\frac{1}{p}]\big)=\mathrm{Lie}(G)[\frac{1}{p}]$. It is "almost" implied by b), cf. [Fa2, rm. iii) of th. 10]: in the loc. cit. we do have to consider Tate-twists as well as. To us, the first (warning: not the second) part of the loc. cit. involving the existence of the line L_0 is not correct as it stands (this is why we have inserted condition a)). To motivate this, let $\mathfrak{C}_0:=(M_0,F_0^1,\varphi_0)$ be a p-divisible object of $\mathcal{MF}_{[0,1]}(W(k))$. Let Γ_k be the Galois group of $B(k)$, and let $\rho_0:\Gamma_k\to GL(N_0)$ be the Galois representation associated to $\mathfrak{g}_0$. Let G_0 be the subgroup of $GL(N_0)$ which is the algebraic envelope of ρ_0. Let G_1 be an arbitrary subgroup of $GL(N_0)$ normalized by G_0. Let $\tilde G_1$ be the subgroup of $GL(M_0)$ corresponding to G_0 via Fontaine's comparison theory. Its Lie algebra defines a p-divisible subobject of $\mathrm{End}(\mathfrak{C}_0)$ (if desired we can assume that $p\ge 5$). It is very easy to adjust the situation such that $\tilde G_1$ has connected fibres. But if L_0 exists as in loc. cit., then G_0 is contained in a subgroup of $GL(N_0)$ which is the extension of $\mathbb{G}_m$ by G_1. Of course this is not always so. So what is missing in loc. cit.? The error consists in: the $\mathbb{Z}_p$-submodules of rank 1 of $\mathcal{T}(N_0)$ and normalized by G_1 are not fixed by G_0, but just permutted. So there are two ingredients missing: condition c) as well as the passage to an algebraic field extension of k (see the paragraph below).

We come back to our situation. The condition c) results from a) and b) (cf. [Wi]). On the other hand, if b) and c) hold and $k=\bar k$, then using a standard argument involving $\mathbb{Z}_p$-structures (it is recalled in 8) below) and the fact that $G_{B(k)}$ is reductive (so we can apply [De3, 3.1 c)]: with the notations of 8), $G_{\mathbb{Q}_p}$ is the subgroup of $GL(M^a[\frac{1}{p}])$ fixing some tensors of $\mathcal{T}(M^a)$), we do get that a) holds automatically. So no doubt, in some sense, the main assumption is b).

$1'$) 1) above suggests the introduction of pseudo Shimura (filtered) σ-crystals: we assume just b) of 2.2.8 1). Similarly, if b) and c) hold, and a) holds after a passage to $W(k_1)$ with k_1 a finite field extension of k (resp. with $k_1 = \bar{k}$), then we speak about quasi (resp. potentially) Shimura (filtered) σ-crystals. Warning: for these notions we still keep G to be a quasi-split reductive group over $W(k)$. The greatest part of the terminology to be introduced below for Shimura crystals makes sense (and so it will be used as well) for these pseudo, quasi or potentially contexts.

We have the following Galois interpretations. Not to introduce extra notations, we refer to the context of a pseudo Shimura filtered σ-crystal $(M_0, F_0^1, \varphi_0, \tilde{G}_1)$ as above (so G_0 normalizes G_1). It is a Shimura (resp. quasi Shimura) filtered σ-crystal iff G_0 (resp. the Zariski closure in G_0 of the the connected component of the origin of $G_{0\mathbb{Q}_p}$) is a subgroup of G_1.

2) If in 2.2.8 1) we have $F^1\big(\mathrm{Lie}(G)\big) \neq 0$, then the pair $(G, [\mu])$ defines a Shimura group pair over $W(k)$.

3) It is an easy exercise to see that the $G(W(k))$-conjugacy class of parabolic Lie subalgebras of $\mathrm{Lie}(G)$ defined by $F^0(\mathrm{Lie}(G))$ is uniquely determined by the Shimura σ-crystal (M, φ, G). We call it the filtration class of the Shimura σ-crystal (M, φ, G). About μ (or the $G(W(k))$-conjugacy class $[\mu]$ defined by μ) we say that it defines the conjugacy class. $[\mu]$ is uniquely determined by (M, φ, G), and so also the $G^{\mathrm{ad}}(W(k))$-conjugacy class $[\mu^{\mathrm{ad}}]$ defined by the cocharacter $\mu^{\mathrm{ad}} : \mathbb{G}_m \to G^{\mathrm{ad}}$ induced by μ, is also uniquely determined by (M, φ, G). In particular, if G is a torus, then μ itself is uniquely determined (this can be easily seen working mod p).

4) The generalized Shimura p-divisible objects over k do not show up in this first part, except breefly in 3.1.7, 3.4.1.5-6 and 4.5.6 8). They will be used in the second part (cf. 4.5.4). We will use freely the terminology to be introduced for Shimura (filtered) σ-crystals and which depends only on their attached (cf. 2.2.13 below) Shimura Lie σ-crystals, in the context of generalized Shimura p-divisible objects over k or of $\mathcal{MF}(W(k))$.

5) When we want to emphasize a family of tensors $(v_\alpha)_{\alpha\in\mathcal{J}}$ as in a) of 2.2.8 1), we denote a Shimura σ-crystal by a quadruple $(M, \varphi, G, (v_\alpha)_{\alpha\in\mathcal{J}})$. Similarly we denote a Shimura filtered σ-crystal by a quintuple $(M, F^1, \varphi, G, (v_\alpha)_{\alpha\in\mathcal{J}})$. Strictly speaking we should refer to $(M, \phi, G, (v_\alpha)_{\alpha\in\mathcal{J}})$ as a Shimura σ-crystal with an emphasized family of tensors, but we rarely do so; the same about the filtered case.

6) If $(M, \varphi, G, (v_\alpha)_{\alpha\in\mathcal{J}})$ and $(M_1, \varphi_1, G_1, (t_\alpha)_{\alpha\in\mathcal{J}_1})$ (resp. $(M, F^1, \varphi, G, (v_\alpha)_{\alpha\in\mathcal{J}})$ and $(M_1, F_1^1, \varphi_1, G_1, (t_\alpha)_{\alpha\in\mathcal{J}_1}))$ are two Shimura σ-crystals (resp. two Shimura filtered σ-crystals) then an isomorphism between them is given by an isomorphism $t : M\tilde{\to}M_1$ taking the triple $(\varphi, G, (v_\alpha)_{\alpha\in\mathcal{J}})$ into $(\varphi_1, G_1, (t_\alpha)_{\alpha\in\mathcal{J}_1})$ (resp. taking the quadruple $(F^1, \varphi, G, (v_\alpha)_{\alpha\in\mathcal{J}})$ into $(F_1^1, \varphi_1, G_1, (t_\alpha)_{\alpha\in\mathcal{J}_1}))$. If $j : \mathcal{J}\tilde{\to}\mathcal{J}_1$ is a bijection, and if the isomorphism t takes v_α into $t_{j(\alpha)}$, then we call t a j-isomorphism. In particular a $1_{\mathcal{J}}$-isomorphism between the two Shimura σ-crystals $(M, \varphi, G, (v_\alpha)_{\alpha\in\mathcal{J}})$ and $(M, \varphi_1, G, (v_\alpha)_{\alpha\in\mathcal{J}})$ is given by an element $g \in G(W(k))$ such that $\varphi_1 = g\varphi g^{-1}$. Of course an isomorphism of (M, φ, G) does not need to fix the family of tensors $(v_\alpha)_{\alpha\in\mathcal{J}}$; easy examples can be constructed for instance if $k = \mathbb{F}_p$, and so, by natural extension over any k.

7) The condition G quasi-split, used in 2.2.8 1-4), is not really needed (cf. 3.8). It is inserted just to be able to give a concrete example of a G-ordinary Shimura filtered σ-crystal (cf. 3.2.3 below). Also proposition 2.3.9 makes it very convenient for applications to Shimura varieties. When we do not know that G is quasi-split, or when it is not, we speak about a Shimura (filtered) σ-crystal of being not necessarily quasi-split.

8) If in 2.2.8 1) we have $k = \bar{k}$, then writing $\varphi = a \circ \mu(\frac{1}{p})$, a becomes a σ-linear automorphism of M and of $\mathrm{Lie}(G)$, and $M^a := \{x \in M | a(x) = x\}$ is a $\mathbb{Z}_p$-free module of rank $\dim_{W(k)}(M)$. Then, using the upper right index a with the same purpose,

$$\mathrm{Lie}(G)^a = \mathrm{Lie}(G) \cap \mathrm{End}(M)^a \subset \mathrm{End}(M^a)$$

is the Lie algebra of a reductive subgroup $G_{\mathbb{Z}_p}$ of $GL(M^a)$. Its extension to $W(k)$ is the reductive subgroup G of $GL(M)$. Let $W(\mathbb{F}_{p^d})[\frac{1}{p}]$ be the field of definition of the conjugacy class of the cocharacter $\mu_{B(k)} : \mathbb{G}_m \to G_{B(k)}$. Here $d \in \mathbb{N}$. We deduce easily the existence of an element $g \in G(W(k))$ such that the Shimura filtered σ-crystal $\big(M, F^1, g\varphi, G, (v_\alpha)_{\alpha \in \mathcal{J}}\big)$ over k is isomorphic to the extension to k of a Shimura σ-crystal

$$\big(M^a \otimes_{\mathbb{Z}_p} W(\mathbb{F}_{p^d}), F^1_{\mu_0}, \varphi_0, G_{W(\mathbb{F}_{p^d})}, (v_\alpha)_{\alpha \in \mathcal{J}}\big)$$

over $\mathbb{F}_{p^d}$, where $F^1_{\mu_0} \subset M^a \otimes_{\mathbb{Z}_p} W(\mathbb{F}_{p^d})$ is the F^1-filtration associated to a cocharacter $\mu_0 : \mathbb{G}_m \to G_{W(\mathbb{F}_{p^d})}$ which over $W(k)$ is $G(W(k))$-conjugate to μ (cf. [Mi3, 4.6-7]). So, from many points of view, for the study of Shimura σ-crystals over a perfect field k_0 of characteristic p, we can assume that k_0 is a finite field (if $k_0 \neq \overline{k_0}$, then passing to a finite field extension k_1 of k_0, M^a, defined as above but working with $g(\varphi \otimes 1)$ instead of φ for a suitable $g \in G(W(k_1))$, is still a free $\mathbb{Z}_p$-module of rank $\dim_{W(k)}(M)$).

9) Any filtered σ-crystal (M, F^1, φ), with F^1 a proper direct summand of M, can be identified with the Shimura filtered σ-crystal $(M, F^1, \varphi, GL(M))$, and any σ-crystal (M, φ) which can be extended to a filtered σ-crystal (M, F^1, φ), with F^1 a proper direct summand of M, can be identified with the Shimura σ-crystal $(M, \varphi, GL(M))$.

10) Any Shimura (filtered) σ-crystal over k gives birth to a Shimura (filtered) isocrystal over k, by making p invertible. But not any Shimura (filtered) isocrystal over k is obtained in this manner: the Shimura isocrystals over k can have slopes which do not belong to the interval $[0, 1]$ and the reductive groups involved are not necessarily quasi-split.

11) The condition F^1 is a proper direct summand of M mentioned in 2.2.8 1) as well as the condition at least two of $\tilde{F}^i$ are different from zero, mentioned in 2.2.8 3), are inserted just to avoid the trivial cases. So the cocharacter μ of 2.2.8 c) and d) does not factor through the center of $GL(M)$. When we have $F^1 = 0$ or $F^1 = M$ in 2.2.8 1) (resp. $M = \tilde{F}^i$ in 2.2.8 3), we speak about a trivial Shimura (filtered) σ-crystals (resp. a trivial generalized p-divisible object of $\mathcal{MF}(W(k))$). Without a special reference, we assume that were are in a non-trivial situation.

12) For the sake of flexibility it is important in 2.2.8 1) not to assume that M has a symplectic perfect form on it, or that $\dim(F^1) = \dim(F^0)$ (as we have in the cases emerging from principally polarized abelian varieties over $W(k)$). So we completely ignore the phenomenon of Tate-twists. However when we have a perfect form $p_M : M \otimes_{W(k)} M \to W(k)(1)$ fixed by G, we speak about a principally quasi-polarized Shimura (filtered) σ-crystal. As notation, we put p_M as part of some n-tuple at the very end; example: $(M, F^1, \varphi, G, (v_\alpha)_{\alpha\in\mathcal{J}}, p_M)$. Moreover we do not assume that $G_{B(k)}$ is the subgroup of $GL(M[\frac{1}{p}])$ fixing v_α, $\forall\alpha \in \mathcal{J}$: we just assume that it is the subgroup of $\mathrm{GSp}(M[\frac{1}{p}], p_M)$ fixing these tensors.

2.2.10. Shimura filtered F-crystals. Let (M, F^1, φ, G) be a Shimura filtered σ-crystal. Let H be a smooth subscheme of G containing the origin, and let R be the ring of the formal completion of H at the origin. There is an isomorphism $\tilde{f} : R\tilde{\to}W(k)[[x_1, \ldots, x_d]]$, with d the relative dimension of H over $W(k)$ in the origin. Let Φ_R be the Frobenius of R extending σ on $W(k)$, and taking x_i to x_i^p, $i = \overline{1, d}$ (we use the identification defined by $\tilde{f}$). Then the filtered F-crystal over R/pR defined by the quadruple

$$(M \otimes_{W(k)} R, F^1 \otimes_{W(k)} R, \Phi, \nabla),$$

with $\Phi := h(\varphi \otimes id)$ defined by the universal element $h \in H(R)$ of H, is called a Shimura filtered F-crystal (defined by (M, F^1, φ, G)) ([Fa2, th. 10] guarantees that there is a unique connection ∇ on $M \otimes_{W(k)} R$ such that Φ is ∇-parallel in the sense of [Fa2, ch. 7]; it is integrable and nilpotent mod p). We denote such a Shimura filtered F-crystal by a sextuple $(M, F^1, \varphi, G, H, \tilde{f})$. When $H = G$, we denote it by a quintuple $(M, F^1, \varphi, G, \tilde{f})$. Usually H is a smooth subgroup of G. When we want to emphasize the family $(v_\alpha)_{\alpha\in\mathcal{J}}$, then we denote a Shimura filtered F-crystal also by a septuple $\big(M, F^1, \varphi, G, H, \tilde{f}, (v_\alpha)_{\alpha\in\mathcal{J}}\big)$, and then, by abuse of notation, we still write $(v_\alpha)_{\alpha\in\mathcal{J}}$ for their images in the essential tensor algebra of $(M \oplus M^*) \otimes_{W(k)} R$. We have:

$\nabla(v_\alpha) = 0$, $\Phi(v_\alpha) = v_\alpha$, and v_α belongs to the F^0-filtration of the essential tensor algebra of $(M \oplus M^*) \otimes_{W(k)} R[\frac{1}{p}]$, $\forall\alpha \in \mathcal{J}$.

This implies: for any point of R/pR with values in a perfect field k_1 we get, by pulling back, a Shimura σ_1-crystal over k_1 (here σ_1 is the Frobenius automorphism of $W(k_1)$).

Similarly we speak about principally quasi-polarized Shimura filtered F-crystals. Moreover, the conventions of 2.2.9 12) apply as well.

2.2.11. Definitions. 1) A non-trivial Shimura filtered Lie σ-crystal is a filtered Lie σ-crystal $\big(\mathfrak{g}, \varphi, F^0(\mathfrak{g}), F^1(\mathfrak{g})\big)$, with $\mathfrak{g}$ the Lie algebra of a reductive group G over $W(k)$, for which there is an injective cocharacter $\mu : \mathbb{G}_m \hookrightarrow G^{\mathrm{ad}}$ such that:

– $(G^{\mathrm{ad}}, [\mu])$ is a Shimura group pair over $\mathrm{Spec}(W(k))$;

– it defines $F^0(\mathfrak{g})$ and $F^1(\mathfrak{g})$: i.e. $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^{-1}$, with $\mathfrak{g}^1 \neq 0$, with $\alpha \in \mathbb{G}_m(W(k))$ acting through μ on $\mathfrak{g}^i$ as the multiplication with α^{-i}, $i = \overline{-1, 1}$, and with $F^1(\mathfrak{g}) = \mathfrak{g}^1$ and $F^0(\mathfrak{g}) = \mathfrak{g}^0 \oplus \mathfrak{g}^1$.

2) A non-trivial Shimura Lie σ-crystal is a Lie σ-crystal $(\mathfrak{g}, \varphi)$ which can be enlarged to a non-trivial Shimura filtered Lie σ-crystal $\big(\mathfrak{g}, \varphi, F^0(\mathfrak{g}), F^1(\mathfrak{g})\big)$. Writing $\varphi = a \circ \mu(\frac{1}{p})$, a is a σ-linear automorphism of $\mathfrak{g}$, whose class $[a] \in \mathrm{Aut}_{\sigma-\mathrm{lin}}(\mathfrak{g})/G^{\mathrm{ad}}(W(k))$ (modulo inner automorphism of $\mathfrak{g}$) does not depend on the choice of μ. We call $[a]$ the automorphism class of the non-trivial Shimura Lie σ-crystal $(\mathfrak{g}, \varphi)$.

3) A Lie σ-crystal $(\mathfrak{g}, \varphi)$, with $\mathfrak{g}$ the Lie algebra of a reductive group G over $W(k)$, and with $\varphi(\mathfrak{g}) = \mathfrak{g}$ is called a trivial Shimura Lie σ-crystal. When we do not bother about trivial or non-trivial, we speak about Shimura (filtered) Lie σ-crystals.

4) If in 1-3) above G is an adjoint group then we speak about a Shimura adjoint (filtered) Lie σ-crystal.

2.2.12. Definitions. A Shimura filtered Lie σ-crystal $\big(\mathfrak{g}, \varphi, F^0(\mathfrak{g}), F^1(\mathfrak{g})\big)$ is said to be:

a) of parabolic type, if the parabolic Lie subalgebra of $\mathfrak{g}$ corresponding to non-negative slopes of $(\mathfrak{g}, \varphi)$ is contained in $F^0(\mathfrak{g})$;

b) of Borel type, if there is a Borel Lie subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ contained in $F^0(\mathfrak{g})$ and such that $\varphi(\mathfrak{b}) \subset \mathfrak{b}$;

b$'$) of strong Borel type, if it is of Borel type and if the parabolic Lie subalgebra of $\mathfrak{g}$ corresponding to non-negative slopes of $(\mathfrak{g}, \varphi)$ is a Borel Lie subalgebra (so there is only one parabolic Lie subalgebra of $\mathfrak{g}$ contained in $F^0(\mathfrak{g})$ and left invariant by φ);

c) cyclic diagonalizable, if there is a $W(k)$-basis $\{e_i | i \in A\}$ of $\mathfrak{g}$ (here A is the set $\{1, \ldots, \dim_{W(k)}(\mathfrak{g})\}$) such that for subsets $C \subset B \subset A$, $\{e_i | i \in C\}$ is a $W(k)$-basis of $F^1(\mathfrak{g})$ and $\{e_i | i \in B\}$ is a $W(k)$-basis of $F^0(\mathfrak{g})$, and there is a permutation π of A for which we have $\varphi(e_i) = p^{\varepsilon_i} e_{\pi(i)}$, with $\varepsilon_i \in \{-1, 0, 1\}$ equal to 1 if $i \in C$, equal to 0 if $i \in B \backslash C$ and equal to -1 if $i \in A \backslash B$; if $\pi = 1_A$, we drop the word cyclic;

d) of toric type, if the Lie subalgebra of $\mathfrak{g}$ corresponding to the slope zero of $(\mathfrak{g}, \varphi)$ is contained in $F^0(\mathfrak{g})$.

2.2.13. Remark. Any Shimura σ-crystal (M, φ, G) has attached to it a Shimura Lie σ-crystal $(\mathrm{Lie}(G), \varphi)$, as well as a Shimura adjoint Lie σ-crystal $\big(\mathrm{Lie}(G^{\mathrm{ad}}), \varphi\big)$, where $\mathrm{Lie}(G^{\mathrm{ad}}) \subset \mathrm{Lie}(G^{\mathrm{der}})[\frac{1}{p}] \subset \mathrm{Lie}(G)[\frac{1}{p}]$ (this results trivially from rm. 8) of 2.2.9, by passage to $W(\bar{k})$). Similarly, for a Shimura filtered σ-crystal we speak about the Shimura (adjoint) filtered Lie σ-crystal attached to it.

2.2.14. Remark. Whenever we want to emphasize k in 2.2.11, we speak about a Shimura (adjoint) (filtered) Lie σ-crystal of being over k. If the reductive group G is split, then we speak about a split Shimura (adjoint) (filtered) (Lie) σ-crystal. If just G^{ad} is split then we speak about a split Shimura (adjoint) (filtered) Lie σ-crystal. We also speak about Shimura (adjoint) (filtered) (Lie) isocrystals (over k), obtained from Shimura (adjoint) (filtered) (Lie) σ-crystals, by making p invertible. For instance with the notations of 2.2.8 1), $(M[\frac{1}{p}], \varphi, G_{B(k)})$ is a Shimura isocrystal.

2.2.15. Remark. Though G. Shimura never dealt with the type of crystals we have introduced, we still feel that the terminology Shimura crystals used in the above part of 2.2 is appropriate, as Shimura was the first one (see the report [Sh]) to consider abelian varieties endowed with (algebraic) cycles, of whose adequate integral crystalline realization, are providing the first examples of the type of σ-crystals we have defined in 2.2.8. Similarly we think that the terminology of a Shimura group or Lie pair is justified. Of course we could argue if 2.2.8 5-6) are justified. We think it is a convenient terminology, very much related to the one introduced in 2.2.8 1-4). The study of such isocrystals, to our knowledge, was started in [Ko1].

2.2.16. Lemma. *We assume that $k = \bar{k}$. Let (M, F^1, φ) be a filtered σ-crystal over k, with F^1 a proper direct summand of M. It is cyclic diagonalizable iff there is a subtorus T of $GL(M)$ such that (M, F^1, φ, T) is a Shimura filtered σ-crystal.*

Proof: If (M, F^1, φ) is cyclic diagonalizable, then we can take T to be the maximal torus of $GL(M)$ which leaves invariant the $W(k)$-submodules of M generated by any element e_i, $i = \overline{1, dim(M)}$, of a basis of M which is as in 2.2.1 b).

If there is a subtorus T of $GL(M)$ such that (M, F^1, φ, T) is a Shimura filtered σ-crystal, then the proof of cyclic diagonalizability of (M, F^1, φ) is entirely as of 4.1.2.

2.2.16.1. Remark. In practice we take T to be the smallest subtorus of $GL(M)$ such that $\mathrm{Lie}(T)$ contains $\mu(\mathrm{Lie}(\mathbb{G}_m))$, where $\mu : \mathbb{G}_m \to GL(M)$ is the canonical split defined in [Wi].

2.2.17. Definitions. Let (M, φ, G) be a Shimura σ-crystal over k. By a lift of it of quasi CM type we mean a direct summand (filtration) F^1 of M such that the quadruple (M, F^1, φ, G) is a Shimura filtered σ-crystal with the property that the intersection of $F^0(\mathrm{Lie}(G_{B(k)}))$ with the maximal Lie subalgebra $\mathfrak{p}_0$ of $\mathrm{Lie}(G_{B(k)})$ left invariant by φ and having the property that the isocrystal $(\mathfrak{p}_0, \varphi)$ has all the slopes zero, contains a φ-invariant Lie subalgebra, which is the Lie algebra of a maximal torus of G. We also say that (M, F^1, φ, G) is a lift of quasi CM type of (M, φ, G).

By a lift of an abelian variety A_k over k of quasi CM type we mean an abelian variety A over $W(k)$ lifting A_k and such that the filtered σ-crystal attached to A is a lift of quasi CM type of the σ-crystal attached to A_k (here we use the identifications of 2.2.9 9)). Here the abbreviation CM refers to complex multiplication.

2.2.17.1. Remark. A Shimura σ-crystal can have more than one lift of quasi CM type. For instance this is the case for the σ-crystal attached to (a product of) supersingular elliptic curves over k. Of course instead of slope $\frac{1}{2}$ we can work equally fast and well with any other slope $\frac{a}{r}$, with $r \in \mathbb{N}$, $r \geq 2$, $(a, r) = 1$, and a σ-crystal of rank r and pure slope $\frac{a}{r}$: we get an infinite number of lifts of quasi CM type. These examples are very simple; but in some sense (see 4.4.13) they are typical.

2.2.18. Remark. There are plenty of examples of filtered σ-crystals (M, F^1, φ), with $\dim(M) = 2\dim(F^1)$, such that the σ-crystal (M, φ) does not have any lift of quasi CM type. The idea of constructing such examples is: if $k = \bar{k}$, then a σ-crystal (M, φ) which has a lift of quasi CM type is a direct sum of σ-crystals having only one slope (cf. 2.2.19). Moreover, the different invariants (like Hodge numbers, the p-adic valuation of determinants, etc.) of these factors are of particular type, and so very easy computable. So we just have to pass to an adequate isogeny. Here is an example with slopes having quite small denominators.

Example 1. Let M be of rank 6 over $W(k)$, and let φ such that it sends the elements of an ordered basis $(e_1, ..., e_6)$ of M into the ordered set $(e_2, pe_3, e_4, pe_5, e_1, pe_6)$. Let M' be the lattice of $M[\frac{1}{p}]$ generated by M and $\frac{e_2+e_4}{p}$. It is easy to see (arguments involving determinants) that (M',φ) is not cyclic diagonalizable. Moreover, if F^1 is the $W(k)$-submodule of M' generated by e_2, e_6 and e_1+e_3, then the triple $\mathfrak{C} := (M',F^1,\varphi)$ is a filtered σ-crystal having one slope 1 and 5 slopes $\frac{2}{5}$. Moreover $\mathfrak{C}\oplus\mathfrak{C}^*(1)$ has a principal quasi-polarization, but it has no lift of quasi CM type.

We would also like to recall that there are filtered σ-crystals (M,F^1,φ) with $\dim(M) = 2\dim(F^1)$ such that there is a lattice M' of $M[\frac{1}{p}]$ with the property that the pair (M',φ) is a σ-crystal which can not be extended to a filtered σ-crystal.

Example 2. Let M be of rank 4 over $W(k)$, and let φ such that it sends the elements e_0, e_1, f_0 and f_1 of a basis of M, into pe_0, e_1, pf_1 and respectively f_0. Let M' be the lattice of $M[\frac{1}{p}]$ generated by e_0, e_1, $\frac{e_0+f_0}{p}$ and f_1. Obviously (M',φ) can not be extended to a filtered σ-crystal.

2.2.19. Proposition. *We assume that $k=\bar{k}$. Let F^1 be a lift of quasi CM type of a Shimura σ-crystal (M,φ,G). Then the filtered σ-crystal (M,F^1,φ) is cyclic diagonalizable.*

Proof: Let $\mu : \mathbb{G}_m \to GL(M)$ be the canonical split of (M,F^1,φ) as defined in [Wi]. From its functoriality we deduce (cf. def. 1) of 2.2.8) that μ factors through G. From the def. 2.2.17 we deduce that there is a $\mathbb{Q}_p$-Lie subalgebra of $F^0(\mathrm{Lie}(G_{B(k)}))$ formed by elements fixed by φ, and which generates the Lie algebra of a maximal torus $T_{B(k)}$ of $G_{B(k)}$. Again from the functoriality of μ we deduce that μ fixes the elements of this $\mathbb{Q}_p$-Lie algebra, and so the generic fibre of μ factors through $T_{B(k)}$. Let $T^1_{B(k)}$ be the smallest subtorus of $T_{B(k)}$ such that the generic fibre of μ factors through it and its Lie algebra is left invariant by φ. It is easy to check that the Zariski closure of $T^1_{B(k)}$ in $GL(M)$ is a subtorus T^1 of G. So μ factors through T^1. We get a Shimura filtered σ-crystal (M,F^1,φ,T^1) (cf. 2.2.9 1)). Now everything results from 2.2.16. This ends the proof of the proposition.

2.2.20. Shimura p-divisible groups. Let X be a regular formally smooth scheme over $W(k)$. A Shimura p-divisible group over X is a pair $(D,(v_\alpha)_{\alpha\in\mathcal{J}})$ comprising from a p-divisible group D over X, and a family of sections $(v_\alpha)_{\alpha\in\mathcal{J}}$ of the essential tensor algebra of the filtered isocrystal (sheaf) on $X^\wedge$ defined by the first crystalline cohomology group sheaf of $D\times D^t$, satisfying the following axiom:

A. There is an open affine covering of X, such that for any member $U=\mathrm{Spec}(R)$ of it whose special fibre is non-empty, denoting $M_R := H^1_{\mathrm{crys}}(D_U/U^\wedge)$, the mentioned sections produce sections (we keep the same notation) of $\mathcal{T}(M_R[\frac{1}{p}])$ satisfying:

i) they are crystalline sections, i.e they are annihilated by the natural connection on M_R, are in the F^0-filtration of $\mathcal{T}(M_R[\frac{1}{p}])$, and are fixed by any Frobenius of M_R defined by a (any) Frobenius lift (it always exists) of $R^\wedge$;

ii) the Zariski closure of the subgroup of $GL(M_R[\frac{1}{p}])$ fixing v_α, $\forall\alpha\in\mathcal{J}$, in $GL(M_R)$ is a reductive group $\tilde{G}_R$ over R;

iii) the F^1-filtration of M_R becomes a proper direct summand in each geometric point of the special fibre of U.

2.2.21. Remarks. 1) Any p-divisible group whose associated crystal is a Shimura filtered σ-crystal or a Shimura filtered F-crystal with an emphasized family of tensors, gets naturally the structure of a Shimura p-divisible group.

2) The importance of this notion springs from 1.16. For the present paper we need it mostly just to simplify the presentation.

3) As in 2.2.9 6) we have similarly the notion of isomorphism between two Shimura p-divisible groups.

4) Sometimes we need to speak about principally quasi-polarized Shimura p-divisible groups. The extra thing we need for this is: D has a principal quasi-polarization p_D, which as a perfect form on M_R is fixed by $\tilde{G}_R$, for any U as above. Notation: $(D,(v_\alpha)_{\alpha\in\mathcal{J}},p_D)$. Moreover we allow 2.2.20 ii) to be modified as in 2.2.9 12).

5) If X_1 is a regular formally smooth scheme over $W(k)$, then the pull back of a Shimura p-divisible group over X through a morphism $X_1 \to X$, is also a Shimura p-divisible group (over X_1).

6) Of course, instead of crystalline sections we can use étale sections. But in the case when X is not (the p-adic completion of) a pro-étale scheme over a smooth scheme over $W(k)$, it is more convenient to work with crystalline sections: see the way [Fa1, 7.1] is stated. Moreover, in §5 we define Shimura p-divisible groups over larger classes of schemes (including ones of positive characteristic), and so it is much more convenient to work in the crystalline context.

7) It is convenient to allow the family $(v_\alpha)_{\alpha\in\mathcal{J}}$ to vary under the constraint that the reductive groups $\tilde{G}_R$ of 2.2.20 ii) remain the same. In other words by enlarging $\mathcal{J}$ or by passing to a subset of $\mathcal{J}$ subject to the mentioned constraint we consider that we have essentially the same Shimura p-divisible group. This leads to the notion of quasi-isomorphism: two Shimura p-divisible groups over X, $E^1 = (D^1,(v_\alpha^1)_{\alpha\in\mathcal{J}})$ and $E^2 = (D^2,(v_\alpha^2)_{\alpha\in\mathcal{J}})$, are said to be quasi-isomorphic if there is an isomorphism $t : D^1 \tilde{\to} D^2$ such that for any U as in 2.2.20, the reductive group $\tilde{G}_R^1$ we get via E^1 is mapped through t isomorphically into $\tilde{G}_R^2$ obtained via E^2. Similarly for the principally quasi-polarized case.

8) We would like to point out that the terminologies of Shimura p-divisible groups and crystals do not quite match: in 2.2.20 we do not require $\tilde{G}_R$ to become quasi-split over all points of R with values in fields of positive characteristic, while in 2.2.8 the accent does not fall on the cycles. The reason we adopted these terminologies is of purely practical matters: in the case of Shimura p-divisible groups we are essentially interested in understanding the cycles and the refined structure of the reductive groups involved, while we use mainly the σ-crystals as a tool for achieving this goal (see §5).

The condition iii) of 2.2.20 A is inserted to match 2.2.9 11). The conventions of 2.2.9 11) extend naturally to the context of Shimura p-divisible groups.

9) If $\mathcal{D} = (D,(v_\alpha)_{\alpha\in\mathcal{J}})$ is a Shimura p-divisible group over X, then for any $n \in \mathbb{N}$, $D[p^n]$ defines an object $\mathcal{D}_n$ of $\mathcal{MF}^\nabla_{[0,1]}(X)$, with the property that the Lie object $\mathrm{End}(\mathcal{D}_n)$ of $\mathcal{MF}^\nabla_{[-1,1]}(X)$ defined by gluing the objects defined naturally by $\mathrm{End}(M_R/p^nM_R)$, has a natural Lie subobject $\mathfrak{g}(\mathcal{D}_n)$ defined naturally by gluing $\mathrm{Lie}(\tilde{G}_R)\otimes_R R/p^nR$. We refer to the pair

$$(\mathcal{D}_n, \mathfrak{g}(\mathcal{D}_n))$$

as the kernel of the multiplication by p^n of $\mathcal{D}$. Similarly, we use the same language in the principally quasi-polarized context. We usually assume that $p \geq 3$ or that $X^\wedge$ is equipped with a Frobenius lift (so that we do not get into complications with [Fa1, 2.3]).

2.2.22. Complements on Shimura σ-crystals. 1) A Shimura σ-crystal (M,φ,G) over k is said to be:

– essentially cyclic diagonalizable, if there is a torus T of G such that (M,φ,T) is a quasi Shimura σ-crystal;

– cyclic diagonalizable, if there is a torus T as above, and we can proceed as in 4.1.2 to show that (M,F^1,φ) is cyclic diagonalizable (here F^1 is the unique direct summand of M such that (M,F^1,φ,T) is a Shimura filtered σ-crystal);

– potentially cyclic diagonalizable if over $\bar{k}$ it is essentially cyclic diagonalizable.

Here is what we mean by "we can proceed as in 4.1.2": if $\mu:\mathbb{G}_m\to T$ is the cocharacter defining the filtration class of (M,F^1,φ,T), then the σ-linear automorphism $a: M\tilde{\to}M$ defined by $a\mu(\frac{1}{p})=\varphi$, defines a $\mathbb{Z}_p$-structure on M. If $k=\bar{k}$ then the essential cyclic diagonalizability implies the cyclic diagonalizability; this is not necessarily true if $k\neq\bar{k}$.

2) Let (M,F^1,φ,G) be a Shimura filtered σ-crystal. By the isogeny class of it we mean the set of isomorphism classes of Shimura filtered σ-crystals of the form $(g(M),g(F^1),\varphi,G)$, with $g\in G(B(k))$ (the Zariski closure of $G_{B(k)}$ in $GL(g(M))$ is a reductive group isomorphic to G -cf. [Ti, end of 2.5]-; so it is appropriate to write the things the way we did). Similarly we define the isogeny class of a Shimura σ-crystal or of a Shimura (adjoint) (filtered) Lie σ-crystal (even more generally: of a generalized Shimura p-divisible object over k or of $\mathcal{MF}(W(k))$).

3) Any isocrystal over $\bar{k}$ can be put in a cyclic diagonalizable form, cf. Dieudonné's classification of isocrystals over $\bar{k}$. So the simplest type of σ-crystals over k are those which can be put similarly in a cyclic diagonalizable form. What we defined (see 2.2.1 b) and 2.2.12 c)) in a context related to p-divisible groups can be defined in the general context of p-divisible objects of $\mathcal{MF}(W(k))$. For simplicity, we keep focussing our attention on filtered σ-crystals.

Let $n\in\mathbb{N}$. Here we consider only n-tuples $(e_1,e_2,...,e_n)$ with $e_i\in\{0,1\}$, $\forall i\in\{1,...,n\}$. We are interested in them only from the circular point of view: for us $(e_1,...,e_n)$ is "equally good" as $(e_j,e_{j+1},...,e_n,e_1,...,e_{j-1})$, where $j\in\{2,...,n\}$. We say that an n-tuple $(e_1,...,e_n)$ is in standard form if

$$\sum_{i=1}^{n} e_i 2^{n-i}\geq\sum_{i=1}^{n} d_i 2^{n-i},$$

for any other n-tuple of the form $(d_1,...,d_n)=(e_j,...,e_n,e_1,...,e_{j-1})$, for some $j\in\{2,...,n\}$.

For any n-tuple $(e_1,...,e_n)$ which is in standard form we can construct a circular diagonalizable filtered σ-crystal (M,F^1,φ) as follows:

M is a free $W(k)$-module of rank n, $\varphi(a_i)=p^{e_i}a_{i+1}$, where $n+1=1$ and $\{a_i|i=\overline{1,n}\}$ is a basis of M, while F^1 is the submodule of M generated by all a_i's with $e_i=1$.

About (M, F^1, φ) we say: it is in standard form associated to $(e_1, ..., e_n)$; about $(e_1, ..., e_n)$ we say: it is a type of (M, F^1, φ). From the very def. 2.2.1 b) we get that any filtered σ-crystal which is cyclic diagonalizable is a direct sum of circular diagonalizable ones; moreover, any filtered σ-crystal which is circular diagonalizable has a type. Warning: there are circular diagonalizable filtered σ-crystals which do not have a unique type. For instance the quadruples $(1, 1, 0, 0)$ and $(1, 0, 1, 0)$ are producing isomorphic filtered σ-crystals.

4) By the deformation dimension of (M, φ, G) we mean the number

$$dd((M, \varphi, G)) := \dim_{W(k)}(\mathrm{Lie}(G)/F^1(\mathrm{Lie}(G))).$$

Here $F^1(\mathrm{Lie}(G))$ is defined as in 2.2.8 1) starting from any (it does not matter which one) direct summand F^1 of M such that (M, F^1, φ, G) is a filtered σ-crystal.

2.3. The standard Hodge situation.

2.3.1. Let (W, ψ) be a symplectic space over $\mathbb{Q}$, and let

$$f : \mathrm{Sh}(G, X) \hookrightarrow \mathrm{Sh}\big(\mathrm{GSp}(W\psi), S\big)$$

be an injective map. Let $p \geq 3$ be a rational prime. We assume the existence of a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W such that ψ induces a perfect form $\psi : L_{(p)} \otimes_{\mathbb{Z}_{(p)}} L_{(p)} \to \mathbb{Z}_{(p)}$ and the Zariski closure $G_{\mathbb{Z}_{(p)}}$ of G in $\mathrm{GSp}(L_{(p)}, \psi)$ is a reductive group over $\mathbb{Z}_{(p)}$. We call such a $\mathbb{Z}_{(p)}$-lattice good with respect to the map f (cf. [Va2, 5.8.3]). This assumption implies that G is unramified over $\mathbb{Q}_p$. Let v be a prime of $E(G, X)$ dividing p. It is unramified over p [Mi3, 4.7]. Let $K := \mathrm{GSp}(W, \psi)(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p)$ and let $H := K \cap G(\mathbb{Q}_p)$. K is a hyperspecial subgroup of $\mathrm{GSp}(W, \psi)(\mathbb{Q}_p)$ (as $\psi : L_{(p)} \otimes_{\mathbb{Z}_{(p)}} L_{(p)} \to \mathbb{Z}_{(p)}$ is a perfect form) and H is a hyperspecial subgroup of $G(\mathbb{Q}_p)$ (as $G_{\mathbb{Z}_{(p)}}$ is a reductive group and as $H = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$). Let

$$e := \frac{\dim_{\mathbb{Q}}(W)}{2}.$$

Let $(v_\alpha)_{\alpha \in \mathcal{J}}$ be a family of homogeneous tensors of the essential tensor algebra of $W \oplus W^*$, such that G is the subgroup of $\mathrm{GSp}(W, \psi)$ fixing the tensors of this family (cf. [De3, 3.1 c)]). Sometimes it is more convenient to work with an enlarged family $(v_\alpha)_{\alpha \in \mathcal{J}'}$ (with $\mathcal{J} \subset \mathcal{J}'$) of such tensors, such that G is the subgroup of $GL(W)$ fixing its members. Also it is convenient to consider that all the elements of $\mathrm{Lie}(Z(G))$ are part of this enlarged family. As $G(\mathbb{Q})$ contains the group of scalar automorphisms of W, we deduce that for any $\alpha \in \mathcal{J}'$, $v_\alpha \in W^{\otimes n_\alpha} \otimes_{\mathbb{Q}} W^{*\otimes n_\alpha}$, with $n_\alpha := \frac{\deg(v_\alpha)}{2}$. Let $G^0_{\mathbb{Z}_{(p)}}$ be the maximal (connected) reductive subgroup of $G_{\mathbb{Z}_{(p)}}$ fixing ψ. So $G^0_{\mathbb{Z}_{(p)}} = \mathrm{Sp}(L_{(p)}, \psi) \cap G_{\mathbb{Z}_{(p)}}$ if the center of $G^0_{\mathbb{Q}}$ has -1_W as a $\mathbb{Q}$–valued point.

To any $x \in X$ it corresponds an injective cocharacter $\mu_x : \mathbb{G}_m \hookrightarrow G_{\mathbb{C}}$ (see [Va2, 2.2]). We often prefer to view G as a subgroup of $GL(W^*)$ or of $\mathrm{GSp}(W^*, \tilde{\psi})$, where $\tilde{\psi}$ is the perfect form on W^* induced by ψ, and then we denote by

$$\mu_x^* : \mathbb{G}_m \hookrightarrow G_{\mathbb{C}}$$

the injective cocharacter (in the dual context) induced by μ_x. So μ_x^* is the cocharacter defined, as in [Va2, 2.2], but for the Hodge structure on W^* defined by $x \in X$. It achieves a decomposition $W^* \otimes_{\mathbb{Q}} \mathbb{C} = F_x^1 \oplus F_x^0$, with $\beta \in \mathbb{G}_m(\mathbb{C})$ acting through it by multiplication with β^{-i} on F_x^i, $i = \overline{0,1}$.

2.3.2. We get an injective map $f : (G, X, H, v) \hookrightarrow \big(\mathrm{GSp}(W, \psi), S, K, p\big)$ of Shimura quadruples. Let $\mathcal{M}$ be the extension to $O_{(v)}$ of the integral canonical model of $\big(\mathrm{GSp}(W, \psi), S, K, p\big)$. It is known (see [Va2, 3.2.9]) that $\mathcal{M}$ parametrizes isomorphism classes of principally polarized abelian schemes of dimension e over $O_{(v)}$-schemes, having level-N symplectic similitude structure for any $N \in \mathbb{N}$ relatively prime to p (the choice of a $\mathbb{Z}$-lattice L of W such that $L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = L_{(p)}$ and $\psi : L \otimes_{\mathbb{Z}} L \to \mathbb{Z}$ is a perfect form is implicit here, cf. [Va2, 4.1]).

2.3.3. Let $\mathcal{N}$ be the integral canonical model of (G, X, H, v) (for $p \geq 5$ cf. [Va2, 6.4.1] and the definitions of [Va2, 3.2.6]; for $p = 3$ cf. §6). Due to the extension property of $\mathcal{M}$, we have a natural morphism $\mathcal{N} \to \mathcal{M}$ (cf. [Va2, rm. 4) of 3.2.7]). From [Va2, 3.2.12] we deduce that $\mathcal{N}$ is the normalization of the Zariski closure of $\mathrm{Sh}(G, X)/H$ in $\mathcal{M}$. From the proof of [Va2, 3.4.1] we deduce that the morphism $\mathcal{N} \to \mathcal{M}$ is finite, and not just pro-finite.

The universal principally polarized abelian scheme over $\mathcal{M}$ gives birth, by pull back, to a principally polarized abelian scheme $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ over $\mathcal{N}$. $\mathcal{A}$ is naturally endowed with a family $(w_\alpha^{\mathcal{A}})_{\alpha \in \mathcal{J}'}$ of Hodge cycles. The points of $\mathcal{N}$ with values in fields $\mathcal{K}$ of characteristic zero, give birth to principally polarized abelian varieties (obtained from $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ by pull back), having a family $(w_\alpha)_{\alpha \in \mathcal{J}'}$ of Hodge cycles (obtained from $(w_\alpha^{\mathcal{K}})_{\alpha \in \mathcal{J}'}$ by pull back; they are defined over $\mathcal{K}$, due to the existence of the level structures), and satisfying some additional conditions and level structures (cf. [Va2, 4.1]).

We denote by H_0 an arbitrary compact open subgroup of $G(\mathbb{A}_f^p)$ such that:

– the subgroup $H_0 \times H$ of $G(\mathbb{A}_f)$ is smooth for (G, X) (so the quotient morphism $\mathcal{N} \to \mathcal{N}/H_0$ is a pro-étale cover, cf. [Va2, 3.4.8 and 6.4.2.1]; loc. cit. are stated for $p = 3$ as well);

– there is a universal principally polarized abelian scheme $(\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ over $\mathcal{N}/H_0$, with $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ obtained from $(\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ through the pro-étale cover $\mathcal{N} \to \mathcal{N}_{H_0}$.

$\mathcal{N}/H_0$ is a quasi-projective smooth scheme over $O_{(v)}$, cf. [Va1, 6.4.1]. Warning: we do not assume that the family $(w_\alpha^{\mathcal{K}})_{\alpha \in \mathcal{J}'}$ is defined over $\mathcal{N}/H_0$.

2.3.3.1. If (G, X) is of compact type, then $\mathcal{N}/H_0$ is a projective smooth scheme over $O_{(v)}$, cf. [Va2, 6.4.1.1 2)] (the proof of loc. cit. applies to the case $p = 3$ as well).

2.3.4. Definition. The $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W is said to be crystalline well positioned for the map f with respect to v, if, besides the properties mentioned in 2.3.1 (i.e. it is good with respect to f), it satisfies the following property:

(∗) *for any perfect field k of characteristic p and for any $z \in \mathcal{N}(W(k))$, the principally polarized abelian variety (A, p_A) over $W(k)$ having a family $(w_\alpha)_{\alpha \in \mathcal{J}}$ of Hodge cycles (obtained from $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ and its family $(w_\alpha^{\mathcal{A}})_{\alpha \in \mathcal{J}}$ of Hodge cycles through the point z), has the* *property that the Zariski closure in $GL(M)$ (with $M := H^1_{\mathrm{crys}}(A/W(k)) = H^1_{dR}(A/W(k)))$ of the subgroup of $GL(M[\frac{1}{p}])$ fixing the perfect alternating form $p_A : M \otimes_{W(k)} M \to W(k)(1)$ (induced by the polarization p_A) and the crystalline (de Rham) component t_α of ω_α (t_α is a homogeneous tensor of $\mathcal{T}(M[\frac{1}{p}])$), $\forall \alpha \in \mathcal{J}$, is a reductive group $\tilde{G}$ over $W(k)$.*

If $L_{(p)}$ is crystalline well positioned for the map f with respect to any prime v of $E(G, X)$ dividing p, then we say that $L_{(p)}$ is crystalline well positioned for the map f.

2.3.5. Definition. A triple $(f, L_{(p)}, v)$ as in 2.3.1, with $L_{(p)}$ a $\mathbb{Z}_{(p)}$-lattice of W which is crystalline well positioned for the map f with respect to v, is called a standard Hodge situation, abbreviated SHS. If moreover there is a $\mathbb{Z}_{(p)}$-subalgebra $\mathcal{B}$ of $\mathrm{End}(L_{(p)})$, which over $W(\mathbb{F})$ is a product of matrix algebras, which is self dual with respect to ψ, and is such that the subgroup of $\mathrm{GSp}(L_{(p)}, \psi)$ fixing its elements has $G_{\mathbb{Z}_{(p)}}$ as its connected component of the origin, then we speak about a standard PEL situation, and denote it by $(f, L_{(p)}, v, \mathcal{B})$ (cf. [Va2, 4.3.11]).

2.3.5.1. Exercise. Let (G_0, X_0) be an adjoint Shimura pair whose simple factors are of A_n type for some $n \in \mathbb{N}$. Let $p > 2$ be a prime such that G_0 is unramified over $\mathbb{Q}_p$. Show that there is an injective map $f\colon (G, X) \hookrightarrow (\mathrm{GSp}(W, \psi), S)$, with $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) = (G_0, X_0)$, and there is a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W good for f, such that for any prime v of $E(G, X)$ dividing p we get a standard PEL situation $(f, L_{(p)}, v, \mathcal{B})$, for a suitable $\mathbb{Z}_{(p)}$-subalgebra $\mathcal{B}$ of $\mathrm{End}(L_{(p)})$. Hint: use [Va2, 6.5 and Case 2 of 6.6.5]; in other words, loc. cit. handles the case $p \geq 5$, and so for $p = 3$ we just have to copy the arguments for $p \geq 5$.

2.3.5.2. Remark. Let $(f, L_{(p)}, v)$ be a SHS as above. Let $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) = \prod_{i \in \mathcal{F}} (G_i, X_i)$ be the direct factor decomposition in simple Shimura varieties. Here $\mathcal{F}$ is a finite set of indices. For any $i \in \mathcal{F}$ there is an injective map $f_i : (G^i, X^i) \hookrightarrow (G, X)$ such that:

– $G^{i\mathrm{der}}$ is the semisimple subgroup of G naturally isogeneous to G_i (so $(G^{i\mathrm{ad}}, X^{i\mathrm{ad}}) = (G_i^{\mathrm{ad}}, X_i^{\mathrm{ad}})$);

– $(f \circ f_i, L_{(p)}, v_i)$ is a SHS, for any prime v_i of $E(G^i, X^i)$ dividing v.

This is an immediate consequence of [Va2, rm. 11) of 3.2.7]. So from many points of view for things involving a SHS $(f, L_{(p)}, v)$, we can assume that G^{ad} is a simple $\mathbb{Q}$–group. However, for the sake of generality, we never assume this a priori.

2.3.6. Example. If the family of tensors $(v_\alpha)_{\alpha \in \mathcal{J}}$ is $\mathbb{Z}_{(p)}$-well positioned with respect to ψ for the group G, if it is enveloped by $L_{(p)}$ (see [Va2, 4.3.4] for the meaning of these), and if it is of degree at most $2(p-2)$, then the triple $(f, L_{(p)}, v)$ is a SHS (cf. [Va2, 5.1 and 5.6.5]). More generally, if there is a family of tensors of $\mathcal{T}(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(\mathbb{F}))$ which is $W(\mathbb{F})$-very well positioned for $G_{W(\mathbb{F})}$ and of degree at most $2(p-2)$, then the triple $(f, L_{(p)}, v)$ is still a SHS, cf. [Va1, 5.6.9].

2.3.7. Convention. Whenever we consider a SHS $(f, L_{(p)}, v)$, we use freely the notations of 2.3.1-3.

2.3.8. Remarks. 1) We do not know when the converse of 2.3.6 is true. That is, we do not know for which SHS $(f, L_{(p)}, v)$ there is a family $(v_\alpha)_{\alpha \in \mathcal{J}}$ of homogeneous tensors of the essential tensor algebra of $W \oplus W^*$, which is $\mathbb{Z}_{(p)}$-well positioned with respect to ψ

for G, is enveloped by $L_{(p)}$, and is of degree at most $2(p-2)$. Obviously for $p=3$ we need a standard PEL situation.

2) A major result of §6 states that any $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W good with respect to f is automatically crystalline well positioned for the map f.

3) Usually when we mention the polarizations we use the set of indices $\mathcal{J}$ (i.e. we would like to "carry" as few Hodge cycles as possible); when we do not, then we use $\mathcal{J}'$. If this is confusing to the reader, the reader can assume that $\mathcal{J}=\mathcal{J}'$.

2.3.8.1. Exercise. Let (G,X,H,v) be a Shimura quadruple of Hodge type, with v dividing $p\geq 3$. Assuming 2.3.8 2) show that there is a SHS $(f,L_{(p)},v)$ for which we do have $H=G(L_{(p)}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Z}_p)$. Hint: just restate the proof of [Va2, 6.7.2].

2.3.9. Proposition. *a) $\tilde{G}$ is an inner form of $G_{W(k)}$; moreover $\tilde{G}^{\text{ab}}$ is a split torus.*

b) The reductive group $\tilde{G}$ mentioned in 2.3.4 splits over a finite Galois cover $\operatorname{Spec}(W(k_1))$ *of* $\operatorname{Spec}(W(k))$ *of order dividing a fixed power of the order q_G of the center of the simply connected group cover of $G^{\text{der}}_{\mathbb{C}}$. Moreover, the Galois group* $\operatorname{Gal}(k_1/k)$ *is nilpotent of index of nilpotency at most two. If for each simple factor (G_i,X_i) of $(G^{\text{ad}},X^{\text{ad}})$ of $D_n^{\mathbb{R}}$ type with $n\in 2\mathbb{N}$, the semisimple subgroups of $G^{\text{der}}_{\mathbb{C}}$ naturally isogeneos to a simply factor of $G_{i\mathbb{C}}$, are simply connected, then this Galois group is abelian.*

c) If all the factors of G^{ad} are of C_n or A_n type for some $n\in\mathbb{N}$, and if for any simple factor $G'_{\mathbb{C}}$ of $G^{\text{ad}}_{\mathbb{C}}$ of A_n type, the semisimple subgroup of $G^{\text{der}}_{\mathbb{C}}$ naturally isogeneous to it is a simply connected semisimple group, then $\tilde{G}$ is isomorphic to $G_{W(k)}$, regardless of the choice of the $W(k)$-valued point z.

d) If G^{der} is a simply connected semisimple group, then the Galois group of b) is abelian of exponent 2.

Proof: First we remark that $k\supset\mathbb{F}=\overline{k(v)}$. This is so due to the existence of level-N structures (of A_k) (so the N-th roots of unity are contained in k), $\forall N\in\mathbb{N}$ with $(N,p)=1$. So $B(k)\supset B(\mathbb{F})$. The proposition is a consequence of this last inclusion and of Fontaine' s comparison theory. We present the instructive details.

a) As G is unramified over $\mathbb{Q}_p$, $G_{B(\mathbb{F})}$ is a split group. So $G_{W(k)}$ is a split group. Fontaine's comparison theory implies that $\tilde{G}_{B(k)}$ is an inner form of $G_{B(k)}$. But $\tilde{G}$ splits over a finite étale extension of $W(k)$. So over this extension it becomes isomorphic to $G_{W(k)}$, cf. the uniqueness of a split reductive group associated to a given root datum (see [SGA3, Vol. 3, p. 305]). We deduce that $\tilde{G}$ is an inner form of $G_{W(k)}$. This implies that $\tilde{G}^{\text{ab}}$ is a split torus.

Now we prove simultaneously the other parts. Any element of $\mathcal{T}(L^*_{(p)}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Q}_p)$ fixed by $G_{\mathbb{Q}_p}$ gives birth (cf. the identification of [Va2, 5.1.2]) through Fontaine's comparison theory to an element of $\mathcal{T}(M[\frac{1}{p}])$ fixed by $\tilde{G}$. As $B(k)\supset B(\mathbb{F})$, we can use $B(\mathbb{F})$-linear combinations of such elements. We denote by $\operatorname{Fix}(G_{B(\mathbb{F})})$ the set of tensors of $\mathcal{T}(L^*_{(p)}\otimes_{\mathbb{Z}_{(p)}} B(\mathbb{F}))$ fixed by $G_{B(\mathbb{F})}$. So any element of $\operatorname{Fix}(G_{B(\mathbb{F})})$ has a counterpart in $\mathcal{T}(M[\frac{1}{p}])$. As $G_{B(\mathbb{F})}$ is a split group, and as all the simple Lie factors of $\operatorname{Lie}(G^{\text{ad}}_{B(\mathbb{F})})$ are classical (split) simple Lie algebras over $B(\mathbb{F})$, we can often "capture" the fact that $G_{B(\mathbb{F})}$ is split using just the set $\mathrm{Fix}(G_{B(\mathbb{F})})$.

We first assume that for any simple factor of $G^{\mathrm{ad}}_{B(\mathbb{F})}$ coming from a simple factor of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ of A_n ($n \in \mathbb{N}$) or $D_n^{\mathbb{R}}$ type ($n \in 2\mathbb{N}$), the semisimple subgroup of $G^{\mathrm{der}}_{B(\mathbb{F})}$ naturally isogeneous to it, is simply connected. This is automatically so in cases c) and d). Using this assumption, [De3, 3.1 (a)], the Weyl complete reductibility theorem, the structure of the irreducible representations of a direct sum of absolutely simple split semisimple Lie algebras over $B(k)$, and the fact that the irreducible subrepresentations of $L_p^* \otimes_{\mathbb{Z}_p} B(k)$ of a simple Lie factor of $\mathrm{Lie}(G^{\mathrm{ad}}_{B(\mathbb{F})})$ are given by minimal weights we deduce:

Fact. *For any simple factor* $\mathfrak{g}_0$ *of* $\mathrm{Lie}(G^{\mathrm{der}}_{B(\mathbb{F})})$*, we can single out the tensor product* $\mathcal{TP}$ *of the classical representation* W_0 *of* $\mathfrak{g}_0$ *(over* $B(\mathbb{F})$*) (its complex version is presented in [He, §8 of ch. 2]) with other representations of the same type of other simple factors of* $\mathrm{Lie}(G^{\mathrm{ad}}_{B(k)})$*, just by using* $\mathrm{Fix}(G_{B(\mathbb{F})})$*; moreover, the bilinear form (if it is not zero, then it is non-degenerate and unique up to a non-zero scalar)* b_0 *on* W_0 *annihilated by* $\mathfrak{g}_0$ *(if* $\mathfrak{g}_0$ *is of* A_n *Lie type,* $n \in \mathbb{N} \setminus \{1\}$*, then this bilinear form is zero), gives birth to a bilinear form on* $\mathcal{TP}$ *fixed by* $G^{\mathrm{der}}_{B(\mathbb{F})}$ *and left invariant by* $G_{B(\mathbb{F})}$*.*

As $\tilde{G}$ is a inner form of $G_{W(k)}$ we deduce that all the factors of $\tilde{G}^{\mathrm{ad}}$ are absolutely simple factors. From the above fact and Fontaine's comparison theory, we get the existence of an absolutely simple factor $\tilde{\mathfrak{g}}_0$ of $\mathrm{Lie}(\tilde{G}^{\mathrm{ad}}_{B(k)})$, which is an inner form of $\mathfrak{g}_0$, and which has a representation on $\tilde{W}_0 := W_0 \otimes_{B(\mathbb{F})} B(k)$, annihilating a bilinear form $\tilde{b}_0$ on $\tilde{W}_0$. $\tilde{b}_0$ is a non-degenerate form iff the Lie type of $\mathfrak{g}_0$ is not A_n, $n \in \mathbb{N}$, $n \geq 2$.

We now treat the case when all the factors of G^{ad} are of A_n or C_n Lie type ($n \in \mathbb{N}$). By reasons of dimensions (resp. due to the existence of the symplectic form $\tilde{b}_0$) in case $\mathfrak{g}_0$ is of A_n Lie type (resp. of C_n Lie type), we get that $\tilde{\mathfrak{g}}_0$ is isomorphic to $\mathfrak{g}_0 \otimes_{B(\mathbb{F})} B(k)$. We deduce that $\tilde{\mathfrak{g}}_0$ is a split Lie algebra over $B(k)$. We conclude that $\tilde{G}^{\mathrm{der}}$ is a split group. So (cf. a)) $\tilde{G}$ is a split group. As its generic fibre is an inner form of $G_{B(k)}$ we deduce that $G_{B(k)}$ is isomorphic to $\tilde{G}_{B(k)}$. We conclude as above that $\tilde{G}$ is isomorphic to $G_{W(k)}$. This ends the proof of c).

Coming back to the general situation (of the assumption), it is enough to show: if $\tilde{b}_0$ is a non-degenerate symmetric form (so $\mathfrak{g}_0$ is of B_n or D_n Lie type), then $\tilde{\mathfrak{g}}_0$ and $\mathfrak{g}_0 \otimes_{B(\mathbb{F})} B(k)$ become isomorphic over a finite unramified abelian extension $B(k_1)$ of $B(k)$ of order a fixed power of 2.

To see this, let $\tilde{G}_0$ be the semisimple group over $W(k)$ whose generic fibre is $\mathrm{Aut}(\tilde{W}_0, \tilde{b}_0)$. We just need to show that $\tilde{G}_{0k}$ splits over a finite abelian extension k_1 of k. We consider a $W(k)$-lattice $\tilde{L}_0$ of $\tilde{W}_0$ such that $\tilde{G}_0$ is a subgroup of $GL(\tilde{L}_0)$, cf. [Ja, 10.4]. The existence of $\tilde{b}_0$ implies (easy exercise) the existence of a non-degenerate bilinear form on $\tilde{L}_0/pL_0$ fixed by $\tilde{G}_{0k}$. Writing it as a sum $\sum_{i=1}^m a_i x_i^2$ with respect to some basis of $\tilde{L}_0/pL_0$ (here $m = \dim(\tilde{W}_0)$, while $a_i \in k$ are non-zero elements), we get immediately that $\tilde{G}_{0k}$ splits over the abelian extension k_1 of k obtained by adjoining square roots of not more than $[\frac{m}{2}]$ elements of k. So $[k_1 : k]$ divides $2^{[\frac{m}{2}]}$. This ends the proof of d), as well as of b) under the above assumption.

To prove b) in the general case we have to use the standard techniques of [Va2, 6.2.3.1, 6.2.3, 6.4.2 and 6.4.5.1] (for the case $p = 3$ cf. §6). In other words, cf. 2.3.6 and loc. cit., there is another SHS $(f^1, L^1_{(p)}, v^1)$ such that:

– the adjoint quadruple of the Shimura quadruples (G^1, X^1, H^1, v^1) and (G, X, H, v) are isomorphic;

– the above assumption is satisfied for it;

– there is an isogeny $G^{1\mathrm{der}} \to G^{\mathrm{der}}$.

As $\mathbb{F} \hookrightarrow k$, we get easily from [Va2, 6.2.3 and 3.2.7 10)] that, we can assume that each connected component of $\mathcal{N}^1_{W(k)}/H^1_0$ is an abelian cover of a connected component of $\mathcal{N}_{W(k)}/H_0$. Moreover the degree $d(H_0, \tilde{H}_0)$ of this cover divides a fixed power of q_G, cf. [Va2, 6.2.3.1]. So the standard trick using Segre's embeddings as in [Va2, Example 3 of 2.5] (it is fully recalled in 4.9), allows us to shift our attention to the new SHS. But for the new situation we can apply the above part. So we get easily b), as the part involving nilpotency is trivial. This ends the proof of the proposition.

2.3.9.1. Remark. We do not know for which SHS $(f, L_{(p)}, v)$ the reductive group $\tilde{G}$ of 2.3.4 is automatically isomorphic to $G_{W(k)}$. The fixed power of q_G referred in 2.3.9 b) is effectively computable, as the degree $d(H_0, \tilde{H}_0)$ is effectively computable. For instance, if G^{der} is a simply connected semisimple group, then the Galois group of 2.3.9 b) we get is the product of at most $r(G^{\mathrm{ad}})$ copies of $\mathbb{Z}/2\mathbb{Z}$. Here $r(G^{\mathrm{ad}})$ is a sum indexed by the simple factors of $G^{\mathrm{ad}}_{\mathbb{C}}$ of B_n or D_n Lie type (like $r(SO(2,m)^{\mathrm{ad}}) = [\frac{m+2}{2}]$, etc.).

2.3.9.2. Convention. From now on, for the sake of simplifying the notations, we assume that whenever we consider a point $\mathrm{Spec}(k) \to \mathcal{N}$ or a point $\mathrm{Spec}(W(k)) \to \mathcal{N}$, the reductive group we get as in 2.3.4 is (isomorphic to) the split group $G_{W(k)}$. This convention is motivated by 2.3.9 and 2.3.9.1. It does not apply to the similar points of a quotient of finite type of $\mathcal{N}$, like $\mathcal{N}/H_0$.

2.3.10. Remark. Let $(f, L_{(p)}, v)$ be a SHS. With the notations of 2.3.4, the triple $(M, \varphi, G_{W(k)})$ (with φ the Frobenius of $M = H^1_{\mathrm{crys}}(A/W(k))$; the family $(t_\alpha)_{\alpha \in \mathcal{J}'}$ is fixed by φ) is a Shimura σ-crystal (see 2.2.8), called the Shimura σ-crystal attached to the point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ defined by z. This means that there is an injective cocharacter $\tilde{\mu} : \mathbb{G}_m \hookrightarrow G_{W(k)}$ which gives birth to a direct sum decomposition $M = F^1 \oplus F^0$ (with $\dim(F^0) = \dim(F^1) = 1/2 \dim(M)$, and with $\beta \in \mathbb{G}_m(W(k))$ acting through $\tilde{\mu}$ on F^i as the multiplication with β^{-i}, $i = \overline{0,1}$), such that $(M, F^1, \varphi, G_{W(k)})$ is a Shimura filtered σ-crystal; the proof of the existence of such a cocharacter $\tilde{\mu}$ is entirely the same as the construction of μ in [Va2, 5.3.1], as 2.3.9.2 tells us that $G_{W(k)}$ is split (cf. also 2.2.9 1)). The cocharacter $\tilde{\mu}$ can be chosen so that F^1 is the Hodge filtration of M defined by A, and then the above Shimura filtered σ-crystal is said to be attached to the point $z \in \mathcal{N}(W(k))$. When we want to emphasize the family of tensors $(t_\alpha)_{\alpha \in \mathcal{J}'}$ then we also write down $(M, \varphi, G_{W(k)}, (t_\alpha)_{\alpha \in \mathcal{J}'})$ and $(M, F^1, \varphi, G_{W(k)}, (t_\alpha)_{\alpha \in \mathcal{J}'})$. We also speak about the Shimura (adjoint) Lie σ-crystal attached to y, and of the Shimura (adjoint) filtered Lie σ-crystal attached to z (cf. 2.2.13). We also speak about Shimura (adjoint) (Lie) isocrystal attached to y, and of Shimura (adjoint) filtered (Lie) isocrystal attached to z.

When we want to emphasize the polarization, we speak about the principally quasi-polarized Shimura σ-crystal attached to y, and denote it by $(M,\varphi,G_{W(k)},p_A)$ or by $(M,\varphi,G_{W(k)},(t_\alpha)_{\alpha\in\mathcal{J}},p_A)$. Similarly for z.

Let now $z:\mathrm{Spec}(W(k))\to\mathcal{N}/H_0$, and let A be the abelian variety over $W(k)$ obtained from $\mathcal{A}_{H_0}$ by pull back through z. Over a finite étale extension $W(k_1)$ of $W(k)$, $A_{W(k_1)}$ gets a family $(w_\alpha)_{\alpha\in\mathcal{J}'}$ of Hodge cycles. As the set of all Hodge cycles of $A_{W(k_1)}$ is invariant under the Galois group $\mathrm{Gal}(W(k_1)/W(k))=\mathrm{Gal}(k_1/k)$, we still get that the (principally quasi-polarized) filtered σ-crystal (M,F^1,φ) attached to A has a natural structure of a (principally quasi-polarized) quasi Shimura filtered σ-crystal $(M,F^1,\varphi,\tilde{G}_{W(k)})$ which is not necessarily quasi-split (cf. 2.2.9 1) for the part involving cocharacters). Warning: we have no reason to consider $\tilde{G}_{W(k)}$ to be an inner form of $G_{W(k)}$. However, if it is an inner form and if k is finite, then we have $\tilde{G}_{W(k)}=G_{W(k)}$.

This allows us to speak about the not necessarily quasi-split Shimura (adjoint) (filtered) (Lie) σ-crystal (isocrystal) attached to points of $\mathcal{N}/H_0$ (with values in perfect fields and in Witt rings of such fields).

2.3.10.1. Comment. The Shimura Lie σ-crystals attached to points $y:\mathrm{Spec}(k)\to\mathcal{N}_{k(v)}$ play a central role on the whole of this paper. We can not refrain from commenting why they have not been used before. We think that mainly due to two reasons. The first reason is: in the case of special fibres of integral canonical models of Siegel modular varieties, all the information obtained from a Shimura Lie σ-crystal attached to points, can be read out from the Shimura σ-crystal producing it. The second reason is: the way of thinking of points of $\mathcal{N}_{k(v)}$ was mainly rational, i.e. working with the isocrystals defined by Shimura σ-crystals attached to these points, even though for the standard PEL situation (see [Ko2, §5]) was quite obvious (cf. [Va2, 4.3.11]) that we get Shimura σ-crystals.

2.3.10.2. Example. We consider a special point $\mathrm{Spec}(B(k))\to\mathcal{N}$. From the extension property of $\mathcal{N}$ we get it extends to a morphism $z_1:\mathrm{Spec}(W(k))\to\mathcal{N}$. Following the proof of 2.2.19 we get immediately that the Shimura filtered σ-crystal attached to z_1 is essentially cyclic diagonalizable.

2.3.11. Let $(f,L_{(p)},v)$ be a SHS. Then all the local deformation theory presented in [Va2, 5.4] can be applied for it (for $\mathcal{N}$). In particular, the fact that for $p\geq 5$ $\mathcal{N}$ is the normalization of the Zariski closure of $\mathrm{Sh}(G,X)/H$ in $\mathcal{M}$, results also from [Va2, 5.4]. For the case $p=3$, cf. also §6.

We have variants of [Va2, 5.4]: we work over an arbitrary perfect field k (instead of $\mathbb{F}$), or we work with $G^0_{W(k)}$ (instead of $G^{\mathrm{der}}_{W(k)}$), or we work with $\mathcal{N}/H_0$ (instead of $\mathcal{N}$; so we work with a finite level symplectic similitude structure, instead of all level-N symplectic similitude structures, $N\in\mathbb{N}$, $(N,p)=1$). This is so due to the fact that [Fa2, th. 10] is true for any perfect field.

Moreover, for any morphism $U\to\mathcal{N}$, with U a regular formally smooth scheme over a Witt ring of a perfect field, we get naturally a principally quasi-polarized Shimura p-divisible group over U. The change of the family $(v_\alpha)_{\alpha\in\mathcal{J}}$ corresponds to a passage to quasi-isomorphic principally quasi-polarized Shimura p-divisible groups, cf. 2.2.21 7).

2.3.12. We start with a closed point $z : \mathrm{Spec}(W(k)) \hookrightarrow \mathcal{N}/H_{0W(k)}$. Let $y : \mathrm{Spec}(k) \hookrightarrow \mathcal{N}/H_{0W(k)}$ be defined by z. We use the notations of 2.3.10. So F^1 is the Hodge filtration of M defined by A. For simplicity let us assume that the family of Hodge cycles $(w_\alpha)_{\alpha\in\mathcal{J}'}$ is defined over $W(k)$, and not only over $W(\bar{k})$, and that the group $\tilde{G}_{W(k)}$ of 2.3.10 is in fact $G_{W(k)}$. We can always achieve this by replacing k by a finite field extension k_1 of it. Let O_y be the local ring of y. So O_y^h and $\widehat{O_y}$ are its henselization and respectively its completion. Let $(A_y^h, p_{A_y^h})$ be the principally polarized abelian scheme over $\mathrm{Spec}(O_y^h)$ obtained from $(\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ by pull back under the natural morphism $\mathrm{Spec}(O_y^h) \to \mathcal{N}/H_0$. Let

$$M_y^h := H^1_{dR}(A_y^h/O_y^h),$$

and let F_y^{1h} be its Hodge filtration defined by A_y^h. Let $(\omega_\alpha^h)_{\alpha\in\mathcal{J}'}$ be the family of Hodge cycles with which $A^h_{yK_y^h}$ is naturally endowed (with K_y^h the field of fractions of O_y^h; $\mathcal{N}$ is formally smooth over $\mathbb{Z}_{(p)}$, and so O_y^h is an integral domain). As we assumed that the family $(w_\alpha)_{\alpha\in\mathcal{J}'}$ is defined over $W(k)$, this implies that the family $(\omega_\alpha^h)_{\alpha\in\mathcal{J}'}$ is defined over O_y^h, and not only over its strict henselization. This is a consequence of the following obvious fact: a Hodge cycle of a deformation which is zero in a point is zero everywhere. Let t_α^h be the de Rham component of ω_α^h. It is a homogeneous tensor of $\mathcal{T}(M_y^h \otimes_{O_y^h} K_y^h)$.

2.3.13. Lemma. *We have an isomorphism $f_y : M_y^h \tilde{\to} M \otimes_{W(k)} O_y^h$ taking F_y^{1h} onto $F^1 \otimes_{W(k)} O_y^h$, $p_{A_y^h}$ into p_A, and t_α^h into t_α, $\forall \alpha \in \mathcal{J}'$ (here we view $p_{A_y^h}$ and p_A as perfect forms on M_y^h and respectively on M).*

Proof: Such an isomorphism exists over $\widehat{O_y}$. This is the same as [Va2, 5.4]. The obstruction to the existence of f_y is measured by an element $\gamma \in H^1_{ff}(O_y^h, P_{O_y^h})$; here P is the parabolic subgroup of $G_{W(k)}$ leaving F^1 invariant, while the lower left index ff refers to the faithfully flat topology. So γ becomes zero over $\widehat{O_y}$. This implies $\gamma = 0$, as O_y^h is the henselization of the excellent ring O_y (so Artin's approximation theorem applies). This proves the lemma.

2.3.13.1. Corollary. *There is an étale morphism $a : Y = \mathrm{Spec}(R) \to \mathcal{N}/H_{0W(k)}$ to which the point y lifts, and there is an isomorphism $M_R \tilde{\to} M \otimes_{W(k)} R$ taking F_R^1 onto $F^1 \otimes_{W(k)} R$, p_{A_Y} into p_A, and t_α^R into t_α, $\forall \alpha \in \mathcal{J}'$ (here M_R, F_R^1, $(t_\alpha^R)_{\alpha\in\mathcal{J}'}$, p_{A_Y} have a meaning similar to the one in the case of O_y^h; for instance $M_R := H^1_{dR}(A_Y/R)$, where $A_Y := \mathcal{A}_{H_0} \times_{\mathcal{N}/H_0} Y$, the morphism $Y \to \mathcal{N}/H_0$ being naturally induced by a).*

2.3.14. Warning: Sp **or** GSp? Once for all we have to decide if we are going to make use of $\mathrm{GSp}(W,\psi)$ or of $\mathrm{Sp}(W,\psi)$ (and so of $G_{W(k)}$ or of $G^0_{W(k)}$). The use of Sp corresponds to a permanent mentioning of the polarizations (and so to a very precise presentation, but more complicated notations). The use of GSp corresponds to a loose treatment of the polarizations (and so to a less precise presentation, but simpler notations). We decided to go along with Sp.

The idea behind the above comment is: we can work out [Va2, 5.4.4-5] using not only Sp and G^{der} (or G^0), but also GSp and G. The use of GSp means that the cycle of the polarization is not exactly p_A, but p_A times some invertible element γ_p. This element γ_p can be made to be 1, by passing to an étale cover defined by the equation $y^2 = \gamma_p$ (as $p > 2$). So the difference is mostly of notations and of preciseness.

2.3.15. Proposition. *There is a smooth morphism $a : Y = \mathrm{Spec}(R) \to \mathcal{N}/H_{0W(k)}$ such that:*

a) the scheme $a^{-1}(y)$ contains a k-valued point y_1 lifting (to Y) y;

b) we have an isomorphism $M_R \tilde{\to} M \otimes_{W(k)} R$ taking F^1_R onto $F^1 \otimes_{W(k)} R$, p_{A_Y} into p_A, and t^R_α into t_α, $\forall\alpha \in \mathcal{J}'$ (here $M_R := H^1_{dR}(A_Y/R)$, with A_Y defined as in 2.3.13.1, etc.; the logical notations);

c) there is a Frobenius lift Φ_R of $R^\wedge$, compatible with the Frobenius automorphism σ of $W(k)$, and which is of the form $z_i \to z_i^p$, $i \in I$, where the elements of the subset $\{z_i | i \in I\} \cup \{p\}$ of R are forming a regular system of parameters of y_1 in Y;

d) the p-divisible object $\mathcal{C}$ of $\mathcal{MF}^\nabla_{[0,1]}(R)$ defined by the p-divisible group of A_Y, can be put in the form $\big(M \otimes_{W(k)} R^\wedge, F^1 \otimes_{W(k)} R^\wedge, g_Y(\varphi\otimes 1), \nabla_{R^\wedge}\big)$, with p_{A_Y} corresponding to p_A. Here $g_Y \in G^0_{W(k)}(R^\wedge)$ is defined by a morphism (still denoted by g_Y) $g_Y : Y^\wedge \to G^{0\wedge}_{W(k)}$, whose special fibre is étale, and for which the origin $\mathrm{Spec}(W(k)) \hookrightarrow G^{0\wedge}_{W(k)}$ can be lifted to a morphism $z_Y : \mathrm{Spec}(W(k)) \to Y^\wedge$ which mod p defines y_1 and which lifts the p-adic completion of z. The connection $\nabla_{R^\wedge}$ of $M \otimes_{W(k)} R^\wedge$ annihilates t_α, $\forall\alpha \in \mathcal{J}'$.

Proof: Let $\hat{G}^0 = \mathrm{Spec}(\hat{R}^0)$ be the completion of $G^0_{W(k)}$ with respect to the origin. As in [Va2, 5.4.4] (see 2.3.11) we construct a (formally smooth) morphism $\hat{G}^0 \overset{\hat{a}}{\to} \mathcal{N}/H_{0W(k)}$ such that the principally quasi-polarized filtered F-crystal with cycles associated to the Shimura principally quasi-polarized p-divisible group over $\hat{G}^0$ obtained (see 2.3.11) by pull back through the composite of $\hat{a}$ with the natural morphism $\mathcal{N}/H_{0W(k)} \to \mathcal{N}/H_0$, is a principally quasi-polarized Shimura filtered F-crystal $(M, F^1, \varphi, G_{W(k)}, G^0_{W(k)}, \tilde{f}, p_A)$. Here $\tilde{f} : W(k)[[x_1, ..., x_{|I|}]] \tilde{\to} \hat{R}^0$.

Now 2.3.15 results from 2.3.13.1 and Artin's approximation theorem. We present the details. Let $n \geq 2$ be an integer, let I be the ideal of $\hat{R}^0$ defining the origin of $\hat{G}^0$ (so $\hat{R}^0/I = W(k)$), and let (cf. [BLR, p. 91]) $Y = \mathrm{Spec}(R)$ be a smooth scheme over $W(k)$ such that:

i) there is a smooth morphism $a : Y \to \mathcal{N}/H_{0W(k)}$ such that the point z lifts to a point $z_1 : \mathrm{Spec}(W(k)) \to Y$;

ii) if I_1 is the ideal of R defining z_1, the morphism $\mathrm{Spec}(R/I_1^n) \to \mathcal{N}/H_{0W(k)}$ defined by a is obtained from the natural morphism $\mathrm{Spec}(\hat{R}^0/I^n) \to \mathcal{N}/H_{0W(k)}$ through an isomorphism $i_n : \hat{R}^0/I^n \tilde{\to} R/I_1^n$.

Replacing, if needed, Y with $Y' = \mathrm{Spec}(R')$ where $Y' \to Y$ is an étale morphism such that z_1 lifts to a $W(k)$-valued point of Y', we can achieve that the morphism a endows the abelian scheme A_Y with a family of Hodge cycles $(w^R_\alpha)_{\alpha\in\mathcal{J}'}$ and that b) and c) are satisfied (here z_i is such that $i_n(x_i)$ is congruent to z_i modulo I_1^n). The argument of this is entirely the same as in 2.3.13. This takes care of a), b) and c): we take y_1 to be defined by z_1.

Now the principally quasi-polarized p-divisible object of $\mathcal{MF}_{[0,1]}^{\nabla}(R)$ defined by the principally quasi-polarized p-divisible group of (A_Y, p_{A_Y}) (cf. [Fa1, 7.1] or 2.2.1.1 2)) can be put (cf. b)) in the form $\big(M \otimes_{W(k)} R^\wedge, F^1 \otimes_{W(k)} R^\wedge, g_Y(\varphi \otimes 1), \nabla_{R^\wedge}, p_A\big)$, with $g_Y \in G^0_{W(k)}(R^\wedge)$ (as p_{A_Y} is fixed by g_Y, and as the Rham cycles t_α^R, $\alpha \in \mathcal{J}'$, are fixed by the Frobenius of this p-divisible object of $\mathcal{MF}_{[0,1]}^{\nabla}(R)$), and with $\nabla_{R^\wedge}$ a connection on $M \otimes_{W(k)} R^\wedge$ annihilating t_α, $\forall \alpha \in \mathcal{J}'$. Moreover $g_Y(\varphi \otimes 1)$ is $\nabla_{R^\wedge}$-parallel. Due to property ii) above, as R is smooth over $W(k)$, we deduce easily that we can assume that g_Y is defined by a morphism $g_Y : Y^\wedge \to G^{0\wedge}_{W(k)}$, étale mod p, and having the property that the composite of the p-adic completion of z_1 with g_Y defines the origin of $G^{0\wedge}_{W(k)}$. This takes care of d), and ends the proof of the proposition.

From the proof of 2.3.15, by removing some extra regular parameters, we get:

2.3.15.1. Corollary. *There is an étale morphism $\tilde{a} : \tilde{Y} = \mathrm{Spec}(\tilde{R}) \to \mathcal{N}/H_{0W(k)}$ such that:*

a) the scheme $\tilde{a}^{-1}(y)$ contains a k-valued point $\tilde{y}_1$ lifting (to $\tilde{Y}$) y;

b) we have an isomorphism $M_{\tilde{R}} \tilde{\to} M \otimes_{W(k)} \tilde{R}$ taking $F^1_{\tilde{R}}$ onto $F^1 \otimes_{W(k)} \tilde{R}$, $p_{A_{\tilde{Y}}}$ into p_A, and $t_\alpha^{\tilde{R}}$ into t_α, $\forall \alpha \in \mathcal{J}'$ (here $M_{\tilde{R}} := H^1_{dR}(A_{\tilde{Y}}/\tilde{R})$, with $A_{\tilde{Y}}$ defined as in 2.3.13.1, etc.; the logical notations);

c) there is a Frobenius lift $\Phi_{\tilde{R}}$ of $\tilde{R}^\wedge$, extending the Frobenius automorphism σ of $W(k)$, and which is of the form $z_i \to z_i^p$, $i \in \tilde{I}$, where the elements of the subset $\{z_i | i \in \tilde{I}\} \cup \{p\}$ of $\tilde{R}$ are forming a system of regular parameters of $\tilde{y}_1$ in $\tilde{Y}$ (so $|\tilde{I}| = d = \dim(Sh(G,X))$);

d) the p-divisible object of $\mathcal{MF}_{[0,1]}^{\nabla}(\tilde{R})$ defined by the p-divisible group of $A_{\tilde{Y}}$, can be put in the form $\big(M \otimes_{W(k)} \tilde{R}^\wedge, F^1 \otimes_{W(k)} \tilde{R}^\wedge, g_{\tilde{Y}}(\varphi \otimes 1), \nabla_{\tilde{R}^\wedge}\big)$ in such a way that $p_{A_{\tilde{Y}}}$ corresponds to p_A. Here $g_{\tilde{Y}} \in G^0_{W(k)}(\tilde{R}^\wedge)$ is defined by a morphism (still denoted by $g_{\tilde{Y}}$) $g_{\tilde{Y}} : \tilde{Y}^\wedge \to G^{0\wedge}_{W(k)}$, for which the origin $\mathrm{Spec}(W(k)) \hookrightarrow G^{0\wedge}_{W(k)}$ can be lifted to a morphism $z_{\tilde{Y}} : \mathrm{Spec}(W(k)) \to \tilde{Y}^\wedge$ which mod p defines $\tilde{y}_1$ and which lifts the p-adic completion of z. Modulo the ideal $J := p((z_i)_{i \in \tilde{I}})^2$, $g_{\tilde{Y}}$ defines an isomorphism from $\mathrm{Spec}(\tilde{R}/J)$ into the completion of a nilpotent subgroup N of $G^0_{W(k)}$ in the origin, taken modulo the ideal defined by p times the second power of the ideal sheaf of N defining its origin; N can be any such subgroup achieving an open embedding $N \hookrightarrow G_{W(k)}/P_{W(k)}$, with $P_{W(k)}$ the parabolic subgroup of $G_{W(k)}$ leaving invariant F^1 (so $g_{\tilde{Y}}$ mod p is a formal closed embedding in $\tilde{y}_1$);

e) The connection $\nabla_{\tilde{Y}^\wedge}$ of $M \otimes_{W(k)} \tilde{Y}^\wedge$ annihilates t_α, $\forall \alpha \in \mathcal{J}'$;

f) The special fibre of $\tilde{Y}$ is connected.

2.3.16. Corollary. *There is a dense open subscheme U_k of G_k such that: for any geometric point $\mathrm{Spec}(k_1) \stackrel{y_1}{\to} U_k$ (with $k_1 = \bar{k}_1 \supset k$), there is a point $z_1 \in \mathcal{N}/H_0(W(k_1))$, lifting a k_1-valued of the same connected component of $\mathcal{N}_{\bar{k}}$ to which the point $y \circ i_k$ belongs, with $i_k : \mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(k)$ the natural map, and having the property that the Shimura* *filtered σ_1-crystal (with σ_1 the Frobenius automorphism of $W(k_1)$) attached to the point z_1 is isomorphic to $\big(M \otimes_{W(k)} W(k_1), F^1 \otimes_{W(k)} W(k_1), g_1(\varphi_0 \otimes 1), G_{W(k_1)}\big)$, where $g_1 \in G_{W(k)}\big(W(k_1)\big)$ mod p is the point y_1.*

Proof: Working with G^0 instead of G, this is a direct consequence of 2.3.15 (we can take as U^0_k the image of the special fibre of $Y^\wedge$ through the étale morphism $g_Y : Y^\wedge \to G^{0\wedge}_{W(k)}$). Now, as U_k, we can take an open subscheme of G_k contained in the image of $U^0_k \times \mathbb{G}_m$ through the natural isogeny $G^0_k \times \mathbb{G}_m \to G_k$ (to be compared with 2.3.14: for any $\alpha \in \mathbb{G}_m(W(k))$, there is $\beta \in \mathbb{G}_m(W(k))$ such that $\alpha = \sigma(\beta)\beta^{-1}$). This proves the corollary.

Warning: working with $G_{W(k)}$ and not with $G^0_{W(k)}$ implies that we have to give up mentioning the principal quasi-polarization. So the isomorphism of 2.3.16 is of non principally quasi-polarized Shimura filtered σ-crystals. For applications to problems involving Newton polygons this is equally good.

2.3.17. Proposition. *With the notations of 2.3.12, the Shimura filtered σ-crystal $(M, F^1_1, \varphi, G_{W(k)})$ attached to another lift $z_1 : \mathrm{Spec}(W(k)) \hookrightarrow \mathcal{N}/H_{0W(k)}$ of y, satisfies $F^1_1 = g(F^1)$, with $g \in G_{W(k)}(W(k))$ leaving invariant F^1/pF^1. Conversely, for any such $g \in G_{W(k)}(W(k))$ there is a lift $z_g : \mathrm{Spec}(W(k)) \hookrightarrow \mathcal{N}_{W(k)}$ of y, whose attached principally quasi-polarized Shimura filtered σ-crystal is $(M, g(F^1), \varphi, G_{W(k)}, p_A)$.*

Proof: The first statement is obvious, as the parabolic subgroups of $G_{W(k)}$ leaving invariant F^1 and respectively F^1_1 are $G_{W(k)}(W(k))$-conjugate, and as $F^1/pF^1 = F^1_1/pF^1_1$. In other words, two parabolic subgroups of $G_{W(k)}$ which are equal mod p, are $G_{W(k)}W(k)$-conjugate (argument: two maximal tori of $G_{W(k)}$ equal mod p are $G_{W(k)}(W(k))$-conjugate by an element equal mod p to the identity; see [SGA3, Vol. 2, p. 48]). Another way to see that these two parabolic groups are $G_{W(k)}(W(k))$-conjugate, is explained in [Va2, 5.3.1].

The converse is a direct consequence of the smoothness of $G_{W(k)}$, of 2.3.11, and of the local deformation theory of abelian varieties (over k) (cf. [Va2, 5.5.1]).

2.3.17.1. Corollary. *The principally quasi-polarized Shimura filtered σ-crystal $(M, g(F^1), \varphi, G_{W(k)}, p_A)$ (here $g \in G_{W(k)}(W(k))$ mod p leaves invariant F^1/pF^1) is induced from the principally quasi-polarized Shimura filtered F-crystal*

$$(M, F^1, \varphi, G_{W(k)}, G^0_{W(k)}, \tilde{f}, p_A)$$

(of the proof of 2.3.15) through a $W(k)$-epimorphism $\hat{R}^0 \twoheadrightarrow W(k)$.

2.3.17.2. Exercise. Show that the above corollary is a property of (principally polarized) Shimura σ-crystals over k. Hint: if we feel more comfortable with the context of principally polarized abelian schemes, follow the ideas of 4.12.12.

2.3.18. The $p = 2$ SHS. The case $p = 2$ presents some particular features, and so has to be singled aside. Working with $p = 2$, we can keep 2.3.1-2. But for 2.3.3 we have to proceed carefully: we define $\mathcal{N}$ as the normalization of $\mathrm{Sh}_H(G, X)$ in $\mathcal{M}$, and we assume that it is a smooth integral model. This implies that it is an integral canonical model of the Shimura quadruple (G, X, H, v). We keep 2.3.4-5 (in particular 2.3.5.1 remains true for

$p = 2$), but we prefer to abbreviate the things differently: the resulting triple $(f, L_{(2)}, v)$ (resp. quadruple $(f, L_{(2)}, v, \mathcal{B})$) is referred to as a $p = 2$ SHS (resp. as a $p = 2$ standard PEL situation). Conventions 2.3.7 and 2.3.9.2 apply, but not paragraphs 2.3.6 or 2.3.8.

The parts a) and c) of 2.3.9 remain valid without any modification. Moreover, 2.3.9 d) applies as well: the part of the proof of 2.3.9 referring to [Ja, 10.4] applies as well. Briefly, this goes as follows. The difference is that, with the notations of the proof of 2.3.9, $\tilde{G}_{0k}$ is the subgroup of $GL(L_0/pL_0)$ fixing a quadratic form $\tilde{Q}_0$ in m variables (argument: this is so over $\bar{k}$, cf. [Bo, 23.6] and its lift version to $W(\bar{k})$; but such a quadratic form is unique modulo an invertible element and so using Hilbert's Theorem 90 we can assume that it is defined over k). Now after a finite (computable) number $n(m)$ of Galois extensions (the first one of k) of degree 2 we can bring $\tilde{Q}_0$ in the standard form $x_0^1 + x_1x_2 + ... + x_{2n-1}x_{2n}$ if $m = 2n + 1$ is odd, and $x_1x_2 + ... + x_{2n-1}x_{2n}$ if $m = 2n$ is even.

Also 2.3.10-16 remain valid: this is so due to the fact that [Fa2, th. 10] and the remarks following it are still true for $p = 2$; so [Va2, 5.4] and its variants (see 2.3.11) remain true for $p = 2$. But in connection to 2.3.17 we have to be more careful: for $p = 2$ [Va2, 5.5.1] has to be modified slightly, as a 2-divisible group over $W_2(k)$ is not determined by its σ-crystal and its filtration mod 4; however it applies to 2.3.17 in the same way, provided k is algebraically closed or (M, φ) does not have slope 0 or 1 (see 2.3.18.1 below).

2.3.18.1. Digression. Here we recall some facts on 2-divisible groups D over a perfect field k of characteristic 2. We write it as a product

$$D = D_1 \times D_2$$

such that D_1 is ordinary, while D_2 does not have slope 0 or 1. Let (M, φ) be its σ-crystal, and let $M = M_1 \oplus M_2$ be the corresponding decomposition. Let k^n be the k-linear space which is the tangent space of the deformation space of its lifts to $W_2(k)$, let T_{def} be the vector group scheme it defines naturally. Let k^n be the k-linear space which parametrizes lifts of the F^1-filtration of $M/2M$ to F^1-filtrations of $M/4M$; let T_{fil} be the vector group scheme it defines naturally. There is a natural morphism (of schemes):

$$m_D : T_{\text{def}} \to T_{\text{fil}}.$$

It is a dominant morphism of degree at most 2^n: using specialization, it is enough to check this for ordinary 2-divisible groups. But this check out, can be performed without difficulty, starting from the theory of canonical crystalline coordinates (in fact [Fa2, th. 10] is enough; see also [Og2]).The same specialization process shows that the resulting map

$$m_D(\bar{k}) : T_{\text{def}}(\bar{k}) \to T_{\text{fil}}(\bar{k})$$

is surjective.

In case D is ordinary, then m_D is a Galois cover of whose Galois group is $(\mathbb{Z}/2\mathbb{Z})^d$. In general, the fibres of $m_D(\bar{k})$ are related to passage to some particular isogenies (involving the 2-torsion). So using Fontaine's comparison theory with rational coefficients in the case of good reduction (see [Fa1]), such a fibre $m_D(\bar{k})(\alpha)$ can have more than one element precisely in the case $\alpha \in T_{\text{fil}}(\bar{k})$ corresponds to a change in a filtration coming from M_1. In particular, if M_1 is zero, than $m_D(\bar{k})$ is a bijection.

§3 The basic results

Let k be a perfect field of characteristic $p \geq 3$, and let σ be the Frobenius automorphism of the Witt ring $W(k)$. Let (M, φ, G) be a Shimura σ-crystal over k (cf. 2.2.8 and 2.2.14). From its definition, we deduce the existence of a cocharacter $\mu : \mathbb{G}_m \to G$ which produces a direct sum decomposition $M = F^1 \oplus F^0$ (with $\beta \in \mathbb{G}_m(W(k))$ acting through μ on F^i as the multiplication with β^{-i}, $i = \overline{0,1}$) such that (M, F^1, φ, G) is a Shimura filtered σ-crystal. The $G(W(k))$-conjugacy class $[\mu]$ of the cocharacter μ is uniquely determined by (M, φ, G) (cf. 2.2.9 3)). From 2.2.8 a) we deduce the existence of a family $(v_\alpha)_{\alpha \in \mathcal{J}}$ of tensors of $\mathcal{T}(M)[\frac{1}{p}]$, fixed by G and φ, and such that $G_{B(k)}$ is the subgroup of $GL\big(M[\frac{1}{p}]\big)$ fixing v_α, $\forall \alpha \in \mathcal{J}$. If desirable, we can assume that the tensors v_α are homogeneous. We also deduce that F^0 and F^1 are proper direct summands of M. Let $\mathfrak{g} := \mathrm{Lie}(G)$.

Fixing the group G, the family of tensors $(v_\alpha)_{\alpha \in \mathcal{J}}$, and the $G(W(k))$-conjugacy class $[\mu]$, any other Shimura σ-crystal (M, φ_1, G), with φ_1 fixing v_α, $\forall \alpha \in \mathcal{J}$, and with $[\mu]$ defining its filtration class, up to isomorphism (cf. 2.2.9 6)), can be put in the form $\varphi_1 = g\varphi$, with $g \in G(W(k))$. To see this, we can consider two Shimura filtered σ-crystals $(M, F^1, \varphi, G, (v_\alpha)_{\alpha \in \mathcal{J}})$ and $(M, F_1^1, \varphi_1, G, (v_\alpha)_{\alpha \in \mathcal{J}})$. Changing the second one by an isomorphism, we can assume that $F^1 = F_1^1$. But then $\varphi_1 = g\varphi$ with $g \in GL(M)(W(k))$. As φ_1 and φ are both fixing v_α, $\forall \alpha \in \mathcal{J}$, we get that actually $g \in G(W(k))$.

3.0. The starting setting. In the whole of §3, we consider only Shimura σ-crystals (M, φ_1, G) or Shimura filtered σ-crystals (M, F^1, φ_1, G), with φ_1 fixing $v_\alpha, \forall \alpha \in \mathcal{J}$, and with $[\mu]$ defining their filtration class.

3.1. The first group of basic results.

3.1.0. Theorem. a) *Among all σ-crystals $(M, g\varphi)$ with $g \in G(W(k))$ (i.e. among all Shimura σ-crystals $(M, \varphi_1, G, (v_\alpha)_{\alpha \in \mathcal{J}})$ whose filtration class is defined by $[\mu]$), the ones which have the smallest Newton polygon $\mathcal{P}$ (in the sense that all others have a Newton polygon strictly above it) are precisely the ones $(M, g_0\varphi)$ for which there is a Shimura filtered σ-crystal $(M, F_0^1, g_0\varphi, G)$ such that the parabolic Lie subalgebra $\mathfrak{p}_0$ of $\mathfrak{g}$, corresponding to non-negative slopes of $(\mathfrak{g}, g_0\varphi)$, is contained in the parabolic Lie subalgebra $F_0^0(\mathfrak{g})$ of $\mathfrak{g}$ leaving invariant the filtration F_0^1 of M.*

b) *For any Shimura σ-crystal $(M, g_0\varphi, G)$ having $\mathcal{P}$ as its Newton polygon, there is a unique filtration F_0^1 of M, such that $(M, F_0^1, g_0\varphi, G)$ is a Shimura filtered σ-crystal satisfying (with the notations of a)) $\mathfrak{p}_0 \subset F_0^0(\mathfrak{g})$.*

c) *Among all σ-crystals $(M, g\varphi)$ with $g \in G(W(k))$ (i.e. among all Shimura σ-crystals $(M, \varphi_1, G, (v_\alpha)_{\alpha \in \mathcal{J}})$ whose filtration class is defined by $[\mu]$), the ones which have the smallest Newton polygon $Lie_G(\mathcal{P})$ (in the sense that all others have a Newton polygon strictly above it) of their attached Shimura Lie σ-crystals $(\mathfrak{g}, g\varphi)$, are precisely the ones having $\mathcal{P}$ as their Newton polygon.*

d) *There is a Shimura filtered F-crystal $\mathfrak{C}_F := (M, F^1, \varphi, G, \tilde{f})$ such that the Shimura σ_1-crystal we get (cf. 2.2.10) over a geometric point over the generic point of the scheme over which this $\mathfrak{C}_F$ is, has $\mathcal{P}$ as its Newton polygon.*

3.1.1. Definitions. Any Shimura σ-crystal $\big(M, \varphi_1, G, (v_\alpha)_{\alpha\in\mathcal{J}}\big)$ whose filtration class is defined by $[\mu]$ and has $\mathcal{P}$ as its Newton polygon, is called a G-ordinary σ-crystal (or a Shimura-ordinary σ-crystal, or, even better, a $(G, [\mu], [a])$-ordinary σ-crystal, as it depends on the filtration class defined by $[\mu]$ and on the automorphism class [a] (cf. def. 2) of 2.2.11)). The formal isogeny type of a G-ordinary σ-crystal $(M, g_0\varphi, G)$ (resp. such a G-ordinary σ-crystal), is called the G-ordinary type (resp. a G-ordinary σ-crystal) produced by (or attached to) the Shimura σ-crystal (M, φ, G) we started with. Any Shimura filtered σ-crystal (M, F_1^1, φ_1, G) with (M, φ_1, G) a $(G, [\mu])$-ordinary σ-crystal, and with F_1^1 such that the parabolic Lie subalgebra of $\mathfrak{g}$ leaving it invariant, contains the parabolic Lie subalgebra of $\mathfrak{g}$ corresponding to non-negative slopes of $(\mathfrak{g}, \varphi_1)$, is called a G-canonical lift (or a Shimura-canonical lift) of (M, φ_1, G).

3.1.1.1. Exercise. If $G = GL(M)$, then the G-ordinary type we get is an ordinary type (i.e. its associated Newton polygon has only the slopes 0 and 1).

3.1.2. Remark. The terminology G-ordinary σ-crystals and G-canonical lifts, has been suggested to us by F. Oort and R. Pink. We have a slightly more preference for the terminology Shimura-ordinary σ-crystals and Shimura-canonical lifts.

3.1.2.1. Convention. Whenever we pass to a perfect field k_1 containing k, for not overloading the notations, we still use the terminology G-ordinary, instead of $G_{W(k_1)}$-ordinary. But we keep using "the phrase" σ_1-crystals, where σ_1 is the Frobenius automorphism of $W(k_1)$.

3.1.3. Reformulation of 3.1. 3.1.0 asserts five things:

a) Among all Newton polygons of σ-crystals $(M, g\varphi)$ with $g \in G(W(k))$, there is one $\mathcal{P}$ which is the smallest; let A_G be the set of σ-crystals $(M, g\varphi)$ having $\mathcal{P}$ as their Newton polygon.

b) Among all Newton polygons of Shimura Lie σ-crystals $(\mathfrak{g}, g\varphi)$ with $g \in G(W(k))$, there is one $\mathrm{Lie}_G(\mathcal{P})$ which is the smallest; let B_G be the set of σ-crystals $(M, g\varphi)$ having $\mathrm{Lie}_G(\mathcal{P})$ as the Newton polygon of their attached Shimura Lie σ-crystal $(\mathfrak{g}, g\varphi)$.

c) $A_G = B_G = C_G$, where C_G is the set of σ-crystals $(M, g\varphi)$, with $g \in G(W(k))$, having the property that there is a Shimura filtered σ-crystal $(M, F^1, g\varphi, G)$ such that the parabolic Lie subalgebra of $\mathfrak{g}$ corresponding to non-negative slopes of $(\mathfrak{g}, g\varphi)$ is contained in the parabolic Lie subalgebra of $\mathfrak{g}$ leaving invariant F^1 (i.e. there is a Shimura filtered σ-crystal $(M, F^1, g\varphi, G)$ such that its Shimura filtered Lie σ-crystal $\big(\mathfrak{g}, g\varphi, F^0(\mathfrak{g}), F^1(\mathfrak{g})\big)$ is of parabolic type; cf. def. 2.2.12 a)).

d) If $(M, g\varphi)$ is an element of C_G, then there is a unique Shimura filtered σ-crystal $(M, F^1, g\varphi, G)$ as in c) (i.e. whose attached Shimura filtered Lie σ-crystal $\big(\mathfrak{g}, g\varphi, F^0(\mathfrak{g}), F^1(\mathfrak{g})\big)$ is of parabolic type).

e) Any Shimura σ-crystal can be deformed (using Shimura F-crystals) so that it is the specialization of a G-ordinary σ_1-crystal over a perfect field k_1 containing k.

3.1.4. Proposition. *For any G-ordinary σ-crystal $(M, \tilde{\varphi}, G)$, there is a unique cocharacter $\tilde{\mu} : \mathbb{G}_m \to G$, with $[\tilde{\mu}] = [\mu]$, producing a direct sum decomposition $M = F^1 \oplus F^0$ (with $\beta \in \mathbb{G}_m(W(k))$ acting through $\tilde{\mu}$ on F^i as the multiplication with β^{-i}, $i = \overline{0,1}$) such that the parabolic Lie subalgebra of $\mathfrak{g}$ corresponding to non-negative slopes (resp. non-positive slopes) of $(\mathfrak{g}, \tilde{\varphi})$ leaves invariant F^1 (resp. leaves invariant F^0).*

For a first proof of this proposition see 3.11.1-2. A second proof of it can be obtained entirely similar to the proof of a) of 4.4.1 3) (where we work in the context of Shimura varieties of Hodge type) and so, for making §4 as little dependent on §3 as possible, it is reproduced only there.

3.1.5. Exercise. The cocharacter $\tilde{\mu}$ is the one defined in [Wi] for the filtered isocrystal $(M[\frac{1}{p}], F^1[\frac{1}{p}], \tilde{\varphi})$.

Solution. We can assume $k = \bar{k}$. From 3.11.1 a) and c) we deduce that we can assume that G is a torus. So the exercise follows from the functoriality of the canonical splits (see [Wi]), and from 2.2.9 3).

A second solution can be obtained by combining 3.11.1 with the proof of b) of 4.4.1 3). Warning: 3.1.4-5 are not referred before 4.5, so no vicious circle is created (the terminology of 3.1.6 is used in 3.11.2 just to introduce some extra terminology in 3.11.3-4).

3.1.6. Definition. The cocharacter $\tilde{\mu}$ of 3.1.4 is called the canonical split of $(M, \tilde{\varphi}, G)$ (or of $(M, F^1, \tilde{\varphi}, G)$).

3.1.7. Remark. It is an easy exercise to see that the whole of 3.1 (as well as the greatest part -i.e. the part which does not mention p-divisible groups or Shimura varieties- of this chapter §3) remains valid in the context of generalized Shimura p-divisible objects over k. We do not stop to reformulate the results in this extended context.

From 3.1.0 d) and 2.2.9 9) we get (cf. 3.1.1.1):

3.1.8. Corollary. *Any p-divisible group D over a perfect field $\tilde{k}$ of characteristic $p > 2$ is the specialization of an ordinary p-divisible group.*

3.1.8.1. Comment. The above corollary, for the case when D has a quasi-polarization over a finite field extension of $\tilde{k}$, was obtained in [NO]. The case when the a-number of D is at most one was handled in [Oo2, 6.7]. The general case, including the case $p = 2$, can be easily deduced from [Fa2, th. 10] through a simple algebraization process as in 3.4.18.4.1 below (i.e., if $GL(M) = \mathrm{Spec}(R_{GL})$, then we work with the Frobenius of $M \otimes_{W(k)} R_{GL}^{\wedge}$ defined by $\mathrm{gl}(\varphi \otimes 1)$, with gl as the universal element of $GL(M)(R_{GL}^{\wedge})$), based on an analysis of the Hasse–Witt invariants and on example 3.2.3 below.

3.1.8.2. Exercise. Using 3.1.8.1, show that any quasi-polarized p-divisible group over $\tilde{k}$ is the specialization of a quasi-polarized ordinary p-divisible group.

This exercise reobtains the first main result of [NO].

3.2. The outline of the proof of 3.1.0.

3.2.1. Lemma. *If (N, φ_i), $i = \overline{1,2}$, are two σ-crystals over k such that $\forall m \in \mathbb{N}$, $\varphi_1^m = h_m \varphi_2^m$, with $h_m \in GL(N)(W(k))$, then they have the same Newton polygon.*

Moreover, for any $\alpha \in [0, \infty)$, the direct summand of N corresponding to slopes of (N, φ_i) bigger or equal to α, does not depend on $i \in \{1, 2\}$.

Proof: The condition $\varphi_1^m = h_m \varphi_2^m$, with $h_m \in GL(N)(W(k))$, implies that the Hodge numbers of the σ^m-crystals (N, φ_1^m) and (N, φ_2^m) are the same. But the Newton polygon of a σ-crystal is determined by the Hodge numbers of its iterates [Ka2, 1.4.4]. This takes care of the first part.

To see the second part we can assume that $k = \bar{k}$. Let $n \in \mathbb{N}$ such that φ_i^n acts diagonally with respect to a basis β_i of $N \otimes_{W(k)} B(k)$ formed by elements of $N \setminus pN$, $i \in \{1, 2\}$. From the hypothesis, applied to all multiplies m of n, we deduce easily by (decreasing) induction (there are only a finite number of slopes of (N, φ_1) and of (N, φ_2)) that for any $\alpha \in [0, \infty)$ the maximal $B(k)$-vector subspace $W_i(\alpha)$ of $N \otimes_{W(k)} B(k)$ with the property that it is left invariant by φ_i, and the slopes of the isocrystal $(W(\alpha), \varphi_i)$ are all greater or equal to α, does not depend on i, i.e. we have $W_1(\alpha) = W_2(\alpha)$. This proves the lemma.

3.2.1.1. Corollary. *If (N, φ_i, G_N) are two Shimura σ-crystals such that $\forall m \in \mathbb{N}$, $\varphi_1^m = h_m \varphi_2^m$, with $h_m \in G_N(W(k))$, then the Lie subalgebra of* $\mathrm{Lie}(G_N)$ *corresponding to non-negative (resp. positive) slopes of* $(\mathrm{Lie}(G_N), \varphi_i)$, *does not depend on $i \in \{1, 2\}$.*

Proof: We just need to apply 3.2.1 to the σ-crystals $(\mathrm{Lie}(G_N), p\varphi_i)$, $i \in \{1, 2\}$.

3.2.2. We come back to the Shimura filtered σ-crystal (M, F^1, φ, G) considered in the beginning of §3. For the meaning of $F^0(\mathfrak{g})$ and $F^1(\mathfrak{g})$ see 2.2.8 1). The following assertions are (obviously) equivalent:

a) $F^1(\mathfrak{g}) = 0$;
b) $F^0(\mathfrak{g}) = \mathfrak{g}$;
c) the cocharacter $\mu : \mathbb{G}_m \to G$ factors through the center of G;
d) the $G(W(k))$-conjugacy $[\mu]$ of μ, has a unique representative $\mu : \mathbb{G}_m \to G$.

If the above four assertions are true (for example in the case when G is a torus), then for any $g \in G(W(k))$, $\varphi \circ g = g_1 \circ \varphi$, with $g_1 \in G(W(k))$. By induction, $(g \circ \varphi)^m = g_m \circ \varphi^m$, with $g_m \in G(W(k))$, $\forall m \in \mathbb{N}$. So in this case, 3.1.0 is a direct consequence of 3.2.1.

3.2.2.1. Convention. In all that follows, till the end of §3, we assume that the assertions a) to d) of 3.2.2 are not true. So $(G, [\mu])$ is a Shimura group pair over $\mathrm{Spec}(W(k))$ (cf. def. 2.2.3).

3.2.3. A special example. Let P be the parabolic subgroup of G, having $F^0(\mathfrak{g})$ as its Lie algebra. Let $T \subset B \subset P$, with T a maximal torus containing the image of μ (so μ can be viewed as a cocharacter $\mu : \mathbb{G}_m \to T$), and with B a Borel subgroup of G (the existence of B is implied by our hypotheses in 2.2.8: G is quasi-split). Let $\mathfrak{t}$ (resp. $\mathfrak{b}$) be the Lie algebra of T ((resp. of B). We have inclusions $\mathfrak{t} \subset \mathfrak{b} \subset F^0(\mathfrak{g})$, and $F^1(\mathfrak{g}) \subset \mathfrak{b}$. We get $\varphi(\mathfrak{t}) \subset \varphi(\mathfrak{b}) \subset \mathfrak{g}$.

Let T_0 be a maximal torus of G having $\varphi(\mathfrak{t})$ as its Lie algebra. Such a torus exists due to the fact that a regular element of $\mathfrak{t} \otimes_{W(k)} W(\bar{k})$ is mapped through $\varphi \otimes 1$ into a regular element of $\mathfrak{g} \otimes_{W(k)} W(\bar{k})$, and a regular element of $\mathfrak{t}[\frac{1}{p}]$ is mapped through φ

into a regular element of $\mathfrak{g}[\frac{1}{p}]$. Let B_0 be the Borel subgroup of G such that its Lie algebra contains $\varphi(\mathfrak{b})$ ($\varphi(\mathfrak{b})[\frac{1}{p}]$ is a Borel Lie subalgebra of $\mathfrak{g}[\frac{1}{p}]$). There is an element $g_0 \in G(W(k))$ such $g_0 B_0 g_0^{-1} = B$ (any two Borel subgroups of G are $G(W(k))$-conjugate). Let $T_1 := g_0 T_0 g_0^{-1} \subset B$. There is $g_1 \in B(W(\bar{k}))$ such that $g_1 T_{1W(\bar{k})} g_1^{-1} = T_{W(\bar{k})}$.

Let $\varphi_0 := g_0\varphi$, and let τ_0 be the formal isogeny type of the σ-crystal (M, φ_0). We consider the following Shimura $\bar{\sigma}$-crystal $\big(M \otimes_{W(k)} W(\bar{k}), \varphi_1, G_{W(\bar{k})}\big)$ defined by

$$\varphi_1 := g_1(\varphi_0 \otimes 1).$$

We have $\varphi_0(\mathfrak{b}) \subset \mathfrak{b}$ and $\varphi_0(\mathfrak{b}[\frac{1}{p}]) = \mathfrak{b}[\frac{1}{p}]$. We have $\varphi_1(\mathfrak{b} \otimes_{W(k)} W(\bar{k})) \subset \mathfrak{b} \otimes_{W(k)} W(\bar{k})$ and $\varphi_1(\mathfrak{t} \otimes_{W(k)} W(\bar{k})) = \mathfrak{t} \otimes_{W(k)} W(\bar{k})$.

3.2.4. The theorem 3.1.0 is a direct consequence of the claims 3.2.5-8 below and of the semicontinuity theorem [Ka2, 2.3.1]: 3.2.5-6 imply 3.1.3 b); 3.2.6 implies 3.1.3 a) and e), and that $\mathcal{P}$ is the Newton polygon attached to the formal isogeny type τ_0; 3.2.5-6 imply $B_G \subset A_G$; 3.2.7 b) implies that $A_G \subset C_G \cap B_G$ and 3.2.7 c) implies $C_G \subset A_G$; so $A_G = B_G = C_G$; 3.2.8 implies 3.1.3 d).

3.2.5. Claim. *Among all Shimura Lie σ-crystals $(\mathfrak{g}, g\varphi)$ with $g \in G(W(k))$, there is no one whose Newton polygon is strictly below the Newton polygon of the Shimura Lie σ-crystal $(\mathfrak{g}, \varphi_0)$. Moreover, any Shimura σ-crystal $(M, g\varphi, G)$, with $g \in G(W(k))$, having the property that the Shimura Lie σ-crystals $(\mathfrak{g}, g\varphi)$ and $(\mathfrak{g}, \varphi_0)$ have the same Newton polygon, has the formal isogeny type τ_0, and its naturally attached Lie F-crystal has the same Newton polygon as $(\mathfrak{g}, \varphi_0)$.*

3.2.6. Lemma. *For any Shimura filtered σ-crystal $(M, F^1, g\varphi, G)$, with $g \in G(W(k))$, there is a Shimura filtered F-crystal $(M, F^1, \varphi, G, \tilde{f})$ such that the resulting F-crystal over the generic point of the special fibre (of the completion of G in the origin) has the formal isogeny type τ_0.*

3.2.7. Proposition. a) *The Shimura filtered σ-crystal (M, F^1, φ_0, G) has the property that the parabolic Lie subalgebra $\mathfrak{p}_0$ of it corresponding to non-negative slopes of $(\mathfrak{g}, \varphi_0)$ is contained in $F^0(\mathfrak{g})$.*

b) *The Shimura σ-crystals (M, φ_2, G) having τ_0 as the formal isogeny type are precisely the ones, which, up to isomorphism, can be extended to a Shimura filtered σ-crystal of the form $(M, F^1, \tilde{p}\varphi_0, G)$, with $\tilde{p}$ a $W(k)$-valued point of the parabolic subgroup P_0 of G satisfying Lie$(P_0) = \mathfrak{p}_0$. For any $\tilde{p} \in P_0(W(k))$, the Lie subalgebra of $\mathfrak{g}$ corresponding to non-negative slopes of $(\mathfrak{g}, \tilde{p}\varphi_0)$ is $\mathfrak{p}_0$ (and so it is included in $F^0(\mathfrak{g})$), and the Shimura Lie σ-crystals $(\mathfrak{g}, \varphi_0)$ and $(\mathfrak{g}, \tilde{p}\varphi_0)$ have the same Newton polygon.*

c) *Any Shimura filtered σ-crystal (M, F_2^1, φ_2, G) having the property that the parabolic Lie subalgebra of $\mathfrak{g}$ corresponding to non-negative slopes of $(\mathfrak{g}, \varphi_2)$ is contained in $F_2^0(\mathfrak{g})$, (as a σ-crystal) has the formal isogeny type τ_0.*

3.2.8. The unicity part. The uniqueness of the filtration mentioned in 3.1.0, results from the fact that any two such filtrations are conjugate by an element of $G(W(k))$, and from the fact that two conjugate parabolic subgroups of a reductive group, containing the same parabolic subgroup, are the same (it is enough to see this over a field; for this case cf. [Bo, 14.22 (iii)]).

3.2.9. Warning. We present two proofs of 3.2.5-7. The first one ends in 3.5.14; it avoids completely the use of 3.4. The second proof ends in 3.7 for 3.2.6-7 and in 3.5.13 for 2.5; it relies on 3.4.0-11.

3.3. About the proof of 3.2.5. We can assume that G is split and that $g_1 \in G(W(k))$ (we achieve this by passing from $W(k)$ to $W(k_1)$, with k_1 a finite extension of k; as G is split, we can choose T in 3.2.3 to be split, and then automatically we can take $g_1 \in G(W(k))$).

3.3.1. The claim 3.2.5 is a direct consequence of the following facts 3.3.2-5.

3.3.2. There is $b \in B(W(k))$ such that $\varphi_0 = b\varphi_1$ (proof: we know that there is $g \in G(W(k))$ such that $\varphi_0 = g\varphi_1$; as $\varphi_0(\mathfrak{b}\otimes_{W(k)} B(k)) = \varphi_1(\mathfrak{b}\otimes_{W(k)} B(k)) = \mathfrak{b}\otimes_{W(k)} B(k)$ we get that g normalizes $\mathfrak{b}\otimes_{W(k)} B(k)$ and so $g \in B(B(k)) \cap G(W(k)) = B(W(k))$).

As $\forall b_1 \in B(W(k))$ we have $\varphi_1 b_1 = b_1'\varphi_1$, with $b_1' \in B(W(k))$, by induction on $n \in \mathbb{N}$, we get that $\varphi_0^n = b(n)\varphi_1^n$, for some $b(n) \in B(W(k))$.

3.3.3. $B \subset P_1$, where P_1 is the parabolic subgroup of G, such that $\mathrm{Lie}(P_1)\otimes_{W(k)} B(k)$ is the parabolic Lie subalgebra of $\mathfrak{g}\otimes_{W(k)} B(k)$ corresponding to non-negative slopes of $(\mathfrak{g}\otimes_{W(k)} B(k), \varphi_1)$ (proof: as $\mathfrak{b} \subset F^0(\mathfrak{g})$, we have $\varphi_1(\mathfrak{b}) \subset \mathfrak{b}$; we deduce that $(\mathfrak{b}, \varphi_1)$ is a σ-crystal, and so its slopes are non-negative). Moreover $P_1 = P_0$ (this is a consequence of 3.2.1.1 and of the second part of 3.3.2).

3.3.4. Among all Shimura filtered σ-crystals $(M, F^1, g_2\varphi_0, G)$ with $g_2 \in G(W(k))$, the ones having a Newton polygon of their attached Shimura Lie σ-crystals $(\mathfrak{g}, g_2\varphi)$ strictly smaller or equal to the one of $(\mathfrak{g}, \varphi_0)$, are precisely the ones of the form $g_2\varphi_0 = g_3p_0\varphi_0g_3^{-1}$, with $g_3 \in G(W(k))$ such that mod p it leaves invariant F^1/pF^1, and with $p_0 \in P_0(W(k))$; this is true without the assumption G is split. For a proof of this see 3.5.13.

3.3.5. We remark that any Shimura Lie σ-crystal $(\mathfrak{g}, g_2\varphi_1)$, with $g_2 \in G(W(k))$ of the form mentioned in 3.3.4, has the same Newton polygon as $(\mathfrak{g}, \varphi_1)$, and that $(M, g_2\varphi_1)$ and (M, φ_1) have the same Newton polygon (proof: we can assume that $g_3 = 1$; then, as in 3.3.2, we have $(p_0\varphi_0)^n = p_{n-1}\varphi_1^n$, with $p_{n-1} \in P_0(W(k)) = P_1(W(k))$, $\forall n \in \mathbb{N}$, and so 3.2.1 and 3.2.1.1 apply).

3.4. The second group of basic results: global deformations and some principles for different categories $\mathcal{MF}(*)$. Let $G = \mathrm{Spec}(R)$ and let $g_2 \in G(W(k))$. Let $U_0 = \mathrm{Spec}(R_0)$ and $U_2 = \mathrm{Spec}(R_2)$ be affine open subschemes of G containing the origin $\mathrm{Spec}(W(k)) \overset{a_0}{\hookrightarrow} U_0$ and respectively containing the point g_2 (defined, for the sake of symmetry, by a morphism $\mathrm{Spec}(W(k)) \overset{a_2}{\hookrightarrow} U_2$), and having the following properties:

i) There are étale morphisms $U_0 \overset{b_0}{\to} Y := \mathrm{Spec}\big(W(k)[z_1, \dots, z_d]\big)$ and $U_2 \overset{b_2}{\to} Y$, with $d = \dim(G)$, such that the $W(k)$-morphisms $b_0 \circ a_0$ and $b_2 \circ a_2$ are defined at the level of rings by: z_i goes to 0, $i = \overline{1, d}$;

ii) There are étale morphisms $W_0 \overset{c_0}{\to} Z = \mathrm{Spec}\big(W(k)[z_1, \ldots, z_e]\big)$ and $W_2 \overset{c_2}{\to} Z$, with $e = \dim\big(GL(M)\big) = (\dim M)^2$, and with $W_j = \mathrm{Spec}(S_j)$ an affine open subscheme of $GL(M)$ containing U_j, $j \in \{0, 2\}$, such that $b_0 = \tilde{d} \circ c_0 \circ i_0$ and $b_2 = \tilde{d} \circ c_2 \circ i_2$, with $i_0 : U_0 \hookrightarrow W_0$ and $i_2 : U_2 \hookrightarrow W_2$ as the natural inclusions, and with $\tilde{d} : Z \to Y$ as the $W(k)$-morphism which at the level of rings takes z_i into z_i, $i = \overline{1, d}$ (we have $e \geq d$).

3.4.0. From now on, till the end of 3.4.8, we assume that k is an infinite field. For the case of a finite field see 3.4.9 3).

We choose a morphism $h : \mathrm{Spec}(W(k)) \hookrightarrow U_0 \cap U_2$ (R/pR is an integral domain cf. [Ti2, 3.8.1]; [Bo, 18.3] guarantees the existence of a point $\mathrm{Spec}(k) \hookrightarrow (U_0 \cap U_2)_k$) such that the $W(k)$-epimorphisms $W(k)[z_1, \ldots, z_d] \twoheadrightarrow W(k)$ associated to $b_0 \circ i_{02} \circ h$ and $b_2 \circ i_{20} \circ h$ (with i_{02} and i_{20} the inclusions of $U_0 \cap U_2$ into U_0 and respectively into U_2) send z_i, $i = \overline{1, d}$, into elements of $\mathbb{G}_m(W(k))$. Composing the morphisms b_0 and b_2 with automorphisms of Y of the form $z_i \to u_i z_i$, with $u_i \in \mathbb{G}_m(W(k))$, $i = \overline{1, d}$, we can assume that these $W(k)$-epimorphisms send z_i to 1, $i = \overline{1, d}$.

Let U_0^0, U_2^0, W_0^0 and W_2^0 be the right translations of U_0, U_2, W_0 and respectively W_2 by h^{-1}. So the morphism $a_0 : \mathrm{Spec}(W(k)) \hookrightarrow GL(M)$ defining the origin (the identity element of $GL(M)(W(k))$) factors through U_0^0, U_2^0, W_0^0 and W_2^0. Let Φ be the Frobenius lift of Y and Z, extending σ on $W(k)$ and taking $z_i - 1$ to $(z_i - 1)^p$, $i = \overline{1, e}$. Let Φ_{R_0}, Φ_{R_2}, Φ_{S_0} and Φ_{S_2} be the Frobenius lifts of $U_0^{0\wedge}$, $U_2^{0\wedge}$, $W_0^{0\wedge}$ and respectively of $W_2^{0\wedge}$ such that the two diagrams below:

$$\begin{array}{ccccc} U_j^{0\wedge} & \overset{r_h^\wedge}{\underset{\sim}{\longrightarrow}} & U_j^\wedge & \overset{b_j^\wedge}{\longrightarrow} & Y^\wedge \\ \downarrow{\scriptstyle \Phi_{R_j}} & & & & \downarrow{\scriptstyle \Phi^\wedge} \\ U_j^{0\wedge} & \overset{r_h^\wedge}{\underset{\sim}{\longrightarrow}} & U_j^\wedge & \overset{b_j^\wedge}{\longrightarrow} & Y^\wedge, \end{array}$$

and

$$\begin{array}{ccccc} W_j^{0\wedge} & \overset{r_h^\wedge}{\underset{\sim}{\longrightarrow}} & W_j^\wedge & \overset{c_j^\wedge}{\longrightarrow} & Z^\wedge \\ \downarrow{\scriptstyle \Phi_{S_j}} & & & & \downarrow{\scriptstyle \Phi^\wedge} \\ W_j^{0\wedge} & \overset{r_h^\wedge}{\underset{\sim}{\longrightarrow}} & W_j^\wedge & \overset{c_j^\wedge}{\longrightarrow} & Z^\wedge, \end{array}$$

$j \in \{0, 2\}$, are cartesian; here r_h is the right translation by h, and $b_j^\wedge$, $r_h^\wedge$, $c_j^\wedge$ and $\Phi^\wedge$ refer to morphisms defined by the p-adic completion of b_j, r_h, c_j and respectively of Φ. The isomorphism r_h allows us to identify $U_j^0 = \mathrm{Spec}(R_j)$ and $W_j^0 = \mathrm{Spec}(S_j)$, $j \in \{0, 2\}$. Let $t_i(j) \in S_j$ such that under the identification $W_j^0 = \mathrm{Spec}(S_j)$, the morphism $c_j^\wedge \circ \tilde{r}_h^\wedge$, at the level of rings, takes $z_i - 1$ into $t_i(j)$, $j \in \{0, 2\}$, $i \in \{1, ..., e\}$. So $\Phi_{S_j}(t_i(j)) = t_i(j)^p$. We can regard $t_i(j)$, $i \in \{1, ..., d\}$, as an element of R_j.

3.4.1. Let

$$M_{R_j} := \big(M \otimes_{W(k)} R_j^\wedge, F^1 \otimes_{W(k)} R_j^\wedge, \Phi_j^r\big)$$

and

$$M_{S_j} := \big(M \otimes_{W(k)} S_j^\wedge, F^1 \otimes_{W(k)} S_j^\wedge, \Phi_j^s\big),$$

with $\Phi_j^r := r_j(h\varphi_0 \otimes 1)$ and $\Phi_j^s := s_j(h\varphi_0 \otimes 1)$, where $r_j \in G(R_j^\wedge)$ and $s_j \in GL(S_j^\wedge)$ are the universal elements of $G^\wedge$ and respectively of $GL(M)^\wedge$, defined by the p-adic completion of the inclusions $U_j^0 \hookrightarrow G$ and $W_j^0 \hookrightarrow GL(M)$, $j \in \{0,2\}$. For any $n \in \mathbb{N}$, $M_{R_j}/p^n M_{R_j}$ is an object of $\mathcal{MF}_{[0,1]}(R_j)$, and $M_{S_j}/p^n M_{S_j}$ is an object of $\mathcal{MF}_{[0,1]}(S_j)$, $j \in \{0,2\}$. So M_{R_j} (resp. M_{S_j}) are p-divisible objects of $\mathcal{MF}_{[0,1]}(R_j)$ (resp. of $\mathcal{MF}_{[0,1]}(S_j)$).

Warning: we often identify a morphism $\mathrm{Spec}(\mathcal{R}) \to G$, with a matrix automorphism of $M \otimes_{W(k)} \mathcal{R}$; here $\mathcal{R}$ is an arbitrary $W(k)$-algebra.

3.4.1.1. For any formally étale morphism $\mathrm{Spec}(\tilde{Q}_j^\wedge) \xrightarrow{d_j} \mathrm{Spec}(R_j^\wedge)$, with $\tilde{Q}_j$ a $W(k)$-algebra, we denote by

$$M_{\tilde{Q}_j} := \big(M \otimes_{W(k)} \tilde{Q}_j^\wedge, F^1 \otimes_{W(k)} \tilde{Q}_j^\wedge, r_j \circ d_j(h\varphi_0 \otimes 1)\big)$$

the pull back of M_{R_j} under d_j, for $\Phi_{\tilde{Q}_j}$ the only Frobenius lift of $\tilde{Q}_j^\wedge$ satisfying $d_j \circ \Phi_{\tilde{Q}_j} = \Phi_{R_j} \circ d_j$. It is a p-divisible object of $\mathcal{MF}_{[0,1]}(\tilde{Q}_j)$. We call it the extension of M_{R_j} to $\mathrm{Spec}(\tilde{Q}_j)$. So for any $n \in \mathbb{N}$, $M_{\tilde{Q}_j}/p^n M_{\tilde{Q}_j}$ is an object of $\mathcal{MF}_{[0,1]}(\tilde{Q}_j)$. In particular, if $\hat{R}_j^0$ is the completion of R_j in the origin, we denote by $M_{\hat{R}_j^0} := (M \otimes_{W(k)} \hat{R}_j^0, F^1 \otimes_{W(k)} \hat{R}_j^0, \hat{\Phi}_j^r)$ the pull back of M_{R_j} under the canonical monomorphism $i_{R_j} : R_j \hookrightarrow \hat{R}_j^0$, for $\Phi_{\hat{R}_j^0}$ the natural Frobenius lift of $\hat{R}_j^0$ induced from Φ_{R_j} by completion, $j \in \{0,2\}$. Here $\hat{\Phi}_j^r = \hat{r}_j(h\varphi_0 \otimes 1)$, with $\hat{r}_j \in G(\hat{R}_j^0)$, obtained from r_j via i_{R_j}. $M_{\hat{R}_j^0}$ is a p-divisible object of $\mathcal{MF}_{[0,1]}(\hat{R}_j^0)$.

Similarly, if k_1 is a perfect field containing k, we denote by $M_{R_{j W(k_1)}}$ its pull back under the canonical morphism $\mathrm{Spec}(R_{j W(k_1)}) \to \mathrm{Spec}(R_j)$, the Frobenius of $R_{j W(k_1)}$ being the logical one. We call it the extension of M_{R_j} to $W(k_1)$.

In a similar manner we define the extension of any object (or p-divisible object) of $\mathcal{MF}_{[\tilde{a},\tilde{b}]}(R_3)$, with R_3 a regular formally smooth $W(k)$-algebra such that $R_3^\wedge$ is endowed with a Frobenius lift, under a formally étale morphism $\mathrm{Spec}(\tilde{Q}_3^\wedge) \to \mathrm{Spec}(R_3)$, or under the natural morphism $\mathrm{Spec}(R_{3W(k_1)}) \to \mathrm{Spec}(R_3)$, with k_1 as above.

3.4.1.1.1. Definitions and notations. 1) Let $d_j : \mathrm{Spec}(\tilde{Q}_j^\wedge) \to \mathrm{Spec}(R_j^\wedge)$ be a formally étale morphism, with $\tilde{Q}_j$ a $W(k)$-algebra. We recall that we have a direct sum decomposition $M = F^1 \oplus F^0$. Let

$$\Phi(\tilde{Q}_j)^0 := r_j \circ d_j(h\varphi_0 \otimes 1)$$

be the Frobenius of $M \otimes_{W(k)} \tilde{Q}_j^\wedge$ and let

$$\Phi(\tilde{Q}_j)^1 : (F^1 + pM) \otimes_{W(k)} \tilde{Q}_j^\wedge \to M \otimes_{W(k)} \tilde{Q}_j^\wedge$$

be defined by the formula $p\Phi(\tilde{Q}_j)^1(m) = \Phi(\tilde{Q}_j)^0(m)$, $m \in (F^1 + pM) \otimes_{W(k)} \tilde{Q}_j^\wedge$. For $n \in \mathbb{N}$ we still denote by $\Phi(\tilde{Q}_j)^i$ its reduction mod p^n ($i \in \{0,1\}$).

2) We say that $M_{\tilde{Q}_j}/p^n M_{\tilde{Q}_j}$ potentially can be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(\tilde{Q}_j)$ if there is a connection

$$\nabla : M \otimes_{W(k)} \tilde{Q}_j/p^n\tilde{Q}_j \to M \otimes_{W(k)} \Omega_{(\tilde{Q}_j/p^n\tilde{Q}_j)/W(k)}$$

such that we have:

$$\nabla \circ \Phi(\tilde{Q}_j)^0(m) = p\Phi(\tilde{Q}_j)^0 \circ d\Phi_{\tilde{Q}_j *}/p \circ \nabla(m), \tag{1}$$

if $m \in F^0 \otimes_{W(k)} \tilde{Q}_j/p^n\tilde{Q}_j$, and

$$\nabla \circ \Phi(\tilde{Q}_j)^1(m) = \Phi(\tilde{Q}_j)^0 \circ d\Phi_{\tilde{Q}_j *}/p \circ \nabla(m), \tag{2}$$

if $m \in F^1 \otimes_{W(k)} \tilde{Q}_j/p^n\tilde{Q}_j$. Here $d\Phi_{\tilde{Q}_j *}/p$ denotes the differential of the Frobenius $\Phi_{\tilde{Q}_j}$ of $\tilde{Q}_j^{\wedge}$ divided by p, and then taking mod p^n. About ∇ we say that it makes (or allows) $M_{\tilde{Q}_j}/p^n M_{\tilde{Q}_j}$ potentially to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(\tilde{Q}_j)$. Warning: the system of equations obtained by putting (1) and (2) together, does not depend on the choice of the direct supplement F^0 of M; moreover as we chose $\Phi(\tilde{Q}_j)^i$ to be Frobenius maps, and not $\tilde{Q}_j^{\wedge}$-linear maps, we got $m \in F^i \otimes_{W(k)} \tilde{Q}_j/p^n\tilde{Q}_j$, and not in $F^i \otimes_{W(k)} {}_{\sigma}\tilde{Q}_j/p^n\tilde{Q}_j$, $i \in \{0,1\}$.

We say that $M_{\tilde{Q}_j}/p^n M_{\tilde{Q}_j}$ potentially is an object of $\mathcal{MF}^{\nabla}_{[0,1]}(\tilde{Q}_j)$ if there is precisely one connection ∇ which makes it potentially to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(\tilde{Q}_j)$; about this connection ∇ we say that it makes (or allows) $M_{\tilde{Q}_j}/p^n M_{\tilde{Q}_j}$ potentially to be an object of $\mathcal{MF}^{\nabla}_{[0,1]}(\tilde{Q}_j)$.

We use the same language for the case of p-divisible objects, i.e. when $n = \infty$. In such a situation we also say that the Frobenius $\Phi(\tilde{Q}_j)^0$ of $M_{\tilde{Q}_j}$ is ∇-parallel.

If in any of the above cases the connection ∇ is integrable, then we drop the word potentially. We prove later on (cf. 3.4.18.4.1), that any connection ∇ which makes $M_{\tilde{Q}_j}/p^n M_{\tilde{Q}_j}$ potentially to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(\tilde{Q}_j)$, is automatically integrable, and so the word potentially is just for the time being.

3) Let $\mathcal{Z}_k$ be the closed subscheme of U_0^0, U_2^0, G, W_0^0, W_2^0 and of $GL(M)$, defining the origin of the special fibre of G_k. So $\mathcal{Z}_k = \mathrm{Spec}(k)$.

4) For an arbitrary regular formally smooth $W(k)$-algebra R_3 as in the end of 3.4.1.1, and for any (p-divisible) object of $\mathcal{MF}_{[\tilde{a},\tilde{b}]}(R_3)$, with $\tilde{a}, \tilde{b} \in \mathbb{Z}$, $\tilde{a} < \tilde{b}$, we are going to use the same language as in 2) above, but always working with connections satisfying the Griffiths transversality condition. The equations needed to be satisfied are similar to the ones of (1) and (2) of 2) above (for instance see [Fa1, p. 33]).

3.4.1.1.2. The convenience assumption. Sometimes it is convenient to make the assumption that the closed subscheme of $\mathrm{Spec}(R_j^0/pR_j^0)$ defined by $t_i(j) = 0$, i running through the elements of an arbitrary subset of $\{1, ..., d\}$, is connected (in particular the origin $\mathrm{Spec}(k)$ of the special fibre of U_j^0 is the closed subscheme defined by $t_i(j) = 0$, $i \in \{1, ..., d\}$).

This can always be achieved by removing a proper closed subscheme of U_j^0. This should be done before the choice of h, so we do not lose track of a_0 and a_2. It can be done as follows: we choose an open affine subscheme $\mathcal{U} = \mathrm{Spec}(R_{\mathcal{U}})$ containing the origin of G, together with an étale morphism $b_{\mathcal{U}} : \mathcal{U} \to Y$, such that the closed subscheme $\mathrm{Spec}(R_{\mathcal{U}}/(p, (z_i)_{i\in I_d}))$ of $\mathcal{U}$, with I_d an arbitrary subset of $\{1, ..., d\}$, is connected, and $\mathrm{Spec}(R_{\mathcal{U}}/((z_i)_{i\in I}))$ is the origin $\mathrm{Spec}(W(k))$ of G. We can take now $U_0 = \mathcal{U}$, and $U_2 = g_2\mathcal{U}$; as b_0 and b_2 we take $b_{\mathcal{U}}$ and respectively its translation by g_2.

Warning: unless specifically stated we do not work under the convenience assumption.

3.4.1.2. Lemma (the unicity principle for Shimura σ-crystals). *For any formally étale morphism $l_j : \mathrm{Spec}(\tilde{Q}_j^{\wedge}) \to \mathrm{Spec}(R_j^{\wedge})$, with $\tilde{Q}_j$ a $W(k)$-algebra such that any connected component of $\mathrm{Spec}(\tilde{Q}_j/p\tilde{Q}_j)$ has a non-empty intersection with $\ell_j^{-1}(\mathcal{Z}_k)$, and for any $n \in \mathbb{N}$, $M_{\tilde{Q}_j}/p^nM_{\tilde{Q}_j}$ potentially can be viewed in at most one way as an object of $\mathcal{MF}_{[0,1]}^{\nabla}(\tilde{Q}_j)$; similarly $M_{\tilde{Q}_j}$ potentially can be viewed in at most one way as a p-divisible object of $\mathcal{MF}_{[0,1]}^{\nabla}(\tilde{Q}_j)$.*

3.4.1.3. Theorem (the ∇, the moduli, and the surjectivity principle for Shimura σ-crystals). 1) *For any $n \in \mathbb{N}$ there is a formally étale morphism $\mathrm{Spec}(Q_{j,n}^{\wedge}) \overset{\ell_{j,n}}{\to} \mathrm{Spec}(R_j^{\wedge})$, $j \in \{0, 2\}$, with $Q_{j,n}$ a smooth $W(k)$-algebra, such that:*

i) $\ell_{j,n}^{-1}(\mathcal{Z}_k) = \mathrm{Spec}(k)$;

ii) $M_{Q_{j,n}}/p^nM_{Q_{j,n}}$ *is an object of* $\mathcal{MF}_{[0,1]}^{\nabla}(Q_{j,n})$, *i.e. there is a unique connection*

$$\nabla_j^n : M \otimes_{W(k)} Q_{j,n}/p^nQ_{j,n} \to M \otimes_{W(k)} \Omega_{(Q_{j,n}/p^nQ_{j,n})/W(k)}$$

which makes $M_{Q_{j,n}}/p^nM_{Q_{j,n}}$ potentially to be viewed as an object of $\mathcal{MF}_{[0,1]}^{\nabla}(Q_{j,n})$; the connection ∇_j^n is integrable (so the word potentially can be disregarded), nilpotent mod p, and respects the G-action;

iii) for any other formally étale morphism $\mathrm{Spec}(\tilde{Q}_j^{\wedge}) \overset{\tilde{\ell}_j}{\to} \mathrm{Spec}(R_j^{\wedge})$, with $\tilde{Q}_j$ a regular formally smooth $W(k)$-algebra, such that $M_{\tilde{Q}_j}/p^nM_{\tilde{Q}_j}$ potentially can be viewed as an object of $\mathcal{MF}_{[0,1]}^{\nabla}(\tilde{Q}_j)$ and any connected component of $\mathrm{Spec}(\tilde{Q}_j/p\tilde{Q}_j)$ has a non-empty intersection with its closed subscheme $\tilde{\ell}_j^{-1}(\mathcal{Z}_k)$, there is a unique $R_j^{\wedge}$-morphism $\mathrm{Spec}(\tilde{Q}_j^{\wedge}) \overset{\tilde{\ell}_j^n}{\to} \mathrm{Spec}(Q_{j,n}^{\wedge})$ such that $\tilde{\ell}_j = \ell_{j,n} \circ \tilde{\ell}_j^n$ and the connection (it is unique cf. 3.4.1.2) on $M \otimes_{W(k)} \tilde{Q}_j/p^n\tilde{Q}_j$ which makes $M_{\tilde{Q}_j}/p^nM_{\tilde{Q}_j}$ potentially to be viewed as an object of $\mathcal{MF}_{[0,1]}^{\nabla}(\tilde{Q}_j)$ is the pull back of ∇_j^n through $\tilde{\ell}_j^n$ mod p^n;

iv) the special fibre of $\mathrm{Spec}(Q_{j,n})$ *is a geometrically connected scheme over* $\mathrm{Spec}(k)$.

2) *The unique $R_j^{\wedge}$-morphism (cf. ii)) $\ell_j^n : \mathrm{Spec}(Q_{j,n+1}^{\wedge}) \to \mathrm{Spec}(Q_{j,n}^{\wedge})$ such that $\ell_{j,n+1} = \ell_{j,n} \circ \ell_j^n$ and the pull back of ∇_j^n under the morphism ℓ_j^n mod p^n is ∇_j^{n+1} mod p^n, when taken mod p (i.e. the resulting morphism at the level of special fibres) is: étale, quasi-finite, generically an étale cover of degree $p^{d(n)}$, with $d(n) \le d^2$. Similarly, the morphism $\ell_{j,1}$ mod p has all these properties.*

3) *The construction of* $\mathrm{Spec}(Q_{j,n}^{\wedge})$ *commutes with (perfect) base field extensions, i.e. if* k_1 *is a perfect field containing* k*, then the* p*-adic completion of the extension of* $\ell_{j,n}$ *to* $W(k_1)$ *is the morphism of 1) obtained for the same* $n \in \mathbb{N}$ *and for the situation where* G *is replaced by* $G_{W(k_1)}$*,* R_j *is replaced by* $R_{j W(k_1)}$*, and* M_{R_j} *is replaced by its extension to* $W(k_1)$*.*

4) *Let* $\mathrm{Spec}(Q_j)$ *be the* p*-adic completion of the* $\mathbb{N}$*-projective limit of* $\mathrm{Spec}(Q_{j,n}^{\wedge})$ *under the transition morphisms* ℓ_j^n*,* $n \in \mathbb{N}$*, and let* $\ell_j : \mathrm{Spec}(Q_j) \to U_j^0 = \mathrm{Spec}(R_j)$ *be the resulting morphism. There is a unique morphism* $a^j : \mathrm{Spec}(W(k)) \to \mathrm{Spec}(Q_j)$ *which composed with* l_j *defines the origin* a_0 *of* U_j^0*.* M_{Q_j} *is a* p*-divisible object of* $\mathcal{MF}_{[0,1]}^{\nabla}(Q_j)$*.* $\mathrm{Spec}(Q_j)$ *has a universal property similar to the one described in iii) of 1). The* k*-scheme* $\mathrm{Spec}(Q_j/pQ_j)$ *is geometrically connected.*

5) *There is a reduced closed subset* $\mathcal{B}_j(k)$ *of* G_k*, not containing the origin, depending on* $h\varphi_0$*, on* Φ_{R_j}*, but not on how small or big* U_j^0 *is, and such that the morphism* ℓ_j *has non-empty fibres over (geometric) points of* $\mathrm{Spec}(R_j/pR_j)$ *which (when viewed as points of* G_k*) do not factor through* $\mathcal{B}_j(k)$*. For any perfect field* k_1 *containing* k*, we get similarly (cf. 3) above) a reduced closed subscheme* $\mathcal{B}_j(k_1)$ *of* G_{k_1} *having similar properties as* $\mathcal{B}_j(k)$*. We can take* $\mathcal{B}_j(k_1) := \mathcal{B}_j(k)_{k_1}$*.*

The proofs of 3.4.1.2-3 are presented in 3.4.8.

3.4.1.4. Remarks. 1) Based on 3.4.18.4.2, it can be checked out that in many cases we can take $Q_{j,n} = R_j$ (i.e. $d(n) = 1$), $\forall n \in \mathbb{N}$. All these cases are going to be listed in §7, after we present (in §7) the classification of Shimura group pairs over $\mathbb{Z}_p$, and introduce (see 3.10) some extra terminology. See also 3.11.4.

2) The morphism $\ell_{j,n}$ is an étale cover iff $Q_{j,n}^{\wedge} = R_j^{\wedge}$ (cf. i) of 3.4.1.3 1)). A simple example when $\ell_{j,1}$ is not an étale cover was constructed by A. Ogus.

Let M be of dimension 2 over $W(k)$, and let $G = GL(M)$. We assume that $(M, F^1, h\varphi_0)$ is the filtered σ-crystal of the canonical lift of an ordinary elliptic curve over k. We assume also the existence of a $W(k)$-basis $\{x, y\}$ of M such that $h\varphi_0$ takes x and y respectively into x and py. We choose a Frobenius of $G^{\wedge}$ compatible with the Frobenius σ of $W(k)$ and such that under the canonical identification $G = \mathrm{Spec}(W(k)[x_{11}, x_{12}, x_{21}, x_{22}][\frac{1}{x_{11}x_{22} - x_{21}x_{12}}])$ defined by the above ordered basis of M, x_{12} goes to x_{12}^p, x_{21} goes to x_{21}^p, while $x_{ii} - 1$ goes to $(x_{ii} - 1)^p$, $i = \overline{1,2}$. We can take $U_0 = U_2 = G$. To see that $l_{j,1}$ is not an étale cover, it is enough to see that the pull back of M_{R_j}/pM_{R_j} under the homomorphism $\mathbb{G}_a \to G$ defining the subgroup of G acting trivially on x and on the quotient $M/xW(k)$, can not get a connection over its pull back under an étale cover of $\mathbb{G}_a^{\wedge}$. This is easy: identifying $\mathbb{G}_a = \mathrm{Spec}(W(k)[t])$, with $t = x_{12}$, the connection we get on $M \otimes_{W(k)} W(k)[[t]]$ annihilates x and takes y into $\alpha(t)xdt$, where

$$\alpha(t) = -\sum_{i=0}^{\infty} t^{p^i - 1}.$$

So $\alpha(t)$ mod p satisfies the equation (in u)

$$t^{p-1}u^p = u + 1$$

(we recall that $p>2$). This equation defines an étale scheme over $\mathbb{G}_{a\,k}=\mathrm{Spec}(k[t])$ which is not an étale cover: it is obtained from the étale cover $\mathrm{Spec}(k[t,v]/(v^p-v-t))$ of $\mathbb{G}_{a\,k}$ by removing its closed subscheme defined by $t=0$, $v\neq 0$. This closed subscheme is a disjoint union of $p-1$ copies of $\mathrm{Spec}(k)$.

In this example, coming back to the general notations, the morphism

$$\ell_j':\ell_j^{-1}(\mathrm{Spec}(Q_j/pQ_j[\frac{1}{\prod_{i=1}^d t_i(j)}])\to \mathrm{Spec}(R_j/pR_j[\frac{1}{\prod_{i=1}^d t_i(j)}]):=\mathcal{W}_j$$

is an $\mathbb{N}$-pro-étale morphism of infinite degree. Here $d=4$, $t_i(j)=x_{ii}-1$ and $t_{i+1}(j)=x_{i,3-i}$, with $i\in\{1,2\}$. It defines an $\mathbb{N}$-pro-étale cover above the open subscheme of $\mathcal{W}_j$ defined by $t_1(j)\neq -1$, as it can be easily checked starting from 3.4.18.4. So if we take U_0^0 to be $\mathcal{W}_0$ and not the whole group G, then we can take $\mathcal{B}_0(k)$ to be defined by $x_{11}=0$.

3) The estimate $d(n)\le d^2$ is not the best possible. For instance if $G=GL(M)$ it can be checked that we can replace d^2 by $\frac{d^2}{4}$. For better estimates of the numbers $d(n)$, see 3.11.4. For the time being, let us just remark that the bound d^2 is independent on n. We do not know (if or) when $d(n)=d(1)$, $\forall n\in\mathbb{N}$.

4) We do not know if (or when) the map ℓ_j is surjective. Similarly about the maps $\ell_{j,n}$ or ℓ_j^n, $n\in\mathbb{N}$, $j\in\{0,2\}$. 3.4.18.5.2 motivates why it is natural to expect (under some conditions) such maps to be surjective.

5) $M_{Q_{j,m}}/p^nM_{Q_{j,m}}$, $m,n\in\mathbb{N}$, $n\le m$, are rigid objects in the sense they have few endomorphisms. For instance, restricting to endomorphism (called G-endomorphism) fixing the family of tensors $(v_\alpha)_{\alpha\in\mathcal{J}}$ introduced at the beginning of §3, we often get only endomorphisms coming from elements of the Lie algebra of $Z(G)$ which are fixed by $h\varphi_0$. This remains true in many cases for the set of all endomorphisms, the group $Z(G)$ being replaced by the centralizer of G in $GL(M)$. This important phenomenon is elaborated in §7, being an essential tool in the strategy of proving the split criterion of 1.19. There are variants of it when we pass to étale schemes over $\mathrm{Spec}(Q_{j,m})$. In these variants extra endomorphisms can show up: like the ones springing from the factors of $G^{\mathrm{ad}}_{W(k_1)}$ having a zero intersection with $F^1(\mathrm{Lie}(G)\otimes_{W(k)}W(k_1))$ and left invariant by $h\varphi_0$; here k_1 is a finite field extension of k.

3.4.1.5. σ-$\mathcal{S}$-crystals. Let $\tilde a,\tilde b\in\mathbb{Z}$, with $\tilde a<\tilde b$. Let $S(\tilde a,\tilde b):=\{\tilde a,...,\tilde b\}$. Let $(\tilde M,(\tilde F^i)_{i\in S(\tilde a,\tilde b)},\tilde\varphi)$ be a p-divisible object of $\mathcal{MF}_{[\tilde a,\tilde b]}(W(k))$. Let $\tilde G$ be a smooth subgroup of $GL(\tilde M)$, having connected fibres, and such that:

– $\mathrm{Lie}(\tilde G)[\frac{1}{p}]$ is stable under the action of $\tilde\varphi$;

– $\mathrm{Lie}(\tilde G)$ together with the induced filtration (from the natural induced filtration of $\mathrm{End}(\tilde M)$) and Frobenius $\tilde\varphi$, is a p-divisible object of $\mathcal{MF}(W(k))$.

The quadruple $(\tilde M,(\tilde F^i)_{i\in S(\tilde a,\tilde b)},\tilde\varphi,\tilde G)$ is called a filtered σ-$\mathcal{S}$-crystal (over k); here $\mathcal{S}$ stands for subobject, and refers to the second of the above two conditions. As in 2.2.8 2) we get σ-$\mathcal{S}$-crystals. As examples we mention the pseudo Shimura σ-crystals, cf. 2.2.9 1'), and the generalized Shimura p-divisible objects over k, cf. def. 4) of 2.2.8. We would like to point out that we did not assume the existence of a cocharacter $\mu : \mathbb{G}_m \to \tilde{G}$, leaving invariant $\tilde{F}^i$, $i \in S(\tilde{a}, \tilde{b})$, with $\beta \in \mathbb{G}_m(W(k))$ acting on $\tilde{F}^i/\tilde{F}^{i+1}$ as the multiplication with β^{-i} (to be compared with 2.2.8, where we assumed this). When such a cocharacter does exist, we speak about (filtered) σ-$\mathcal{S}$-crystals for which the W-condition is satisfied (the letter W stands to honor [Wi]). It is easy to see that 2.2.9 1) and 1') apply to the present context; keeping in mind what $\mathcal{S}$ stands for, we felt it appropriate not to use the word "pseudo" in front of the above terminology, in case the W-condition is not satisfied.

3.4.1.6. Remark. It is expected that everything in 3.4.1.3 (when properly formulated; see the paragraph below) remains true in the context of σ-$\mathcal{S}$-crystals. This is true if $\tilde{a} = 0$ and $\tilde{b} = 1$, as well as for generalized Shimura p-divisible objects of $\mathcal{MF}(W(k))$. The proof in the case when $\{\tilde{a}, \tilde{b}\} = \{0, 1\}$ is entirely the same. In the other case we have to interpret 3.4.1.3 as a property of Shimura filtered Lie σ-crystals, and to construct a Shimura filtered σ-crystal whose Shimura adjoint filtered Lie σ-crystal is isomorphic to the one of our Shimura p-divisible object of $\mathcal{MF}(W(k))$; for details see §7. In these two situations we can introduce as in 2.2.10 the notion of filtered F-$\mathcal{S}$-crystals and respectively of generalized Shimura filtered p-divisible objects of some category $\mathcal{MF}_{[\tilde{a},\tilde{b}]}(*)$; here $*$ stands for regular rings of formal power series with coefficients in $W(k)$.

What we mean by "properly formulated": following the discussion of 3.4.8.9, we might have to work with a different Frobenius of the module underlying the p-divisible objects involved (i.e. we do not use r_j of 3.4.1, but some other element of $\tilde{G}(*)$, carefully chosen).

3.4.2. Some consequences. Before proving 3.4.1.2-3, let us see some of their immediate implications. It is known [Fa1, 7.1] (here is the first place where we need $p \geq 3$) that the category $\mathcal{MF}^{\nabla}_{[0,1]}(Q_{j,n})$ is antiequivalent to the category of finite flat commutative group schemes of p-power rank on $\mathrm{Spec}(Q^{\wedge}_{j,n})$, $j \in \{0, 2\}$. So corresponding to $M_{Q_{j,n}}/p^n M_{Q_{j,n}}$ we get a finite flat commutative group scheme $D_{j,n}$ on $\mathrm{Spec}(Q^{\wedge}_{j,n})$ of rank $p^{n \dim(M)}$. Using the uniqueness of ∇^n_j we get that the pull back of $D_{j,n}$ under the morphism ℓ^n_j is isomorphic to the kernel of the multiplication with p^n on $D_{j,n+1}$, $\forall n \in \mathbb{N}$, $j \in \{0, 2\}$. So the objects (of $\mathcal{MF}^{\nabla}_{[0,1]}(Q_j)$) $M_{Q_j}/p^n M_{Q_j}$, $n \in \mathbb{N}$, put together give birth to a p-divisible group D_j over $\mathrm{Spec}(Q_j)$. Let ∇_j be the connection on $M \otimes_{W(k)} Q_j$ which makes M_{Q_j} to be a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(Q_j)$. As the connections ∇^n_j, $n \in \mathbb{N}$, respect the G-action, we deduce that ∇_j also respects the G-action. We conclude: the pair

$$D(j) := (D_j, (v_\alpha)_{\alpha \in \mathcal{J}})$$

is a Shimura p-divisible group over Q_j.

3.4.2.1. For simplifying the notations we assume now, till the end of 3.4.5, that $k = \bar{k}$. We also assume that h^{-1} mod p does not define a point of $\mathcal{B}_0(k)$, and that $g_2 h^{-1}$ mod p does not define a point of $\mathcal{B}_2(k)$. We have:

a) The pull back of $D(j)$ under the morphism a^j is the Shimura p-divisible group over $W(k)$ associated to the Shimura filtered σ-crystal $(M, F^1, h\varphi_0, G, (v_\alpha)_{\alpha \in \mathcal{J}})$, $j \in \{0, 2\}$.

b) The pull back of $D(0)$ under a morphism $\mathrm{Spec}(W(k)) \to \mathrm{Spec}(Q_0)$ which through ℓ_0 produces the morphism $\mathrm{Spec}(W(k)) \hookrightarrow U^0_0$ defining h^{-1}, is the Shimura p-divisible group over $W(k)$ associated to the Shimura filtered σ-crystal $(M, F^1, \varphi_0, G, (v_\alpha)_{\alpha \in \mathcal{J}})$.

c) The pull back of $D(2)$ under a morphism $\mathrm{Spec}(W(k)) \to \mathrm{Spec}(Q_2)$ which through ℓ_2 produces the morphism $\mathrm{Spec}(W(k)) \hookrightarrow U_2^0$ defining g_2h^{-1}, is the Shimura p-divisible group over $W(k)$ associated to the Shimura filtered σ-crystal $(M, F^1, g_2\varphi_0, G, (v_\alpha)_{\alpha\in\mathcal{J}})$.

d) We denote by

$$O_0 := W(k)[[\omega_1, \ldots, \omega_d]]$$

the ring of the completion of G (or of U_0^0, or of U_2^0) in the origin. It is the same as the ring of the completion of Q_j in a^j, $j \in \{0, 2\}$; so we have $O_0 = \hat{R}_0^0 = \hat{R}_2^0$. The pull backs of $D(0)$ and $D(2)$ to $\mathrm{Spec}(O_0)$ give us two Shimura p-divisible groups: D_0^0 and respectively D_2^0. The p-divisible objects with cycles of $\mathcal{MF}_{[0,1]}^{\nabla}(O_0)$ they define are both Shimura filtered F-crystals: $(M, F^1, h\varphi_0, G, \tilde{f}_0)$ and respectively $(M, F^1, h\varphi_0, G, \tilde{f}_2)$. Warning: the Frobenius of O_0 corresponding to D_0^0 is different from the Frobenius of O_0 corresponding to D_2^0. But these two Shimura filtered F-crystals are induced one from each other, cf. [Fa2, rm. iii) after th. 10] (the two Frobenius of O_0 are as needed for [Fa2, th. 10], cf. the way we defined Φ_{R_j}). This means that there are $W(k)$-homomorphisms

$$O_0 \underset{\omega_2}{\overset{\omega_0}{\rightrightarrows}} O_0$$

such that the Shimura filtered F-crystal associated to the pull back of D_j^0 under the morphism associated to ω_j is (isomorphic to) the Shimura filtered F-crystal associated to D_{2-j}^0, $j \in \{0, 2\}$.

3.4.2.2. Remark. We call this last fact, that D_0^0 and D_2^0 are induced one from each other, the very weak gluing principle. It is trivial to check that we can actually choose ω_j, $j \in \{0, 2\}$, to be isomorphisms of $W(k)$-algebras, and we call this the weak gluing principle.

3.4.3. We consider (cf. 3.4.1.3) the following p-divisible object of $\mathcal{MF}_{[-1,1]}^{\nabla}(Q_j)$:

$$\mathrm{End}(M_{Q_j}) := \big(\mathrm{End}(M)\otimes_{W(k)}Q_j, F^0(\mathrm{End}(M))\otimes_{W(k)}Q_j, F^1(\mathrm{End}(M))\otimes_{W(k)}Q_j, r_j\circ\ell_j(h\varphi_0\otimes 1), \nabla_j\big).$$

Similarly, we define $\mathfrak{g}_{Q_j}$ by working with $\mathfrak{g} = \mathrm{Lie}(G)$ instead of $\mathrm{End}(M)$. For any $n \in \mathbb{N}$, $\mathrm{End}(M_{Q_j}/p^nQ_j) := \mathrm{End}(M_{Q_j})/p^n\mathrm{End}(M_{Q_j})$ is an object of $\mathcal{MF}_{[-1,1]}^{\nabla}(Q_j)$, having (as ∇_j respects the G-action) $\mathfrak{g}_{Q_j}/p^n\mathfrak{g}_{Q_j}$ as a subobject in $\mathcal{MF}_{[-1,1]}^{\nabla}(Q_j)$, $j \in \{0, 2\}$. We deduce: for any map $z : \mathrm{Spec}(W(k_1)) \to \mathrm{Spec}(Q_j)$, with k_1 a perfect field, the pull back of M_{Q_j} through z gets a natural structure of a Shimura filtered σ_1-crystal $\big(M \otimes_{W(k)} W(k_1), F^1 \otimes_{W(k)} W(k_1), \varphi_1^1, G_{W(k_1)}, (v_\alpha)_{\alpha\in\mathcal{J}}\big)$, with $\varphi^1 = g^1\varphi_0$ for some element $g^1 \in G(W(k_1))$ (this is just a particular case of the principle of 2.2.20 5); here σ_1 is the Frobenius automorphism of $W(k_1)$). So we can speak about its Shimura filtered Lie σ_1-crystal defined by $\mathrm{Lie}(G_{W(k_1)}) = \mathfrak{g} \otimes_{W(k)} W(k_1) \subset \mathrm{End}(M \otimes_{W(k)} W(k_1))$.

3.4.4. Taking a point $z^j : \mathrm{Spec}(W(k_j) \to \mathrm{Spec}(Q_j)$, with k_j a perfect field, such that the induced point $\mathrm{Spec}(k_j) \to \mathrm{Spec}(Q_j/pQ_j)$ sits over the generic point of $\mathrm{Spec}(Q_j/pQ_j)$, we get a Shimura filtered σ_j-crystal

$$\mathcal{C}_j := \big(M \otimes_{W(k)} W(k_j), F^1 \otimes_{W(k)} W(k_j), \varphi^j, G_{W(k_j)}\big),$$

with $\varphi^j = g^j\varphi_0$ for some element $g^j \in G(W(k_j))$; here σ_j is the Frobenius automorphism of $W(k_j)$. The Shimura filtered Lie σ_j-crystal

$$\mathrm{Lie}(\mathcal{C}_j) := \big(\mathfrak{g}_{W(k_j)}, \varphi^j, F^0(\mathfrak{g})_{W(k_j)}, F^1(\mathfrak{g})_{W(k_j)}\big)$$

has a Newton polygon which does not depend on $j \in \{0, 2\}$ (cf. 3.4.2 d)) and is below the Newton polygon of the Shimura Lie σ-crystal $(\mathfrak{g}, \varphi_0)$ (as $\mathcal{C}_0$ specializes to (M, F^1, φ_0), cf. 3.4.2 b)). Moreover $\mathcal{C}_2$ specializes to $(M, F^1, g_2\varphi_0)$, cf. 3.4.2 c), and $\mathrm{Lie}(\mathcal{C}_j)$ specializes to $(\mathfrak{g}, g_2\varphi_0)$.

3.4.5. Remarks. 1) We could have worked to achieve the connection of the Shimura σ-crystal $(M, g_2\varphi_0, G)$ to the Shimura σ-crystal (M, φ_0, G) (expected to be G-ordinary) working with just one deformation, cf. 3.4.7 below. We preferred to work with two global deformations (over Q_0 and Q_2) as we hope that this method will be useful in other contexts (we have in mind applications to the construction of integral models of Shimura varieties of special type by making use of moduli schemes of p-divisible groups, as well as applications to p-adic uniformizations of Shimura varieties), cf. also 3.4.19.

2) It can happen that h^{-1} mod p belongs to $\mathcal{B}_0(k)$, or that g_2h^{-1} mod p belongs to $\mathcal{B}_2(k)$ (see 3.4.1.4 2)). Then we can not go ahead with 3.4.2-4. So we have to proceed more carefully, cf. 3.4.7 below.

3) We denote the p-divisible group D_j as well as the Shimura p-divisible group $D(j)$ over $\mathrm{Spec}(Q_j)$ by

$$\mathcal{D}(U_j^0, \Phi_{R_j}, M, F^1, h\varphi_0, G),$$

in order to emphasize the data needed to construct them.

3.4.6. Lemma. *The set of elements $\tilde{h} \in G(W(k))$ with the property that $\tilde{h}\varphi_0 = gp_0\varphi_0 g^{-1}$, with $g \in G(W(k))$ leaving invariant F^1/pF^1, and with $p_0 \in P_0(W(k))$ (P_0 was defined in 3.2.7), maps into a Zariski dense set $\mathcal{L}$ of k-valued points of G_k.*

Proof: Let H be the Zariski closure of $\mathcal{L}$ in G_k. For a Galois extension k_1 of k, we denote by $\mathcal{L}_1 \subset G(k_1)$ and $H_1 \hookrightarrow G_{k_1}$ the analogues of $\mathcal{L}$ and H obtained by working over $W(k_1)$ instead of $W(k)$. It is an easy exercise to see that $\mathcal{L}_1$ is stable under the natural action of the Galois group $\mathrm{Gal}(k_1/k)$ on $G(k_1)$ and that $\mathcal{L}$ is the subset of $\mathcal{L}_1$ fixed by the Galois action of $\mathrm{Gal}(k_1/k)$. So if $H_1 = G_{k_1}$, from 3.4.0 and [Bo, 18.3] we deduce that $H = G$. So we can assume that $k = \bar{k}$ (in fact it is enough to assume that G is split). This allows us to replace φ_0 by φ_1 (cf. 3.2.3 and 3.3.2-3).

We use the notations of 3.5 below. It is enough to work with a subset I_0 of I corresponding to a cycle γ_0 of γ (obviously H is $Z(G_k)$-invariant and $Z(G_k) \subset H$). We want to show that H contains the subgroup of G generated by G_i, $i \in I_0$.

We have to consider the following two cases: 1) $|I_0| = 1$, and 2) $|I_0| \geq 2$. Let N_i be the abelian subgroup of G_i having $F^1(\mathfrak{g}_i)$ as its Lie algebra, and let N_{-i} be the abelian subgroup of G_i whose Lie algebra is

$$\mathrm{Lie}(N_{-i}) = \bigoplus_{\substack{\beta \in \Phi_i^+ \\ \mathfrak{g}_\beta \subset \mathrm{Lie}(N_i)}} \mathfrak{g}_{-\beta}.$$

We first treat the case 1). So $I_0 = \{1\} = \{i\}$. Working with an arbitrary element $p_0 \in P_0(i)(W(k))$, and with

$$g \in \exp\bigl(p\mathrm{Lie}(N_{-i})\bigr) \times \prod_{\substack{\alpha\in\Phi_1^-\\ \mathfrak{g}_\alpha\not\subset\mathrm{Lie}(N_{-i})\oplus\mathrm{Lie}(P_0(i))}} \mathbb{G}_{a,\alpha}(W(k)),$$

(here exp is the exponential map, $P_0(i) := G_i \cap P_0$, and $\mathbb{G}_{a,\alpha}$ is the $\mathbb{G}_a$-subgroup of G_i having $\mathfrak{g}_\alpha$ as its Lie algebra) we deduce that H contains an open subscheme of G_{ik}, and so it contains G_{ik}.

The case 2) is very much the same. We also use the notations of 3.7.2 (they depend just on the previous notations of 3.5). We choose $g = \prod_{i\in I_0} g_i$, with $g_i \in G_i(W(k))$ of the form $g_i = v_i w_i$, where $v_i \in \exp\bigl(p\mathrm{Lie}(N_{-i})\bigr)$ and

$$w_i \in \prod_{\substack{\alpha\in\Phi_i^-\\ \tilde{g}_i(\alpha)\not\subset\mathrm{Lie}(N_{-i})\oplus\mathrm{Lie}(P_0(i))}} \mathbb{G}_{a,\alpha(i)}(W(k)),$$

$\forall i \in I_0$. Here $\mathbb{G}_{a,\alpha(i)}$ is the $\mathbb{G}_a$-subgroup of G having $\tilde{g}_i(\alpha)$ as its Lie algebra, while $P_0(i) := G_i \cap P_0$. We choose $p_0 = \prod_{i\in I_0} p_0^i$, with $p_0^i \in P_0(i)(W(k))$. The equality $\tilde{h}\varphi_1 = gp_0\varphi_1 g^{-1}$ gives us an element $\tilde{h} = \prod_{i\in I_0} h_i$ of $G(W(k))$, with $h_i \in G_i(W(k))$ of the form

$$h_i := w_i p_0^i \varphi_1(w_{i-1}^{-1})\varphi_1(v_{i-1}^{-1}),$$

where for $\tilde{g} \in G(W(k))$ leaving invariant F^1/pF^1, $\varphi_1(\tilde{g}) := \varphi_1\tilde{g}\varphi_1^{-1} \in G(W(k))$. It is enough to show that the Zariski closure Z_H of the k-valued points of the form $\prod_{i\in I_0}\tilde{h}_i$ (where $\tilde{h}_i$ is the image of h_i in $G_i^{\mathrm{ad}}(k)$) in $\prod_{i\in I_0} {G_i^{\mathrm{ad}}}_k$, is $\prod_{i\in I_0} {G_i^{\mathrm{ad}}}_k$.

Let $\tilde{N}_i$ be the connected subgroup of G_i having $\varphi_1(\mathrm{Lie}(N_{-(i-1)}))$ as its Lie algebra. The role of $\varphi_1(v_{i-1}^{-1})$ is just that of an element of $\tilde{N}_i(W(k))$. Using [Bo, 14.14] we get that $h_i = w_i\varphi_1(w_{i-1}^{-1})\tilde{p}_0^i\tilde{n}_i$, with $\tilde{n}_i \in \tilde{N}_i(W(k))$, and $\tilde{p}_0^i \in P_0(i)(W(k))$. For this we need p_0^i to be a $W(k)$-valued point of an open subscheme of $P_0(i)$, containing the origin, and small enough.

The key point is: the one dimensional $B(k)$-vector spaces $\tilde{g}_i(\alpha)\otimes_{W(k)} B(k)$, $i \in I_0$, which are not included in $\mathrm{Lie}(P_1)\otimes_{W(k)} B(k) = \mathrm{Lie}(P_0)\otimes_{W(k)} B(k)$ (see 3.3.3 for this identification), are permuted by φ_1 in such a way that in any resulting cycle there is such a $B(k)$-vector space contained in $\mathrm{Lie}(N_{-i})\otimes_{W(k)} B(k)$, for some $i \in I_0$.

Writing $w_i = \tilde{w}_i u_i \varphi_1(w_{i-1})$, with $\tilde{w}_i \in \prod \mathbb{G}_{a,\alpha(i)}(W(k))$, where the product runs through all $\alpha \in \Phi_i$ such that $\tilde{g}_i(\alpha)$ is not included in $\mathrm{Lie}(P_1)\oplus\mathrm{Lie}(N_{-i})\oplus\varphi_1(p\mathrm{Lie}(N_{-(i-1)}))$, and with $u_i \in \prod \mathbb{G}_{a,\alpha(i)}(W(k))$, where the product runs through all $\alpha \in \Phi_i$ such that $\tilde{g}_i(\alpha)$ is not included in $\mathrm{Lie}(P_1)\oplus\varphi_1(p\mathrm{Lie}(N_{-(i-1)}))$ but $\tilde{g}_i(\alpha) \subset \mathrm{Lie}(N_{-i})$, we get that

$$h_i = \tilde{w}_i u_i \tilde{p}_i \tilde{n}_i.$$

The above key point implies: $\tilde{w}_i$, u_i, $\tilde{p}_i$ and $\tilde{n}_i$ mod p are allowed to run independently through the k-valued points of open subschemes of subgroups of G_{ik} whose Lie algebras put together are forming a direct sum decomposition of $\mathfrak{g}_i \otimes_{W(k)} k$, $\forall i \in I_0$. This implies (again cf. [Bo, 14.14]) that Z_H is $\prod_{i\in I_0} G_i^{\mathrm{ad}}{}_k$, ending the proof of the lemma.

3.4.6.0. Exercise. If $\tilde{h} \in G(W(k))$ is obtained as in 3.4.6, and if $g_3 \in G(W(k))$ mod p is the identity element, then $(M, g_3\tilde{h}\varphi_0)$ has τ_0 as its formal isogeny type, and $(\mathfrak{g}, g_3\tilde{h}\varphi_0)$ has the same Newton polygon as $(\mathfrak{g}, g_3\tilde{h}\varphi_0)$. (Hint: use the ideas of the proof of 3.4.6 and of b) of 4.4.1 2); by induction on $m \in \mathbb{N}$ show that we can assume that g_3 mod p^m is the identity element, and so, choosing m big enough [Ka2, 1.4.4] applies).

3.4.6.1. Remarks. 1) Another proof of 3.4.6 (and actually that $\mathcal{L} \supset U_1(k)$, with U_1 an open subscheme of G_k; warning: the above proof of 3.4.6 also gives this result) can be obtained using 3.5.11 (the whole of 3.5-6 is independent of 3.4) and 3.4.10.

2) A fast proof of 3.1.3 a) and b) together with the inclusion $A_G \subset B_G$ can be obtained by putting together 3.4.6, 3.3.5, 3.5.11 and 3.4.10. In fact 3.5.11 can be substituted by 3.4.6.0.

3.4.7. Section 3.4 started with the desire of connecting the (arbitrary) Shimura filtered σ-crystal $(M, F^1, g_2\varphi_0, G)$ with the Shimura filtered σ-crystal (M, F^1, φ_0, G) (or with another such Shimura filtered σ-crystal which is -supposed to be- a G-ordinary σ-crystal). This is achieved as follows.

Let $\mathcal{U} = \mathrm{Spec}(R_1)$ be an open affine subscheme of G, with $\mathcal{U}(W(k))$ containing the identity element of $G(W(k))$. We assume that there is an étale morphism $b_{\mathcal{U}} : \mathcal{U} \to Y$, with the origin of G factoring through the closed subscheme of $\mathcal{U}$ obtained as the inverse image through $b_{\mathcal{U}}$ of the closed subscheme of Y defined by $z_i = 0$, $i = \overline{1, d}$. If needed, we can shrink it a little bit, so that the condition ii) of the beginning of 3.4 is satisfied. Let Φ_{R_1} be the Frobenius of $\mathcal{U}^\wedge$, obtained through $b_{\mathcal{U}}$, from the Frobenius of Y which takes z_i to z_i^p, $i = \overline{1, d}$.

Let $D_1 := \mathcal{D}(\mathcal{U}, \Phi_{R_1}, M, F^1, g_2\varphi_0, G)$ be the p-divisible group over the p-adic completion $\mathrm{Spec}(Q_1)$ of an affine $\mathbb{N}$-pro-étale scheme over $\mathcal{U}$, obtained as in 3.4.1.3, cf. 1) and 3) of 3.4.5. Sometimes, not to complicate the notation, we refer to D_1 as a Shimura p-divisible group, the Shimura structure being the natural one. From 3.4.1.3 5) and 3.4.6 we deduce (as in 3.4.3-4 and 2.3.16) that:

3.4.7.1. Proposition. *There is a Shimura σ_1-crystal $\mathcal{C}_1$ over the algebraic closure of the field of fractions of Q_1/pQ_1, specializing to $(M, g_2\varphi_0, G)$ and to a Shimura $\bar{\sigma}$-crystal of the form $(M \otimes_{W(k)} W(\bar{k}), g_3\varphi_0, G_{W(\bar{k})})$, with $g_3 \in G(W(\bar{k}))$ which mod p belongs to the set $\mathcal{L}$ of 3.4.6, but defined for $\bar{k}$ (here σ_1 is the Frobenius automorphism of $W(k_1)$).*

3.4.8. The proofs of 3.4.1.2-3. We work with just one index: 0. We have the following 14 steps.

1) We recall that $\Phi_{\hat{R}_0^0}$ is induced from Φ_{R_0}. There is a unique connection $\nabla_{\hat{R}_0^0}$ on $M \otimes_{W(k)} \hat{R}_0^0$ which makes $M_{\hat{R}_0^0}$ potentially to be viewed as a p-divisible object of $\mathcal{MF}_{[0,1]}^{\nabla}(\hat{R}_0^0)$; it is integrable and nilpotent mod p. For these facts cf. [Fa2, th. 10].

2) $\nabla_{\hat{R}_0^0}$ respects the G-action, i.e. is of the form $\delta + \beta$, with

$$\beta \in \hat{R}_0^0 \otimes_{R_0} (\mathrm{Lie}(G) \otimes_{W(k)} \Omega_{R_0/W(k)})$$

(cf. [Fa2, rm. ii) of th. 10]). Here δ is the connection on $M \otimes_{W(k)} \hat{R}_0^0$ that takes any element of M into 0.

3) Let $S(M) := \{1, ..., dim(M)\}$, and let $S(G) := \{1, ..., d\}$. We choose a basis $\{e_i | i \in S(M)\}$ of M, and let $\{e_{is} | i, s \in S(M)\}$ be the canonical basis of End(M) with respect to the chosen basis of M. So $e_{is}(e_s) = e_i$. For the sake of generality, for the time being, we make the choice arbitrarily. Later on we specialize to a proper choice (see 3.4.8.5) suitable for computations. $\Omega_{R_0/W(k)}$ is a free R_0-module, having $\{dz_i | i \in S(G)\}$ as a basis.

Let $\tilde{l}_0 : \mathrm{Spec}(\tilde{Q}_0^\wedge) \to \mathrm{Spec}(R_0)$ be a formally étale morphism, with $\tilde{Q}_0$ a $W(k)$-algebra. An arbitrary connection ∇ on $M \otimes_{W(k)} \tilde{Q}_0^\wedge$ can be written in the form

$$(0) \qquad \nabla = \delta + \sum_{(i,s,l) \in S(M) \times S(M) \times S(G)} x_{isl} e_{is} \otimes dz_l,$$

with $x_{isl} \in \tilde{Q}_0^\wedge$, $\forall (i, s, l) \in S(M) \times S(M) \times S(G)$. Similarly, if $n \in \mathbb{N}$, and ∇ is a connection on $M \otimes_{W(k)} \tilde{Q}_0/p^n\tilde{Q}_0$, then we have the same expression for it, but with $x_{isl} \in \tilde{Q}_0/p^n\tilde{Q}_0$.

4) We first treat the case $n = 1$. The condition that a connection ∇ on $M \otimes_{W(k)} \tilde{Q}_0/p\tilde{Q}_0$ makes $M_{\tilde{Q}_0}/pM_{\tilde{Q}_0}$ potentially to be viewed as an object of $\mathcal{MF}^\nabla_{[0,1]}(\tilde{Q}_0)$, is expressed by the following equations

$$(1) \qquad \nabla \circ \Phi(\tilde{Q}_0)^0(m) = p\Phi(\tilde{Q}_0)^0 \circ d\Phi_{\tilde{Q}_0 *}/p \circ \nabla(m),$$

if $m \in F^0 \otimes_{W(k)} \tilde{Q}_0/p\tilde{Q}_0$, and

$$(2) \qquad \nabla \circ \Phi(\tilde{Q}_0)^1(m) = \Phi(\tilde{Q}_0)^0 \circ d\Phi_{\tilde{Q}_0 *}/p \circ \nabla(m),$$

if $m \in F^1 \otimes_{W(k)} \tilde{Q}_0/p\tilde{Q}_0$.

5) The above equations are replaced, using the chosen form (0) of ∇ (with $x_{isl} \in \tilde{Q}_0/p\tilde{Q}_0$), by equations involving x_{isl}, $(i, s, l) \in S(M) \times S(M) \times S(G)$, and the Frobenius transforms of some lifts of them to $\tilde{Q}_0$ taken mod p (i.e. and their p-powers). These equations have coefficients in R_0/pR_0 and so they define an affine scheme $\mathcal{S}^1$ of finite type over $\mathcal{S}^0 := \mathrm{Spec}(R_0/pR_0)$. $\mathcal{S}^1$ is a moduli scheme of connections. Let

$$\ell(0) : \mathcal{S}^1 \to \mathcal{S}^0$$

be the morphism we get.

The nice thing is that these equations involving x_{isl} take the form

$$(3) \qquad x_{isl} = L_{isl}(x_{111}^p, x_{112}^p, ..., x_{\dim(M)\dim(M)d}^p) + a_{isl},$$

where the expression L_{isl} is homogeneous and linear in each variable and has coefficients in R_0/pR_0, and where $a_{isl} \in R_0/pR_0$. We have such an equation for any triple $(i, s, l) \in S(M) \times S(M) \times S(G)$. This nice shape of the equations involving x_{isl} is obtained as follows.

Let $i \in S(M)$. There are $e_i(0) \in F^0 \otimes_{W(k)} \tilde{Q}_0/p\tilde{Q}_0$ and $e_i(1) \in F^1 \otimes_{W(k)} \tilde{Q}_0/p\tilde{Q}_0$ such that

$$e_i = \Phi(\tilde{Q}_0)^0(e_i(0)) + \Phi(\tilde{Q}_0)^1(e_i(1))$$

(we recall: $M_{\tilde{Q}_0}$ is a p-divisible object of $\mathcal{MF}_{[0,1]}(\tilde{Q}_0)$). Summing up the equations obtained by plugging $e_i(0)$ and $e_i(1)$ in (1) and respectively in (2) of 4) above, we get the mentioned shape for the equations involving x_{isl}, $(s,l) \in S(M) \times S(G)$, in the left hand side of (3). By doing this for any $i \in S(M)$, we obtain "all" the equations in the variables x_{isl} (in the sense that any other equation is a linear combination of these ones).

6) The existence of another connection on $M \otimes_{W(k)} \tilde{Q}_0/p\tilde{Q}_0$ making $M_{\tilde{Q}_0}/pM_{\tilde{Q}_0}$ potentially to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(\tilde{Q}_0)$ corresponds to a non-trivial solution (with values in $\tilde{Q}_0/p\tilde{Q}_0$) of the system of equations

$$x_{isl} = L_{isl}(x^p_{111}, x^p_{112}, ..., x^p_{\dim(M)\dim(M)d}), \tag{4}$$

$(i,s,l) \in S(M) \times S(M) \times S(G)$. Let $\mathcal{I}$ be the ideal of R_0/pR_0 generated by $t_i(0)$, $i \in \{1,...,d\}$. The coefficients of the linear forms L_{isl} are elements of $\mathcal{I}^{p-1}$. This is so due to the fact that $\Phi_{R_0}(t_i(0)) = t_i(0)^p$.

To prove 3.4.1.2 for $n=1$, we can assume that $\tilde{Q}_0$ is a complete ring with the maximal ideal generated by $\mathcal{I}$ and residue field $\bar{k}$. By induction on $m \in \mathbb{N}$, we get that any solution of the above system (4) of equations, belongs to $\mathcal{I}^{p-1+pm}\tilde{Q}_0/p\tilde{Q}_0$. So we get that any such solution must be the trivial one, given by $x_{isl} = 0$, $\forall(i,s,l) \in S(M) \times S(M) \times S(G)$. This proves 3.4.1.2, for $n=1$.

7) From the shape of the equations of (3) we deduce easily that $\Omega_{\mathcal{S}^1/\mathcal{S}^0} = 0$ and that the criterion of formal smoothness is satisfied for $\ell(0)$. This implies that $\mathcal{S}^1$ is étale over $\mathcal{S}^0$. Let $\mathcal{S}^1_0$ be the maximal open closed subscheme of $\mathcal{S}^1$ with the property that any connected component of it has a non-empty intersection with the closed subscheme $\ell(0)^{-1}(\mathcal{Z}_k)$ of $\mathcal{S}^1$. From 6) above we get that $\ell(0)^{-1}(\mathcal{Z}_k)$ is either empty or $\mathrm{Spec}(k)$. So $\mathcal{S}^1_0$ is connected; below we show that it is non-empty. Lifting $\mathcal{S}^1_0$ to an affine smooth $W(k)$-scheme (this is standard), we get a formally étale morphism $\ell_{0,1} : \mathrm{Spec}(Q^{\wedge}_{0,1}) \to \mathrm{Spec}(R^{\wedge}_0)$, with $Q_{0,1}$ a smooth $W(k)$-algebra such that $\mathrm{Spec}(Q_{0,1}/pQ_{0,1}) = \mathcal{S}^1_0$. From its construction and from 6) above we deduce that ii) and iii) of 1) of 3.4.1.3 are satisfied for $n=1$ (the fact that the connection ∇^1_0 on $M \otimes_{W(k)} Q_{0,1}/pQ_{0,1}$ we get is integrable, nilpotent mod p and respects the G-action results from the fact that there is a morphism $\mathrm{Spec}(\hat{R}^0_0/p\hat{R}^0_0) \to \mathcal{S}^1$ factoring through $\mathcal{S}^1_0$, cf. 1) above). Next we show that $\mathcal{S}^1_0$ is non-empty.

8) M_{R_0} is the pull back of M_{S_0} under the natural inclusion $i^0_0 : U^0_0 \hookrightarrow W^0_0$ respecting the Frobenius lifts. So to prove that $\mathcal{S}^1_0$ is non-empty, we can work with M_{S_0} instead of M_{R_0}, i.e. we can assume that $G = GL(M)$ and that $R_0 = S_0$. So $d = \dim(M)^2$. But in this case in 5) above we have $d^2 = \dim(M)^4$ equations in d^2 variables; so the fact that S^1_0 is non-empty is handled by the simple lemma 3.4.8.1 below. We return to the proofs of 3.4.1.2-3 in 3.4.8.2, after a short intermezzo.

3.4.8.1. Lemma. *Let $m, l \in \mathbb{N}$, and let k_1 be a perfect field of characteristic $p > 0$. We consider an affine scheme $\mathcal{Y}_0$ over k_1 defined by a system of m equations in $m+s$*

variables $x_1, ..., x_{m+s}$, $s \in \mathbb{N} \cup \{0\}$, *of the form*

$$x_i = L_i(x_1, ..., x_{m+s})^{p^l} + c(i), \tag{5}$$

$i = \overline{1, m}$, *where* $\forall i \in \{1, ..., m\}$, L_i *is a homogeneous polynomial of degree one with coefficients in* k_1, *and* $c(i) \in k_1$. *Then* $\mathcal{Y}_0$ *is a non-empty smooth scheme over* k_1.

If $s = 0$ *then* $\mathcal{Y}_0$ *is étale over* k_1 *and the number of elements of the set* $\mathcal{Y}_0(\overline{k_1})$ *is finite and equal to* p^{lm_1}, *with* $m_1 \in \mathbb{N} \cup \{0\}$, $m_1 \leq m$. *Moreover* m_1 *depends only on the coefficients of the homogeneous linear forms* L_i, $i \in \{1, ..., m\}$, *and not on the free coefficients* $c(i)$, $i \in \{1, ..., m\}$. *We have:* m_1 *is less or equal to the rank* r_L *of the* $m \times m$ *matrix obtained using the coefficients of* L_i, $i \in \{1, ..., m\}$.

Proof: It is trivial to check the criterion of formal smoothness for the morphism $\mathcal{Y}_0 \to \mathrm{Spec}(k_1)$. As $\mathcal{Y}_0$ is of finite type over k_1 we deduce that $\mathcal{Y}_0$ is smooth over k_1. For the other things we can assume that $s = 0$; this implies that $\mathcal{Y}_0$ is étale over k_1, as the sheaf of relative differentials $\Omega_{\mathcal{Y}_0/k_1}$ is zero. We can assume that k_1 is algebraically closed. The case $m = 1$ is obvious. We can introduce a new variable x_0, so that the equations become homogeneous:

$$x_i x_0^{p^l - 1} = L_i(x_1, ..., x_m)^{p^l} + c(i) x_0^{p^l}. \tag{6}$$

These equations are defining a closed subscheme $\mathcal{Y}$ of $\mathbb{P}^m_{k_1}$. By repeatedly using the projective dimension theorem (cf. [Ha, p. 48]), we deduce that it is non-empty. If it has no point in the hypersurface defined by $x_0 = 0$, then we are done: $\mathcal{Y}_0 = \mathcal{Y}$ and $|\mathcal{Y}_0(k_1)| = p^{ml}$, cf. the intersection theory. If $\mathcal{Y}$ has a point with $x_0 = 0$, then we get that the homogeneous polynomials L_i of degree 1, $i = \overline{1, m}$, are linearly dependent, i.e. there are constants $a(i) \in k_1$, not all zero, such that $\sum_{i=1}^m a(i) L_i = 0$. So $\sum_{i=1}^m a(i)^{p^l} L_i^{p^l} = 0$. We can assume that $a(m)$ is different from zero, and so we can assume that it is 1.

So the initial system of (non-homogeneous) equations is equivalent to the system of equations, where we preserve the first $m - 1$ equations, while we replace the last equation (obtained for $i = m$) by the linear equation

$$x_m + \sum_{i=1}^{m-1} a(i)^{p^l} x_i = c(m) + \sum_{i=1}^{m-1} c(i) a(i)^{p^l}.$$

This last equation allows us to eliminate x_m. We come across a system of $m - 1$ equations in $m - 1$ variables of the same type as the original one. Moreover the coefficients of the new homogeneous linear forms in the variables x_1, x_2,..., x_{m-1} depend on L_i, $i \in \{1, ..., m\}$, but do not depend on $c(i)$, $i \in \{1, ..., m\}$. So we can proceed by induction. The last part about the estimate of m_1 is obvious: as above, we can eliminate $m - r_L$ variables at once. This proves the lemma.

3.4.8.1.0. More on m_1. m_1 can be smaller than r_L. In fact easy examples show that, if $r_L < m$, then m_1 can take any value of the set $\{0, ..., r_L\}$. Here, we just mention one example to illustrate this phenomenon. We take $m = 2$ and we consider the system defined by the following two equations $x_1 = x_1^{p^l} - x_2^{p^l}$ and $x_2 = x_1^{p^l} - x_2^{p^l} + 1$. Obviously $r_L = 1$. But $m_1 = 0$ as the only solution of this system is $(x_1, x_2) = (-1, 0)$.

On the other hand, if $r_L = m$ then the above system (5) for $s = 0$ is equivalent to a system of the form

$$y_i^{p^l} = \tilde{L}_i(y_1, ..., y_m), \tag{7}$$

$i = \overline{1, m}$, with $\tilde{L}_i$ as non-homogeneous linear forms; so in this case $\mathcal{Y} = \mathcal{Y}_0$ and $m_1 = m$.

3.4.8.1.1. Corollary. *Let $m \in \mathbb{N}$ and let (E, τ) be a pair comprising from a field E and an automorphism τ of it. Let $L_i(x_1, ..., x_m)$ be linear forms in m variables with coefficients in E, $i = \overline{1, m}$. We consider the difference variety defined by the system of equations*

$$y_i = L_i(\tau(y_1), ..., \tau(y_m)), \tag{8}$$

$i = \overline{1, m}$. The conclusion is: it is non-empty, i.e. it has a solution in the difference closure of (E, τ).

Proof: We can assume that (E, τ) is equal to its difference closure. The imerging philosophy of E. Hrushovski and A. Macintyre (see [Hr]) allows us to assume that $E = \mathbb{F}$ and that τ is a power of its Frobenius automorphism having $\mathbb{F}_p$ as its fixed field. But this case is handled by the above lemma.

3.4.8.1.2. Corollary. *Let $m, n \in \mathbb{N}$. Let $R(0)$ be a reduced $\mathbb{F}_p$-algebra. Let $L_i(x_1, x_2, ..., x_m)$ be homogeneous linear forms in m variables with coefficients in $R(0)$, $i = \overline{1, m}$.*

a) (the abstract surjectivity principle) *There is a Zariski dense open subscheme $U(0)$ of $\mathrm{Spec}(R(0))$, such that for any m-tuple $(c(1), ..., c(m))$ of elements of $R(1)$, with $R(1)$ a reduced $\mathbb{F}_p$-algebra containing $R(0)$, the system of equations*

$$x_i = L_i(x_1^{p^n}, x_2^{p^n}, ..., x_m^{p^n}) + c(i), \tag{9}$$

$i = \overline{1, m}$, defines an étale cover of $U(1) := \mathrm{Spec}(R(1)) \times_{\mathrm{Spec}(R(0))} U(0)$.

b) (the abstract constructibility property) *Let $R(\infty)$ be an inductive limit of $\mathbb{F}_p$-algebras $R(q)$, $q \in \mathbb{N}$, with $R(q)$ as the $R(q-1)$-algebra defined by a system of equations of the type (9) above, but with $c(i)$, $i = \overline{1, m}$, replaced by some $c(i)_q \in R(q-1)$. Then the image of any connected component of $\mathrm{Spec}(R(\infty))$ in $\mathrm{Spec}(R(0))$ is a constructible dense subset. Moreover the image of any open closed subscheme of $\mathrm{Spec}(R(\infty))$ in $\mathrm{Spec}(R(0))$ is an open subscheme.*

c) *If $R(0)$ is a strictly henselian local ring, then the additive map*

$$\mathcal{L}(n) : R(0)^m \to R(0)^m$$

defined by

$$\mathcal{L}(n)(x_1, ..., x_m) := \big(x_1 - L_1(x_1^{p^n}, ..., x_m^{p^n}), ..., x_m - L_m(x_1^{p^n}, ..., x_m^{p^n})\big)$$

is surjective.

Proof: a) As the number of the coefficients of the forms $L_1, ..., L_m$ are finite, we can assume that $R(0)$ is a finitely generated reduced $\mathbb{F}_p$-algebra. In particular $R(0)$ is a noetherian ring. We can assume that $R(0)$ is an integral domain. We already know that the system (9) of equations defines an étale scheme $\mathrm{Spec}(R(2))$ over $\mathrm{Spec}(R(1))$. So we just have to show that $U(2) := \mathrm{Spec}(R(2)) \times_{\mathrm{Spec}(R)} U(0)$ is finite over $U(1)$, with $U(0)$ a Zariski dense open affine subscheme of $R(0)$ depending only on the forms L_i, $i = \overline{1, m}$. We can assume that the p^n-th roots of the coefficients of the linear forms L_i are in $R(0)$. So we can write $L_i(x_1^{p_n}, ..., x_m^{p^n}) = \tilde{L}_i(x_1, ..., x_m)^{p^n}$, with $\tilde{L}_i$ as homogeneous linear forms. Now we can follow the proof of 3.4.8.1. We proceed by induction on $m \in \mathbb{N}$. The case $m = 1$ is obvious. If the linear forms $\tilde{L}_i$ are linearly dependent over the field of fractions $K(0)$ of $R(0)$, by localizing $R(0)$ with respect to some non-zero element of it, we can eliminate one variable; so the induction applies. If the linear forms $\tilde{L}_i$ are linearly independent over $K(0)$, then they are linearly independent, over any point of an open (non-empty) affine subscheme $U(0)$ of $R(0)$. From 3.4.8.1.0 we get: $U(2)$ is an étale cover of $U(0)$. This proves the a) part.

b) This is a consequence of a) using standard noetherian induction on the topological space underlying the spectrum of the $\mathbb{F}_p$-subalgebra of $R(0)$ generated by the coefficients of L_1,..., L_m.

c) This is a direct consequence of 3.4.8.1. This ends the proof of the corollary.

3.4.8.1.3. Definitions. The largest open subscheme $U(0)$ of $\mathrm{Spec}(R(0))$ having the property of 3.4.8.1.2 a) is called the Artin–Schreier open subscheme defined by the linear homogeneous forms L_1,..., L_m. Using the restrictions of L_1,..., L_m to the reduced subscheme defined by $\mathrm{Spec}(R(0)) \setminus U(0)$, we get similarly an Artin–Schreier open subscheme $U(1)$ of $\mathrm{Spec}(R(0)) \setminus U(0)$. In this way, by induction, we get a finite stratification of $\mathrm{Spec}(R(0))$ in locally closed subschemes. We call it the Artin–Schreier stratification of $\mathrm{Spec}(R(0))$ defined by the linear forms L_1,..., L_m.

We similarly define the refined Artin–Schreier stratification of $\mathrm{Spec}(R(0))$ defined by the linear homogeneous forms L_1,..., L_m. The only difference: instead of $U(0)$, we take its open closed subscheme $U(0)'$ such that the number of geometric points of (let's say) $\mathrm{Spec}(R(1))$ above geometric points of $U(0)'$ is maximal. So it is the stratification indexed by the number of geometric points of $\mathrm{Spec}(R(1))$ we get above geometric points of $\mathrm{Spec}(R(0))$.

3.4.8.1.4. Lemma (the purity property). *All Artin–Schreier stratifications and all refined Artin–Schreier stratifications satisfy the purity condition.*

Proof: We use the notations of 3.4.8.1.2. We can assume that $R(0)$ is a finitely generated integral $\mathbb{F}_p$-algebra. So $U(0) = U(0)'$. Let $R(0)^n$ (resp. $R(1)^n$) be the normalization of $R(0)$ in its field of fractions (resp. in the ring of fractions of $R(1)$). $\mathrm{Spec}(R(1))$ is an affine open subscheme of $\mathrm{Spec}(R(1)^n)$. So its complement is either empty or of pure codimension 1, as we have $R(1) = R(1)^n[\frac{1}{a(1)}]$, for some $a(1) \in R(1)^n$. Let $b(1) \in R(0)^n$ be the norm of $a(1)$. Let $P(x)$ be a monic polynomial with non-zero free coefficient $a(0)$ such that $P(b(1)) = 0$. Obviously $\mathrm{Spec}(R(0)) \setminus U(0)$ is the locus of the zeroes of $a(0)$.

This implies that either $U(0)$ is $\mathrm{Spec}(R(0))$, or its complement in $\mathrm{Spec}(R(0))$ is of pure codimension one and defined by a single equation. This proves the lemma.

3.4.8.1.5. Remark. All the above part of 3.4.8.1 remains valid if L_i's are not necessarily (homogeneous) linear forms (in m variables), but sums of monomials which are some p-powers of the variables involved. The reason is: we can introduce new variables and equations of the form $y_i = x_i^p$, so that the new system of equations is of the same form as the one of 3.4.8.1.2 a), with $n = 1$ and with an increased number of variables.

3.4.8.1.6. Remark. In 3.4.8.1.2 we do not need to assume that we are dealing with reduced $\mathbb{F}_p$-algebras.

We come back to the proof of 3.4.1.2-3.

3.4.8.2. Lemma (the explicit form of the surjectivity principle). *There is a reduced closed subscheme $\mathcal{B}_0(k)$ of G_k not containing the origin, depending only on the linear forms L_{isl}, but not on the coefficients a_{isl}, $(i,s,l) \in S(M) \times S(M) \times S(G)$, and such that the fibres of $\ell_{0,1}$ above points of* $\mathrm{Spec}(R_0/pR_0)$ *not belonging to $\mathcal{B}_0(k)$, are non-empty.*

Proof: All the schemes to be introduced below are reduced. First of all let us remark that there is a closed subscheme $\mathcal{R}(0)$ of $\mathrm{Spec}(R_0/pR_0)$, different from $\mathrm{Spec}(R_0/pR_0)$, and such that denoting by $\mathcal{T}(0)$ its open complement, the morphism $\ell_{0,1}^{-1}(\mathcal{T}(0)) \to \mathcal{T}(0)$ is an étale cover. By elementary matrix operations we see that $\mathcal{R}(0)$ depends only on the linear forms L_{isl}, and not on the coefficients a_{isl} (cf. 3.4.8.1.2 a)). Now we are going to construct $\mathcal{B}_0(k)$.

In what follows $\mathcal{J}(\tilde{i})$ are finite set of indices, $\tilde{i} = \overline{0, d-1}$. All the connected components of $\mathcal{R}(0)$ not passing through the origin of G_k should be subschemes $S_l(0)$, $l \in \mathcal{J}(0)$, of $\mathcal{B}_0(k)$. For any connected component of $\mathcal{R}(0)$ passing through the origin of G_k, we have to repeat the argument. Let $\mathcal{C}^1$ be the union of such connected components. There is (cf. 3.4.8.1.2 a)) a Zariski dense open subscheme $\mathcal{T}(1)$ of $\mathcal{C}^1$, again depending only on the restriction of the forms L_{isl} to $\mathcal{C}^1$, and not on the restriction of the coefficients a_{isl} to $\mathcal{C}^1$, $(i,s,l) \in S(M) \times S(M) \times S(G)$, such that the morphism $\ell_{0,1}^{-1}(\mathcal{T}(1)) \to \mathcal{T}(1)$ is an étale cover. Let $\mathcal{R}(1)$ be the subscheme of $\mathcal{C}^1$ defined by the complement of $\mathcal{T}(1)$ in $\mathcal{C}^1$. The connected components of $\mathcal{R}(1)$ not passing through the origin of G_k, should be subschemes $S_l(1)$, $l \in \mathcal{J}(1)$, of $\mathcal{B}_0(k)$. We keep going with those connected components of $\mathcal{R}(1)$ passing through the origin of G_k.

By induction on $\tilde{i} \in \{0, ..., d-1\}$ we build up closed schemes $S_l(\tilde{i})$, $l \in \mathcal{J}(\tilde{i})$, not containing the origin of G_k, and which are connected components of a closed subscheme $\mathcal{R}(\tilde{i})$ of $\mathrm{Spec}(R_0/pR_0)$. $\mathcal{B}_0(k)$ is defined as the Zariski closure in G_k of the (finite) union of these subschemes $S_l(\tilde{i})$, $l \in \mathcal{J}(\tilde{i})$, with $\tilde{i}$ running from 0 to $d-1$. It does not contain the origin of G_k.

From its construction we deduce that it does not depend on the coefficients a_{isl}, $(i,s,l) \in S(M) \times S(M) \times S(G)$, and that for any perfect field k_1 containing k we have $\mathcal{B}_0(k_1) = \mathcal{B}_0(k)_{k_1}$. This ends the proof of the lemma.

9) From the fact that the coefficients of L_{isl} are in $\mathcal{I}$ we deduce that the closed subscheme $\ell_{0,1}^{-1}(\mathcal{Z}_k)$ of $\mathcal{S}_0^1$ is $\mathrm{Spec}(k)$. This implies iv) of 3.4.1.3 1). The part of 3.4.1.3 2)

referring to $\ell_{0,1}$ is a consequence of the proof of 3.4.8.1 (cf. also 13) below). This ends the proof of 3.4.1.3 1) and 2) involving $n = 1$.

10) By induction on $n \in \mathbb{N}$ we prove 3.4.1.3 1). Let $n \in \mathbb{N}$. We assume that we managed to build up $Q_{0,m}$, for any $m \in \mathbb{N}$ smaller than $n+1$. We now construct $Q_{0,n+1}$.

11) First we arbitrarily lift ∇_0^n to a connection respecting the G-action, still denoted by ∇_0^n, on $M \otimes_{W(k)} Q_{0,n}/p^{n+1}Q_{0,n}$. Let $\tilde{\ell}_0 : \mathrm{Spec}(\tilde{Q}_0^\wedge) \to \mathrm{Spec}(Q_{0,n}^\wedge)$ be a formally étale morphism, with $\tilde{Q}_0$ a $W(k)$-algebra. The general form of a connection on $M \otimes_{W(k)} \tilde{Q}_0/p^{n+1}\tilde{Q}_0$ lifting the pull back of ∇_0^n, is of the form

$$\nabla = \nabla_0^n + \sum_{(i,s,l)\in S(M)\times S(M)\times S(G)} p^n x_{isl} e_{is} \otimes dz_l, \tag{10}$$

with all $x_{isl} \in \tilde{Q}_0/p\tilde{Q}_0$.

12) The condition that this connection ∇ on $M \otimes_{W(k)} \tilde{Q}_0/p^{n+1}\tilde{Q}_0$ makes $M_{\tilde{Q}_0}/p^{n+1}M_{\tilde{Q}_0}$ potentially to be viewed as an object of $\mathcal{MF}_{[0,1]}^{\nabla}(\tilde{Q}_0)$ is expressed by a system of equations involving x_{isl}, which is of the same type as the system (3) of 5) above. We get the same linear forms (!) L_{isl}, $(i,s,l) \in S(M) \times S(M) \times S(G)$. The only difference is that the free coefficients a_{isl} of 5) above are replaced by new coefficients $a_{isl}(n)$. The fastest way to see this, is to use the same trick as in 8) above (for $G = GL(M)$ this is obvious).

As the coefficients a_{isl} have played no role in the steps 5-8) above, we can repeat the arguments of 5-8) above. So we get a moduli scheme $\mathcal{S}^{n+1}$ of connections, which is an étale scheme over $\mathcal{S}_0^n := \mathrm{Spec}(Q_{0,n}/pQ_{0,n})$. We define similarly $\mathcal{S}_0^{n+1}$. This takes care of 3.4.1.2 and of 1) of 3.4.1.3 for $n+1$ (again iv) is implied by i), while 1) above implies that ∇_0^{n+1} is integrable, nilpotent mod p, and respects the G-action).

13) 3.4.1.3 4) results from 3.4.1.3 1). 3.4.1.3 3) is obvious. The fact that the morphism ℓ_j^n mod p is:

– étale, results from the same argument used in 7) above (cf. also 3.4.8.1);

– of finite type results from the construction of $\mathrm{Spec}(Q_{0,n+1}/pQ_{0,n+1})$ (it is a closed subscheme of a scheme of finite type over $\mathcal{S}_0^n$);

– quasi-finite, with geometric fibres of cardinality at most p^{d^2}, results from the fact that ∇_0^{n+1} respects the G-action, and so we actually have only d^2 variables, and from the intersection theory (cf. the proof of 3.4.8.1).

This takes care of 3.4.1.3 2).

14) 3.4.1.3 5) is a consequence of the proof of 3.4.8.2 and of 12) above (cf. 3.4.8.1.2 a)). More precisely, for any $n \in \mathbb{N}$ we construct an open subscheme $\mathcal{S}_1^n$ of $\mathcal{S}_0^n$, such that, under the morphism ℓ_j^n, $\mathcal{S}_1^{n+1}$ projects surjectively onto $\mathcal{S}_1^n$, as well as $\mathcal{S}_1^1$, under the morphism $\ell_{0,1}$, projects surjectively onto the open subscheme of $\mathrm{Spec}(R_0/pR_0)$ defined by the complement of $\mathcal{B}_0(k)$.

$\mathcal{S}_1^n$ is formed by removing from $\mathcal{S}_0^n$ the closed subscheme formed by the union of all connected components of $\ell_{0,n}^{-1}(\mathcal{C}_{\tilde{i}}^1)$ not containing the closed subscheme $\ell_{0,n}^{-1}(\mathcal{Z}_k) = \mathrm{Spec}(k)$, for $\mathcal{C}^1_{\tilde{i}}$ running through all connected components not containing $\mathcal{Z}_k$ of the scheme $\mathcal{R}(\tilde{i})$ defined in 3.4.8.2, with $\tilde{i}$ as its turn running through the set $\{0, ..., d-1\}$.

This ends the proof of 3.4.1.2-3.

3.4.8.3. A digression. A natural question arises: how to compute the connections ∇^n_j, and so how to construct the morphisms ℓ^n_j (write down the equations defining $Q_{j,n}$)? This is not difficult (cf. 3.4.8 5); see also 3.4.8.5 and 3.4.18.4). Here, as a digression, we show how using Shimura varieties of A_n type ($n \in \mathbb{N}$) we can construct a formally étale morphism $\tilde{Q}^{\wedge}_{j,1} \to \mathrm{Spec}(R^{\wedge}_j)$ which factors through $\ell_{j,1}$. We can assume that k is the algebraic closure of a countably generated field over $\mathbb{F}_p$; so in particular we have $k = \bar{k}$. We are going to work again with only one index: 0.

1) M_{R_0} is the pull back of M_{S_0} under the inclusion $i^0_0 : U^0_0 \hookrightarrow W^0_0$. So we can work with M_{S_0} instead of M_{S_0}, i.e. we can assume from now on that $G = GL(M)$ and that $R_0 = S_0$.

2) Let us first remark that there is a SHS $(f, L_{(p)}, v)$ defined by an injective map $f : (G_0, X) \hookrightarrow \big(\mathrm{GSp}(W, \psi), S\big)$ (here we put exceptionally a lower left index 0, i.e. we write G_0 and $2e_0 = \dim_{\mathbb{Q}}(W)$, not to create confusion with the meaning of $G = GL(M)$ and $e = \dim(G) = \dim(GL(M)))$ such that $G_{0W(k)} \supset G = GL(M)$ has the property that $G^{\mathrm{der}}_{0\ W(k)} = SL(M)$ (the confusion between G and $G_{0W(k)}$ would have not been too serious) and the Shimura Lie pairs attached to the Shimura group pairs $(G, [\mu])$ and $(G_1, [\tilde{\mu}^*_x])$, are isomorphic (cf. def. 3-4) of 2.2.5). Here G_1 is the subgroup of $G_{0W(k)}$ generated by $G^{\mathrm{der}}_{0W(k)}$ and by the image of any cocharacter $\tilde{\mu}^*_x : \mathbb{G}_m \to G_{0W(k)}$, which, under an embedding of $W(k)$ in $\mathbb{C}$, extending the composite inclusion $O_{(v)} \subset E(G_0, X) \subset \mathbb{C}$, is $G_0(\mathbb{C})$-conjugate to the cocharacters μ^*_x, $x \in X$.

We need to have $G^{\mathrm{der}}_{0\mathbb{R}} = SU(p_0, q_0)_{\mathbb{R}}$, with $p_0 = \dim(F^1)$ and $q_0 = \dim(F^0) = \dim(M/F^1)$. We can take G^{der}_0 to be an arbitrary $\mathbb{Q}$–form of $G^{\mathrm{der}}_{0\mathbb{R}}$ which splits over $\mathbb{Q}_p$ (to be compared with 4.6.4). [Va2, 6.4.2] guarantees the existence of a SHS as above if $p \geq 5$; the existence of such a SHS for $p = 3$ is not a problem: we can assume that we have a standard PEL situation $(f, L_{(p)}, v, \mathcal{B})$, cf. 2.3.5.1 (see also §6). For example, if $p_0 = 1$ and $q_0 = 2$, we can take a Picard surface (G_0, X) (cf. [Go]) having a reflex field $E(G_0, X)$ in which p is totally split. The general case is just an obvious extension of the case $p_0 = 1$, $q_0 = 2$ (we have $p_0 q_0 \neq 0$ due to 3.2.2.1). In what follows we assume that $(f, L_{(p)}, v)$ is obtained as in [Va2, 6.6.5], or, in case p does not divide $\dim(M)$, as in [Va2, 6.5.1.1].

3) Secondly, we assume the existence of a $W(k)$-point z of $\mathcal{N}/H_0$ (cf. the standard notations 2.3.1-3 for the SHS introduced in 2) above), such that its Shimura filtered σ-crystal $\big(L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(k), F^1_z, \varphi_z, G_{0W(k)}\big)$ (cf. 2.3.10 and the assumption $k = \bar{k}$) has the property that the induced Shimura Lie filtered σ-crystal $\big(\mathfrak{g}_1, \varphi_z, F^0_z(\mathfrak{g}), F^1_z(\mathfrak{g})\big)$, with $\mathfrak{g}_1 := \mathrm{Lie}(G_1) \subset \mathrm{Lie}(G_{0W(k)})$, is isomorphic to the Shimura Lie filtered σ-crystal $(\mathrm{End}(M), h\varphi_0, F^0(\mathrm{End}(M)), F^1(\mathrm{End}(M)))$.

4) It is easy to see that we have $G_1 = \mathbb{G}_m \times SL(M)$. So we have a natural isogeny $G_1 \to G = GL(M)$; it is not an étale morphism if p divides $\dim(M)$. Substituting M with $\tilde{M} = M \oplus W(k)$, and the direct sum decomposition $M = F^1 \oplus F^0$ with $\tilde{M} = F^1 \oplus \tilde{F}^0$,

where $\tilde{F}^0 = F^0 \oplus W(k)$, we replace the Shimura σ-crystal $(M, \varphi, GL(M))$ with the Shimura σ-crystal $(\tilde{M}, \tilde{\varphi}, GL(\tilde{M}))$ (with $\tilde{\varphi}$ acting as φ on M, and as the Frobenius σ on the $W(k)$ summand of $\tilde{M}$). This is achieved in the following 4 items.

a) We choose arbitrarily a $W(k)$-basis $\{v_1, ..., v_{\dim(M)}\}$ of M; it gives birth to a $W(k)$-basis $\{v_1,...,v_{\dim(M)+1}\}$ of $\tilde{M}$, where $v_{\dim(M)+1}$ is 1 of the summand $W(k)$ of $\tilde{M}$.

b) We view $GL(M)$ as a subgroup of $GL(\tilde{M})$ (with $b_M \in GL(M)(W(k))$ acting as b_M on the summand M of $\tilde{M}$, and as identity on the other summand $W(k)$ of $\tilde{M}$).

c) We choose $\tilde{S}_0$ to be an open affine subscheme of $GL(\tilde{M})$ of the form $S_0 \times \mathbb{G}_a^{2\dim(M)} \times \mathbb{G}_m$ (the factors $\mathbb{G}_a$ correspond to elements of $GL(\tilde{M})(W(k))$, viewed as matrices, having 1 on the diagonal, and 0 anywhere else, except at the intersection of an i_1-th row with an i_2-th column, where i_1 and i_2 are distinct elements of the set $S_M = \{1, 2, ..., \dim(M) + 1\}$ such that the intersection $\{i_1, i_2\} \cap \{\dim(M) + 1\}$ is non-empty; the factor $\mathbb{G}_m$ is the subgroup of $GL(\tilde{M})$ leaving invariant the summand $W(k)$ of $\tilde{M}$ and fixing M).

d) We choose a Frobenius lift of the p-adic completion of $\tilde{S}_0$ of the form $\Phi_{S_0} \times \Phi_1$ (corresponding to the above product decomposition of $\tilde{S}_0$), with Φ_1 as the Frobenius lift of the p-adic completion of $\mathbb{G}_a^{2\dim(M)} \times \mathbb{G}_m = \mathrm{Spec}(W(k)[x_1, ..., x_{2\dim(M)+1}][\frac{1}{x_{2\dim(M)+1}}])$ which takes x_s to x_s^p, $\forall s \in \{1, ..., 2\dim(M)\}$, and $x_{2\dim(M)+1} - 1$ to $(x_{2\dim(M)+1} - 1)^p$, and which acts as σ on $W(k)$.

5) The same argument which allowed us to assume that G is $GL(M)$, allows us to replace $GL(M)$ with $GL(\tilde{M})$, if p divides $\dim(M)$. So we can assume that p does not divide $\dim(M)$. So $G_1 \to G$ is an étale cover and $\mathfrak{g}_1 = \mathfrak{g}$. This allows us to identify the completion of G_1 in the origin with the completion $\mathrm{Spec}(O_0)$ of G in the origin.

6) From the assumed existence of the point $z : \mathrm{Spec}(W(k)) \to \mathcal{N}/H_0$, we deduce that $\mathfrak{g}_{R_0} = \mathrm{End}(M)_{S_0}$ is (p-adically) algebraizable. To explain what we mean by this, we start with a formally étale morphism $\mathrm{Spec}(O_0) \to \mathcal{N}/H_0$ lifting z and such that the pull back through it of the abelian scheme $\mathcal{A}_{H_0}$ over $\mathcal{N}/H_0$ gives birth to a Shimura filtered F-crystal

$$\mathfrak{C} := \big(L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(k), F^1_z, \varphi_z, G_1, \bar{f}\big)$$

over O_0 (with $\bar{f}$ induced by the morphism c_0 of ii) before 3.4.0). Using Artin's approximation theorem (cf. 2.3.15), we get:

7) a smooth morphism $\mathrm{Spec}(S_1) \xrightarrow{\ell_1} \mathcal{N}/H_0$ and a formally étale morphism $\mathrm{Spec}(S_1^\wedge) \xrightarrow{\ell} W_1^0 := \mathrm{Spec}(S_0) \times_G G_1$ such that the p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(S_1)$ defined by the pull back of $\mathcal{A}_{H_0}$ through ℓ_1 can be put in the form

$$\mathfrak{C}_{\ell_1} = \big(L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} S_1^\wedge, F^1_z \otimes_{W(k)} S_1^\wedge, h_1(\varphi_z \otimes 1), \nabla_z\big),$$

with $h_1 \in G_{0W(k)}(S_1^\wedge)$ which mod p is defined by ℓ (we view $\mathrm{Spec}(S_0) \times_G G_1$ as an open subscheme of G_1, and so as a locally closed subscheme of $G_{0W(k)}$), and with Φ_{S_1} as the

Frobenius of $S_1^\wedge$ for which the diagram

$$\begin{array}{ccc} \mathrm{Spec}(S_1^\wedge) & \xrightarrow{pr_1} & \mathrm{Spec}(S_0^\wedge) \\ \big\downarrow \Phi_{S_1} & & \big\downarrow \Phi_{S_0} \\ \mathrm{Spec}(S_1^\wedge) & \xrightarrow{pr_1} & \mathrm{Spec}(S_0^\wedge) \end{array}$$

is cartesian (the fact that we chose Φ_{S_1} to be obtained via ℓ, means that we know that h_1 only mod p is defined by ℓ); here $pr_1 : \mathrm{Spec}(S_1^\wedge) \to \mathrm{Spec}(S_0^\wedge)$ is the p-adic completion of the composite of ℓ with the projection $W_1^0 \to \mathrm{Spec}(S_0)$;

8) a point $\mathrm{Spec}(W(k)) \overset{d_0}{\hookrightarrow} \mathrm{Spec}(S_1)$ such that $l_1 \circ d_0 = z$, and $\ell \circ d_0^\wedge$ is the p-adic completion of the origin of $W_1^0 \subset G_1$;

9) modulo a power $I_{d_0}^n (n \in \mathbb{N}\backslash\{1\})$ of the ideal I_{d_0} defining d_0, $\mathfrak{C}_{\ell_1}$ is isomorphic to $\mathfrak{C}$ modulo the n-th power of the ideal of O_0 defining the origin of $\mathrm{Spec}(O_0)$.

10) We deduce that the p-divisible object of $\mathcal{MF}^\nabla_{[-1,1]}(S_1)$ defined by the Lie part of $\mathfrak{C}_{\ell_1}$ corresponding to $\mathfrak{g}$ is $\mathfrak{g}_{S_1} := \big(\mathfrak{g}\otimes_{W(k)} S_1^\wedge, h_1(\varphi_z \otimes 1), F^0(\mathfrak{g})\otimes_{W(k)} S_1^\wedge, F^1(\mathfrak{g})\otimes_{W(k)} S_1^\wedge, \nabla_z\big)$.

11) So the object $\mathrm{End}(M_{S_0})/p\mathrm{End}(M_{S_0}) = \mathfrak{g}_{S_0}/p\mathfrak{g}_{S_0}$ of $\mathcal{MF}_{[-1,1]}(S_0)$ becomes by extension to $\mathrm{Spec}(S_1)$, through the morphism ℓ, an object of $\mathcal{MF}^\nabla_{[-1,1]}(S_1)$. This easily implies the existence of an étale morphism $\mathrm{Spec}(S_1/pS_1) \to \mathcal{S}_0^1$ which composed with $\ell_{0,1}$ is the morphism $\mathrm{Spec}(S_1/pS_1) \to \mathrm{Spec}(S_0/pS_0)$ naturally induced by $pr_1 \circ \ell$.

In other words, we can regain the connection of M_{S_1}/pM_{S_1} from ∇_z and from the one of $\mathrm{End}(M_{S_1})/p\mathrm{End}(M_{S_1})$ and from ∇_z. In fact, from the proof of [Va2, 6.4.2] (as p does not divide $\dim(M)$ we can refer to [Va2, 6.5.1.1]), as $G_{0\mathbb{Z}_p}$ is a split group, we get that we can assume that there is an injective map $i_z : (M, F^1, h\varphi_0, G) \hookrightarrow (L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(k), F_z^1, \varphi_z, G_{0W(k)})$, with $i_z(M)$ as a direct summand of $L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(k)$ left invariant by $G_{0W(k)}$, and with $i_z(F^1) = i_z(M) \cap F_z^1$.

12) If there is no $W(k)$-point z of $\mathcal{N}/H_0$ for which 6) above takes place, we have to choose a $W(k)$-point z_0 of $\mathcal{N}/H_0$ such that its attached Shimura filtered σ-crystal $\big(L_p^* \otimes_{\mathbb{Z}_p} W(k), F_{z_0}^1, \varphi_{z_0}, G_{0W(k)}\big)$ has the property that its corresponding Shimura Lie σ-crystal $\big(\mathfrak{g}, \varphi_{z_0}, F_{z_0}^0(\mathfrak{g}), F_{z_0}^1(\mathfrak{g})\big)$ is isomorphic to a Shimura Lie filtered σ-crystal

$$\big(\mathrm{End}(M), g_3\varphi_0, F^0(\mathrm{End}(M)), F^1(\mathrm{End}(M))\big),$$

with $g_3 \in GL(M)(W(k))$ a $W(k)$-valued point of a "small" open subscheme U_3 of $U_0 \cap U_2$ containing h (cf. 2.3.16). Let U_3^0 be the right translate of U_3 by h^{-1}. Repeating the arguments of 3-11) we conclude that the spectrum $\mathcal{Z}$ of the normalization of the S_0/pS_0-subalgebra of $\hat{R}_0^0$ (we recall that $R_0 = S_0$) generated by the coefficients of β of 3.4.8 1) taken mod p is generically étale over $\mathrm{Spec}(S_0/pS_0)$. As some trouble might result if $\mathcal{S}^1$ is not connected, this can be seen in two ways: either by remarking that the system (4) of equations of 3.4.8 6) gives birth to an étale scheme over S_0/pS_0 (so somehow we go back to the general ideas), or by using the Kodaira-Spencer map (here is the place where we need U_3 to be "small": we can assume that this map defined using the coefficients of the mentioned β still has a image of the right rank in g_3 mod p) and an argument of gluing entirely similar to the one of 3.4.2.1 d), but performed mod p. The need of working mod p is implied by the fact that we do not know a priori that the spectrum of the S_0-algebra generated by the coefficients of β, is smooth over $W(k)$ in a (suitable) $W(k)$-valued point of it lifting g_3; of course we can assume that g_3 mod p lifts to a smooth point of $\mathcal{Z}$. So the lifting property of 3.4.18.5.2 used in a relative context (see 3.4.18.7) implies that the gluing argument of 3.4.2.1 d) can be performed mod p as well. We do not detail fully this second approach here, as actually a point z as in 3) always exists: as $k = \bar{k}$, 3.4.15 A below applies, as we can assume that $\mathcal{N}$ has the completion property (cf. the proof of 4.12.12). We will just point out that it is based on the fact that we can regain an étale cover of $\mathrm{Spec}(S_0/pS_0)$, from the (coefficients with respect to a $W(k)$-basis of M of the) Frobenius of M_{S_0}/pM_{S_0}.

3.4.8.4. Remarks. 1) The special form of the Frobenius of $M \otimes_{W(k)} R_0$ played no role in the proof of 3.4.1.2-3. This allows us to use the method employed in 3.4.8 in (all) other situations (for instance in 3.4.18.4).

2) In the proof of the surjectivity principle, the special shape of U_0 played no role. So it can be used in all other situations (for instance in 3.4.11).

3) The (ring) Frobenius Φ_{R_0} did play a role in the above proof of 3.4.1.2-3. However, this role is not essential. For the case of a general type of (ring) Frobenius see 3.4.18.8.3.

4) Let $(f, L_{(p)}, v)$ be a SHS defined by an injective map $f : (G', X) \hookrightarrow (\mathrm{GSp}(W, \psi), S)$ (here exceptionally we put a right upper index $'$ for G and e, so we can make use of the other notations introduced in §3, without creating confusion). If we already know that the Shimura filtered σ-crystal $(M, F^1, h\varphi, G)$ is isomorphic to the Shimura filtered σ-crystal attached to a point $z : \mathrm{Spec}(W(k)) \to \mathcal{N}/H_0$, then 3.4.8.3 is significantly simplified: the replacement of G by $GL(M)$ is not needed. This implies that (for 3.4.8.3) we do not need the property ii) of the beginning of 3.4, as well as in 3.4.0 we do not need the "complication" with W_0, W_2, W_0^0, W_2^0, Φ_{S_0}, Φ_{S_2}, etc. Warning: 3.4.8.3 is not used anywhere in the paper.

5) We do not know if (or when) we have $\mathrm{Spec}(Q_{0,n}/pQ_{0,n}) = \mathcal{S}^n$. We do expect this equality (at least in the majority of cases).

3.4.8.5. The closed subscheme $\mathcal{B}_0(k)$. It is very much desirable to have a good description of the closed subscheme $\mathcal{B}_0(k)$ of G_k introduced in 3.4.8.2. In what follows we explain a simpler approach for determining its intersection with U_0^0.

From 3.4.8.2 we deduce that it is enough to determine $\mathcal{B}_0(\bar{k})$. So we can assume that $k = \bar{k}$. We choose the basis $\{e_i | i \in S(M)\}$ of M in such a way that:

There is a permutation $\tilde{\pi}$ of the set $S(M)$ with the property that $h\varphi_0(e_i)$ mod p is either zero or $e_{\tilde{\pi}(i)}$ mod p.

The existence of such a basis is guaranteed by [Mu, cor. p. 143], applied to the σ-linear map $M/pM \to M/pM$ defined by $h\varphi_0$ mod p, and by the standard Jordan form of a nilpotent σ-linear map (this is the same as in the case of a linear map).

1) Rewriting the equations of 3.4.8 5) skillfully, for the new basis, we come across equations of the form

$$t_l(0)^{p-1}x_{isl}^p = \tilde{L}_{isl}(x_{11l}, x_{12l}, ..., x_{\dim(M)\dim(M)l}) + \tilde{b}_{isl}, \tag{11}$$

$\forall (i,s,l) \in S(M)\times S(M)\times S(G)$ such that $h\varphi_0(e_s)$ is zero mod p, and of the form

$$\tilde{L}_{isl}(x_{11l}, ..., x_{\dim(M)\dim(M)l}) = \tilde{b}_{isl}, \tag{12}$$

$\forall (i,s,l) \in S(M)\times S(M)\times S(G)$ such that $h\varphi_0(e_s)$ is non-zero mod p. Here $\tilde{L}_{isl}$ is a homogeneous linear form, while $\tilde{b}_{isl}\in R_0/pR_0$, $\forall (i,s,l) \in S(M)\times S(M)\times S(G)$.

These equations are obtained by plugging e_i in the equations (1) and (2) of 3.4.8 4), and identifying the coefficients of $r_0\circ\tilde{l}_0(e_s)dz_l$ (we recall that $\Phi(\tilde{Q}_0)^0 = r_0\circ\tilde{l}_0(h\varphi_0\otimes 1)$).

2) Denoting $y_{isl} := t_l(0)x_{isl}$, $(i,s,l) \in S(M)\times S(M)\times S(G)$, this system over $R_0/pR_0[\frac{1}{\prod_{i=1}^d t_i(0)}]$ is equivalent to a system of equations of the form

$$y_{isl}^p = \tilde{L}_{isl}(y_{11l}, ..., y_{\dim(M)\dim(M)l}) + t_l(0)\tilde{b}_{isl}, \tag{13}$$

if $s\in S(M)$ is such that $h\varphi_0(e_s)$ mod p is zero, and of the form

$$\tilde{L}_{isl}(y_{11l}, ..., y_{\dim(M)\dim(M)l}) = t_l(0)\tilde{b}_{isl}, \tag{14}$$

if $s\in S(G)$ is such that $h\varphi_0$ mod p is non-zero.

This system is much simpler than the one in 3.4.8 5). It can be used to get a very good description of the intersection of $\mathcal{R}(0)$ (of the proof of 3.4.8.2) with $\mathrm{Spec}(R_0/pR_0[\frac{1}{\prod_{i=1}^d t_i(0)}])$.

3) Similarly, for any subset $S(t)$ of $S(G)$, restricting the system of equations of 1) above to $R_0/(p,(t_i(0))_{i\in S(t)})R_0$, we get a system of equations similar to the one in 2) above, involving y_{isl}, $(i,s,l)\in S(M)\times S(M)\times \big(S(G)\setminus S(t)\big)$. It is easier to determine the intersection of $\mathcal{B}_0(k)$ with $\mathrm{Spec}(R_0/(p,(t_i(0))_{i\in S(t)})[\frac{1}{\prod_{i\in S(G)\setminus S(t)} t_i(0)}])$ using it instead of another one, similar to the one of 3.4.8 5). This motivates why we get simplifications if we work under 3.4.1.1.2.

4) Let now $\tilde{G}$ be a smooth subgroup of G having connected fibres and such that its Lie algebra is stable under the action of the σ-linear map $h\varphi_0$. The whole of 3.4.1.3 can be redone in this context (cf. 3.4.1.6), with the group G replaced by $\tilde{G}$. We are going to use the same notations, not to complicate the story, with G being replaced everywhere by $\tilde{G}$.

We assume now that $\tilde{G}$ is an abelian group. We speak about the abelian situation. The nice thing in this case is:

Fact. $\forall (i,s,l)\in S(M)\times S(M)\times S(\tilde{G})$, *the coefficients of* $\tilde{L}_{isl}$ *are elements of* $W(k)$.

This is so due to the fact that $\tilde{G}$ is abelian, and the connections ∇_j^n we get respect the $\tilde{G}$-action. So in the abelian situation $\mathcal{T}(0)$ (of the proof of 3.4.8.2) contains

$\mathrm{Spec}(R_0/pR_0[\frac{1}{\prod_{i\in S(\tilde{G})} t_i(0)}])$. The argument can be repeated for the closed subscheme $\mathrm{Spec}(R_0/(p,(t_i(0))_{i\in S(t)})$, with $S(t)$ an arbitrary subset of $S(G)$, to get the following result:

3.4.8.7. Proposition. *Under the convenience assumption, in the abelian situation, the morphisms $\ell_{j,n}$ mod p, $n\in\mathbb{N}$, are defining étale covers of the locally closed subscheme of $\mathrm{Spec}(R_j/pR_j)$ defined by making some of the variables $t_i(j)$ to be zero, and inverting all others. In particular $\mathcal{B}_j(k)$ is empty.*

3.4.8.8. Remark. In the abelian situation $\mathcal{S}_0^n$ can be different from $\mathcal{S}^n$, i.e. $\mathcal{S}^n$ is not connected. This can be seen by examples similar to the one in 3.4.1.4 2), but working with σ-crystals of ordinary abelian varieties of dimension at least three.

3.4.8.9. The constancy property. We call the fact that when we pass in the above proof (of 3.4.1.2-3) from things mod p to things mod p^n, we get the same homogeneous forms (see (3), (4), (11), (12), (13) and (14) above, as well as 3.4.8 12)), as the constancy property. The system of equations defined by (3) and (4) is said to be of the first type. The system of equations defined by (11) and (12) is said to be of additive second type. The system of equations defined by (13) and (14) is said to be of adjusted additive second type. For extra terminology see 3.4.18.4.6.

As mentioned in 3.4.8.4, this constancy property remains true in the context of a (truncation of a) p-divisible object of $\mathcal{MF}_{[0,1]}(R)$, with R a regular formally smooth $W(k)$-algebra. Of course, we can replace anywhere $\mathcal{MF}_{[0,1]}$ by $\mathcal{MF}_{[a,a+1]}$, $a\in\mathbb{Z}$. So, the natural question arises:

Q. *What about (truncations of) p-divisible objects of $\mathcal{MF}_{[a,b]}(R)$, with $a,b\in\mathbb{Z}$, $b\geq a+2$?*

Due to the fact that the connections have to satisfy the Griffiths transversality condition, the counting argument (of equations and variables) performed before 3.4.8.1, gives us in this situation, more equations than variables. We obtain systems of equations of the following form:

$$x_i+c_i=L_i(x_1^p,...,x_m^p), \tag{15}$$

for $i\in\{1,...,m\}$, and

$$c_i=L_i(x_1^p,...,x_m^p), \tag{16}$$

for $i\in\{m+1,...,m+s\}$. Here $c_i\in R/pR$, L_i are homogeneous linear forms with coefficients in R/pR, $i\in\{1,...,m+s\}$, while $m,s\in\mathbb{N}\cup\{0\}$. If $s>0$ we say that the above system of equations is of the third type.

In general $ms>0$, and so it easy to see that there are systems of the above type which do not have any solution, or which do not define an étale scheme over $\mathrm{Spec}(R/pR)$. However the constancy property remains. This is a very useful fact: for many rings R, treating c_{m+1},..., c_{m+s} as variables, it is easy to put conditions on the coefficients of L_i, $i\in\{1,...,m+s\}$, so that the above system (viewed in $m+s$ variables) has a solution over any geometric point of R/pR (i.e. c_{m+1},..., c_{m+s} do belong to R/pR), regardless of the values of the coefficients c_i, $i \in \{1, ..., m\}$. These conditions can be interpreted in terms of a chosen Frobenius of R, and of the Frobenius (of a projective $R^\wedge$-module) used in defining the (truncation) of the p-divisible object of $\mathcal{MF}_{[a,b]}(R)$ we started with. We get nice criteria of when 3.4.8 above can be worked out for such a (truncation of a) p-divisible object, after suitable modifications (corresponding to c_{m+1},..., c_{m+s}) of the Frobenius of the projective $R^\wedge$-module involve. We hope to come back to these ideas in a future paper.

3.4.8.9.1. It is worth commenting why initially we thought that we can work out the things using (mod p) just étale morphisms instead of $\mathbb{N}$-pro-étale morphisms in 3.4.1.3. Referring to 3.4.1.3, once we obtained $Q_{j,1}$, we thought that the free coefficients $a_{isl}(n)$ of 3.4.8 12) we obtain for $n \in \mathbb{N}$, $n \geq 2$, do not depend on $n \geq 2$. But this is not the case.

However, in many cases, proceeding as follows, we can avoid the passage (mod p) to $\mathbb{N}$-pro-étale morphisms. Warning: what follows has to be interpreted up to a passage to an étale scheme over $\mathrm{Spec}(Q_{j,n})$, for some $n \in \mathbb{N}$; so $\mathrm{Spec}(Q_{j,m})$, with $m \geq n$, are replaced by the resulting pull back. We have to make a decision:

Either we work with "wild" connections (their behaviour modulo high powers of the prime p is uncontrollable) but with very nice Frobenius of the module underlying M_{R_j} (as in 3.4.1), and so later on we hope to get (sort of) a completion property (see 3.4.15 A) and a slice principle (see § 7), or we work with very nice connections and with very "wild" Frobenius of the underlying modules.

We explain this. The homogeneous linear forms L_{isl} of 3.4.8 (3) depend only on the expressions of $\Phi(R_j)^0$ and $\Phi(R_j)^1$ mod p and of the expression of Φ_{R_j} mod p^2. But the coefficients $a_{isl}(n)$ depend on the expressions of these three maps modulo higher powers of p. But, in many cases, we can modify $\Phi(Q_{j,1})^0$ and $\Phi(Q_{j,1})^1$ with something which is zero mod p (so we do not change the forms L_{isl}, and so we do not affect the étale property expressed in 3.4.1 through r_j mod p being a universal element), to make $a_{isl}(n) = 0$, $\forall(i, s, l) \in S(M) \times S(M) \times S(G)$, and $\forall n \in \mathbb{N}$, $n \geq m$, for some $m \in \mathbb{N}$: first we try to modify $\Phi(Q_{j,1})^0$ and $\Phi(Q_{j,1})^1$ modulo p^2 to get all $a_{isl}(1)$ to be zero, and then, what results, we try to modify it mod p^3 by something which is zero p^2 to get all $a_{isl}(2)$ to be zero, etc.

Why should this be possible? Answer: there are two reasons. First, $\forall(l, n) \in S(G) \times \mathbb{N}$, the endomorphism of $M \otimes_{W(k)} Q_{j,n}/pQ_{j,n}$ defined by the matrix whose coefficients are $a_{isl}(n)$, $i, s \in S(M)$, is an element of $\mathrm{Lie}(G) \otimes_{W(k)} Q_{j,n}/pQ_{j,n}$. Second, any time we have to pass from things mod p^n to things mod p^{n+1}, we lift the connection (we got mod p^n) (as in 3.4.8 11)), not blindly, but in such a way that the resulting coefficients $a_{isl}(n)$ are linear combinations (with fixed coefficients of R_j/pR_j) of p-power elements of $Q_{j,n}/pQ_{j,n}$; after this careful lift we start to modify $\Phi(Q_{j,n})^i$, $i = \overline{0, 1}$, by trying to achieve all $a_{ijl}(n)$ to be zero.

So the situation should get stabilized: we expect (at least if $k = \bar{k}$) the existence of $m \in \mathbb{N}$ such that we can take $Q_{j,n} = Q_{j,m}$ $\forall n \in \mathbb{N}$, $n \geq m$. Sometimes it is preferable to have (if possible) $a_{isl}(n) = 0$, $\forall n \in \mathbb{N}$, $n \geq m$. If such stabilization is possible, then we do not have to pass (mod p) to $\mathbb{N}$-pro-étale covers. This is so due to the fact that $Q_{j,n}/pQ_{j,n}$,

$n \geq m$, is a connected component of the étale scheme over $\mathrm{Spec}(Q_{j,m}/pQ_{j,m})$ defined by the system of equations (4) of 3.4.8. This gives a very nice expression (fully controllable) of the resulting connections ∇_j^n, but the control on $\Phi(Q_{j,n})^0$ and $\Phi(Q_{j,n})^1$ is much less. Still the situation looks better from many points of view, but we can not regain the (sort of) completion property or the slice principle. Once we gained the completion property and the slice principle, the second approach (trying to avoid the use of $\mathbb{N}$-pro-étale morphisms), looks, from many points of view, more convenient. In all the concrete cases we analyzed (for instance, this is automatically so if we are in a context involving abelian varieties, or which can be reduced to such a context: see 4.12.12) the situation gets stabilized. We will come back to this in §10.

3.4.9. Remarks. 1) We could have worked 3.4.1 with U_0 and U_2 open subschemes of G^{der}. This is the most common case we need for applications. For example, a σ-crystal $(M, tg_2\varphi_0)$, with $t \in Z(G)(W(k))$ and $g_2 \in G(W(k))$, is very much the same as the σ-crystal $(M, g_2\varphi_0)$; so we often (for instance when $k = \bar{k}$ and p does not divide the order of the center of G^{der}) can consider only elements $g_2 \in G^{\mathrm{der}}(W(k))$.

2) It can be checked easily, that the property ii) of the beginning of 3.4, is implied by the property i). This makes life easier (cf. 3.4.9.1 and 3.4.10 below), and in particular 3.4.11 below can be expressed in a simpler form, without mentioning U or b.

3) Looking back to 3.4.1-8 we see that the only places we used the fact that k is an infinite field are:

– 3.4.0, where we needed the existence of an element $h \in \big(U_0 \cap U_2\big)(k)$, which as a point of U_0 is different from the point obtained by taking the special fibre of a_0, and as a point of U_2 is different from the point obtained by taking the special fibre of g_2;

– 3.4.6, where we needed $\mathcal{L}$ to be infinite.

So if we are not bothered about gluing things ("over $W(k)$") (cf. 3.4.5), we can work with only one deformation, cf. 3.4.7. So, in this context of just one deformation, the whole of 3.4.1 makes sense and remains true even in the case when k is a finite field.

3.4.9.1. Potential-deformation sheets. Let k be a perfect field of characteristic $p > 2$. An affine locally closed subscheme U of a reductive group G over $W(k)$ (resp. an affine scheme U over $W(k)$), with $U(W(k))$ containing the identity element of $G(W(k))$ (resp. such that there is a point $y \in U(W(k))$), is said to be a potential-deformation sheet, if there is an étale morphism (resp. a formally étale morphism)

$$b_U : U \to \mathrm{Spec}(W(k)[x_1, ..., x_{d(U)}]),$$

for some $d(U) \in \mathbb{N}$, such that the identity element of $G(W(k))$ (resp. y) factors through the closed subscheme of U defined by $x_i = 0$, $i = \{1, ..., d(U)\}$. If moreover we have:

– the closed subscheme of U defined by the identity element of $G(W(k))$ (resp. defined by y) is the closed subscheme of U defined by $x_i = 0$, $i = \overline{1, d}$, and

– the closed subscheme of U_k defined by $x_i = 0$, with i running through an arbitrary subset of $\{1, ..., d(U)\}$, is connected, then we speak about a convenient potential-deformation sheet.

Very often we denote a potential-deformation sheet by a pair (U, b_U) (resp. by a triple (U, b_U, y)). For any potential-deformation sheet we denote by Φ_U the Frobenius of $U^\wedge$ which takes x_i into x_i^p, $\forall i \in \{1, ..., d(U)\}$, and is compatible with σ.

This definition of a (convenient) potential-deformation sheet is suggested by 3.4.9 2) and the convenience assumption 3.4.1.1.2.

3.4.10. Corollary. *For any $g_3 \in G(W(k))$ and for any open affine subscheme $U_3 = \mathrm{Spec}(R_3)$ of G which is a potential-deformation sheet, there is an $\mathbb{N}$-pro-étale affine morphism*

$$\ell_4 : U_4 = \mathrm{Spec}(R_4) \to U_3$$

such that:

– there is a closed subscheme of G_k not containing the origin of G_k, and having the property that the fibres of ℓ_4 above points of the special fibre of U_3 not belonging to it, are non-empty;

– the special fibre of U_4 is geometrically connected over k;

– $\ell_4^{-1}(\mathcal{Z}_k) = \mathrm{Spec}(k)$, and so the origin of U_3 can be lifted uniquely to a point $a_4 \in U_4(W(k)) = U_4^\wedge(W(k))$;

– there is a p-divisible group $\mathcal{D}(U_3, \Phi_{U_3}, M, F^1, g_3\varphi_0, G)$ over $U_4^\wedge$, whose corresponding p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(U_4)$ (the Frobenius on $U_4^\wedge$ being the one induced from a fixed Frobenius Φ_{U_3} of $U_3^\wedge$ as in 3.4.9.1), is isomorphic to

$$(M_{R_4}, \nabla_4) = \big(M \otimes_{W(k)} R_4^\wedge, F^1 \otimes_{W(k)} R_4^\wedge, h_4(g_3\varphi_0 \otimes 1), \nabla_4\big),$$

with h_4 as the universal element of G defined by $U_4^\wedge$, and with ∇_4 an integrable connection on $M \otimes_{W(k)} R_4^\wedge$ respecting the G-action;

Moreover we have the following universal property: For any formally étale morphism $\ell_5 : U_5 = \mathrm{Spec}(R_5) \to U_3$, such that the closed subscheme $\ell_5^{-1}(\mathcal{Z}_k)$ has a non-empty intersection with any connected component of the special fibre of U_5, and for which we get a p-divisible group $\mathcal{D}_5$ over $U_5^\wedge$, defining (by forgetting the connection) a p-divisible object of $\mathcal{MF}_{[0,1]}(R_5)$ isomorphic to M_{R_5}, there is a unique $U_3^\wedge$-morphism $\ell : \mathrm{Spec}(U_5^\wedge) \to \mathrm{Spec}(U_4^\wedge)$ such that $\ell_4^\wedge = \ell_5^\wedge \circ \ell$ and $\mathcal{D}_5$ is isomorphic to $\ell^*(\mathcal{D}(U_3, \Phi_{U_3}, M, F^1, g_3\varphi_0, G))$.

Here M_{R_3} is defined as in 3.4.1.1, and allows us to get M_{R_4} and M_{R_5} as in 3.4.1.1.

3.4.11. Corollary. *Let $\tilde{H}$ be a smooth subgroup of G, having a connected special fibre, and let $U_{\tilde{H}} = \mathrm{Spec}(R_5)$ be an open affine subscheme of $\tilde{H}$ containing the origin a_0 of $G(W(k))$, and having the property that there is an open subscheme U of G containing $U_{\tilde{H}}$, and such that there are étale morphisms b and c making the diagram*

$$\begin{array}{ccc} U_{\tilde{H}} & \xrightarrow{i_{\tilde{H}}} & U \\ \Big\downarrow b & & \Big\downarrow c \\ \mathrm{Spec}\big(W(k)[z_1, \ldots, z_{e_{\tilde{H}}}]\big) & \xrightarrow{a} & \mathrm{Spec}\big(W(k)[z_1, \ldots, z_d]\big) \end{array}$$

to be cartesian (with $e_{\tilde{H}} := \dim(\tilde{H})$, with $i_{\tilde{H}}$ as the natural inclusion, with a as the natural closed embedding defined by: z_i goes to z_i, for $i \in \{1, ..., e_{\tilde{H}}\}$, and to 0 otherwise), and the origin a_0 factors through the closed subscheme of $U_{\tilde{H}}$ defined by $z_i = 0$, $i = \overline{1, e_{\tilde{H}}}$. Let the Frobenius Φ_{R_5} of $U_{\tilde{H}}^{\wedge}$ (compatible with σ) be defined by: z_i goes to z_i^p, $i = \overline{1, e_{\tilde{H}}}$. Then for any $\tilde{g} \in G(W(k))$, there is a formally étale morphism

$$\ell_6 : U_6 = \mathrm{Spec}(R_6) \to U_{\tilde{H}},$$

with the special fibre of U_6 a geometrically connected k-scheme, with $\ell_6^{-1}(\mathcal{Z}_k) = \mathrm{Spec}(k)$ (so the origin of $U_{\tilde{H}}$ lifts uniquely to a morphism $a_6 : \mathrm{Spec}(W(k)) \to U_6^{\wedge}$), and such that there is a p-divisible group over $U_6^{\wedge}$ whose corresponding p-divisible object of $\mathcal{MF}_{[0,1]}^{\nabla}(R_6)$ is isomorphic to

$$(M_{R_6^{\wedge}}, \nabla_6) = \big(M \otimes_{W(k)} R_6^{\wedge}, F^1 \otimes_{W(k)} R_6^{\wedge}, h_6(\tilde{g}\varphi_0 \otimes 1), \nabla_6\big),$$

with h_6 as the universal element of $\tilde{H}$ defined by the p-adic completion of ℓ_6. ∇_6 respects the G-action (in fact, here we can replace G by the smallest smooth subgroup of G, containing $\tilde{H}$, and whose Lie algebra is left invariant by $p\tilde{g}\varphi_0$). We have a universal property similar to the one described in end of 3.4.10. Moreover, there is a closed subscheme of $\tilde{\mathcal{B}}_k$, not containing the origin of $\tilde{H}_k$, and such that the fibres of ℓ_6 above points of the special fibre of $U_{\tilde{H}}$ not belonging to it, are non-empty.

The results 3.4.10-11 are a direct consequence of 3.4.1.3 and its proof, of 3.4.9 2-3), and of 3.4.8.4 2).

3.4.11.1. Notation. We denote by $\mathcal{D}(U_{\tilde{H}}, \Phi_{R_5}, M, F^1, \tilde{g}\varphi_0, G)$ the p-divisible group over $U_6^{\wedge}$ mentioned in 3.4.11. For any $n \in \mathbb{N}$ we denote by $U_{6,n} = \mathrm{Spec}(Q_{6,n}^{\wedge})$ the schemes constructed as in 3.4.1.3, starting from the data of the sextuple $(U_{\tilde{H}}, \Phi_{R_5}, M, F^1, \tilde{g}\varphi_0, G)$. We are going to use morphisms $\ell_{6,n}$, and ℓ_6^n, $n \in \mathbb{N}$, having the same significance as in 3.4.1.3. We can take as U_6 the limit of the $\mathbb{N}$-projective system ℓ_6^n, $n \in \mathbb{N}$. Let Q_6 be the $W(k)$-algebra such that $U_6^{\wedge} = \mathrm{Spec}(Q_6)$.

3.4.12. Remark. In 3.4.11 it is enough to assume that $U_{\tilde{H}}$ is a locally closed subscheme of G containing the origin a_0 of $G(W(k))$, having a geometrically connected special fibre, and being smooth over $W(k)$, without assuming that it is the open subscheme of a smooth subgroup $\tilde{H}$ of G^{der}. In our work either $\tilde{H}$ is G^{der} (or G) or $\tilde{H}$ is an abelian subgroup of G isomorphic to $\mathbb{G}_a^{e_{\tilde{H}}}$, and in this situation we can take $U_{\tilde{H}} = \tilde{H}$. We can also work out the case when the special fibre of $U_{\tilde{H}}$ is just connected, but then U_6 is not going to have a geometrically connected special fibre: the number of (geometrically) connected components of $U_{6\bar{k}}$ is the same as the number of (geometrically) connected components of $U_{\tilde{H}} \times_k \mathrm{Spec}(\bar{k})$ (this can be deduced from 3.4.1.2).

3.4.13. Let $(f, L_{(p)}, v)$ be a SHS defined by an injective map $f : (\tilde{G}, X) \hookrightarrow \big(\mathrm{GSp}(W, \psi), S\big)$. We use the standard notation of 2.3.1-3, but with G replaced everywhere by $\tilde{G}$. Let k be a perfect field of characteristic p and let

$$z : \mathrm{Spec}(W(k)) \to \mathcal{N}_{W(k)}/H_0.$$

Let $(M, F^1, g\varphi_0, G', \tilde{\psi}_M)$ be its attached principally quasi-polarized (not necessarily quasi-split) quasi Shimura filtered σ-crystal (cf. 2.3.10). Here G' is a form (not necessarily inner) of $\tilde{G}_{W(k)}$. For simplicity of presentation we assume that $G' = \tilde{G}_{W(k)}$; so we do not have to mention any more "not necessarily quasi-split". Let G be the subgroup of G' such that under the previous identification it corresponds to $\tilde{G}^0_{W(k)}$ (see 2.3.1 for the meaning of $\tilde{G}^0_{\mathbb{Z}_{(p)}}$). Here $g \in G(W(k))$; strictly speaking φ_0 is obtained from $g\varphi_0$, as in 3.2.3. We are going to use the previous notations of this 3.4. In particular we have a direct sum decomposition $M = F^1 \oplus F^0$, and a_0 denotes the origin of G. Let U_3 be an affine open subscheme of $G = \tilde{G}^0_{W(k)}$ (containing the origin) which is a potential-deformation sheet. Let $U_3 = \mathrm{Spec}(R_3)$ and let $(M_{R_4}, \nabla_4) = \bigl(M \otimes_{W(k)} R_4^\wedge, F^1 \otimes_{W(k)} R_4^\wedge, h_4(g\varphi_0 \otimes 1), \nabla_4\bigr)$ be the p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(R_4)$ mentioned in 3.4.10 for $g_3 = g \in G(W(k))$. We use the notations of 3.4.10; strictly speaking $(M, F^1, g\varphi_0, G, \tilde{\psi}_M)$ is not a principally quasi-polarized Shimura filtered σ-crystal, but for the sake of not complicating the notations, it is "close enough" (cf. 2.2.9 1) and 1': we are in a pseudo context) to a principally quasi-polarized quasi Shimura filtered σ-crystal, so that 3.4.10 makes sense (cf. 3.4.1.6 and 3.4.11). Also, for a considerable simplification of the presentation, we assume that the subgroup H_0 of $G(\mathbb{A}_f^p)$ is small enough (with respect to z) so that the family of Hodge cycles $(\omega_\alpha)_{\alpha \in \mathcal{J}'}$ with which the abelian variety $A_z \times_{W(k)} \mathrm{Spec}(W(\bar{k}))$ over $W(\bar{k})$ is naturally endowed (through any lift $\mathrm{Spec}(W(\bar{k})) \to \mathcal{N}_{W(k)}$ of z), where $A_z := \mathrm{Spec}(W(k))\ {}_z\!\times_{\mathcal{N}_{W(k)}/H_0} \mathcal{A}_{H_0}$, is defined over $W(k)$, and so it gives birth to a family $(t_\alpha)_{\alpha \in \mathcal{J}'}$ of de Rham components of the F^0-filtration of $\mathcal{T}(M)$, and implicitely of the F^0-filtration of $\mathcal{T}(M \otimes_{W(k)} R_4)$. This assumption is not really needed (cf. 3.4.14.2 below); however, it allows us to to eliminate the word "quasi".

We have: $\nabla_4(t_\alpha) = 0$, and $h_4(g\varphi_0 \otimes 1)(t_\alpha) = t_\alpha$, $\forall \alpha \in \mathcal{J}'$. The fact that G fixes $\tilde{\psi}_M$, implies that $\tilde{\psi}_M$ defines a principal quasi-polarization of M_{R_4}. Let

$$\tilde{M}_{R_4} := (M_{R_4}, \nabla_4, \tilde{\psi}_M).$$

It is a principally quasi-polarized p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(R_4)$. Sometimes we prefer to regard it as being with cycles (they are t_α, $\alpha \in \mathcal{J}'$).

We denote by m_{nat} the natural morphism $(\mathcal{N}_{W(k)}/H_0)^\wedge \to \mathcal{N}_{W(k)}/H_0$.

SA1. Let $\widehat{G} = \mathrm{Spec}(\widehat{R_3}) = \mathrm{Spec}(\widehat{R_4})$ be the completion of G in the origin (or the completion of U_4 in a_4). From [Va2, 5.4.4] (see the proof of 2.3.15) we deduce the existence of a morphism $z_{\widehat{G}} : \widehat{G} \to \mathcal{N}_{W(k)}/H_0$ such that the completion of a_4 is mapped into z, and the two principally quasi-polarized Shimura p-divisible groups over $\widehat{G}$ obtained by pulling back the ones over $\mathcal{N}_{W(k)}/H_0$ and $\mathrm{Spec}(R_4^\wedge)$ are isomorphic, through an isomorphism lifting the identity in (the completion of) a_4. We assume that $z_{\widehat{G}}$ mod p factors through the henselization of the localization of $\mathrm{Spec}(R_4/pR_4)$ in the k-valued point defined by a_4.

3.4.14. Theorem (the deformation principle: the expanded form). *There is an integral affine scheme $V_G = \mathrm{Spec}(S_G)$, which is the p-adic completion of an $\mathbb{N}$-pro-étale scheme over a smooth scheme over $W(k)$, and there are morphisms $a : \mathrm{Spec}(W(k)) \hookrightarrow V_G$, $b : V_G \to (\mathcal{N}_{W(k)}/H_0)^\wedge$, $c : V_G \to U_4^\wedge$ such that:*

a) $a_4 = c \circ a$ *and the* p*-adic completion of* z *is* $b \circ a$*;*

b) b *is obtained by taking the* p*-adic completion of the composite of an* $\mathbb{N}$*-pro-étale morphism with a smooth morphism, and* c *is the* p*-adic completion of an* $\mathbb{N}$*-pro-étale morphism;*

c) The special fibre of V_G *is connected, and the image of* c *mod* p *contains an open dense subset of the special fibre of* $U_4^\wedge$*;*

d) If $N_{S_G} := (N, F^1_{S_G}, \Phi_N, \nabla_N, \tilde{\psi}_N)$ *is the principally quasi-polarized* p*-divisible object of* $\mathcal{MF}^\nabla_{[0,1]}(S_G)$ *defined by the principally quasi-polarized* p*-divisible group of the principally polarized abelian scheme* $V_{G\, b_0} \times_{\mathcal{N}_{W(k)}/H_0} (\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ *over* V_G*, then there is an isomorphism of* N_{S_G} *into* $\tilde{M}_{S_G}$ *(the pull back of* $\tilde{M}_{R_4}$ *through the morphism* c*) taking the de Rham component* $t_\alpha^{S_G}$ *of the Hodge cycle* $\omega_\alpha^{S_G}$ *of* $V_{G\, b_0} \times_{\mathcal{N}_{W(k)}/H_0} \mathcal{A}_{H_0}$ *into* t_α*,* $\forall \alpha \in \mathcal{J}'$ *(here* $b_0 := m_{\mathrm{nat}} \circ b$*).*

Let $\tilde{H}$ be the unipotent radical of the parabolic subgroup of $\tilde{G}^0_{W(k)}$ leaving F^0 invariant. Let $U_{\tilde{H}} = \mathrm{Spec}(R_5)$ be an open affine subscheme of $\tilde{H}$ such that the conditions of 3.4.11 are satisfied. We use the notations of 3.4.11 and 3.4.11.1, with $\tilde{g} = g$. Similarly we define a principally quasi-polarized p-divisible object (sometimes viewed with cycles) of $\mathcal{MF}^\nabla_{[0,1]}(R_6)$

$$\tilde{M}_{R_6} := (M_{R_6}, \nabla_6, \tilde{\psi}_M).$$

SA2. As above we assume that the similarly defined morphism from the completion of $U_{\tilde{H}}$ in the origin into $\mathcal{N}_{W(k)}/H_0$, modulo p, it factors through the henselization of the localization of $\mathrm{Spec}(Q_6/pQ_6)$ in its k-valued point defined by a_6.

We refer to this assumption (as well as of the one of SA1) as the starting assumption.

3.4.14.1. Theorem (the deformation principle). *There is an integral affine scheme* $V_{\tilde{H}} = \mathrm{Spec}(S_{\tilde{H}})$*, which is the* p*-adic completion of an* $\mathbb{N}$*-pro-étale scheme over a smooth scheme over* $W(k)$*, and there are morphisms* $a : \mathrm{Spec}(W(k)) \hookrightarrow V_{\tilde{H}}$*,* $b : V_{\tilde{H}} \to (\mathcal{N}_{W(k)}/H_0)^\wedge$*,* $c : V_{\tilde{H}} \to U_6^\wedge$ *such that:*

a) $a_6 = c \circ a$ *and the* p*-adic completion of* z *is* $b \circ a$*;*

b) b *and* c *are* p*-adic completions of* $\mathbb{N}$*-pro-étale morphisms;*

c) The special fibre of $V_{\tilde{H}}$ *is connected, and the image of* c *mod* p *contains an open dense subset of the special fibre of* $U_6^\wedge$*;*

d) If $N_{S_{\tilde{H}}} := (N, F^1_{S_{\tilde{H}}}, \Phi_N, \nabla_N, \tilde{\psi}_N)$ *is the principally quasi-polarized* p*-divisible object of* $\mathcal{MF}^\nabla_{[0,1]}(S_{\tilde{H}})$ *defined by the principally quasi-polarized* p*-divisible group of the principally polarized abelian scheme* $V_{\tilde{H}\, b_0} \times_{\mathcal{N}_{W(k)}/H_0} (\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ *over* $V_{\tilde{H}}$*, then there is an isomorphism of* $N_{S_{\tilde{H}}}$ *into* $\tilde{M}_{S_{\tilde{H}}}$ *(the pull back of* $\tilde{M}_{R_6^\wedge}$ *through the morphism* c*) taking the de Rham component* $t_\alpha^{S_{\tilde{H}}}$ *of the Hodge cycle* $\omega_\alpha^{S_{\tilde{H}}}$ *of* $V_{\tilde{H}\, b_0} \times_{\mathcal{N}_{W(k)}/H_0} \mathcal{A}_{H_0}$ *into* t_α*,* $\forall \alpha \in \mathcal{J}'$ *(here* $b_0 := m_{\mathrm{nat}} \circ b$*).*

Proof: It is easy to see that 3.4.14.1 can be deduced from 3.4.14 (also 3.4.14 can be deduced from 3.4.14.1). The proof of 3.4.14 is entirely similar to the proof of 3.4.14.1, just the presentation in the case of 3.4.14.1 is slightly more complicated. So we prove here only

3.4.14.1. We do not hesitate (though this makes the proof longer) to take the necessary time to explain in detail what the systems of equations of first type defined in 3.4.8.9 (can) bring new (besides 3.1.3.1) to the deformation theory of Shimura p-divisible groups. As mentioned in 1.15, we do not hesitate to refer to parts of 3.5-12.

Let $Y_1 = \mathrm{Spec}(S^1_{\tilde{H}})$ be an affine smooth scheme over $W(k)$, with a connected special fibre, and such that there are morphisms $a_1 : \mathrm{Spec}(W(k)) \to Y_1$, $b_1 : Y_1 \to \mathcal{N}_{W(k)}/H_0$, $c_1 : Y_1^{\wedge} \to U_{\tilde{H}}$ satisfying the properties:

A) $a_6 = c_1 \circ a_1$ and $z = b_1 \circ a_1$;

B) c_1 mod p and b_1 are both étale morphisms;

C) The principally polarized abelian scheme $(A_{Y_1}, p_{A_{Y_1}}) := Y_1\, {}_{b_1}\!\times_{\mathcal{N}_{W(k)}/H_0} (\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ has the family of Hodge cycles $(\omega_\alpha^{S^1_G})_{\alpha\in\mathcal{J}'}$ (with which it is naturally endowed) defined over Y_1 (and not only over an étale scheme over Y_1), and there is an isomorphism

$$d_1 : N := H^1_{dR}(A_{Y_1}/S^1_{\tilde{H}}) \tilde{\to} M \otimes_{W(k)} S^1_{\tilde{H}}$$

taking the Hodge filtration of N defined by A_{Y_1} into $F^1 \otimes_{W(k)} S^1_{\tilde{H}}$, $p_{A_{Y_1}}$ into $\tilde{\psi}_M$, and the de Rham component $t_\alpha^{S^1_{\tilde{H}}}$ of $\omega_\alpha^{S^1_{\tilde{H}}}$ into t_α, $\forall \alpha \in \mathcal{J}'$;

C') There is a Frobenius of $S^1_{\tilde{H}}$ compatible with σ and taking z_i into z_i^p, $i \in \{1, ..., e\}$, where the elements of the set $\{z_1, ..., z_e\}$ are forming a system of regular parameters of a_1 in Y_1, such that the resulting Frobenius of $M \otimes_{W(k)} S^{1\wedge}_{\tilde{H}}$ (cf. the identification d_1 of C) above) is $g_{Y_1}(g\varphi_0 \otimes 1)$, where $g_{Y_1} \in \tilde{G}^0_{W(k)}(S^{1\wedge}_{\tilde{H}})$ is defined by c_1;

D) Denoting by I_z the ideal of $S^1_{\tilde{H}}$ defining the point a_1, the principally quasi-polarized p-divisible group we get over $\mathrm{Spec}(S^1_{\tilde{H}}/I_z^2)$ together with its de Rham cycles is isomorphic to the principally quasi-polarized p-divisible group over $\mathrm{Spec}(R_5/I_0^2) = \mathrm{Spec}(Q_6/I_0^2Q_6)$ together with its cycles, where I_0 is the ideal of $U_{\tilde{H}}$ defining a_6.

The existence of such a scheme Y_1 and of morphisms a_1, b_1 and c_1 with the above properties is just 2.3.15.1.

Over

$$Y_{16} := Y_1 \times_{W(k)} \mathrm{Spec}(Q_6)$$

we have two principally quasi-polarized p-divisible groups $(\mathcal{D}_1, \mathcal{P}_{\mathcal{D}_1})$ and $(\mathcal{D}_2, \mathcal{P}_{\mathcal{D}_2})$, obtained by pulling back under the natural projections from this product into its factors, the principally quasi-polarized p-divisible group of the principally polarized abelian scheme $(A_{Y_1}, p_{A_{Y_1}})$ over Y_1, and respectively the principally quasi-polarized p-divisible group

$$\big(\mathcal{D}(U_{\tilde{H}}, \Phi_{R_5}, M, F^1, g\varphi_0, G), \tilde{\psi}_M\big)$$

over $\mathrm{Spec}(Q_6)$ (cf. 3.4.11.1). They can be viewed naturally as principally quasi-polarized Shimura p-divisible groups. For any $n \in \mathbb{N}$, let $Y_{16,n}$ be the moduli scheme over Y_{16} parametrizing isomorphisms between the two principally quasi-polarized finite flat commutative group schemes over Y_{16} defined by the kernels of the multiplication with p^n on them. It is an affine scheme of finite type over Y_{16}. We get an $\mathbb{N}$-projective system defined by the morphisms

$$t^n : Y_{16,n+1} \to Y_{16,n}.$$

Let $Y_{16,\infty}$ be its projective limit. Property D) implies: there is a morphism

$$h_Z : Z := \mathrm{Spec}(W(k)[x_1, ..., x_e]/(x_1, ..., x_e)^2) \to Y_{16,\infty}$$

inducing isomorphisms $Z \tilde{\to} \mathrm{Spec}(S^1_{\tilde{H}}/I_z^2)$ and $Z \tilde{\to} \mathrm{Spec}(Q_6/I_0^2 Q_6)$.

From D) and the proof of 2.3.15 (in other words, from [Fa2, th. 10 and the remarks after]) we deduce the existence of a unique $U_{\tilde{H}}$-morphism

$$m_{\tilde{H}} : \tilde{H}^0 = \mathrm{Spec}(S^0_{\tilde{H}}) \to Y_{16,\infty},$$

where $\tilde{H}^0$ is the completion of $\tilde{H}$ (or of $U_{\tilde{H}}$) in the origin; $Y_{16,\infty}$ is naturally a $\mathrm{Spec}(Q_6)$-scheme, and so a $U_{\tilde{H}}$-scheme. The key point is:

E) The morphism $m_{\tilde{H}}$ mod p factors through the Y_{16}-scheme $\tilde{Y}_1 \times_{Y_{16}} Y_{16,\infty}$, where the Y_{16}-scheme $\tilde{Y}_1$ is defined by the following fibre product:

$$\begin{array}{ccc} \tilde{Y}_1 & \xrightarrow{\tilde{c}_1} & \mathrm{Spec}(Q_6) \\ \downarrow{\scriptstyle \tilde{\ell}_6} & & \downarrow{\scriptstyle \ell_6} \\ Y_1 & \xrightarrow{c_1} & U_{\tilde{H}}. \end{array}$$

This is a consequence (in fact a restatement) of the starting assumption. We stop to digress why the statement of the starting assumption is not automatically satisfied. Let us first remark that we have a canonical closed embedding $a_Z : Z \hookrightarrow \tilde{Y}_1$ (cf. D)). The completion of $\tilde{Y}_1$ with respect to the ideal defining this embedding, can be identified with $\tilde{H}^0$ (cf. B)). Let $\tilde{c}_1^0$ and $\tilde{\ell}_6^0$ be the natural morphisms (obtained through this identification) from $\tilde{H}^0$ to $\mathrm{Spec}(Q_6)$ and respectively to Y_1. Let $m \in \mathbb{N}$, $m \geq 2$. Referring to [Fa2, th. 10], when we have to pass from things modulo $I_z^m S^0_{\tilde{H}}$ to things modulo $I_z^{m+1} S^0_{\tilde{H}}$, we do not know a priori that the morphism $\tilde{c}_1^0$ mod p is not changed. To argue this, we recall that P is the parabolic subgroup of G leaving invariant F^1. So $\mathrm{Lie}(P) = F^0(\mathrm{Lie}(G))$. The obstruction (as in loc. cit.) of lifting things modulo I_z^m to things modulo I_z^{m+1}, is given by some element

$$\gamma_m \in p\mathrm{Lie}(G) \otimes_{W(k)} I_z^m / I_z^{m+1}$$

(cf. C') and D) for the case $m = 2$). We can modify (as in loc. cit.) $\tilde{c}_1^0$ by something which is zero mod p so that we have $\gamma_m \in pF^0(\mathrm{Lie}(G)) \otimes_{W(k)} I_z^m / I_z^{m+1}$. But this element corresponds (as in loc. cit.) to a replacement of a Frobenius $\Phi^0_{\tilde{H}}$ of $M \otimes_{W(k)} S^0_{\tilde{H}}$ (defined for instance via $\mathcal{D}_1$) by another one $(1 - \gamma_m)\Phi^0_{\tilde{H}}(1 - \gamma_m)^{-1}$ isomorphic to it. The new obstruction we get is given by some new element in

$$\gamma_{m+1} \in \mathrm{Lie}(G) \otimes_{W(k)} I_z^{m+1} / I_z^{m+2}.$$

But γ_{m+1} is not a priori divisible by p (however: if γ_m is divisible by p^2, then γ_{m+1} is divisible by p). So the induction does not apply. That is why we needed the starting assumption.

Let now $n \in \mathbb{N}$. From [BLR, Th. 12 of p. 83 and Th. 16 of p. 91], and from the fact that $Y_{16,n}$ is obtained from a scheme of finite type over $Y_1 \times_{W(k)} \operatorname{Spec}(Q^{\wedge}_{6,n})$ by pull back through the natural morphism $\operatorname{Spec}(Q_6) \to \operatorname{Spec}(Q^{\wedge}_{6,n})$, we deduce that the morphism $\tilde{H}^0 \to Y_{16,n}$ we get factors through an affine $Y_{16,n}$-scheme which (under the natural projection $\operatorname{Spec}(Y_{16,n}) \to \operatorname{Spec}(Q_6)$) is smooth over $\operatorname{Spec}(Q_6)$; we have no problems in applying loc. cit. with $Q^{\wedge}_{6,n}$, as the finite flat commutative group scheme annihilated by p^n on $\operatorname{Spec}(Q^{\wedge}_{6,n})$ is defined over the spectrum of a $Q_{6,n}$-subalgebra of $Q^{\wedge}_{6,n}$ of finite type. From this and D), using a slice, we get an affine morphism

$$t(n) : \operatorname{Spec}(T_n) \to Y_{16,n}$$

such that, when composed with the projections of $Y_{16,n}$ on Y_1 and respectively on $\operatorname{Spec}(Q_6)$, it gives birth to a formally étale affine morphism $q(n)$ and respectively an affine étale morphism $s(n)$. Moreover we can assume that the induced (by h_Z) morphism $Z \to Y_{16,n}$ factors through $\operatorname{Spec}(T_n)$, producing an affine morphism

$$h_n : Z \to \operatorname{Spec}(T_n),$$

and that $t(n)$ mod p factors through $\tilde{Y}_1 \times_{Y_{16}} Y_{16,n}$. Also we can assume that $\operatorname{Spec}(T_n/pT_n)$ is a connected scheme.

By induction on $n \in \mathbb{N}$ we check that we can choose the things in such a way that there is an étale affine morphism

$$t_n : \operatorname{Spec}(T_{n+1}) \to \operatorname{Spec}(T_n)$$

for which we have:

F) $s(n+1) = s(n) \circ t_n$, $t_{n+1} \circ h_{n+1} = h_n$ and $t^n \circ t(n+1)$ is congruent to $t(n) \circ t_n$ mod p^{n+1}.

The proof of F) in the ordinary case. To prove F) we just need to check that we can not have two formally étale morphisms from $\operatorname{Spec}(T_{n+m})$ into Y^1 which are different mod p^n, but identical mod p, and whose images mod p contain an open subset U^1 of the special fibre of Y^1, and such that the pull backs of $A_{Y_1}[p^{n+m}]$ (and its cycles, including its principal quasi-polarization) under these two morphisms are the same. We refer to this statement as the statement (n, m) $(m \in \mathbb{N} \cup \{0\})$.

We first prove F) in the case the ordinary points of $\mathcal{N}_{k(v)}$ are Zariski dense in $\mathcal{N}_{k(v)}$ (see 4.2.1; or just 2.3.15 and 3.12), based on [Ka3, ch. 2] .

The idea is: it is enough to check a statement (n, m) at $\tilde{G}_{W(k)}$-ordinary points of U^1 (due to the fact that $\operatorname{Spec}(T_n/pT_n)$ is connected) using (good) $\tilde{G}_{W(k)}$-multiplicative coordinates. As we did not develop this theory yet (see 4.1-4 and 4.7) we have to work around. Such a theory of ordinary points and canonical coordinates is available (cf. [De2] and [No1]) for the particular case in which the Shimura-ordinary type associated to $\mathcal{N}$ (see 4.1) is an ordinary type. So we first prove statement (n,n) and F) in this particular case of Zariski dense set of ordinary points of $\mathcal{N}_{k(v)}$.

Obviously we can assume that $s(n+1) = s(n) \circ t_n$ and that $t_{n+1} \circ h_{n+1} = h_n$. For the congruence mod p^n of $t^n \circ t(n+1)$ and $t(n) \circ t_n$, it is enough to check that $q(n+1)$ is congruent to $q(n) \circ t_n$ mod p^n. So we first check that $q(n) \circ t_{n+1}$ is congruent to $q(n+1)$ modulo $p^{m(n)}$ for a sufficiently large $m(n) \in \mathbb{N}$. From [Ka3, p. 151] we deduce that, at the ring levels, $q(n) \circ t_n$ is congruent to $q(n+1)$ modulo the $[\frac{n+1}{2}]$-power of any maximal ideal of T_n corresponding to an ordinary point $\mathrm{Spec}(T_n)(\bar{k})$ (with the notations of the loc. cit., if $x \in A(k)[p^n]$, then $p^{2n}\tilde{x} = 0$). But the ordinary points of $\mathrm{Spec}(T_n/pT_n)$ are Zariski dense (cf. 3.12 and the fact that ordinary points can be recognized working mod p). So we can take $m(n) = [\frac{n+1}{2}]$. This takes care of the statement (n,n). After a relabeling (like T_{2^n} becomes T_n), we get F). This ends the proof of F) in the ordinary case.

We come back to the general case, and we assume (for the time being) that property F) holds. Let $\mathrm{Spec}(T_\infty)$ be the limit of the $\mathbb{N}$-projective system defined by t_n, $n \in \mathbb{N}$. From F) above, we deduce the existence of a morphism

$$\mathrm{Spec}(T_\infty^\wedge) \to Y_{16,\infty}$$

such that h_Z factors through it. Moreover $\mathrm{Spec}(T_\infty^\wedge)$ together with the logical morphisms $\mathrm{Spec}(T_\infty^\wedge) \to U_6^\wedge$, $\mathrm{Spec}(T_\infty^\wedge) \to (\mathcal{N}_{W(k)}/H_0)^\wedge$ and the morphism $\mathrm{Spec}(W(k)) \to \mathrm{Spec}(T_\infty^\wedge)$ defined via h_Z, satisfies all the requirements of 3.4.14.1, except c). So to get 3.4.14.1 c), we have to work in a more refined way: we have to show that $\mathrm{Spec}(T_\infty/pT_\infty)$ can be obtained from $\mathrm{Spec}(T_1/pT_1)$ using systems of equations "closed in spirit" to those of the first type, in the same manner as in 3.4.8.1.2 b); on the way we prove F) in the general case.

The needed refinement. We work in the general case. We reached the situation: over $\mathrm{Spec}(T_1)$ there are two principally quasi-polarized Shimura p-divisible groups such that their kernels under the multiplication by p are isomorphic; as $\mathrm{Spec}(T_1)$ is a formally smooth regular $W(k)$-scheme it makes sense to speak about isomorphisms preserving the cycles: by this we mean filtered isomorphisms of the filtered projective $T_1^\wedge$-modules underlying the filtered F-crystals of these two p-divisible groups, preserving the quasi-polarizations and the cycles, and which mod p result in an isomorphism between truncations mod p of principally quasi-polarized p-divisible objects of $\mathcal{MF}^\nabla_{[0,1]}(T_1)$. Let $\mathcal{I}_1$ be this last isomorphism between truncations mod p. We first remark that:

G) There is a formally étale affine $\mathrm{Spec}(T_1)$-scheme $\tilde{Z}_\infty$ through which the morphism h_1 factors, such that over the special fibre $\tilde{Y}_\infty$ of $\tilde{Z}_\infty$, the two principally quasi-polarized p-divisible groups we get naturally over $\mathrm{Spec}(T_1/pT_1)$, are isomorphic, under an isomorphism, which viewed as a non-filtered isomorphism between p-divisible objects of $\mathcal{MF}^\nabla_{[0,1]}(\tilde{Z}_\infty)$, lifts the extension of $\mathcal{I}_1$ to $\tilde{Z}_\infty$, and which in the Z-valued point defined by the factorization of h_1, is the one defined via h_Z.

This is a consequence of F). As we proved property F) only in the ordinary case, we need to point out that G) is also a consequence of SA2. To see this we just need to remark that the integral closed subscheme of the $\mathrm{Spec}(T_1/pT_1)$-scheme parametrizing isomorphisms between the kernels of the multiplication by p^n on the two p-divisible groups we got over $\mathrm{Spec}(T_1/pT_1)$, through which the morphism $\tilde{m}_H$ mod p factors naturally, is generically finite over $\mathrm{Spec}(T_1/pT_1)$, and so étale over $\mathrm{Spec}(T_1/pT_1)$ in its k-valued point obtained from the one of $\tilde{H}^0$, via this factorization.

This results from 3.1.0 a) and d), and from 3.11.1 c), being a property of Shimura-ordinary σ-crystals. In other words, we just need to remark: if $k_1 \subset k_2$ are two algebraically closed fields containing k, then any isomorphism IS over $W(k_2)$ between the truncations mod p^n of two Shimura-canonical p-divisible groups over $W(k_1)$, is automatically defined over $W(k_1)$; here IS is preserving the cycles, in the sense that it lifts to an $1_{\mathcal{J}'}$-isomorphism (in the sense of 2.2.9 6)) of Shimura p-divisible groups. Moreover the number of such isomorphisms is obviously finite. Obviously G) implies F), as we have a logical (defined by changing the filtration of the $\tilde{Z}_\infty^\wedge$-module underlying the F-crystal of the special fibre of $\mathcal{D}1_{\tilde{Z}_\infty}$) morphism

$$\tilde{Z}_\infty^\wedge \to Y_{16,\infty}$$

through which h_Z factors. This ends the proof of F) in the general case.

To explain in detail this property G), we need some notations. Let Φ_2 be the Frobenius of $M \otimes_{W(k)} T_1^\wedge$ induced from the Frobenius of $M \otimes_{W(k)} Q_6$, and let Φ_1 be the Frobenius of $M \otimes_{W(k)} T_1^\wedge$ defined by the pull back of the natural p-divisible group on Y_1 (through the resulting morphism $\mathrm{Spec}(T_1) \to Y_1$) (cf. C)). For computing Φ_1 and Φ_2, we choose the Frobenius Φ_{T_1} of the ring $T_1^\wedge$ to be the one induced from the one of Q_6. By property G), we mean: there is $\tilde{h}_\infty \in G(T_1^{h\wedge})$ (here $\mathrm{Spec}(T_1^h)$ is the henselization of $\mathrm{Spec}(T_1)$ with respect to the k-valued point defined naturally by h_1) such that mod p is the identity and over $T_1^{h\wedge}$ we have the identity

$$\tilde{h}_\infty \circ (\Phi_1 \otimes 1) = (\Phi_2 \otimes 1) \circ \tilde{h}_\infty. \tag{17}$$

Working as in 3.4.1.3, modulo powers of p, this equation (17) defines an $\mathbb{N}$-pro-étale scheme over $\mathrm{Spec}(T_1/pT_1)$ as follows. We start working mod p^2. So we are trying to find an element $\tilde{h}_1 \in G(T_1^\wedge)$ which mod p is the identity, and for which we have:

$$\Phi_2 \circ \tilde{h}_1(m/p) = \tilde{h}_1 \circ \Phi_1(m/p) \mod p^2, \tag{18}$$

for $m \in F^1$, and

$$\Phi_2 \circ \tilde{h}_1(m) = \tilde{h}_1 \circ \Phi_1(m) \mod p^2, \tag{19}$$

if $m \in F^0$. The above congruencies are between elements of $M \otimes_{W(k)} T_1^\wedge$.

For solving this system, only the expression of $\tilde{h}_1$ mod p^3 is important. Moreover as we already know the existence of $\mathcal{I}_1$, we can assume that $\tilde{h}_1$ mod p^2 defines an element of $P(T_1/pT_1)$. So let $\gamma_{\tilde{h}_1} \in \mathrm{Lie}(P) \otimes_{W(k)} T_1/pT_1$ such that $\tilde{h}_1$ is congruent to $1 + p\gamma_{\tilde{h}_1}$ mod p^2. We choose arbitrarily a basis $\{\tilde{e}_1, ..., \tilde{e}_{\dim(P)}\}$ of $\mathrm{Lie}(P)$, and a basis $\{n_1, ..., n_{\dim(\tilde{H})}\}$ of $\mathrm{Lie}(\tilde{H})$. We can assume that $\{\tilde{e}_1, ...\tilde{e}_{\dim(\tilde{H})}\}$ is a basis of $F^1(\mathrm{Lie}(P))$. We write

$$\gamma_{\tilde{h}_1} = \sum_{i=1}^{\dim(P)} e_i x_i,$$

with x_i as variables. Moreover we can write $\tilde{h}_1$ mod p^3 as follows

$$1+p\gamma_{\tilde{h}_1}+p^2\sum_{j=1}^{\dim(\tilde{H})} n_j y_j,$$

with y_j as extra variables. The above equations (18) and (19) get translated in a system of equations involving x_i's and y_j's. It defines an affine $\mathrm{Spec}(T_1/pT_1)$-scheme $\mathrm{Spec}(T_1')$.

3.9.3 allows us to speak about $\tilde{G}_{W(k)}$-ordinary points of $\mathrm{Spec}(T_1/pT_1)$. Their definition does not depend on which of the two p-divisible groups over $\mathrm{Spec}(T_1)$ we use to define them. 3.12 implies that they are Zariski dense. We have:

H) $\mathrm{Spec}(T_1')$ has finite fibres over the $\tilde{G}_{W(k)}$-ordinary points of $\mathrm{Spec}(T_1/pT_1)$ with values in $\bar{k}$.

We get this by looking at the system of equations in x_i's and y_j's as in 3.4.8.5; it is obtained by combining the equations of (20) and (21) below

$$x_i = L_i(x^p_{\dim(\tilde{H})+1},...,x^p_{\dim(P)},y_1^p,...,y^p_{\dim(\tilde{H})}), \tag{20}$$

$i\in\{1,...,\dim(P)\}$, and

$$L_i(x^p_{\dim(\tilde{H})+1},...,x^p_{\dim(P)},y_1^p,...,y^p_{\dim(\tilde{H})})=0, \tag{21}$$

$i\in\{\dim(P)+1,...,\dim(G)\}$. Here L_i are non-homogeneous linear forms with coefficients in T_1/pT_1, $i\in\{1,...,\dim(G)\}$. Over the perfection $\mathrm{Spec}(T_1/pT_1)^{\mathrm{perf}}$ (of $\mathrm{Spec}(R_1/pR_1)$), the last $\dim(P)-\dim(\tilde{H})$ equations of (20) together with the p-th root of the equations of (21), define a system of equations SE, which is very close to one of the first type.

In fact, in any $\tilde{G}_{W(\bar{k})}$-ordinary point of $\mathrm{Spec}(T_1/pT_1)$ with values in $\bar{k}$, SE becomes a system of equations (with coefficients in $\bar{k}$) which is equivalent to one of the first type. Using a Φ_{T_1}-Teichmüller lift $\mathrm{Spec}(W(\bar{k}))\to\mathrm{Spec}(T_1^\wedge)$ of such a point $y_G:\mathrm{Spec}(\bar{k})\to\mathrm{Spec}(T_1/pT_1)$, this statement is a consequence of 3.11.1 and 3.7.2. In other words, replacing the old variables by new once, "modelled" on the filtration $F^1(y_G)$ of $M\otimes_{W(k)}W(\bar{k})$ such that we get, by pulling back $\mathcal{D}_2$ via y_G, a Shimura-canonical lift $(M\otimes_{W(k)}W(\bar{k}),F^1(y_G),g(y_G)(\varphi_0\otimes 1),\tilde{G}_{W(\bar{k})})$, we get a system of equations equivalent (i.e. it defines the same affine scheme over $\bar{k}$) to the one obtained from SE via y_G, and which is of the first type; this last property can be easily checked (the argument is very similar to the one used in the proof of 3.4.18.4 B)), using the notations of 3.7.2 (3.11.1 c) allows us to refer to 3.7.2). So 3.4.8.1 applies to end the proof of property H).

Moreover G) guarantees that $\mathrm{Spec}(T_1')$ is not an empty scheme, and that h_1 mod p factors through an integral closed subscheme $\mathrm{Spec}(T_1^{0\prime})$ of $\mathrm{Spec}(T_1')$ which is a connected scheme, étale over an open subscheme $\mathcal{U}_1$ of $\mathrm{Spec}(T_1/pT_1)$ (cf. also H)). But the same argument as the one used in 3.4.8.1.2 a) applies to give us that we can choose $\mathcal{U}_1$ to depend only on the coefficients of the homogeneous parts of the forms L_i, $i=\overline{1,\dim(G)}$. So we can repeat the arguments, for things modulo p^n ($n\in\mathbb{N}$, $n>1$). We obtain new systems of equations, involving the same type of linear forms $L_i(n)$; moreover the homogeneous part of $L_i(n)$ is the same as of L_i, $\forall i \in \{1, ..., \dim(G)\}$ and $\forall n \in \mathbb{N} \setminus \{1\}$. From 3.4.8.1.2 a) we deduce that we can assume that $\tilde{Y}_\infty$ is an affine $\mathrm{Spec}(T_1/pT_1)$-scheme such that its fibres over points of $\mathcal{U}_1$ are non-empty. In other words, coming back to the last paragraph before the needed refinement, we can assume that the morphism $\mathrm{Spec}(T_\infty/pT_\infty) \to \mathrm{Spec}(Q_6/pQ_6)$ has an image which contains an open subscheme of $\mathrm{Spec}(Q_6/pQ_6)$. This takes care of 3.4.14.1 c), ending the proof of 3.4.14.1.

3.4.14.2. Exercise. Show that 3.4.14 and 3.4.14.1 remain true, as mentioned in 3.4.13, without the assumptions made in 3.4.13. Hint: the only difference is that we have to work with different de Rham cycles, which become linear combination of de Rham components of Hodge cycles, after passage to an étale morphism.

3.4.14.3. Exercise. Restate 3.4.14.1 and 3.4.14.1 in terms of a k-valued point y of $\mathcal{N}_{W(k)}/H_0$, instead of a $W(k)$-valued point z of $\mathcal{N}_{W(k)}/H_0$.

3.4.14.4. Remark. We do not know when the starting assumptions SA1-2 are satisfied. Of course, instead of open subschemes of $G = G^0_{W(k)}$ or of $\tilde{H}$, we can work with étale schemes over them; so the chances of checking out such starting assumptions, in the new context are much higher. When there are no hopes to get a starting condition, in 3.4.14 d) and 3.4.14.4.1 d) we have to work with truncations mod p^n of the p-divisible objects involved ($n \in \mathbb{N}$); of course in such a situation we do not need to pass to $\mathbb{N}$-pro-ètale morphisms.

It seems to us possible to prove directly, using just the techniques of 3.4.14 (at least in the case (G, X) is of compact type), the following expected result.

Expectation. *We assume $k = \bar{k}$. There is a $W(k)$-valued point of $\mathcal{N}_{W(k)}/H_0$ such that the kernel of the multiplication by p^n of the Shimura p-divisible group we get over $W(k)$ through it, is isomorphic to the same type of kernel of the Shimura p-divisible group we get through an arbitrary $W(k)$-valued point of* $\mathrm{Spec}(Q_6)$.

3.4.15. For the sake of presenting the group of ten fundamental principles in a reasonable compact form, we include the following expected result (it is proved in §11; see also 4.12) here.

Let $y : \mathrm{Spec}(k) \to \mathcal{N}_{W(k)}/H_0$ be defined by z. We continue to assume that $k = \bar{k}$.

A. Expectation (the completion property). *Giving $g_1 \in \tilde{G}^0_{W(k)}(W(k))$, there is a point $z_1 : \mathrm{Spec}(W(k)) \to \mathcal{N}_{W(k)}/H_0$ lifting a k-valued point of the same connected component $\mathcal{C}^0$ of $\mathcal{N}_k/H_0$ through which y factors, and such that its attached principally quasi-polarized Shimura filtered σ-crystal $\left(M_1, F^1_1, \varphi_1, \tilde{G}_{W(k)}, \tilde{\psi}_{M_1}\right)$ is isomorphic to $\left(M, F^1, g_1\varphi_0, \tilde{G}_{W(k)}, \tilde{\psi}_M\right)$.*

The motivation for this expectation is based on 3.4.14.4 and on the following exercise.

B. Exercise. Let (M, φ, G) be an arbitrary Shimura σ-crystal. Show that there is $n \in \mathbb{N}$, such that for any $h_n \in G(W(k))$ congruent to 1 mod p^n, $(M, h_n\varphi, G)$ is isomorphic to (M, φ, G). The same remains true in a principally quasi-polarized context.

3.4.16. Remarks. 1) We say that $\mathcal{N}$ has the completion property, if 3.4.15 A holds.

2) We think it is an interesting problem to determine in 3.4.15 B, the smallest value of n (or at least sharp estimates of it, perhaps in terms of the Hodge numbers of the iterates of (M,φ) and of $(\mathfrak{g},\varphi)$).

3.4.17. Terminology. We propose that all connections used in 3.4.1-16 as well as in 2.2-3 to be called Faltings connections, in the honor of their source [Fa2, th. 10 and the remarks after].

3.4.18. Universal principles for connections in Fontaine's categories $\mathcal{MF}$. Let $m \in \mathbb{N}$. The notations to be used in 3.4.18-20 are entirely independent from the previous notations of §3. In this section

$$R := W(k)[[z_1, ..., z_m]]$$

is the ring of formal power series in m variables with coefficients in $W(k)$. Let Φ_R be an arbitrary Frobenius of R, compatible with the Frobenius σ on $W(k)$.

As any object of a category $\mathcal{MF}_{[a,b]}(R)$ lifts to a p-divisible object (simple argument at the level of lifting matrices, based on the fact that R is a local ring), we prefer to look at such objects as being obtained from truncations (mod p^n, for some $n \in \mathbb{N} \cup \{0\}$) of p-divisible objects, by a repeated finite process of extensions (via short exact sequences). So we start with a p-divisible object

$$\mathfrak{C} := (M \otimes_{W(k)} R, F^1 \otimes_{W(k)} R, \Phi),$$

of $\mathcal{MF}_{[0,1]}(R)$. Here M is a free $W(k)$-module of finite rank, F^1 is a direct summand of M, while Φ is a Φ_R-linear endomorphism of $M \otimes_{W(k)} R$ producing an isomorphism (still denoted by Φ) of R-modules

$$\Phi : (M + \frac{1}{p}F^1) \otimes_{R\ \Phi_R} R \tilde{\to} M \otimes_{W(k)} R.$$

A natural question arises:

Q Is there a p-divisible group over R, whose corresponding p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$, when viewed as a p-divisible object of $\mathcal{MF}_{[0,1]}(R)$, is $\mathfrak{C}$?

If the answer is yes, we say that $\mathfrak{C}$ is induced from a p-divisible group. Similarly we speak about $\mathfrak{C}$ being induced from a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R_1)$, with R_1 a regular ring of formal power series with coefficients in $W(k)$.

3.4.18.0. Types of Frobenius of R. [Fa2, th. 10] can be restated: the answer to **Q** is yes provided the Frobenius Φ_R is of the form $\Phi_R(z_i) = z_i^p$. This assumption (on the special shape of the Frobenius of R) has been made in [Fa2, ch. 7] (quoting) "for technical reasons".

Let $S(m) := \{1, ..., m\}$ and let $S(M) := \{1, ..., \dim(M)\}$. An arbitrary Frobenius Φ_R of R compatible with σ of $W(k)$ is of the form

$$\Phi_R(z_i) = z_i^p + pP_i(z_1, ..., z_m) + pL_i(z_1, ..., z_m) + p\gamma_i,$$

where $P_i = \sum_{j=2}^{\infty} Q_{ij}$, with Q_{ij} a homogeneous polynomial of degree $j \geq 2$, where L_i is a linear homogeneous polynomial, and where $\gamma_i \in W(k)$, $\forall i \in S(m)$. It is an easy exercise to see that by replacing z_i by $z_i + p\delta_i$, with $\delta_i \in W(k)$ conveniently chosen, $i \in S(m)$, we can assume that $\gamma_i = 0$, $\forall i \in S(m)$: by induction on $l \in \mathbb{N}$ we can check that we can assume that p^l divides $p\gamma_i$; this is the general argument showing the existence of Teichmüller lifts. We assume this from now on.

We can write the Frobenius Φ of $M \otimes_{W(k)} R$ in the form

$$\Phi = g_R(\varphi \otimes 1),$$

where $g_R \in GL(M)(R)$ modulo the ideal $(z_1, ..., z_m)$ of R is the identity element of $G(W(k))$, and where φ is the σ-linear endomorphism of M obtained from Φ by making $z_i = 0$, $i \in S(m)$. So what we are trying to do (for answering **Q**), is nothing else but to deform over R the p-divisible group over $W(k)$ corresponding to the filtered σ-crystal (M, F^1, φ), in the way "prescribed" by $\mathfrak{C}$.

We write

$$L_i(z_1, ..., z_m) = \sum_{j=1}^{m} a_{ij} z_j,$$

with $a_{ij} \in W(k)$, $\forall i, j \in S(m)$. We obtain a matrix $A(\Phi_R) \in M_m(W(k))$, having a_{ij} as its entries. Let $\bar{A}(\Phi_R) \in M_m(k)$ be the matrix defined by $A(\Phi_R)$ mod p. We call $A(\Phi_R)$ (resp $\bar{A}(\Phi_R)$) the linear part (resp. the k-linear part) of Φ_R with respect to $z_1, ..., z_m$. Replacing z_i by $\sum_{j=1}^{m} b_{ij} z_j$, with $b_{ij} \in W(k)$ defining an invertible matrix $B \in GL_m(W(k))$, the matrix $A(\Phi_R)$ is replaced by $\sigma(B) A(\Phi_R) B^{-1}$. In what follows it is important only how $\bar{A}(\Phi_R)$ looks like.

3.4.18.0.1. Terminology. If $\Phi_R(z_i) = z_i^p$, $\forall i \in S(m)$, we speak about an additive type Frobenius of R. If $\Phi(z_i) = (z_i + 1)^p - 1$, then we speak about a multiplicative type Frobenius of R. If $A(\Phi_R)$ is an invertible matrix then we speak about an essentially multiplicative type Frobenius of R. If $\bar{A}(\Phi_R)$ is a nilpotent matrix, then we speak about an essentially additive type Frobenius of R.

3.4.18.1. Notations. Let $f(0) := \dim(M/F^1)$, and $f(1) := \dim(F^1)$. We denote by $s(i)$ the multiplicity of the slope i for (M, φ), $i = \{0, 1\}$. So $s(0)s(1)$ is the multiplicity of the slope -1 of $(\operatorname{End}(M), \varphi)$. Let r be the number of non-zero eigenvalues (counted with their multiplicities) in $\bar{k}$ of $\bar{A}(\Phi_R)$.

3.4.18.1.1. Remarks. It is easy to see that r is well defined regardless of all choices (cf. also 3.4.18.4). In particular the notion of essentially additive type Frobenius of R (case $r = 0$) is well defined. We can have $r = 0$ without $\bar{A}(\Phi_R)$ being zero. So we have plenty of essentially additive type Frobenius of R which are not of additive type. On the other, if $k = \bar{k}$ it can be easily check that the notions of multiplicative type and essentially of multiplicative type coincide. But this is not so if $k \neq \bar{k}$; example: if $k = \mathbb{F}_p$ then $\Phi_R(z_i) = (z_i + 1)^p - 1 + p^2 z_i$ is not of multiplicative type.

3.4.18.2. The generic situation. This is obtained by requiring that:

a) (M, φ) to be an ordinary σ-crystal, and (M, F^1, φ) to be diagonalizable;

b) $\Phi_R(z_i) = (z_i+1)^p - 1$, $\forall i \in S(m)$;

c) $k = \bar{k}$.

It is a simple exercise to see that for any $n \in \mathbb{N}$, there are precisely

$$p^{nmf(0)f(1)}$$

connections on $M \otimes_{W(k)} R/p^n R$ which make $\mathfrak{C}/p^n\mathfrak{C}$ potentially to be viewed as an object of $\mathcal{MF}_{[0,1]}(R)$, regardless of the shape of g_R. Of course, with respect to an appropriately chosen R-basis of $M \otimes_{W(k)} R$, we can assume that g_R is the identity element. Moreover all these connections are integrable, and (cf. also [Fa1, p. 34]) nilpotent mod p. The above number for connections mod p^n, for instance, can be obtained using 3.4.8 (3.4.8.9 and 3.4.8.1 show that it is enough to check it only for $n = 1$).

The reason we call this the generic situation is related to [De2, 1.4.7]. There are two philosophies at play:

– any p-divisible group over k is the specialization of an ordinary p-divisible group (cf. 3.1.8 and 3.1.8.1);

– whatever we can prove for the standard situation, and does not involve the precise power of p of the number of solutions we get for different system of equations (in the other cases we get smaller powers of p), can be proved in the general case (in particular we always get solutions).

As examples of this second philosophy see 3.4.18.4. It works also in the more general context where the category $\mathcal{MF}_{[0,1]}(R)$ is replaced by another one $\mathcal{MF}_{[a,b]}(R)$, with $a, b \in \mathbb{Z}$, $a < b$ (of course we keep b) and c), while a) is modified accordingly).

3.4.18.3. Example. It is easy to see that the question **Q** does not always have a positive answer. Here is a simple example for which the answer to **Q** is no. Let $R = \mathbb{Z}_p[[t]]$ with $\Phi_R(t-1) = (t-1)^p$. We take $\mathfrak{C}$ to be defined as follows. Let M be a free $\mathbb{Z}_p$-module of dimension 2, and let $\{x, y\}$ be a basis of M such that $\{y\}$ a basis of F^1. Let $\Phi(x) = x$ and $\Phi(y) = py + px + pty$. It is easy to see that there is no connection ∇ on $M \otimes_{\mathbb{Z}_p} R$ for which Φ is ∇-parallel. However it can be checked that, moving from $\mathbb{Z}_p$ to $W(\mathbb{F})$ (i.e. we replace $\mathbb{F}_p$ by its algebraic closure), $(M \otimes_{\mathbb{Z}_p} W(\mathbb{F})[[t]], F^1 \otimes_{\mathbb{Z}_p} W(\mathbb{F})[[t]], \Phi \otimes 1)$ is induced from a p-divisible group (over $W(\mathbb{F})[[t]]$). So the obstruction is: the field $\mathbb{F}_p$ is not algebraically closed.

3.4.18.3.1. Lemma. *If there is an integrable connection ∇ on $M \otimes_{W(k)} R$ which makes $\mathfrak{C}$ to be viewed as a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$, then $\mathfrak{C}$ is induced from a p-divisible group.*

Proof: The connection ∇ is also nilpotent mod p, cf. [Fa1, p. 34]. So it allows us to change the Frobenius of R to one of additive type, cf. [De2, (1.1.2.1)]. Now everything results from [Fa2, th. 10].

3.4.18.4. Theorem (the universal local ∇ principle for $\mathcal{MF}_{[0,1]}$). *We assume $k = \bar{k}$.*

A) There are connections ∇ on $M \otimes_{W(k)} R$ which make $\mathfrak{C}$ potentially to be viewed as a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$.

B) Let $n \in \mathbb{N}$. The number of connections which make $\mathfrak{C}/p^n\mathfrak{C}$ potentially to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$ is $p^{nrs(0)s(1)}$. This number is smaller or equal to $p^{nmf(0)f(1)}$.

Proof: The proof of part A) is contained in the proof of 3.4.1.3, cf. 3.4.8.4 and 3.4.8.9. From 3.4.8.1 and 3.4.8.9 we deduce that for the counting part of B) it is enough to consider the case $n = 1$. As $k = \bar{k}$, from 3.4.18.0 we deduce that we can assume that $\bar{A}(\Phi_R)$ is upper triangular. The existence of a connection on $M \otimes_{W(k)} R/pR$ making $\mathfrak{C}/p\mathfrak{C}$ potentially to be viewed as an object of $\mathcal{MF}_{[0,1]}(R)$ is given (see 3.4.8 (3)) by a system of equations of the form

$$x_{ijl} = L_{ijl}(x^p_{111}, ..., x^p_{\dim(M)\dim(M)m}) + c_{ijl},$$

$(i, j, l) \in S(M) \times S(M) \times S(m)$, where L_{ijl} are linear homogeneous forms in $m\dim(M)^2$ variables, and where $c_{ijl} \in R/pR$, $\forall (i, j, l) \in S(M) \times S(M) \times S(m)$.

This system defines an affine étale scheme over $\mathrm{Spec}(R/pR)$, cf. 3.4.8.1. We consider the ideal $I := (z_1, ..., z_m)$ of R/pR. As R/pR is a strictly henselian ring, it is enough to look at this system only modulo I. Due to the upper triangular form of $\bar{A}(\Phi_R)$, this system modulo I can be "separated":

Fact. *In the expression of L_{ijl} modulo I only those variables $x_{ijl'}$ show up for which $l' \leq l$.*

So this allows us to assume that $m = 1$: first we solve the system we get involving just x_{ij1}'s; we use any solution we get for it to get a new system involving just x_{ij2}'s (in other words x_{ij1}'s become constants for this new system involving the x_{ij2}'s); etc. (the induction applies; warning: all these are modulo I). From now on, as $m = 1$, we drop the index l. So modulo I we get a system $\mathcal{S}$ of $\dim(M)^2$ equations in $\dim(M)^2$ variables with coefficients in k.

If $\bar{A}(\Phi_R) \in k$ is zero, then the system $\mathcal{S}$ is of the form $x_{ij} = d_{ij}$, $(i, j) \in S(M) \times S(M)$, with $d_{ij} \in k$. Obviously it has a unique solution. So we can assume that $\beta := \bar{A}(\Phi_R) \in k$ is non-zero (in fact we can assume that $\beta = 1$).

Modulo I, g_R contributes only to the coefficients d_{ij} (obtained by taking c_{ij} modulo I). From 3.4.8.1 we deduce that we can assume that g_R is the identity element of $GL(M)(R)$.

Now we are essentially done. We present first the particular case when (M, F^1, φ) is cyclic diagonalizable. We consider a basis $\{e_i | i \in S(M)\}$ of M. From def. 2.2.1 b) we deduce that we can assume that there is a subset $S(F^1)$ of $S(M)$ such that $\{e_i | i \in S(F^1)\}$ is a basis of F^1, and that there is a permutation π of $S(M)$ such that $\varphi(e_i) = pe_{\pi(i)}$, if $i \in S(F^1)$, and $\varphi(e_i) = e_{\pi(i)}$, if $i \notin S(F^1)$. So the system $\mathcal{S}$ is of the form

$$x_{\pi(i)\pi(j)} + d_{ij} = \beta\alpha_{ij}x^p_{ij},$$

$i, j \in S(M)$, where α_{ij} is 1 or 0 depending on the fact that (i, j) is or is not in $(S(M) \setminus S(F^1)) \times S(F^1)$, and where $d_{ij} \in k$. So if $\pi(i) \neq i$ or if $\pi(j) \neq j$, or if $\pi(i) = i$ and

$\pi(j) = j$, but $(i,j) \notin (S(M) \setminus S(F^1)) \times S(F^1)$, then in the sequence of elements α_{ij}, $\alpha_{\pi(i)\pi(j)}$,..., $\alpha_{\pi^d(i)\pi^d(j)}$, at least one is zero; so x_{ij} is uniquely determined. For any pair (i,j), with $\pi(i) = i$ and $\pi(j) = j$, and with $i \notin S(F^1)$ and $j \in S(F^1)$, we have precisely p (independent) choices for x_{ij}. The number of such pairs is precisely $s(0)s(1)$. This handles the particular case.

We come back to the general case. Let $B(\varphi)$ be a $\dim(M) \times \dim(M)$-matrix with coefficients in k formed naturally (i.e. it acts naturally on M/pM) by a solution of the system $\mathcal{S}$. From the shape of the equations of 3.4.1.1.1 2) we deduce that $B(\varphi)$ annihilates the image of $\varphi(M)$ in M/pM. This easily implies that $B(\varphi)$ annihilates the image in M/pM of the maximal σ-subcrystal (M,φ) corresponding to slopes smaller than 1. The fastest way to see this: we can assume that $\Phi_R(z_1) = z_1^p + pz_1$; so working modulo (z_1) instead of modulo (p, z_1), we get similarly a matrix $\tilde{B}(\varphi)$ lifting $B(\varphi)$ to $W(k)$, such that, when viewed as an endomorphism of M, we have

$$(1), \qquad p\varphi(\tilde{B}(\varphi)) = \tilde{B}(\varphi),$$

cf. 3.4.1.1.1 2).

So we just need to look at the action of $B(\varphi)$ on the image in M/pM of elements $e \in M$ for which $\varphi(e) = pe$: (M,φ) is a direct sum of three σ-crystals having respectively only the slope 1, only the slope 0, only slopes of the interval $(0,1)$ (cf. the existence of the filtration F^1 of M). From the same equations we deduce (the argument is as in the above particular case) that $B(\varphi)(e)$ belongs to the image in M/pM of the maximal σ-subcrystal of (M,φ) corresponding to the slope 0. So we have at most $p^{s(0)s(1)}$ possibilities for $B(\varphi)$. From this we deduce, cf. 3.4.18.2, that there are precisely $p^{s(0)s(1)}$ solutions of $\mathcal{S}$. Another way (shorter) is to remark that (1) above says that $\tilde{B}(\varphi)$ is an element corresponding to slope -1 of $(\mathrm{End}(M),\varphi)$. This ends the proof of the B) part, and so of the theorem.

3.4.18.4.1. Theorem (the integrability principle for $\mathcal{MF}_{[0,1]}(R)$). *All the connection ∇ proved to exist in 3.4.18.4 are integrable and nilpotent mod p.*

Proof: Let $R^{\mathrm{al}} := W(k)[z_1, ..., z_m]$. Here al stands for algebraization. The fact that a connection is integrable is expressed through some algebraic equations. In our case it is enough to check (regardless of which $n \in \mathbb{N}$ we work with) that these equations are satisfied modulo the ideal $J_r := (p, z_1, ..., z_m)^r$ of R, for any $r \in \mathbb{N}$. But these equations modulo J_r depend only on the expressions of Φ and Φ_R modulo J_{r+2}. This allows us to algebraize the things. From the Chinese remainder theorem we deduce the existence:

– of an element $g_{R^{\mathrm{al}}} \in GL(M)(R^{\mathrm{al}})$ such that modulo the ideal $J^{\mathrm{al}}_{r+2} := (p, z_1, ..., z_m)^{r+2}$ of R^{al} it is g_R modulo J_{r+2}, and modulo the ideal I_1^{r+2} of R^{al}, with $I_1 := (p, z_1 - 1, ..., z_m - 1)$, is an element $g_1 \in GL(M)(W_n(k)) \subset GL(M)(R^{\mathrm{al}}/I_1^{r+2})$ for which the Frobenius $g_1(\varphi \otimes 1)$ of $M \otimes_{W(k)} R^{\mathrm{al}}/I_1^{r+2}$ is isomorphic to one coming from a generic situation, as in 3.4.18.2 above;

– of a Frobenius $\Phi_{R^{\mathrm{al}}}$ of R^{al} compatible with σ, which modulo J^{al}_{r+2} is the same as the Frobenius Φ_R modulo J_{r+2}, and which modulo I_1^{r+2} is isomorphic to the one coming from a generic situation as in 3.4.18.2 above.

Let $n \in \mathbb{N}$, and let ∇_n be a connection on $M \otimes_{W(k)} R/p^nR$ which makes $\mathfrak{C}/p^n\mathfrak{C}$ potentially to be viewed as an object of $\mathcal{MF}_{[0,1]}(R)$. As in 3.4.1.3 we construct for the object

$$\mathfrak{C}^{\mathrm{al}} := (M \otimes_{W(k)} R^{\mathrm{al}}/p^nR^{\mathrm{al}}, F^1 \otimes_{W(k)} R^{\mathrm{al}}/p^nR^{\mathrm{al}}, g_{R^{\mathrm{al}}}(\varphi \otimes 1))$$

of $\mathcal{MF}_{[0,1]}(R^{\mathrm{al}})$, where the Frobenius of R^{al} is $\Phi_{R^{\mathrm{al}}}$, a moduli affine scheme

$$M_n(\mathfrak{C}^{\mathrm{al}}) = \mathrm{Spec}(R_n)$$

of connections which make the extension of $\mathfrak{C}^{\mathrm{al}}/p^n\mathfrak{C}^{\mathrm{al}}$ to $*$ potentially to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(*)$. Here $*$ stands for a p-adically complete formally étale R^{al}-algebra. Mod p it defines an étale affine scheme $\mathcal{R}(n) := \mathrm{Spec}(R_n/pR_n)$ over $\mathcal{R}(0) := \mathrm{Spec}(R^{\mathrm{al}}/pR^{\mathrm{al}})$. Moreover the fibres of $\mathcal{R}(n)$ over k-valued points of $\mathcal{R}(0)$ have a number of points less or equal to the number of points of the fibre $\mathcal{F}_1$ above the k-valued point of $\mathcal{R}(0)$ defined by $z_i = 1$, $i \in S(m)$, cf. 3.4.18.4 B). From this and the surjectivity principle 3.4.8.2 (cf. 3.4.8.4) we deduce that:

Fact 1. *Any connected component of $\mathcal{R}(n)$ contains points of $\mathcal{F}_1$.*

From this and from 3.4.18.2 we get:

Fact 2. *$M_n(\mathfrak{C}^{\mathrm{al}})$ is a moduli scheme of integrable connections, nilpotent mod p.*

From Fact 2 and the fact that ∇_n modulo the ideal $(p, z_1, ..., z_m)^r$ of R/p^nR, is equal to a connection obtained from the universal one on $M \otimes_{W(k)} R_n/p^nR_n$, through a morphism $\mathrm{Spec}(R/p^nR) \to M_n(\mathfrak{C}^{\mathrm{al}})$, we get that ∇_n is integrable and nilpotent mod p (we recall that $r \in \mathbb{N}$ was arbitrary). This ends the proof of the theorem.

3.4.18.4.2. Corollary (the moduli principle). *For any separated regular scheme (resp. regular affine scheme) S which is formally smooth over $W(k)$ and has a Frobenius compatible with σ of $W(k)$, any object (resp. any p-divisible object) $\mathfrak{C}$ of $\mathcal{MF}_{[0,1]}(S)$ defines a moduli scheme $M(\mathfrak{C})$ which is the p-adic completion of an affine étale scheme over S (resp. of an $\mathbb{N}$-pro-étale scheme over S) of connections (over p-adically complete formally étale affine schemes over S) which makes the extension of $\mathfrak{C}$ to such a scheme S_1 to be viewed as an object (resp. as a p-divisible object) of $\mathcal{MF}^{\nabla}_{[0,1]}(S_1)$. Moreover the fibres of $M(\mathfrak{C})$ over points with values in fields of the special fibre of S are not empty. In the case of an object of $\mathcal{MF}_{[0,1]}(S)$, these fibres are étale schemes over fields defined by algebras of dimension (as vector spaces over these fields) a power of p.*

Proof: Working with "potentially to be viewed" instead of "to be viewed" this is a direct consequence of 3.4.8.4 1) and 3.4.8.1 (with $l = 1$), cf. also the extra feature below. But 3.4.18.4.1 allows us to drop the word "potentially".

When we pass from p-divisible objects to objects, there is an extra feature showing up, which is worth being pointed out. If $S = \mathrm{Spec}(R_1)$ is an affine scheme, if $\mathcal{C}$ is an object of $\mathcal{MF}_{[0,1]}(S)$, then the underlying R_1-module M_1 of $\mathcal{C}$ might not be a projective R_1/p^nR_1-module, for some $n \in \mathbb{N} \cup \{0\}$. So sometimes, when we pass from things mod p^n to things mod p^{n+1}, we get system of equations in less variables, and so of completely different type of what we got working mod p^m, $m \in \mathbb{N} \cup \{0\}$, $m \leq n$. But the situation is the same: we still get system of equations of the first type. This is an easy consequence of the structure of the R_1-module M_1, cf. [Fa1, ii) of th. 2.1] (the proof in the loc. cit. works in the case of regular formally smooth schemes over $W(k)$).

The fact that for the case of an object of $\mathcal{MF}_{[0,1]}(S)$, we can work out the things just for S being a separated regular scheme (not necessarily affine) is a consequence of the fact that the things we obtain for an affine open cover of S can be glued together, cf. [Fa1, 2.3]. Here is the second place where we need $p > 2$. This ends the proof of the corollary.

As examples of such moduli schemes for connections, we have the schemes $\mathcal{S}^n$, $n \in \mathbb{N}$, of 3.4.8. As in Fact 2 of the proof of 3.4.18.4.1 we get:

3.4.18.4.3. Corollary (the touching property). *We assume that the special fibre of S is connected and that $\mathfrak{C}$ is an object of $\mathcal{MF}_{[0,1]}(S)$. Let p^m be the maximal number of points* $\mathrm{Spec}(k_1) \to M(\mathfrak{C})$ *lifting the same k_1-valued point of S, with k_1 an algebraically closed field containing k. Then any connecting component of $M(\mathfrak{C})$ intersects all the fibres of $M(\mathfrak{C})$ which are defined by étale algebras of dimension, as vector spaces over residue fields of the special fibre of S, equal to p^m.*

3.4.18.4.4. Remarks. 1) 3.4.18.4 remains true for the case when R/pR is a local strictly henselian ring and R is a regular formally smooth $W(k)$-algebra.

2) From now on, in connection to the categories $\mathcal{MF}_{[0,1]}(*)$, we do not need (cf. 3.4.18.4.1) to use anymore the word potentially (see 3.4.1.1.1).

From 3.4.18.4.2, the construction of the schemes $M(\mathfrak{C})$ (see 3.4.8), and 3.4.8.1.2 b) we get immediately:

3.4.18.4.5. Corollary (the constructibility property). *Under the assumptions of 3.4.18.4.2, if the special fibre of S is a noetherian scheme, then any connected component of the special fibre of $M(\mathfrak{C})$, with $\mathfrak{C}$ a p-divisible object, maps into a constructible dense subset of a connected component of the special fibre of S.*

3.4.18.4.6. Extra terminology. What follows in this section is an algebraic digression supplementing the first paragraph of 3.4.8.9. In that paragraph we defined systems of equations of the second type, while working with an additive type ring Frobenius. Similarly, if the ring Frobenius involved is of multiplicative (resp. essentially multiplicative or additive) type, we define systems of equations of multiplicative (resp. essentially multiplicative or additive) second type. In the proof of 3.4.18.4 we used such systems. The difference from what we got in 3.4.8.5 is: the coefficients of x_{isl}^p we get are not any more some $p-1$ power of a non-invertible element (see 3.4.8.5 (11)), but they are of a more general nature; in particular they can be invertible.

A. Let now $\bar{R}$ be an arbitrary $\mathbb{F}_p$-algebra. A system of equations of the form

$$(*) \qquad x_i = L_i(x_1^p, x_2^p, ..., x_n^p) + c_i,$$

$i = \overline{1,n}$, where L_i is a linear homogeneous form with coefficients in $\bar{R}$, and $c_i \in \bar{R}$, $\forall i \in \{1, ..., n\}$, is said to be of first type with coefficients in $\bar{R}$ (cf. terminology of 3.8.9);

we also call it a quasi Artin–Schreier system in n variables with coefficients in $\bar{R}$. It defines naturally an affine étale $\mathrm{Spec}(\bar{R})$-scheme (cf. 3.4.8.1). If this system defines a finite étale morphism (at the level of schemes), then we omit the word quasi. The $n \times n$ matrix obtained naturally (as in 3.4.18) from the coefficients of the linear forms L_i, $i = \overline{1, n}$, is called the matrix of the system $(*)$.

So (see 3.4.8) the special fibres of the schemes $M(\mathfrak{C})$ of 3.4.18.4.2 are obtained as (the $\mathbb{N}$-projective limit of) a sequence of morphisms defined by quasi Artin–Schreier systems; in the case of an object $\mathfrak{C}$ the sequence is finite, while in the case of a p-divisible $\mathfrak{C}$ it is infinite.

B. Artin–Schreier fundamental groups. Let $n \in \mathbb{N}$. We say that an étale cover $q_1 : \mathrm{Spec}(\bar{R}_1) \to \mathrm{Spec}(\bar{R})$ is crystalline elementary n-admissible if it is defined by an open closed subscheme of an étale cover of $\mathrm{Spec}(\bar{R})$ obtained using an Artin–Schreier system in at most n variables; here the "at most" part is motivated by the extra feature of the proof of 3.4.18.4.2. We say that q_1 is crystalline n-admissible if it is obtained through a finite sequence of consecutive crystalline elementary n-admissible étale covers.

We assume that $\mathrm{Spec}(\bar{R})$ is connected. The Galois group of the Galois pro-étale cover of $\mathrm{Spec}(\bar{R})$ generated by the connected components of crystalline n-admissible étale covers of $\mathrm{Spec}(\bar{R})$, is called the level n Artin–Schreier fundamental group of $\bar{R}$ or of $\mathrm{Spec}(\bar{R})$, and is denoted by $\Pi_n^{AS}(R)$. We have natural epimorphisms

$$\Pi_{n+1}^{AS}(\bar{R}) \twoheadrightarrow \Pi_n^{AS}(\bar{R}).$$

Variant: above we consider only finite sequences of length at most m, where $m \in \mathbb{N}$ is fixed; we obtain pro-finite Galois groups $\Pi_{n,m}^{AS}(\bar{R})$, and natural epimorphisms

$$\Pi_{n_1,m_1}^{AS}(\bar{R}) \twoheadrightarrow \Pi_{n,m}^{AS}(\bar{R}),$$

with $n_1, m_1 \in \mathbb{N}$, $n_1 \geq n$, and $m_1 \geq m$. We denote by $\Pi_\infty^{AS}(\bar{R})$ the projective limit of $\Pi_n^{AS}(\bar{R})$. It is a quotient of the first fundamental group of $\mathrm{Spec}(\bar{R})$ (in all above, we can fix a geometric point of $\mathrm{Spec}(\bar{R})$ in the usual way; not to complicate the notations we do not mention it explicitly). We call it the Artin–Schreier fundamental group of $\bar{R}$ or of $\mathrm{Spec}(\bar{R})$. We have:

C. Lemma (the fundamental lemma of the π_1-theory in positive characteristic). *If k_1 is a field (not necessarily perfect) containing $\mathbb{F}_p$, then $\Pi_\infty^{AS}(k_1)$ is exactly the Galois group of k_1.*

Proof: We can assume that k_1 is perfect (this is standard). Let k_2 be a finite Galois extension of k_1. Let $n_0 := [k_2 : k_1]$ and let $E := \{e_1, ..., e_{n_0}\}$ be an arbitrary basis of k_2 over k_1. Let σ_2 be the Frobenius automorphism of k_2. An arbitrary element $x \in k_2$ can be written uniquely as $x = \sum_{i=1}^{n_0} x_i e_i$, with $x_i \in k_1$, $i = \overline{1, n_0}$. We look at the equation $\sigma_2(x) = x$. This equation gets translated into an Artin–Schreier system of the form

$$x_i = L_i(x_1^p, ..., x_{n_0}^p),$$

$i = \overline{1, n_0}$, with L_i as linear homogeneous forms. The matrix A_E formed by the coefficients of these linear forms is invertible. So 3.4.8.1 and 3.4.8.1.0 (with $l = 1$) apply. We get that k_2 splits over the Galois extension of k_1 generated by the connected k-schemes obtained using this Artin–Schreier system. This ends the proof of the lemma.

D. The global context. In the above lemma the condition k_1 is a field can be weaken: we just need that any projective k_1-module of finite rank is free. In particular it holds for local rings. So to understand the first fundamental groups of arbitrary connected $\mathbb{F}_p$-schemes we just need to understand how to "glue" Artin–Schreier systems. This goes as follows.

The proof of C above suggests looking only at Artin–Schreier systems which are defined by homogeneous linear forms whose coefficients are producing an invertible matrix. Referring to the proof of C, the change of the basis E, results in a replacement of A_E by $B^{[p]} A_E B^{-1}$, where $B \in GL_{n_0}(k_1)$, and where $B^{[p]}$ is the matrix whose entries are the p-th power of the entries of B. In general, two invertible matrices A_1 and A_2 of rank n with coefficients in $\bar{R}$, are called F-equivalent, if there is an invertible matrix B with coefficients in $\bar{R}$ such that $A_2 = B^{[p]} A_1 B^{-1}$, with $B^{[p]}$ defined similarly.

Now for an arbitrary separated connected $\mathbb{F}_p$-scheme X_p we can define a Galois–Artin–Schreier (or an Artin–Schreier homogeneous invertible) system in n variables over X_p as follows: there is an open cover of X_p by affine schemes $X_p^i = \mathrm{Spec}(R^i)$, $i \in \mathcal{I}$, and there invertible $n \times n$ matrices A^i with coefficients in R^i, such $\forall i, j \in \mathcal{I}$, the matrices A_i and A_j are F-equivalent over $\mathrm{Spec}(R^{ij}) := X_p^i \cap X_p^j$. The equivalence of two such Galois–Artin–Schreier systems in n variables over X_p is defined in the standard manner. As above, we can use them to define pro-finite Galois groups $\Pi_n^{GAS}(X_p)$, $n \in \mathbb{N}$, as well as their projective limit $\Pi_\infty^{GAS}(X_p)$ (obtained by identifying an $n \times n$ matrix with entries $(a_{ij})_{i,j \in \{1,\dots,n\}}$ with a $n+1 \times n+1$ matrix whose entries $(b_{ij})_{i,j \in \{1,\dots,n+1\}}$ are defined by $b_{ij} = a_{ij}$ and $b_{n+1,i} = b_{j,n+1} = 0$ if $i, j \leq n$, and by $b_{n+1,n+1} = 1$). Following the proof of C, we get directly:

Lemma (the fundamental lemma of the π_1-theory in positive characteristic: the global form). *The fundamental group of X_p is $\Pi_\infty^{GAS}(X_p)$.*

E. Remarks. 1) From the proof of C above we deduce that $\mathrm{Gal}(k_2/k_1)$ is naturally a quotient of $\Pi_{n_0,1}^{AS}(k_1)$. It is very much desirable to replace here the lower index $n_0, 1$ with another one $n, 1$, with n as small as possible. We do not know how to get better upper bounds for such an n. Of course, if $n_0 \geq 2$, we can replace it by $n_0 - 1$, by using 3.4.8.1.5 and the fact that 1, x, x^p,...$x_{p^{n_0-1}}$ are linearly dependent over k_1, for any $x \in k_2$; in this way we can obtain a second proof of C (but not of D).

2) Let $l \in \mathbb{N}$. Using p^l-powers instead of p-powers (as in 3.4.8.1) we define similarly groups $\Pi_{n,m,l}^{AS}(\bar{R})$. Obviously (see 3.4.8.1.5) we have a natural epimorphism:

$$\Pi_{nl,m}^{AS}(\bar{R}) \twoheadrightarrow \Pi_{n,m,l}^{AS}(\bar{R}).$$

3) There are $n(n+1)$ coefficients involved in the definition of a quasi Artin–Schreier system in n variables. Of course they are entirely independent. One can learn a lot by

looking at Artin–Schreier systems in a small number of variables, like 2 or 3. For $n=2$ we have 6 coefficients, while for $n=3$ we have 12.

4) If a quasi Artin–Schreier system with coefficients in $\bar{R}$ defines a Galois cover of an open dense subscheme of $\mathrm{Spec}(\bar{R})$, then the étale scheme over $\mathrm{Spec}(\bar{R})$ naturally associated to it is an open subscheme of a Galois cover of $\mathrm{Spec}(\bar{R})$, cf. 3.4.8.1.

F. Definition. The Artin–Schreier dimension of $\bar{R}$, denoted $d^{AS}(\bar{R})$, is the smallest number $q \in \{0\} \cup \mathbb{N} \cup \{\infty\} \cup \{\infty+1\}$ such that $\Pi_q^{AS}(\bar{R})$ is the whole fundamental group of $\mathrm{Spec}(\bar{R})$ (here $\Pi_0^{AS}(\bar{R})$ is the trivial group, while $\Pi_{\infty+1}^{AS}(\bar{R})$ is the fundamental group of $\mathrm{Spec}(\bar{R})$.

G. Exercise. Compute the level n Artin–Schreier fundamental groups and the Artin–Schreier dimensions of algebraic field extensions of $\mathbb{F}_p$.

H. Artin–Schreier systems in mixt characteristic. We will not stop to present here a full theory: we will just be interested to point out the ideas. Let O be a DVR of characteristic zero having k as its residue field. Let R be an integral regular faithfully flat formally smooth O-algebra. We assume that the smallest open subscheme of Spec containing $\mathrm{Spec}(R \otimes_O k)$ is $\mathrm{Spec}(R)$ itself. We situate ourselves to the context of A above, with $c_i \in R$ and L_i having coefficients in R, $\forall i \in \{1,...,n\}$. We assume that the matrix of coefficients of L_i's is invertible. Then, as usual, we get an étale cover of R. We can imitate B above, and we can define the Artin–Schreier fundamental groups $\Pi^{AS}(R)$ of R. We do believe that, in this way, one can learn a lot about the Galois groups of number fields. This is our first approach towards the understanding of finite quotients of such Galois groups.

I. Artin–Schreier systems in unequal positive characteristic. H above suggest treating the situation where we deal with a system (*) as in A above (involving invertible matrices), but over affine schemes which are of characteristic $l \neq p$. Of course, the first thing to be done is to understand criteria when such systems (defining finite flat schemes) do define étale covers.

We hope to come back in a future paper with the main (expected or known to be true) properties of these Artin–Schreier fundamental groups and dimensions. We end my mentioning that we do believe that there is a deep connection between I above, and [Bu2] and the theory of ordinary reductions of a proper scheme over a number field.

3.4.18.5. Theorem (the inducing property). *The answer to the question* **Q** *is yes if one of the following conditions is satisfied:*

a) $k=\bar{k}$*;*

b) Φ_R *is essentially of additive type;*

c) the Newton polygon of (M,φ) *does not have either the slope 0 or the slope 1.*

Proof: This is a trivial consequence of 3.4.18.3-4.

3.4.18.5.1. Corollary. *We assume that the Frobenius* Φ_R *is essentially of additive type. Then we have an equivalence of categories between the category of p-divisible objects of* $\mathcal{MF}_{[0,1]}(R)$ *and the category of p-divisible objects of* $\mathcal{MF}_{[0,1]}^{\nabla}(R)$*, and we have an* *antiequivalence of categories between these two categories and the category of p-divisible groups over R.*

Proof: The antiequivalence part is implied by the equivalence part and by 3.4.18.5: we can change the Frobenius Φ_R (cf. [De2, (1.1.2.1)]) to one of additive type $\Phi_R(z_i) = z_i^p$; so the Grothendieck-Messing theory (see [Me] and [BBM) and [Fa2, th. 10] apply. In other words, we construct uniquely, starting from a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$, by induction on $n \in \mathbb{N}\cup\{0\}$, a p-divisible group over $R/(z_1, ...z_m)^n$; moreover this construction is functorial.

For the equivalence part, from 3.4.18.4 (we have $r = 0$) we deduce that there is only one way to associate to a p-divisible object of $\mathcal{MF}_{[0,1]}(R)$ a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$. So we just need to check that this association is well behaved with respect to maps (morphisms) between p-divisible objects. Everything boils down to (keeping the previous notations): if $f \in F^0(\mathrm{End}(M)) \otimes_{W(k)} R$ is fixed by Φ, then it is annihilated by the (unique) connection ∇ on $M \otimes_{W(k)} R$ for which Φ is ∇-parallel. Let $\nabla(f) = f_1 \in \mathrm{End}(M) \otimes_{W(k)} \bar{\Omega}_{R/W(k)}$. It satisfies the equation

$$f_1 = p\Phi \circ d\Phi_{R*}/p(f_1) = \Phi \circ d\Phi_{R*}(f_1).$$

We need to show that $f_1 = 0$. It is enough to show that f_1 mod p is zero. But the above equation mod p defines a system of equations of the first type. As in the proofs of 3.4.1.2 and of 3.4.18.4 (see also 3.4.18.8), due to the fact that Φ_R is essentially of additive type, we get that f_1 mod p is zero. This ends the proof of the corollary.

3.4.18.5.2. Corollary (the liftability property). *We assume that $k = \bar{k}$ or that Φ_R is essentially of additive type. Let $n \in \mathbb{N}$ and let $\mathfrak{C}$ be an object (resp. a p-divisible object) of $\mathcal{MF}_{[0,1]}(R)$. Then any connection which makes $\mathfrak{C}/p^n\mathfrak{C}$ to be an object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$, can be lifted (not necessarily uniquely for the case $k = \bar{k}$) to a connection making $\mathfrak{C}$ to be an object (resp. a p-divisible object) of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$.*

Proof: This is a consequence of the proof of 3.4.18.4.2.

3.4.18.5.3. Corollary. *We assume that the Frobenius lift Φ_R is essentially of additive type. Then we have an equivalence of categories between $\mathcal{MF}_{[0,1]}(R)$ and $\mathcal{MF}^{\nabla}_{[0,1]}(R)$, and we have an antiequivalence of categories between these two categories and the category of finite flat commutative group schemes of p-power rank over R.*

Proof: The equivalence part is obtained as in the proof of 3.4.18.5.1. For the antiequivalence part, as we have 3.4.18.5.1, we just have to remark two things. First we need to use the second paragraph of 3.4.18: any object of $\mathcal{MF}_{[0,1]}(R)$ lifts to a p-divisible object of $\mathcal{MF}_{[0,1]}(R)$. Second any finite flat commutative group scheme over R of rank a power of p, is a closed subgroup of a p-divisible group over R (for instance, cf. [Fa1, proof of 7.1], Artin's approximation theorem and the liftability property of 3.4.18.5.2). The rest is as in loc. cit. This proves the corollary.

3.4.18.5.4. Remark. It is easy to see that the results 3.4.18.5.1-3 extend to a more general context, inspired from 3.4.18.4.2 (cf. also 3.4.18.4.4). We leave this to the reader (to be compared to 3.4.18.8). 3.4.18.5.1 and 3.4.18.5.3 are the third place where we need $p > 2$.

3.4.18.5.5. The case of categories $\mathcal{MF}_{[a,b]}(R)$. A natural question arises. What about the categories $\mathcal{MF}_{[a,b]}(R)$, with $a, b \in \mathbb{Z}$, $b > a+1$? In what follows, if the reader desires, can assume $a = 0$.

A. First of all let us point out that as in 3.2.3 and 3.1.1.1 we can speak about a generic situation. The second philosophy of 3.4.18.2 says that it is enough to look at the case of such a generic situation. The conclusion is: due to the reasons explained in 3.4.8.9, none of the results 3.4.18.4-5 above remain true (in general), except one. We start explaining this exception.

Let $\mathfrak{C}$ be a p-divisible object of $\mathcal{MF}_{[a,b]}(R)$. As in 3.4.18.1 we define numbers $f(i) \in \mathbb{N} \cup \{0\}$, $\forall i \in \{a, ..., b\}$. Then the number of connections that make $\mathfrak{C}/p^n\mathfrak{C}$ potentially to be viewed as an object of $\mathcal{MF}^{\nabla}_{[a,b]}(R)$ is at most

$$p^{nm \sum_{i=a}^{b-1} f(i)f(i+1)},$$

cf. 3.4.8.9 and the estimate of m_1 in 3.4.8.1. Moreover this number is attained for the generic situation.

But if $m \geq 2$ and the sum

$$\sum_{i=a}^{b-2} f(i)f(i+1)f(i+2)$$

is not zero, not all of them are integrable, and this creates problems as well as beauty. Of course, if $f(i)f(i+1)f(i+2) = 0$, $\forall i \in \{a, ..., b-2\}$, then all these connections are integrable.

Let

$$N(n, m, f(a), f(a+1), ..., f(b))$$

be the number of integrable connections (mod p^n) for the generic situation (modelled on $\mathfrak{C}$). Despite the fact that none of the others results remain true in general, many of them have weaker forms, which are also very useful. We explain them one by one.

B. The role of $s(0)s(1)$ in 3.4.18.4 has to be replaced by the number $s(-1)$ of slopes -1 of $\mathrm{End}(\mathfrak{C})$. As in the proof of 3.4.18.4, we get that the number $\mathrm{CONN}(\mathfrak{C}, n)$ of connections which make $\mathfrak{C}/p^n\mathfrak{C}$ potentially to be viewed as an element of $\mathcal{MF}^{\nabla}_{[a,b]}(R)$, is at most

$$p^{nrs(-1)}.$$

Even if $k = \bar{k}$ it can happen that actually we have less than predicted such connections. We do not know when $\mathrm{CONN}(\mathfrak{C}, n) = p^{nf(\mathfrak{C})}$, for any $n \in \mathbb{N}$. Warning: in general there is no such constant $f(\mathfrak{C}) \in \mathbb{N} \cup \{0\}$.

C. Similarly we define the number $\mathrm{INT}(\mathfrak{C}, n)$ counting the integrable connections. It seems to us that we always have $\mathrm{INT}(\mathfrak{C}, n) \leq N(n, m, f(a), f(a+1), ..., f(b))$, but we do know how to prove this using the ideas of the proof of 3.4.18.8.1. However, the mentioned ideas give us immediately: If $\mathrm{CONN}(\mathfrak{C}, n) = p^{nm \sum_{i=a}^{b-1} f(i)f(i+1)}$, then $\mathrm{INT}(\mathfrak{C}, n) = N(n, m, f(a), f(a+1), ..., f(b))$; of course we can reobtain this by remarking (based on [De2, 1.2.4] and standard induction; cf. also [De2, 1.3.2]) that we are actually in the context of the generic situation.

D. 3.4.18.4.2 has a weaker version. We use its notations, but replacing the lower right index $[0,1]$ everywhere by $[a,b]$. For $m \geq 2$ we have two moduli schemes: $M(\mathfrak{C})$ of integrable connections and $M_1(\mathfrak{C})$ of connections; this is a consequence of 3.4.8.9 (15) and (16). Always $M(\mathfrak{C})$ is an open closed subscheme of $M_1(\mathfrak{C})$. But we can not say anything about their fibres. In particular they can be empty. Also, in case of an object, there is no a priori reason to have fibres with a number of elements which is a power of p.

E. In accordance to D, if $M_1(\mathfrak{C})$ or $M(\mathfrak{C})$ is not empty and the special fibre of S is noetherian and connected, then the constructibility property of 3.4.18.4.5 is preserved. Obviously, in the same context, the purity property of 3.4.8.1.4, the touching property of 3.4.18.4.3, and the surjectivity principle of 3.4.18.6 b) below are preserved. All these can be read out from 3.4.8.9 (15) and (16).

F. The inducing property is a much subtler thing, as it can happen to have $\mathrm{INT}(\mathfrak{C}, n) = 0$ for some $n \in \mathbb{N}$. We do not know any simple and sufficiently general criteria which could guarantee that $\mathrm{INT}(\mathfrak{C}, n) \geq 1$, $\forall n \in \mathbb{N}$.

3.4.18.5.6. Problem. With the notations of 3.4.18.0-2, if $r \geq 1$, study the behaviour of connections (obtained as in 3.4.18.4) with respect to exact (short) sequences of objects.

3.4.18.6. Theorem. *Let now* $\mathrm{Spec}(R_1)$ *be a formally étale affine scheme over a smooth scheme over* $W(k)$*. We assume that its special fibre is connected, and that it is part of a triple* $(\mathrm{Spec}(R_1), \Phi_{R_1}, y)$ *defining a potential-deformation sheet. Let* $\mathfrak{C}_1$ *be a* p*-divisible object of* $\mathcal{MF}_{[0,1]}(R_1)$.

a) (the universal global ∇ principle for $\mathcal{MF}_{[0,1]}(*)$). *The whole of 3.4.1.3 remains true (except the fact that we get connections respecting the* G*-action which does not make any sense here) in this context. In particular* $\mathfrak{C}_1$ *becomes a* p*-divisible object of* $\mathcal{MF}^{\nabla}_{[0,1]}(Q_1)$*, with* Q_1 *a* $W(k)$*-algebra having a connected special fibre, and such that there is a morphism* $d_1 : \mathrm{Spec}(Q_1^{\wedge}) \to \mathrm{Spec}(R_1^{\wedge})$*, which is the* p*-adic completion of an* $\mathbb{N}$*-pro-étale morphism* ℓ_1*. Moreover the point* y *lifts to a point* $\mathrm{Spec}(W(k)) \to \mathrm{Spec}(Q_1^{\wedge})$.

b) (the surjectivity principle) *The fibres of* ℓ_1 *over an open subscheme of the special fibre of* $\mathrm{Spec}(R_1)$ *containing the* k*-valued point defined by* y*, are not empty.*

The same remains true for objects: we just need to replace the words $\mathbb{N}$*-pro-étale by the word étale (the b) part becomes trivial).*

The proof of this is entirely the same as the proof of 3.4.1.3 (cf. 3.4.18.4.1 for the integrability part), and so it is omitted.

3.4.18.7. The relative situation. 3.4.18.4-6 works in the relative context, i.e. when we start with a Shimura filtered σ-crystal, or even more generally with a filtered

σ-$\mathcal{S}$-crystal of $\mathcal{MF}_{[0,1]}(W(k))$. So here we assume that there is a smooth subgroup $\tilde{G}$ of $GL(M)$ having connected fibres and the property that there is a family $(v_\alpha)_{\alpha\in\mathcal{J}}$ of the F^0-filtration of $\mathcal{T}(M[\frac{1}{p}])$ fixed by φ and such that $\tilde{G}_{B(k)}$ is the subgroup of $GL(M[\frac{1}{p}])$ fixing its members. We have:

Fact. *If $g_R \in \tilde{G}(R)$, then there is an integrable connection ∇ on $M\otimes_{W(k)} R$, which respects the $\tilde{G}$-action and makes $\mathfrak{C}$ to be viewed as a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$. Moreover, this p-divisible object is induced from a filtered F-$\mathcal{S}$-crystal of the form $(M,F^1,\varphi,\tilde{G},\tilde{f})$ (defined as in 2.2.10).*

Proof: This is a consequence of 3.4.18.4.1 and [Fa2, rm, iii) after th. 10] (to be compared with 3.4.18.3), once we show that there is a connection on $M\otimes_{W(k)} R$ which makes $\mathfrak{C}$ to be viewed as a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$ and which respects the $\tilde{G}$-action. To see this, following the ideas of 3.4.8 4-7) and 11-12), we get that it is enough to show that there is a connection on $M\otimes_{W(k)} R/pR$ which makes $\mathfrak{C}/p\mathfrak{C}$ to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$ (i.e. the liftability property works in this relative context also, cf. the paragraph below).

We need to start with a conection $\nabla=\delta+\beta$, where δ is the connection on $M\otimes_{W(k)} R/pR$ annihilating M, and where

$$\beta\in LIE:=\mathrm{Lie}(\tilde{G})\otimes_{W(k)}\Omega_{(R/pR)/W(k)}.$$

The equations of 3.4.1.1.1 2), in this case, express the equality between two elements of LIE. So we get a system of $m\dim((\mathrm{Lie}(\tilde{G}))$ equations in the same number of variables; the resulting system of equations is of the first type, and so 3.4.8.1 applies. This ends the proof of the fact.

Warning: not all connections making $\mathfrak{C}/p\mathfrak{C}$ to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$ respect the $\tilde{G}$-action; simple examples are provided by the generic situation 3.4.18.2. However we have:

3.4.18.7.0. Formula. *The number of connections that respect the $\tilde{G}$-action and make $\mathfrak{C}/p^n\mathfrak{C}$ to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$ is p to the power nr times the number of slopes -1 of the Lie σ-crystal* $(\mathrm{Lie}(\tilde{G}),\varphi)$.

The proof of this formula, is entirely the same as the proof of 3.4.18.4 B).

3.4.18.7.1. An estimate. In the case of the filtered σ-$\mathcal{S}$-crystal of 3.4.1.5, denoting it by $\mathfrak{C}$, we similarly get that the number of connections that respect the $\tilde{G}$-action and make $\mathfrak{C}/p^n\mathfrak{C}$ potentially to be viewed as an object of $\mathcal{MF}^{\nabla}_{[a,b]}(R)$ is at most p to the power nr times the number of slopes -1 of the Lie σ-crystal $(\mathrm{Lie}(\tilde{G}),\varphi)$.

3.4.18.7.2. It is worth detailing what 3.4.18.4.6 becomes in the relative context. There are two variants: one modelled on the faithful representation ρ of $\tilde{G}_k$ on M/pM, and another one modelled on $(M,F^1,\varphi,\tilde{G})$. We first detail the first one, then we hint the idea of the second one.

Let $\tilde{m} := \dim(M)$. Let $\bar{R}$ be a k-algebra. In the proof of 3.4.18.4.6 C we came across Artin–Schreier systems in n variables ($n \in \mathbb{N}$) with coefficients in $\bar{R}$, whose $n \times n$ matrix A_L is invertible. So A_L defines an element of $GL_n(\bar{R})$. If $\tilde{m} = n$, and if $A_L \in \tilde{G}_k(\bar{R})$ (under the representation ρ), then we say that such an Artin–Schreier system is modelled on ρ, and any connected component of the $\mathrm{Spec}(\bar{R})$-scheme defined by it is said to be a crystalline elementary ρ-admissible étale cover of $\mathrm{Spec}(\bar{R})$. As in 3.4.18.4.6 we define crystalline ρ-admissible étale covers of $\mathrm{Spec}(\bar{R})$, as well as Galois groups

$$\Pi_{\tilde{m},n}^{\rho-AS}$$

and

$$\Pi_{\tilde{m}}^{\rho-AS},$$

by using only Artin–Schreier systems which are modelled on ρ. Warning: all the Artin–Schreier systems modelled on ρ are in precisely $\tilde{m}$ variables. We refer to $\Pi_{\tilde{m}}^{\rho-AS}$ as the ρ Artin–Schreier fundamental group of $\mathrm{Spec}(\bar{R})$ (the lower right index $\tilde{m}$ just having the role of keeping track of the dimension of the faithful representation ρ). Similarly, following the pattern of 3.4.18.4.6, we can define Galois–Artin–Schreier fundamental groups $\Pi_n^{\rho-GAS}(X_p)$ and $\Pi_n^{\rho-GAS}(X_p)$, for any separated connected $\mathbb{F}_p$-scheme X_p.

The most interesting cases are when ρ is an irreducible representation. For instance, we would like to understand the case when ρ is the spinorial (resp. semi-spinorial) representation of a split semisimple group over k of B_l (resp. of D_l) Lie type, with $l \in \mathbb{N}$, $l \geq 3$ (resp. $l \in \mathbb{N}$, $l \geq 4$).

The second variant imitates the first variant as well as the case described in the equations (3) and (4) of 3.4.18; in other words we require that some specific entries of A_L are zero, without requiring that A_L is invertible or that it defines an element of $\tilde{G}_k$. We intend to come back in a future paper to detail this second variant, as we think it is equally important as the first one.

3.4.18.8. Theorem (the universal unicity principle). *Let $\mathrm{Spec}(R_1)$ be a formally étale affine scheme over a smooth scheme over $W(k)$ such that $\mathrm{Spec}(R_1/pR_1)$ is a connected scheme. Let Φ_{R_1} be a Frobenius of R_1 compatible with the Frobenius σ of $W(k)$, and such that there is a maximal ideal m_{R_1} of R_1, with R_1/m_{R_1} a finite field extension of k, and with the property that the resulting Frobenius of the completion $\widehat{R_1}$ of R_1 with respect to it is essentially of additive type. Then for any object (or p-divisible object) $\mathfrak{C}_1$ of $\mathcal{MF}_{[\tilde{a},\tilde{b}]}(R_1)$, there is at most one connection which makes this object (resp. this p-divisible object) potentially to be viewed as an object (resp. as a p-divisible object) of $\mathcal{MF}_{[\tilde{a},\tilde{b}]}^{\nabla}(R_1)$ (here $\tilde{a}, \tilde{b} \in \mathbb{Z}$, with $\tilde{a} < \tilde{b}$).*

The proof of this is the same as the proof of 3.4.1.2: We work inductively modulo p^n, $n \in \mathbb{N}$. The difference from 3.4.8 6) is that we need to apply the same upper triangular trick used in the proof of 3.4.18.4 (for the resulting Frobenius of $\widehat{R_1}$) for "separating" the variables. Warning: as we can have $\tilde{b} \geq \tilde{a} + 2$, we need to keep track of the word "potentially", cf. 3.4.18.5.5.

3.4.18.8.1. Theorem (the universal slope type unicity principle). *Let* $\mathrm{Spec}(R_1)$ *be a formally étale affine scheme over a smooth scheme over* $W(k)$ *such that* $\mathrm{Spec}(R_1/pR_1)$ *is a connected scheme. Let* Φ_1 *be a Frobenius of* R_1*. Let* $m_1 : R_1 \twoheadrightarrow W(k_1)$ *be a fixed* Φ_1*-Teichmüller lift* $W(k)$*-epimorphism, with* k_1 *an algebraic field extension of* k*.*

a) Let $\mathfrak{C}_1$ *be a p-divisible object of* $\mathcal{MF}_{[0,1]}(R_1)$ *such that the p-divisible object* $m_1(\mathfrak{C}_1)$ *of* $\mathcal{MF}_{[0,1]}(W(k_1))$ *obtained from it through this* $W(k)$*-epimorphism does not have slope zero or slope one. Then for any object (or p-divisible object)* $\mathfrak{C}_1'$ *of* $\mathcal{MF}_{[0,1]}(R_1)$ *which mod p is isomorphic to* $\mathfrak{C}_1/p\mathfrak{C}_1$*, there is at most one connection which makes it to be an object (resp. a p-divisible object) of* $\mathcal{MF}_{[0,1]}(R_1)$*.*

b) Let $\mathfrak{C}_1$ *be a p-divisible object of* $\mathcal{MF}_{[\tilde{a},\tilde{b}]}(R_1)$ *such that the "End" of the p-divisible object* $m_1(\mathfrak{C}_1)$ *of* $\mathcal{MF}_{[\tilde{a},\tilde{b}]}(W(k_1))$ *obtained from it through this* $W(k)$*-epimorphism does not have slope* -1*. Then for any object (or p-divisible object)* $\mathfrak{C}_1'$ *of* $\mathcal{MF}_{[\tilde{a},\tilde{b}]}(R_1)$ *which mod p is isomorphic to* $\mathfrak{C}_1/p\mathfrak{C}_1$*, there is at most one connection which makes it potentially to be an object (resp. a p-divisible object) of* $\mathcal{MF}_{[\tilde{a},\tilde{b}]}(R_1)$*.*

Proof: This is a consequence of the proof 3.4.18.4: everything (see the second paragraph of 3.4.18 and the proof of 3.4.18.4.2) gets reduced to the case of p-divisible objects. We just need to remark: in case $\mathfrak{C}_1'$ is just an object, it is obtained by using a sequence of short exact sequences, from $\mathfrak{C}_1/p\mathfrak{C}_1$ and some of its subobjects. The fact that $m_1(\mathfrak{C}_1)$ (resp. its "End") does not have slope 0 or 1 (resp. slope -1), can be read out from its truncation $\mathcal{T}$ mod p; so what we know for $\mathcal{T}$ remains true for all its subobjects. This proves the theorem.

3.4.18.8.2. Remark. We refer to 3.4.18.6. As usual we can speak about the p-rank of $\mathfrak{C}_1/p\mathfrak{C}_1$ in any maximal (or geometric) point of $\mathrm{Spec}(R_1/pR_1)$. The p-rank is zero in such a point iff the F-crystal we get over a point of $\mathrm{Spec}(Q_1/pQ_1)$ lifting it does not have slope zero. So if $\mathfrak{C}_1$ has p-rank zero with respect to any maximal point of $\mathrm{Spec}(R_1/pR_1)$ then we do not need to pass to $\mathbb{N}$-pro-étale covers (or in the case of an object to étale covers). This is a consequence of 3.4.18.4 and of 3.4.18.4.2. Similarly, we can "detect" the number of slopes one by just looking at $\mathfrak{C}_1/p\mathfrak{C}_1$; their number is equal to the p-rank of $(\mathfrak{C}/p\mathfrak{C})^*(1)$.

3.4.18.8.3. Remark. 3.4.1.2-3 remain true for any Frobenius Φ_{R_j} of R_j such that $\Phi_{\hat{R}_j^0}$ is essentially of additive type. This gives us more freedom: for instance, we can always adjust the things (in a way similar to the algebraization process of the proof of 3.4.18.4.1) so that in 3.4.2 we "end up" over k (i.e. the morphisms of 3.4.2.1 b) and c) are defined over k and not only over $\bar{k}$).

For an arbitrary Frobenius Φ_{R_j}, we lose the unicity part, so 3.4.1.2-3 have to be modified to a significant extend (for instance not all the connections we get are G-invariant, $\ell_{j,n}^{-1}(\mathcal{Z}_k)$ is not necessarily $\mathrm{Spec}(k)$, etc.). We do not stop to rewrite what remains true out of 3.4.1.2-3, in this general context. We will just point out that we still have a universal property with respect to G-invariant connections, but the corresponding p-adically complete moduli $R_j^\wedge$-schemes, can have non-connected special fibres.

3.4.18.9. Some estimates. Let R_1 be a regular formally smooth $W(k)$-algebra. We assume that $Y_1 := \mathrm{Spec}(R_1/pR_1)$ is connected and that the only open subscheme of Y_1 containing all the $\bar{k}$-valued points of Y_1, is Y_1 itself. We also assume the existence of a Frobenius Φ_{R_1} of $R_1^\wedge$ compatible with σ. Let $\mathcal{C}$ be an object of $\mathcal{MF}_{[0,1]}(R_1)$. We denote by $\nabla(\mathcal{C})$ the number of connections which make $\mathcal{C}$ to be viewed as an object of $\mathcal{MF}^\nabla_{[0,1]}(R_1)$.

Let $y \in Y_1(\bar{k})$. Either from the very definitions of 3.4.18.4.0-2, or from 3.4.18.4, we deduce that a similar number $r_y \in \mathbb{N} \cup \{0\}$ as in 3.4.18.1, can be defined starting just from Φ_{R_1}. As in 3.4.18.1 we define uniquely numbers $s_y(0), s_y(1) \in \mathbb{N} \cup \{0,1\}$. Let $n \in \mathbb{N} \cup \{0\}$ be the smallest number such that $p^n\mathcal{C} = 0$. Let

$$d_y(\mathcal{C}) := nr_ys_y(0)s_y(1).$$

From 3.4.18.4 we get directly:

$$\nabla(\mathcal{C}) \leq min\{d_y(\mathcal{C}) | y \in Y_1(\bar{k})\}.$$

From 3.4.18.4.2 and 3.4.8.1.2 a) we get directly:

Fact. *If the number $r_ys_y(0)s_y(1)$ does not depend on $y \in Y_1(\bar{k})$, and if the underlying module of $\mathcal{C}$ is a projective R_1/p^nR_1-module, then the special fibre of the moduli scheme $M(\mathcal{C})$ is an étale cover of Y_1.*

3.4.18.10. p-divisible objects of bounded ∇-deviations. We assume that R_1, Y_1 and Φ_{R_1} are as in 3.4.18.9. Warning: we have no restriction on the "shape" of Φ_{R_1}. Let $\mathcal{C}$ be a p-divisible object of $\mathcal{MF}_{[0,1]}(R_1)$. We say that $\mathcal{C}$ is of bounded (resp. totally bounded) ∇-deviation if at least one connected component (resp. if all connected components) of the special fibre of $M(\mathcal{C})$ is (resp. are) of finite type over Y_1. The interesting cases are when Y_1 is of finite type over k, or when $R_1 = W(k)[[z_1, ..., z_m]]$, with $k \neq \bar{k}$. Plenty of examples of p-divisible objects of bounded ∇-deviation can be obtained following the strategy of 3.4.8.9.1. Similarly, using 3.4.18.8-9, we get plenty of examples of p-divisible objects of totally bounded ∇-deviation. Two problems are dear to us.

P1. In the interesting cases mentioned above, determine all p-divisible objects of bounded ∇-deviation.

P2. Determine conditions under which, in the case when R_1 is a smooth $W(k)$-algebra, a p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(R_1)$:

a) comes from a p-divisible group over R_1 (not only from one over $R_1^\wedge$);

b) viewed as an object of $\mathcal{MF}_{[0,1]}(R_1)$ is of totally bounded ∇-deviation.

3.4.19. The gluing principle. 3.4.14 and 3.4.14.1 are particular cases of a more general principle pertaining to Shimura p-divisible groups, which we call the gluing principle. However, we singled them out, as they are in the context of abelian varieties, are the most important cases needed in applications, and moreover we had to give a motivation for the expectation of 3.4.15.

A. Let $\mathrm{Spec}(R^{i\wedge})$ be the p-adic completion of a pro-étale affine scheme $\mathrm{Spec}(R^i)$ over a smooth scheme over $W(k)$, $i \in \overline{1,2}$. Let $z^i : \mathrm{Spec}(W(k)) \to \mathrm{Spec}(R^i)$, and let

$$\mathcal{D}^i = (D^i, (v^i_\alpha)_{\alpha \in \mathcal{J}})$$

be a Shimura p-divisible group over $\mathrm{Spec}(R^{i\wedge})$. We denote the p-adic completion of z^i still by z^i. We assume that:

i) the two Shimura p-divisible groups over $W(k)$ obtained from $\mathcal{D}^1$ and $\mathcal{D}^2$ through z^1 and respectively through z^2 are isomorphic;

ii) the relative dimension d_1 of $\mathrm{Spec}(R^1)$ in z^1 is bigger or equal to the relative dimension d_2 of $\mathrm{Spec}(R^2)$ in z^2; moreover $\mathcal{D}^i$ is a versal deformation in z^i, $i = \overline{1,2}$;

iii) $\mathrm{Spec}(R^i/pR^i)$ is a connected scheme, $i = \overline{1,2}$.

Here the versality condition is defined as follows.

B. Definitions. Let $\mathrm{Spec}(R)$ be a formally étale scheme over a smooth scheme over $W(k)$. Let Φ_R be a Frobenius of $R^\wedge$ compatible with σ. A Shimura p-divisible group $\mathcal{D}_R$ over $\mathrm{Spec}(R)$ is said to be a versal deformation, if for each point $y_R : \mathrm{Spec}(k_1) \to \mathrm{Spec}(R/pR)$, with k_1 an algebraically closed field, the dimension of the image of the Kodaira-Spencer map attached to $\mathcal{D}_R$ (and so computed relative to $W(k)$) in this point is equal to the deformation dimension of the Shimura σ_1 crystal we get naturally over k_1 through this point.

Similarly, an object or a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$ is said to be a quasi-versal deformation, if the Kodaira-Spencer map in each point y_R as above is injective, having an imagine whose dimension depends only on the connected component of $\mathrm{Spec}(R)$ through which y_R factors.

C. If $\mathcal{D}_i$ has a principal quasi-polarization $\mathcal{P}_{\mathcal{D}_i}$, then we say that we are in a principally quasi-polarized context; if we are in such a context then we assume that the isomorphism of i) respects the principal quasi-polarizations involved.

D. Let (M, F^1, φ, G) be the Shimura filtered σ-crystal associated to the two Shimura p-divisible groups obtained in i). Let $\mathrm{Spec}(\widehat{R^1})$ (resp. $\mathrm{Spec}(R^{1h})$) be the completion of $\mathrm{Spec}(R^1)$ (resp. the henselization of $\mathrm{Spec}(R^1)$) in z^1. We still denote by z^i the resulting $W(k)$-valued point of $\mathrm{Spec}(\widehat{R^i})$ or of $\mathrm{Spec}(R^{ih})$. We assume the existence of a formally smooth morphism $f_{12} : \mathrm{Spec}(\widehat{R^1}) \to \mathrm{Spec}(R^2)$ such that:

– $f_{12} \circ z^1 = z^2$;

– the image of R^2/pR^2 in $\widehat{R^1}/p\widehat{R^2}$ is contained in R^{1h}/pR^{1h};

– the pull back of $\mathcal{D}^2$ through f_{12} is isomorphic to the pull back of $\mathcal{D}^1$ through the natural morphism $\mathrm{Spec}(\widehat{R^1}) \to \mathrm{Spec}(R^1)$, through an isomorphism, lifting the one of i). Following the proofs of 2.3.15 and 3.4.14.1 we can state (cf. also 3.4.2.2):

Theorem (the gluing principle: the global form). *There is an affine scheme $\mathrm{Spec}(R^3)$ over $W(k)$, with $R^{3\wedge} = R^3$, and there is a formally étale morphism $q_1 : \mathrm{Spec}(R^3) \to \mathrm{Spec}(R^{1\wedge})$ and a formally smooth morphism $q_2 : \mathrm{Spec}(R^3) \to \mathrm{Spec}(R^{2\wedge})$, $i = \overline{1,2}$, and there is a morphism $a^3 : \mathrm{Spec}(W(k)) \to \mathrm{Spec}(R^3)$, such that:*

a) $q_i \circ a^3 = z^i$, $i = \overline{1,2}$;

b) The pull back of $\mathcal{D}^1$ through q_1 is isomorphic to the pull back of $\mathcal{D}^2$ through q_2, through an isomorphism lifting the one of i) (cf. a));

c) $\mathrm{Spec}(R^3/pR^3)$ *is a connected scheme; it is a pro-étale scheme over an affine smooth k-scheme.*

If R^i is a smooth $W(k)$-algebra ($i \in \{1, 2\}$), then we can also assume that:

d) the image of $\mathrm{Spec}(R^3/pR^3)$ *in* $\mathrm{Spec}(R^{3-i}/pR^{3-i})$ *contains an open dense subscheme of it.*

Proof: The proof of this theorem is contained in the proof 3.4.14.1. Only two extra things we need to add. First, [BLR, Theorem 12, p. 83] applies as well, due to the following fact.

Fact. *If X is a projective limit of quasi-compact separated étale schemes X_α, $\alpha \in \mathcal{J}(X)$, over a smooth scheme over $W(k)$, then the category $\mathcal{MF}^{\nabla}_{[0,1]}(X)$ is the projective limit of the categories $\mathcal{MF}^{\nabla}_{[0,1]}(X_\alpha)$. If all these schemes are affine, and if we fix Frobenius lifts compatible with σ and with the projective system, then the same thing can be said for the category $\mathcal{MF}(X)$.*

The proof of this fact is standard (see [EGA]), based on [Fa1, 2.1 (ii)] and on the fact that any finitely generated projective module over a ring is finitely presented.

The second thing we need to add is related to the fact that d_i can be bigger than the deformation dimension d of the Shimura p-divisible groups of i) above. But the refinement part of the proof of 3.4.14.1 applies entirely in this situation. Using these two additions we get immediately this theorem.

E. Variants. There are many variants of the gluing principle:

iv) we work in a context involving Shimura F-crystals instead of Shimura p-divisible groups;

v) we weaken the part of ii) above involving the versality condition;

vi) we work in a principally quasi-polarized context;

vii) we work modulo a power of p.

We detail what we mean by this v). Let Q^i be the completion of R^i in z^i. Let $\tilde{\mathcal{D}}^i$ be the pull back of $\mathcal{D}^i$ through the natural morphism $\mathrm{Spec}(Q^i) \to \mathrm{Spec}(R^i)$. Let Q^0 be the regular ring of the universal deformation space of the Shimura p-divisible $\mathcal{D}(W(k))$ showing up in i) above (see also 3.11 and [Va2, 5.4-5]). Corresponding to $\tilde{\mathcal{D}}^i$ we naturally get $W(k)$-homomorphisms $q_i : Q^0 \to Q^i$, $i = \overline{1,2}$. What we want to say in v): it is enough to assume that $\ker(q_1) = \ker(q_2)$, that over dominant geometric points $\mathrm{Spec}(\tilde{k}) \to \mathrm{Spec}(Q^i/pQ^i)$ we get naturally Shimura-ordinary $\tilde{\sigma}$-crystals, and that the monomorphism $Q^0/\ker(q_i) \to Q^i$ is formally smooth. We express this fact (meant by v)) by: $\mathcal{D}^1$ and $\mathcal{D}^2$ unfold the same quasi-versal deformation of (the Shimura filtered σ-crystal underlying) $\mathcal{D}(W(k))$, which generically (in the special fibre) is Shimura-ordinary.

3.4.20. Final remarks for 3.4. 1) We might wonder why we preferred to work 3.4 with open subschemes of the group G, and not with open subschemes of the (abelian) subgroup of G acting trivially on F^0 and on M/F^1. The reason is: the full generality is useful in other situations, gives us more flexibility, and as we have seen a couple of times, the particular case (of an abelian group) can be obtained from the general case, by taking slices (cf. also §7).

2) We needed (3.1.8 and 3.5.14 are afterthoughts) the ∇ principle for Shimura σ-crystals for proving 3.1.0 and the conjecture of Milne (cf. 1.16) in their full generality (not only in the context of a SHS $(f, L_{(p)}, v)$, where it is not needed: 2.3.16 is enough, as the scheme $\mathcal{N}$ itself provides the needed deformation). The other principles (of 3.4.18.4-8) are just a natural extension of it. The deformation principle is a logical continuation of the ∇-principle. Its importance stems from the fact that, when applies, we can take slices (cf. §7; see also the proof of 3.4.14.1) and its version 3.4.14.4 is very convenient for different global and local computations, cf. §10. Also the rigidity property (see 3.4.1.4 5) and §7) is a useful tool. We will use it in §8, in connection to 1.19. The gluing principle is our straightforward approach to prove the existence of integral canonical models of Shimura varieties of special type (cf. also [Va2, rm. 8) of 3.2.7]). No doubt we have been influenced (and inspired) by [Fa1, 7.1] and [Fa2, th. 10].

3) In parts of 3.4 we used [Fa2, th. 10 and the remarks after], but not in 3.4.18.4. It is an easy exercise to check that using 3.4.18.4.1 and [Fa1, 7.1] we obtain another (completely different) proof of [Fa2, th. 10 and the remarks after] for $p \geq 3$. But the proof in the loc. cit. (works for $p = 2$ as well and) is of unsurpassable beauty. Also [Fa1, 7.1] can be deduced from [Fa2, th. 10] (Hint: follow the proof of 3.4.14.1; another approach is via 3.4.19 vii)). As [Fa1, 7.1] is stated for $p \geq 3$, in 3.14 we will not come back to it.

4) There are variants of many of the above principles. For instance, by working with quasi p-divisible objects (these are extensions of non-trivial p-divisible objects by objects), or working with finitely generated modules, which are not necessarily projective (see 2.2.1 c)), or working with (some) rings (algebras) which are not necessarily regular formally smooth (over $W(k)$). We do not stop to state the results in this generality, as anyone can state them easily, when needed. We will just point out that in the case of an $W(k)$-algebra R, we always have to work with connections which are with respect to an a priori specified locally free R-submodule of $\Omega_{R/W(k)}$, left invariant by a fixed Frobenius lift of R.

5) It seems to us possible to use the ideas of 3.4 and to reobtain the result stated in [Va2, 3.2.2 1)], at least when we deal with a DVR faithfully flat over $\mathbb{Z}_{(p)}$, and of index of ramification 1. Even for $p = 2$ we hope to gain in this way a good understanding of which (we expect all of them) regular formally smooth scheme over such a DVR are healthy regular schemes.

6) It seems to us (see also 3.4.18.10) that there is a great similarity between 3.4.1.3, the theory of p-jets of A. Buium (see [Bu1]), and the theory of almost étale extensions of G. Faltings (see [Fa1] and [Fa3]). We think that it is possible to obtain significant progress in the related areas by “fitting” these three theories together.

7) We view 3.4.8.1 (the part $s = 0$) and 3.4.8.1.2 c) as the algebraic analogue of the well known theorem of S. Lang on connected algebraic groups over finite fields (see [Bo, 16.3-4]).

8) There are many other ways (besides 3.4.19 B) to express the versality condition of 3.4.19 ii). One way is to use the proof of 3.12.1. Another way (more fashionable) is inspired from 2.3.17.2: any Shimura p-divisible group over $W(k)$, whose Shimura σ-crystal over k is isomorphic to the one obtained from $\mathcal{D}^i$ through the k-valued point of $\mathrm{Spec}(R^{i\wedge})$ defined by z^i, is induced from $\mathcal{D}^i$ through a $W(k)$-valued point of $\mathrm{Spec}(R^{i\wedge})$. The fact that all these definitions are equivalent is proved in the same manner as for the case of a SHS (see 2.3.17.2 and [Va2, 5.4-5]).

9) In [Va6] and §6 we use the gluing principle (see 3.4.2.2 and 3.4.19 D) to construct the theory of Shimura envelopes. This theory generalizes the theory of (quotients of) integral canonical models of Shimura varieties.

10) In positive characteristic, regardless of the fact that we are interested in some fundamental groups (see 3.4.18.4.6 C), or on some connections (see the whole of 3.4.8), or on some stratifications (see 3.4.8.1.4 and 4.5.17), or on some deformations (see 3.4.14), or on some homomorphisms between two p-divisible groups (see the proofs of 3.4.14.1 and 3.11.1), the quasi Artin–Schreier systems are at the root of the subject.

11) In 3.4.18 we fixed Φ and Φ_R, and varied the connections, subject to the equations of 3.4.1.1.1 2). It would be interesting (though not so useful) to study what happens if we fix an integrable nilpotent mod p connection and one of the two Φ_R and Φ, and allow the other one to vary. The problem is: the connections involve two few variables (to be compared with 3.4.18.2), to truly determine (up to étale extensions) what has been allowed to vary.

3.5. Split Shimura Lie σ-crystals. We come back to the situation mentioned at the beginning of 3.3: G is split and $g_1 \in G(W(k))$. We have:

$$\mathfrak{g}[\frac{1}{p}] = \mathfrak{z}[\frac{1}{p}] \bigoplus \oplus_{i\in I}\mathfrak{g}_{\mathfrak{i}}[\frac{1}{p}]$$

as Lie algebras, where $\mathfrak{z}$ is the Lie algebra of the connected component $Z(G)^0$ of the origin of the center $Z(G)$ of G, and where $\mathfrak{g}_{\mathfrak{i}}$ is the Lie algebra of a semisimple subgroup G_i of G, such that

$$G^{\mathrm{ad}} = \prod_{i\in I} G_i^{\mathrm{ad}}$$

(cf. [Ti1, p. 46]), with G_i^{ad} absolutely simple split adjoint groups, $\forall i \in I$. We have an injective linear map

$$\mathfrak{z} \oplus \bigoplus_{i\in I} \mathfrak{g}_{\mathfrak{i}} \hookrightarrow \mathfrak{g}$$

which becomes an isomorphism by inverting p, but which is not always surjective (as we have assumed $p \geq 3$, the failure of surjectivity can happen only when $p = 3$ and there are factors G_i of E_6 Lie type or when p is arbitrary and there are factors G_i of A_ℓ Lie type, with $p|\ell+1$). We have an isogeny

$$Z(G)^0 \prod \prod_{i\in I} G_i \to G.$$

3.5.0. The σ-linear endomorphism φ_1 of M leaves invariant $\mathfrak{z}$ and permutes the factors $\mathfrak{g}_i \otimes_{W(k)} B(k)$, $i \in I$, producing a permutation γ of I. We write

$$\gamma = \prod_{s\in \mathcal{J}_0} \gamma_s$$

as a product of disjoint cyclic permutations γ_s. The σ-crystal $(\mathfrak{z},\varphi_1)$ has all the slopes zero. We start working with just one cycle γ_0 of γ (so $0\in\mathcal{J}_0$) permuting cyclically a subset I_0 of I. We can assume that $I_0=\{1,2,\ldots,n\}$ and that γ_0 is the permutation: $i\to i+1$ (with $n+1=1$). Let

$$\mathfrak{g}_0:=\bigoplus_{i\in I_0}\mathfrak{g}_i.$$

Let us fix an element $g_2\in G(W(k))$ and let $\varphi_2:=g_2\circ\varphi_1$.

Let $\bar{h}\in\mathfrak{t}$ be the element defined by: $\bar{h}(x)=0$ if $x\in F^0$, and $\bar{h}(x)=x$ if $x\in F^1$. We recall that the cocharacter $\mu:\mathbb{G}_m\to G$ produces a direct sum decomposition $M=F^1\oplus F^0$, with $\beta\in\mathbb{G}_m(W(k))$ acting through μ on F^i as the multiplication with β^{-i}, $i=\overline{0,1}$; so

$$\bar{h}\in\mu(\operatorname{Lie}(\mathbb{G}_m))\subset\operatorname{Lie}(T)=\mathfrak{t}.$$

Let $\bar{h}_i$ be the component of $\bar{h}$ in $\mathfrak{g}_i\left[\frac{1}{p}\right]$ (corresponding to the direct sum decomposition $\mathfrak{g}[\frac{1}{p}]=\mathfrak{z}[\frac{1}{p}]\bigoplus\oplus_{i\in I}\mathfrak{g}_i[\frac{1}{p}]$). If $\bar{h}_i=0$ then $\mathfrak{g}_i\subset F^0(\mathfrak{g})$. If $\bar{h}_i\neq 0$ then $\bar{h}_i$ is a semisimple element of $\mathfrak{g}_i\left[\frac{1}{p}\right]$. When viewed as an endomorphism of $\mathfrak{g}_i[\frac{1}{p}]$ (via $z_i\in\mathfrak{g}_i$ is mapped into $[\bar{h}_i,z_i]$), its eigenvalues are precisely -1, 0 and 1.

We have $F^1(\mathfrak{g})=\{x\in\mathfrak{g}\mid[\bar{h},x]=x\}$, and so

$$F^1(\mathfrak{g}_i):=F^1(\mathfrak{g})\cap\mathfrak{g}_i=\{x\in\mathfrak{g}_i\mid[\bar{h}_i,x]=x\}.$$

Similarly, $F^0(\mathfrak{g})=\{x\in\mathfrak{g}\mid[\bar{h},x]=0\text{ or }x\}$, and so

$$F^0(\mathfrak{g}_i):=\mathfrak{g}_i\cap F^0(g)=\{x\in\mathfrak{g}_i\mid[\bar{h}_i,x]=0\text{ or }x\}.$$

Let $F^j(\mathfrak{g}_0):=\bigoplus_{i\in I_0}F^j(\mathfrak{g}_i)$, $j=\overline{0,1}$. We get:

3.5.1. Lemma. *The quadruple $(\mathfrak{g}_0,\varphi_j,F^0(\mathfrak{g}_0),F^1(\mathfrak{g}_0))$ is a Shimura filtered Lie σ-crystal, $j=\overline{1,2}$.*

Proof: $\varphi_j\left(\frac{1}{p}F^1(\mathfrak{g}_i)+F^0(\mathfrak{g}_i)+p\mathfrak{g}_i\right)$ is a direct summand of $\mathfrak{g}$ of rank $\dim_{W(k)}(\mathfrak{g}_i)=\dim_{W(k)}(\mathfrak{g}_{i+1})$, and is included in $\mathfrak{g}_{i+1}\left[\frac{1}{p}\right]$; so it is $\mathfrak{g}_{i+1}$ ($\forall i\in I_0$, $\forall j\in\{1,2\}$). So $(\mathfrak{g}_0,\varphi_j,F^0(\mathfrak{g}_0),F^1(\mathfrak{g}_0))$ is a p-divisible object of $\mathcal{MF}_{[-1,1]}(W(k))$. It is a Shimura filtered Lie σ-crystal, as the cocharacter of G^{ad}, obtained by composing μ with the natural quotient homomorphism $G\to G^{\mathrm{ad}}$, shows that the conditions of 2.2.11 1) are satisfied. This ends the proof.

3.5.1.1. Remark. We can work equally well 3.5.1 (as well as all that follows) with $\operatorname{Lie}(\sum_{i\in I_0}G_i)$ (we have $\mathfrak{g}_0\subset\operatorname{Lie}(\sum_{i\in I_0}G_i)\subset\mathfrak{g}_0[\frac{1}{p}]$), or with $\operatorname{Lie}((\sum_{i\in I_0}G_i)^{\mathrm{ad}})=\oplus_{i\in I_0}\operatorname{Lie}(G_i^{\mathrm{ad}})$ instead of $\mathfrak{g}_0$ (cf. 2.2.13) (here $\sum_{i\in I_0}G_i$ is the subgroup of G generated by G_i, $i\in I_0$).

3.5.1.2. We assume that $F^1(\mathfrak{g}_0)\neq 0$, otherwise there is nothing to be done (cf. 3.2.2).

3.5.1.3. Let $\tilde{G}_0$ be the subgroup of G generated by the torus T and the semisimple subgroups G_i, $i \in I_0$. So $\tilde{G}_0^{\mathrm{ad}} = \times_{i\in I_0} G_i^{\mathrm{ad}}$. We have: $\mathfrak{g}_0$ is a Lie subalgebra of $\mathrm{Lie}(\tilde{G}_0)$. Moreover a $G(W(k))$-conjugate μ_1 of μ factors through $\tilde{G}_0$, giving birth to a Shimura group pair $(\tilde{G}_0, [\mu_1])$ over $W(k)$, and to a Shimura σ-crystal $(M, \varphi_1, \tilde{G}_0)$ (cf. 2.2.9 1)).

3.5.2. Let $\mathfrak{H}_2$ (resp. $\mathfrak{H}_1$) be the set of slopes of the isocrystal $\big(\mathfrak{g}_0[\frac{1}{p}], \varphi_2\big)$ (resp. of the isocrystal $\big(\mathfrak{g}_0[\frac{1}{p}], \varphi_1\big)$). The sets $\mathfrak{H}_2$ and $\mathfrak{H}_1$ are subsets of the real interval $[-1, 1]$. We have direct sum decompositions

$$\mathfrak{g}_0[\frac{1}{p}] = \bigoplus_{\alpha\in\mathcal{H}_2} g_2(\alpha)$$

and

$$\mathfrak{g}_0[\frac{1}{p}] = \bigoplus_{\alpha\in\mathcal{H}_1} g_1(\alpha),$$

with $g_j(\alpha)$ as the maximal $B(k)$-vector subspace of $\mathfrak{g}_0\big[\frac{1}{p}\big]$ left invariant by φ_j, and such that the isocrystal $\big(g_j(\alpha), \varphi_j\big)$ has pure slope α, $j = \overline{1,2}$. As φ_1^n and φ_2^n leave invariant any one of $\mathfrak{g}_i$, $i \in I_0$, we deduce that:

$$g_j(\alpha) = \bigoplus_{i\in I_0} g_j(\alpha) \cap \mathfrak{g}_i\big[\frac{1}{p}\big], \tag{3.5.2.1}$$

$\forall j \in \{1, 2\}$, and $\forall \alpha \in \mathcal{H}_j$. As φ_j maps $g_j(\alpha) \cap \mathfrak{g}_i[\frac{1}{p}]$ bijectively onto $g_j(\alpha) \cap \mathfrak{g}_{i+1}[\frac{1}{p}]$, $i \in I_0$, we have:

(3.5.2.2) *For $j \in \{1, 2\}$, the dimension of $g_j(\alpha) \cap \mathfrak{g}_i[\frac{1}{p}]$ does not depend on $i \in I_0$.*

3.5.3. Let $\mathfrak{t}_i$ ($\mathfrak{b}_i$) be the component of $\mathfrak{t}$ (resp. of $\mathfrak{b}$) in $\mathfrak{g}_i$, i.e. $\mathfrak{t}_i := \mathfrak{g}_i \cap \mathfrak{t}\big[\frac{1}{p}\big]$ (resp. $\mathfrak{b}_i := \mathfrak{g}_i \cap \mathfrak{b}\big[\frac{1}{p}\big]$). Let Δ_i be the base of roots of $\mathfrak{g}_i$ with respect to the maximal torus T_i of G_i having $\mathfrak{t}_i$ as its Lie algebra, corresponding to the Borel Lie subalgebra $\mathfrak{b}_i$ of $\mathfrak{g}_i$. Let Φ_i be the set of roots of $\mathfrak{g}_i$ with respect to the torus T_i. Then Φ_i^+ (the set of positive roots) and Φ_i^- (the set of negative roots) are defined with respect to Δ_i. Here $i \in I$.

3.5.3.1. φ_1^n leaves invariant $\mathfrak{g}_1[\frac{1}{p}]$, $\mathfrak{b}_1$ and $\mathfrak{t}_1$, and so it induces a permutation π_1 of Δ_1. Let e be the order of π_1. It is very convenient to work simultaneously with the Shimura Lie σ-crystal

$$(\mathfrak{g}_0^e, \varphi_j^{(e)}),$$

where $\mathfrak{g}_0^e := \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_0$ (e-factors), and where $\varphi_j^{(e)}$ is the Frobenius automorphism of $\mathfrak{g}_0^e[\frac{1}{p}]$ acting like φ_j on any direct summand $\mathfrak{g}_i$, $i \in I_0$, of a $\mathfrak{g}_0$ copy of $\mathfrak{g}_0^e$, but sending the $\mathfrak{g}_n$ direct summand of the s-th $\mathfrak{g}_0$ copy to the $\mathfrak{g}_1$ direct summand of the $(s+1)$-th $\mathfrak{g}_0$ copy, $\forall s \in \mathbb{Z}/e\mathbb{Z}$. The set of slopes of $(\mathfrak{g}_0^e, \varphi_j^{(e)})$ is the set of slopes of $(\mathfrak{g}_0, \varphi_j)$, all multiplicities being e-times more, $j = \overline{1,2}$.

The permutation of Δ_1 defined by $(\varphi_1^{(e)})^{ne}$ is the trivial one.

e can be 1, 2 or 3. It can be 3 only in the case when $\mathfrak{g}_1$ is of D_4 Lie type, and it can be to 2 only when $\mathfrak{g}_1$ is of A_ℓ ($\ell \geq 2$), D_ℓ ($\ell \geq 4$) or E_6 Lie type (and then π_1 is the usual involution of Δ_1).

Let $\tilde{I}_0 := \{1, ..., en\}$. We like to think of $\mathfrak{g}_0^e$ as a direct sum $\oplus_{i\in\tilde{I}_0}\mathfrak{g}_i$. Here for $s \in \{1, ..., e-1\}$ and $i \in I_0$, $\mathfrak{g}_{i+sn} := \mathfrak{g}_i$; similarly $\mathfrak{t}_{i+sn} := \mathfrak{t}_i$, $\Delta_{i+sn} := \Delta_i$, $\mathfrak{b}_{i+sn} := \mathfrak{b}_i$, and $\bar{h}_{i+sn} := (\varphi_1^e)^{sn}(\bar{h}_i) \in \mathfrak{g}_{i+sn}[\frac{1}{p}]$.

3.5.3.2. Let ℓ be the rank of $\mathfrak{g}_1$. For $i \in \tilde{I}_0$ let $\alpha_1(i), \ldots, \alpha_\ell(i)$ be the simple roots of Δ_i, numbered in such a way that φ_1 sends the (one dimensional) $B(k)$-vector subspace of $\mathfrak{g}_i[\frac{1}{p}]$ corresponding to $\alpha_s(i)$, $s \in \{1, ..., \ell\}$, to the (one dimensional) $B(k)$-vector subspace of $\mathfrak{g}_{i+1}[\frac{1}{p}]$ corresponding to $\alpha_s(i+1)$ (with $en+1=1$), and such that the numbering for $\mathfrak{g}_1$ is the one from [Bou, planche I-VI]. The maximal root of Δ_1 (dropping the lateral right index (1) from now on) depends on the Lie type of $\mathfrak{g}_1$. It is (cf. loc. cit.):

a) $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell$ if $\mathfrak{g}_1$ is of A_ℓ Lie type ($\ell \geq 2$);

b) $\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_\ell$ if $\mathfrak{g}_1$ is of B_ℓ Lie type ($\ell \geq 1$);

c) $2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$ if $\mathfrak{g}_1$ is of C_ℓ Lie type ($\ell \geq 2$);

d) $\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$ if $\mathfrak{g}_1$ is of D_ℓ Lie type ($\ell \geq 4$);

e6) $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ if $\mathfrak{g}_1$ is of E_6 Lie type;

e7) $2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$ if $\mathfrak{g}_1$ is of E_7 Lie type.

$\mathfrak{g}_1$ is one of these Lie types, cf. 2.2.7 and 3.5.1.2-3. We usually prefer to consider $B_\ell(\ell \geq 1)$ and $A_\ell(\ell \geq 2)$, i.e. we prefer to consider B_1 instead of A_1; but we are not bothered if we have B_2 or C_2.

3.5.3.3. Let $\mathfrak{g}_1 = \mathfrak{t}_1 \oplus \bigoplus_{\beta\in\Phi_1}\mathfrak{g}_\beta$ be the Weyl direct sum decomposition associated to the maximal split torus T_1 of G_1. So T_1 acts on $\mathfrak{g}_\beta$ through the character defining the root β.

In cases b), c) and e7), $\bar{h}_1 \in \mathfrak{t}_1$ is either zero or is defined by:

– in case b), $\bar{h}_1$ acts as zero on $\mathfrak{g}_{\alpha_s}$, $s \in \{2, ..., \ell\}$, and as identity on $\mathfrak{g}_{\alpha_1}$;

– in case c), $\bar{h}_1$ acts as zero on $\mathfrak{g}_{\alpha_s}$, $s \in \{1, ..., \ell-1\}$, and as identity on $\mathfrak{g}_{\alpha_\ell}$;

– in case e7), $\bar{h}_1$ acts as zero on $\mathfrak{g}_{\alpha_s}$, $s \in \{1, ..., 7\}$, and as identity on $\mathfrak{g}_{\alpha_7}$.

In case a), $\bar{h}_1 \in \mathfrak{t}_1$ is either zero or there is $s_1 \in \{1, 2, \ldots, \ell\}$ such that it acts as zero on $\mathfrak{g}_{\alpha_s}$, $s \in \{1, ..., \ell\} \setminus \{s_1\}$, and as identity on $\mathfrak{g}_{\alpha_{s_1}}$ (we have ℓ non-zero possibilities for $\bar{h}_1$).

In case d), $\bar{h}_1 \in \mathfrak{t}_1$ is either zero or there is $s_1 \in \{1, \ell-1, \ell\}$ such that it acts as zero on $\mathfrak{g}_{\alpha_s}$, $s \in \{1, ..., \ell\} \setminus \{s_1\}$, and as identity on $\mathfrak{g}_{\alpha_{s_1}}$ (we have 3 non-zero possibilities for $\bar{h}_1$).

In case e6), $\bar{h}_1$ is either zero or there is $s_1 \in \{1, 6\}$ such that it acts as zero on $\mathfrak{g}_{\alpha_s}$, $s \in \{1, ..., 6\} \setminus \{s_1\}$, and as identity on $\mathfrak{g}_{\alpha_{s_1}}$ (we have 2 non-zero possibilities for $\bar{h}_1$).

3.5.3.4. For $i \in \tilde{I}_0$, we define $\varepsilon_i = 0$ if $\bar{h}_i = 0$, and $\varepsilon_i = s$ if $\bar{h}_i$ acts as identity on the rank one direct summand of $\mathfrak{g}_i$ corresponding to the root $\alpha_s(i)$; here $s \in \{1, ..., \ell\}$. So, fixing i, ε_i can take more than one non-zero value only if we are in the cases a), d) or e6).

The assumption $F^1(\mathfrak{g}_0) \neq 0$ implies that the set

$$I_1 := \{i \in I_0 \mid \varepsilon_i \neq 0\}$$

is not empty. Let

$$\tilde{I}_1 := \{i \in \tilde{I}_0 \mid \varepsilon_i \neq 0\}.$$

3.5.4. We have:

Fact. *The sets $\mathcal{H}_1$ and $\mathcal{H}_2$ are subsets of $\left[-\frac{|I_1|}{n}, \frac{|I_1|}{n}\right]$. The smallest element of $\mathcal{H}_1$ is*

$$\delta := -\frac{1}{n}\,|I_1|\,.$$

For instance $\mathfrak{g}_{-\beta}$, with $\beta \in \Phi_1^+$ the maximal root with respect to the base Δ_1, is included in $g_1(\delta)$.

3.5.5. Let $\Psi_j^0 : \mathfrak{g}_0 \to \mathfrak{g}_0$, $j = \overline{1,2}$, be the σ-linear maps defined by: if $x \in \mathfrak{g}_i$, then

i) $\Psi_j^0(x) = \varphi_j(x)$ if $\varepsilon_i = 0$;

ii) $\Psi_j^0(x) = p\varphi_j(x)$ if $\varepsilon_i \neq 0$.

Taking everything mod p, we get maps $\bar{\psi}_j^0 : \mathfrak{g}_0/p\mathfrak{g}_0 \to \mathfrak{g}_0/p\mathfrak{g}_0$, $j = \overline{1,2}$. We have $\Psi_2^0 = g_2\Psi_1^0$ and $\bar{\psi}_2^0 = g_2\bar{\psi}_0^0$.

We can assume that $\varepsilon_1 \neq 0$. Let

$$N_2 := g_2(\delta) \cap \mathfrak{g}_1.$$

For any $i \in \tilde{I}_1$, let $\mathfrak{p}_{\varepsilon_i}$ be the parabolic Lie subalgebra of $\mathfrak{g}_1$ containing $\mathfrak{b}_1$, and such that its nilpotent radical $\mathfrak{s}_{\varepsilon_i}$ is the $W(k)$-submodule of $\mathfrak{g}_1$ generated by $\mathfrak{g}_\beta$, with $\beta \in \Phi_1^+$ having $+\alpha_{\varepsilon_i}$ in their expression. Let $\mathfrak{q}_{\varepsilon_i}$ be the parabolic Lie subalgebra of $\mathfrak{g}_1$, opposite to $\mathfrak{p}_{\varepsilon_i}$ (with respect to $\mathfrak{t}_i$ and Δ_i), and let $\mathfrak{r}_{\varepsilon_i}$ be its nilpotent radical. The multiplicity m_δ of the slope δ for $(\mathfrak{g}_0, \varphi_1)$ is n times the rank of the intersection $\bigcap_{i\in I_1} \mathfrak{r}_{\varepsilon_i}$ (as $(\mathfrak{g}_0, \varphi_1)$ gives birth to a 0-symmetric isocrystal). So we have:

$$m_\delta = \dim_{B(k)} g_1(\delta).$$

The multiplicity $\dim_{B(k)} g_2(\delta)$ of the slope δ for $(\mathfrak{g}_0, \varphi_2)$ is n times the rank of N_2 (cf. 3.5.2.2). We denote by $\mathfrak{p}_{\varepsilon_i}(i)$, $\mathfrak{q}_{\varepsilon_i}(i)$, $\mathfrak{r}_{\varepsilon_i}(i)$, respectively $\mathfrak{s}_{\varepsilon_i}(i)$ the intersection of $\mathfrak{g}_i$ with $\varphi_1^{i-1}(\mathfrak{p}_{\varepsilon_i}[\frac{1}{p}])$, $\varphi_1^{i-1}(\mathfrak{q}_{\varepsilon_i}[\frac{1}{p}])$, $\varphi_1^{i-1}(\mathfrak{r}_{\varepsilon_i}[\frac{1}{p}])$, and respectively with $\varphi_1^{i-1}(\mathfrak{s}_{\varepsilon_i}[\frac{1}{p}])$, $i = \overline{1, en}$.

3.5.6. Theorem. *If $\delta \in \mathcal{H}_2$, then the multiplicity of the slope δ of the Shimura Lie σ-crystal $(\mathfrak{g}_0, \varphi_2)$ is less or equal to m_δ. If it is m_δ, then $\forall i \in I_1$, we have*

$$\Big(\Big(\bigoplus_{\alpha\in\mathcal{H}_2\cap[0,1]} g_2(\alpha)\Big) \cap \mathfrak{g}_i\Big) \otimes_{W(k)} k \subset \mathfrak{p}_{\varepsilon_i}(i) \otimes_{W(k)} k. \qquad (3.5.7)$$

Moreover, the fact that this multiplicity is m_δ or not depends only on the expression of g_2 mod p.

Before proving this, let us see first some of its immediate implications. Not to complicate the notations by introducing extra indices, in 3.5.8-12 we allow g_2 to vary as needed.

3.5.8. Proposition. *The Newton polygon of the Shimura Lie σ-crystal $(\mathfrak{g}_0, \varphi_1)$ is not strictly above the Newton polygon of any other Shimura Lie σ-crystal of the form $(\mathfrak{g}_0, g_2\varphi_1)$, with $g_2 \in \tilde{G}_0(W(k))$.*

Proof: We assume that 3.5.8 is false; let $g_2 \in \tilde{G}_0(W(k))$ for which 3.5.8 is not true. As $\delta \in \mathcal{H}_1$, from 3.5.4 and 3.5.6 we deduce that the multiplicity of the slope $\delta \in \mathcal{H}_2$ for $(\mathfrak{g}_0, \varphi_2)$ is m_δ.

Let P_2 be the parabolic subgroup of $\tilde{G}_0$ whose Lie algebra is the parabolic Lie subalgebra of $\mathrm{Lie}(\tilde{G}_0)$ corresponding to non-negative slopes of $(\mathrm{Lie}(\tilde{G}_0), \varphi_2)$. $p\varphi_2$ leaves invariant $\mathrm{Lie}(P_2)$. From 3.5.7 we deduce that P_{2k} is included in P^0_k (with P^0 as the parabolic subgroup of $\tilde{G}_0$ having $F^0(\mathfrak{g}) \cap \mathrm{Lie}(\tilde{G}_0)$ as its Lie algebra). We deduce the existence of an element $g_3 \in \mathrm{Ker}\big(\tilde{G}_0(W(k)) \to \tilde{G}_0(k)\big)$ such that $P_3 = g_3P_2g_3^{-1} \subset P^0$. We can assume that $\mathfrak{b} \cap \mathrm{Lie}(\tilde{G}_0)$ is included in $\mathrm{Lie}(P_3)$.

Let $\varphi_3 := g_3\varphi_2g_3^{-1}$. We can write $\varphi_3 = g_4\varphi_1$, with $g_4 \in \tilde{G}_0(W(k))$. Obviously $(\mathfrak{g}_0, \varphi_3)$ and $(\mathfrak{g}_0, \varphi_2)$ have the same Newton polygon. Let B_0 be the Borel subgroup of $\tilde{G}_0$ whose Lie algebra contains $\oplus_{i\in I_0}\mathfrak{b}_i$. We have $B_0 \subset P^0$. Repeating the argument of 3.2.3, we deduce the existence of an element $g_5 \in P_3(W(k))$ such that $\varphi_4 = g_5\varphi_3$ takes $\mathfrak{t}$ onto $\mathfrak{t}$ and $\mathrm{Lie}(B_0)$ into $\mathrm{Lie}(B_0)$. We get $\varphi_4^s = p_{3,s}\varphi_3^s$, $\forall s \in \mathbb{N}$, with $p_{3,s} \in P_3(W(k))$, and that $\varphi_4^s = b_s\varphi_1^s$, $\forall s \in \mathbb{N}$, with $b_s \in B_0(W(k))$. This implies that $\mathrm{Lie}(P_3) = \mathrm{Lie}(\tilde{G}_0 \cap P_1)$, cf. 3.2.1.1; so $P_3 = \tilde{G}_0 \cap P_1$.

From 3.2.1.1 we get also that $(\mathfrak{g}_0, \varphi_2)$ and $(\mathfrak{g}_0, \varphi_1)$ have the same Newton polygon. Contradiction. This ends the proof of the proposition.

3.5.9. Corollary. *If $g_2 \in \tilde{G}_0(W(k))$ is such that the Shimura Lie σ-crystals $(\mathfrak{g}_0, g_2\varphi_1)$ and $(\mathfrak{g}_0, \varphi_1)$ have the same Newton polygon, then $(M, g_2\varphi_1)$ and (M, φ_1) have the same Newton polygon, and $g_2\varphi_1$ is of the form $g_3p_1\varphi_1g_3^{-1}$, with $p_1 \in P_{10}(W(k))$ where P_{10} is the parabolic subgroup of $\tilde{G}_0$ having $\mathrm{Lie}(\tilde{G}_0) \cap \mathrm{Lie}(P_1)[\frac{1}{p}]$ as its algebra, and with $g_3 \in \tilde{G}_0(W(k))$ such that it leaves invariant F^1/pF^1.*

3.5.10. Corollary. *If $h_0 \in \tilde{G}_0(W(k))$, then the multiplicity of the slope δ for $(\mathfrak{g}_0, h_0\varphi_1)$ is less or equal to m_δ. If we have equality, then $(\mathfrak{g}_0, h_0\varphi_1)$ and $(\mathfrak{g}_0, \varphi_1)$ have the same Newton polygon.*

3.5.9-10 are consequences of the proof of 3.5.8. 3.5.11-12 below result from 3.5.6-10, by working with all cyclic permutations γ_s, $s \in \mathcal{J}_0$, at once.

3.5.11. Corollary. *If $g_2 \in G(W(k))$ is such that the Shimura Lie σ-crystals $(\mathfrak{g}, g_2\varphi_1)$ and $(\mathfrak{g}, \varphi_1)$ have the same Newton polygon, then the σ-crystals $(M, g_2\varphi_1)$ and (M, φ_1) have the same Newton polygon, and $g_2\varphi_1 = g_3p_1\varphi_1g_3^{-1}$, with $g_3 \in G(W(k))$ leaving invariant F^1/pF^1, and with $p_1 \in P_1(W(k))$. Moreover the fact that $(\mathfrak{g}, g_2\varphi)$ has the same Newton polygon as $(\mathfrak{g}, \varphi_1)$ depends only on the expression of g_2 mod p.*

3.5.12. Corollary. *The Newton polygon of the Shimura Lie σ-crystal $(\mathfrak{g}, \varphi_1)$ is not strictly above the Newton polygon of any other Shimura Lie σ-crystal of the form $(\mathfrak{g}, \mathfrak{g}_2\varphi_1)$, with $g_2 \in G(W(k))$.*

3.5.13. Remark. 3.5.11-12 remain true if we replace φ_1 by φ_0, and do not assume that k is algebraically closed (assumption made at the beginning of 3.5). This is an easy consequence of 3.3.2 and of the fact that two parabolic subgroups of G which over $W(\bar{k})$ are $G(W(\bar{k}))$-conjugate are $G(W(k))$-conjugate (this results easily from [Bo, 20.9] and [Ti, 3.3.2]). So in 3.5.11 (as every parabolic subgroup is its own normalizer), if $g_2 \in G(W(k))$, then we can choose $g_3 \in G(W(k))$ (not just $g_3 \in G(W(\bar{k}))$); so we get $g_2\varphi_0 = g_3 p_0 \varphi_0 g_3^{-1}$, with g_3 mod p leaving invariant F^1/pF^1, and so $p_0 \in G(W(k)) \cap P_0(W(\bar{k})) = P_0(W(k))$. This proves 3.3.4 and ends the proof of 3.2.5.

3.5.14. Exercise. Using just the ideas and results of 3.1.8.1 and 3.5.6-13 prove the whole of 3.1.0 (i.e. end the proof of 3.2.6-7).

This exercise ends the first proof of 3.2.6-7, and so (cf. 3.2.4) of 3.1.0.

3.6. The proof of 3.5.6. With the notations of 3.5.4-5, for proving the part of 3.5.6 not involving the expression of g_2 mod p, it is enough to show that

$$\dim_{W(k)}(N_2) \leq \dim_{W(k)}\Big(\bigcap_{i\in\tilde{I}_1} \mathfrak{r}_{\varepsilon_i}\Big) = \frac{m_\delta}{n}, \tag{0}$$

and, if equality holds, then (3.5.7) is true, with $i = 1$.

For $i \in \tilde{I}_0$ let $\bar{\mathfrak{g}}_i := \mathfrak{g}_i \otimes_{W(k)} k$. For every $i \in \tilde{I}_1$, let Π_{ε_i} be the projector of $\bar{\mathfrak{g}}_1$ of whose kernel is $\bar{\mathfrak{p}}_{\varepsilon_i} := \mathfrak{p}_{\varepsilon_i} \otimes_{W(k)} k$ and of whose image is $\bar{\mathfrak{r}}_{\varepsilon_i} := \mathfrak{r}_{\varepsilon_i} \otimes_{W(k)} k$. Let $\bar{\mathfrak{s}}_{\varepsilon_i} := \mathfrak{s}_{\varepsilon_i} \otimes_{W(k)} k$, and let $\bar{\mathfrak{q}}_{\varepsilon_i} := \mathfrak{q}_{\varepsilon_i} \otimes_{W(k)} k$.

Following the image of $\bar{\mathfrak{g}}_1$ under the s-th iterate of $\bar{\psi}_2^0$, and identifying $\bar{\mathfrak{g}}_i$ and $\bar{\mathfrak{g}}_i \otimes_k k$ (the map $k \to k$ being the Frobenius automorphism σ) with $\bar{\mathfrak{g}}_1$, we come across k-vector spaces of the form

$$^{h_s}\Pi_{\varepsilon_{i_s}}\big(^{h_{s-1}}\Pi_{\varepsilon_{i_{s-1}}}\big(\cdots\big(\Pi_{\varepsilon_{i_2}}(^{h_1}\bar{\mathfrak{r}}_{\varepsilon_1})\cdots\big)\big)\big), \tag{1}$$

where $h_1, \ldots, h_s \in G_1(k)$. Here the left upper index ^{h_l}, $l = \overline{1, s}$, means conjugation with h_l. Taking $s \in \mathbb{N}$ big enough and a multiple of $\left|\tilde{I}_1\right|$, the k-vector space of (1) becomes nothing else but $N_2 \otimes_{W(k)} k$. We allow such an s to vary. We first prove (0), i.e. that the dimension of the k-vector spaces of (1) computing $\dim_k(N_2 \otimes_{W(k)} k)$, is less or equal to $\dim_{W(k)}(\bigcap_{i\in\tilde{I}_1} \mathfrak{r}_{\varepsilon_i})$.

3.6.1. Let $A := \{\varepsilon_i | i \in \tilde{I}_1\}$. So $\cap_{i\in\tilde{I}_1}\mathfrak{r}_{\varepsilon_i} = \cap_{a\in A}\mathfrak{r}_a$. The set A can have more than two elements only when $\mathfrak{g}_1$ is in case a), d) or e6). In case d) A has at most 3 elements. In case e6) A has at most 2 elements. In case a) A can have more than 3 elements, but in this case

$$\bigcap_{a\in A} \mathfrak{r}_a = \mathfrak{r}_{a_0} \cap \mathfrak{r}_{a_1},$$

with a_0 as the smallest element of A, and with a_1 as the greatest element of A. This is so because any positive root containing α_{a_0} and α_{a_1} in their expression, automatically contains α_{a_2} in their expression, for any $a_2 \in \mathbb{N}$ such that $1 \leq a_0 < a_2 < a_1 \leq \ell$ (see [Bou, planche I]).

3.6.2. Let $\bar{G}_1 := G_{1k}$ and let $\bar{T}_1 := T_{1k}$. We call an element $\omega \in \bar{G}_1(k)$ normalizing $\bar{T}_1$, a Weyl element. For any $\varepsilon_i \in A$, let $\bar{Q}_{\varepsilon_i}$ (resp. $\bar{P}_{\varepsilon_i}$) be the parabolic subgroup of $\bar{G}_1$ having $\bar{\mathfrak{q}}_{\varepsilon_i}$ (resp. $\bar{\mathfrak{p}}_{\varepsilon_i}$) as its Lie algebra.

3.6.3. Let us study closer a k-linear endomorphism of $\bar{\mathfrak{g}}_1$ of the form $\Pi_a\, {}^g\Pi_b$, with $a, b \in A$. The Bruhat decomposition allows us to write $g = q_a w q_b$, with $q_b \in \bar{Q}_b(k)$, with $w \in \bar{G}_1(k)$ a Weyl element, and with $q_a \in \bar{Q}_a(k)$. We get:

$$\mathrm{Im}(\Pi_a\, {}^g\Pi_b) = \mathrm{Im}(\, {}^{\tilde{g}}\Pi_a\, {}^w\Pi_b), \quad \text{with} \quad \tilde{g} \in \bar{G}_1(k). \tag{2}$$

We also have the following inversion property:

$$\Pi_a\, {}^w\Pi_b = \, {}^w\Pi_b\, {}^w\Pi_a\, {}^{w^{-1}}, \tag{3}$$

for any Weyl element $w \in \bar{G}_1(k)$.

For any $h \in G_1(k)$ we have the following elimination property

$$\mathrm{Im}(\Pi_a\, {}^g\Pi_a) \subset \mathrm{Im}(\Pi_a); \tag{4}$$

So for proving (0) we can assume that any two consecutive ε_{i_j} and $\varepsilon_{i_{j+1}}$ in (1) above are distinct.

We are now ready to prove the first statement of 3.5.6.

1) If $|A| = 1$, we have nothing to prove.

2) If $|A| = 2$, then in any sequence ${}^{h_s}\Pi_{\varepsilon_{i_s}} \ldots {}^{h_1}\Pi_{\varepsilon_1}$ "computing" $N_2 \otimes_{W(k)} k$, we get a subsequence $\Pi_a\, {}^g\Pi_b$, with $A = \{a, b\}$ and $g \in \bar{G}_1(k)$. We deduce that $\dim_{W(k)}(N_2) \leq \dim_{W(k)} \mathfrak{r}_a \cap \mathfrak{r}_b$, as

$$\dim_k(\Pi_a({}^g\bar{\mathfrak{r}}_b)) \leq \dim_k(\bar{\mathfrak{r}}_a \cap \bar{\mathfrak{r}}_b).$$

This can be proved in many ways. Using property (2), this can be reduced to the case when g is a Weyl element, and then by a simple computation

$$\dim(\Pi_a({}^g\bar{\mathfrak{r}}_b)) = \dim(\bar{\mathfrak{r}}_b) - \dim({}^g\bar{\mathfrak{r}}_b \cap \bar{\mathfrak{p}}_a) = \dim(\bar{\mathfrak{r}}_b) - \dim(\bar{\mathfrak{p}}_a) + \dim({}^g\bar{\mathfrak{p}}_b \cap \bar{\mathfrak{p}}_a)$$

(the second equality due to the fact that g is a Weyl element). And so the dimension of $\Pi_a({}^g\bar{\mathfrak{r}}_b)$ is maximal when the dimension of the intersection ${}^g\bar{\mathfrak{p}}_b \cap \bar{\mathfrak{p}}_a$ of parabolic Lie subalgebras of $\bar{\mathfrak{g}}_1$ is maximal, i.e. when the intersection ${}^g\bar{\mathfrak{p}}_b \cap \bar{\mathfrak{p}}_a$ contains a Borel Lie subalgebra of $\bar{\mathfrak{g}}_1$. This can be checked very easily, case by case, starting from the fact that we can assume that g is a Weyl element, and from [Bou, planche I, IV, and V]; the most involved case (i.e. the case $A = \{1, \ell - 1\}$, with $\mathfrak{g}_1$ of D_ℓ lie type) is treated in 4) below (the other cases can be treated similarly). From this and [Bo, 14.22] we get $\dim(\, {}^g\bar{\mathfrak{p}}_b \cap \bar{\mathfrak{p}}_a) \leq \dim(\bar{\mathfrak{p}}_b \cap \bar{\mathfrak{p}}_a)$.

3) If $|A| \geq 3$, then using repeatedly properties (2) and (3), we can bring together a short sequence of the form $\Pi_a\, {}^g\Pi_b$, if $\mathfrak{g}_1$ is in case a), with a the smallest element of A and with b the greatest element of A, or of the form $\Pi_a\, {}^g\Pi_b\, {}^h\Pi_c$ if $\mathfrak{g}_1$ is in case d), with $A = \{a, b, c\}$. This proves (0) if $\mathfrak{g}_1$ is in case a) (cf. 2) above).

4) Let us assume that we are in case d) with $|A| = 3$.

Let $\mathfrak{g}_1$ be of type D_ℓ ($\ell \geq 4$). Then $A = \{1, \ell - 1, \ell\}$. Using property (2), we can assume that $\Pi_a\, {}^g\Pi_b\, {}^h\Pi_c$ is such that h is a Weyl element. Due to the circular form of the expressions of (1) above, we can assume that $c = 1$ and $b = \ell - 1$. But in this case the dimension of the image of $\Pi_a\, {}^g\Pi_b\, {}^h\Pi_c$ is less or equal to $\ell - 2$.

This can be checked using [Bou, planche IV]. Using the notations of the loc. cit. for this and the next paragraphs (so ε_i are not any more some integers in the set $\{0, 1, ..., \ell\}$, but some vectors in $\mathbb{R}^\ell$), we get

– the roots of Φ_1 corresponding to $\bar{\mathfrak{r}}_1$ are of the form $-\varepsilon_1 \pm \varepsilon_s$, with $2 \leq s \leq \ell$;

– the roots corresponding to $\mathfrak{r}_{\ell-1}$ are of the form $-\varepsilon_i + \varepsilon_\ell$, with $i < \ell$, or of the form $-(\varepsilon_i + \varepsilon_s)$, with $1 \leq i < s < \ell$.

So if the element h of the Weyl group of loc. cit. sends ε_1 into an element of the set $\{\varepsilon_1, \varepsilon_2, \ldots \varepsilon_{\ell-1}, -\varepsilon_\ell\}$, then $\bar{\mathfrak{r}}_{\ell-1} \cap {}^h\bar{\mathfrak{r}}_1$ has dimension exactly $\ell - 1$. If h sends ε_1 into an element of the set $\{-\varepsilon_1, -\varepsilon_2, \ldots, -\varepsilon_{\ell-1}, \varepsilon_\ell\}$ then we have $\bar{\mathfrak{r}}_{\ell-1} \cap {}^h\bar{\mathfrak{r}}_1 = 0$. If $\bar{\mathfrak{r}}_{\ell-1} \cap {}^h\bar{\mathfrak{r}}_1 = 0$, we are done. If $\dim(\bar{\mathfrak{r}}_{\ell-1} \cap {}^h\bar{\mathfrak{r}}_1) = \ell - 1$, then the intersection $\bar{\mathfrak{q}}_{\ell-1} \cap {}^h\bar{\mathfrak{q}}_1$ contains a Borel subalgebra of $\bar{\mathfrak{g}}_1$, and we can apply the same argument as in case 2) above to conclude that the k-vector space $\bar{\mathfrak{r}}_\ell \cap {}^g(\bar{\mathfrak{r}}_{\ell-1} \cap {}^h\bar{\mathfrak{r}}_1)$ has dimension $\leq \ell - 2$ (the arguments of (2) above show that we can assume as well that g is a Weyl element).

This ends the proof of (0), and so of the first statement of 3.5.6.

3.6.4. We assume now that $\dim(N_2) = \frac{m_\delta}{n}$. To prove (3.5.7) we first remark:

Fact. *The Lie subalgebra* $\big(g_2(-\delta) \cap \mathfrak{g}_1\big) \otimes_{W(k)} k$ *of* $\bar{\mathfrak{g}}_1$ *is of the form* ${}^g\big(\cap_{a\in A}\bar{\mathfrak{s}}_a\big)$, *with* $g \in \bar{P}_{\varepsilon_1}(k)$, *and so its normalizer in* $\bar{G}_1$ *is* ${}^g\big(\bigcap_{a\in A} \bar{P}_a\big) \subset \bar{P}_{\varepsilon_1}$.

Proof: We start pointing out: if the Lie subalgebra $\big(g_2(-\delta) \cap \mathfrak{g}_1\big) \otimes_{W(k)} k$ of $\bar{\mathfrak{g}}_1$ is of the form ${}^g\big(\cap_{a\in A}\bar{\mathfrak{s}}_a\big)$, with $g \in \bar{G}_1(k)$ and if its normalizer in $\bar{G}_1$ is ${}^g\big(\bigcap_{a\in A} \bar{P}_a\big)$, then automatically $g \in \bar{P}_{\varepsilon_1}(k)$. This is a consequence of [Bo, 14.22 (i)] and of the fact that $\big(g_2(-\delta) \cap \mathfrak{g}_1\big) \otimes_{W(k)} k$ is included in $\bar{\mathfrak{s}}_{\varepsilon_1} = F^1(\bar{\mathfrak{g}}_1)$, and so $F^1(\bar{\mathfrak{g}}_1)$ (being abelian) is included in the normalizer of $\big(g_2(-\delta) \cap \mathfrak{g}_1\big) \otimes_{W(k)} k$ in $\bar{\mathfrak{g}}_1$. So we are left to show that $\big(g_2(-\delta) \cap \mathfrak{g}_1\big) \otimes_{W(k)} k$ of $\bar{\mathfrak{g}}_1$ is of the form ${}^g\big(\cap_{a\in A}\bar{\mathfrak{s}}_a\big)$ and that its normalizer in $\bar{G}_1$ is ${}^g\big(\bigcap_{a\in A} \bar{P}_a\big)$

This statement is obvious if $|A| = 1$ (cf. a) below). If $|A| \geq 2$ we prove it working with notations involving negative roots. If $|A| = 2$, with $A = \{a, b\}$, we just have to show: if the image of the endomorphism $\Pi_a\, {}^w\Pi_b$, with $w \in G_1(k)$ a Weyl element, has dimension $\dim(\bar{\mathfrak{r}}_a \cap \bar{\mathfrak{r}}_b)$, then $\bar{\mathfrak{r}}_b \cap {}^{w^{-1}}\bar{\mathfrak{r}}_a$ is $G_1(k)$-conjugate to $\bar{\mathfrak{r}}_a \cap \bar{\mathfrak{r}}_b$, and the normalizer of $\bar{\mathfrak{r}}_a \cap \bar{\mathfrak{r}}_b$ is $\bar{Q}_a \cap \bar{Q}_b$.

The first part is obvious as $\bar{Q}_a$ and ${}^w\bar{Q}_b$ must contain a Borel subgroup of $\bar{G}_1$ (cf. 3.6.3 2)). Let $\bar{\mathfrak{g}}_\alpha := \mathfrak{g}_\alpha \otimes_{W(k)} k$. For the second part, as any parabolic group is its own normalizer, it is enough to show that the Lie subalgebra of $\bar{\mathfrak{g}}_1$ normalizing $\bar{\mathfrak{r}}_a \cap \bar{\mathfrak{r}}_b$ is $\mathrm{Lie}(\bar{Q}_a \cap \bar{Q}_b)$. We use the following facts:

a) $[\bar{\mathfrak{g}}_\alpha, \bar{\mathfrak{g}}_{-\alpha}] \neq 0$ *(this holds even if* $|A| = 1$*; this is the third place where we need* $p > 2$*);*

b) *If* $\alpha, \beta, \alpha+\beta \in \Phi_1$*, then* α *and* β *generate a root system of rank 2 of type* A_2 *(the inequality* $|A| \geq 2$ *implies that all roots of* Φ_1 *have the same length, cf. [Bou, planche I, IV, and V]).*

So everything results from [BT, 4.2-3]. What [BT, 4.2-3] is saying, due to b) above, we have the following formula

$$[\bar{\mathfrak{g}}_\alpha, \bar{\mathfrak{g}}_\beta] = \bar{\mathfrak{g}}_{\alpha+\beta}, \tag{5}$$

where we have $\alpha, \beta, \alpha+\beta \in \Phi_1$. Of course, this formula can be checked directly, inside the SL-group of a three dimensional k-vector space; it is true even if $p = 2$.

If $|A| \geq 3$ and $\mathfrak{g}_1$ is of A_ℓ Lie type, then we have to work with $\Pi_a{}^w\Pi_b$, with a and b the smallest and respectively the biggest element of A, and with $w \in \bar{G}_1(k)$ a Weyl element. So this case is reduced to the case when $|A| = 2$.

If $|A| = 3$ and $\mathfrak{g}_1$ is of D_ℓ Lie type, then the same argument at the level of Lie algebras applies: we have to work with $\Pi_a{}^w\Pi_b{}^{w_0}\Pi_c$, where $A = \{a, b, c\}$, and $w, w_0 \in G_1(k)$ are two Weyl elements (cf. end of 3.6.3 4)). This ends the proof of the fact.

3.6.4.1. We are now ready to prove (3.5.7). We recall that $(\mathfrak{g}_0, \varphi_j)$, $j = \overline{1,2}$, are giving birth to 0-symmetric isocrystals and so we can work equally well with the slope $-\delta$.

The parabolic subgroup of $\bar{G}_1$ having $\big(\bigoplus_{\alpha \in \mathcal{H}_1 \cap [0,1]} g_0(\alpha) \cap \mathfrak{g}_1\big) \otimes_{W(k)} k$ as its Lie algebra obviously normalizes $\big(g_2(-\delta) \cap \mathfrak{g}_1\big) \otimes_{W(k)} k$. So 3.6.4 implies that it is included in $\bar{P}_{\varepsilon_1}$. This ends the proof of (3.5.7). The last part of 3.5.6 (involving g_2 mod p) is a consequence of the fact that the σ-linear map $\bar{\psi}_2^0$ used in computing the multiplicity of m_δ of $(\mathfrak{g}_0, \varphi_2)$ depends only on g_2 mod p. This ends the proof of the Theorem 3.5.6.

3.6.5. Remark. It is easy to see that 3.5.1-8, 3.5.10 as well as the whole of 3.6 remain true if, instead of $g_2 \in G(W(k))$ (resp. $h_0 \in \tilde{G}_0(W(k))$ for 3.5.10) we work with any inner automorphism of $G_{W(k)}$ (resp. of $\tilde{G}_0$). Warning: this is not so if we use an outer automorphism, as such an automorphism changes the automorphism class of (M, φ, G), producing a different G-ordinary type τ.

In particular we get: if $g \in G^{\mathrm{ad}}(W(k))$ is such that $(\mathfrak{g}, \varphi)$ and $(\mathfrak{g}, g\varphi_0)$ have the same Newton polygon, then at the level of Lie automorphisms of $\mathfrak{g}$ we have $g\varphi_0 = g_3 p_4 \varphi_0 g_3^{-1}$, where g_3 is a $W(k)$-valued point of G^{ad} which mod p defines a k-valued point of the parabolic subgroup of G^{ad} having $F^0(\mathfrak{g})[\frac{1}{p}] \cap \mathrm{Lie}(G^{\mathrm{ad}})$ as its Lie algebra, and where p_4 is a $W(k)$-valued point of the parabolic subgroup of G^{ad} having $\mathrm{Lie}(P_1)[\frac{1}{p}] \cap \mathrm{Lie}(G^{\mathrm{ad}})$ as its Lie algebra. The same remains true if instead of $\mathfrak{g}$ we work with $\mathrm{Lie}(G^{\mathrm{ad}})$; moreover we have a variant of this when we work with $\mathfrak{g}_0$ instead of $\mathfrak{g}$ (resp. with $\oplus_{i \in I} \mathrm{Lie}(G_i^{\mathrm{ad}})$ instead of $\mathrm{Lie}(G^{\mathrm{ad}})$).

3.6.6. We expect that the following statement is true:

Expectation. *Let H be a semisimple split group H over a field k, and let $P_H(1),\ldots,$ $P_H(n)$ be n parabolic subgroups of it ($n \in \mathbb{N}$). Then the dimension of the intersection of the Lie algebras of n parabolic subgroups $\tilde{P}_1,\ldots, \tilde{P}_n$ of H which are respectively $H(k)$-conjugate to $P_H(1),\ldots, P_H(n)$, is maximal precisely in the case when there is a Borel subgroup of H contained in $\cap_{i=1}^n \tilde{P}_i$.*

The case $n = 2$ is easy (to be compared with 3.6.3 2)). For $n > 2$ we can prove it in many situations, but we do not have a proof which works in the general case.

3.7. The proofs of 3.2.6-7.

3.7.1. Let $g_2 \in G(W(k))$. We use the Shimura σ_1-crystals $\mathcal{C}_1$ considered in 3.4.7.1. As $\mathcal{C}_1$ specializes to $(M\otimes_{W(k)} W(\bar{k}), g_3(\varphi_0\otimes 1), G_{W(\bar{k})})$, with $g_3 \in G_{W(\bar{k})}(W(\bar{k}))$ such that mod p belongs to the subset $\mathcal{L}$ of $G(\bar{k})$ defined in 3.4.6, but for $\bar{k}$, we deduce from 3.5.11-13 that the Shimura Lie σ-crystal attached to $\mathcal{C}_1$ has the same Newton polygon as $(\mathfrak{g}, \varphi_1)$, and so as $(\mathfrak{g}, \varphi_0)$ (cf. 3.3.2 and 3.3.5). From the fact that $\mathcal{C}_1$ specializes to $(M, g_2\varphi_0)$, cf. 3.4.7.1, we deduce that the Newton polygon of $(\mathfrak{g}, g_2\varphi_0)$ is above the Newton polygon of $(\mathfrak{g}, \varphi_0)$. The same remains true working just with $\mathfrak{g}_0$.

From 3.3.4-5, and from the fact that Shimura Lie σ-crystal attached to $\mathcal{C}_1$ and $(\mathfrak{g}, \varphi_1)$ have the same Newton polygon, we get (cf. 3.5.11) that the formal isogeny type of $\mathcal{C}_1$ is τ_0. As $\mathcal{C}_1$ specializes to $(M, g_2\varphi_0)$, and as $g_2 \in G(W(k))$ was arbitrary, we get 3.2.6.

3.7.2. We assume here that $k = \bar{k}$. Let

$$\mathfrak{t} \oplus \bigoplus_{\substack{i\in I\\ \alpha\in\Phi_i}} \tilde{g}_i(\alpha) \subset \mathrm{Lie}(G)$$

be a Weyl direct sum decomposition of $\mathrm{Lie}(Z(G)^0 \times \times_{i\in I} G_i)$ associated to the maximal torus T of G. As $\varphi_1(\mathfrak{t}) = \mathfrak{t}$ we deduce that φ_1 permutes the $B(k)$-vector spaces $\tilde{g}_i(\alpha)\otimes_{W(k)} B(k)$. We get a permutation γ_Φ of the disjoint union $\cup_{i\in I}\Phi_i$. So φ_1 takes any $p\tilde{g}_i(\alpha)$ into $\tilde{g}_{\gamma(i_0)}\big(\gamma_\Phi(\alpha)\big)$. More precisely we have

$$\varphi_1\big(\tilde{g}_i(\alpha)\big) = p^s \tilde{g}_{\gamma(i)}\big(\gamma_\Phi(\alpha)\big),$$

where $s \in \{-1, 0, 1\}$ is such that the cocharacter $\mu : \mathbb{G}_m \to T$ has the property that $\mu(\beta)$ acts on $\tilde{g}_i(\alpha)$ as the multiplication with β^{-s}, $\forall \beta \in \mathbb{G}_m(W(k))$. From this we deduce that $\mathrm{Lie}(P_1) \subset F^0(\mathfrak{g})$.

For any $p_1 \in P_1(W(k))$, the parabolic subalgebra $\mathfrak{p}^1_{\geq 0}$ of $\mathfrak{g}$ corresponding to non-negative slopes of $(\mathfrak{g}, p_1\varphi_1)$ is $\mathrm{Lie}(P_1)$. To see this we first remark that $(p_1\varphi_1)^s = p_s\varphi_1^s$, with $p_s \in P_1(W(k))$, $\forall s \in \mathbb{N}$. From this and 3.2.1 we get that $\mathrm{Lie}(P_1) \subset \mathfrak{p}^1_{\geq 0}$. The same argument applies to get the reverse inclusion. So $\mathrm{Lie}(P_1) = \mathfrak{p}^1_{\geq 0}$ (of course we can also quote directly 3.2.1.1).

3.7.3. 3.2.7 a) results directly from 3.7.2 and 3.3.2.

3.7.4. Let (M, F^1_2, φ_2, G) be a Shimura filtered σ-crystal with the property that the parabolic subalgebra ${}_2\mathfrak{p}$ of $\mathfrak{g}$ corresponding to non-negative slopes of $(\mathfrak{g}, \varphi_2)$, is contained in

$F_2^0(\mathfrak{g})$. Let ${}_2B$ be a Borel subgroup of G containing the image of a cocharacter $\mu_2 : \mathbb{G}_m \to G$ of the form $g\mu g^{-1}$ (with $g \in G(W(k))$, and such that $g(F^1) = F_2^1$, and $F_2^1(\mathfrak{g}) \subset \mathrm{Lie}({}_2B) \subset {}_2\mathfrak{p} \subset F_2^0(\mathfrak{g})$. Let $g_3 \in {}_2P(W(k))$, with ${}_2P$ as the parabolic subgroup of G having ${}_2\mathfrak{p}$ as its Lie algebra, such that $g_3\varphi_2\big(\mathrm{Lie}({}_2B)\big) \subset \mathrm{Lie}({}_2B)$ (as in 3.2.3). But now $(M, F_2^1, g_3\varphi_2, G)$ has the same properties as (M, F^1, φ_0, G) which allowed us to prove 3.2.5-6. We get that $(M, g_3\varphi_2)$ has the formal isogeny type τ_0. As $(g_3\varphi_2)^s = g_{3,s}\varphi_2^s$, with $g_{3,s} \in {}_2P(W(k))$, $\forall s \in \mathbb{N}$, we deduce from 3.2.1, that (M, φ_2) has the formal isogeny type τ_0. This proves 3.2.7 c).

3.7.5. Let (M, F_2^1, φ_2, G) be a Shimura filtered σ-crystal having τ_0 as the formal isogeny. From 3.2.6, we deduce that the Lie σ-crystal $\big(\mathrm{End}(M), \varphi_2\big)$ has the smallest Newton polygon $\mathrm{End}(\mathcal{P})$ among all Lie σ-crystals $\big(\mathrm{End}(M), g\varphi_2\big)$, with $g \in G(W(k))$. But for any $g \in G(W(k))$, the Newton polygon of $\big(\mathrm{End}(M), g\varphi_2\big)$ is obtained from the Newton polygon of $(\mathfrak{g}, g\varphi_2)$ and from the Newton polygon of $\big(\mathrm{End}(M)/\mathfrak{g}, g\varphi_2\big)$. Again from 3.2.6 we get that the smallest Newton polygon $\mathrm{End}(\mathcal{P})$ is obtained from the smallest Newton polygon $\mathrm{Lie}_G(\mathcal{P})$ of all Lie σ-crystals $(\mathfrak{g}, g\varphi_2)$, $g \in G(W(k))$, and from the smallest Newton polygon $\mathrm{Lie}_G^{\perp}(\mathcal{P})$ of all quotients $\big(\mathrm{End}(M)/\mathfrak{g}, g\varphi_2\big)$, $g \in G(W(k))$; these quotients can be extended to p-divisible objects

$$\big(\mathrm{End}(M)/\mathfrak{g}, F^0(\mathrm{End}(M)/\mathfrak{g}), F^1(\mathrm{End}(M)/\mathfrak{g}), g\varphi_2\big)$$

of $\mathcal{MF}_{[-1,1]}(W(k))$, but they are not Lie σ-crystals. We deduce that $(\mathfrak{g}, \varphi_2)$ has the same Newton polygon as $(\mathfrak{g}, \varphi_0)$. Changing everything by an isomorphism (given by an element of $G(W(k))$) we can take $F_2^1 = F^1$. But ${}_2P$ and P_0 are conjugate by an element $g_3 \in G(W(k))$ leaving invariant F^1/pF^1 (this is so over $W(\bar{k})$, cf. 3.3.4, and so over $W(k)$ as it can be easily checked starting from [Bo, 20.9]; see 3.5.13). So, up to isomorphism, we can assume that ${}_2P = P_0$. But then obviously $\varphi_2 = p_0\varphi_0$, with $p_0 \in P_0(W(k))$. This, together with 3.3.2 and 3.7.2, ends the proof of 3.2.7 b), and so it ends the proof of 3.2.7.

3.7.6. End of the second proof of 3.1.0. From 3.2.4 we deduce 3.1.0.

3.8. The case when the reductive group G is not quasi-split. The basic results of 3.1 remain true, if we do not assume that G is quasi-split (assumption made in the definition 2.2.8). To see this, we first remark that if G is not quasi-split, then the field k must be infinite (any reductive group over a finite field is quasi-split, cf. [Bo, 16.6]). Let k_1 be a finite Galois extension of k such that G_{k_1} is quasi-split. Let σ_1 be the Frobenius automorphism of $W(k_1)$. We know that 3.1.0 is true for $G_{W(k_1)}$ (i.e. when we work with the Shimura σ_1-crystals $(M \otimes_{W(k)} W(k_1), g(\varphi \otimes 1), G_{W(k_1)})$, with $g \in G(W(k_1))$). So to get 3.1.0 for G we just have to remark two things:

3.8.1. There is $g \in G(W(k))$ such that $(M \otimes_{W(k)} W(k_1), g(\varphi \otimes 1), G_{W(k_1)})$ is a $G_{W(k_1)}$-ordinary σ_1-crystal.

3.8.2. If $(M, g_0\varphi)$, with $g_0 \in G(W(k))$, is a G-ordinary σ-crystal, due to the uniqueness assertion of b) of 3.1.0, the filtration F_0^1 of $M \otimes_{W(k)} W(k_1)$ describing the $G_{W(k_1)}$-canonical lift of the $G_{W(k_1)}$-ordinary σ_1-crystal $\big(M \otimes_{W(k)} W(k_1), g_0(\varphi \otimes 1), G_{W(k_1)}\big)$, is definable over $W(k)$, once it is defined over $W(k_1)$ (i.e. $F_0^1 = \tilde{F}_0^1 \otimes_{W(k)} W(k_1)$, with $\tilde{F}_0^1 \subset M$).

3.8.2 is trivial. 3.8.1 results from the fact that $G(k)$ is Zariski dense in G_{k_1} (cf. [Bo, 18.3]), and from the fact that there is (cf. 3.4.6.1 1)) an open subscheme U of G_{k_1} such that for any $g \in G(W(k_1))$ lifting a k_1-valued point of U, $\big(M \otimes_{W(k)} W(k_1), g(\varphi_0 \otimes 1)\big)$ is a $G_{W(k_1)}$-ordinary σ_1-crystal.

3.9. A supplement to 3.1.

3.9.1. The Lie stable p-rank. Let (M, φ_2, G) be a Shimura σ-crystal. We use the previous notations of §3. For not overloading the notation, we assume $k = \bar{k}$. So G is split.

Let $\Psi : \bigoplus_{i\in I} \mathfrak{g}_i \to \bigoplus_{i\in I} \mathfrak{g}_i$ be the σ-linear map defined by the rule:

$$\Psi(x) := \begin{cases} p\varphi_2(x) & \text{if } x \in \mathfrak{g}_i \text{ and } F^1(\mathfrak{g}_i) \neq 0 \\ \varphi_2(x) & \text{if } x \in \mathfrak{g}_i \text{ and } F^1(\mathfrak{g}_i) = 0\,. \end{cases}$$

Let $\bar{\Psi} : \bigoplus_{i\in I} \mathfrak{g}_i \otimes_{W(k)} k \to \bigoplus_{i\in I} \mathfrak{g}_i \otimes_{W(k)} k$ be obtained by taking $\bar{\Psi}$ mod p. Let

$$N_{\varphi_2} := \bigcap_{m\in\mathbb{N}} \mathrm{Im}(\bar{\Psi}^m).$$

We define

$$R_{\varphi_2} := \dim_k(N_{\varphi_2}).$$

We call N_{φ_2} the Lie stable k-vector space of (M, φ_2, G) and we call R_{φ_2} the Lie stable p-rank of (M, φ_2, G).

Similarly we define the Lie stable k-vector space and the Lie stable p-rank of any Shimura Lie σ-crystal over k.

3.9.1.1. Remark. The Lie stable p-ranks and the Lie stable k_1-vector spaces are well defined for any Shimura σ_1-crystal (M_1, φ^1, G_1) over a perfect field k_1. This is so due to the fact that the map Ψ_1 defined as above but starting from the extension to $\overline{k_1}$ of (M_1, φ^1, G_1) is in fact defined over $W(k_1)$, cf. the existence of a cocharacter $\mu_1 : \mathbb{G}_m \to G_1$ defining the filtration class of (M_1, φ^1, G_1) and of the structure (see [Ti1]) of $W(k_1)$-simple adjoint groups. In other words: $\mathrm{Lie}(G_1^{\mathrm{ad}})$ is a direct sum of $W(k_1)$-simple factors, each of them having the property that all its simple factors over $W(\bar{k}_1)$ are producing (as in 3.5.3.4) either only positive or only zero ε_i's numbers.

3.9.2. Theorem. *A Shimura σ-crystal (M, φ_2, G) is a G-ordinary σ-crystal iff its Lie stable p-rank is maximal (i.e. $R_{\varphi_2} \geq R_{\varphi_3}$, for any $\varphi_3 = g_3\varphi_2$, with $g_3 \in G(W(k))$), and so equal to R_{φ_0}.*

Proof: We remark that $(\mathfrak{g}, \varphi_2)$ has the smallest Newton polygon among all Shimura Lie σ-crystals $(\mathfrak{g}, g\varphi_2)$, iff for any cycle γ_0 of the permutation γ of I, $(\mathfrak{g}_0, \varphi_2)$ has the smallest Newton polygon among all Shimura Lie σ-crystals $(\mathfrak{g}_0, g\varphi_2)$ (here $g \in G(W(k))$, γ_0 is a cyclic permutation of a subset I_0 of I, and $\mathfrak{g}_0 = \oplus_{i\in I_0}\mathfrak{g}_i$). So everything results from 3.5.10, once we interpret R_{φ_2} as the sum indexed by elements of $\mathcal{J}_0$, each element of the sum being a suitable multiplicity of some slope δ_s, defined for each $s \in \mathcal{J}_0$, in the same manner as δ; for instance $\delta_0 = \delta$ and the corresponding multiplicity is the multiplicity of the slope δ for $(\mathfrak{g}_0, \varphi_2)$.

3.9.3. Corollary. *The fact that a Shimura σ-crystal $(M, g\varphi, G)$ is G-ordinary depends only on the value of g mod p.*

Proof: This results from 3.9.2 once we remark that the Lie stable p-rank of $(M, g\varphi)$ depends only on the value of g module p.

3.9.3.1. We refer to 3.4.1.3. From 3.4.18.4 B), 3.1.0 and 3.9.3 we get directly:

Corollary. *If $y \in \mathrm{Spec}(R_j)(\bar{k})$ gives birth to a G-ordinary σ-crystal (3.9.3 guarantees that this makes sense), and if $r_y = \dim(G)$ (see 3.4.18.9 for the definition of r_y), then y lifts to a point $y_\infty \in \mathrm{Spec}(Q_j)(\bar{k})$.*

3.9.4. The refined Lie stable p-rank. We use the notations from the beginning of 3.5. We choose a bijection $f_{\mathcal{J}_0}$ between $\mathcal{J}_0$ and the set $\{1, 2, \ldots, |\mathcal{J}_0|\}$. Then for any $s \in \mathcal{J}_0$, we can define the Lie stable p-rank $R_{\varphi_2}(s)$ of (M, φ_2, G) with respect to s. For instance, if we work with the cycle γ_0, $R_{\varphi_2}(f_{\mathcal{J}_0}(0))$ is nothing else but the Lie stable p-rank of $(\mathfrak{g}_0, \varphi_2)$. We denote by $R^r_{\varphi_2}$ the sequence $(a_1, \ldots, a_{|\mathcal{J}_0|})$ of $|\mathcal{J}_0|$ non-negative integers defined by the formula:

$$a_l := \frac{R_{\varphi_2}\big(f_{\mathcal{J}_0}^{-1}(l)\big)}{\mathrm{length}\big(\gamma_{f_{\mathcal{J}_0}^{-1}(l)}\big)},$$

for any $l \in \{1, 2, \ldots, |\mathcal{J}_0|\}$ (a_l are non-negative integers cf. (3.5.2); for instance if $l = f_{\mathcal{J}_0}(0)$, then $R_{\varphi_2}(0)$ is the multiplicity of the slope δ of $(\mathfrak{g}_0, \varphi_2)$, and so is a multiple of $|I_0|$).

On the set of sequences of $|\mathcal{J}_0|$ non-negative integers we have a partial order defined by $(b_1, b_2, \ldots, b_{|\mathcal{J}_0|}) \leq (c_1, c_2, \ldots, c_{|\mathcal{J}_0|})$ if $b_i \leq c_i$, $\forall i \in \{1, 2, \ldots, |\mathcal{J}_0|\}$. From the proof of 3.9.2 we get:

3.9.5. Theorem. *A Shimura σ-crystal (M, φ_2, G) is a G-ordinary σ-crystal iff its refined Lie stable p-rank is maximal, equal to $R^r_{\varphi_0}$.*

3.9.6. Remark. It is an easy exercise to see that the refined Lie stable p-rank of a Shimura σ-crystal or of a Shimura Lie σ-crystal over k is the same, if instead of working with Shimura Lie σ-crystals of the above form $(\mathfrak{g}_0, \varphi_2)$, we work with their adjoint Lie σ-crystal $(\mathfrak{g}_0^{\mathrm{ad}}, \varphi_2)$, where $\mathfrak{g}_0^{\mathrm{ad}} := \mathrm{Lie}(\tilde{G}_0^{\mathrm{ad}}) \subset \mathrm{Lie}(\tilde{G}_0)[\frac{1}{p}]$ (cf. 2.2.13).

3.9.7. The zero, the positive, and the non-negative rank and type. Let k again be an arbitrary perfect field. By the non-negative rank (resp. type) of a Shimura σ-crystal (M, φ, G) over k (or of its attached Shimura Lie σ-crystal) we mean the rank (resp. the $G(W(k))$-conjugacy class) of the parabolic Lie subalgebra $\mathfrak{p}_{\geq 0}$ of $\mathrm{Lie}(G)$ corresponding to non-negative slopes of $(\mathrm{Lie}(G), \varphi)$. Similarly we define the positive and the zero ranks and types, by using positive slopes and respectively the slope zero (of course the wording "parabolic" has to be replaced accordingly). This definition extends to Shimura Lie σ-crystals, or to a filtered context.

3.10. Terminology and formulae.

3.10.0. Cyclic Shimura Lie σ-crystals. The notations that follows are similar to the previous ones, but, without specifically stating, they are independent. A Shimura adjoint Lie σ-crystal $(\mathfrak{g}, \varphi)$ is said to be cyclic, if $\mathfrak{g}$ is the Lie algebra of an adjoint group G over $W(k)$ such that $G_{W(\bar{k})} = \prod_{i\in\tilde{I}} G'_i$, with all ${G'_i}^{\mathrm{ad}}$ as simple adjoint groups over $W(\bar{k})$, and with φ permuting cyclically the direct summands Lie $(G'_i)[\frac{1}{p}]$ of $\mathfrak{g} \otimes_{W(k)} W(\bar{k})[\frac{1}{p}]$.

$(\mathfrak{g}, \varphi)$ is said to be trivial if $\varphi(\mathfrak{g}) = \mathfrak{g}$; otherwise it is said to be non-trivial. $(\mathfrak{g}, \varphi)$ is said to be of A_ℓ, B_ℓ, C_ℓ, D_ℓ, E_6 or E_7 Lie type if the factors G'_i are of this Lie type. We do not bother here about the other Lie types, as they force $(\mathfrak{g}, \varphi)$ to be trivial, cf. 2.2.7.

Let $g \in G(W(\bar{k}))$ such that $g\varphi$ leaves invariant the Lie algebras $\mathfrak{b}$ and $\mathfrak{t}$ of a Borel subgroup $B_{W(\bar{k})}$ of $G_{W(\bar{k})}$, and respectively of a maximal torus $T_{W(\bar{k})}$ of $B_{W(\bar{k})}$ (cf. 3.2.3). The permutation of the basis of roots of $\mathrm{Lie}(G'_i)$ with respect to the maximal torus T_i of G'_i contained in $T_{W(\bar{k})} \cap G'_i$, and corresponding to the Borel subgroup $B'_i := B_{W(\bar{k})} \cap G'_i$ of G'_i, induced by $(g\varphi)^{|\tilde{I}|}$, can be non-trivial. This permutation can be:

– an involution (this can happen only when $\mathfrak{g}$ is of A_ℓ, D_ℓ or E_6 Lie type);

– a cycle of order 3 (this can happen only when $\mathfrak{g}$ is of D_4 Lie type);

– trivial.

We say respectively that $(\mathfrak{g} \otimes_{W(k)} W(\bar{k}), g\varphi)$ (or $(\mathfrak{g}, \varphi)$) is with involution, or with a 3-cycle, or without a cycle.

The above definitions will be used as well in a filtered context, or in a non-necessarily adjoint context (i.e. when we work with a semisimple group over $W(k)$ which is not necessarily adjoint).

3.10.0.1. Remark. If we require just that the simple factors of $\mathrm{Lie}(G)$ to be permuted cyclically by φ, then we say that $(\mathfrak{g}, \varphi)$ is weakly cyclic. The notions of weakly cyclic and cyclic coincide over finite fields, but not in general. In what follows, without specifically stating, we will deal only with the cyclic notion.

3.10.1. Definitions: cyclic factors. Any split Shimura adjoint (filtered) Lie σ-crystal $\mathfrak{C}^{\mathrm{ad}}$ is isomorphic to a direct sum of cyclic Shimura adjoint (filtered) Lie σ-crystals, which are called the cyclic factors of $\mathfrak{C}^{\mathrm{ad}}$. The direct sum of the non-trivial cyclic factors is called the non-trivial part of $\mathfrak{C}^{\mathrm{ad}}$, while the direct sum of the trivial ones is called the trivial part of $\mathfrak{C}^{\mathrm{ad}}$.

Similarly, for any Shimura σ-crystal (M, φ, G) or Shimura filtered σ-crystal (M, F^1, φ, G) (resp. for any Shimura Lie σ-crystal $(\mathrm{Lie}(G), \varphi)$, or any Shimura filtered Lie σ-crystal $(\mathrm{Lie}(G), \varphi, F^0(\mathrm{Lie}(G)), F^1(\mathrm{Lie}(G))))$, with G^{ad} a split group, we define its cyclic Lie factors (resp. its cyclic factors), as well as its cyclic adjoint factors. For instance, the Shimura filtered Lie σ-crystal $(\mathfrak{g}_0, \varphi_2, F^0(\mathfrak{g}_0), F^1(\mathfrak{g}_0))$ of 3.5.1 is cyclic, and is called a cyclic Lie factor of (M, F^1, φ_2) (or a cyclic factor of $(\mathfrak{g}, \varphi_2, F^0(\mathfrak{g}), F^1(\mathfrak{g})))$; its adjoint (defined by the Lie algebra $\oplus_{i\in I_0}\mathrm{Lie}(G_i^{\mathrm{ad}})$) is called a cyclic adjoint factor of (M, F^1, φ_2) or of $(\mathfrak{g}, \varphi_2, F^0(\mathfrak{g}), F^1(\mathfrak{g}))$. Similarly we speak about the non-trivial part (or the trivial part) of a split Shimura (filtered) Lie σ-crystal.

Similarly, for any Shimura (adjoint) (filtered) Lie σ-crystal (resp. any Shimura (filtered) σ-crystal), we speak about its weakly cyclic (adjoint) factors (resp. about its weakly cyclic Lie factors and its weakly cyclic adjoint factors), and about these factors being trivial or non-trivial. The difference is: the adjoint groups we get are not always product of absolutely simple adjoint groups (cf. 3.10.0.1).

3.10.2. From now on we work with a cyclic Shimura adjoint filtered Lie σ-crystal $\big(\mathfrak{g}, \varphi, F^0(\mathfrak{g}), F^1(\mathfrak{g})\big)$ which is non-trivial, and has the property that there are Lie algebras $\mathfrak{b}$ and $\mathfrak{t}$ of a Borel subgroup B of G and respectively of a maximal torus T of B, such that $F^1(\mathfrak{g}) \subset \mathfrak{b} \subset F^0(\mathfrak{g})$, $\varphi(\mathfrak{t}) = \mathfrak{t}$, and $\varphi(\mathfrak{b}) \subset \mathfrak{b}$ (cf. 3.2.3). Here G is as in 3.10.0.

All the terminology to be introduced below for the case G split adjoint will be also used in the context of 3.10.2 (by passage to $W(\bar{k})$), for the cyclic factors of split Shimura filtered Lie σ-crystals defined by Shimura-canonical lifts (i.e. for factors like $\big(\mathfrak{g}_0, \varphi_1, F^0(\mathfrak{g}_0), F^1(\mathfrak{g}_0)\big)$ of 3.2.3), as well as for the adjoints of such factors.

3.10.3. From now on we also assume that G is a split group. For not overloading the notations, we assume that $G = \times_{i \in I_0} G_i$, $\mathfrak{g} = \mathfrak{g}_0$, $F^0(\mathfrak{g}) = F^0(\mathfrak{g}_0)$, $F^1(\mathfrak{g}) = F^1(\mathfrak{g}_0)$, and $\varphi = \varphi_1$ (of 3.2.3). So we use fully the notations and conventions introduced in 3.5.0-3 and 3.5.5. So $\tilde{I} = I_0$ and $G'_i = G_i$, $\forall i \in I_0$. Warning: here G_i are adjoint groups; so only the notations are as in the mentioned places.

3.10.4. The ε-type. The sequence $(\varepsilon_1, \ldots, \varepsilon_m)$ is called the ε-type of $(\mathfrak{g}_0, \varphi_1)$. Here

$$m := \left|\tilde{I}_0\right| = e\left|I_0\right| \in \{\left|I\right|_0, 2\left|I_0\right|, 3\left|I_0\right|\}.$$

If $(\mathfrak{g}_0, \varphi_1)$ is with involution or with a 3-cycle, then we replace $(\mathfrak{g}_0, \varphi_1)$ by $(\oplus_{j \in \{1,\ldots,e\}} \mathfrak{g}_0, \varphi_1^{(e)})$ (cf. 3.5.3.2) (e being 2 or 3, depending on π_1 being an involution or a 3-cycle). So $\varepsilon_i \in \{0, 1, \ldots, \ell\}$.

3.10.4.1. Warning. We replace $(\mathfrak{g}_0, \varphi_1)$ by $\big(\oplus_{j \in \{1,\ldots,e\}} \mathfrak{g}_0, \varphi_1^{(e)}\big)$ only for defining the ε-type of $(\mathfrak{g}_0, \varphi_1)$. For the formulae that follows this is not allowed.

3.10.4.2. If $(\mathfrak{g}_0, \varphi_1)$ is without a cycle, then $(\varepsilon_1, \ldots, \varepsilon_m)$ is uniquely defined up to a cyclic permutation, and up to a renumbering of the roots of Δ_1 (cf. 3.5.3.2). We have $m = n = |I_0|$.

If $(\mathfrak{g}_0, \varphi_1)$ is with involution then $(\varepsilon_1, \ldots, \varepsilon_m)$ is uniquely determined up to a cyclic permutation, and, in the case when $\mathfrak{g}_0$ is of D_4 Lie type, up to a renumbering of the roots of Δ_1. We have $m = 2n$.

If $(\mathfrak{g}_0, \varphi_1)$ is with a 3-cycle then $(\varepsilon_1, \ldots, \varepsilon_m)$ is uniquely determined up to cyclic permutation, and up to a renumbering of the roots of Δ_1. We have $m = 3n$.

3.10.4.3. Convention. We say that an ε-type $(\varepsilon_1, \ldots, \varepsilon_m)$ is in standard form if $\sum_{i=1}^m \varepsilon_i(\ell+1)^{m-i}$ is the biggest number among all the numbers obtained in this manner using an ε-type obtained from $(\varepsilon_1, \ldots, \varepsilon_m)$ through the rules of this 3.10.4.2 (i.e. up to renumbering and cyclic permutation). Here the role of $\ell+1$ is irrelevant: we just need an integer bigger then all ε_i's. We always prefer to use ε-types which are in standard form. For instance we prefer to work with the ε-type $(1, 1, 0, 0)$ rather than the ε-type $(0, 0, 1, 1)$.

3.10.4.4. Spreadings. We define $A := \{\varepsilon_i | i \in I_1\}$ and $B := \{\varepsilon_i | i \in \tilde{I}_1\}$. We have $|A| \leq |B| \leq 2\,|A|$. The numbers $|A|$ and $|B|$ are called the apparent spreading and respectively the spreading of the cyclic Shimura Lie σ-crystal $(\mathfrak{g}_0, \varphi_1)$.

3.10.5. Definitions. If $\varepsilon_i \neq 0$, $\forall i \in \tilde{I}_0$, then $(\mathfrak{g}_0, \varphi_1)$ is said to be of totally non-compact type. If there is $i \in I_0$ such that $\varepsilon_i = 0$, then we say that $(\mathfrak{g}_0, \varphi_1)$ has compact factors or is of compact type.

If all ε_i, $i \in \{1, ..., m\}$, are equal and different from zero, we say that $(\mathfrak{g}_0, \varphi_1)$ is totally non-compact of constant type, or totally non-compact of type:

i) 1 if $\mathfrak{g}_0$ is of B_ℓ, C_ℓ, D_4, E_6 or E_7 Lie type, or of D_ℓ Lie type with $\varepsilon_1 = 1$.

ii) 2 if $\mathfrak{g}_0$ is of D_ℓ Lie type, with $\ell \geq 5$, and $\varepsilon_1 \in \{\ell - 1, \ell\}$.

iii) $\min\{\varepsilon_i, 1 + \ell - \varepsilon_i\}$ if $\mathfrak{g}_0$ is of A_ℓ Lie type.

In case i) for $\mathfrak{g}_0$ of type D_ℓ, with $\ell \geq 4$, we also say that $(\mathfrak{g}_0, \varphi_1)$ is totally non-compact of $D_\ell^{\mathbb{R}}$ type. In case ii) we also say that $(\mathfrak{g}_0, \varphi_1)$ is of simple $D_\ell^{\mathbb{H}}$ type. Similarly if all non-zero ε_i are equal ($i \in \{1, ..., m\}$) but there is an $\varepsilon_{i_0} = 0$ ($i_0 \in I_0$), we say that $(\mathfrak{g}_0, \varphi_1)$ is of constant type, or of type 1, or 2, or $\min\{\varepsilon_i, 1 + \ell - \varepsilon_i\}$ (or of $D_\ell^{\mathbb{R}}$ type or of simple $D_\ell^{\mathbb{H}}$ type) depending on which of the situations i)-iii) above we are, but dropping (sometimes replacing by the expression compact) the expression of totally non-compact type.

If $|B| \geq 2$ we say that $(\mathfrak{g}_0, \varphi_1)$ is of non-constant type. This can happen only when $\mathfrak{g}_0$ is of A_ℓ, D_ℓ or E_6 Lie type.

If $\mathfrak{g}_0$ is of D_ℓ Lie type, with $B = \{\ell - 1, \ell\}$, or if $\mathfrak{g}_0$ is of D_4 Lie type, with $|B| = 2$, we say that $(\mathfrak{g}_0, \varphi_1)$ is of (non-simple) $D_\ell^{\mathbb{H}}$ type. In the cases when $\mathfrak{g}_0$ is of D_4 Lie type with $|B| = 3$, or of D_ℓ Lie type, with $\ell \geq 5$ and $|B| \geq 2$, but $B \neq \{\ell - 1, \ell\}$, we say that $(\mathfrak{g}_0, \varphi_1)$ is of mixed D_ℓ type. If $\mathfrak{g}_0$ is of D_ℓ Lie type with $|B| = 3$, that is $B = \{1, \ell - 1, \ell\}$, we say that $(\mathfrak{g}_0, \varphi_1)$ is of strong mixed D_ℓ type.

If $\mathfrak{g}_0$ is of A_ℓ Lie type with $|B| \geq 2$, we say that $(\mathfrak{g}_0, \varphi_1)$ is of $\{a, b\}$ A_ℓ type, where $1 \leq a < b \leq \ell$ and $a \leq \ell + 1 - b$, and, if potentially changing $\ell - i$ with $i + 1$, $i = \overline{0, \ell - 1}$, a is the smallest element of B and b is the biggest element of B.

$q_1 = |I_1|$ is called the non-compact length of $(\mathfrak{g}_0, \varphi_1)$, and $q_0 = n = |I_0|$ is called the cyclic length of $\mathfrak{g}_0$. As we have assumed that $(\mathfrak{g}_0, \varphi_1)$ is non-trivial, we have $q_1 \geq 1$. By the rank of $(\mathfrak{g}_0, \varphi_1)$ we mean the rank $\ell \in \mathbb{N}$ of any simple Lie factor of $\mathfrak{g}_0$.

Most of the above terminology is inspired from and conforms with [De1].

3.10.6. Formulae. The formulae below are based on the list of hermitian symmetric domains and of their real dimension (see [He, p. 518]), and on the standard tables of roots of the simple split semisimple Lie algebras over a field of characteristic zero (see [Bou, planche I-VI]). There is nothing deep about the computations involved, so we just list the results.

i) Let $\mathfrak{g}_0$ be of B_ℓ, C_ℓ or E_7 Lie type.

The slopes of $(\mathfrak{g}_0, \varphi_1)$ are precisely $-\frac{q_1}{q_0}$, 0 and $\frac{q_1}{q_0}$. The multiplicity of $\frac{q_1}{q_0}$ is $q_0 \dim(X)$, where X is a hermitian symmetric domain associated to an $\mathbb{R}$-simple adjoint group of the same Lie type as $\mathfrak{g}_0$.

If $\mathfrak{g}_0$ is of C_ℓ Lie type, then $\dim(X) = \frac{\ell(\ell+1)}{2}$.
If $\mathfrak{g}_0$ is of B_ℓ Lie type, then $\dim(X) = 2\ell - 1$.
If $\mathfrak{g}_0$ is of E_7 Lie type, then $\dim(X) = 27$.

ii) Let $\mathfrak{g}_0$ be of E_6 Lie type.

Case 1. $(\mathfrak{g}_0, \varphi_1)$ is of type 1; then everything is as in i), with $\dim(X) = 16$.

Case 2. $(\mathfrak{g}_0, \varphi_1)$ is of non-constant type, but without a cycle, and with $|I_2| \neq |I_1 \backslash I_2|$, where $I_2 := \{i \in I_1 | \varepsilon_i = 1\}$. We can assume that $|I_2| > |I_1 \backslash I_2|$. Let $q_2 := |I_2|$. The slopes of $(\mathfrak{g}_0, \varphi_1)$ are precisely $-\frac{q_1}{q_0}$, $-\frac{q_2}{q_0}$, $\frac{-q_1+q_2}{q_0}$, 0, $\frac{q_1-q_2}{q_0}$, $\frac{q_2}{q_0}$ and $\frac{q_1}{q_0}$.

The multiplicities of the slopes $\frac{q_1}{q_0}$, $\frac{q_2}{q_0}$ and $\frac{q_1-q_2}{q_0}$ are all equal to $8q_0$.

Case 3. $(\mathfrak{g}_0, \varphi_1)$ is of non-constant type with involution, or is without a cycle, but with $|I_2| = |I_1 \backslash I_2|$ (with I_2 as above). The slopes of $(\mathfrak{g}_0, \varphi_1)$ are precisely $\frac{-q_1}{q_0}$, $\frac{-q_1}{2q_0}$, 0, $\frac{q_1}{2q_0}$ and $\frac{q_1}{q_0}$.

The multiplicity of $\frac{q_1}{2q_0}$ is $16q_0$. The multiplicity of $\frac{q_1}{q_0}$ is $8q_0$.

iii) Let $\mathfrak{g}_0$ be of D_ℓ Lie type.

Case 1. $(\mathfrak{g}_0, \varphi_1)$ is of type 1 (i.e. of $D_\ell^{\mathbb{R}}$ type); then the situation is as in i), with $\dim(X) = 2(\ell - 1)$.

Case 2. $(\mathfrak{g}_0, \varphi_1)$ is of type 2 (i.e. it is of simple $D_l^{\mathbb{H}}$ type); then the situation is as in i), with $\dim(X) = \frac{\ell(\ell-1)}{2}$.

Case 3. $(\mathfrak{g}_0, \varphi_1)$ is of $D_\ell^{\mathbb{H}}$ type. We can assume that $B = \{\ell - 1, \ell\}$ even for $\ell = 4$. Let $I_2 := \{i \in I_1 | \varepsilon_i = \ell - 1\}$. We can assume that $|I_2| \geq |I_1 \backslash I_2|$. Let $q_2 := |I_2|$.

Subcase 1. $(\mathfrak{g}_0, \varphi_1)$ is without a cycle and $q_2 \neq q_1 - q_2$. The slopes are like in case 2 of ii).

The multiplicity of the slope $\frac{q_1}{q_0}$ is $\frac{(\ell-1)(\ell-2)}{2} q_0$.

The multiplicity of the slope $\frac{q_2}{q_0}$ is $(\ell - 1)q_0$ and is equal to the multiplicity of $\frac{-q_2+q_1}{q_0}$.

Subcase 2. $(\mathfrak{g}_0, \varphi_1)$ is without a cycle and $|I_2| = |I_1 \backslash I_2|$, or is with involution. The slopes are like in case 3 of ii).

The multiplicity of $\frac{q_1}{q_0}$ is $\frac{(\ell-1)(\ell-2)}{2} q_0$.

The multiplicity of $\frac{q_1}{2q_0}$ is $2(\ell - 1)q_0$.

Case 4. $(\mathfrak{g}_0, \varphi_1)$ is of mixed type.

Subcase 1. $(\mathfrak{g}_0, \varphi_1)$ is of strong mixed type. Let $I_2 := \{i \in I_1 | \varepsilon_i = 1\}$, $I_3 := \{i \in I_1 | \varepsilon_i = \ell - 1\}$, and $I_4 := \{i \in I_1 | \varepsilon_i = \ell\}$. Let $q_j := |I_j|$ for $j = \overline{2, 4}$. We have $q_1 = q_2 + q_3 + q_4$. We can assume $q_3 \geq q_4$.

Subcase 1.1. $(\mathfrak{g}_0, \varphi_1)$ is without a cycle. The set of slopes is formed from 0 and $\pm a$, where a is of the form $\frac{\sum_{j \in \mathcal{J}_0} q_j}{q_0}$, with $\mathcal{J}_0$ a non-empty subset of $\{q_2, q_3, q_4\}$.

The multiplicity of

$$a = \frac{\sum_{j \in \mathcal{J}_0} q_j}{q_0}$$

is equal to

$$\Big(\sum_{\mathcal{J} \in \mathcal{J}_1} r_{\mathcal{J}}\Big) q_0,$$

where $\mathcal{J}_1$ is the set of non-empty subsets of $\{2,3,4\}$, with the property that $\sum_{j \in \mathcal{J}_1} q_j = a q_0$. Here $r_{\{2,3,4\}} = \ell - 2$, $r_{\{2,3\}} = r_{\{2,4\}} = 1$, $r_{\{3,4\}} = \frac{(\ell-3)(\ell-2)}{2}$, and $r_{\{2\}} = r_{\{3\}} = r_{\{4\}} = \ell - 2$.

Subcase 1.2. $(\mathfrak{g}_0, \varphi_1)$ is with a 3-cycle. This implies $\ell = 4$. The slopes are $-\frac{q_1}{q_0}, \frac{-2q_1}{3q_0}, \frac{-q_1}{3q_0}, 0, \frac{q_1}{3q_0}, \frac{2q_1}{3q_0}$ and $\frac{q_1}{q_0}$.

The multiplicity of the slope $\frac{q_1}{q_0}$ is $2q_0$.

The multiplicity of the slope $\frac{2q_1}{3q_0}$ is $3q_0$.

The multiplicity of the slope $\frac{q_1}{3q_0}$ is $6q_0$.

Subcase 1.3. $(\mathfrak{g}_0, \varphi_1)$ is with involution. If $\ell = 4$, then we assume that the involution of the basis of roots of $\mathfrak{g}_1$ takes $\alpha_{\ell-1}$ into α_ℓ (to fit the assumption $q_3 \geq q_4$). The slopes are $0, \pm a$, with a of the form $\big(\sum_{j \in \mathcal{J}_0} q_j\big)\frac{1}{2q_0}$, for $\mathcal{J}_0$ one of the sets $\{2,3,4,5,6,7\}, \{2,3,4,5\}, \{3,4,6,7\}, \{2,5\}$ or $\{3,4\}$ (here $q_{3+i} = q_i$ for $i = \overline{2,4}$).

The multiplicity of

$$a = \frac{1}{2q_0}\Big(\sum_{j \in \mathcal{J}_0} q_j\Big)$$

is

$$\Big(\sum_{\mathcal{J}_1 \in \mathcal{J}_2} r_{\mathcal{J}_1}\Big) q_0,$$

where $\mathcal{J}_2$ is the set of those sets $\mathcal{J}$ as above for which $\sum_{j \in \mathcal{J}} q_j = 2q_0 a = \sum_{j \in \mathcal{J}_0} q_j$. Here $r_{\mathcal{J}_1}$ is $\ell - 2$ if $\mathcal{J}_1 = \{2,3,4,5,6,7\}$ or $\mathcal{J}_1 = \{2,5\}$, is 2 if $\mathcal{J}_1 = \{2,3,4,5\}$, is $\frac{(\ell-3)(\ell-2)}{2}$ if $\mathcal{J}_1 = \{3,4,6,7\}$, and is $2(\ell-2)$ if $\mathcal{J}_1 = \{3,4\}$.

Subcase 2. $(\mathfrak{g}_0, \varphi_1)$ is not of strong mixed type. This implies that $(\mathfrak{g}_0, \varphi_1)$ is without an involution and that $\ell \geq 5$. This subcase is entirely analogous to the subcase 1.1, for the value $q_4 = 0$.

iv) Let $\mathfrak{g}_0$ be of A_ℓ type ($\ell \geq 2$).

Case 1. $|B| = 1$. This happens when $|A| = 1$ and $(\mathfrak{g}_0, \varphi_1)$ is without a cycle, or when $(\mathfrak{g}, \varphi_1)$ is with involution and $A = \{\frac{\ell+1}{2}\}$ (in this second situation ℓ must be odd).

The slopes are $-\frac{q_1}{q_0}$, 0 and $\frac{q_1}{q_0}$.

The multiplicity of the slope $\frac{q_1}{q_0}$ is $i_0(\ell + 1 - i_0)q_0$, where $B = \{i_0\}$.

Case 2. $|B| = 2$. We distinguish three subcases.

Subcase 1. $A = \{i_0\}$ with $i_0 \neq \frac{\ell+1}{2}$, and $(\mathfrak{g}_0, \varphi_1)$ is with involution. We can assume $i_0 < \frac{\ell+1}{2}$. The slopes are $-\frac{q_1}{q_0}, \frac{-q_1}{2q_0}, 0, \frac{q_1}{2q_0}$ and $\frac{q_1}{q_0}$.

The multiplicity of $\frac{q_1}{q_0}$ is $i_0^2 q_0$.

The multiplicity of $\frac{q_1}{2q_0}$ is $2i_0(\ell+1-2i_0)q_0$.

Subcase 2. $A = \{i_0, i_1\}$ with $1 \leq i_0 \leq \frac{\ell+1}{2}$, $i_0 < i_1$, and $(\mathfrak{g}_0, \varphi_1)$ is without a cycle. Let $I_2 := \{j \in I_1 | \varepsilon_j = i_0\}$ and let $q_2 := |I_2|$.

Subcase 2.1. $2q_2 \neq q_1$. The slopes are $\pm\frac{q_1}{q_0}$, $\pm\frac{q_2}{q_0}$, $\pm\frac{q_1-q_2}{q_0}$ and 0.

The multiplicity of $\frac{q_1}{q_0}$ is $i_0(\ell+1-i_1)q_0$.

The multiplicity of $\frac{q_2}{q_0}$ is $i_0(1+i_1-i_0)q_0$.

The multiplicity of $\frac{q_1-q_2}{q_0}$ is $(\ell+1-i_1)(-i_0+i_1)q_0$.

Subcase 2.2. $2q_2 = q_1$. The slopes are $\pm\frac{q_1}{q_0}, \pm\frac{q_1}{2q_0}$, and 0.

The multiplicity of $\frac{q_1}{q_0}$ is $i_0(\ell+1-i_1)q_0$.

The multiplicity of $\frac{q_1}{2q_0}$ is $\big[i_0(i_1-i_0+1)+(i_1-i_0)(\ell+1-i_1)\big]q_0$.

Subcase 3. $A = \{i_0, \ell+1-i_0\}$ and $(\mathfrak{g}_0, \varphi_1)$ is with involution. This is entirely similar to the subcase 1. The slopes are $-\frac{q_1}{q_0}, -\frac{q_1}{2q_0}, 0, \frac{q_1}{2q_2}$ and $\frac{q_1}{q_0}$.

The multiplicity of $\frac{q_1}{q_0}$ is $i_0^2 q_0$.

The multiplicity of $\frac{q_1}{2q_0}$ is $2i_0(\ell+1-2i_0)q_0$.

Case 3. $|B| \geq 3$. So $|A| \geq 2$. We do not treat this case. We just mention that the multiplicity of the biggest positive slope $\frac{q_1}{q_0}$ is $i_0(\ell+1-i_1)$, where i_0 (resp. i_1) is the smallest (resp. the biggest) element of B.

3.10.7. Remark. If the ε-type of $(\mathfrak{g}_0, \varphi_1)$ is $(\varepsilon_1, \ldots, \varepsilon_m)$, then the parabolic Lie subalgebra $\mathfrak{p}_0$ of $\mathfrak{g}_0$ corresponding to non-negative slopes of $(\mathfrak{g}_0, \varphi_1)$ is a direct sum of its intersection with the factors $\mathfrak{g}_i$ of $\mathfrak{g}_0$ ($i \in I_0$), while its intersection with $\mathfrak{g}_i$ is the $W(k)$-module generated by $\mathfrak{t}_i$, and by the (one dimensional) $W(k)$-submodules corresponding to all positive roots (of Φ_i^+), and to those negative roots (of Φ_i^{-1}), not having $-\alpha_{\varepsilon_j}(i)$ in their expression, for any $j \in \{1, \ldots, m\}$ such that $\varepsilon_j \neq 0$.

Using this we get directly the following facts:

F1. $\mathfrak{p}_0 \cap \mathfrak{g}_i$ *is a maximal parabolic Lie subalgebra of* $\mathfrak{g}_i$ *if and only if* $(\mathfrak{g}_0, \varphi_1)$ *is of constant type.*

F2. $\mathfrak{p}_0 = F^0(\mathfrak{g}_0)$ *if and only if* $(\mathfrak{g}_0, \varphi_1)$ *is totally non-compact of constant type.*

F3. $\mathfrak{p}_0$ *is a Borel Lie subalgebra of* $\mathfrak{g}$ *if and only if* $\mathfrak{g}_0$ *is of* A_ℓ *Lie type with* $B = \{1, 2, \ldots, \ell\}$. *Here* $\ell \geq 1$.

When the situation of $F3$ takes place, we say that $\big(\mathfrak{g}_0, \varphi_1, F^0(\mathfrak{g}_0), F^1(\mathfrak{g}_0)\big)$ is of strong Borel type (cf. c) of 2.2.12), or that it has maximal A-spreading.

3.11. The case $k = \bar{k}$. We use the notations of 3.1-2. We assume that $(M, \varphi, G, (v_\alpha)_{\alpha \in \mathcal{J}})$ is a G-ordinary σ-crystal.

3.11.1. Theorem. *Let $\mathfrak{C} = (M, F^1, \varphi, G, (v_\alpha)_{\alpha \in \mathcal{J}})$ be the G-canonical lift of $(M, \varphi, G, (v_\alpha)_{\alpha \in \mathcal{J}})$. We have:*

a) *it is cyclic diagonalizable;*

b) *its Shimura filtered Lie σ-crystal is cyclic diagonalizable, of Borel and parabolic type;*

c) *it is isomorphic to $\big(M, F^1, \varphi_1, G, (v_\alpha)_{\alpha \in \mathcal{J}}\big)$ mentioned in 3.2.3.*

Proof: There are many ways in proving c). One way is entirely the same as the proof of b) of 4.4.1 2) (see the extra details needed for the proof of b) of 4.4.1 3)), made in the context of a SHS. As the referred proof is self contained (based on 3.5.11), this way is not repeated here (cf. the policy of 1.15).

For the convenience of the reader we present another way, based on 3.5.11 and 3.1.0 c). So we can assume that $\varphi = g\varphi_1$, with $g \in P_1(W(k)))$ (see 3.3.3 for the meaning of P_1). We need to find $h \in P_1(W(k))$ such that $hg\varphi_1 = \varphi_1 h$. We first treat the case when g is congruent to 1 mod p. As φ_1 leaves invariant $\mathrm{Lie}(P_1)$ and as P_1 leaves invariant F^1, we can use induction, working modulo powers of p: inductively on $m \in \mathbb{N}$ we can assume that g is congruent to 1 mod p^m. To perform this induction we come across quasi Artin–Schreier systems of equations. So 3.4.8.1.2 c) applies, as $k = \bar{k}$, to give us the existence of such an element h.

To treat the general case, let us first remark that P_1 is the semidirect product of its unipotent radical R_U and of the reductive subgroup of R_1 of P_1 whose Lie algebra is the Lie subalgebra of $\mathfrak{g}$ corresponding to slope zero for $(\mathfrak{g}, \varphi_1)$ (cf. 3.7.2). So we can write

$$g = g_1 n_1$$

with $g_1 \in R_1(W(k))$ and $n_1 \in R_U(W(k))$. But R_U is unipotent and so it can be identified with the group scheme defined by its Lie algebra. So the above argument applies: we can assume that n_1 is the identity element. Moreover, R_U has a canonical $\mathbb{Z}_p$-structure; so [Bo, 16.4] applies to give us that we can assume that g_1 is congruent to 1 mod p.

a) results from c) and 2.2.19 (cf. the def. of φ_1 in 3.2.3). b) results from c) (cf. 3.7.2). This proves the theorem.

3.11.2. Remark. From 3.11.1 c) we get immediately 3.1.4. This goes as follows. The unicity part is implied by 3.2.8; it allows us to assume that in 3.1.4 we have $k = \bar{k}$. So referring to (M, F^1, φ_1, G) (of 3.2.3) we have $\tilde{\mu} = \mu$.

So let now $\mu_y : \mathbb{G}_m \to G$ be the canonical split of (M, φ, G) (cf. 3.1.6). We write $\varphi = a\mu_y(\frac{1}{p})$. Then a is a σ-linear automorphism of M and $M^a := \{x \in M | a(x) = x\}$ defines a $\mathbb{Z}_p$-structure on M. Let $\bar{h}_1 \in \mathrm{End}(M)$ be defined by: $\bar{h}_1(x) = x$ if $x \in F^1$, and $\bar{h}_1(x) = 0$ if $\mathbb{G}_m$ through μ_y acts trivially on x. We have $\bar{h}_1 \in \mathrm{End}\big(M^a \otimes_{\mathbb{Z}_p} W(\mathbb{F})\big)$ (cf. 3.11.1 c)). Let $d \in \mathbb{N}$ be such that $\varphi^d(\bar{h}_1) = \bar{h}_1$, but $\forall i \in \mathbb{N}$, $i < d$, we have $\varphi^i(\bar{h}_1) \neq \bar{h}_1$. So μ_y can be viewed as a cocharacter $\mu_y : \mathbb{G}_m \to GL\big(M^a \otimes_{\mathbb{Z}_p} W(\mathbb{F}_{p^d})\big)$. Let $\varphi^i(\bar{h}_1) = \bar{h}_{i+1}$, $\forall i \in \mathbb{N}$. So $\bar{h}_{d+1} = \bar{h}_1$. The $\bar{h}_i$ are commuting among themselves. Let $G_{\mathbb{Z}_p}$ be the subgroup of $GL(M^a)$ which over $W(k)$ becomes G. So (M, φ, G) is isomorphic to the extension to $W(k)$ of the following Shimura σ-crystal $\big(M^a \otimes_{\mathbb{Z}_p} W(\mathbb{F}_{p^d}), \sigma\mu_y(\frac{1}{p}), G_{W(\mathbb{F}_{p^d})}\big)$ (σ acts trivially on M^a and as Frobenius on $W(\mathbb{F}_{p^d})$).

Let ${}_A\bar{h}_1$ (resp. ${}_T\bar{h}_1$) be the image of $\bar{h}_1$ in $\mathrm{Lie}(G^{\mathrm{ad}})$ (resp. in $\mathrm{Lie}(G^{\mathrm{ab}})$), and let $d_1 \in \mathbb{N}$ (resp. $d_2 \in \mathbb{N}$) be the smallest positive integer such that ${}_A\bar{h}_1$ (resp. ${}_T\bar{h}_1$) is the image of $\bar{h}_{d_1+1}$ in $\mathrm{Lie}(G^{\mathrm{ad}})$ (resp. the image of $\bar{h}_{d_2+1}$ in $\mathrm{Lie}(G^{\mathrm{ab}})$). Then d_1 and d_2 divide d, and in fact d is the least common multiple of d_1 and d_2.

All the above facts are a consequence of 3.11.1 c) (cf. 3.7.2).

3.11.3. Degrees of definition. The number d of 3.11.2 is called the degree of definition of (M, φ, G), d_1 is called the degree of definition of $\big(\mathrm{Lie}(G^{\mathrm{ad}}), \varphi\big)$, or the A-degree of definition of (M, φ, G), and d_2 is called the degree of definition of $\big(\mathrm{Lie}(G^{\mathrm{ab}}), \varphi\big)$, or the T-degree of definition of (M, φ, G). Here A- and T- stand for adjoint and respectively toric.

3.11.4. An application: more on the numbers $d(n)$ of 3.4.1.3. We come back to the notations of 3.1-4. 3.11.1 allows us to define the number $r((M, \varphi, G)) \in \mathbb{N} \cup \{0\}$, defined as the number of slopes -1 of any G-canonical lift produced by (M, φ, G). Directly from 3.4.18.4 B) we get: $\forall n \in \mathbb{N}$ we have the estimate

$$d(n) \leq \dim(G) r((M, \varphi, G)).$$

The numbers $r((M, \varphi, G))$ can be easily computed using 3.10.6. As a gross estimate we get $d(n) \leq \frac{\dim(G)^2}{2}$. In particular, if $r((M, \varphi, G)) = 0$ (there are plenty of such examples, even in the context of a SHS; see 4.6.8 1)), then $d(n) = 1$, $\forall n \in \mathbb{N}$. This result is a consequence of 3.4.18.8.1 a). So in such a case we can take $Q_{j,n} = R_j$, $\forall n \in \mathbb{N}$.

3.12. Shimura F-crystals and generic points. Let $\big(M, F^1, \varphi, G, (v_\alpha)_{\alpha \in \mathcal{J}}\big)$ be a Shimura filtered σ-crystal over k. Let $\mathfrak{C}_H := (M, F^1, \varphi, G, H, \tilde{f})$ be a Shimura filtered F-crystal with H a locally closed subscheme of G such that the morphism $H \to G/P$ is smooth in a neighborhood of the origin of H (here P is the parabolic subgroup of G leaving invariant F^1). With the notations of 2.2.10, let $\eta : \mathrm{Spec}(k_\eta) \to \mathrm{Spec}(R)$ be a geometric point sitting over the generic point of the special fibre of $\mathrm{Spec}(R)$. Let

$$\mathfrak{C}_\eta := \big(M \otimes_{W(k)} W(k_\eta), \varphi_\eta, G_{W(k_\eta)}\big)$$

be the Shimura σ_η-crystal obtained from $(M, F^1, \varphi, G, H, \tilde{f})$ by pull back through η (here σ_η is the Frobenius automorphism of $W(k_\eta)$).

3.12.1. Theorem. *$\mathfrak{C}_\eta$ is a $G_{W(k_\eta)}$-ordinary σ_η-crystal.*

Proof: There is a Shimura filtered F-crystal $\mathfrak{C}_G := (M, F^1, \varphi, G, \tilde{f}_1)$ with the property that the Shimura σ-crystals obtained from it by pull back through geometric points over the generic point of the special fibre of the scheme over which $\mathfrak{C}_G$ is defined are G-ordinary σ-crystals (cf. 3.2.6 and 3.1.0 d)).

Let $\ell := dd((M, \varphi, G))$ (see 2.2.22 4)). We consider a filtered F-crystal $\mathfrak{C}_1$ with cycles over the special fibre of $\mathrm{Spec}(R_1)$, where $R_1 := W(k)[[x_1, ..., x_\ell]]$, which is a versal deformation of $(M, F^1, \varphi, G, (v_\alpha)_{\alpha\in\mathcal{J}})$. It can be constructed as follows. Let N be a smooth abelian subgroup of G such that the natural morphism $N \to G/P$ is an open embedding. Then $\mathfrak{C}_1$ (cf. the remarks following [Fa2, th. 10]) is isomorphic to any Shimura filtered F-crystal $\mathfrak{C}_2 = (M, F^1, \varphi, G, N, \tilde{f}_2)$ (i.e. it is induced from $\mathfrak{C}_2$ through an isomorphism $R_2 \tilde{\to} R_1$; here $\mathrm{Spec}(R_2)$ is the completion of N at the origin).

Now the theorem results from the fact that $\mathfrak{C}_H$ and $\mathfrak{C}_G$ are both induced from $\mathfrak{C}_1$ through (uniquely determined) formally smooth $W(k)$-homomorphisms $R_1 \hookrightarrow R$ and respectively $R_1 \hookrightarrow R_G$ (as the natural morphism $H \to G/P$ is smooth around the origin and as the natural morphism $G \to G/P$ is smooth) (to be compared with [Va2, 5.4.4-8]). This ends the proof of the theorem.

3.13. Deviations of Shimura (filtered) (Lie) σ-crystals. In this section we express our hopes for building up a reasonable theory of good "G-pseudo-canonical lifts" of Shimura σ-crystals (M, φ, G) which are not potentially cyclic diagonalizable. We restrict to present some definitions, to restate some of the previous results and problems in terms of them, and to list some of the most interesting problems arising.

Let $\mathfrak{C}_0 := (M, \varphi, G)$ be a Shimura σ-crystal over k. Let $\mathcal{F}$ be the set of direct summands F^1 of M such that the quadruple (M, F^1, φ, G) is a Shimura filtered σ-crystal. From 2.2.9 3) we deduce that $\mathcal{F}$ is the $G(W(k))$-conjugacy class of any $F^1 \in \mathcal{F}$. To fix the notations we choose (arbitrarily) an element $F^1 \in \mathcal{F}$. We get a Shimura filtered σ-crystal $\mathfrak{C} := (M, F^1, \varphi, G)$. As usual, let $F^0(\mathrm{Lie}(G))$ be the parabolic Lie subalgebra of $\mathrm{Lie}(G)$ leaving invariant F^1. We assume that G is not a torus. Let $\mathfrak{p}$ be the parabolic Lie subalgebra of $\mathfrak{sl}(M))$ defined by the property that $\mathfrak{p}[\frac{1}{p}]$ is left invariant by φ and the isocrystal $(\mathfrak{p}[\frac{1}{p}], \varphi)$ is the maximal isocrystal of $(\mathfrak{sl}(M[\frac{1}{p}]), \varphi)$ having only non-negative slopes. Let

$$\mathfrak{p}_G := \mathrm{Lie}(G^{\mathrm{der}}) \cap \mathfrak{p}.$$

Let π be a projector of $\mathfrak{gl}(M[\frac{1}{p}])$ on $\mathrm{Lie}(G_{B(k)})$ fixed by φ and G. For instance we can take π such that its kernel is the $B(k)$-vector subspace of $\mathfrak{gl}(M[\frac{1}{p}])$ perpendicular on $\mathrm{Lie}(G_{B(k)})$ with respect to the trace form on $\mathfrak{gl}(M[\frac{1}{p}])$ (see [Va2, 4.2]). In what follows, except the undetailed second part of 3.13.3 3), it is irrelevant which π we choose. Let $\mathfrak{p}_G^0 := \mathfrak{p} \cap \ker(\pi)$. The $B(k)$-vector spaces $\mathfrak{p}_G^0[\frac{1}{p}]$ and $\mathfrak{p}_G[\frac{1}{p}]$ are left invariant by φ.

3.13.1. Definitions and notations. 1) By the deviation of $\mathfrak{C}$ we mean the rational number

$$d(\mathfrak{C}) := \frac{\dim\big(\mathfrak{p}_G \cap F^0(\mathrm{Lie}(G))\big)}{\dim(\mathfrak{p}_G)}.$$

2) By the complementary deviation of $\mathfrak{C}$ we mean the rational number

$$d^\perp(\mathfrak{C}) := \frac{\dim\big(\mathfrak{p}_G^0 \cap F^0(\mathfrak{sl}(M))\big)}{\dim(\mathfrak{p}_G^0)}.$$

3) By the global deviation of $\mathfrak{C}$ we mean the rational number

$$gd(\mathfrak{C}) := \frac{\dim\big(\mathfrak{p} \cap F^0(\mathfrak{sl}(M))\big)}{\dim(\mathfrak{p})}.$$

4) By the deviation of $\mathfrak{C}_0$ we mean the rational number $d(\mathfrak{C}_0)$ defined as the maximum of all numbers $d(\mathfrak{C})$, with the filtration F^1 running through the set $\mathcal{F}$.

5) By the complementary deviation of $\mathfrak{C}_0$ we mean the rational number $d^{\perp}(\mathfrak{C}_0)$ defined as the maximum of all rational numbers $d^{\perp}(\mathfrak{C})$, with the filtration F^1 of M running as in 4) above.

6) By the global deviation of $\mathfrak{C}_0$ we mean the rational number $gd(\mathfrak{C}_0)$ defined as the maximum of all numbers $gd(\mathfrak{C})$, with the filtration F^1 running as in 4) above.

7) By a G-pseudo-canonical lift of $\mathfrak{C}_0$ we mean a Shimura filtered σ-crystal $\tilde{\mathfrak{C}} = (M, \tilde{F}^1, \varphi, G)$ such that $d(\mathfrak{C}_0) = d(\tilde{\mathfrak{C}})$.

8) By the CM-deviation of $\mathfrak{C}_0$, we mean the number $CM - d(\mathfrak{C}_0) \in \mathbb{N} \cup \infty$, defined as follows: if there is a finite flat DVR extension V of $W(\bar{k})$ such that there is a maximal $\mathbb{Q}_p$-subtorus of $G_{B(\bar{k})}$ formed by elements of $\mathrm{Lie}(G_{B(\bar{k})})$ fixed by φ, and included in a parabolic subgroup of G_V, lifting the parabolic subgroup of $G_{\bar{k}}$ leaving invariant $F^1 \otimes \bar{k}$, then $CM - d(\mathfrak{C}_0)$ is the smallest possible value of $[V : W(\bar{k})]$ for such a V; if such a V does not exist, then $CM - d(\mathfrak{C}_0) = \infty$.

3.13.2. Remarks. 1) $\mathfrak{C}_0$ is a G-ordinary σ-crystal iff $d(\mathfrak{C}_0) = 1$. $\mathfrak{C}$ is a G-canonical lift iff $d(\mathfrak{C}) = 1$. This motivates our terminology: deviations of Shimura (filtered) σ-crystals. So we view the numbers introduced in 1-6) of 3.13.1 as a measure of how far a Shimura σ-crystal (M, φ, G) (resp. a Shimura filtered σ-crystal (M, F^1, φ, G)) is from being a G-ordinary σ-crystal (resp. a G-canonical lift).

2) A G-pseudo-canonical lift is not unique. This can be seen looking at σ-crystals associated to supersingular abelian varieties over k.

3) The CM-deviations are measuring how far a Shimura σ-crystal is from being potentially cyclic diagonalizable: the CM-deviation of a Shimura σ-crystal over $\bar{k}$ is 1 iff it has a lift of quasi CM type (cf. 2.2.19).

3.13.3. Variants. 1) 3.13.1 has been worked out using non-negative slopes. The same thing can be worked out using positive slopes, or negative slopes, or non-positive slopes. In such a case, as well as in others (to be defined below), we use a left upper index of the form $+$, $-$, $0+$, or $0-$ for the deviations introduced, to emphasize that they have been defined using positive, negative, non-negative, or respectively non-positive slopes. For instance we get rational numbers ${}^{+}d(\mathfrak{C})$, $p^n - {}^{0-}d(\mathfrak{C})$, etc.

2) In some cases it is useful to define deviations of higher order. By this we mean that in 1-6) of 3.13.1 we replace $F^0(\mathrm{Lie}(G^{\mathrm{der}}))$ or $F^0(\mathfrak{sl}(M))$, with their iterates under the action of φ (for instance, we replace $F^0(\mathfrak{sl}(M))$ by

$$\mathfrak{sl}(M) \cap \varphi^i(F^0(\mathfrak{sl}(M[\frac{1}{p}]))),$$

$i \in \mathbb{N} \cup \{0\}$). So we obtain sequences of deviation numbers $d_i(\mathfrak{C})$, $d_i^{\perp}(\mathfrak{C}_0)$, $gd_i(\mathfrak{C})$, $i \in \mathbb{N} \cup \{0\}$, which for $i = 0$ become the numbers introduced in 1-6) of 3.13.1.

$2'$**)** The same as above but we replace $\varphi^i(F^0(\mathfrak{sl}(M[\frac{1}{p}])))$ by

$$\cap_{n=0}^{i} \varphi^n(F^0(\mathfrak{sl}(M[\frac{1}{p}]))).$$

We obtain $\cap$ (intersection) type deviations of higher order. The notations to be used: $d_{\cap i}(\mathfrak{C})$, $d_{\cap i}^{\perp}(\mathfrak{C}_0)$, etc.

3) All the definitions of 3.13.1 make sense for Shimura (filtered) isocrystals over k. Of course, for a Shimura isocrystal (M^1, φ^1, G^1) we need to choose a set $\mathcal{F}^1$ of "admissible" filtrations of M^1. Usually it is a $G^1(B(k))$-conjugacy class of a fixed filtration. Warning: an element of $\mathcal{F}^1$ is not always just a $B(k)$-subspace of M^1. So we speak about deviations of Shimura isocrystals.

Sometimes, for the sake of refinement, we work also with p^n-deviations of Shimura (filtered) σ-crystals, $n \in \mathbb{N}$. Not to make the story too long, here we define just the p^n-deviation of $\mathfrak{C}$, $n \in \mathbb{N}$. It is the rational number defined by:

$$p^n - d(\mathfrak{C}) := \frac{\text{length}\big(\mathfrak{p}_G \otimes_{W(k)} W_n(k) \cap F^0(\text{Lie}(G) \otimes_{W(k)} W_n(k))\big)}{\text{length}\big(\mathfrak{p}_G \otimes_{W(k)} W_n(k)\big)}.$$

Here the length$(*)$ is the usual length of finitely generated modules over the artinian local ring $W_n(k)$.

4) All the deviations introduced above which are not of global or complementary type, make sense for Shimura (filtered) Lie σ-crystals and (the "conventions" of 3.13.3 3) apply) for Shimura Lie isocrystals.

5) We can define isogeny deviations. They will be denoted by putting $iso-$ in front of the notation of the involved deviation. For instance, by the isogeny deviation of $\mathfrak{C}_0$, denoted $iso - d(\mathfrak{C}_0)$, we mean the biggest number $d(\mathfrak{C}_0^i)$, with $\mathfrak{C}_0^i$ running through the representatives of the isogeny class of $\mathfrak{C}_0$. Of course, for $iso - CM - (\mathfrak{C}_0)$, the word "biggest" has to be replaced by the word "smallest".

6) We can define strong CM-deviations, denoted by replacing CM by SCM, by replacing (in 3.13.1 8)) the expression "there is a maximal" by: "for any maximal".

3.13.3.1. Remarks 1). $\mathfrak{C}_0$ is a G-ordinary σ-crystal iff $p - d(\mathfrak{C}_0) = 1$. $\mathfrak{C}$ is a G-canonical lift iff $p^n - d(\mathfrak{C}) = 1$ for all $n \in \mathbb{N}$. These two facts are a consequence of the proof of 3.5.8.

2) We can introduce also the numbers

$$p^n - d_f(\mathfrak{C}) := \frac{\text{length}\big(\mathfrak{p}_G \otimes_{W(k)} W_n(k) \cap F^0(\text{Lie}(G) \otimes_{W(k)} W_n(k))\big)}{\text{length}\big(F^0(\text{Lie}(G)) \otimes_{W(k)} W_n(k)\big)}.$$

But these numbers can be bigger or smaller than the ones attached to a G-canonical lift. Using this we can easily construct examples for which these numbers for a G-canonical lift are not 1, and do not characterize the G-canonical lifts.

3) In 1995 we conjectured: all the $iso - SCM$-deviations are finite numbers. This conjecture is very much related to the Langlands-Rapoport conjecture referred to in 1.24, and so it will play a central role in §14.

4) We do believe that the deviations introduced in this 3.13 allow a uniform treatment of all Shimura (adjoint) (filtered) (Lie) σ-crystals, regardless of the fact they are or are not cyclic diagonalizable.

3.13.4. Problems. Many problems (questions) are arising. We mention just few of them.

1) Can a G-pseudo-canonical lift be defined in terms of $d^{\perp}(*)$, or in terms of $gd(*)$?

2) Fixing $\mathfrak{C}_0$, can we determine all the possible values of the numbers $d(\mathfrak{C})$, $d^{\perp}(\mathfrak{C})$, $gd(\mathfrak{C})$, where the filtration F^1 of M runs as in 3.13.1 4)?

3) Determine all Shimura σ-crystals $\mathfrak{C}_0$ for which we can define uniquely a G-pseudo-canonical lift of them, by using the numbers introduced in 2) and 3) of 3.13.3.

4) How are p^n-deviations ($n \in \mathbb{N}$) related to deviations?

5) How deviations (p^n-deviations, CM-deviations) vary when we replace φ by $g\varphi$ with $g \in G(W(k))$? What are their possible values?

6) What is the connection between CM-deviations and deviations?

7) Let A be an abelian variety over a number field E. For any unramified prime v of E of good reduction for A, we can define a CM-deviation and a SCM-deviation (of the σ-crystal of the reduction of A with respect to v; here σ is the Frobenius of $W(k(v))$). How are they varying in terms of v? Is it true that all these CM-deviations are 1, except a finite number? We have similar questions in the relative context, where we keep track of the Mumford–Tate group of A.

3.13.5. Wintenberger groups and deviations. We consider the canonical split (cocharacter) $\mu(F^1) : \mathbb{G}_m \to GL(M)$ defined (cf. [Wi]) by the filtered σ-crystal (M, F^1, φ). Due to its functoriality (see loc. cit.) $\mu(F^1)$ factors through G. Let $\mathcal{W}_1$ be the subgroup of G generated by $\mu(F^1)$, with F^1 running as in 3.13.1 4). Let $\mathcal{W}_2$ be the Zariski closure in G of the smallest (i.e. connected) subgroup of $G_{B(k)}$ with the property that its Lie algebra is stable under φ and contains $\mathrm{Lie}(\mathcal{W}_1)$. Let $\mathcal{W}_{i+2}$ be the Zariski closure in $GL(M)$ of the centralizer of the generic fibre of $\mathcal{W}_i$ in $GL(M[\frac{1}{p}])$.

We call $\mathcal{W}_i$ as the i-th Wintenberger group of $\mathfrak{C}_0$. Similarly, the Zariski closure in G of the smallest subgroup of $G_{B(k)}$ whose Lie algebra is invariant under φ and contains the Lie algebra of $\mu(F^1)(\mathbb{G}_m)$, is called the Wintenberger group of (M, F^1, φ); from the proof of 2.2.19 we deduce that it is a subtorus of G iff (M, F^1, φ, G) is potentially cyclic diagonalizable.

Exercise. If (M, φ) is an ordinary σ-crystal, show that $\mathcal{W}_1$ is the extension of $\mathbb{G}_m$ by a smooth commutative unipotent group of dimension equal to $dd(\mathfrak{C}_0)$.

3.13.5.1. Versal endomorphisms of $\mathfrak{C}_0$. We consider an endomorphism $h : M \to M$ fixed by φ. We say that h is a versal endomorphism of $\mathfrak{C}_0$ if one of the following equivalent conditions is satisfied:

– h leaves invariant any element of $\mathcal{F}$;

– h is fixed by $\mathcal{W}_1$;

– h is fixed by $\mathcal{W}_2$;

– h defines naturally an endomorphism of the universal deformation (here viewed just as a p-divisible group) of the Shimura p-divisible group over $W(k)$ defined by $\mathfrak{C}$ over a formal power series ring in $dd(\mathfrak{C}_0)$ variables with coefficients in $W(k)$.

Similarly we speak about versal automorphisms of $\mathfrak{C}_0$; in this case the above equivalent conditions are also equivalent to:

– $h \in \mathcal{W}_3(W(k))$;

– $h \in \mathcal{W}_4(W(k))$.

Warning: a versal automorphism is not necessarily an automorphism in the sense of 2.2.9 6).

3.13.5.2. Comments. We do not know when $\mathcal{W}_i$ is a smooth group over $W(k)$. We do not know when the special fibre of $\mathcal{W}_i$ is a connected group. We do not know when the generic fibre of $\mathcal{W}_3$ or $\mathcal{W}_4$ is a connected group. We do not know how to compute the relative dimension w_i of $\mathcal{W}_i$. We do not know how to determine the structure of the generic fibre of $\mathcal{W}_i$. However, it is easy to check (starting from [Wi]) that w_i is an isogeny invariant.

3.13.5.3. $\mathcal{W}$-deviations. Let $g \in G(W(k))$ such that $(M, g\varphi, G)$ is a G-ordinary σ-crystal. We similarly define $w_i(G-ord)$ as the relative dimension of the i-th Wintenberger group of $(M, g\varphi, G)$. We call the rational number

$$\frac{w_i}{w_i(G-ord)}$$

as the i-th $\mathcal{W}$-deviation or the i-th Wintenberger deviation of $\mathfrak{C}_0$. Some of the previous questions and problems can be easily adapted to the context of $\mathcal{W}$-deviations.

3.13.5.4. Relative rigidity. Let G_1 be a reductive subgroup of $GL(M)$ containing G and such that (M, φ, G_1) is a Shimura σ-crystal. We say that $\mathfrak{C}_0$ is rigid with respect to G_1 if for any versal automorphism of $\mathfrak{C}_0$ defined by an $h \in G_1(W(k))$ we have: h centralizes $\mathrm{Lie}(G^{\mathrm{der}})$. When $G_1 = GL(M)$ we drop the words "with respect to G_1".

A. Exercise. We assume that $\dim(M) = 8$, that G contains the group of scalar automorphisms of M, and that G is a split GL_2 group whose representation on M is a direct sum of 4 copies of the standard two-dimensional irreducible representation of G. If $dd(\mathfrak{C}_0) = 1$ (i.e. if it is non-zero), then $\mathfrak{C}_0$ is rigid.

3.14. The case $p = 2$. Looking back to 3.1-13 we see that we used the fact that $p > 2$ in precisely five places: in 3.4.2, in 3.4.18.4.2, in 3.4.18.5.1 and 3.4.18.5.3, and twice in 3.6.4.1.

A. In 3.4.18.4.2, for the case $p = 2$, we have to restrict to the case: S is affine regular scheme.

B. The equivalence parts of 3.4.18.5.1 and 3.4.18.5.3 remain true for $p = 2$, but not the antiequivalence parts. However, due to the fact that [Fa2, th. 10] is true as well for $p = 2$, the antiequivalence parts can be reformulated in weaker forms:

Fact. *We refer to 3.4.18.5.1 and 3.4.18.5.3 with $p = 2$. We have: a quotient of the category of 2-divisible groups (resp. of finite flat commutative group schemes of 2-power order) over R is antiequivalent to the category of 2-divisible objects (resp. objects) of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$.*

In other words, we have: any object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$ is induced (in the same sense as in paragraph before 3.4.18.0) by a finite flat commutative group scheme of 2-power order over R, and the same can be stated for morphisms. So in the above fact, by a quotient of the category of 2-divisible groups (resp. of finite flat commutative group schemes of 2-power order) over R we mean that some 2-divisible groups are "forced" to be isomorphic (resp. that some finite flat commutative group schemes of 2-power order are forced to be isomorphic, and some morphisms between them are "forced" to be zero).

C. It is easy to see that in 3.6.4 it was not at all essential that $p > 2$. To see this we use the notations of 3.6.4, with $p = 2$. The formula (5) of 3.6.4 is true for $p = 2$ (it can be checked immediately inside the SL group of a three dimensional vector space over k), but 3.6.4 a) is not. So we have to replace the argument based on 3.6.4 a) and made at the level of Lie algebras in characteristic p, either by a refined one or by an argument at the level of reductive groups in characteristic zero. In particular the Fact of 3.6.4 has to be restated as follows. We have:

Fact. *The Lie subalgebra* $\bigl(g_2(-\delta)\cap\mathfrak{g}_1\bigr)\otimes_{W(k)} k$ *of* $\bar{\mathfrak{g}}_1$ *is of the form* ${}^{g}\bigl(\cap_{a\in A}\bar{\mathfrak{s}}_a\bigr)$, *with* $g\in\bar{P}_{\varepsilon_1}(k)$, *and* $\bigl(\bigl(\bigoplus_{\alpha\in\mathcal{H}_2\cap[0,1]} g_2(\alpha)\bigr)\cap\mathfrak{g}_i\bigr)\otimes_{W(k)} k$ *is* ${}^{g}\bigl(\bigcap_{a\in A}\bar{P}_a\bigr)\subset\bar{P}_{\varepsilon_1}$.

Proof: The case $|A| = 1$ is obvious: we apply [Bo, 14.22 (iii)] and the fact that $\bigl(g_2(-\delta)\cap\mathfrak{g}_1\bigr)\otimes_{W(k)} k\subset\bar{\mathfrak{s}}_{\varepsilon_1} = F^1(\bar{\mathfrak{g}}_1)$.

The case $|A|\ge 2$ can be treated similarly. First, as in 3.6.4 we get that $\bigl(g_2(-\delta)\cap\mathfrak{g}_1\bigr)\otimes_{W(k)} k$ is of the form ${}^{g}\bigl(\cap_{a\in A}\bar{\mathfrak{s}}_a\bigr)$ for some $g\in\bar{G}_1(k)$. Using formula (5) of 3.6.4, the only situation which is not treatable as in 3.6.4, is when $\bigl(g_2(-\delta)\cap\mathfrak{g}_1\bigr)\otimes_{W(k)} k$ is a one dimensional k-vector space: in such a case, if $\bigl(g_2(-\delta)\cap\mathfrak{g}_1\bigr)\otimes_{W(k)} k = \mathfrak{g}_\beta\otimes_{W(k)} k$ with $\beta\in\Phi_1$ (see 3.5.3.3 for notation), then the normalizer of $\bigl(g_2(-\delta)\cap\mathfrak{g}_1\bigr)\otimes_{W(k)} k$ can (theoretically) happen to contain $\mathfrak{g}_{-\beta}\otimes_{W(k)} k$.

But $\bigl(g_2(-\delta)\cap\mathfrak{g}_1\bigr)\otimes_{W(k)} k$ is a one dimensional k-vector space iff G_1 is of A_ℓ Lie type, and $\{1,\ell\}\subset A$. If we are in such a case then the representation of $\bar{G}_1$ on M/pM is via one or both of the minimal weights ω_1 and ω_ℓ. So $\bar{G}_1$ is the SL-group of a $1+\ell$ dimensional vector space over k. But in such a case it is easy to see that $\mathfrak{g}_{-\beta}\otimes_{W(k)} k$ does not normalize $\mathfrak{g}_\beta\otimes_{W(k)} k$. This ends the proof of the fact.

The above proof give us more information than the statement of the fact, in the case $|A|\ge 2$.

D. In 3.4.2 it was crucial that $p > 2$. So everything in 3.1-15, which is not based on 3.4.2 or on the equivalence parts of 3.4.18.5.1 and 3.4.18.5.3, remains true for the case $p = 2$. In other words, everything in 3.1-15 except 3.4.2, the equivalence parts of 3.4.18.5.1 and 3.4.18.5.3, and 3.4.14-15, remains true for the case $p = 2$.

E. Despite D above, it is possible to modify the statement and the proofs of 3.4.14.1-4 accordingly (i.e. without relying on 3.4.2). For this we just have to replace the assumptions SA1-2, by the restatement of property G) of the proof of 3.4.14.1, where we do not mention anything about 2-divisible groups. So 3.4.14 and 3.4.14.1, under this restatement, are true as well for a $p = 2$ SHS.

3.15. Some conclusions. 3.4.1.3 allows us (cf. 3.4.9 3)) to strengthen the result of 3.1.8. Here $p \geq 2$ is an arbitrary prime. We have:

Fact 1. *Any p-divisible group over a perfect field $\tilde{k}$ of characteristic p admits a global deformation over the p-adic completion of an $\mathbb{N}$-pro-étale scheme* $\mathrm{Spec}(R_1)$ *over a smooth affine $W(\tilde{k})$-scheme* $\mathrm{Spec}(R)$ *(and so also over an $\mathbb{N}$-pro-étale scheme $\bar{R}_1$ over a smooth $\tilde{k}$-scheme $\bar{R}$), of whose Kodaira-Spencer map is an isomorphism. We can assume that* $\mathrm{Spec}(R_1/pR_1)$ *is connected, and that the morphism* $\mathrm{Spec}(R_1/pR_1) \to \mathrm{Spec}(R/pR)$ *is surjective. Moreover the p-divisible group over the generic point of* $\mathrm{Spec}(R_1/pR_1)$ *(or of $\bar{R}$) is ordinary.*

Similarly, in the relative context, we have:

Fact $1'$**.** *Let $\mathcal{D}(W(\tilde{k}))$ be a Shimura p-divisible group over a perfect field $\tilde{k}$ of characteristic p. Then there is a smooth $W(\tilde{k})$-algebra R, an $\mathbb{N}$-pro-étale affine R-scheme* $\mathrm{Spec}(R_1)$*, and a Shimura p-divisible group $\mathcal{D}$ over* $\mathrm{Spec}(R_1^{\wedge})$ *such that:*

a) in a $W(\tilde{k})$-valued point z of $\mathrm{Spec}(R_1^{\wedge})$*, $\mathcal{D}$ becomes $\mathcal{D}(W(\tilde{k}))$;*

b) $\mathcal{D}$ is a versal deformation in each geometric point of $\mathrm{Spec}(R_1/pR_1)$*;*

c) $\mathrm{Spec}(R_1/pR_1)$ *is connected, and the morphism* $\mathrm{Spec}(R_1/pR_1) \to \mathrm{Spec}(R/pR)$ *is surjective.*

Moreover, the resulting Shimura σ-crystal $\mathcal{C}$ over the algebraic closure of the field of fractions of R_1/pR_1 is Shimura-ordinary.

Proof: The parts involving (Shimura) ordinary properties of the above facts are a consequence of 3.12.1. The fact that the above facts are true also for $p = 2$, can be checked as follows. The trouble with $p = 2$ is in lifting things modulo 2 to things modulo 4. We can refer just to Fact $1'$. So using [Fa2, th. 10] and Artin's approximation theorem, we need to start with a smooth $W(\tilde{k})$-algebra R such that:

i) it has a connected special fibre;

ii) its spectrum has a $W(\tilde{k})$-valued point z;

iii) its p-adic completion is equipped with a Frobenius lift Φ_R such that (in z) we have a property similar to one of 2.3.15 c);

iv) there is an object $\mathcal{C}(4)$ of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$ whose underlying module is a free $R/4R$-module of rank equal to the rank r of $\mathcal{D}(W(\tilde{k}))$, and corresponds to a finite flat commutative group scheme $\mathcal{G}(4)$ over R of rank 4^r;

v) $\mathcal{G}(4)$ in z is the kernel of the multiplication by 4 of $\mathcal{D}(W(\tilde{k}))$;

vi) $\mathcal{C}(4)$ is naturally endowed with some crystalline cycles.

We lift $\mathcal{C}(4)$, viewed just as an object (with cycles) of $\mathcal{MF}_{[0,1]}(R)$, to a p-divisible object (with cycles) $\mathcal{C}$ of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$ in such a way that in z it corresponds to the p-divisible object (with cycles) $\mathcal{C}_z$ of $\mathcal{MF}_{[0,1]}(W(\tilde{k}))$ defined by $\mathcal{D}(W(\tilde{k}))$, and we can keep track of the relative situation (corresponding to the extra Shimura structure). So we can perform 3.4.1.3 to get the necessary R_1 such that $\mathcal{C}_{R_1}$ is a p-divisible object with cycles of $\mathcal{MF}^{\nabla}_{[0,1]}(R_1)$. We conclude: $\mathcal{G}(4)_{R_1^{\wedge}}$ lifts, as prescribed by $\mathcal{C}_{R_1}$, to a 2-divisible group $\mathcal{G}_{R_1^{\wedge}}$

over $\mathrm{Spec}(R_1^\wedge)$ in such a way that in the factorization of z through $\mathrm{Spec}(R_1^\wedge)$ (cf. 3.4.1.3) it is $\mathcal{D}(W(\tilde{k}))$.

In fact, the statement of Fact 1' is such that, we do not need to bother about lifting $\mathcal{G}(4)_{R_1^\wedge}$ as well: we can just remark (easy exercise of descent based on [Fa2, th. 10]) that the F-crystal over $R_1/2R_1$ defined by $\mathcal{C}_{R_1}$ is the F-crystal defined by a 2-divisible group over $R_1/2R_1$. So based on the freeness part of iv), it lifts (see [Me] and 2.3.18.1), up to a replacement of $\mathrm{Spec}(R_1/pR_1)$ by an étale cover of it, to a 2-divisible group over $R_1^\wedge$ such that its filtered F-crystal is $\mathcal{C}_{R_1}$. This proves the above two facts.

3.15.1. Comment. The expectation of 3.4.8.9.1 gets translated: in the above facts, (at least if $k = \bar{k}$) it should be possible to avoid the complication with $\mathbb{N}$-pro-étale morphisms (and so to avoid introducing R_1). For the case $k = \bar{k}$, in connection to Fact 1, this is achieved in 4.12.12 and 4.12.12.2.

§4 Applications of the basic results to integral canonical models of Shimura varieties of preabelian type, to p-divisible groups, and to abelian varieties

§3 contains much more than what is needed for applications to integral canonical models of Shimura varieties of preabelian type. In 4.1-8 and 4.11 which form a sequence, continuing 2.3, we try to make as little use of §3 as possible. In particular, we avoid the use of 3.4, except of its independent lemma 3.4.6; even the use of 3.4.6 is in fact entirely avoidable, cf. 3.5.14.

We start with a SHS $(f, L_{(p)}, v)$.

4.1. The G-ordinary type. Let $L_p^* := L_{(p)}^* \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$. We view $G_{\mathbb{Z}_p}$ also as a subgroup of $GL(L_p^*)$. Let T be a maximal torus of $G_{\mathbb{Z}_p}$ such that:

1) there is a cocharacter $\mu : \mathbb{G}_m \hookrightarrow T_{W(k(v))}$ which over an embedding of $W(k(v))$ in $\mathbb{C}$, extending the composite inclusion $O_{(v)} \subset E(G, X) \subset \mathbb{C}$, is $G(\mathbb{C})$-conjugate to the cocharacters $\mu_x^* : \mathbb{G}_m \hookrightarrow G_{\mathbb{C}}$, $x \in X$;

2) there is a Borel subgroup B of $G_{\mathbb{Z}_p}$, whose Lie algebra leaves invariant the F^1 filtration of $L_p^* \otimes_{\mathbb{Z}_p} W(k(v))$ defined by μ, i.e. we have $\mathrm{Lie}(B) \subset F^0\big(\mathrm{Lie}(G_{W(k(v))})\big)$.

For instance, we can take T to contain a maximal split $\mathbb{Z}_p$-torus of $G_{\mathbb{Z}_p}$, and over $W(k(v))$ to contain a maximal split $W(k(v))$-torus of $G_{W(k(v))}$, and then we can take B to be a Borel subgroup of $G_{\mathbb{Z}_p}$, containing T (cf. the definition of the reflex field: $W(k(v))\left[\frac{1}{p}\right]$ is the field of definition of the conjugacy class of cocharacters of $G_{W(\overline{k(v)})}$ which under an embedding of $W(\overline{k(v)})[\frac{1}{p}]$ into $\mathbb{C}$, extending the inclusion $E(G, X) \subset \mathbb{C}$, are $G(\mathbb{C})$-conjugate to the cocharacters μ_x, $x \in X$; see [Mi3, 4.6-7]).

B not necessarily contains T, but it must contain the smallest subtorus T_μ of T with the property that the cocharacter μ of $T_{W(k(v))}$ factors through $T_{\mu W(k(v))}$.

We recall that μ gives birth to a direct sum decomposition $L_{(p)}^* \otimes_{\mathbb{Z}_{(p)}} W(k(v)) = F^1 \oplus F^0$, with $\beta \in \mathbb{G}_m\big(W(k(v))\big)$ acting through μ on F^i as the multiplication with β^{-i}, $i = \overline{0, 1}$.

4.1.1. The G-ordinary type τ associated to (or defined by) the SHS $(f, L_{(p)}, v)$ is the formal isogeny type associated to the σ-crystal

$$(L_{(p)}^* \otimes_{\mathbb{Z}_{(p)}} W(k(v)), \varphi)$$

defined by $\varphi := \sigma \circ \mu\big(\frac{1}{p}\big)$. Here σ acts as identity on L_p^* and as Frobenius on $W(k(v))$, and $\frac{1}{p} \in \mathbb{G}_m(W(k(v)\left[\frac{1}{p}\right]))$.

4.1.1.0. Convention. We usually write $\sigma\mu(\frac{1}{p})$ instead of $\sigma \circ \mu(\frac{1}{p})$.

4.1.1.1. Let $M := L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(k(v))$, and let $\mathfrak{t} := \mathrm{Lie}\big(T_{W(k(v))}\big)$. Let $\bar{h}_1 : M \to M$ be the endomorphism defined by $\bar{h}_1(x) = 0$ if $x \in F^0$, and by $\bar{h}_1(x) = x$ if $x \in F^1$. $\bar{h}_1$ is an element of $\mathfrak{t}$. Let $d \in \mathbb{N}$ such that $|k(v)| = p^d$; so $k(v) = \mathbb{F}_{p^d}$. Then $\varphi^d(\bar{h}_1) = \bar{h}_1$, and $\forall i \in \{1, 2, \ldots, d-1\}$, $\varphi^i(\bar{h}_1) := \bar{h}_{i+1} \neq \bar{h}_1$. Let

$$h^0 := \frac{1}{d} \sum_{i=1}^{d} \bar{h}_i.$$

We have $\varphi(h^0) = h^0$. Then the slopes of the σ-crystal (M, φ) are the eigenvalues of h^0 (acting on M), and the multiplicities are the same. This results from the fact that if $x \in M_i := \big\{x \in M | h^0(x) = \frac{i}{d}x\big\}$, with $i \in \{0, 1, \ldots, d\}$, and $x \notin pM$, then $\varphi^d(x) = p^i y$, with $y \in M_i$, but $y \notin pM$. We have $M = \bigoplus_{0 \leq i \leq d} M_i$ (as $\bar{h}_i \in \mathfrak{t}$, $\forall i \in \{1, ..., d\}$).

4.1.1.2. Let $\mathrm{Lie}_G(\tau)$ be the formal isogeny type associated to the Lie σ-crystal

$$\big(\mathrm{Lie}\big(G_{W(k(v))}\big), \varphi\big).$$

As above, the slopes of it are the eigenvalues of h^0 (acting on $\mathrm{Lie}\big(G_{W(k(v))}\big)$), and the multiplicities are the same. We call it the Lie G-ordinary type associated to (or defined by) the SHS $(f, L_{(p)}, v)$.

4.1.1.3. The G-ordinary type τ depends on f and v, but not on $L_{(p)}$, or on the choice of T or B. Also the Lie G-ordinary type $\mathrm{Lie}_G(\tau)$ depends only on the Shimura pair (G, X) and on v.

4.1.1.4. For any $i \in \{1, \ldots, d\}$ and $j \in \{0, 1\}$ we denote by ${}^iF^j$ the free $W(k(v))$-submodule of M on which $\bar{h}_i$ acts trivially if $j = 0$, and identically if $j = 1$. For any function $\bar{f} : \{1, \ldots, d\} \to \{0, 1\}$ we denote by

$$F_{\bar{f}} := \bigcap_{i \in \{1, \ldots, d\}} {}^iF^{\bar{f}(i)}.$$

Let $\mathcal{L}$ be the set of functions $\bar{f}$ as above for which $F_{\bar{f}} \neq 0$. We get a direct sum decomposition

$$M = \oplus_{\bar{f} \in \mathcal{L}} F_{\bar{f}}$$

due to the fact that all endomorphisms $\bar{h}_i$ of M are commuting among themselves. Let

$$\bar{\varphi} : \mathcal{L} \to \mathcal{L}$$

be the bijection defined by the rule: $\bar{\varphi}(\bar{f})(i) = \bar{f}(i-1)$; here $\bar{f}(0) := \bar{f}(d)$. We get: $\varphi(F_{\bar{f}}) \subset F_{\bar{\varphi}(\bar{f})}$, and $\varphi(F_{\bar{f}}) = F_{\bar{\varphi}(\bar{f})}$ if $\bar{f}(1) = 0$, and $\varphi(F_{\bar{f}}) = pF_{\bar{\varphi}(\bar{f})}$ if $\bar{f}(1) = 1$.

Let $\bar{\varphi} = \prod_{i \in I_{\bar{\varphi}}} \bar{\varphi}_i$ be written as a product of disjoint cyclic permutations. We allow trivial cyclic permutations. So we have a disjoint union $I_{\bar{\varphi}} = I^1_{\bar{\varphi}} \cup I^0_{\bar{\varphi}}$; $i \in I_{\bar{\varphi}}$ belongs to $I^1_{\bar{\varphi}}$ iff $\bar{\varphi}_i$ is a non-trivial permutation (and then any $\bar{f} \in \mathcal{L}$ such that $\bar{\varphi}_i(\bar{f}) \neq \bar{f}$ is said to be associated to $\bar{\varphi}_i$). So $I^0_{\bar{\varphi}} := \{\bar{f} \in \mathcal{L} | \bar{\varphi}(\bar{f}) = \bar{f}\}$; any $\bar{f} \in I^0_{\bar{\varphi}}$ is said to be associated to $\bar{\varphi}_{\bar{f}}$. $I^0_{\bar{\varphi}}$ has either zero or two elements. As $\varphi^d = 1_{\mathcal{L}}$, we get that the order d_i of the cyclic permutation $\bar{\varphi}_i$ divides d. For any $i \in I_{\bar{\varphi}}$, we choose arbitrarily an element $\bar{f}_i$ of $\mathcal{L}$ which is associated to $\bar{\varphi}_i$. Then

$$\varphi^{d_i}(F_{\bar{f}_i}) = p^{\frac{d_i}{d}\ell(\bar{f}_i)} F_{\bar{f}_i},$$

with

$$\ell(\bar{f}_i) := \sum_{s=1}^{d} \bar{f}_i(s),$$

and φ^{d_i} acts on $F_{\bar{f}_i}$ as $p^{\frac{d_i}{d}\ell(\bar{f}_i)}\sigma^{d_i}$. Let

$${}_pF_{\bar{f}_i} := \big\{x \in F_{\bar{f}_i} | \sigma^{d_i}(x) = x\big\}.$$

It is a $W(\mathbb{F}_{p^{d_i}})$-free submodule of $F_{\bar{f}_i}$ such that $F_{\bar{f}_i} = {}_pF_{\bar{f}_i} \otimes_{W(\mathbb{F}_{p^{d_i}})} W(k(v))$.

4.1.2. Proposition. *The filtered σ-crystal (M, F^1, φ) is cyclic diagonalizable.*

Proof: We choose arbitrarily a $W(\mathbb{F}_{p^{d_i}})$-basis $\{\ell_s | s \in \mathcal{J}_i\}$ of ${}_pF_{\bar{f}_i}, \forall i \in I_{\bar{\varphi}}$. We view it as a $W(k(v))$-basis of $F_{\bar{f}_i}$. For any cyclic permutation $\bar{\varphi}_i$ of length ≥ 2 (i.e. when we deal with an $i \in I^1_{\bar{\varphi}}$), and for any element $\bar{f} \in \mathcal{L}$ associated to $\bar{\varphi}_i$, but different from $\bar{f}_i$, let j be the smallest natural number such that $\bar{f} = \bar{\varphi}^j(\bar{f}_i)$. We get a $W(k(v))$-basis of $F_{\bar{f}}$ by taking $\frac{1}{p^{e_{ji}}}$ of the images of $\varphi^j(\ell_s)$, $s \in \mathcal{J}_i$, with

$$e_{ji} := \sum_{n=1}^{j} \bar{f}_i(n).$$

With respect to the $W(k(v))$-basis of M obtained by putting together the chosen bases of $F_{\bar{f}}$, $\bar{f} \in \mathcal{L}$, (M, F^1, φ) gets the desired cyclic diagonalizable form. This ends the proof of the proposition.

4.1.2.1. Remark. By very definition the Shimura filtered σ-crystal $(M, F^1, \varphi, T_{W(k(v))})$ is cyclic diagonalizable.

4.1.3. Example. $\mathcal{L}$ has two elements iff τ is an ordinary type, i.e. iff $d = 1$.

4.1.4. Proposition. *The Shimura filtered σ-crystal*

$$\big(\mathrm{Lie}(G_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} W(k(v)), \varphi, F^0\big(\mathrm{Lie}(G_{W(k(v))})\big), F^1\big(\mathrm{Lie}(G_{W(k(v))})\big)\big)$$

and its adjoint are cyclic diagonalizable and of Borel type.

Proof: The fact that the mentioned two Shimura filtered Lie σ-crystals are of Borel type results from the fact that $\varphi(\mathrm{Lie}(B_{W(k(v))})) \subset \mathrm{Lie}(B_{W(k(v))})$ (we recall that B is a Borel subgroup of $G_{\mathbb{Z}_p}$). The proof of their cyclic diagonalizability is entirely analogous to the proof of 4.1.2, and so it is omitted.

4.1.5. Let $N \subset G_{\mathbb{Z}_p}$ be the normalizer of T. Let

$$W_G := (N/T)\big(W(\overline{k(v)})\big)$$

be the Weyl group (of $G_{W\left(\overline{k(v)}\right)}$) with respect to $T_{W(\overline{k(v)})}$. It depends only on G. For any $\omega \in W_G$, let $g_\omega \in N(W(\overline{k(v)}))$ be such that its image in W_G is ω. Let

$$\mathfrak{C}_\omega := (M \otimes_{W(k(v))} W(\overline{k(v)}), F^1 \otimes_{W(k(v))} W(\overline{k(v)}), g_\omega \sigma \mu(\frac{1}{p}), G_{W(\overline{k(v)})})$$

be the Shimura filtered σ-crystal defined by g_ω. We denote by $\mathcal{P}_\omega$ the Newton polygon of $\mathfrak{C}_\omega$, and by $\mathrm{Lie}_G(\mathcal{P}_\omega)$ the Newton polygon of the Shimura filtered Lie σ-crystal attached to $\mathfrak{C}_\omega$. We have:

4.1.6. Proposition. *$\mathfrak{C}_\omega$ is cyclic diagonalizable, its attached Shimura filtered (adjoint) Lie σ-crystal is cyclic diagonalizable, and $\mathcal{P}_\omega$ and $\mathrm{Lie}_G(\mathcal{P}_\omega)$ depend only on ω and not on the choice of g_ω.*

Proof: The proof of the cyclic diagonalizability of $\mathfrak{C}_\omega$ or of its attached Shimura filtered (adjoint) Lie σ-crystal is entirely analogous to the proof of 4.1.2, and so it is omitted. If $t \in T\big(W(\overline{k(v)})\big)$, then $\big(tg_\omega\sigma\mu(\frac{1}{p})\big)^n = t_n\big(g_\omega\sigma\mu(\frac{1}{p})\big)^n$, $\forall n \in \mathbb{N}$, with $t_n \in T\big(W(\overline{k(v)})\big)$, so the last part of 4.1.6 results from 3.2.1.

4.1.6.1. Remark. In 4.1.5-6, instead of $\overline{k(v)}$, we can work as well with the smallest finite field extension $k(v^{\mathrm{sp}})$ of $k(v)$ over which $G_{k(v)}$ splits.

4.1.7. We view the alternating form $\tilde{\psi}$ of 2.3.1 also as a perfect alternating form $\tilde{\psi} : M \otimes_{W(k(v))} M \to W(k(v))(1)$.

4.2. G-ordinary points. In all that follows, k is a perfect field containing $k(v)$.

4.2.1. Theorem. *There is a $G(\mathbb{A}_f^p)$-invariant Zariski dense open subscheme $\mathcal{U}$ of $\mathcal{N}_{k(v)}$ such that:*

a) a point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ has τ as the formal isogeny type of the Shimura σ-crystal attached to it (cf. 2.3.10), if and only if y factors through $\mathcal{U}$;

b) a point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ has $\mathrm{Lie}_G(\tau)$ as the formal isogeny type of the Shimura Lie σ-crystal attached to it (cf. 2.3.10), if and only y factors through $\mathcal{U}$.

Proof: Let $\mathbb{F} := \overline{k(v)}$ and let $V_0 := W(\mathbb{F})$. We still denote by σ the Frobenius automorphism of V_0. Let $\mathcal{N}_{V_0} := \mathcal{N} \times_{O(v)} V_0$ and let $\mathcal{N}_{\mathbb{F}}$ be its special fibre. Let $y_0 : \mathrm{Spec}(\mathbb{F}) \hookrightarrow \mathcal{N}_{\mathbb{F}}$ be a closed point and let $z_0 : \mathrm{Spec}(V_0) \hookrightarrow \mathcal{N}_{V_0}$ be a lift of it. Let $\mathcal{N}_{\mathbb{F}}^0$ be the connected component of $\mathcal{N}_{\mathbb{F}}$ containing y_0. Let (A_0, p_{A_0}) be the principally polarized abelian variety over V_0 obtained from $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ by pull back via z_0. Let $\big(M_0, F_0^1, \varphi_0, G_{V_0}, (t_\alpha)_{\alpha\in\mathcal{J}'}, p_{A_0}\big)$ be the principally quasi-polarized Shimura filtered σ-crystal attached to the point z_0 (cf. 2.3.10). So $M_0 := H^1_{dR}(A_0/V_0)$, F_0^1 is its Hodge filtration defined by A_0, and $(t_\alpha)_{\alpha\in\mathcal{J}'}$ is the family of de Rham components of the family $(\omega_\alpha)_{\alpha\in\mathcal{J}'}$ of Hodge cycles with which A_0 is naturally endowed (cf. 2.3.3). From 2.3.13.1

we deduce that for any point $z_1 : \mathrm{Spec}(V_0) \hookrightarrow \mathcal{N}_{V_0}$ lifting a point $y_1 : \mathrm{Spec}(\mathbb{F}) \hookrightarrow \mathcal{N}^0_{\mathbb{F}}$, the Shimura filtered σ-crystal attached to it is of the form $\big(M_0, F^1_0, \varphi_1, G_{V_0}, (t_\alpha)_{\alpha\in\mathcal{J}'}\big)$. We have $\varphi_1 = g_1\varphi_0$, with $g_1 \in G_{V_0}(V_0)$ (cf. the beginning of §3). From 2.3.16 we deduce that:

(4.2.1.1) *For any $\mathbb{F}$-valued point $\bar{g}_1$ of an open subscheme of $G_{\mathbb{F}}$ there is $g_1 \in G_{V_0}(V_0)$ lifting it, and there is an V_0-valued point z_1 of $\mathcal{N}_{V_0}$ lifting an $\mathbb{F}$-valued point y_1 of $\mathcal{N}^0_{\mathbb{F}}$, such that we have $\varphi_1 = g_1\varphi_0$.*

Let $\mathcal{U}_0$ (resp. $\mathcal{U}_1$) be the open subscheme of $\mathcal{N}^0_{\mathbb{F}}$ of whose geometric points have the smallest Newton polygon of their attached Shimura σ-crystals (resp. Lie σ-crystals). Such open subschemes due exist, cf. the semicontinuity theorem [Ka2, 2.3.1].

As all σ-crystals (M_0, φ_1) give birth to a 1/2-symmetric isocrystals we deduce:

(4.2.2) *The fact that the σ-crystal (M_0, φ_1) attached to the point $y_1 \in \mathcal{N}^0_{\mathbb{F}}(\mathbb{F})$ has the smallest Newton polygon among all σ-crystals attached to $\mathbb{F}$-valued points of $\mathcal{N}^0_{\mathbb{F}}$, is equivalent to the fact that the Lie σ-crystal $\big(\mathrm{End}(M_0), \varphi_1\big)$ has the smallest Newton polygon among all Lie σ-crystals $\big(\mathrm{End}(M_0), \tilde{\varphi}\big)$ defined naturally by the σ-crystals attached to $\mathbb{F}$-valued points of $\mathcal{N}^0_{\mathbb{F}}$.*

As the Newton polygon of $\big(\mathrm{End}(M_0), \varphi_1\big)$ is obtained from the Newton polygon of $\big(\mathrm{Lie}(G_{V_0}), \varphi_1\big)$ and from the Newton polygon of $\big(\mathrm{End}(M_0)/\mathrm{Lie}(G_{V_0}), \varphi_1\big)$, we deduce that $\mathcal{U}_0 \subset \mathcal{U}_1$. $\big(\mathrm{End}(M_0)/\mathrm{Lie}(G_{V_0}), \varphi_1\big)$ is not a Lie σ-crystal, but

$$\big(\mathrm{End}(M_0)/\mathrm{Lie}(G_{V_0}), F^0(\mathrm{End}(M_0))/F^0(\mathfrak{g}), F^1(\mathrm{End}(M_0))/F^1(\mathfrak{g}), \varphi_1\big)$$

is a p-divisible object of $\mathcal{MF}_{[-1,1]}(V_0)$.

4.2.3. Let $K_0 := V_0[\frac{1}{p}]$. It is known that there is an isomorphism

$$H^1_{\acute{e}t}(A_{0\bar{K}_0}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K_0 \tilde{\to} M_0 \otimes_{V_0} K_0$$

taking the p-component of the étale component u_α of ω_α into t_α, $\forall \alpha \in \mathcal{J}'$, and taking $\tilde{\psi}$ into p_{A_0} (cf. [Va2, 5.1.4 and 5.2.17.2]). But $H^1_{\acute{e}t}(A_{0\bar{K}_0}, \mathbb{Q}_p)$ and the tensors $(u_\alpha)_{\alpha\in\mathcal{J}'}$ of $\mathcal{T}\big(H^1_{\acute{e}t}(A_{0\bar{K}_0}, \mathbb{Q}_p)\big)$ can be identified (non-canonically) with $L^*_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and the tensors $(v_\alpha)_{\alpha\in\mathcal{J}'}$ of $\mathcal{T}(L^*_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ (cf. [Va2, 5.1.2]). So we get an isomorphism

$$\rho : M \otimes_{W(k(v))} K_0 \tilde{\to} M_0 \otimes_{V_0} K_0$$

taking v_α into t_α, $\forall \alpha \in \mathcal{J}'$, and $\tilde{\psi}$ into p_{A_0}.

The isomorphism ρ allows us to identify the Frobenius automorphism φ of $M[\frac{1}{p}]$ defined in 4.1 with a σ-linear automorphism of $M_0 \otimes_{V_0} K_0$. So we have $\varphi = \tilde{g}\varphi_0$ with $\tilde{g} \in G^0(K_0)$. There is a cocharacter $\mu_0 : \mathbb{G}_m \to G_{V_0}$ which produces a direct sum decomposition $M_0 = F^1_0 \oplus F^0_0$, with $\beta \in \mathbb{G}_m(V_0)$ acting through μ on F^i_0 as the multiplication with β^{-i} (cf. 2.3.10). Under an embedding of V_0 in $\mathbb{C}$, extending the composite inclusion $O_{(v)} \subset E(G, X) \subset \mathbb{C}$, μ_0 becomes $G(\mathbb{C})$-conjugate to the cocharacters μ^*_x, $x \in X$. We deduce that μ_0 and μ are conjugate over K_0, as G_{K_0} is a split group. So we can assume that

$$\rho(F^1 \otimes_{W(k(v))} K_0) = F_0^1 \otimes_{V_0} K_0.$$

If G is a torus, 4.2 is trivial: we have $\mathcal{U} = \mathcal{N}_{k(v)}$ (cf. [Va2, 3.2.8 and 3.3.1-2]). So we can assume that G is not a torus.

4.2.4. We got two particular hyperspecial subgroups of $G(K_0)$: $G_{V_0}(V_0)$ (defined by the V_0-lattice M_0 of $M_0 \otimes_{V_0} K_0$) and $G_{\mathbb{Z}_{(p)}}\big(M \otimes_{W(k(v))} V_0\big)$ (defined by the V_0-lattice $M \otimes_{W(k(v))} V_0$ of $M \otimes_{W(k(v))} K_0$). But any two hyperspecial subgroups of $G(K_0)$ are conjugate by an element $g_0 \in G^{\mathrm{ad}}(K_0)$ (cf. [Ti, p. 47]). Changing the Shimura adjoint filtered Lie σ-crystal attached to z_0 by the inner automorphism defined by g_0, we deduce that it can be put in the form $\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0}), \varphi_0', F_0^0\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big), F_0^1\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big)\big)$. Warning: here $G^{\mathrm{ad}}_{V_0}$ is with respect to the second hyperspecial subgroup; we felt it is unmotivately to write $\mathrm{Lie}(G_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} V_0$ instead of $\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})$.

As μ and μ_0 are conjugate over K_0, changing everything by an isomorphism (defined by an element of $G^{\mathrm{ad}}_{V_0}(V_0)$), we can assume that $F_0^0\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big) = F^0\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big)$ and $F_0^1\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big) = F^1\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big)$. As $\varphi = \tilde{g}\varphi_0$ with $\tilde{g} \in G^0(K_0)$, we deduce that the Shimura adjoint filtered Lie σ-crystal attached to z_0 is isomorphic to $\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0}), g\varphi, F^0\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big), F^1\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big)\big)$, with $g \in G^{\mathrm{ad}}_{V_0}(K_0)$. As g leaves invariant $\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})$, and as $G^{\mathrm{ad}}_{V_0}(V_0)$ is a maximal compact subgroup of $G^{\mathrm{ad}}_{V_0}(K_0)$ (being a hyperspecial subgroup of it), we deduce that $g \in G^{\mathrm{ad}}_{V_0}(V_0)$.

This last fact allows us, via 3.6.5, to appeal to the results of 3.1-3, 3.4.6 and 3.5.

4.2.5. 4.2.1.1 implies: there is an open subscheme U of $G^{\mathrm{ad}}_{\mathbb{F}}$ with the property that for any element $\bar{g}_0 \in U(\mathbb{F})$, there is a V_0-valued point of $\mathcal{N}_{V_0}$ lifting an $\mathbb{F}$-valued point of $\mathcal{N}^0_{\mathbb{F}}$ such that its attached Shimura adjoint filtered Lie σ-crystal is isomorphic to

$$\Big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0}), g_0 g\varphi, F^0\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big), F^1\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big)\Big),$$

with g_0 mod p equal to $\bar{g}_0$.

4.2.6. Now 4.2.1 for $\mathcal{N}^0_{\mathbb{F}}$ is a direct consequence of 4.2.5, of the proof of 3.1.0, of 3.4.6, of 3.5.11, of 3.6.5, and of 4.2.7 below. We present the details. From §3 we need:

– 3.3.4 which implies that $\big(M, \varphi, G_{W(k(v))}\big)$ is a $G_{W(k(v))}$-ordinary σ-crystal.

– 3.4.6 and 3.5.11 which together with 4.2.5 imply (via 3.6.5) that there are $\mathbb{F}$-valued points of $\mathcal{N}^0_{\mathbb{F}}$ whose attached Shimura Lie σ-crystals have $\mathrm{Lie}_G(\tau)$ as their formal isogeny type.

– 3.2.5 and 3.6.5 which together with 4.2.5 imply that the $\mathbb{F}$-valued points of $\mathcal{U}_1$ have $\mathrm{Lie}_G(\tau)$ as the formal isogeny type of their attached Shimura Lie σ-crystals; we recall that $\mathrm{Lie}\big(Z(G_{V_0})\big)$ contributes only with slopes zero to the formal isogeny type of $\big(\mathrm{Lie}(G_{V_0}), g\varphi_0\big)$, $\forall g \in G_{V_0}(V_0)$, cf. 3.5.0.

– 3.2.5 which implies that $\mathcal{U}_0 = \mathcal{U}_1$.

In fact the use of 3.4.6 can be avoided, cf. 3.5.14. So we need to have $\mathcal{U}_{\mathbb{F}} \cap \mathcal{N}^0_{\mathbb{F}} = \mathcal{U}_1 = \mathcal{U}_0$.

4.2.7. The fact that the $\mathbb{F}$-valued points of $\mathcal{U}_1$ give birth to Shimura σ-crystals having precisely τ as their formal isogeny type results form the fact that under the isomorphism ρ of 4.2.3, the slopes of (M, φ) together with their multiplicities, and the slopes of the Shimura σ-crystal attached to a point of $\mathcal{U}_1(\mathbb{F})$ together with their multiplicities, are both computed using the eigenvalues (and their multiplicities) of two semisimple elements s_1 and respectively s_2 of $G_{K_0}(K_0)$ which are conjugate by an element of $G^{\mathrm{ad}}_{V_0}(K_0)$, and so are conjugate by an element of $G_{V_0}(\overline{K_0})$ as the homomorphism $G_{V_0}(\overline{K_0}) \to G^{\mathrm{ad}}_{V_0}(\overline{K_0})$ is surjective.

To see this, for being able to use the previous notations, we assume that y_0 factors through $\mathcal{U}_0$. A sufficiently high power n of φ (resp. of φ_0) acts diagonally with respect to a $W(\mathbb{F})$-basis of $M \otimes_{W(k(v))} W(\mathbb{F})$ (resp. of M_0), producing (via 3.11.2) the semisimple element s_1 (resp. s_2) of $G_{K_0}(K_0)$ (we use the identification achieved by ρ). Choosing n big enough and a multiple of d times the exponent of p of the number of elements of the finite field over which $A_{0\mathbb{F}}$ is defined, these semisimple elements are uniquely determined, up to $G_{K_0}(K_0)$-conjugation. As the cocharacter of $G^{\mathrm{ab}}_{K_0}$ obtained by composing μ with the quotient homomorphism $m_q : G_{K_0} \to G^{\mathrm{ab}}_{K_0}$, is the same as the one obtained by composing μ_0 with m_q (as μ and μ_0 are $G_{K_0}(K_0)$-conjugate), we deduce that the images of s_1 and s_2 in $G^{\mathrm{ab}}_{K_0}(K_0)$ are the same. Moreover the images of s_1 and s_2 in $G^{\mathrm{ad}}_{K_0}(K_0)$ are $G^{\mathrm{ad}}_{K_0}(K_0)$-conjugate: this is an immediate consequence of 4.2.4-5, via 3.5.11 and 3.6.5. Let q be the order of the isogeny $G^{\mathrm{der}} \to G^{\mathrm{ad}}$. We deduce that s_1^q and s_2^q are $G^{\mathrm{ad}}_{K_0}(K_0)$-conjugate.

4.2.8. The formal isogeny types τ and $\mathrm{Lie}_G(\tau)$ depend only on f and v, and so what we got for $\mathcal{N}^0_{\mathbb{F}}$ remains true for all other connected components of $\mathcal{N}_{\mathbb{F}}$. This implies that the open subscheme $\mathcal{U}$ of $\mathcal{N}_{k(v)}$ defined by 4.2.1 a) exists, is also defined by 4.2.1 b), and moreover is Zariski dense in $\mathcal{N}_{k(v)}$. As the Newton polygon is an isogeny invariant we get that $\mathcal{U}$ is $G(\mathbb{A}^p_f)$-invariant. This ends the proof of 4.2.1.

Another way of passing from one connected component of $\mathcal{N}_{\mathbb{F}}$ to all others is to use the following fact:

4.2.8.1.* Fact. *The connected components of $\mathcal{N}_{\mathbb{F}}$ are permuted transitively by $G(\mathbb{A}^p_f)$ (cf. [Va2, 3.3.2] and the existence of smooth toroidal compactifications of $\mathcal{N}$, see [Va3]; if (G, X) is of compact type, then this fact is a consequence of 2.3.3.1).*

4.2.9. Definition. The points of $\mathcal{U}_{k(v)}$ (resp. of $\mathcal{U}_{k(v)}/\tilde{H}_0$) with values in fields are called G-ordinary points, or Shimura-ordinary points of $\mathcal{N}_{k(v)}$ (resp. of $\mathcal{N}_{k(v)}/\tilde{H}_0$). Here $\tilde{H}_0$ is an arbitrary compact (not necessarily open) subgroup of $G(\mathbb{A}^p_f)$.

4.2.10. Corollary. *The automorphism class (see 2.2.11 for a definition of this) of the Shimura adjoint σ-crystals attached to points of $\mathcal{N}_{k(v)}$ with values in a perfect field k is given by the Frobenius σ-automorphism of* $\mathrm{Lie}(G^{\mathrm{ad}}_{W(k)}) = \mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} W(k)$ *acting trivially on* $\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p})$.

Proof: This is nothing else but 4.2.5 (or 4.2.4). The fact that we can assume that k

is just perfect is implied by the convention 2.3.9.2, and so by the fact that $G_{W(k)}$ is split; so the isomorphism ρ of 4.2.3 exists as well in this case.

4.3. The (refined) Lie stable p-rank.

4.3.1. In what follows we apply 3.9-10 to the context of 4.2. Let

$$G^{\mathrm{ad}}_{\mathbb{Z}_p} = \times_{i\in\mathcal{H}} G_i$$

be the product decomposition of $G^{\mathrm{ad}}_{\mathbb{Z}_p}$ in $\mathbb{Z}_p$-simple factors. So any $G_{i\mathbb{Q}_p}$ is a $\mathbb{Q}_p$-simple adjoint group. Without specifically stating, we assume that $k = \bar{k}$. For any point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ the Shimura adjoint Lie σ-crystal $\big(\mathrm{Lie}(G^{\mathrm{ad}}_{W(k)}), \varphi_y\big)$ attached to y is a product of $|\mathcal{H}|$-factors $\big(\mathrm{Lie}(G_{iW(k)}), \varphi_y\big)$, $i \in \mathcal{H}$, each one being a cyclic Shimura adjoint Lie σ-crystal. In other words φ_y permutes (cyclically) transitively the Lie algebras of the $W(k)$-simple factors of $G_{iW(k)}$ (this is a consequence of 4.2.10 and of the fact that for any $i \in \mathcal{H}$, $G_{i\mathbb{Q}_p}$ is the Weil restriction of an absolutely simple adjoint group over an unramified finite field extension of $\mathbb{Q}_p$). We call these factors the cyclic adjoint factors attached to the point y. We also speak about the cyclic Lie factors attached to y (cf. def. 3.10.1). We naturally extend the terminology to points $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$, or to the context of the quotient $\mathcal{N}/H_0$, or to the case when k is not algebraically closed, but just perfect (in this last case the adjoint groups involved are still products of absolutely simple adjoint groups; this is a consequence of 4.2.5 and of 3.10.0.1).

4.3.1.1. We write

$$G_{iW(k)} = \times_{j\in\mathcal{H}_i} G_j,$$

with G_j an absolutely simple adjoint $W(k)$-group. So

$$G^{\mathrm{ad}}_{W(k)} = \times_{\substack{i\in\mathcal{H}\\ j\in\mathcal{H}_i}} G_j = \times_{\ell\in I} G_\ell,$$

where I is the (disjoint) union of $\mathcal{H}_i$, $i \in \mathcal{H}$. So φ_y permutes the factors $\mathrm{Lie}(G_\ell)$, $\ell \in I$, achieving a permutation γ of I. γ is a product of $|\mathcal{H}|$-disjoint cyclic permutations γ_i, with γ_i the cyclic permutation of $\mathcal{H}_i$ defined by φ_y.

4.3.1.2. Notations. We denote by R_y (resp. R^r_y) the Lie stable p-rank (resp. the refined Lie stable p-rank) of the Shimura adjoint σ-crystal attached to y (cf. def. 3.9.1 and 3.9.4).

We denote by $R(\tau)$ (resp. $R^r(\tau)$) the Lie stable p-rank (resp. the refined Lie stable p-rank) of the Shimura adjoint filtered Lie σ-crystal attached to the Shimura σ-crystal $\big(M, F^1, \varphi, G_{W(k(v))}\big)$ defined in 4.1.

4.3.2. R_y is a non-negative integer. Fixing a bijection $\mathcal{H} \overset{f_{\mathcal{H}}}{\to} \{1, 2, \ldots, |\mathcal{H}|\}$, R^r_y is a sequence of non-negative integers, indexed by the elements of the set $\{1, ..., |\mathcal{H}|\}$. These elements index the $\mathbb{Q}_p$-simple factors of $G^{\mathrm{ad}}_{\mathbb{Q}_p}$. The n-th number of the sequence is the Lie stable p-rank of the cyclic Shimura adjoint Lie σ-crystal $\Big(\mathrm{Lie}\big(G_{f^{-1}_{\mathcal{H}}(n)}W(k)\big), \varphi_y\Big)$, divided by $\left|\mathcal{H}_{f_{\mathcal{H}}^{-1}(n)}\right|$ (here $n \in \{1, \ldots, |\mathcal{H}|\}$). If $k \neq \bar{k}$, we still denote by R_y (resp. R_y^r) the Lie stable p-rank (resp. the refined Lie stable p-rank) of the point $\bar{y}$ defined by the composite morphism $\mathrm{Spec}(\bar{k}) \xrightarrow{i_k} \mathrm{Spec}(k) \xrightarrow{y} \mathcal{N}_{k(v)}$, with i_k defined by the natural inclusion $k \hookrightarrow \bar{k}$ (see also 3.9.1.1).

4.3.3. Remark. The set of Lie stable p-ranks of points of $\mathcal{N}_{k(v)}$ is not a sequence of consecutive non-negative integers. For example in the case of special fibres of integral canonical models of a Siegel modular variety $\mathrm{Sh}\big(\mathrm{GSp}(W, \psi), S\big)$, the set of values of Lie stable p-ranks is $\{1, 3, 6, \ldots, \frac{e(e+1)}{2}\}$ (we recall that $\dim_{\mathbb{Q}}(W) = 2e$).

4.3.4. Theorem. *Let $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ be an arbitrary point. The following statements are equivalent:*

a) *y factors through $\mathcal{U}$;*

b) *$R_y = R(\tau)$;*

c) *$R_y^r = R^r(\tau)$.*

Proof: This is a direct consequence of 3.9.2, 3.9.4, 4.2.1 and 4.2.4, applied to any connected component of $\mathcal{N}_{\mathbb{F}}$ (cf. also 3.9.6).

4.3.5. Corollary. *$R(\tau)$ (resp. $R^r(\tau)$) is the biggest number (resp. the biggest sequence) among all numbers R_y (resp. among all sequences of numbers $R_{\tilde{y}}^r$) associated to points $\tilde{y} \in \mathcal{N}_{k(v)}(\tilde{k})$) (with $\tilde{k}$ a perfect field).*

Proof: This is a consequence of 4.2.1.1, 4.2.4 and 3.5.6.

4.3.6. Corollary. *The complement $\mathcal{N}_{k(v)} \backslash \mathcal{U}$ of $\mathcal{U}$ in $\mathcal{N}_{k(v)}$, is of pure codimension 1 in $\mathcal{N}_{k(v)}$.*

Proof: The proof of this is entirely analogous to the one in the case of the open subscheme of ordinary points of the special fibre of an integral canonical model of a Siegel modular variety. If $\dim(X) = 0$, then the corollary is obvious: we have $\mathcal{U} = \mathcal{N}_{k(v)}$.

We assume now that $\dim(X) \geq 1$. 4.3.4-5 assures (to be compared with 4.3.7) that $\mathcal{N}_{k(v)} \backslash \mathcal{U}$ is of pure codimension 1, if it is non void.

To see that $\mathcal{N}_{k(v)} \backslash \mathcal{U}$ is non void, we first treat the case when $\mathcal{N}$ has the completion property, just to emphasize how 3.4.15 can be applied in practice. So, assuming that $\mathcal{N}$ has the completion property, the fact that $\mathcal{N}_{k(v)} \backslash \mathcal{U}$ is non void can be checked, by just remarking: If $\big(M_y, F_z^1, \varphi_y, G_{W(k)}\big)$ is the Shimura filtered σ-crystal attached to a G-canonical lift z (defined in 4.4.2 below; its definition relies just on 2.3.17 and 3.1.0) of a G-ordinary point $y \in \mathcal{U}(k)$, then $F_z^1\big(\mathrm{Lie}(G_{W(k)}\big) \neq 0$ (corresponding to the fact $\dim(X) > 0$), and there is $g \in G_{W(k)}^0(W(k))$ such that $\big(M_y, g\varphi_y, G_{W(k)}\big)$ is not a $G_{W(k)}$-ordinary σ-crystal.

To see this let G_{ℓ_0} be a $W(k)$-simple factor of $G_{W(k)}^{\mathrm{ad}}$, with $\ell_0 \in I$, such that $F_z^1\big(\mathrm{Lie}(G_{l_0})\big) \neq 0$. Let T and B be as in 4.1, and let $\mathrm{Lie}(B) = \mathfrak{b}$. Let $\omega \in G_W^0(k)(W(k))$ be a Weyl element of $G_{W(k)}$ with respect to $T_{W(k)}$ (i.e. $\omega T_{W(k)} \omega^{-1} = T_{W(k)}$) such that its action on the G_{ℓ_0} factor of $G_{W(k)}^{\mathrm{ad}}$ takes the nilpotent radical of the opposite Borel Lie subalgebra of $\mathfrak{b}_{\ell_0} := \mathfrak{b} \otimes_{\mathbb{Z}_p} W(k)[\frac{1}{p}] \cap \mathrm{Lie}(G_{\ell_0})$ with respect to T_{ℓ_0} (defined as the image of $T_{W(k)}$ in G_{l_0}) into the nilpotent radical of $\mathfrak{b}_{\ell_0}$. In other words ω takes the negative roots of the Weyl decomposition of $\mathrm{Lie}(G_{\ell_0})$ with respect to T_{ℓ_0} and $\mathfrak{b}_{\ell_0}$, into positive roots.

We can assume that $\mathfrak{b}$ leaves invariant F_z^1, and that φ_y leaves invariant $\mathrm{Lie}(T_{W(k)})$ and $\mathfrak{b} \otimes_{\mathbb{Z}_p} W(k)$ (cf. 3.4.15 and 3.2.3, or cf. 3.11.1 c)). Obviously the Lie stable p-rank of the cyclic Shimura adjoint Lie σ-crystal $\big(\mathrm{Lie}(G_{iW(k)}), \omega\varphi_1\big)$ (with G_i a $\mathbb{Z}_p$-simple factor of $G^{\mathrm{ad}}_{\mathbb{Z}_p}$ such that $G_{\ell_0} \subset G_{iW(k)}$) is zero (cf. also 4.3.7 below), and so (cf. 4.3.4) it is not a $G_{W(k)}$-ordinary σ-crystal.

To treat the general case, we use the above notations. We use special quadruples (cf. [Va2, 3.2.10]). Let $\tilde{T}$ be a maximal $\mathbb{Q}$–torus of G such that:

i) over $\mathbb{Z}_p$ it is a maximal torus of $G_{\mathbb{Z}_p}$ with the property that its image in G_i has rank 0 over $W(k(v_i))$ (here $k(v_i)$ is a subfield of $k(v)$ such that $B(k(v_i))$ is the field of definition of the conjugacy class of the cocharacter of $G_{iB(k(v))}$ obtained by taking the generic fibre of the composite of μ (of 4.1) with the natural quotient map $G_{W(k(v))} \to G_{iW(k(v))}$);

ii) over $\mathbb{R}$ it is the extension of a compact torus by a one dimensional split torus.

The existence of such a torus is implied by [Har, 5.5.3], by the structure of simple adjoint groups over fields (see [Ti1]), and by the following simple fact

Fact. *Any absolutely simple adjoint group of classical Lie type $\tilde{S}$ over a finite field $\tilde{k}$ has a maximal torus whose rank over $\tilde{k}_1$ is zero; here $\tilde{k}_1$ is $\tilde{k}$ or the field extension of $\tilde{k}$ of degree two, depending on the fact that $\tilde{S}$ is split or splits over $\tilde{k}_1$.*

This Fact can be proved easily, case by case. As in [Va2, 3.2.11], we deduce the existence of a special quadruple $(\tilde{T}, \{\tilde{h}\}, \tilde{H}_{\tilde{T}}, \tilde{v})) \hookrightarrow (G, X, H, v)$. Obviously the $\tilde{T}$-ordinary type we get using special quadruple is not the G-ordinary type of 4.4.1, as its A-degree of definition is bigger (to be compared with 4.11). This implies that $\mathcal{N}_{k(v)} \setminus \mathcal{U}$ is non-empty. This ends the proof of the corollary.

4.3.6.1. Remark. Using the fact that any maximal compact torus T_1^{ad} of a non-compact factor $\mathcal{F}$ of $G^{\mathrm{ad}}_{\mathbb{R}}$ is generated by the images in $\mathcal{F}$ of homomorphisms from the real standard torus $\mathbb{S}$ (see [Va2, 2.2]) into $G_{\mathbb{R}}$ corresponding to elements of X, we deduce that in the proof of 4.3.6 it is enough to assume that the image of $\tilde{T}_{\mathbb{Q}_p}$ in (the simple factor $G_{i\mathbb{Q}_p}$ of) $G^{\mathrm{ad}}_{\mathbb{R}}$ (corresponding to $\mathcal{F}$) is not split: we can choose $\tilde{h}$ accordingly.

4.3.7. Examples. We concentrate just on one $\mathbb{Z}_p$-simple factor G_i of $G^{\mathrm{ad}}_{\mathbb{Z}_p}$. We can assume that $\mathcal{H}_i = \{1, 2, \ldots, d_i\}$, with $d_i \in \mathbb{N}$. So $|\mathcal{H}_i| = d_i$. Let $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$ be a lift of y and let

$$ {}_i\mathcal{C}_y := \Big(\mathrm{Lie}\big(G_{iW(k)}\big), \varphi_y, F^0\big(\mathrm{Lie}(G_{iW(k)})\big), F^1\big(\mathrm{Lie}(G_{iW(k)})\big)\Big) $$

be the i-th cyclic factor of the Shimura adjoint filtered Lie σ-crystal attached to z. We can assume that $F^1\big(\mathrm{Lie}(G_1)\big) \neq 0$ (if $F^1\big(\mathrm{Lie}(G_{iW(k)}\big) = 0$, there is nothing to be done). Let $\bar{\psi} : \mathrm{Lie}(G_{ik}) \to \mathrm{Lie}(G_{ik})$ (defined as in 3.9.1) be the map used in computing the Lie stable p-rank of ${}_i\mathcal{C}_y$.

Case 1. ${}_i\mathcal{C}_y$ **is without involution.** The σ^{d_i}-linear map

$$ {}_1\overline{\psi_i}^{d_i} : \big(\mathrm{Lie}(G_1)/F^0(\mathrm{Lie}(G_1))\big) \otimes_{W(k)} k \to \big(\mathrm{Lie}(G_1)/F^0(\mathrm{Lie}(G_1))\big) \otimes_{W(k)} k $$

obtained from $\psi_i^{d_i}$ by passage to quotients, still computes (in the sense of 3.6) $\frac{1}{d_i}$ times the Lie stable p-rank of ${}_i\mathcal{C}_y$.

If moreover ${}_i\mathcal{C}_y$ is of B_ℓ, C_ℓ or $D_\ell^{\mathbb{R}}$ type, then this Lie stable p-rank is d_i times the $f_{\mathcal{H}}(i)$-element of the sequence describing the refined Lie stable p-rank of a G-ordinary point of $\mathcal{N}_{k(v)}$ iff the σ^{d_i}-linear map ${}_1\overline{\psi_i}^{d_i}$ has a non-zero determinant (cf. with the formulae of 3.10.6 i)). So if all the simple factors of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ are of B_ℓ, C_ℓ or $D_\ell^{\mathbb{R}}$ type, then the fact that for a point $y \in \mathcal{N}_{k(v)}(k)$ we have $R_y = R(\tau)$, can be expressed in terms of some determinants being non-zero. This is very close to the intuition provided by the Hasse–Witt matrices.

Case 2. ${}_i\mathcal{C}_y$ **is with involution.** So ${}_i\mathcal{C}_y$ is of A_ℓ or of $D_\ell^{\mathbb{H}}$ type (cf. the classification of adjoint Shimura varieties of abelian type [De1]). The σ^{2d_i}-linear map

$$ {}_1\overline{\psi_i}^{2d_i} : \big(\mathrm{Lie}(G_1)\big)/F^0\big(\mathrm{Lie}(G_1)\big) \otimes_{W(k)} k \to \big(\mathrm{Lie}(G_1)\big)/F^0\big(\mathrm{Lie}(G_1)\big) \otimes_{W(k)} k $$

obtained from $\psi_i^{2d_i}$ by passage to quotients, still computes $\frac{1}{d_i}$ times the Lie stable p-rank of ${}_i\mathcal{C}_y$.

So in the cases when ${}_i\mathcal{C}_y$ is of A_ℓ or of $D_\ell^{\mathbb{H}}$ type, with or without involution, the fact that the Lie stable p-rank of this σ^{m_i}-linear map ${}_1\overline{\psi_i}^{m_i}$, with $m_i \in \{d_i, 2d_i\}$, is d_i times the $f_{\mathcal{H}}(i)$-element of the sequence describing the refined Lie stable p-rank of a G-ordinary point of $\mathcal{N}_{k(v)}$, can be expressed (cf. 4.3.5) in terms of some matrices of rank at most 1 being non-zero (such matrices are obtained by taking an adequate exterior power of ${}_1\overline{\psi_i}^{m_i}$).

As a conclusion we can always express the fact that for a point $y \in \mathcal{N}_{k(v)}(k)$ we have $R_y = R(\tau)$, in terms of some matrices of rank at most 1 being non-zero.

4.3.8. Remarks. 1) For any point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$, we denote by $p - \mathrm{Lie}_G(y)$, the Lie p-rank, defined as the multiplicity of the slope -1 of the Shimura adjoint Lie σ-crystal attached to y. If $p - \mathrm{Lie}_G(y) \neq 0$, then the Lie stable p-rank of y is different from zero, and the p-rank of the abelian variety A_y over k, obtained from $\mathcal{A}$ by pull back through y, is different from zero. But the converse of the first implication is not true, i.e. there are examples ($G_{\mathbb{R}}^{\mathrm{ad}}$ must have compact factors) when the Lie stable p-rank of y (or the p-rank of A_y) is different from zero, but the Lie p-rank of y is zero.

2) In the case of special fibres of integral canonical models of Siegel modular varieties, to know the Lie p-rank r^y of a k-valued point y (of such a fibre) is equivalent to knowing the p-rank r_y of A_y: we have

$$ r^y = \frac{r_y(r_{y+1})}{2}. $$

But this is not true in the general context of Shimura varieties of Hodge type.

3) In the computations of 4.3.7 it is irrelevant which G_1 factor $G_{iW(k)}$ we choose, subject to $F^1(\mathrm{Lie}(G_1)) \neq 0$.

4) For each $i \in \mathcal{H}$ we can define as in 3.11.3, an A-degree of definition $d_A(i)$: it is defined by the property (with the notations of the proof of 4.3.6) that $[k(v_i) : \mathbb{F}_p] = p^{d_A(i)}$. We have $[k(v^{\mathrm{ad}}) : \mathbb{F}_p] = p^{d_A}$ (resp. $[k(v^{\mathrm{ab}}) : \mathbb{F}_p] = p^{d_T}$), where d_A (resp. d_T is the A-degree (resp. the T-degree) of definition of the G-ordinary σ-crystals attached to G-ordinary points of $\mathcal{N}_{k(v)}$. d_A is the smallest common multiple of $d_A(i)$, $i \in \mathcal{H}$, while $d = [d_A, d_T]$. We refer to $d_A(i)$ as the i-th A-degree of definition of (the G-ordinary type produced by) the SHS $(f, L_{(p)}, v)$.

4.3.9. Problem. For any abelian variety over a perfect field $\tilde{k}$, interpret the a-number of it in terms of the Lie $\tilde{\sigma}$-crystal attached to it (defined by End( ) of its $\tilde{\sigma}$-crystal), and then use this to define G-a-numbers for abelian varieties over $\tilde{k}$, obtained from $\mathcal{A}$ by pull back through points $\mathrm{Spec}(\tilde{k}) \to \mathcal{N}_{k(v)}/H_0$ (here $\tilde{\sigma}$ is the Frobenius automorphism of $W(\tilde{k})$).

4.3.10. Definition. The σ^{m_i}-linear maps ${}_1\bar{\psi}_i^{m_i}$, $m_i \in \{d_i, 2d_i\}$, defined in 4.3.7, as well as their exterior powers are called Shimura–Hasse–Witt maps, and the matrices defined by them are called Shimura–Hasse–Witt matrices. Similarly, for any cyclic Shimura Lie σ-crystal (so we are not necessarily over an algebraically closed field; see 3.9.1.1) we define such maps and matrices. The only difference: m_i can be also $3d_i$ (cf. 3.10.0).

4.4. G-canonical lifts.

4.4.1. Theorem. 1) *A point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ factors through $\mathcal{U}$ iff y is a parabolic point, i.e. iff there is a lift $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$ of y with the property that the Shimura filtered Lie σ-crystal attached to z is of parabolic type.*

2) *We assume now that y factors through $\mathcal{U}$. We have:*

a) There is a unique lift $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$ of y with the property that the Shimura filtered Lie σ-crystal attached to z is of parabolic type;

b) If $k = \bar{k}$, then the Shimura adjoint filtered Lie σ-crystal attached to this unique lift z is isomorphic to $\Big(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} W(k), \sigma\mu(\frac{1}{p}), F^0\big(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} W(k)\big), F^1\big(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} W(k)\big)\Big)$, and so it is cyclic diagonalizable and of Borel type (μ being as in 4.1);

c) If $k = \bar{k}$ then the Shimura filtered Lie σ-crystal attached to z is isomorphic to

$$\Big(\mathrm{Lie}(G_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} W(k), \sigma\mu(\frac{1}{p}), F^0\big(\mathrm{Lie}(G_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} W(k)\big), F^1\big(\mathrm{Lie}(G_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} W(k)\big)\Big),$$

and so it is cyclic diagonalizable and of Borel type.

3) *Let $\big(M_y, F^1_z, \varphi_y, G_{W(k)}, (t_\alpha)_{\alpha \in \mathcal{J}'}, \psi_y\big)$ be the principally quasi-polarized Shimura filtered σ-crystal attached to z. We have:*

a) There is a unique cocharacter $\mu_y : \mathbb{G}_m \hookrightarrow G_{W(k)}$, with $[\mu_y] = [\mu]$, which produces a direct sum decomposition $M_y = F^1_z \oplus F^0_z$ such that the parabolic Lie subalgebra of $\mathrm{Lie}(G_{W(k)})$ corresponding to non-negative (resp. non-positive) slopes of $\big(\mathrm{Lie}(G_{W(k)}), \varphi_y\big)$ leaves invariant F^1_z (resp. F^0_z);

b) If $k=\bar{k}$, then the filtered σ-crystal (M_y,F^1_z,φ_y) is cyclic diagonalizable, and $\big(M_y,F^1_z,\varphi_y,G_{W(k)},(t_\alpha)_{\alpha\in\mathcal{J}'},\psi_y\big)$ is isomorphic to any other principally quasi-polarized Shimura filtered σ-crystal $\big(M_{y_1},F^1_{z_1},\varphi_{y_1},G_{W(k)},(t^1_\alpha)_{\alpha\in\mathcal{J}'},\psi_{y_1}\big)$ defined similarly but for another G-ordinary point $y_1:\mathrm{Spec}(k)\to\mathcal{U}$ factoring through the same connected component of $\mathcal{U}_k$ through which y factors;

c) If $k=\bar{k}$, then there is $n\in\mathbb{N}$ such that the principally quasi-polarized Shimura filtered σ^n-isocrystal $\big(M_y[\frac{1}{p}],F^1_z[\frac{1}{p}],\varphi^n_y,G_{B(k)},(t_\alpha)_{\alpha\in\mathcal{J}'},\psi_y\big)$ is isomorphic to the principally quasi-polarized Shimura filtered σ^n-isocrystal $\big(M\otimes_{W(k(v))}W(k)[\frac{1}{p}],F^1\otimes_{W(k(v))}W(k)[\frac{1}{p}],(\varphi\otimes 1)^n,G_{B(k)},(v_\alpha)_{\alpha\in\mathcal{J}'},\tilde{\psi}\big)$;

d) $\big(M_y,F^1_z,\varphi_y,G_{W(k)},(t_\alpha)_{\alpha\in\mathcal{J}'},\psi_y\big)$ is isomorphic to $\big(M\otimes_{W(k(v))}W(k),F^1\otimes_{W(k(v))}W(k),\varphi\otimes 1,G_{W(k)},(v_\alpha)_{\alpha\in\mathcal{J}'},\tilde{\psi}\big)$. So in c) above we can take $n=1$.*

Proof: The part 1) is a direct consequence of 3.1.0 and 2.3.17. What is actually needed from §3 is 3.5.11 and 3.2.7 c) (the proof of 3.2.7 c), cf. 3.7.4, does not use 3.4). Part a) of 2) follows from 3.2.8 and 2.3.17. For the part b) of 2) we can assume already (cf. 4.2.4, 3.5.11 and 3.6.5) that the Shimura adjoint filtered Lie σ-crystal attached to z is

$$\Big(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p})\otimes_{\mathbb{Z}_p}W(k),g\sigma\mu(\frac{1}{p}),F^0\big(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p})\otimes_{\mathbb{Z}_p}W(k)\big),F^1\big(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p})\otimes_{\mathbb{Z}_p}W(k)\big)\Big),$$

with $g\in\tilde{P}_0(W(k))$, where $\tilde{P}_0$ is the parabolic subgroup of $G^{\mathrm{ad}}_{W(k)}$ having as its Lie algebra the Lie subalgebra $\tilde{\mathfrak{p}}_0$ of $\mathrm{Lie}\big(G^{\mathrm{ad}}_{W(k)}\big)$ corresponding to non-negative slopes of $\Big(\mathrm{Lie}\big(G^{\mathrm{ad}}_{W(k)}\big),\sigma\mu(\frac{1}{p})\Big)$. Let $\varphi_y:=g\sigma\mu(\frac{1}{p})$. $\tilde{\mathfrak{p}}_0$ is also the Lie subalgebra of $\mathrm{Lie}\big(G^{\mathrm{ad}}_{W(k)}\big)$ corresponding to non-negative slopes of $\Big(\mathrm{Lie}\big(G^{\mathrm{ad}}_{W(k)}\big),\varphi_y)\Big)$. Let $\mathfrak{n}_0$ be the nilpotent radical of $\tilde{\mathfrak{p}}_0$. Then $(\mathfrak{n}_0,\varphi_y)$ and $\big(\mathfrak{n}_0,\sigma\mu(\frac{1}{p})\big)$ are Lie σ-crystals as well as p-divisible objects of $\mathcal{MF}_{[0,1]}(W(k))$. Let $\mathfrak{p}^0_0$ be the Lie subalgebra of $\tilde{\mathfrak{p}}_0$ corresponding to the slope zero for $\big(\tilde{\mathfrak{p}}_0,\sigma\mu(\frac{1}{p})\big)$, and let $\mathfrak{p}^1_0$ be the Lie subalgebra of $\tilde{\mathfrak{p}}_0$ corresponding to slope zero for $(\tilde{\mathfrak{p}}_0,\varphi_y)$. Now $\mathfrak{p}^0_0$ and $\mathfrak{p}^1_0$ are both Lie algebras of two Levi subgroups $\tilde{P}^0_0$ and respectively $\tilde{P}^1_0$ of $\tilde{P}_0$. For $(\tilde{\mathfrak{p}}_0,\sigma\mu(\frac{1}{p}))$ this is obvious, cf. its definition; for $(\tilde{\mathfrak{p}}_0,\varphi_y)$ this is a consequence of the fact that for any $n\in\mathbb{N}$, $\varphi^n_y=a_n\varphi^n$, with $a_n\in\tilde{P}_0(W(k))$. So both these two Lie algebras project bijectively onto $\tilde{\mathfrak{p}}_0/\mathfrak{n}_0$.

But any two Levi subgroups are conjugate by an element $g_0\in\tilde{P}_0(W(k))$. Changing φ_y with $g_0^{-1}\varphi_y g_0$ (i.e. changing it by an isomorphism, as $\tilde{\mathfrak{p}}_0\subset F^0(\mathrm{Lie}(G^{\mathrm{ad}}_{W(k)})))$, we can assume that $\mathfrak{p}^0_0=\mathfrak{p}^1_0$. But this implies that $g\in\tilde{P}^0_0$. This means that g commutes with the images of the elements (introduced in 4.1) $\bar{h}_i$ in $\mathrm{Lie}\big(G^{\mathrm{ad}}_{W(k)}\big)$, $i\in\{1,\ldots,d\}$. Let $\sigma_1:=g\sigma$, and let $\tilde{G}^{\mathrm{ad}}_{\mathbb{Z}_p}$ be the adjoint group over $\mathbb{Z}_p$ having

$$\mathrm{Lie}(G^{\mathrm{ad}}_{W(k)})^{\sigma_1}:=\big\{x\in\mathrm{Lie}(G^{\mathrm{ad}}_{W(k)})|\sigma_1(x)=x\big\}$$

as its Lie algebra (the fact that $k=\bar{k}$, implies that $\mathrm{Lie}(G^{\mathrm{ad}}_{W(k)})^{\sigma_1}$ has the right dimension, and that a $\mathbb{Z}_p$-basis of it is a $W(k)$-basis of $\mathrm{Lie}(G^{\mathrm{ad}}_{W(k)})$).

Claim. *$\tilde{G}^{\mathrm{ad}}_{\mathbb{Z}_p}$ is isomorphic to $G^{\mathrm{ad}}_{\mathbb{Z}_p}$.*

Proof: Let $\tilde{\mu}$ be the composite of μ with the quotient homomorphism $G_{W(k(v))} \twoheadrightarrow G^{\mathrm{ad}}_{W(k(v))}$. As $\tilde{\mu}\big(\mathrm{Lie}(\mathbb{G}_m)\big) \subset \tilde{G}^{\mathrm{ad}}_{V_0}$ (we recall that $V_0 = W(\mathbb{F})$), we can assume that $k = \mathbb{F}$. As the map $\tilde{P}^0_0(\mathbb{F}) \to \tilde{P}^0_0(\mathbb{F})$ defined by $\tilde{g} \to \tilde{g}\sigma(\tilde{g})^{-1}$ is surjective (cf. [Bo, 16.4]), we can assume that g mod p is the identity element of $\tilde{P}^0_0(\mathbb{F})$ (as $\tilde{P}^0_0$ commutes with $\tilde{\mu}$, when we replace $g\sigma\tilde{\mu}(\frac{1}{p})$ with $g_1 g\sigma\tilde{\mu}(\frac{1}{p})g_1^{-1}$, with $g_1 \in \tilde{P}^0_0(V_0)$, we get $g_1 g\sigma(g_1)^{-1}\sigma\tilde{\mu}(\frac{1}{p})$; moreover $\tilde{P}^0_0$ is defined over $\mathbb{Z}_p$). But $g \equiv 1$ mod p implies $\tilde{G}^{\mathrm{ad}}_{\mathbb{F}_p} = G^{\mathrm{ad}}_{\mathbb{F}_p}$. As two adjoint semisimple groups over $\mathbb{Z}_p$ having isomorphic special fibres, are isomorphic, we get that $\tilde{G}^{\mathrm{ad}}_{\mathbb{Z}_p}$ is isomorphic to $G^{\mathrm{ad}}_{\mathbb{Z}_p}$. This proves the claim.

But now $\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0}), \sigma_1\tilde{\mu}(\frac{1}{p})\big)$ with respect to the new $\mathbb{Z}_p$-structure defined by $\mathrm{Lie}(\tilde{G}^{\mathrm{ad}}_{\mathbb{Z}_p})$, is exactly $\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0}), \sigma\tilde{\mu}(\frac{1}{p})\big)$ with the old $\mathbb{Z}_p$-structure defined by $\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p})$. This ends the proof of part b) of 2), as obviously $\big(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} V_0, \sigma\tilde{\mu}(\frac{1}{p})\big)$ is of Borel type and cyclic diagonalizable (cf. 4.1.4). c) of 2) is a consequence of b) of 2).

For the proof of part a) of 3) we first remark that μ_y, if exists, is unique (F^0_z is uniquely determined, cf. the argument of 3.2.8 for the uniqueness of F^1_z), and due to the uniqueness assertion, we can assume $k = \bar{k}$. But in this case, it results directly from the part b) of 2) (i.e. μ_y is determined once we know $\tilde{\mu}_y : \mathbb{G}_m \to G^{\mathrm{ad}}_{W(k)}$, obtained by composing μ_y with the natural quotient homomorphism $G_{W(k)} \twoheadrightarrow G^{\mathrm{ad}}_{W(k)}$).

The proof of part b) of 3) not involving the cyclic diagonalizable part is entirely analogous to the proof of part b) of 2), starting from 3.5.11 and 2.3.13.1: a reductive group over $\mathbb{Z}_p$ is uniquely determined by its special fibre, while a torsor of a reductive group over $\mathbb{Z}_p$ which is trivial over $\mathbb{F}_p$, is trivial. It is an easy exercise to verify that the Shimura filtered σ-crystal $(M_y, F^1_z, \varphi_y, G_{W(k)})$ is cyclic diagonalizable (hint: $\mu_y : \mathbb{G}_m \to G_{W(k)}$ defines an element $\bar{h}^z_1 \in \mathrm{End}(M_y)$ by $\bar{h}^z_1(x) = x$ if $x \in F^1_z$, and $\bar{h}^z_1(x) = 0$ if $x \in F^0_z$; denoting $\bar{h}^z_i = \varphi_y^{i-1}(\bar{h}^z_1)$, $i \in \mathbb{N}$, we again get that $\bar{h}^z_{d+1} = \bar{h}^z_1$, i.e. $\varphi^d_y(\bar{h}^z_1) = \bar{h}^z_1$, and the whole argument of the proof of 4.1.2 can be repeated).

To see c) we just have to remark that there is an isomorphism of quadruples

$$\big(M \otimes_{W(k(v))} B(k), F^1 \otimes_{W(k(v))} B(k), (v_\alpha)_{\alpha\in\mathcal{J}'}, \tilde{\psi}\big) \tilde{\to} \big(M_y[\frac{1}{p}], F^1_z[\frac{1}{p}], (t_\alpha)_{\alpha\in\mathcal{J}'}, \psi_y\big),$$

taking $\bar{h}_i$ into $\bar{h}^z_i$, $i = \overline{1,d}$ (for the first part cf. 4.2.3; for the second part cf. part b) of 2), as the fact that μ and μ_y are $G_{W(k)}(W(k))$-conjugate takes care of $G^{\mathrm{ab}}_{W(k)}$).

In other words, writing $\varphi_y = a_y\mu_y(\frac{1}{p})$, a_y becomes a σ-linear automorphism of M_y; M_y and $G_{W(k)}$ get a natural $\mathbb{Z}_p$-structure $M_y(\mathbb{Z}_p)$ and respectively $\tilde{G}_{\mathbb{Z}_p}$, by considering the elements fixed by a_y (for $G_{W(k)}$, we look at the elements of $\mathrm{Lie}(G_{W(k)})$ fixed by a_y: they form $\mathrm{Lie}(\tilde{G}_{\mathbb{Z}_p})$). Moreover, $t_\alpha \in \mathcal{T}(M_y(\mathbb{Z}_p)[\frac{1}{p}])$, $\forall \alpha \in \mathcal{J}'$. It is easy to see that in fact $\tilde{G}_{\mathbb{Z}_p}$ is isomorphic to $G_{\mathbb{Z}_p}$ (cf. b) of 2) for the adjoint parts; the fact that $G^{\mathrm{ab}}_{\mathbb{Z}_p} = \tilde{G}^{\mathrm{ab}}_{\mathbb{Z}_p}$ is standard). This is not important here; what we need: there is $n \in \mathbb{N}$ such that over $W(\mathbb{F}_{p^n})[\frac{1}{p}]$ the representation of $G_{\mathbb{Q}_p}$ on L^*_p becomes isomorphic to the representation of the generic fibre $\tilde{G}_{\mathbb{Q}_p}$ of $\tilde{G}_{\mathbb{Z}_p}$ on $M_y(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and under this isomorphism $\rho(y,n)$, the cycles (tensors) are preserved, $\bar{h}_i$ goes into $\bar{h}_i^z$, etc., as mentioned above. For instance we need n such that both $G_{\mathbb{Z}_p}$ and $\tilde{G}_{\mathbb{Z}_p}$ split over $W(\mathbb{F}_{p^n})$. The fact that $\rho(y,n)$ takes φ^n into φ_y^n is obvious.

The proof of part d) of 3) will be presented in §5 as it is related to the Milne's conjecture referred to in 1.16. Here we just remark that we do not need to assume that $k = \bar{k}$ (cf. 2.3.9.2).

4.4.2. Definition. The unique lift $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$ (resp. $z : \mathrm{Spec}(W(k)) \to \mathcal{N}/H_0$) of a G-ordinary point $y : \mathrm{Spec}(k) \to \mathcal{U}$ (resp. of a G-ordinary point $y : \mathrm{Spec}(W(k)) \to \mathcal{N}/H_0$) with the property that its attached Shimura filtered Lie σ-crystal is of parabolic type is called the G-canonical lift of y, or the Shimura-canonical lift of y.

4.4.2.1. Remark. The translation of a G-canonical lift by an element of $G(\mathbb{A}_f^p)$ is again a G-canonical lift. This allows to replace H_0 in the above definition with any compact subgroup $\tilde{H}_0$ of $G(\mathbb{A}_f^p)$.

4.4.3. Definition. Let $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}/H_0$ (resp. $z : \mathrm{Spec}(W(k)) \to \mathcal{N}/H_0$) be an arbitrary point, and let A_y (resp. A_z) be the abelian variety over k (resp. over $W(k)$) obtained from $\mathcal{A}_{H_0}$ by pull back through y (resp. through z). An endomorphism f_y (resp. f_z) of A_y (resp. of A_z) is said to be a G-endomorphism if, as an element of $\mathrm{End}\big(H^1_{\mathrm{crys}}(A_y/W(k))\big)$, annihilates the de Rham components of the Hodge cycles $(\omega_\alpha)_{\alpha\in\mathcal{J}'}$ of A_z (i.e. it belongs to the Lie algebra $\mathrm{Lie}(G_{W(k)})$ defining the Shimura Lie σ-crystal attached to y). We denote the set of G-endomorphisms of A_y (A_z) by $\mathrm{End}_G(A_y)$ (resp. by $\mathrm{End}_G(A_z)$). $\mathrm{End}_G(A_y)$ and $\mathrm{End}_G(A_z)$ have a natural structure of Lie algebras over $\mathbb{Z}$.

An automorphism a_y (resp. a_z) of A_y (resp. A_z) is said to be a G-automorphism if, as an automorphism of $H^1_{\mathrm{crys}}(A_y/W(k))$, is an element of $G_{W(k)}(W(k))$. Notation $\mathrm{Aut}_G(A_y)$ (resp. $\mathrm{Aut}_G(A_z)$). Obviously $\mathrm{Aut}_G(A_y)$ and $\mathrm{Aut}_G(A_z)$ have a group structure. Warning: $\mathrm{Aut}_G(*)$, as a set, is not a subset of $\mathrm{End}_G(*)$; here $* \in \{A_y, A_z\}$.

Similarly we speak about G-$\mathbb{Q}$–endomorphisms or about G-$\mathbb{Q}$–automorphisms.

4.4.4. Corollary. *Let $y : \mathrm{Spec}(k) \to \mathcal{U}/H_0$ be a G-ordinary point and let $z : \mathrm{Spec}(W(k)) \to \mathcal{N}/H_0$ be its G-canonical lift. Then*

$$\mathrm{End}_G(A_y) = \mathrm{End}_G(A_z)$$

and

$$\mathrm{Aut}_G(A_y) = \mathrm{Aut}_G(A_z)$$

(i.e. any G-endomorphism and any G-automorphism of A_y lifts to an endomorphism of A_z).

Proof: This is a direct consequence of a) of 4.4.1 2).

4.4.4.1. Remark. 4.4.3-4 can be easily extended to the case of Shimura-ordinary σ-crystals and of their Shimura-canonical lifts. The essence behind them is 3.1.0 a) and b).

4.4.5. Corollary. *If $(f, L_{(p)}, v, \mathcal{B})$ is a standard PEL situation (cf. [Ko2, §5]), then any G-canonical lift $z : \mathrm{Spec}\big(W(\mathbb{F})\big) \to \mathcal{N}/H_0$ of a G-ordinary point $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{U}$ gives birth to a special point $z_0 : \mathrm{Spec}\big(W(\mathbb{F})[\frac{1}{p}]\big) \to \mathrm{Sh}(G, X)/H_0$ (i.e. A_z has complex multiplication).*

Proof: With the notations of 2.3.5 (see also [Ko2, §5]), $\mathrm{End}_G(A_y)$ is the intersection of the centralizer of $\mathcal{B}$ in $\mathrm{End}(A_y)$ with the normalizer of the $B(\mathbb{F})$-vector space of the principal polarization p_{A_y} of A_y, obtained from $\mathcal{P}_{\mathcal{A}}$ by pull back through y, and viewed as a cycle in the crystalline context. So the $\mathbb{Q}$–group defined by the invertible elements of $\mathrm{End}_G(A_y) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains a maximal torus of rank equal to the rank of G. Now everything results from the identity $\mathrm{End}_G(A_y) = \mathrm{End}_G(A_z)$ (cf. 4.4.4).

4.4.6. Remark. In §12 we will see that 4.4.5 remains true without assuming that $(f, L_{(p)}, v)$ comes from a standard PEL situation $(f, L_{(p)}, v, \mathcal{B})$. This implies that $\mathcal{U}$ is a locally closed subscheme of $\mathcal{M}_{k(v)}$ (cf. §13).

4.4.7. Remark. Let $\bar{k} = k$. With the notations of c) of 4.4.1 3), $M_y(\mathbb{Z}_p) := \{x \in M_y | a_y(x) = x\}$ is a $\mathbb{Z}_p$-structure on M with respect to which μ_y can be viewed as an injective cocharacter $\mu_y : \mathbb{G}_m \hookrightarrow GL\big(M^a \otimes_{\mathbb{Z}_p} W(k(v))\big)$, and $t_\alpha \in \mathcal{T}(M^a)$, $\forall \alpha \in \mathcal{J}$. So $\big(M_y, F_z^1, \varphi_y, (t_\alpha)_{\alpha \in \mathcal{J}'}\big)$ is isomorphic to $(M_y(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k), {}'F_z^1, \sigma_y \mu_y(\frac{1}{p}), (t_\alpha)_{\alpha \in \mathcal{J}'})$, with σ_y as the σ-linear automorphism of $M^a \otimes_{\mathbb{Z}_p} W(k)$ fixing $M_y(\mathbb{Z}_p)$, and with ${}'F_z^1 := F_z^1 \cap M_y(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k(v))$.

Moreover, the subgroup of $GL(M^a \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ fixing t_α, $\forall \alpha \in \mathcal{J}$, is $G_{\mathbb{Q}_p}$, and the degree of definition of $\big(M_y, F_z^1, \varphi_y, G_{W(k)}\big)$ is d (we recall that $k(v) = \mathbb{F}_{p^d}$). This is an easy consequence of b) of 4.4.1 2) and of the fact that the cocharacters of $G_{W(k)}^{\mathrm{ab}}$ defined naturally by μ and μ_y are the same (the resulting $\mathbb{Z}_p$-structures of $G_{W(k)}^{\mathrm{ab}}$ are as well the same).

4.4.8. Exercises. 1) We assume 4.4.6. If $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{U}$, prove that there is a unique lift (the G-canonical lift of y, cf. 4.4.6) $z : \mathrm{Spec}\big(W(\mathbb{F})\big) \to \mathcal{N}$ such that A_z has complex multiplication. Hint: if $k(v) = \mathbb{F}_p$ this is well known, cf. property P1 of 4.6, and cf. the theory of CM-lifts of ordinary points; if $k(v) \neq \mathbb{F}_p$, first copy the proof of 2.2.19, and then use 3.7.2.

2) Show that the property express in 1) is a property of Shimura-ordinary σ-crystals (of course we need to work in the context of lifts of quasi CM type). Hint: use 2.2.19 and the proof of b) of 4.4.1 2).

3) This is a particular case of 2). We assume that $k = \bar{k}$. Let $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$ be the G-canonical lift of a point $y : \mathrm{Spec}(k) \to \mathcal{U}$. Let $\big(\mathrm{Lie}\big(G_{W(k)}\big), \varphi_y\big)$ be the Shimura Lie σ-crystal attached to y. If the parabolic Lie subalgebra of $\mathrm{Lie}(G_{W(k)})$ corresponding to non-negative slopes of $(\mathrm{Lie}(G_{W(k)}), \varphi_y)$ is a Borel Lie subalgebra, then z is the unique lift of y to $W(k)$, whose attached Shimura filtered σ-crystal is a lift of quasi CM type (cf. def. 2.3.17) of the Shimura σ-crystal attached to y. (Hint: use property P7 of 4.6).

For a theory behind (and a solution of) 4.4.8 see 4.4.13.3.

4.4.9. Remark. Coming back to 4.4.4, we would like to mention that not always we have $\mathrm{End}(A_y) = \mathrm{End}(A_z)$. This is always the case when $k(v) = \mathbb{F}_p$, cf. properties

$P1$ and $P2$ of 4.6. But when $k(v) \neq \mathbb{F}_p$, we can have $\mathrm{End}(A_y) \underset{\neq}{\supset} \mathrm{End}(A_z)$ for some G-ordinary points y. For instance this is so when G is a torus and the G-ordinary type is a supersingular type (concrete example: when $\mathrm{Sh}\big(\mathrm{GSp}(W,\psi),S\big)$ is the elliptic modular curve, G is a torus and the prime v is such that the G-ordinary type is (1,1)). There are many examples with G a torus such that $\mathrm{End}(A_y) \neq \mathrm{End}(A_z)$, and starting from this we can build up examples with G a reductive group which is not a torus and when not always $\mathrm{End}(A_y)$ is equal to $\mathrm{End}(A_z)$.

However, in §14, we will see that

$$\mathrm{End}(A_y) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{End}_G(A_y) \otimes_{\mathbb{Z}} \mathbb{Q} \oplus \mathrm{End}(A_y)^{\perp_G} \otimes_{\mathbb{Z}} \mathbb{Q}$$

and similarly

$$\mathrm{End}(A_z) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{End}_G(A_z) \otimes_{\mathbb{Z}} \mathbb{Q} \oplus \mathrm{End}(A_z)^{\perp_G} \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $\mathrm{End}(A_y)^{\perp_G}$ (resp. $\mathrm{End}(A_z)^{\perp_G}$) is the set of endomorphisms of A_y (resp. of A_z) with the property that inside $\mathrm{End}\Big(H^1_{\mathrm{crys}}\big(A_y/W(k)\big)\big[\frac{1}{p}\big]\Big)$ they are perpendicular to $\mathrm{Lie}(G_{W(k)})$ with respect to the trace form on $\mathrm{End}\big(H^1_{\mathrm{crys}}(A_y/W(k))\big)$.

4.4.10. Problem. Referring again to 4.4.4, study the structure of the Lie algebras $\mathrm{End}_G(A_y) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{End}_G(A_y)$ for various G-ordinary points $y : \mathrm{Spec}(k) \to \mathcal{U}$ or $y : \mathrm{Spec}(k) \to \mathcal{U}/H_0$. The most interesting case is when $k = \mathbb{F}$ or when k is a finite field.

4.4.11. Remark. Not only $d(\mathfrak{C})$, but also the numbers $d^{\perp}(\mathfrak{C})$ and $gd(\mathfrak{C})$ (defined in 3.13.1), with $\mathfrak{C}$ the Shimura filtered σ-crystal attached to a G-canonical lift $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$, do not depend on the choice of the G-canonical lift z (cf. c) of 4.4.1 3)).

4.4.12. Remark. We assume d) of 4.4.1 3) and that $k = \bar{k}$. Let $V(k)$ be a complete DVR which is a finite faithfully flat extension of $W(k)$. Let $m_{V(k)}\colon \mathrm{Spec}(V(k)) \to \mathcal{N}$ be a morphism. Let $(A_{V(k)}, p_{A_{V(k)}}) = m^*_{V(k)}(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$, and let $(t_\alpha^{V(k)})_{\alpha\in\mathcal{J}'}$ be the family of Hodge cycles with which $A_{V(k)}$ is naturally endowed. From 2.3.13.1 and the density part of 4.2.1, we deduce the existence of a $V(k)$-isomorphism

$$L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} V(k) = M \otimes_{W(k(v))} V(k) \tilde{\to} H^1_{dR}(A_{V(k)}/V(k))$$

taking $\tilde{\psi}$ into $p_{A_{V(k)}}$, and v_α into $t_\alpha^{V(k)}$, $\forall \alpha \in \mathcal{J}'$.

4.4.13. The U-ordinariness. 2.2.17.1 raises the question: when a lift of quasi CM type of a Shimura σ-crystal is unique? To answer this question we start with some definitions.

4.4.13.1. Definitions. a) A potentially cyclic diagonalizable Shimura σ-crystal $(\tilde{M}, \tilde{\varphi}, \tilde{G})$ over k is called a U-ordinary σ-crystal, if its extension to $\bar{k}$ has a unique lift of quasi CM type.

b) Due to this uniqueness, this lift of quasi CM type is defined by a direct summand $\tilde{F}^1$ of $\tilde{M}$. The Shimura filtered σ-crystal $(\tilde{M}, \tilde{F}^1, \tilde{\varphi}, \tilde{G})$ is called the U-canonical lift of $(\tilde{M}, \tilde{\varphi}, \tilde{G})$.

c) A U-ordinary Shimura σ-crystal $(\tilde{M}, \tilde{\varphi}, \tilde{G})$ is called T-ordinary if the Lie subalgebra of $\mathrm{Lie}(\tilde{G})$ corresponding to slope zero for $(\mathrm{Lie}(\tilde{G}), \tilde{\varphi})$ is abelian (equivalently is the Lie subalgebra of a maximal torus of $\tilde{G}$).

d) In the same way we defined G-ordinary points of $\mathcal{N}_{k(v)}$, we define U-ordinary and T-ordinary points of it, as well as U-canonical lifts (to $\mathcal{N}$) of U-ordinary points of $\mathcal{N}_{k(v)}$ with values in perfect fields.

e) a) to c) above extend naturally to the context of Shimura (filtered) Lie σ-crystals.

To characterize the U-ordinary σ-crystals we can work over an algebraically closed field. We have:

4.4.13.2. Proposition. *A Shimura σ-crystal $(\tilde{M}, \tilde{\varphi}, \tilde{G})$ over k is U-ordinary iff it can be extended to a Shimura filtered σ-crystal $(\tilde{M}, \tilde{F}^1, \tilde{\varphi}, \tilde{G})$ whose Shimura filtered Lie σ-crystal is of toric type.*

Proof: We can assume that $k = \bar{k}$. We first assume the existence of $\tilde{F}$ such that the Shimura filtered Lie σ-crystal $\big(\mathrm{Lie}(\tilde{G}), \tilde{\varphi}, \tilde{F}^0(\mathrm{Lie}(\tilde{G})), \tilde{F}^1(\mathrm{Lie}(\tilde{G}))\big)$ is of toric type. Let $\tilde{Z}$ be the connected subgroup of $\tilde{G}$ whose Lie algebra is the Lie subalgebra of $\mathrm{Lie}(\tilde{G})$ corresponding to the slope zero for $(\mathrm{Lie}(\tilde{G}), \tilde{\varphi})$. It has a natural $\mathbb{Z}_p$-structure $Z_{\mathbb{Z}_p}$, defined by the elements of its Lie algebra fixed by $\tilde{\varphi}$. Let $\tilde{\mu} : \mathbb{G}_m \to \tilde{G}$ be the canonical split of $(\tilde{M}, \tilde{F}^1, \tilde{\varphi}, \tilde{G})$. As in the proof of 2.2.19 we get that $\tilde{F}^1$ is a lift of quasi CM type, and that $\tilde{\mu}$ factors through the center of Z. Let now $\tilde{F}_1^1$ be another lift of quasi CM type of $(\tilde{M}, \tilde{\varphi}, \tilde{G})$. Let $\tilde{\mu}_1 : \mathbb{G}_m \to \tilde{G}$ be the canonical split of $(\tilde{M}, \tilde{F}_1^1, \tilde{\varphi}, \tilde{G})$. Its generic fibre factors through a maximal torus of $Z_{B(k)}$ and so it commutes with $\tilde{\mu}$. This obviously implies that $\tilde{\mu} = \tilde{\mu}_1$ (to be compared with 2.2.9 3)) and so that $\tilde{F}_1^1 = \tilde{F}^1$.

To prove the other implication, we just have to show that for any cyclic diagonalizable Shimura filtered σ-crystal $(\tilde{M}, \tilde{F}^1, \tilde{\varphi}, \tilde{G})$ whose Shimura filtered Lie σ-crystal is not of toric type, there is a lift of quasi CM type of $(\tilde{M}, \tilde{\varphi}, \tilde{G})$ which is different than $\tilde{F}_1$. It is easy to see that there is $g \in Z_{\mathbb{Z}_p}(\mathbb{Z}_p)$ which leaves invariant $\tilde{F}^1/p\tilde{F}^1$ and which does not leave invariant $\tilde{F}^1$. But now, from the very definition 2.2.17, we get that $g(\tilde{F}^1)$ is another lift of quasi CM type of $(\tilde{M}, \tilde{\varphi}, \tilde{G})$. This proves the proposition.

4.4.13.3. Examples. From 4.4.13.2 we get immediately that any Shimura-ordinary σ-crystal is U-ordinary, and that any Shimura-canonical lift is a U-canonical lift. This solves the three exercises of 4.4.8. But there are plenty of U-ordinary σ-crystals which are not Shimura-ordinary. For simplicity, we will present the simplest type of such examples directly in the adjoint context.

Let $\tilde{G}_0$ be a split simple group over $W(\mathbb{F}_p)$ of some C_n, B_n, E_7, or (with $n \geq 5$) D_n Lie type. Let $\tilde{G}$ be the Weil restriction from $W(\mathbb{F}_{p^3})$ to $W(\mathbb{F}_p)$ of $\tilde{G}_{0W(\mathbb{F}_{p^3})}$; $\tilde{G}_0$ is naturally a subgroup of $\tilde{G}$. So $\tilde{G}_{W(\mathbb{F}_{p^3})}$ is a product of three copies $\tilde{G}_1$, $\tilde{G}_2$ and $\tilde{G}_3$ of $\tilde{G}_{0W(\mathbb{F}_{p^3})}$. Let $\tilde{\mu} : \mathbb{G}_m \hookrightarrow \tilde{G}$ be an injective cocharacter factoring through $\tilde{G}_0$ such that the pair $(\tilde{G}, [\tilde{\mu}])$ is a Shimura group pair. In the D_n case we assume that it is modelled on the $D_n^{\mathbb{R}}$ type. Let $\tilde{T}$ (resp. $\tilde{B}$) be a maximal torus of $\tilde{G}$ through which $\tilde{\mu}$ factors (resp. a Borel subgroup of $\tilde{G}$ containing $\tilde{T}$ an such that its Lie algebra is included in the $\tilde{F}^0$-filtration of

$\mathrm{Lie}(\tilde{G})$ produced as usual by $\tilde{\mu}$). Let $w \in \tilde{G}(W(\mathbb{F}_{p^3}))$ be a Weyl element (with respect to $\tilde{T}$) such that its component in $\tilde{G}_3$ is trivial, while its component in $\tilde{G}_i$ ($i \in \{1,2\}$) takes $\tilde{B}_{W(\mathbb{F}_{p^3})} \cap \tilde{G}_i$ into its opposite (with respect to $\tilde{T}_{W(\mathbb{F}_{p^3})} \cap \tilde{G}_i$).

As in 4.1.5 we build up a Shimura adjoint filtered Lie σ-crystal over $\mathbb{F}_{p^3}$

$$\big(\mathrm{Lie}(\tilde{G}_{W(\mathbb{F}_{p^3})}), w\sigma\tilde{\mu}(\frac{1}{p}), F^0(\mathrm{Lie}(\tilde{G}_{W(\mathbb{F}_{p^3})})), F^1(\mathrm{Lie}(\tilde{G}_{W(\mathbb{F}_{p^3})}))\big)$$

which is cyclic diagonalizable, is of toric type, but it is not of parabolic type. So any Shimura filtered σ-crystal having it as its adjoint is a U-canonical lift, without being a Shimura-canonical lift; if $\tilde{G}$ is of A_1 Lie type, then the U-ordinary σ-crystal underlying it is in fact a T-ordinary σ-crystal.

Similar examples can be build up modelled on some A_n, $D_n^{\mathbb{H}}$, E_6 or D_n^{mixt} (Shimura) type. There are plenty of examples involving also adjoint factors which are included in the corresponding F^0-filtration, as well as examples which involve other Weyl elements. See [Va8] for a minute analysis and classification of such examples.

4.4.13.4. Remarks. 1) From 4.4.13.2 we deduce that the U-ordinary points do not show up in the classical setting of Siegel modular varieties. But they do show up in some classical PEL situations (for instance, cf. the C_n Lie type part of 4.4.13.3 and the completion property; to be compared with the proof of 4.12.12). To our knowledge, not even one example of U-ordinary points which are not G-ordinary had been previously identified.

2) 4.4.3-4, 4.4.5-6 and 4.4.9-10 extend immediately to the context of U-ordinary points. In particular 4.4.5-6 give us new ways to construct abelian varieties having with complex multiplication.

3) The letter T in 4.4.13.1 stands for toric, while the letter U stands for unicity as well as for the way the Weyl elements (see 4.4.13.3 for a sample) have to act on different Lie algebras of (some) adjoint groups in order to produce U-ordinary σ-crystals.

4.5. Different stratifications of $\mathcal{N}_{k(v)}$.

4.5.1. Definition. The stratification of $\mathcal{N}_{k(v)}$ in $G(\mathbb{A}_f^p)$-invariant locally closed (reduced) subschemes indexed and defined by the Newton polygons of the Shimura Lie σ-crystals attached to points $\mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ is called the canonical Lie stratification.

4.5.2. Definition. The stratification of $\mathcal{N}_{k(v)}$ in $G(\mathbb{A}_f^p)$-invariant locally closed (reduced) subschemes, indexed and defined by sequences of length $|\mathcal{H}|$ formed by the Newton polygons of the cyclic factors (their number is $|\mathcal{H}|$) of the Shimura adjoint σ-crystals attached to points $\mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ is called the refined canonical Lie stratification.

4.5.3. Remark. $\mathcal{U}$ is the Zariski dense open stratum of both Lie stratifications of $\mathcal{N}_{k(v)}$ defined above (cf. 4.2 and 4.3.4).

4.5.4. Remark. We assume $k = \bar{k}$. Any representation $\rho : G_{\mathbb{Z}_p} \to GL(N)$, with N a free $\mathbb{Z}_p$-module of finite rank, allows us to attach to any point $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$, a p-divisible object ${}_\rho\mathfrak{C}_z$ of $\mathcal{MF}_{[a_\rho,b_\rho]}(W(k))$. Here $a_\rho, b_\rho \in \mathbb{Z}$ are defined by the fact that

$$\mu_\rho := \rho \circ \mu : \mathbb{G}_m \to GL\big(N \otimes_{\mathbb{Z}_p} W(k(v))\big)$$

produces a direct sum decomposition

$$N \otimes_{\mathbb{Z}_p} W(k(v)) = \oplus_{i \in \mathcal{I}_\rho} F^i_\rho$$

(with all $F^i_\rho \neq 0$, with $a_\rho, b_\rho \in \mathcal{I}_\rho$ such that $\mathcal{I}_\rho \subset \{a_\rho, \ldots, b_\rho\}$, and where $\beta \in \mathbb{G}_m(W(k))$ acts through μ_ρ on F^i_ρ as the multiplication with β^{-i}. This is done as follows.

We make use of an isomorphism of the quadruple $\bigl(M \otimes_{W(k(v))} W(k), \mu, (v_\alpha)_{\alpha \in \mathcal{J}'}, \tilde{\psi}\bigr)$ with the quadruple $\bigl(M_y, \mu_y, (t_\alpha)_{\alpha \in \mathcal{J}'}, p_{A_y}\bigr)$ (cf. d) of 4.4.1 3) and 4.4.12; if we do not want to use this d) of 4.4.1 3), then we can use 4.4.7: at the end we get the same thing). Here (A_z, p_{A_z}) is the principally polarized abelian variety over k obtained from $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ through z, A_y is the special fibre of A_z, while $M_y := H^1_{\mathrm{crys}}\bigl(A_y, W(k)\bigr)$; $(t_\alpha)_{\alpha \in \mathcal{J}'}$ are de Rham components of the Hodge cycles with which A_z is naturally endowed. $\mu_y : \mathbb{G}_m \to GL(M_y)$ is a cocharacter fixing t_α, $\forall \alpha \in \mathcal{J}'$; it produces a direct sum decomposition $M_y = F^1_z \oplus F^0_z$, with F^1_z as the Hodge filtration of M_y defined by A_z; $\beta \in \mathbb{G}_m(W(k))$ acts through μ_y on F^i_z as the multiplication with β^{-i}.

We can write the Frobenius φ_y of M_y, under this isomorphism, as $g\sigma\mu(\frac{1}{p})$ with $g \in G_{\mathbb{Z}_p}(W(k)) \subset GL\bigl(M \otimes_{W(k(v))} W(k)\bigr)(W(k))$. Now

$$\bigl(N \otimes_{\mathbb{Z}_p} W(k), (F^i_\rho \otimes_{W(k(v))} W(k))_{i \in \mathcal{I}_\rho}, \varphi_{y,\rho}\bigr),$$

with

$$\varphi_{y,\rho} := \rho(g)\sigma\mu_\rho(\frac{1}{p}),$$

is the p-divisible object of $\mathcal{MF}_{[a_\rho,b_\rho]}(W(k))$ attached to ρ and z. In §5 we will see that there is a p-divisible object ${}_\rho\mathfrak{C}\mathcal{N}$ of $\mathcal{MF}^\nabla_{[a_\rho,b_\rho]}(\mathcal{N})$ attached to ρ in such a manner that in z it becomes ${}_\rho\mathfrak{C}_z$. ${}_\rho\mathfrak{C}_z$ is called the ρ p-divisible object of $\mathcal{MF}_{[a_\rho,b_\rho]}(W(k))$ attached to z. The p-divisible object ${}_\rho\mathfrak{C}\mathcal{N}$ defines a Newton polygon stratification of $\mathcal{N}_{k(v)}$ in locally closed subschemes (cf. §5). We call it the ρ-stratification of $\mathcal{N}_{k(v)}$.

4.5.5. Definition. The intersection of all ρ-stratifications of $\mathcal{N}_{k(v)}$ is called the absolute stratification of $\mathcal{N}_{k(v)}$.

4.5.6. Exercises. 0) The ρ-stratification of $\mathcal{N}_{k(v)}$ is trivial (i.e. formed just by one stratum) iff μ_ρ factors through the center of $GL(N \otimes_{\mathbb{Z}_p} W(k(v)))$.

1) Any ρ-stratification is $G(\mathbb{A}^p_f)$-invariant.

2) The canonical Lie stratification is the ρ^{ad}-stratification, where $\rho^{\mathrm{ad}} : G_{\mathbb{Z}_p} \to GL(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p}))$ is the adjoint representation.

3) If $\rho^{\mathrm{ad}}_i : G_{\mathbb{Z}_p} \to GL(\mathrm{Lie}(G_i))$, $i \in \mathcal{H}$, is the (irreducible) subrepresentation of ρ^{ad} induced on the $\mathbb{Z}_p$-direct summand $\mathrm{Lie}(G_i)$ of $\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p})$ (we recall that $G^{\mathrm{ad}}_{\mathbb{Z}_p} = \times_{i \in \mathcal{H}} G_i$), then the refined canonical Lie stratification is the intersection of all the ρ^{ad}_i-stratifications, $i \in \mathcal{H}$.

4) The absolute stratification is well-defined, i.e. its strata are locally closed subschemes, and their number is finite.

5) In the case of special fibres of integral canonical models of Siegel modular varieties, any non-trivial ρ-stratification coincides with the usual Newton polygon stratification.

6) If $\rho_0 : G_{\mathbb{Z}_p} \hookrightarrow GL(L_p^*)$ is the faithful representation introduced in 4.1, then the ρ_0-stratification is the stratification of $\mathcal{N}_{k(v)}$ defined by the Newton polygons of the Shimura σ-crystals attached to points of $\mathcal{N}_{k(v)}$ with values in perfect fields.

7) Show by examples that 5) above is not true for an arbitrary SHS $(f, L_{(p)}, v)$ (Hint: either make use of products of Shimura varieties of Hodge type or look at the case when $|\mathcal{H}| \geq 2$).

8) $\mathcal{U}$ is an open subscheme of the (generic) Zariski dense open stratum of any ρ-stratification, and so it is the (generic) open dense stratum of the absolute stratification (Hint: use 3.1.0 in the context of generalized Shimura p-divisible objects (cf. 3.1.7); another way -a simpler one- is to make use of b) of 4.4.1 2)).

All the above exercises are very easy.

4.5.6.1. Proposition. *The absolute stratification coincides with the refined Lie canonical stratification. Moreover the statement of 5) above remains true if* $|\mathcal{H}| = 1$.

Proof: We assume $k = \bar{k}$. Let $y_i : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$, $i = \overline{1,2}$, be two points of the same stratum of the refined canonical Lie stratification of $\mathcal{N}_{k(v)}$. We need to show that they belong to the same stratum of any ρ-stratification of $\mathcal{N}_{k(v)}$. Let $(M_i, \varphi_i, G_{W(k)})$ be the Shimura σ-crystal attached to y_i. Let $m \in \mathbb{N}$ such that φ_i^m acts diagonally on the elements of a $W(k)$-basis of M_i. We can assume that $k = \mathbb{F}$, and so that the Shimura σ-crystals involved are defined over finite fields; so we can always choose m so that the semisimple elements to be introduced are uniquely determined (however, for future references and full generality, in what follows we will still use the notation k instead of $\mathbb{F}$).

We get a semisimple element $b_i \in G(B(k))$, whose eigenvalues, as an endomorphism of M_i are p to powers m times the slopes of (M, φ_i), multiplicities corresponding. We choose m to be a multiple of 6 times the product of the length of the cyclic adjoint factors of the Shimura Lie σ-crystal attached to y_1 (or y_2; 4.4.1 2) implies that this number does not depend on the k-valued point of $\mathcal{N}_{k(v)}$). The key fact is:

Claim. *Replacing m by a suitable multiple of itself, b_1 and b_2 become conjugate by an element of $G(B(k))$.*

To prove this, let $\tilde{G}$ be the product of the simple connected semisimple group cover of G^{der} and of the toric part of $Z(G)$. Replacing m by a multiple of itself, b_i is replaced by a power of itself; so we can assume that b_i lifts to an element $\tilde{b}_i \in \tilde{G}(B(k))$. We consider a representation $\tilde{G}_{B(k)} \to GL(V_{B(k)})$ such that restricted to $\tilde{G}^{\mathrm{der}}_{B(k)}$ is a direct sum, indexed by the simple factors of $G^{\mathrm{ad}}_{B(k)}$, of the standard embeddings (see [He, §8]) of the classical groups defined by the simple connected group covers of these adjoint factors (these group covers are quotients as well as factors of $\tilde{G}^{\mathrm{der}}_{B(k)}$). For instance, if we have a factor $SL(\tilde{V}_{B(k)})$ of $\tilde{G}^{\mathrm{der}}_{B(k)}$, with $\tilde{V}_{B(k)}$ a $B(k)$-vector space of finite dimension, then we get the natural representation of it on $\tilde{V}_{B(k)}$. Similarly, if we have factors of B_ℓ, C_ℓ or D_ℓ Lie type, we get respectively the classical representation of the split group $SO(2\ell+1)$, $\mathrm{Sp}(\ell)$ or $SO(2\ell)$ (over $B(k)$), described at the level of Lie algebras, in the loc. cit.

$\tilde{b}_i$ acts diagonally on $V_{B(k)}$; replacing m again by a multiple of itself, we can choose $\tilde{b}_i$ so that its eigenvalues, as an endomorphism of $V_{B(k)}$, are integral powers of p. It also acts diagonally on the Lie algebra of the image of $\mathrm{Lie}(\tilde{G}^{\mathrm{der}}_{B(k)})$ in $\mathrm{End}(\tilde{V}_{B(k)})$. With its eigenvalues, as an endomorphism of this image, we build up a Newton polygon NP_i. The fact that y_1 and y_2 are points of the same stratum of the refined canonical Lie stratification of $\mathcal{N}_{k(v)}$, gets restated in:

$$NP_1 = NP_2.$$

This equality (together with the principle expressed in 3.5.1 and 3.5.2.2) implies that the images of $\tilde{b}_1$ and $\tilde{b}_2$ in $GL(\tilde{V}_{B(k)})$ are conjugate by an element of the group of $B(k)$-valued points of the image of $\tilde{G}^{\mathrm{der}}_{B(k)}$ in $GL(\tilde{V}_{B(k)})$. This is a simple exercise (for $\tilde{G}^{\mathrm{der}}$ having only simple factors of A_ℓ Lie type this is well known; to be compared with 4.2.2). 3.5.1 and 3.5.2.2 imply that the images of b_i in the simple factors of $G^{\mathrm{ad}}_{B(k)}$ of whose Lie algebras are permuted by φ_i, up to canonical identification of these groups, are the same. So the above exercise can be restated:

Fact. *Let C be a classical semisimple group over $B(k)$ (so C is $SO(n)$, $SL(n)$ or $\mathrm{Sp}(n)$, for some adequate $n \in \mathbb{N}$). Let b^1 and b^2 be semisimple elements of $C(B(k))$ such that their eigenvalues (under the classical representation of C) are integral powers of p. If the Newton polygons obtained with them acting on $\mathrm{Lie}(C)$ are the same; then b^1 and b^2 are $C(B(k))$-conjugate.*

The simple proof of this fact is left to the reader.

We can assume that the images of $\tilde{b}_1$ and $\tilde{b}_2$ in $\tilde{G}^{\mathrm{ab}}$ are the same. From the above fact we deduce that $\tilde{b}_1^{m^2_{\mathrm{iso}}}$ and $\tilde{b}_2^{m^2_{\mathrm{iso}}}$ are $\tilde{G}(B(k))$-conjugate; here m_{iso} is the degree of the natural isogeny from $\tilde{G}$ into the product of $\tilde{G}^{\mathrm{ab}}$ with the image of $\tilde{G}^{\mathrm{der}}$ in $GL(V_{B(k)})$. This proves the claim.

The claim obviously implies that y_1 and y_2 are points of the same stratum of any ρ-stratification of $\mathcal{N}_{k(v)}$ (cf. the constructions in 4.5.4).

The second part (of when $|\mathcal{H}| = 1$) is trivial: for any representation $\rho : G_{\mathbb{Z}_p} \to GL(N)$ over $\mathbb{Z}_p$, either ρ factors through $G^{\mathrm{ab}}_{\mathbb{Z}_p}$ (and so the ρ-stratification is trivial), or the restriction of ρ to $G^{\mathrm{der}}_{\mathbb{Z}_p}$ is an isogeny (and so the above part of the proof applies). This ends the proof of the proposition.

4.5.7. Some sets of Newton polygons. Let $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}_{k(v)}$ be an arbitrary point and let $\big(M_0, \varphi, G_{W(\mathbb{F})}\big)$ be its attached Shimura σ-crystal. Let $NP(\mathcal{N}_{k(v)})$ be the set of Newton polygons defined by the σ-crystals $(M_0, g\varphi)$ with $g \in G^0_{W(\mathbb{F})}\big(W(\mathbb{F})\big)$, and let $LNP(\mathcal{N}_{k(v)})$ be the set of Newton polygon defined by the Shimura adjoint Lie σ-crystals $\big(\mathrm{Lie}(G^{\mathrm{ad}}_{W(\mathbb{F})}), g\varphi\big)$ with $g \in G^0_{W(\mathbb{F})}\big(W(\mathbb{F})\big)$. Similarly we can define $RLNP(\mathcal{N}_{k(v)})$ as the set of sequences of length $|\mathcal{H}|$ of Newton polygons attached to the cyclic factors (their number is $|\mathcal{H}|$) of the above Shimura adjoint Lie σ-crystals.

Let v^{ad} be the prime of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ divided by v. Then $N(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}}) := \big|LNP(\mathcal{N}_{k(v)})\big|$ is the number of the strata of the canonical Lie stratification of $\mathcal{N}_{k(v)}$, and $N_1(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}}) := \big|RLNP(\mathcal{N}_{k(v)})\big|$ is the number of the strata of the refined canonical

Lie stratification of $\mathcal{N}_{k(v)}$, provided $\mathcal{N}$ has the completion property. This is a consequence of b) of 4.4.1 2) and of 3.4.15. We also write $N(\mathcal{N}_{k(v)}) = N(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$, and $RLNP(\mathcal{N}_{k(v)}) = RLNP(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$.

4.5.8. Remarks. *1)* The natural numbers $N(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$ and $N_1(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$ depend only on the isomorphism class of the Shimura group pair $(G^{\mathrm{ad}}_{\mathbb{Z}_p}, \mu^{\mathrm{ad}})$, where $\mu^{\mathrm{ad}} : \mathbb{G}_m \to G^{\mathrm{ad}}_{W(k(v^{\mathrm{ad}}))}$ has the similar meaning as μ of 4.1, but for Shimura variety $\mathrm{Sh}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ (cf. b) of 4.4.1 2)). The same about the set $RLNP(\mathcal{N}_{k(v)})$. This justifies our notations.

2) in §10 we will use different ρ-stratifications of $\mathcal{N}_{k(v)}$ to compute the numbers $N(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$ and $N_1(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$.

4.5.9. For more about the values of $N(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$ and $N_1(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$, and the dimensions and local geometry of strata of different stratifications of $\mathcal{N}_{k(v)}$ see §10.

4.5.10. Remark. The canonical stratification of the special fibre $\mathcal{M}_{k(v)}$ of $\mathcal{M}$ (cf. [EO]) gives birth (by pull back) to an f-canonical stratification of $\mathcal{N}_{k(v)}$, in $G(\mathbb{A}_f^p)$-invariant locally closed (reduced) quasi-affine subschemes (as $\mathcal{N} \to \mathcal{M}$ is a finite morphism, $\mathcal{N}_{k(v)} \to \mathcal{M}_{k(v)}$ is a finite morphism). We do not deal here with the interplay between the f-canonical stratification of $\mathcal{N}_{k(v)}$ and the canonical Lie stratification of $\mathcal{N}_{k(v)}$. We just mention that $\mathcal{U}$ is the generic (open dense) stratum of the f-canonical stratification of $\mathcal{N}_{k(v)}$ (cf. b) or d) of 4.4.1 3) and the definition of the canonical stratification of $\mathcal{M}_{k(v)}$). It is expected that the interpretation of the f-canonical stratification of $\mathcal{N}_{k(v)}$ in terms of Shimura Lie σ-crystals attached to points of $\mathcal{N}_{k(v)}$ with values in perfect fields leads to a refined f-canonical stratification.

4.5.11. Remark. We do not stop here to prove that if $y_1, y_2 \in \mathcal{N}_{k(v)}(k)$ are two distinct points giving birth to the same point $y \in \mathcal{M}_{k(v)}(k)$, then y_1 and y_2 are always in the same stratum, for any ρ-stratification of $\mathcal{N}_{k(v)}$ (defined by a $\mathbb{Z}_p$-representation ρ of $G_{\mathbb{Z}_p}$), as a proof of the Langlands–Rapoport conjecture for $\mathcal{N}_{k(v)}$ in the context provided by $(f, L_{(p)}, v)$ (see [Mi2] and [Va2, 1.7]) implies that such a situation $y_1 \to y \leftarrow y_2$ does not occur (cf. §14; to be compared with [Va2, 5.6.4] and 4.4.6).

4.5.12. Toric points. Here k is again an arbitrary perfect field. The isomorphism of b) of 4.4.1 2) leads us:

a) to define a map $f_G : W_G \to NP(\mathcal{N}_{k(v)})$ from the Weyl group W_G of $G_{W(\mathbb{F})}$ (introduced in 4.1.5) into $NP(\mathcal{N}_{k(v)})$, associating to $\omega \in W_G$, $\mathcal{P}_\omega$ (of 4.1.5);

b) to define for any $\omega \in W_G$ the set of $G(\omega)$-ordinary points $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$, defined by the property that the Shimura adjoint Lie σ-crystal attached to y, over $\bar{k}$ it is isomorphic to the extension to $\bar{k}$ of the Shimura adjoint Lie σ-crystal of some $\mathfrak{C}_\omega$ of 4.1.5.

If $\mathcal{N}$ has the completion property, then the set of $G(\omega)$-ordinary points is non-empty.

4.5.12.1. We assume that the set of $G(\omega)$-ordinary points is non-empty. The $G(\omega)$-ordinary points are points of a stratum $\mathfrak{s}_\omega$ of the refined canonical Lie stratification of $\mathcal{N}_{k(v)}$. We do not know (if or) when all the points of $\mathfrak{s}_\omega$ are $G(\omega)$-ordinary points. We do not know when the $G(\omega)$-ordinary points are points of an open dense subscheme of $\mathfrak{s}_\omega$, or (at the opposite pole) when any connected component of $\mathfrak{s}_\omega$ has $G(\omega)$-ordinary points.

4.5.12.2. Definition. The points of $\mathcal{N}_{k(v)}$ with values in perfect fields which are $G(\omega)$-ordinary, for some $\omega \in W_G$, are called the toric points of $\mathcal{N}_{k(v)}$.

4.5.12.2.1. Exercise. Show that the toric points can be easily defined in terms of Galois representations: we can assume that $k = \bar{k}$; a point $\mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ is a toric point, iff it admits a $W(k)$-lift, whose attached Galois representation $\rho : \mathrm{Gal}(\overline{B(k)}/B(k)) \to G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ (we view $G_{\mathbb{Z}_p}$ as a subgroup of $GL(L_p^*)$, and identify, cf. [Va2, 5.1.2], $L_{(p)}^* \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ with a first group of the étale cohomology with $\mathbb{Z}_p$ coefficients) factors through the group of $\mathbb{Z}_p$-valued points of a maximal torus of $G_{\mathbb{Z}_p}$.

4.5.12.3. Remark. We will see in §10 that the toric points are the "pillars" of the (refined) canonical Lie stratification of $\mathcal{N}_{k(v)}$. The most important ones are the U-ordinary points introduced in 4.4.13.

4.5.13. Problems. 1) The map f_G is usually not injective. Compute the number of elements of the fibres of f_G and of its image $f_G(W)$.

2) Compute the refined Lie stable p-ranks of toric points.

4.5.14. Problem. Compute the number of the strata and the dimensions of the connected components of the strata of the f-canonical stratification of $\mathcal{N}_{k(v)}$.

4.5.15. Other Lie type stratifications. Using the (refined) Lie stable p-ranks attached to points of $\mathcal{N}_{k(v)}$ with values in fields, we obtain a $G(\mathbb{A}_f^p)$-invariant (refined) Lie-stable stratification of $\mathcal{N}_{k(v)}$ in locally closed subschemes. We can use also the Lie p-ranks to define another Lie type stratification of $\mathcal{N}_{k(v)}$, but the refined Lie-stable stratification is a more refined one (in general it is a much more refined one). Variants: using the maps ${}_1\bar{\psi}_i^{m_i}$ of 4.3.7, or their exterior or symmetric powers, or the σ-linear maps of 3.9.1, we can define a whole bunch of Lie type $G(\mathbb{A}_f^p)$-invariant stratifications of $\mathcal{N}_{k(v)}$. For instance: using the images of the maps of 4.3.7, and not their stable images, we get the Lie non-stable stratification. There are many questions arising, and which involve these Lie type stratifications (for instance how are they related?, etc.). To us, the most interesting ones are:

Q_1) When are the connected components of a stratum of the Lie-stable stratification of $\mathcal{N}_{k(v)}$ (or of $\mathcal{N}_{\mathbb{F}}$) permuted transitively by $G(\mathbb{A}_f^p)$?

Q_2) Using all the stratifications hinted at in this 4.5.15, we can define their intersection. We call it the ultra stratification of $\mathcal{N}_{k(v)}$. It is easy to see that in general it is a refinement of the absolute stratification. How more refined it is? How is it related to the f-canonical stratification?

Q_3) These stratifications are very much related to the incipient theory of deviations of 3.13: the σ-linear maps of 3.9.1, and other ones constructed naturally from them - like the ones of 4.3.7- give birth to a big bunch of stable or non-stable -or mixed-stable- deviations of Shimura (Lie) (filtered) σ-crystals. Which other type of "points" (besides the G-ordinary ones), and of their lifts, can be singled out using the σ-linear maps of 3.9.1

(and eventually some deviations similar to the ones introduced in 3.13)? Can we in this way single out all toric points (which are not U-ordinary)?

4.5.16. Non Lie type stratifications. Of course, forgetting the extra (Shimura) structure of the σ-crystals attached to points of $\mathcal{N}_{k(v)}$ with values in perfect fields, we can define other (not necessarily of Lie type!) stratifications of $\mathcal{N}_{k(v)}$, which are closer to the spirit coming from the case of Siegel modular varieties (like the one using the p-rank, see [NO]). The slight unpleasant feature of such stratifications of $\mathcal{N}_{k(v)}$: very often they are not canonical (i.e. they depend on the SHS producing them; on the contrary the Lie type stratifications are canonical, see 4.9), or they are not refined enough, as it can be seen easily through examples (like when $|\mathcal{H}| \geq 2$).

4.5.17. Artin–Schreier stratifications. Let X_p be an arbitrary $\mathbb{F}_p$-scheme. A stratification of it in locally closed reduced subschemes is called an Artin–Schreier stratification (to be abbreviated as: an AS stratification), if there is $m \in \mathbb{N}$, such that locally in the Zariski topology of X_p, it can be defined as in 3.4.8.1.3, using quasi Artin–Schreier systems in m variables (see 3.4.18.4.6 A for the definition of these systems). Similarly we define a refined Artin–Schreier stratification (to be abbreviated as a RAS stratification) of X_p.

Example 1. We assume that X_p is a smooth connected k-scheme, and that there is a principally quasi-polarized p-divisible group (D_p, p_{D_p}) over X_p. We consider the stratification of X_p defined by the p-ranks of D_p over geometric points of X_p. From 3.4.18.4 B) we deduce immediately that it is a RAS stratification (we can take m to be $\dim(X_p)\mathrm{rk}(D_p)^2$). In the case X_p is a connected component of $\mathcal{M}_{k(v)}$ and we are dealing with the standard principally quasi-polarized p-divisible group over it, from [NO, 4.1] we deduce that this RAS stratification is in fact an AS stratification.

Example 2. The Lie p-rank stratification of $\mathcal{N}_{k(v)}$ is a RAS stratification (cf. 3.4.18.7.0). Similarly, using the proof of 3.4.18.4.6 C we get that the refined Lie-stable stratification of $\mathcal{N}_{k(v)}$ is a RAS stratification.

The above two examples are particular samples: 3.4.20 4) shows that in fact always a Lie p-rank (resp. p-rank) stratification in a context involving Shimura p-divisible groups (resp. polarized Shimura p-divisible groups), are RAS stratifications. Warning: not always a RAS stratification is an AS stratification.

Example 3. Let $R := k[x_1, x_2]$. We consider the quasi Artin–Schreier system in two variables with coefficients in R defined by the equations $z_1 = x_1x_2z_1^p + x_1z_2^p$ and $z_2 = x_1x_2z_1^p + 2x_1z_2^p$. Obviously the RAS stratification of $\mathrm{Spec}(R)$ defined by it can not be an AS stratification.

4.5.17.1. Expectation. *We expect that (at least in many cases) a Newton polygon stratification associated to p-divisible groups is a RAS stratification.*

The motivation of this expectation is based on examples, on 3.4.20 4) and on the fact that taking different exterior powers of objects of $\mathcal{MF}_{[0,1]}(*)$, we expect formulas similar to the ones of 3.4.18.4 B) and 3.4.18.7, and which involves slopes which are not just integral values. These different exterior powers should be able to “capture” the shape of the Frobenius of the vector bundles involved modulo higher powers of p (we recall that in getting the formula of 3.4.18.4 B) only the value of Φ mod p was important). In particular using 3.4.8.1.4 one should be able to regain the purity of Newton polygon stratifications of de Jong and Oort.

4.5.17.2. Remark. One of the advantages of AS stratifications in comparison to the Newton polygon ones is: using different determinants or norm maps, we can put different natural scheme structures on the strata, which are not a priori reduced (there is always logic in the choice of m and of the quasi Artin–Schreier systems involved).

We hope to come back to the ideas of this paragraph in a future paper.

4.6. Examples and main properties. Here we list the main properties of the theory of G-ordinary types, of G-ordinary points, and of their G-canonical lifts, and present some examples to illustrate them and the complexities which are arising.

P1 *The G-ordinary type τ is the (usual) ordinary type iff $k(v) = \mathbb{F}_p$.*

Proof: From the description of the G-ordinary type given in 4.1, we deduce (with the notations of 4.1.1.1) that h^0 has only the eigenvalues 0 and 1 iff $\bar{h}_i = \bar{h}_1$, $\forall i \in \{1, \ldots, d\}$, and this is equivalent to $\sigma(\bar{h}_1) = \bar{h}_1$, i.e. it is equivalent to μ being defined over $\mathbb{Z}_p$. But $W(k(v))[\frac{1}{p}]$ is the field of definition of the $G_{\mathbb{Z}_p}(\mathbb{Z}_p^{sh})$-conjugacy class of cocharacters of $G_{\mathbb{Z}_p^{sh}}$ (which over an embedding of the field of fractions of $\mathbb{Z}_p^{sh}$ in $\mathbb{C}$, extending the inclusion $E(G, X) \subset \mathbb{C}$, are $G(\mathbb{C})$-conjugate to $\mu_x^*, x \in X$). Now everything results from [Mi3, 4.7].

4.6.1. Example 1. In the case of a standard PEL situation $(f, L_{(p)}, v, \mathcal{B})$, if $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is of simple C_ℓ type, we have $E(G, X) = \mathbb{Q}$. So we get only usual ordinary types. Similarly, for the classical Spin modular varieties of odd dimension and rank 2 considered in [Va2, 5.7.5], we get only usual ordinary types.

P2 *If $k(v) = \mathbb{F}_p$, then the abelian variety A_z over $W(k)$ obtained from $\mathcal{A}$ by pull back through a G-canonical lift z of a G-ordinary point $y : \mathrm{Spec}(k) \to \mathcal{U}$ (i.e. of an ordinary point of $\mathcal{N}_{k(v)}$, cf. P1) is the (usual) canonical lift of the (ordinary) abelian variety over k, defined by the special fibre of A_z.*

Proof: This is a direct consequence of c) of 4.4.1 3) and of the description of the σ-crystal (M, φ) of 4.1.1 giving birth to the G-ordinary type.

P3 *If the G-ordinary type τ is $e(1, 1)$, i.e. if τ is the formal isogeny type associated to supersingular abelian varieties of dimension e, then G is a torus.*

Proof: If τ is $e(1, 1)$, then the Shimura Lie σ-crystal attached to a G-ordinary point y of $\mathcal{N}_{k(v)}$ has only slopes zero. Now everything results from the fact that the Shimura filtered Lie σ-crystal of a G-canonical lift z of y is of parabolic type, and from 3.2.2.

P4 *The slopes of the Shimura σ-crystals attached to G-ordinary points of $\mathcal{N}_{k(v)}$ are of the form $\frac{i}{d}$, with $i \in \{0, 1, \ldots, d\}$.*

Proof: This is a consequence of the form of h_0 in 4.1. We recall that the eigenvalues of h_0 are the slopes of the G-ordinary σ-crystal $(M, \varphi, G_{W(k(v))})$ defined in 4.1.1.

4.6.2. Example 2. If $k(v) = \mathbb{F}_{p^2}$ and if G is not a torus, then τ is of the form $r(1,0) + s(1,1) + r(0,1)$, with $r, s \in \mathbb{N}$ (i.e. the slopes of a Shimura σ-crystal attached to a G-ordinary point of $\mathcal{N}_{k(v)}$ are precisely 0, 1/2 and 1).

Proof: This is a direct consequence of the properties P1, P3 and P4.

4.6.3. Example 3. We consider now the case of Picard surfaces, with f the map defined in [Go]. If $k(v) = \mathbb{F}_{p^2}$, then τ is $2(1,0) + (1,1) + 2(0,1)$.

4.6.4. Example 4. This is the generalization of 4.6.3. Let E be an imaginary quadratic field, let V be a $(m+n)$-dimensional E-vector space (with $m, n \in \mathbb{N}$), and let $\mathcal{J}_V : V \times V \to E$ be a non-degenerate hermitian form on V which has signature (m,n) over $\mathbb{R}$. We can assume $m \geq n$. Let $\mathrm{Sh}(G,X)$ be the Shimura variety defined as in [Go], for this situation. Similarly to [Go] it can be embedded in a Siegel modular variety $f : \mathrm{Sh}(G,X) \hookrightarrow \mathrm{Sh}\big(\mathrm{GSp}(W,\psi), S\big)$ with $\dim_{\mathbb{Q}}(W) = 2(m+n)$. We have $E(G,X) = E$ if $m > n$, and $E(G,X) = \mathbb{Q}$, if $m = n$. Moreover $G^{\mathrm{der}}_{\mathbb{R}} = SU(m,n)_{\mathbb{R}}$. If v is a prime of E relatively prime with 2 and such that $k(v) = \mathbb{F}_{p^2}$, and if $(f, L_{(p)}, v)$ is a SHS, then the G-ordinary type we get is $2n(1,0) + (m-n)(1,1) + 2n(0,1)$.

P5 *The Lie G-ordinary type $\mathrm{Lie}_G(\tau)$ is an ordinary type (i.e. it has precisely only the slopes $-1, 0$ and 1) iff G is not a torus and $k(v^{\mathrm{ad}}) = \mathbb{F}_p$, where v^{ad} is the prime of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ divided by v.*

Proof: This is a direct consequence of b) of 4.4.1 2), and of 3.2.2.

Let $\mathfrak{g} := \mathrm{Lie}(G_{W(k)})$ and let $\big(\mathfrak{g}, \varphi_0, F^0(\mathfrak{g}), F^1(\mathfrak{g})\big)$ be the Shimura filtered Lie σ-crystal attached to the G-canonical lift of a G-ordinary point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$.

P6 *The parabolic Lie subalgebra $\mathfrak{p}_0$ of $\mathfrak{g}$ corresponding to non-negative slopes of $(\mathfrak{g}, \varphi_0)$ is $F^0(\mathfrak{g})$ iff $k(v^{\mathrm{ad}}) = \mathbb{F}_p$.*

Proof: This is a direct consequence of c) of 4.4.1 3) and of the description of $\big(M, F^1, \sigma\mu(\frac{1}{p})\big)$ in 4.1 (cf. also with the proof of P1).

P7 *$\mathfrak{p}_0$ can be the Lie subalgebra of a Borel subgroup of $G_{W(k)}$.*

Proof: This is equivalent to: the rank of G is equal to the number of slopes zero of $(\mathfrak{g}, \varphi_0)$. From the formulae of 3.10.6 we deduce that, if this is so, then all simple adjoint factors of $G^{\mathrm{ad}}_{\mathbb{C}}$ are of A_ℓ Lie type, $\ell \in \mathbb{N}$. What we actually need is: any cyclic adjoint factor of $(\mathfrak{g}, \varphi_0, F^0(\mathfrak{g}), F^1(\mathfrak{g}))$ has maximal A-spreading (cf. def. 3.10.7).

Concrete example: Let F be a totally real number field, and let $\tilde{G}$ be an absolutely simple adjoint group over F of A_ℓ Lie type, such that $\tilde{G}_{\mathbb{R}}$ is the adjoint group of $\prod_{i=1}^{[\frac{\ell+1}{2}]} SU(i, \ell+1-i)_{\mathbb{R}}$. We assume the existence of a prime w of F unramified over a rational prime $p > 2$, and such that: $\tilde{G}$ is unramified over $\mathbb{Q}_p$, w is the only prime of F over p, and $\tilde{G}$ over the completion F_w of F with respect to w has non-trivial involution. Then for any SHS $(f, L_{(p)}, v)$, with $G^{\mathrm{ad}} = \mathrm{Res}_{F/\mathbb{Q}}\tilde{G}$, and with v dividing w (we have $F \subset E(G,X)$) and such that $k(v) = k(w)$, P7 above holds. Such a SHS does exist cf. [Va2, 6.5.1.1].

4.6.5. Example 5. P7 holds for the case considered in 4.6.3.

4.6.6. Example 6. If in 4.6.4 we have $m+n \geq 4$, then $\mathfrak{p}_0$ is not the Lie subalgebra of a Borel subgroup. If $m=n$, then $\mathrm{Lie}_G(\tau)$ is always an ordinary type.

P8 *For any rational number $r = a/d$, with $a \in \{1, \ldots, d\}$, there are situations when the slopes of the Shimura Lie σ-crystals attached to G-ordinary points of $\mathcal{N}_{k(v)}$ are precisely $-r, 0$ and r.*

Proof: We take (G,X) of B_ℓ (or C_ℓ or $D_\ell^{\mathbb{R}}$) type and such that $G^{\mathrm{ad}} = \mathrm{Res}_{F/\mathbb{Q}} G^1$, with F a totally real number field satisfying $[F:\mathbb{Q}] = d$, and with G^1 an absolutely simple F-group. We assume that for precisely a-embeddings of F into $\mathbb{R}$, the Lie group defined by the $\mathbb{R}$-values points of G^1 is non-compact. We also assume the existence of a prime v of $E(G,X) = F$ (cf. [De2, 2.3.12]) such that $k(v) = \mathbb{F}_{p^d}$, with $p > 2$ a prime for which G is unramified over $\mathbb{Q}_p$. Now for a SHS $(f, L_{(p)}, v)$ (cf. [Va2, 6.4.2]) defined by an injective map $f : (G,X) \hookrightarrow (\mathrm{GSp}(W,\psi), S)$, the slopes of the Shimura Lie σ-crystals attached to G-ordinary points of $\mathcal{N}_{k(v)}$ are precisely $-\frac{a}{d}$, 0 and $\frac{a}{d}$ (cf. 3.10.6 i) and iii)).

Obviously for any given $r \in \mathbb{Q} \cap (0,1]$ we can choose a number field F and a reductive group G^1 over F such that the above assumptions are satisfied. Of course there are examples for P8 with (G,X) of A_ℓ or $D_\ell^{\mathbb{H}}$ type (starting from 3.10.6 iii) and iv)). We leave this to the reader.

P9 *If the slopes of the Shimura σ-crystal attached to G-ordinary points of $\mathcal{N}_{k(v)}$ are rational numbers of the interval $(0,1)$, then no cyclic factor of the Shimura adjoint Lie σ-crystal attached to a G-ordinary point of $\mathcal{N}_{k(v)}$ is totally non-compact (see def. 3.10.5). The converse of this not true.*

Proof: The first part is a direct consequence of the formulae of 3.10.6. The second part can be seen through trivial examples.

As a corollary we get:

P10 *If there is a totally non-compact cyclic factor of the Shimura adjoint Lie σ-crystal attached to a G-ordinary point of $\mathcal{N}_{k(v)}$, then the abelian varieties obtained by pulling back $\mathcal{A}$ through G-ordinary points with values in algebraically closed fields, have non-trivial p-torsion.*

For future references we include the following elementary property which can be read out immediately from 4.1.1.

P11 *We assume that the representation $G_{\mathbb{C}} \to GL(W \otimes_{\mathbb{Q}} \mathbb{C})$ is irreducible. Then the abelian varieties obtained by pulling back $\mathcal{A}$ through G-ordinary points with values in algebraically closed fields, have non-zero p-ranks. These p-ranks are one precisely in the case when $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ is a simple $\mathbb{Q}_p$-group of A_1 Lie type.*

4.6.7. Example 7: The case of curves. We assume now that $\dim(X) = 1$, i.e. that $G^{\mathrm{ad}} = \mathrm{Res}_{F/\mathbb{Q}} G^1$, with F a totally real number field, and with G^1 a form of PSL_2 such that for exactly one embedding of F into $\mathbb{R}$, $G^1_{\mathbb{R}}$ is a non-compact (so split) group (cf. [De1, 2.3.4]). We make no restrictions on $(G^{\mathrm{ab}}, X^{\mathrm{ab}})$. Let $(f, L_{(p)}, v)$ be a SHS defined by an injective map $f : (G,X) \hookrightarrow (\mathrm{GSp}(W,\psi), S)$, and let v^{ad} be the prime of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}}) = F$ (cf. [De2, 2.3.12]) divided by v.

Let $G^{\mathrm{ad}}_{\mathbb{Q}_p} = \prod_{i\in I_F} \mathrm{Res}_{F^i/\mathbb{Q}_p} G^1_i$, where $F \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{i\in I_F} F_i$, with F_i an unramified finite field extension of $\mathbb{Q}_p$, and where G^1_i is an absolutely simple F_i-group. Let $i_0 \in I_F$ corresponding to the non-compact factor of $G^{\mathrm{ad}}_{\mathbb{R}}$, and let $d_0 = [F_{i_0} : \mathbb{Q}_p] \geq 1$. Then $k(v^{\mathrm{ad}}) = \mathbb{F}_{p^{d_0}}$, and the slopes of the adjoint Lie σ-crystal attached to a G-ordinary point of $\mathcal{N}_{k(v)}$ are precisely $-\frac{1}{d_0}$, 0 and $\frac{1}{d_0}$; the multiplicity of $-1/d_0$ (or of $1/d_0$) is d_0, while the multiplicity of 0 is $3\big(\dim_{\mathbb{Q}}(F) - d_0\big) + d_0 = 3\dim_{\mathbb{Q}}(F) - 2d_0$ (cf. 3.10.6 i)). So if $d_0 = 1$ P10 applies.

We assume now that the representation $G_{\mathbb{C}} \to GL(W \otimes_{\mathbb{Q}} \mathbb{C})$ is irreducible. Then the p-ranks of the abelian varieties obtained by pulling back $\mathcal{A}$ through G-ordinary points with values in algebraically closed fields are $2^{\dim_{\mathbb{Q}}(F)-d_0}$.

4.6.8. Remarks. 1) There are plenty of examples when the Newton polygon of the G-ordinary type of a SHS $(f, L_{(p)}, v)$ does not have the slopes 0 or 1. Such examples can be easily constructed starting from [Va2, 6.5.1.1] and its proof. The examples we hinted at are such that the irreducible representations of the faithful representation $G^{\mathrm{der}}_{\mathbb{C}} \hookrightarrow GL(W\otimes_{\mathbb{Q}}\mathbb{C})$ involve no tensors products. So, in some sense, we are at the opposite pole than the situation of P11.

2) In [Va5] we also show that the results of 4.1-6 remain true (under proper formulation) for the integral canonical models to be constructed in §6 for quotients of Shimura varieties of Hodge type with respect to large classes of non-hyperspecial subgroups.

3) We reobtained the "ordinary theory" for special fibres of integral canonical models of Siegel modular varieties, with respect to primes $p \geq 2$ (for $p = 2$ cf. 4.14.3). In particular we obtained a completely new proof of the Zariski density (see [NO], [FC] and [EO]) of ordinary points in these special fibres.

4) The whole of 4.1 as well as properties P1 to P10 remain true under proper formulation in the context of Shimura σ-crystals (cf. 3.11.2). The only difference is: we have to change $k(v) = \mathbb{F}_{p^d}$ (with $d \in \mathbb{N}$) with the degree of definition ($= d$), and $k(v^{\mathrm{ad}}) = \mathbb{F}_{p^{d_0}}$, with the A-degree of definition ($= d_0$). Besides we have to change the condition G is a torus, of P3 (resp. G is not a torus, of P5), to any one of the equivalent conditions a) to d) of 3.2.2 (resp. with the condition that 3.3.2 a) to d) does not hold).

5) In particular 4.1 and P1 to P10 can be applied to the context of abelian varieties with Hodge cycles without an a priori defined principal polarization (for instance, there is no problem if we have a polarization which is not principal), as long as we are in a context of Shimura σ-crystals.

6) It can happen that we have two standard Hodge situations $(f, L_{(p)}, v_i)$, $i = \overline{1,2}$, with v_1 and v_2 dividing the same rational prime $p \geq 3$, such that their attached G-ordinary types are different. For instance, this is the case if $k(v_1) = \mathbb{F}_p$, while $k(v_2) = \mathbb{F}_{p^d}$, with $d \geq 2$, cf. P1.

4.6.9. An application. Let E be a number field and let A be an abelian variety over E. Let $\mathrm{Spec}(O^r_E) \to \mathrm{Spec}(O_E)$ be the maximal open subscheme of the affine scheme defined by the ring of integers O_E of E, over which A extends to an abelian scheme A^r. Let $E(A)$ be the reflex field of the Shimura variety associated to the Mumford–Tate group of A (cf. [Va2, 2.12.3]).

4.6.9.1. Theorem *(The non-ordinary reduction criterion). If $E(A) \neq \mathbb{Q}$, then A^r has a non-ordinary reduction with respect to an infinite set of primes of O_E^r.*

Proof: We consider an embedding $E \hookrightarrow \mathbb{C}$, and let $\tilde{L}_{\mathbb{Z}} := H_1(A_{\mathbb{C}}, \mathbb{Z})$ be the first group of the Betti homology of $A_{\mathbb{C}}$. The Mumford–Tate group of A (of $A_{\mathbb{C}}$) is a reductive subgroup G_A of $GL(\tilde{L}_{\mathbb{Z}} \otimes_{ZZ} \mathbb{Q})$. Now there is $N_0(A) \in \mathbb{N}$ such that for any prime $p \geq N_0(A)$, the Zariski closure of G_A in $GL(\tilde{L}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$ is a reductive group over $\mathbb{Z}_{(p)}$. If also $p \geq \dim(A)$ and $p \geq 5$, then there is a $\mathbb{Z}_{(p)}$-well positioned family of tensors (of $\mathcal{T}(\tilde{L}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$) for the group G_A (cf. [Va2, 5.8.6]) enveloped by $\tilde{L}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Let now v be a prime of O_E^r dividing a rational prime $p \geq \max\big(N_0(A), 5, \dim(A)\big)$. Let V be the completion of the maximal unramified cover of the completion of $O_{(v)}$. So V is a finite extension of $V_0 = W(\mathbb{F})$. Let A_V be the abelian variety over V, obtained from A^r through the logical morphism $\mathrm{Spec}(V) \to \mathrm{Spec}(O_E^r)$. All the Hodge cycles of A_V are defined over $V_1[\frac{1}{p}]$, with V_1 a finite DVR cover of V (cf. [De3]; in fact we can take $V_1 = V$, but this is irrelevant here). We choose a family $(t_\alpha)_{\alpha \in \mathcal{J}(V_1)}$ of Hodge cycles of A_{V_1} such that G_A is the subgroup of $GL(\tilde{L}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q})$ fixing the Betti realizations of the members of the family $(t_\alpha)_{\alpha \in \mathcal{J}(V_1)}$.

Let $\bar{e} := [V_1 : V_0]$. In what follows, to simplify the presentation, we use similar notations as before, though the situation is entirely unrelated to a fixed SHS. Repeating the arguments of [Va2, 5.2-3], we get an abelian scheme $A_{\tilde{R}_{\bar{e}}}$ over the ring $\tilde{R}_{\bar{e}}$ defined in [Va2, 5.2.1] endowed with a family $({t_\alpha}^{\tilde{R}_{\bar{e}}})_{\alpha \in \mathcal{J}(V_1)}$ of Hodge cycles such that:

i) under the canonical V_0-epimorphism $\tilde{R}_{\bar{e}} \twoheadrightarrow V_1$ defined by a uniformizer of V_1 (see [Va2, 5.2]), it becomes A_{V_1}, together with $(t_\alpha)_{\alpha \in \mathcal{J}(V_1)}$;

ii) under the canonical V_0-epimorphism $\tilde{R}_{\bar{e}} \twoheadrightarrow V_0$, it becomes an abelian variety A_{V_0} for which $M := H^1_{\mathrm{crys}}(A_{\mathbb{F}}, V_0)$, together with its Hodge filtration F^1 associated to A_{V_0}, and with its Frobenius φ and the family of de Rham components $(u^0_\alpha)_{\alpha \in \mathcal{J}(V_1)}$ of the Hodge cycles (obtained naturally from $(t_\alpha)_{\alpha \in \mathcal{J}(V_1)}$) $(t^0_\alpha)_{\alpha \in \mathcal{J}(V_1)}$ of A_{V_0}, is a Shimura filtered σ-crystal $\big(M, F^1, \varphi, G_{V_0}, (u^0_\alpha)_{\alpha \in \mathcal{J}(V_1)}\big)$ (here G_{V_0} is the Zariski closure in $GL(M)$ of the subgroup of $GL(M[\frac{1}{p}])$ defined by fixing u^0_α, $\forall \alpha \in \mathcal{J}(V_1)$).

Let $\mathcal{P}$ be the Newton polygon of the G_{V_0}-ordinary type associated to the G_{V_0}-ordinary σ-crystals produced by (M, φ, G_{V_0}). From 3.1.0 we get:

(4.6.9.2) *The Newton polygon of the abelian variety over $k(v)$ obtained from A by reduction with respect to v, is above $\mathcal{P}$.*

Let E_1 be the composite field of E and $E(A)$, and let ω be a prime of E_1 dividing v. Let ω_A be the prime of $E(A)$ divided by ω. We have:

(4.6.9.3) *If $k(\omega_A) \neq \mathbb{F}_p$, then $\mathcal{P}$ is not the Newton polygon of an ordinary type.*

This results from property $P1$ of 4.6: the principal polarization existing in the context of a SHS played no role in obtaining the theory of 4.1-6 (cf. rm. 5) of 4.6.8); of course [Mu, cor. 1, p. 234] allows us to assume that A is principally polarized.

Now 4.6.9.1 is a direct consequence of 4.6.9.2-3, as there are infinitely many primes of E_1 dividing primes of $E(A)$ not having a residue field with a prime number of elements.

4.6.9.4. Remark. We can make 4.6.9.1 more precise. Let v be a prime of O_E^r such that:

a) it divides a prime $p \geq 3$ such that G_A is unramified over $\mathbb{Q}_p$;

b) there is a prime ω of E_1 dividing v and dividing a prime ω_A of $E(A)$ for which $k(\omega_A)$ has more than p elements.

Then A has a non-ordinary reduction with respect to v (and actually there is a Newton polygon $\mathcal{P}$ of an abelian variety over $\mathbb{F}$, of dimension equal to $\dim(A)$, which is the Newton polygon of a G_A-ordinary type which is not an ordinary type, and such that the Newton polygon of the reduction of A with respect to v is above $\mathcal{P}$).

This will become transparent in the light of §5-8, cf. 2.3.8 2). In [Va5] we will see that in a) above, in most situations, we can allow p to be 2 as well.

4.6.9.5. Question. Let $O_{E_1}^r$ be the normalization of O_E^r in E_1, and let $A_{E_1}^r$ be the abelian scheme over $O_{E_1}^r$, obtained from A^r by pull back. We do not assume that $E(A)$ is not $\mathbb{Q}$. Is it true that for an infinite set of primes $\mathcal{L}$ of $O_{E_1}^r$, $A_{E_1}^r$ has a G_A-ordinary reduction which is not ordinary?

4.6.9.6. Remark. Property P1 supports the well-known conjecture (for instance cf. [Oo1, p. 11]) concerning the existence of ordinary reductions of an abelian variety over a number field E with respect to primes of E.

4.6.9.7. Remark. We regard the above non-ordinary reduction criterion as the natural generalization of the (well-known) fact that any abelian variety over a number field E, having complex multiplication over $\bar{E}$, has a non-ordinary reduction with respect to an infinite set of primes of E.

4.6.9.8. Exercise. Let $(f, L_{(p)}, v)$ be a SHS. Let τ be the G-ordinary type attached to it. Show that there is an infinite number of standard Hodge situations $(f, L_{(q)}, v_q)$ (with v_q a prime of $E(G, X)$ dividing an odd rational prime q) which have τ as their attached G-ordinary type. (Hint: If the reflex field $E(G, X)$ is a Galois extension of $\mathbb{Q}$, this is obvious; now consider the Galois extension $E_1(G, X)$ of $\mathbb{Q}$ generated by $E(G, X)$, and make use of 4.1 and of Tchebotarev density theorem).

This simple fact makes 4.6.9.5 an interesting question, and suggests that the conjectures referred to in 4.6.9.6 should admit variants for G-ordinary types, which are not ordinary.

4.6.10. The case when the degree of definition is 1. For the sake of future references we point out the following two particular cases of P1 and P2.

A. The split case. *If $(f, L_{(p)}, v)$ is a SHS, and if $G_{\mathbb{Q}_p}$ is a split reductive group, then the G-ordinary type we obtain is the ordinary type, and the abelian varieties over Witt rings of perfect fields, obtained from $\mathcal{A}$ by pull back through G-canonical lifts of $\mathcal{N}$, are (usual) (ordinary) canonical lifts.*

B. *If $(f, L_{(p)}, v)$ is a SHS, and if a $G_{W(\mathbb{F})}$-canonical lift of a $G_{W(\mathbb{F})}$-ordinary point of $\mathcal{N}_{k(v)}$ gives birth to a Shimura σ-crystal whose degree of definition is 1, then all abelian* *varieties over $W(\mathbb{F})$ obtained from $\mathcal{A}$ by pull back through any $G_{W(\mathbb{F})}$-canonical lift of $\mathcal{N}$, are (usual) canonical lifts, and so have complex multiplication.*

4.6.11. An application to [Va2, 6.8.6]. We consider the situation and notations described in [Va2, 6.8.6], but we do not assume that we have a non-trivial involution. So $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is a totally imaginary quadratic extension of a totally real number field (resp. a totally real number field) if the involution is non-trivial (resp. trivial). We assume that we are dealing with a prime v^{ad} of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ dividing a prime $p > 2$.

A. Exercise. Show that the criterion b) of [Va2, 6.8.2] is satisfied. Hint: the situation gets reduced to a standard PEL situation; so we can use 4.4.5, the density part of 4.2.1 and [Mi1, 4.12].

B. Corollary. *Everywhere in [Va2] the stars can be removed.*

Proof: The cases postponed in [Va2] (cf. [Va2, 6.8.6]), are exactly the ones treated in A above. This proves the corollary.

4.6.12. Exercise. Prove d) of 4.4.1 3) for the case $k(v) = \mathbb{F}_p$. Hint: use P1, P2 and Fontaine's comparison theory.

4.7. Crystalline coordinates.

4.7.1. Let $y : \mathrm{Spec}(k) \to \mathcal{U}/H_0$ be a G-ordinary point. For simplifying the notations we assume that attached to y we do have a principally quasi-polarized Shimura σ-crystal, and that it is of the form $\big(M_y, \varphi_y, G_{W(k)}, (t_\alpha)_{\alpha\in\mathcal{J}'}, \psi_y\big)$ (cf. 2.3.10). Let $\mu_y : \mathbb{G}_m \to G_{W(k)}$ be its canonical split. It produces a direct sum decomposition $M_y = F^1_y \oplus F^0_y$, with $\beta \in \mathbb{G}_m(W(k))$ acting through μ_y on F^i_y as the multiplication with β^{-i}. Let N be the subgroup of $G^{\mathrm{der}}_{W(k)}$ which acts trivially on F^0_y and on M_y/F^0_y. Let R be the ring of the formal completion of N at the origin. We choose an isomorphism $\tilde{f} : R\tilde{\to}W(k)[[x_1, \ldots, x_{e_0}]]$, with $e_0 = \dim(X)$ (see [Va2, 5.4.7]). We are mainly interested in the case when $k = \bar{k}$; but for a great part of what follows this is not needed. It is easy to see that the group N is isomorphic to $\mathbb{G}_a^{e_0}$ (i.e. it can be identified with the group scheme defined by its Lie algebra). We can choose the coordinates $x_1, \ldots, x_{e_0}$ (i.e. we can choose $\tilde{f}$) such that the i-th $\mathbb{G}_a$ copy of N is $\mathrm{Spec}\big(W(k)[x_i]\big)$. So N as a scheme is $\mathrm{Spec}\big(W(k)[x_1, \ldots, x_{e_0}]\big)$. We take the Frobenius lift of N to be compatible with σ, and such that at the rings level it takes $x_i \to x_i^p$, $i = \overline{1, e_0}$. It induces a Frobenius Φ_R on R.

As the case of a torus (of a zero dimensional Shimura variety) is trivial, we assume from now on that G is not a torus. Let N_0 be the smallest connected subgroup of $G_{W(k)}$ such that:

i) it contains N;

ii) its Lie algebra is $p\varphi_y$-invariant.

For all that follows, we have to rely heavily on the terminology and formulae of 3.10.6. It is easy to see (to be compared also with 3.7.2) that:

0) N_0 is a product indexed by the number of simple factors of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$.

1) N_0 is a nilpotent smooth subgroup of $G_{W(k)}$.

2) We get a Lie σ-crystal $(\mathrm{Lie}(N_0), \varphi_y, F^0(\mathrm{Lie}(N_0)), F^1(\mathrm{Lie}(N_0)))$, where $F^i(\mathrm{Lie}(N_0)) := F^i(\mathrm{End}(M)) \cap \mathrm{Lie}(N_0)$, $i = \overline{0,1}$.

3) $N = N_0$ iff the Lie G-ordinary type $\mathrm{Lie}_G(\tau)$ we get is ordinary (i.e. it has precisely the slopes -1, 0 and 1), cf. properties P5 and P6 of 4.6.

4) N_0 is an abelian group iff the spreading of any cyclic adjoint factor attached to y is 0 or 1, cf. def. in 3.10.4.4 and 4.3.1. This automatically happens if all the simple factors of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ are of B_l, C_l or $D_l^{\mathbb{R}}$ type, but it is not necessarily true if there are simple factors of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ of A_ℓ or $D_\ell^{\mathbb{H}}$ type ($\ell \in \mathbb{N}$).

5) More generally, N_0 is a nilpotent group of order of nilpotency not bigger than some $m \in \mathbb{N}$, if the spreading of any cyclic adjoint factor attached to y is not bigger than m.

So if $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ has only simple factors of $D_\ell^{\mathbb{H}}$ type ($\ell \in \mathbb{N}$), then N_0 has the order of nilpotency at most 2. If $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is a simple adjoint Shimura variety of A_ℓ type, then the order of nilpotency of N_0 is smaller or equal to ℓ, and equality can take place (cf. property P7 of 4.6); in fact the order of nilpotency of N_0 can be any number of the set $\{1, ..., \ell\}$, as it can be read out from the definition of spreadings of 3.10.4.4.

We prefer to regard y as a closed embedding $y : \mathrm{Spec}(k) \hookrightarrow \mathcal{N}_{W(k)}/H_0$. Let $\hat{O}_y$ be the completion of the local ring of y in $\mathcal{N}_{W(k)}/H_0$. Let $(A_{\hat{y}}, p_{A_{\hat{y}}})$ be the principally polarized abelian scheme over $\mathrm{Spec}(\hat{O}_y)$ obtained from $(\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ through the canonical morphism $\mathrm{Spec}(\hat{O}_y) \to \mathcal{N}/H_0$. We consider the following principally quasi-polarized Shimura filtered F-crystal

$$_N M_{\hat{y}} := (M_y, F_y^1, \varphi_y, G_{W(k)}, N, \tilde{f}, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_y)$$

over R/pR. So there is a unique connection $_N\nabla_{\hat{y}}$ on $M_y \otimes_{W(k)} R$ which is integrable, nilpotent mod p, and such that the Frobenius $\Phi_y := n(\varphi_y \otimes 1)$ of $M_y \otimes_{W(k)} R$ is $_N\nabla_{\hat{y}}$-parallel, cf. 2.2.10 (here $n \in N(R)$ is the universal element defined by the natural morphism $\mathrm{Spec}(R) \to N$).

Similarly we define $_{G^{\mathrm{der}}_{W(k)}} M_{\hat{y}}$, starting from a Frobenius Φ_{R_1} of the ring R_1 of the completion of $G^{\mathrm{der}}_{W(k)}$ at the origin, leaving invariant the kernel of the canonical $W(k)$-epimorphism $R_1 \twoheadrightarrow R$, and inducing Φ_R on R).

Let $\tilde{e}_0$ be the dimension of N_0. As a scheme $N_0 = \mathrm{Spec}(W(k)[x_1, ..., x_{\tilde{e}_0}])$, cf. the above assumption on k. Such an identification can be chosen so that each closed subscheme $\mathrm{Spec}(W(k)[x_i])$, $i \in \{1, ...\tilde{e}_0\}$, is a $\mathbb{G}_a$ subgroup of N_0, to be referred as the i-th $\mathbb{G}_a$ copy of N_0. Let $\mathrm{Spec}(R_0)$ be the completion of the localization of N_0 in the origin. So $R_0 = W(k)[[x_1, ..., x_{\tilde{e}_0}]]$, with $\mathrm{Spec}(W(k)[[x_i]])$ being the completion of the i-th $\mathbb{G}_a$ copy of N_0, $i = \overline{1, \tilde{e}_0}$. We choose such an identification, so that the natural epimorphism $R_0 \twoheadrightarrow R$ is defined by x_i goes to x_i, if $i \in \{1, ..., e_0\}$, and into zero, if $i \in \{1 + e_0, ...\tilde{e}_0\}$. We assume that the Frobenius Φ_{R_1} of R_1 is such that it leaves invariant the kernel of the natural epimorphism $R_1 \twoheadrightarrow R_0$, inducing a Frobenius of R_0 which takes x_i into x_i^p, $i = \overline{1, \tilde{e}_0}$.

4.7.2. Lemma. *The principally quasi-polarized filtered F-crystal over* $\mathrm{Spec}(\hat{O}_y/p\hat{O}_y)$ *endowed with a family of crystalline tensors, defined by $(A_{\hat{y}}, p_{A_{\hat{y}}})$ and de Rham components of the Hodge cycles with which $A_{\hat{y}}$ is naturally endowed, is isomorphic to ${}_N M_{\hat{y}}$.*

Proof: In [Va2, 5.4.4-5] we saw that there is a morphism $\mathrm{Spec}(R_1) \xrightarrow{a_1} \mathcal{N}_{W(k)}/H_0$ lifting y, and such that the principally quasi-polarized filtered F-crystal over R_1/pR_1 endowed with tensors, and induced by the pull back through a_1 of $(A_{\hat{y}}, p_{A_{\hat{y}}})$ and de Rham components of the mentioned Hodge cycles of $A_{\hat{y}}$, is isomorphic to ${}_{G^{\mathrm{der}}_{W(k)}} M_{\hat{y}}$. But the pull back of ${}_{G^{\mathrm{der}}_{W(k)}} M_{\hat{y}}$ under the canonical morphism $\mathrm{Spec}(R) \overset{b_1}{\hookrightarrow} \mathrm{Spec}(R_1)$ is ${}_N M_{\hat{y}}$ (due to the compatibility of the two Frobenius lifts of R_1 and R with the $W(k)$-epimorphism $R_1 \twoheadrightarrow R$). But the morphism $c_1 := b_1 \circ a_1 : \mathrm{Spec}(R) \to \mathrm{Spec}(\hat{O}_y)$ is an isomorphism, cf. [Va2, 5.4.7-8]. This proves the lemma.

We denote by $\bar{\Omega}_{R/W(k)}$ the free R-module generated by dx_i, $i = \overline{1, e_0}$. $\bar{\Omega}_{R/W(k)}$ is naturally a direct summand of the R-module $\Omega_{R/W(k)}$.

4.7.3. Lemma. *The connection ${}_N\nabla_{\hat{y}}$ is of the form $d + \beta_N$, where $\beta_N \in \mathrm{Lie}(N_0) \otimes_{W(k)} \Omega_{N/W(k)} \otimes_{W(k)} R = \mathrm{Lie}(N_0) \otimes_{W(k)} \bar{\Omega}_{R/W(k)}$.*

Proof: We define similarly ${}_{N_0}M_{\hat{y}}$. ${}_N M_{\hat{y}}$ is obtained from ${}_{N_0}M_{\hat{y}}$ by pull back through the $W(k)$-epimorphism $R_0 \twoheadrightarrow R$. So the lemma is a consequence of [Fa2, rm. ii) of th. 10] applied to the subgroup N_0 of $GL(M_y)$.

4.7.3.1. We write

$$\beta_N = \sum_{j=1}^{\tilde{e}_0} n_j \otimes \varepsilon_j,$$

with $\{n_j\}$ a $W(k)$-basis of the Lie algebra of the j-th copy $\mathbb{G}_a$ of N_0, and with $\varepsilon_j \in \bar{\Omega}_{R/V_0}$ (cf. 4.7.3). The elements n_j, $j \in \{1, ..., \tilde{e}_0\}$, are endomorphisms of $M_y \otimes_{W(k)} R$.

4.7.4. Levels of complexity. Let v^{ad} be the prime of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ divided by v. To build up G-multiplicative coordinates, and then to extend this to the general case of 4.7.33.1, we distinguish five different cases (levels of complexity). They are:

A. $N = N_0$.

B. N_0 is an abelian group.

C. N_0 is a nilpotent group of order of nilpotency two.

D. N_0 is a nilpotent group of order of nilpotency greater than two.

E. The general case referred to in 4.7.33.1 below.

The condition $N = N_0$ is equivalent to $k(v^{\mathrm{ad}}) = \mathbb{F}_p$, as it can be easily checked starting from 3.10.6 and property P2 of 4.6. First we describe the case **A** (so $\tilde{e}_0 = e_0$). Warning: 7.5-29 below refer to this case only. The general theory of crystalline coordinates is elaborated in 4.7.30. In 4.7.31-37 we deal with different aspects of it.

From now on we assume $k = \bar{k}$.

4.7.5. Theorem. *We assume that $N = N_0$.*

i) *There is a basis* $\{a_i | 1 \leq i \leq e\}$ *of* F_y^0 *and a basis* $\{b_i | 1 \leq i \leq e\}$ *of* F_y^1 *and a permutation* π *of* $\{a_1, \ldots, a_e, b_1, \ldots, b_e\}$ *satisfying the conditions:*

(4.7.6) ${}_N\nabla_{\hat{y}} a_i = 0$, $\forall i \in \{1, \ldots, e\}$;

(4.7.7) ${}_N\nabla_{\hat{y}} b_i = \sum_{j=1}^{e_0} n_j(b_i) \otimes \varepsilon_j$;

(4.7.8) $\Phi_y(a_i) = \pi(a_i)$ *if* $\pi(a_i) \in \{a_1, \ldots, a_e\}$;

(4.7.9) $\Phi_y(a_i) = n\pi(a_i)$ *if* $\pi(a_i) \in \{b_1, \ldots, b_e\}$;

(4.7.10) $\Phi_y(b_i) = p\pi(b_i)$ *if* $\pi(b_i) \in \{a_1, \ldots, a_e\}$;

(4.7.11) $\Phi_y(b_i) = pn\pi(b_i)$ *if* $\pi(b_i) \in \{b_1, \ldots, b_e\}$.

Furthermore ε_i*'s are closed 1-forms verifying the equations:*

(4.7.12) *If* $i \in \{1, \ldots, e\}$ *is such that* $\pi(b_i) \in \{a_1, \ldots, a_e\}$, *then*

$$0 = \sum_{j=1}^{e_0} \varphi(n_j)\big(\pi(b_i)\big) \otimes \Phi_R^* \varepsilon_j\,;$$

(4.7.13) *If* $i \in \{1, \ldots, e\}$ *is such that* $\pi(b_i) \in \{b_1, \ldots, b_e\}$, *then*

$$pdn\big(\pi(b_i)\big) + \sum_{j=1}^{e_0} p\, n_j\big(\pi(b_i)\big) \otimes \varepsilon_j = \sum_{j=1}^{e_0} \varphi(p\, n_j)\big(\pi(b_i)\big) \otimes \Phi_R^*(\varepsilon_j);$$

(4.7.14) *If* $i \in \{1, \ldots, e\}$ *is such that* $\pi(a_i) \in \{b_1, \ldots, b_e\}$ *then*

$$dn\,\big(\pi(a_i)\big) + \sum_{j=1}^{e_0} n\big(\,n_j\big(\pi(a_i)\big)\big) \otimes \varepsilon_j = 0.$$

ii) *There are formal power series* $\mu_j \in W(k)[\frac{1}{p}][[x_1, \ldots, x_{e_0}]]$ *such that:*

(4.7.15) $\varepsilon_j = d\mu_j\ \forall j \in \{1, \ldots, e\}$;

(4.7.16) $\forall i \in \{1, \ldots, e\}$ *such that* $\pi(b_i) \in \{a_1, \ldots, a_e\}$ *we have*

$$\big(\sum_{j=1}^{e_0} \varphi(n_j)\big(\pi(b_i)\big)\Phi_R^* \mu_j\big)(0) = 0;$$

(4.7.17) $\forall i \in \{1, \ldots, e\}$ *such that* $\pi(b_i) \in \{b_1, \ldots, b_e\}$ *we have*

$$\big((n - 1_{M_y \otimes_{W(k)} R})(\pi(b_i)) + p \sum_{j=1}^{e_0} n_j\big(\pi(b_i)\big)\mu_j\big)(0) = \sum_{j=1}^{e_0} \big(\varphi(p\, n_j)\big(\pi(b_i)\big)\Phi_R^* \mu_j\big)(0);$$

(4.7.18) *$\forall i \in \{i, \ldots, e\}$ such that $\pi(a_i) \in \{b_1, \ldots, b_e\}$ we have*

$$\big((\log n)\big(\pi(a_i)\big) + \sum_{j=1}^{e_0} n_j\big(\pi(a_i)\big)\mu_j\big)(0) = 0.$$

We have $\mu_j(0) \in pW(k)$ and the series $q_j = \exp(\mu_j)$, $j \in \{1, \ldots \tilde{e}_0\}$, belongs to R and satisfies $q_j(0) = 1$ modulo $pW(k)$.

4.7.19. Corollary. *The elements $q_j - 1 \in R$, $j = \overline{1, e_0}$, form together with p a regular system of parameters of R, i.e. the $W(k)$-homomorphism $R \to R$ sending x_j to $q_j - 1$, $j = \overline{1, e_0}$, is an isomorphism. Furthermore if we take the Frobenius of R to be defined by $q_j \to q_j^p$, $j = \overline{1, e_0}$, then the Frobenius of ${}_N M_{\hat{y}}$ takes b_j into $p\pi(b_j)$, and takes a_j into $\pi(a_j)$, $\forall j \in \{1, \ldots, e\}$.*

4.7.20. Definition. The elements $q_j - 1 \in R$ form, due to 4.7.19, the G-canonical coordinates of R, or the Shimura-canonical coordinates of R.

4.7.21. Corollary. *The formal moduli scheme $\mathcal{M}_G(y)$ of G-deformations (i.e. deformations factoring through $\mathcal{N}/H_0$) of the principally polarized abelian variety (A_y, p_{A_y}), over artinian local $W(k)$-algebras having k as their residue field, is canonically isomorphic to the formal torus $\mathbb{G}_m^{e_0}{}^\wedge/W(k)$; the principally polarized abelian variety over $W(k)$, obtained from $(\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ by pull back through the G-canonical lift of y, corresponds to the unit element of $\mathbb{G}_m^{e_0}{}^\wedge/W(k)(W(k))$.*

4.7.22. Corollary. *The permutation π is trivial if and only if $k(v) = \mathbb{F}_p$, i.e. A_y is an ordinary abelian variety. If this is the case then the formal torus of G-deformations of (A_y, p_{A_y}) is a formal subtorus of the formal torus of deformations of A_y, or of GSp-deformations of (A_y, p_{A_y}) (i.e. of deformations of the principal polarized abelian variety (A_y, p_{A_y})).*

4.7.23. About the proofs of 4.7.5-22. The proofs of 4.7.5 and of 4.7.19 are entirely analogous to the proofs of [De2, 1.4.2 and 1.4.7]. As this is the only part of our work which is similar to the theory of ordinary points of special fibres of integral canonical models of Siegel modular varieties, here we just indicate which are the differences occurring. We do not bother to mention what happens when we choose different Frobenius lifts on R (see [De2]).

The first part b) of 4.4.1 3) guarantees that the filtered σ-crystal (M_y, F_y^1, φ_y) is cyclic diagonalizable. From the proof of it, we deduce the existence of a basis $\{a_i | 1 \leq i \leq e\}$ of F_y^0 and a basis $\{b_i | 1 \leq i \leq e\}$ of F_y^1, and of a permutation π of $B := \{a_1, \ldots, a_e, b_1, \ldots, b_e\}$ such that $\varphi_y(b_i) = p\pi(b_i)$ and $\varphi_y(a_i) = \pi(a_i)$, $i = \overline{1, e}$.

The difference from [De2, 1.4] comes from the fact that that π is not always the trivial permutation 1_B of B.

Now (4.7.6-7) just say what is the action of ${}_N\nabla_{\hat{y}}$ on the basis B of M_y, using the expression of ${}_N\nabla_{\hat{y}}$ given in 4.7.3.1. (4.7.8-11) just makes explicit the action of Φ_y on elements of B, starting from the mentioned expression of φ_y. The proof of the fact that ε_j

are closed is as in [De2, 1.4.2], starting from the fact that the connection ${}_N\nabla_{\hat{y}}$ is integrable (this is also implied by the assumption $N = N_0$). (4.7.12-14) just writes down what means that Φ_y is ${}_N\nabla_{\hat{y}}$-parallel (with respect to Φ_R) (cf. [De2, 1.1.3.3 and 1.4.2.5]).

To see 4.7.5 ii), we follow closely the proof of [De2, 1.4.2.5 ii)]. First the Poincaré lemma assures the existence of formal power series $\mu_j \in W(k)[\frac{1}{p}][[x_1, \ldots, x_{e_0}]]$ such that $d\mu_j = \varepsilon_j$, unique up to a constant. Now (4.7.12-14) can be rewritten as:

(4.7.24) $d\Big(\sum_{j=1}^{e_0} \varphi(n_j)\big(\pi(b_i)\big)\Phi_R^*\mu_j\Big) = 0;$

(4.7.25) $d\Big(pn\big(\pi(b_i)\big) + p\sum_{j=1}^{e_0} n_j\big(\pi(b_i)\big)\mu_j - \varphi(p\,n_j)\big(\pi(b_i)\big)\Phi_R^*\mu_j\Big) = 0;$

(4.7.26) $d\Big(\log\, n\pi(a_i) + \sum_{j=1}^{e_0} n_j\big(\pi(a_i)\big)\mu_j\Big) = 0.$

We remark that $\varphi(p\,n_j) \in M_y$ is non-zero mod p, and that N is an abelian group, and so we have no problems with $\log n$. The equations (4.7.24-26) are of similar type: they are obtained by taking the d-operator of linear combinations of functions; (4.7.25) is an equation similar to the ones obtained by taking the d-operator of [De2, 1.4.2.7]. The values of μ_j at zero are determined by the equations (4.7.16-18). To see why this is so, let $t_j := n_j\mu_j(0)$, $j \in \{1, ..., e_0\}$, and let $t := \sum_{j=1}^{e_0} t_j$, We have $t \in \mathrm{Lie}(N)[\frac{1}{p}]$. To determine $\mu_j(0)$, $j \in \{1, \ldots, e_0\}$, is equivalent to determine t. The equations (4.7.16-18) can be rewritten respectively as:

(4.7.27) $\varphi(t)\big(\pi(b_i)\big) = 0;$

(4.7.28) $\varphi(pt)\big(\pi(b_i)\big) = pt\big(\pi(b_i)\big);$

(4.7.29) $t\big(\pi(a_i)\big) = 0.$

Let $B_0 := \{a_1, \ldots, a_e\} \cup \big\{\pi(a_i), \ldots, \pi(a_e)\big\}$. Now t acts trivially on the subspace of M_y generated by B_0, i.e. on $F_y^0 \cup \varphi_y(F_y^0)$. This implies that if there is $t \in \mathrm{Lie}(N)[\frac{1}{p}]$ such that (4.7.27-29) holds, then t is unique, and $t \in p\mathrm{Lie}(N)$ (as $\frac{t}{p}$ takes M_y into M_y). It is easy to see that there is a unique $t \in \mathrm{End}(M_y)$ such that $t|F_y^0 = 0$ and 4.7.27-29 hold.

So we need to show that such a t annihilates t_α, $\forall \alpha \in \mathcal{J}'$. This is easy: repeating the arguments for $(M_y \otimes_{W(k)} M_y^*)^{n_\alpha}$ (with $\alpha \in \mathcal{J}'$), instead of M_y, we get equations similar to the ones of 4.7.24-26. As the connection ${}_N\nabla_{\hat{y}}$ of $(M_y \otimes_{W(k)} M_y^*)^{n_\alpha} \otimes_{W(k)} R$ is uniquely determined (cf. 3.4.18.8), the new equations we get are linear combinations of those of 4.7.24-25 and respectively of the ones of 4.7.26. So 4.7.8 can be interpreted as $t(u_\alpha) = 0$ if $\varphi(u_\alpha) = u_\alpha$ and $u_\alpha \in F^0\big((M_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} M_{\mathbb{Z}_p}^*)^{\otimes n_\alpha} \otimes_{\mathbb{Z}_p} W(k)\big)$, with $M_{\mathbb{Z}_p}$ the $\mathbb{Z}_p$-submodule of M_y generated by the base B). So $t(t_\alpha) = 0$, $\forall \alpha \in \mathcal{J}'$. In fact, due to the special form of Φ_y and of Φ_R, we can easily check that $t = 0$ (cf. also 4.7.30 1)).

The last fact of ii), to prove that $q_j \in R$ and that $q_j(0) = 1 \bmod pW(k)$, is similar to [De2, 1.4.5], starting from (4.7.23-25) and using the element

$$t_R := \sum n_j u_j \in \mathrm{Lie}(N) \otimes_{W(k)} W(k)[\frac{1}{p}][[x_1, \ldots, x_{e_0}]].$$

The proof of 4.7.19 is entirely analogous to the proof of [De2, 1.4.7]: we are in the context of principally polarized abelian schemes and so the Kodaira–Spencer map

$$gr_N\nabla_{\hat{y}} : \bar{T}_{R/W(k)} \to \mathrm{Hom}_R(F_y^1 \otimes_{W(k)} R, F_y^0 \otimes_{W(k)} R)$$

(with $\bar{T}_{R/W(k)} := \bar{\Omega}^*_{R/W(k)}$) is injective, having $\mathrm{Lie}(N) \otimes_{W(k)} R$ as its image (we have a canonical injective $W(k)$-linear map $\mathrm{Lie}(N) \hookrightarrow \mathrm{Hom}_{W(k)}(F_y^1, F_y^0)$).

Now 4.7.21 is obvious. 4.7.22 is a particular case of the principle expressed in 4.7.36; it reobtains in the case of a SHS $(f, L_{(p)}, v)$, with $k(v) = \mathbb{F}_p$, the main result of [No1].

4.7.30. The general theory of crystalline coordinates for p-divisible groups over k. We itemize the ideas.

1) Based on [De2, 1.4] and on the results pertaining to a SHS and presented till 4.7.4, the above part 4.7.5-29 is essentially trivial. What we did: we interpreted [De2, 1.4] as a property of deformations of Shimura adjoint Lie σ-crystals. In other words, the p-divisible group associated to (M_y, φ_y) is a direct sum of two p-divisible group D_y^1 and D_y^2 such that:

– the slopes of D_y^1 are precisely 0 and 1, while D_y^2 does not have any one of these slopes;

– the G-deformations of (A_y, p_{A_y}) referred above correspond to deformations of D_y^1, while "keeping" D_y^2 fixed.

So, if

$$M_y = M_y^1 \oplus M_y^2$$

is the direct sum decomposition corresponding to D_y^1 and D_y^2, then

$$\mathrm{Lie}(N) \subset \mathrm{End}(M_y^1).$$

The main goal of including 4.7.5-29 here is: to get familiar with the new type of equations (differential or not) showing up when we pass from (Shimura) ordinary σ-crystals to cyclic diagonalizable Shimura σ-crystals. The equations 4.7.9-10, 4.7.12, 4.7.14, 4.7.16, 4.7.18, 4.7.24 and 4.7.26 are typical to the cyclic diagonalizable situation: they do not show up in [De2, 1.4].

2) Let D be a p-divisible group over k. Let m be the product of the dimensions of D and of its dual D^t. It is known that D has a versal deformation $\mathcal{D}$ over $R_m := W(k)[[z_1, ..., z_m]]$. For any $W(k)$-epimorphism $q : W(k)[[z_1, ..., z_m]] \twoheadrightarrow W(k)[[x_1, ..., x_n]]$, with $n \in \mathbb{N}$, $n \leq m$, we get a p-divisible group $\mathcal{D}_q$ over $R_n := W(k)[[x_1, ..., x_n]]$. The first goal of the general problem of crystalline coordinates (for p-divisible groups) is to find a good description (if possible canonical) of the p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R_n)$ defined by $\mathcal{D}_q$, for epimorphisms q of interest; we have freedom (as part of the good description) for the choice of a Frobenius Φ_{R_n} of R_n. If Φ_{R_n} is compatible with σ and satisfies $\Phi_{R_n}(x_i) = x_i^p$, $i = \overline{1, n}$, we speak about an additive type situation. If Φ_{R_n} is compatible with σ and satisfies $\Phi_{R_n}(x_i) = (x_i - 1)^p$, we speak about a multiplicative type situation. These are the most common (and handable) type of situations. The second goal of the problem of crystalline coordinates is to compare, in the case we do not have a canonical description, the different descriptions obtained in the process of achieving the first goal. In what follows we essentially concentrate in achieving the first goal.

Very often the σ-crystal $(\tilde{M}, \tilde{\varphi})$ defined by D gets naturally the structure of a Shimura σ-crystal $(\tilde{M}, \tilde{\varphi}, \tilde{G})$, and then we take q to be the maximal quotient of R_m preserving the cycles (cf. 2.2.8 1)) involved (for concrete examples see [Va2, 5.4-5]). So we get a p-divisible object with cycles $\mathcal{C}_q$ of $\mathcal{MF}^{\nabla}_{[0,1]}(R_n)$, defined by q. It is associated to a Shimura p-divisible group over R_n. The theorem 3.4.18.5 (as well as its Shimura context counterpart 3.4.18.7), in essence, achieves the first goal of the theory of crystalline coordinates. What we have to do is:

a) First we choose, according to the needs and desires, a Frobenius of R_n leaving invariant its ideal $I_n := (x_1, ..., x_n)$, and a Frobenius Φ of $\tilde{M} \otimes_{W(k)} R_n$ of the form $g_{R_n}(\tilde{\varphi} \otimes 1)$, with $g_{R_n} \in \tilde{G}(R_n)$ which modulo I_n is zero. We choose also a filtration $\tilde{F}^1$ of $\tilde{M}$ such that $(\tilde{M}, \tilde{F}^1, \tilde{\varphi}, \tilde{G})$ is a Shimura filtered σ-crystal. We get, as in 3.4.18, a p-divisible object with cycles $\mathcal{C}$ of $\mathcal{MF}_{[0,1]}(R_n)$.

b) We determine the connections which make $\mathcal{C}/p\mathcal{C}$ to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(R_n)$. We look at the resulting Kodaira-Spencer maps (one for each such connection), and we pick up one such connection of whose Kodaira-Spencer map is injective (an isomorphism on its image) modulo the maximal ideal of R_n; the choices in a) above should be made in such a way that such a connection does exist. So $\mathcal{C}$ and $\mathcal{C}_q$, as p-divisible objects with cycles of $\mathcal{MF}^{\nabla}_{0,1]}(R_n)$, are isomorphic (cf. [Fa2, th. 10 and the remarks after]).

Using the steps a-b) above, we regain (cf. 3.4.18.2) the results of [De2, 2.1], without any computational effort. Moreover, the case of ordinary $K3$ surfaces (see [De2, 2.1 B]) for any prime p (this is not covered in loc. cit.), is settled (for $p = 2$ in the sense of the first goal mentioned above) by [Fa2, th. 10]. Loc. cit. is not stated as such, but starting from [De2, 2.1.6] it can be easily checked that it applies: with the notations of [De2, p. 109] we apply [Fa2, th. 10] to U_0 first, then to U_1, and then to U_2/U_0, and finally we use the fact that we have a perfect symmetric polarization. We end with a principally (symmetric) quasi-polarized versal filtered F-crystal over $k[[x_1, ..., x_{20}]]$ (it is a principally (symmetric) quasi-polarized p-divisible object of $\mathcal{MF}^{\nabla}_{[0,2]}(W(k)[[x_1, ..., x_{20}]]))$ in the desired (predicted) shape. The case of polarized ordinary K3 surfaces of degree relatively prime to p (this excludes entirely the case $p = 2$) can be deduced as well from 4.7.5, by a natural process of moving (i.e lifting things) from the context of $SO(2, 19)$-groups to the context of $\mathrm{Spin}(2, 19)$-groups. A. Ogus pointed out to us that his unpublished work (see [Og2]) on ordinary elliptic curves can also be used to settle the case $p = 2$.

3) Let us look closer at [De2] to understand the real principles underlying the main results of the loc. cit. We see that there are three ideas. We formulate them in the language of this paper.

a) We have a Shimura σ-crystal which admits a canonical split: with the notations of 2) above, this is a cocharacter $\mu : \mathbb{G}_m \to \tilde{G}$ with special properties; in particular it produces a direct sum decomposition $\tilde{M} = \tilde{F}^0 \oplus \tilde{F}^0$, with $\beta \in \mathbb{G}_m(W(k))$ acting through μ on $\tilde{F}^i$ as the multiplication with β^{-i}, $i = \overline{0,1}$ (to be compared with 3.1.4).

b) The resulting Shimura filtered σ-crystal $(\tilde{M}, \tilde{F}^1, \tilde{\varphi}, \tilde{G})$ is (cyclic) diagonalizable.

c) We can build up good additive coordinates. By this we mean [De2, 1.4.2] (see also 5) below).

$3')$ Using the cocharacter μ above, we get a natural $\mathbb{Z}_p$-structure of $(\tilde{M}, \tilde{F}^1, \tilde{\varphi}, \tilde{G})$, as in 4.4.7. So $\tilde{M} = M_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W(k)$, with $M_{\mathbb{Z}_p}$ a free $\mathbb{Z}_p$-module; under this identification, $\tilde{G}$ is obtained by extension of scalars of a reductive subgroup $\tilde{G}_{\mathbb{Z}_p}$ of $GL(M_{\mathbb{Z}_p})$. Moreover μ is defined over a finite unramified DVR extension $W(k_\mu)$ of $\mathbb{Z}_p$. Let T be the smallest subtorus of $\tilde{G}_{\mathbb{Z}_p}$ such that μ factors through $T_{W(k_\mu)}$. Let C be the centralizer of T in $\tilde{G}_{\mathbb{Z}_p}$. It is a reductive group over $\mathbb{Z}_p$.

The principle behind the choice of bases in [De2, 1.4.2] and in 4.7.5 is: we pick up a maximal torus T_C of C^{der}; only the $C(\mathbb{Z}_p)$-conjugacy class of T_C matters. Of course, if C^{der} is split, then the most natural choice for T_C is: a split maximal torus of C^{der}. When C^{der} is not split, still we have quite natural choices for T_C. For instance we can take T_C to contain a maximal split torus of C^{der} and to split over the smallest unramified extension of $\mathbb{Z}_p$ over which C^{der} splits (cf. [Ti2, 1.10]). For the most general applications it is important to take T_C arbitrarily. Let T_C^1 be the maximal torus of $\tilde{G}_{\mathbb{Z}_p}$ generated by T_C and the toric part of $Z(C)$.

4) It is easy to see that choosing a multiplicative type Frobenius of R_n, we can not, for an arbitrary cyclic diagonalizable Shimura σ-crystal, choose g_{R_n} to be the identity element of $\tilde{G}(R_n)$. Even for the case of the p-divisible group of a supersingular elliptic curve over k (so $n = 1$) we can not choose $g_{R_1} = 1$. To get out from this deadlock, we have to either use the additive situation (see 5) below), or work with the multiplicative situation, with a carefully chosen g_{R_n} (see 6) below).

By $\tilde{G}$-canonical (multiplicative) coordinates for $(\tilde{M}, \tilde{\varphi}, \tilde{G})$ we mean that, referring to a-b) of 2), we can choose:

– a lift $\tilde{F}^1$ of it of quasi CM type;

– $n = dd((\tilde{M}, \tilde{\varphi}, \tilde{G}))$ (see 2.2.22 4));

– a Frobenius of R_n of multiplicative type;

– $g_{R_n} = 1$;

– a connection which respects the $\tilde{G}$-action and of whose Kodaira-Spencer map modulo the maximal ideal of R_n is injective.

From the proof of 3.4.18.4 B) we get (cf. also 2) above) directly:

Fact. *A cyclic diagonalizable Shimura σ-crystal $(\tilde{M}, \tilde{\varphi}, \tilde{G})$ has $\tilde{G}$-canonical (multiplicative) coordinates iff its Shimura adjoint Lie σ-crystal is ordinary, with the multiplicity of the slope -1 being maximal, i.e. equal to $dd((\tilde{M}, \tilde{\varphi}, \tilde{G}))$.*

The above iff part can be reformulated: iff it is a $\tilde{G}$-ordinary σ-crystal and its A-degree of definition (cf. 3.11.3) is 1. In particular we reobtain, in a much faster manner, the results of 4.7.19-20.

5) Let $\tilde{N}$ be the (abelian) subgroup of $\tilde{G}$ acting trivially on $\tilde{F}^0$ and on $\tilde{M}/\tilde{F}^0$. We identify R_n with the completion of $\tilde{N}$ in the origin (obviously $\dim(\tilde{N}) = dd((\tilde{M}, \tilde{\varphi}, \tilde{G}))$).

$\tilde{N}$ has a natural $W(k_\mu)$-structure; let $\tilde{N}_{W(k_\mu)}$ be the subgroup of $\tilde{G}_{W(k_\mu)}$ which over $W(k)$ is $\tilde{N}$. If the Frobenius Φ_{R_n} is of additive type, then we speak about additive coordinates. Due to condition on the Kodaira-Spencer map, we do not have too many choices: the p-divisible object with cycles $\mathcal{C}_q$ of $\mathcal{MF}^{\nabla}_{[0,1]}(R_n)$ is a Shimura filtered F-crystal of the form $(\tilde{M}, \tilde{F}^1, \tilde{\varphi}, \tilde{G}, \tilde{N}, \tilde{f})$; so the only choice we have is in choosing (the isomorphism) $\tilde{f}$ (of R_n). The choice of $\tilde{f}$ corresponds to a choice of an additive type Frobenius of R_n. [Fa2, rm. iii) after th. 10] tells us that it does not matter which choice we make. We propose one such choice, which to us looks convenient (at least from the point of view of 8) below).

T^1_C might not be split. It splits over an unramified finite DVR extension $W(k_1)$ of $\mathbb{Z}_p$. We choose the smallest such extension. Let $\mathbb{G}_a(i)$, $i = \overline{1, \dim(\tilde{N})}$, be the different subgroups of $\tilde{N}_{W(k_1)}$, whose Lie algebras are left invariant by T^1_C. We feel inclined of choosing Φ_{R_n} such that it leaves invariant the subring $R(i) = W(k_1)[[z_i]]$ of R_n defined by the projection (it is a group homomorphism) of $\tilde{N}_{W(k_1)}$ on $\mathbb{G}_a(i)$ having $\prod_{j\in\{1,\dots,\dim(\tilde{N})\}\setminus\{i\}} \mathbb{G}_a(j)$ as its kernel; of course $\Phi_{R_n}(z_i) = z_i^p$, $i = \overline{1, \dim(\tilde{N})}$. We can use T^1_C to determine uniquely z_n, through different considerations, but this (presently) looks irrelevant to us.

6) We move now to the case when Φ_{R_n} is of multiplicative type. A natural question arises: What happens if the conditions of the fact of 4) above are not satisfied? Is there a best choice of g_{R_n} in such a case? We propose here one good choice of g_{R_n} working out in all cases (i.e. in the case of the above mentioned fact, g_{R_n} becomes the identity). Let $g^1_{R_n} \in \tilde{N}(R_n)$ be the universal element; so

$$g^1_{R_n} := 1_{\tilde{M}\otimes_{W(k)} R_n} + \sum_{i=1}^{\dim(\tilde{N})} a_i z_i,$$

where $a_i \in \mathrm{Lie}(\mathbb{G}_a(i))$ generates this Lie algebra. Let

$$h_{R_n} := 1_{\tilde{M}\otimes_{W(k)} R_n} + \sum_{i=1}^{\dim(\tilde{N})} \tilde{\varphi}(pa_i) z_i \in \tilde{G}(R_n).$$

We take

$$g_{R_n} := g^1_{R_n} h^{-1}_{R_n}.$$

It is not difficult to check that for such a g_{R_n} we can choose a connection on $\tilde{M} \otimes_{W(k)} R_n$ whose Kodaira-Spencer map is an injection, having $\mathrm{Lie}(\tilde{N})$ as its image (cf. the proof of 3.4.18.4 B)). We usually choose a_i such that $\tilde{\varphi}^{j_i}(a_i)$ is a p-power multiple of a_i, j_i being the smallest positive integer such that $\tilde{\varphi}^{j_i}(\mathrm{Lie}(\mathbb{G}_a(i)) \subset \mathrm{Lie}(\mathbb{G}_a(i))[\frac{1}{p}]$. So in the case of the fact of 4) above, we can choose a_i such that $\tilde{\varphi}(pa_i) = a_i$; for such a choice we get $g_{R_n} = 1$ (as $\tilde{N}$ is an abelian group). In the general case it is still arguable which is the best choice for a_i and for z_i. Of course a_i and z_i are interrelated. We leave this problem of the best choice to get matured on its due time.

7) The ideas of 3.4 (for instance see 3.4.1.3 and 3.4.8.1.2) suggest that rings of the form $W(k)[[z_1, ..., z_m]]$ are not always the right rings of building up crystalline coordinates. Often they should be replaced by the p-adic completion of ind-étale affine algebras over smooth $W(k)$-algebras. For instance we can work with the p-adic completion of the henselization of the localization of $W(k)[z_1, ..., z_m]$ with respect to its maximal ideal $(p, z_1, ..., z_m)$. But this rings are still "loosing" a lot of information. Very often the rings we get have plenty of k-valued points. For instance, working with Shimura σ-crystals which are ordinary, using the fact of 3.4.18.9, we get plenty of rings whose spectra are the p-adic completion of $\mathbb{N}$-pro-étale covers of the p-adic completion of affine open subschemes of $\mathrm{Spec}(W(k)[z_1, ..., z_m])$ (like the one where we make all z_i to be invertible), such that we have versal deformations of Shimura p-divisible groups over them, in a "prescribed way" (see 3.4.1.3 and 3.4.11). This opens a completely new horizon: variation of crystalline coordinates in families.

When we are in the context of a principally polarized abelian variety, we do not know for which "prescribed ways" we still get principally polarized abelian schemes over the spectra of these rings; we have variants when we have polarizations which are not necessarily principal. Of course we do need a polarization in order to (hope to) get abelian schemes over such spectra.

We do expect that the study of the variation of coordinates leads to nice modular properties. We hope to come back to this idea in a future paper.

8) In 3-6) we had some choices: of μ (and so of $\tilde{F}^1$ and of $\tilde{N}$), of T_C, of z_i, and of a_i, $i = \overline{1, \dim(\tilde{N})}$. We do not stop to see if we can make all these choices to result in something canonical. However, the $\tilde{G}$-canonical multiplicative coordinates of the fact of 4) above, are canonical (this can be checked in the same manner as in the case of an ordinary p-divisible group: there is a unique multiplicative type Frobenius of R_n for which g_{R_n} is the identity element); this motivates the terminology: G-canonical.

So 7) above gives us natural good structures of a formal torus of the formal moduli scheme $\mathsf{M}(\tilde{M}, \tilde{\varphi}, \tilde{G})$ of deformations of $(\tilde{M}, \tilde{\varphi}, \tilde{G})$ (see the proof of 3.11.1): we do not stop to see if there is a unique group structure on $\mathsf{M}(\tilde{M}, \tilde{\varphi}, \tilde{G})$ (the origin being fixed; it is defined by the F^1-filtration $\tilde{F}^1$ of $\tilde{M}$).

We assume now that there is a reductive subgroup $\tilde{G}_0$ of $\tilde{G}$ such that $(\tilde{M}, \tilde{\varphi}, \tilde{G}_0)$ is also a Shimura σ-crystal. It is not difficult to see that we can make all the choices, so that, the formal moduli scheme of deformations of $(\tilde{M}, \tilde{\varphi}, \tilde{G}_0)$ is a subtorus of the formal torus (scheme) of deformations of $(\tilde{M}, \tilde{\varphi}, \tilde{G}_0)$.

9) We come back to the general case of 3). Let r be the Lie p-rank of $(\tilde{M}, \tilde{\varphi}, \tilde{G})$. We have $r \leq dd((\tilde{M}, \tilde{\varphi}, \tilde{G}))$. If equality holds then we are in the situation of the above fact. We assume now that $1 \leq r < dd((\tilde{M}, \tilde{\varphi}, \tilde{G}))$. In this case there is a quotient $R_n \twoheadrightarrow R_r := W(k)[[x_1, ..., x_r]]$ such that, in the sense of 4) above we can take (the same argument applies) g_{R_r} to be 1. So, the choices of 8) above can be made, so that we get a canonical subtorus of $\mathsf{M}(\tilde{M}, \tilde{\varphi}, \tilde{G})$ of dimension r.

10) Of course all the above 9) points can be entirely adapted (limitations do apply) to the case when k is just perfect.

4.7.31. Definition. We come back to the situation described in 4.7.5. The lifts of y (to DVR which are finite extensions of $W(k)$) corresponding to torsion points of the formal torus describing the G-deformations of A_y, are called G-quasi-canonical lifts of y.

4.7.32. Exercise. We assume 4.4.6. If $k = \mathbb{F}$, prove that the G-quasi-canonical lifts of a G-ordinary point $y : \mathrm{Spec}(k) \to \mathcal{U}$ are giving birth to special points of $\mathcal{N}$ (i.e. the abelian varieties obtained from $\mathcal{A}$ by pull back through these G-quasi-canonical lifts have complex multiplication). Show that they are the only lifts of y to DVR's of mixed characteristic, which are special points of $\mathcal{N}$ (Hint: if $k(v) = \mathbb{F}_p$, this is well known; if $k(v) \neq \mathbb{F}_p$ then use 4.4.6, 4.7.30 1) and just imitate the argument of the case $k(v) = \mathbb{F}_p$).

4.7.33. Remark. We do not assume anymore that $N = N_0$. From 4.7.30 3-6) we deduce that for any toric point $y_1 : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ there are good G-multiplicative coordinates of the formal moduli scheme of G-deformations of $(A_{y_1}, p_{A_{y_1}}) := \mathrm{Spec}(k)_{y_1} \times_{\mathcal{N}} (\mathcal{A}, \mathcal{P}_{\mathcal{A}})$. The same thing remains valid if we do not work in the Shimura (i.e. having a principal polarization, etc.) context. In particular:

a) For any point $y_1 : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ whose attached Shimura adjoint Lie σ-crystal is isomorphic to the extension to k of the Shimura adjoint Lie σ-crystal of $\mathfrak{C}_\omega$ of 4.1.5 (i.e. for any $G(\omega)$-ordinary point y_1), we get good $G(\omega)$-multiplicative coordinates describing the formal moduli scheme of G-deformations of $(A_{y_1}, p_{A_{y_1}})$.

b) For any point $y : \mathrm{Spec}(k) \to \mathcal{U}$, we get good GSp-multiplicative coordinates (resp. good multiplicative coordinates) describing the formal moduli scheme of deformations of the principally polarized abelian variety (A_y, p_{A_y}) (resp. of deformations of A_y).

c) For any supersingular elliptic curve A_{y_1} over k (so we reobtain an unpublished result of A. Ogus), we get a good multiplicative coordinate describing its formal moduli scheme of deformations.

d) For any abelian variety (over an algebraically closed field of characteristic $p > 2$), which is a product of abelian varieties A_{y_1} as in a) to c) above, we get good multiplicative coordinates describing its formal moduli scheme of deformations.

More generally we have (for instance using multiplicative coordinates):

4.7.33.1. Proposition. *If A is an abelian variety over k of dimension d, and if its attached σ-crystal has a lift (over $W(k)$) of quasi CM type, then the formal moduli scheme of deformations of A is naturally isomorphic to the formal torus $\mathbb{G}_m^{d^2}{}^\wedge/W(k)$, with the abelian variety defined by the lift of pseudo CM type as its origin.*

Proof: This is a direct consequence of 2.2.19 and of 4.7.33.

4.7.33.2. Remarks. 4.7.33.1 is the most general context in which presently we can prove the existence of useful (crystalline) (additive or multiplicative) coordinates. From 2.2.18 we deduce easily that it does not handle the case of all abelian varieties over k. In the cases left, the hopes of building up any theory of multiplicative coordinates which has any significance at all (i.e. it is useful and motivated), are based on 3.13. So the problem 13 from [Oo1, p. 13] is still open for the cases left. We leave it to the reader to formulate 4.7.33.1 in the relative context (of abelian varieties over k whose attached σ-crystals have the extra structure of a cyclic diagonalizable Shimura σ-crystals) (cf. 4.7.30 5)).

4.7.33.3. In what follows we refer to 4.7.33 a). For any good $G(\omega)$-multiplicative coordinates, we can define $G(\omega)$-quasi-canonical lifts of y_1. It looks to us interesting to study them. In particular, if y_1 is a G-ordinary point, and if $N \neq N_0$, we do not know if these G-quasi-canonical lifts are uniquely determined (we had some choices in 4.7.30), and if the facts of 4.7.32 remain true.

4.7.34. Exercise. Coming back to 4.7.3, prove that $\forall n \in \mathbb{N}$, ${}_N\nabla_{\hat{y}}$ mod p^n is algebraic, i.e. the functions u_j, $j = \overline{1, e_0}$, are algebraic over $N_{W_n(k)}$ (Hint: use 4.4.7 to reduce to the case $k = \overline{k(v)}$; then use the fact that everything, like the bases $(a_i)_{1\le i\le e}$ and $(b_i)_{1\le i\le e}$, μ_y, etc., are definable over a finite extension of $k(v)$; another way: cf. 3.4.1.3).

4.7.35. Exercise. Show that all of [No2, section 2] remains valid in the context of G-ordinary points (Hint: if $N = N_0$ use 4.7.36 below; if $N \neq N_0$ use 4.7.30 6) and work formally).

4.7.36. Remark. Every formal subscheme of $\mathcal{M}_G(y)$ of 4.7.21 which can be "defined" starting from a smooth subgroup N_1 of N whose Lie-algebra is left invariant by φ_y is a formal subtorus. This can be checked using 4.7.5-29 or 4.7.30 8). The same thing remains true in the general context of 4.7.30 2). Here, for the sake of future reference, we state this principle explicitly, in the context of Shimura σ-crystals. With the notations of 4.7.30 2) to 4), we have:

Fact. *If there is a reductive subgroup $\tilde{G}_1$ of $\tilde{G}$ such that $(\tilde{M}, \tilde{\varphi}, \tilde{G}_1)$ is also a Shimura σ-crystal, and if the filtration defining its $\tilde{G}_1$-canonical lift is the same as the one defining the $\tilde{G}$-canonical lift of $(\tilde{M}, \tilde{\varphi}, \tilde{G})$, then the formal torus of dimension $n_1 := dd((\tilde{M}, \tilde{\varphi}, \tilde{G}_1))$ of deformations of $(\tilde{M}, \tilde{\varphi}, \tilde{G}_1)$ is canonically a formal subtorus of the formal torus of deformations of $(\tilde{M}, \tilde{\varphi}, \tilde{G})$.*

Proof: From [Fa2, th.10 and the remarks after] and 4.7.30 2) we deduce that it is enough to show that there is a filtered F-crystal over the special fibre $\mathrm{Spec}(k[[x_1, ..., x_n]])$ of $\mathrm{Spec}(R_n)$, with n as in 4.7.30 4), such that:

– it is described by a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R_n)$ of the form $(\tilde{M} \otimes_{W(k)} R_n, \tilde{F}^1 \otimes_{W(k)} R_n, \tilde{\varphi} \otimes 1, \tilde{\nabla})$, with $\tilde{\nabla}$ as in 4.7.30 4);

– the Frobenius of R_n is compatible with σ and takes $x_i - 1$ to $(x_i - 1)^p$;

– $\tilde{\nabla}$ modulo the ideal of R_n generated by $x_{n_1+1}, ..., x_n$ is $\tilde{G}_1$-invariant.

The construction of such a filtered F-crystal is trivial (cf. 4.7.30 1) to 4) and 3.4.18.2).

4.7.37. Problem. Obtain the general form of the main result of [No1] and of [No2] in the context of G-ordinary points (Hint: if $N = N_0$ use 4.7.5-29 and 4.7.32, and just imitate the arguments of [No1-2]; if $N \neq N_0$ then use 4.7.30).

4.7.38. Problem. State a theorem similar to 4.7.5 in the general case, i.e. when $k(v^{\mathrm{ad}})$ is not $\mathbb{F}_p$. Hint: use 4.7.4.

4.8. The Galois property of G-ordinary points.

4.8.1. We continue to assume that $k = \bar{k}$. We start with a morphism $y : \mathrm{Spec}(k) \to \mathcal{U}/H_0$. Let $\mathrm{Spec}(V) \overset{z_V}{\to} \mathcal{N}/H_0$ be a lift of y, with V a DVR which is a finite flat extension of $W(k)$. Let $K := V[\frac{1}{p}]$ and let (A_K, p_{A_K}) be the principally polarized abelian variety over K obtained from $(\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ by pull back through z_V. Let

$$\rho : \mathrm{Gal}(\bar{K}/K) \to GL\big(H^1_{\acute{e}t}(A_{\bar{K}}, \mathbb{Q}_p)\big)(\mathbb{Q}_p)$$

be the p-adic Galois representation associated to A_K. Let $G_{\mathbb{Q}_p} \hookrightarrow GSp\big(H^1_{\acute{e}t}(A_{\bar{K}}, \mathbb{Q}_p), p_{A_K}\big)$ be the subgroup fixing the p-components of the étale components of the family of Hodge cycles $(t_\alpha)_{\alpha \in \mathcal{J}'}$ with which $A_{\bar{K}}$ is naturally endowed.

4.8.2. Theorem. *There is a finite extension K_1 of K such that the restriction $\rho_1 := \rho|\mathrm{Gal}(\bar{K}/K_1)$ factors through the group of $\mathbb{Q}_p$-valued points of a connected solvable subgroup of $G_{\mathbb{Q}_p}$.*

Proof: We can assume that K_1 is big enough so that the family $(t_\alpha)_{\alpha \in \mathcal{J}'}$ is defined over K_1. In fact, as we have assumed $k = \bar{k}$, for this part we can take $K_1 = K$. So ρ_1 factors through $G_{\mathbb{Q}_p}$. As $k = \bar{k}$ we have:

i) the residue field of the ring of integers of K_1 is k;

ii) there is a morphism $z : \mathrm{Spec}(W(k)) \to \mathcal{N}/H_0$ lifting y, and such that the Shimura filtered σ-crystal attached to it is a $G_{W(k)}$-canonical lift;

iii) $G_{W(k)}$ splits (so 4.7.1-3 makes sense).

In fact the condition ii) is satisfied for any k (cf. 4.4.3).

Let $(M_y, F^1_y, \varphi_y, G_{W(k)})$ be the Shimura filtered σ-crystal attached to z. Let $\mu_y : \mathbb{G}_m \to GL(M_y)$ be its canonical split, cf. 3.1.4. Let $\bar{h}_i$, $i = \overline{1,d}$, be the elements of $\mathrm{End}(M_y)$ defined as in 4.1 but working with μ_y.

Let N_1 be the smallest subgroup of $G_{W(k)}$ such that $\mathrm{Lie}(N_1)$ contains $\mathrm{Lie}(N)$ (N was def. in 4.7.1), and the Lie algebra generated by $\bar{h}_i$, $i = \overline{1,d}$. N_1 is a connected solvable subgroup of $G_{W(k)}$. Let B_{N_1} be a Borel subgroup of $G_{W(k)}$ containing N_1 and such that its Lie algebra is left invariant by $p\varphi_y$. The existence of such Borel subgroups is a consequence of b) 4.4.1 3) (to be compared with 3.11.1 c) and 3.7.2). Let $\tilde{N}_1$ be the connected component of the origin of the intersection of all such Borel subgroups B_{N_1} of $G_{W(k)}$ containing N_1. Now $\mathrm{Lie}(N_1)$, $\mathrm{Lie}(\tilde{N}_1)$ and $\mathrm{Lie}(B_{N_1})$ are left invariant by $p\varphi_y$.

With the notations of 4.7.1 we get that $\mathrm{Lie}(N_1) \otimes_{W(k)} R$ and $\mathrm{Lie}(B_{N_1}) \otimes_{W(k)} R$ are left invariant by $n(\varphi_y \otimes 1)$. So for the morphism $\mathrm{Spec}(V_1) \hookrightarrow \mathrm{Spec}(R)$ (with V_1 the ring of integers of K_1), corresponding to A_{V_1} (cf. 4.7.2), we get that the Galois representation ρ_1 leaves invariant $\mathrm{Lie}(N_2)$, $\mathrm{Lie}(\tilde{N}_2)$ and $\mathrm{Lie}(B_{N_2})$, with $N_2 \subset \tilde{N}_2 \subset B_{N_2} \subset G_{\mathbb{Q}_p}$ such that B_{N_2} is a Borel subgroup of $G_{\mathbb{Q}_p}$, and with N_2 and $\tilde{N}_2$ as solvable groups (N_2, $\tilde{N}_2$ and B_{N_2} are respectively the $\mathbb{Q}_p$-étale correspondents of the crystalline ones N_1, $\tilde{N}_1$ and B_{N_1}) . But as B_{N_2} is its own normalizer in $G_{\mathbb{Q}_p}$, we do get that ρ_1 factors through B_{N_2}. Repeating the argument for any Borel subgroup of $G_{W(k)}$ as above, we do get that, by extending again K_1 by a finite cover (as this intersection of all such Borel subgroups B_{N_1} might not be connected), ρ_1 factors through the group of $\mathbb{Q}_p$-valued points of $\tilde{N}_2$. This proves the theorem.

4.8.3. Remarks. **a)** 4.8.2 solves a question of C.-L. Chai. We would like to thank R. Pink who pointed out to us the construction of N_1 (of the proof), provided 4.7.2 and c) of 4.4.1 2) were known. We do not know when $\mathrm{Lie}(\tilde{N}_1) = \mathrm{Lie}(N_1) + \mathrm{Lie}(Z(G_{W(k)}))$. We just remark that the unipotent radical of $\tilde{N}_1$ is the unipotent radical of N_1.

b) Using [De3, 3.1 and 3.2 a)] it can be checked out easily that we do not need to use $\tilde{N}_1$, and so that ρ_1 factors through $N_2(\mathbb{Q}_p)$, without having to replace K by K_1.

c) There are plenty of examples in which $\tilde{N}_1$ is a Borel subgroup of $G_{W(k)}$ (for instance if the parabolic Lie subalgebra $\mathfrak{p}_0$ of 4.6.4 is of Borel type, see P7 of 4.6; if moreover $G^{\mathrm{ad}}_{\mathbb{Z}_p}$ is a simple group then $N_1 \cap G^{\mathrm{der}}_{W(k)}$ is a Borel subgroup $G^{\mathrm{der}}_{W(k)}$) and plenty of examples when it is not (for instance if $G^{\mathrm{ad}}_{\mathbb{Z}_p}$ is a simple split group of C_ℓ Lie type, with $\ell \geq 2$).

d) A similar result as 4.8.1 can be proved for an arbitrary toric point $\mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$. The only difference is: we need to state explicitly that $\mathrm{Lie}(N_1)$ has to be also $p\varphi_y$-invariant. Moreover, if we still want $\tilde{N}_1$, then we have to define it using parabolic subgroups containing N_1. Warning: N_1 is not necessarily a solvable group. There are plenty of cases when its image in $G^{\mathrm{ad}}_{W(k)}$ is a parabolic subgroup of $G^{\mathrm{ad}}_{W(k)}$ which is not a Borel subgroup.

e) 4.8.2 and d) above are properties of Shimura-ordinary σ-crystals (instead of 4.7.2 we have to use 3.11.1). Moreover the condition $k = \bar{k}$ can be weaken: we just need the existence of a Borel subgroup B_{N_1} of $G_{W(k)}$ of the type mentioned above; but in this case we can not (in general) replace $\tilde{N}_2$ by N_2.

We assume now that k is a finite field, that we are in the context of 4.8.1, and that 4.4.6 is satisfied. Then, up to a passage to a finite field extension, the existence of such a Borel subgroup B_{N_1} is automatic. This can be very easily checked starting from 4.4.6, 3.3.4 and the proof of the Claim of the proof of 4.4.1. In this way we obtain plenty of quasi-ordinary crystalline representations, in the sense of [Pi, 2.10].

f) We stated this section 4.8 in terms of $\mathbb{Q}_p$-valued points; but obviously everything can be restated in terms of $\mathbb{Z}_p$-valued points.

4.9. The passage from Shimura varieties of Hodge type to Shimura varieties of preabelian type.

4.9.1. We start with an arbitrary Shimura variety $\mathrm{Sh}(G_0, X_0)$ of adjoint abelian type. Let (G_0, X_0, H_0, v_0) be a Shimura quadruple with v_0 dividing a rational prime $p \geq 3$. We fix a connected component X_0^0 of X_0 (it does not matter which one, cf. [Va2, 3.3.3]). Let (G_i, X_i, H_i, v_i), $i = \overline{1,3}$, be Shimura quadruples having (G_0, X_0, H_0, v_0) as their adjoint quadruples, and such that $X_0^0 \subset X_i$ (cf. [Va2, 2.4]). We assume the existence of two standard Hodge situations $(f_i, L_{i(p)}, v_i)$ defined by $f_i : (G_i, X_i) \hookrightarrow \big(\mathrm{GSp}(W_i, \psi_i), S_i\big)$, $i = \overline{1,2}$, such that $H_i = G_i(L_{i(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p)$ (for $p \geq 5$ cf. [Va2, 6.4.2]; for $p = 3$ cf. §6]). For the two standard Hodge situations considered we use the standard notations of 2.3.1-3, except that we put a right lower index i everywhere, $i = \overline{1,2}$.

We denote by $\mathcal{N}_0$ and $\mathcal{N}_3$ the integral canonical models of (G_0, X_0, H_0, v_0) and respectively of (G_3, X_3, H_3, v_3). We get natural morphisms $\mathcal{N}_i \xrightarrow{j_i} \mathcal{N}_0$ (cf. [Va2, rm. 4) of 3.2.7]), $i = \overline{1,3}$.

4.9.1.1. Let $V_0 := W(\mathbb{F})$ and let $\mathfrak{C}_0$ be the open closed subscheme of $\mathcal{N}_{0V_0}$ defined as the intersection of the images (through j_3) of $\mathcal{N}_{3V_0}$ into $\mathcal{N}_{0V_0}$ associated to all finite maps $i_3 : (G_3, X_3, H_3, v_3) \to (G_0, X_0, H_0, v_0)$ such that $X_0^0 \subset X_3$. The fact that $\mathfrak{C}_0$ is non-empty is implied by [Va2, rm. 3) of 3.2.7] and by the fact that $\mathfrak{C}_0 \times_{V_0\, j} \mathbb{C}$ is non-empty (with $j : V_0 \hookrightarrow \mathbb{C}$ an embedding such that restricted to $O_{(v)}$ is the composite inclusion $O_{(v)} \subset E(G, X) \subset \mathbb{C}\big)$. The fact that $\mathfrak{C}_0$ is open closed is implied for $p \geq 5$ by [Va2, 6.4.5.1] (for $p = 3$ cf. §6). Similarly we define $\mathfrak{C}_i \hookrightarrow \mathcal{N}_{iV_0}$, $i = \overline{1,3}$ (by working with the connected component X_0^0 of X_i).

4.9.2. Let (G_4, X_4, H_4, v_4) be a Shimura quadruple which together with maps $h_i : (G_4, X_4, H_4, v_4) \to (G_i, X_i, H_i, v_i)$, $i = \overline{1,2}$, is a quasi fibre product of the finite maps q_1 and q_2, i.e. we have a commutative diagram

$$\begin{array}{ccc} (G_4, X_4, H_4, v_4) & \xrightarrow{h_1} & (G_1, X_1, H_1, v_1) \\ \Big\downarrow{\scriptstyle h_2} & & \Big\downarrow{\scriptstyle q_1} \\ (G_2, X_2, H_2, v_2) & \xrightarrow{q_2} & (G_0, X_0, H_0, v) \times \big(\mathbb{G}_m, \{x_0\}, \mathbb{G}_m(\mathbb{Z}_p), p\big) \end{array}$$

in the category $qf - Sh$ defined in [Va2, rm. 3) of 3.2.7], satisfying a quasi universal property as in [Va2, 2.4.0]. Here q_i is defined by the canonical map $(G_i, X_i, H_i, v_i) \to (G_0, X_0, H_0, v_0)$ and by the canonical map $h_i : (G_i, X_i) \to (GSp^{\mathrm{ab}}, S^{\mathrm{ab}})$ defined by f_i, $i = \overline{1,2}$, where $\{x_0\} := S^{\mathrm{ab}}$ and $GSp^{\mathrm{ab}} = \mathbb{G}_m$. We assume that $X_0^0 \subset X_4$, cf. loc. cit. and [Va2, 2.4.0]. We call (G_4, X_4) as a Hodge twist of (G_1, X_1) and (G_2, X_2) and denote it by $(G_1, X_1)\tilde{\times}(G_2, X_2)$.

We consider a Hodge quasi product $(G_1 \times_{\mathcal{H}} G_2, X_1 \times_{\mathcal{H}} X_2)$ of (G_1, X_1) and (G_2, X_2), and a Hodge quasi product $(\mathrm{GSp}(W_1, \psi_1) \times_{\mathcal{H}} \mathrm{GSp}(W_2, \psi_2), S_1 \times_{\mathcal{H}} S_2)$ of $(\mathrm{GSp}(W_i, \psi_1), S_1)$ and $(\mathrm{GSp}(W_2, \psi_2), S_2)$, cf. [Va2, Example 3 of 2.5]. We use a similar notation for quadruples. We can assume that $X_4 \subset X_1 \times_{\mathcal{H}} X_2 \subset S_1 \times_{\mathcal{H}} S_2$. So we get a SHS $(f_4, L_{1(p)} \oplus L_{2(p)}, v_4)$ defined by

$$f_4 : (G_4, X_4) \overset{\tilde{i}_4}{\hookrightarrow} \big(\mathrm{GSp}(W_1, \psi_1) \times_H \mathrm{GSp}(W_2, \psi_2), S_1 \times_H S_2\big) \overset{i}{\hookrightarrow} (\mathrm{GSp}(W_1 \oplus W_2, \psi_1 \oplus \psi_2), S),$$

where $\tilde{i}_4 := (f_1 \times_{\mathcal{H}} f_2) \circ i_4$, with $i_4 : (G_4, X_4) \hookrightarrow (G_1 \times_H G_2, X_1 \times_H X_2)$ and

$$f_1 \times_{\mathcal{H}} f_2 : (G_1 \times_{\mathcal{H}} G_2, X_1 \times_{\mathcal{H}} X_2) \hookrightarrow (\mathrm{GSp}(W_1, \psi_1) \times_{\mathcal{H}} \mathrm{GSp}(W_2, \psi_2), S_1 \times_{\mathcal{H}} S_2)$$

as the natural inclusions, and where i is the Segre embedding, as defined in [Va2, Example 3 of 2.5].

Let $\mathcal{N}_4$ be the integral canonical model of (G_4, X_4, H_4, v_4), and let $\mathfrak{C}_4 \subset \mathcal{N}_{4V_0}$ be defined similarly as $\mathfrak{C}_0$.

The fact that the triple $(f_4, L_{1(p)} \oplus L_{2(p)}, v_4)$ is indeed a SHS, is related to the proof of Milne's conjecture (see 1.16, d) of 4.4.1 3) and 4.4.12). Let $\mathcal{M}_4$ be the integral canonical model associated to the SHS $(1_{(\mathrm{GSp}(W_1 \oplus W_2), \psi_1 \oplus \psi_2), S)}, L_{1(p)} \oplus L_{2(p)}, p)$. From [Va2, rm. 4) of 3.2.7] we get a natural morphism $\tilde{m} : \mathcal{N}_4 \to \mathcal{M}_4$. We need to show that for any morphism $z : \mathrm{Spec}(W(k)) \to \mathcal{N}_4$, working with $\tilde{m}$, the condition $(*)$ of 2.3.4 is satisfied. The idea behind checking this is rooted in [BLR, th. 1 of p. 109] and the density part of 4.2.1.

To explain it, let us mention that 2.3.12-13 makes sense even without knowing that we are in a reductive context: using canonical splits (see [Wi]) of filtered σ-crystals endowed with cycles, we can perform [Va2, 5.4-5] directly in the context of abelian unipotent groups; in other words, groups like N in 4.7.1 are always smooth, even if by chance $G_{W(k)}$ is not a subgroup of a GL-group involved. We deduce that it is enough to check condition $(*)$ of 2.3.4 for a point z lifting a G_4-ordinary point of $\mathcal{N}_4$ (the set of such points is the same, regardless of which morphism $\mathcal{N}_4 \to \mathcal{N}_1$ or $\mathcal{N}_4 \to \mathcal{N}_2$ we use to define them -by being mapped into G_i-ordinary points of $\mathcal{N}_i$, $i \in \{1,2\}$-: the proof of 4.9.4 below, at the level of isocrystals, applies). If we desire to simplify the things, we can define such G_4-ordinary points, as points mapping into G_i-ordinary points of $\mathcal{N}_i$, $\forall i \in \{1,2\}$ (sort of intersection: [Va2, 6.4.5.1] and 4.2.1 guarantee that this intersection is an open dense subscheme of $\mathcal{N}_{k(v_4)}$). Moreover we can assume that z maps into a G_i-canonical lift of $\mathcal{N}_i$, $\forall i \in \{1,2\}$ (cf. 4.9.4, performed with filtered isocrystals over k; b) of 4.4.1 2) implies that a Shimura-canonical lift can be read out from the attached Shimura adjoint filtered Lie isocrystal). This implies that the Galois representation defined by z (working with $\tilde{m}$) factors through the $\mathbb{Z}_p$-valued points of a torus of $G_{\mathbb{Z}_p}$ (see also 4.5.12.2.1). It is in this context where we can prove directly Milne's conjecture referred in 1.16 (for details see §5). So condition $(*)$ of 2.3.4 is satisfied for z.

As we did not include a proof of d) of 4.4.1 3), here we present an alternative (independent) proof of why $(*)$ of 2.3.4 is satisfied for z. We consider the Lie filtered isocrystal $LI \otimes_{W(k)} B(k)$ associated to z. It is a Lie filtered subisocrystal of the direct sum of $(LI_1 \oplus LI_2) \otimes_{W(k)} B(k)$, where LI_i is the Lie filtered σ-crystal associated to the morphism $\mathrm{Spec}(W(k)) \to \mathcal{N}_i$ defined by z ($i \in \{1,2\}$). We just need to show that $LI := LI \otimes_{W(k)} B(k) \cap (LI_1 \oplus LI_2)$ is the Lie algebra of a reductive subgroup of $G_{1V_0} \times G_{2V_0}$. [Va2, 4.3.13] takes care of the toric part, while [Va2, 3.1.6] allows us to concentrate just on the "der" parts, and so on the adjoint parts. But for adjoint groups everything results from 4.2.3-5, once we remark that $G^{\mathrm{ad}}_{i\mathbb{Z}_{(p)}}$ does not depend on $i \in \{1,2,4\}$. In other words, the Galois representation

$$\mathrm{Gal}(\overline{B(k)}/B(k)) \to G^{\mathrm{ad}}_{4\mathbb{Z}_p}(\mathbb{Z}_p) \hookrightarrow \mathrm{Aut}(\mathrm{Lie}(G^{\mathrm{ad}}_{4\mathbb{Z}_p}))(\mathbb{Z}_p)$$

defined naturally by the $W(k)$-valued point of $\mathcal{M}_1 \times \mathcal{M}_2$ obtained naturally from z (we recall that, cf. 2.3.7, $\mathcal{M}_i$ is the integral canonical model of the SHS $(1_{\mathrm{GSp}(W_i,\psi_i),S_i}, L_{i(p)}, p))$, defines, via Fontaine's comparison theory, an adjoint Lie isocrystal

$$(\mathrm{LIE}, \tilde{\varphi}) := (\mathrm{Lie}(G^{\mathrm{ad}}_{4B(k)}), \varphi_{\mathrm{ad}})$$

over k. Moreover, using Fontaine's comparison theory for the $W(k)$-valued point of $\mathcal{N}_i$ defined by z, we deduce the existence of a (natural) $W(k)$-lattice LIE^i_W of LIE, such that $(\mathrm{LIE}^i_W, \tilde{\varphi})$ is a Shimura adjoint Lie σ-crystal, $i = \overline{1,2}$. But $\mathrm{LIE}^1_W = \mathrm{LIE}^2_W$.

This last fact, for $p \geq 5$ is a consequence of [Fa2, th. 7 of §6], while for $p = 3$ is a consequence of loc. cit. and of the parabolic type property of 4.4.1 1) and of a) of 4.4.1

2). So, if $B^+(V)$ and β have the same significance as in [Fa2, §6], for $V := W(k)$, then we have (cf. loc. cit.) regardless of p the following inclusions of $B^+(V)$-modules:

$$\mathrm{LIE}^i_W \otimes_{W(k)} \beta^2 B^+(V) \subset \mathrm{Lie}(G^{\mathrm{ad}}_{4\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} B^+(V) \subset \mathrm{LIE}^i_W \otimes_{W(k)} \beta^{-2} B^+(V).$$

The part of 4.4.1 mentioned above implies directly that we can replace β^2 and β^{-2} by β and respectively by β^{-1}; so the things are fine for $p = 3$ as well.

4.9.2.1. For the SHS $(f_4, L_{1(p)} \oplus L_{2(p)}, v_4)$ we are going to use the standard notations of 2.3.1-3, except that we put a right lower index 4 everywhere. We can assume that $H_4 = G_4(\mathbb{Q}_p) \cap \mathrm{GSp}(W_1 \oplus W_2, \psi_1 \oplus \psi_2)\big((L_{1(p)} \oplus L_{2(p)}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p\big)$.

We (naturally) get a commutative diagram of pro-étale covers:

$$\begin{array}{ccc} \mathfrak{C}_4 & \xrightarrow{m_2} & \mathfrak{C}_1 \\ \big\downarrow{\scriptstyle m_1} & & \big\downarrow{\scriptstyle n_1} \\ \mathfrak{C}_2 & \xrightarrow{n_2} & \mathfrak{C}_0 \end{array}$$

(cf. the definition of $\mathfrak{C}_i$, $i \in \{1, 2, 4\}$, and [Va2, rm. 4) of 3.2.7 and 6.4.5.1]).

4.9.3. Let $z_0 \in \mathfrak{C}_0(V_0)$ be a lift of $y_0 \in \mathfrak{C}_0(\mathbb{F})$, and let $z_i \in \mathfrak{C}_i(V_0)$ such that $n_i \circ z_i = z_0$, $i = \overline{1,2}$. Let $z_4 \in \mathcal{C}_4(V_0)$ such that $m_1 \circ z_4 = z_2$, and let $m_2 \circ z_4 = z_3$. We denote by $y_i \in \mathfrak{C}_{s_i}(\mathbb{F})$ the point defined by $z_i \in \mathfrak{C}_{s_i}(V_0)$, with $i = \overline{0,4}$, with $s_i = i$ if $i \neq 3$, and with $s_3 = 1$. Let $\mathfrak{C}^{\mathrm{ad}}_{y_i}$ be the Shimura adjoint Lie σ-crystal attached to y_i, $i = \overline{1,4}$, obtained using the SHS $(f_{s_i}, L_{s_i(p)}, v_{s_i})$, with $L_{4(p)} := L_{1(p)} \oplus L_{2(p)}$. From the construction of f_4, we deduce the existence of a map $\mathfrak{C}^{\mathrm{ad}}_{y_4} \to \mathfrak{C}^{\mathrm{ad}}_{y_3} \oplus \mathfrak{C}^{\mathrm{ad}}_{y_2}$ between Shimura adjoint Lie σ-crystals, which produces an isomorphism of $\mathfrak{C}^{\mathrm{ad}}_{y_4}$ with $\mathfrak{C}^{\mathrm{ad}}_{y_2}$ and $\mathfrak{C}^{\mathrm{ad}}_{y_3}$.

4.9.4. Lemma. *$\mathfrak{C}^{\mathrm{ad}}_{y_i}$, $i = \overline{1,4}$, are all isomorphic.*

Proof: We have just seen that $\mathfrak{C}^{\mathrm{ad}}_{y_4}$ is isomorphic to $\mathfrak{C}^{\mathrm{ad}}_{y_2}$ and $\mathfrak{C}^{\mathrm{ad}}_{y_3}$. The fact that $\mathfrak{C}^{\mathrm{ad}}_{y_1}$ and $\mathfrak{C}^{\mathrm{ad}}_{y_3}$ are isomorphic results from the fact that z_1 and z_3 are two lifts to $\mathfrak{C}_1$ of z_0, and so $z_3 = a(z_1)h$, with $h \in G_1(\mathbb{A}^p_f)$, and with $a : (G_1, X_1, H_1, v_1) \tilde{\to} (G_1, X_1, H_1, v_1)$ an automorphism defined by an element of $G_0(\mathbb{Z}_{(p)}) = G_0(\mathbb{Q}) \cap H_0$ (cf. [Va2, 3.3.1]). But the Shimura adjoint Lie σ-crystal attached to y_1 is isomorphic to the one attached to $a(y_1)$ (the argument of this is entirely analogous to the one which produced an isomorphism of $\mathfrak{C}^{\mathrm{ad}}_{y_4}$ with $\mathfrak{C}^{\mathrm{ad}}_{y_2}$ and $\mathfrak{C}^{\mathrm{ad}}_{y_3}$), and so it is isomorphic to the one attached to $y_3 = a(y_1)h$.

4.9.5. We call $\mathfrak{C}^{\mathrm{ad}}_{y_1}$ the Shimura adjoint Lie σ-crystal attached to y_0. The above lemma guarantees that it is well defined.

4.9.5.1. For any algebraically closed field k of characteristic p, and for any $y_{01} \in \mathfrak{C}_0(k)$, we define similarly the Shimura adjoint Lie σ-crystal attached to y_{01}.

4.9.6. If now $z^0_0 \in \mathcal{N}_0(V_0)$ is the translate of z_0 by an element of $G_0(\mathbb{A}^p_f)$, and we work with (the assumed to exist) $z^0_1, z^0_2, z^0_3, z^0_4$, and $y^0_1, y^0_2, y^0_3, y^0_4$ having a similar meaning, we get similarly that $y^0_i = a^0_i(y_i)h_i$, with $h_i \in G_i(\mathbb{A}^p_f)$ and with $a^0_i : (G_i, X_i, H_i, v_i) \tilde{\to}$

(G_i, X_i, H_i, v_i) an automorphism defined by an element of $G_0(\mathbb{Z}_{(p)})$, $\forall i \in \{1, 2, 4\}$. As in 4.9.4 we get:

4.9.7. Fact. *The Shimura adjoint Lie σ-crystals attached to y_1^0, y_2^0, y_3^0 and y_4^0 are all isomorphic to $\mathfrak{C}_{y_4}^{\mathrm{ad}}$.*

4.9.8. Theorem. *For any integral canonical model $\mathcal{N}$ of a Shimura quadruple (G, X, H, v) of preabelian type, with v diving a rational prime $p \geq 3$, there is a unique $G(\mathbb{A}_f^p)$-invariant stratification (called the refined canonical Lie stratification) of $\mathcal{N}_{k(v)}$ in locally closed (reduced) subschemes such that:*

a) the strata are indexed by a subset of the set $RLNP(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$ (defined in 4.5.7);

b) it is functorial with respect to finite morphism between quadruples;

c) there is a unique open stratum; it is Zariski dense in $\mathcal{N}_{k(v)}$;

d) any point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$, with k a perfect field, of the open dense stratum has a uniquely defined G-canonical lift $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$;

e) the G-canonical lifts are functorial with respect to finite maps between quadruples;

f) if $\mathcal{N}$ and (G, X, H, v) are related to a SHS $(f, L_{(p)}, v)$ as in 4.1, then we reobtain the refined canonical Lie stratification defined in 4.5 and the G-canonical lifts defined in 4.4;

g) it is invariant under isomorphisms, i.e. it is invariant under the natural action of the group $\mathrm{Aut}((G, X, H))$ *(defined in [Va2, 9) of 3.2.7]) on $\mathcal{N}_{k(v)}$.*

Proof: For the case $p = 3$ we refer to §6. We assume now that $p \geq 5$. Let (G_0, X_0, H_0, v_0) be the adjoint quadruple of (G, X, H, v). We use the notations of 4.9.1-7. A point $y_0 \in \mathfrak{C}_0(\mathbb{F})$ belongs to the stratum of some $s \in RLNP(G_0, X_0, v_0)$ iff the sequence of Newton polygons defined by the cyclic factors of the Shimura adjoint Lie σ-crystal attached to y_0, is the same as that of s. The Hecke operators $G_0(\mathbb{A}_f^p)$ allows us to define the stratum of belongings for any point $y \in \mathcal{N}_0(\mathbb{F})$ (cf. [Va2, 3.3.2]). 4.9.4 and 4.9.7 guarantees that this is well defined. The fact that the things do not depend on the connected component X_0^0 of X_0 we chose is implied by [Va2, 3.3.3].

To see that the partition of $\mathcal{N}_0(\mathbb{F})$ in disjoint subsets defines locally closed subschemes we just have to remark that the pull back (of these disjoint sets) to $\mathcal{N}_1(\mathbb{F})$ through the morphism j_1 (of 4.9.1) are forming the partition of $\mathcal{N}_1(\mathbb{F})$ in subsets corresponding to the refined canonical Lie stratification of $\mathcal{N}_{1k(v_1)}$ defined by the SHS $(f_1, L_{1(p)}, v_1)$. So [Va2, 6.4.5.1] applies. This takes care of a) and f).

The stratification of $\mathcal{N}_{k(v)}$ is obtained from the one of $\mathcal{N}_{0k(v_0)}$ by the pull back operation under the natural morphism $\mathcal{N} \to \mathcal{N}_0$ (defined by the extension property of $\mathcal{N}_0$).

Now 4.9.4 and 4.9.7 guarantees that everything is well defined. This takes care of b), while c) results from 4.2.1, via [Va2, 6.4.5.1]. The existence and uniqueness of G-canonical lifts is a direct consequence of the following remark (it is just a particular case of the filtered version of 4.9.4):

(4.9.8.1) If in 4.9.4 z_2 is a G-canonical lift, then z_1 and z_3 are also G-canonical lifts.

So it makes sense to say that z_0 is a G_0-canonical lift of y_0 (4.9.7 guarantees that z_0 is well defined and unique). For a point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ of the open stratum, with k a perfect field, we first get a G-canonical lift $\bar{z} : \mathrm{Spec}(W(k_1)) \to \mathcal{N}$ (with k_1 a finite Galois extension of k) of the point $\bar{y} \in \mathcal{N}(k_1)$ defined by y, which due to uniqueness property is defined over $W(k)$, i.e. we get a G-canonical lift $z : \mathrm{Spec}(W(k)) \to \mathcal{N}_{k(v)}$ of y. This takes care of d) and e).

The argument why the refined canonical Lie stratification of $\mathcal{N}_{k(v)}$ is invariant under the action of $\mathrm{Aut}((G, X, H))$ has been presented in the proof of 4.9.4 (cf. the definition of $\mathrm{Aut}((G, X, H))$). This takes care of g), ending the proof of the theorem.

4.9.8.2. Corollary. *If $\dim(X) > 0$, then there is a $G(\mathbb{A}_f^p)$-invariant closed subscheme of $\mathcal{N}_{k(v)}$ different from $\mathcal{N}_{k(v)}$.*

Proof: [Va2, 6.4.2 and 6.4.5.1] reduces easily (cf. the proof of 4.9.8) the situation to the case when we have a SHS. But this case is handled by 4.3.6. This ends the proof of the corollary.

This corollary fulfills the promise of [Va2, 3.2.12].

4.9.9. Remark. A result similar to 4.9.8 can be obtained for the canonical Lie stratification, or for the Lie-stable stratification (defined for a SHS in 4.5).

4.9.10. Definitions. The points with values in perfect fields of the open dense stratum of $\mathcal{N}_{k(v)}/\tilde{H}_0$ (with $\tilde{H}_0$ a compact subgroup of $G(\mathbb{A}_f^p)$) are called G-ordinary points or Shimura-ordinary points.

4.9.11. Corollary. *We have a natural functor from the full subcategory of qf-Sh of whose objects are Shimura quadruples (G, X, H, v) of preabelian type, with $(v, 2) = 1$, to the category of stratified schemes, taking such a quadruple into the scheme defined by the special fibre of its integral canonical model, and endowed with the refined canonical Lie stratification (the action of the functor on morphisms being the logical one, cf. [Va2, rm. 4) of 3.2.7]).*

4.9.12. Corollary. *Under the Hecke operators of $G(\mathbb{A}_f^p)$, a G-canonical lift is taken into a G-canonical lift.*

4.9.13. Remark. Let $\tilde{H}_0$ be an arbitrary compact subgroup of $G(\mathbb{A}_f^p)$. As in 4.4.2.1 we define uniquely the G-canonical lift $\mathrm{Spec}(W(k)) \to \mathcal{N}/\tilde{H}_0$ of a G-ordinary point $\mathrm{Spec}(k) \to \mathcal{N}_{k(v)}/\tilde{H}_0$. In §12 we will see that a G-canonical lift $\mathrm{Spec}\big(W(\mathbb{F})\big) \to \mathcal{N}/\tilde{H}_0$ gives birth to a special point $\mathrm{Spec}\big(W(\mathbb{F})[\frac{1}{p}]\big) \to \mathrm{Sh}(G, X)/\tilde{H}_0$ (cf. 4.4.6).

4.9.14. Question. With the notations of 4.5, is it true that the f-canonical stratification of $\mathcal{N}_{k(v)}$ defined in 4.5.10 is indeed canonical, i.e. it does not depend on the SHS $(f, L_{(p)}, v)$ producing it?

The expectation (cf. 4.5.10) of the existence of a refined f-canonical stratification of $\mathcal{N}_{k(v)}$, suggests (in general) a negative answer to this question. We expect that the supposed existing refined f-canonical stratification of $\mathcal{N}_{k(v)}$ (cf. 4.5.10), does not depend on the SHS producing it (and so that we can drop mentioning "f-").

4.9.15. Problem. Prove a result similar to 4.9.8, but for the (refined) canonical stratifications.

4.9.16. Exercise. Show that the theory of G-multiplicative coordinates presented in 4.7.33 extends to the case of Shimura-ordinary points of an integral canonical model of a Shimura quadruple (G, X, H, v) of preabelian type. Moreover the G-multiplicative coordinates we get are uniquely determined, provided the prime v^{ad} of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ divided by v has $\mathbb{F}_p$ as its residue field. Hint: for the uniqueness part, use the idea of the proof of 4.9.4, and 4.7.30 4).

4.9.17. Definition. Let $\mathcal{N}$ be the integral canonical model of a Shimura quadruple (G, X, H, v) of preabelian type, with $(v, 2) = 1$. Let p be the characteristic of $k(v)$. Let $y \in \mathcal{N}(k)$, with k an algebraically closed field of characteristic p. Let $\mathcal{N}_0$ be the integral canonical model of the quadruple $(G_0, X_0, H_0, v_0) := (G^{\mathrm{ad}}, X^{\mathrm{ad}}, H^{\mathrm{ad}}, v^{\mathrm{ad}})$, and let $y_0 \in \mathcal{N}_0(k)$ be the composite of y with the natural morphism $\mathcal{N} \to \mathcal{N}_0$. Let $h \in G^{\mathrm{ad}}(\mathbb{A}_f^p)$ such that $y_{01} = y_0 h \in \mathfrak{C}_0(k)$ (cf. [Va2, 3.3]). The Shimura adjoint Lie σ-crystal $\mathfrak{C}_{y_{01}}^{\mathrm{ad}}$ (cf. 4.9.5.1) is called the Shimura adjoint Lie σ-crystal attached to y.

The reason why is this well defined is nothing else but 4.9.4 and 4.9.7 (performed for $W(k)$, instead of V_0). In the case of a SHS $(f, L_{(p)}, v)$ we recover the definition of a Shimura adjoint Lie σ-crystal attached to a point $y \in \mathcal{N}(k)$.

4.9.17.1. Remark. In the situation of 4.9.17, we similarly define the Shimura adjoint filtered Lie σ-crystal attached to a point $\mathrm{Spec}(W(k)) \to \mathcal{N}$. So the canonical lifts $\mathrm{Spec}(W(k)) \to \mathcal{N}$ can be recognized (see 4.4.1 2)) as follows: their Shimura adjoint filtered Lie σ-crystals are (cyclic diagonalizable and) of parabolic (and Borel) type.

4.9.17.2. Definition. A point $y : \mathrm{Spec}(k_1) \to \mathcal{N}_{k(v)}$, with k_1 a perfect field, is called a toric point if the Shimura adjoint Lie $\bar{\sigma}_1$-crystal attached to the $\overline{k_1}$-valued point of $\mathcal{N}_{k(v)}$ naturally defined by y is isomorphic to the extension to $\overline{k_1}$ of the Shimura adjoint Lie $\bar{\sigma}_1$-crystal defined by a Shimura σ-crystal $\mathfrak{C}_\omega$ showing up in 4.1.5, by working with an adequately chosen SHS. Here σ_1 is the Frobenius automorphism of k_1 and of $W(k_1)$. If $\mathfrak{C}_\omega$ is a U-ordinary (resp. T-ordinary) σ-crystal, then we call the corresponding points U-ordinary (resp T-ordinary).

4.9.17.3. Remark. 4.9.4 shows that the notion of toric points is intrinsic (canonical). In particular, if (G, X, H, v) is obtained from a SHS $(f, L_{(p)}, v)$ then we "regain" the toric points defined in 4.5.12.2. Moreover, 4.12.3 and 4.12.6 show that there are plenty of toric points (cf. as well to [Va2, 6.4.2] and 4.12.5). As in 4.5.12.2.1, the toric points can be defined in terms of Galois representations.

4.9.17.4. Remark. Based on 4.9.8, it makes sense to speak, as in 3.4.15-16, about $\mathcal{N}_0$ having the completion property (we just rephrase 3.4.15 in terms of Shimura adjoint Lie σ-crystals).

4.9.17.5. U-canonical lifts. From 4.4.13 we get immediately, as in 4.9.8, that any U-ordinary point $\mathrm{Spec}(\mathrm{k}_1) \to \mathcal{N}_{k(v)}$ with k_1 a perfect field, has a uniquely determined lift $\mathrm{Spec}(W(k_1)) \to \mathcal{N}$ (called a U-canonical lift) with the property that the Shimura adjoint filtered Lie σ_1-crystal attached to its extension to $W(\bar{k_1})$ is of toric type (in the sense of 2.2.12 d)).

4.9.18. Remark. We think that the terminology (refined) canonical Lie stratification is justified. The word Lie should make distinction from other ρ-stratifications introduced in 4.5.4 (for a SHS), while the word canonical should emphasize that in the case of a Shimura quadruple (G, X, H, v) emerging from a SHS $(f, L_{(p)}, v)$, the (refined) (canonical) Lie stratification we get for $\mathcal{N}_{k(v)}$ does not depend on the choice of the SHS $(f, L_{(p)}, v)$. We might argue that the terminology the absolute stratification might have been better, cf. 4.5.6.1. But the proof of 4.5.6.1 works only in the context of deformations of Shimura σ-crystals whose attached cyclic adjoint factors are of classical Lie types. So the cases of E_6 and E_7 Lie types are left unsolved (see also 4.14.2).

4.9.19. Exercise. Let (G_i, X_i, H_i, v_i), $i = \overline{1,2}$, be two Shimura quadruples of preabelian type, with v_1 and v_2 dividing the same prime $p \geq 3$. Let w be a prime of $E(G_1 \times G_2, X_1 \times X_2)$ dividing v_1 and v_2. So we get a Shimura quadruple $(G_1 \times G_2, X_1 \times X_2, H_1 \times H_2, w)$. Let $\mathcal{M}_i$ be the extension to $\mathrm{Spec}(O_{(w)})$ of the integral canonical model $\mathcal{N}_i$ of (G_i, X_i, H_i, v_i), $i = \overline{1,2}$. It is known that $\mathcal{M}_1 \times_{O_{(w)}} \mathcal{M}_2$ is the integral canonical model of $(G_1 \times G_2, X_1 \times X_2, H_1 \times H_2, w)$ (cf. [Va2, 3.2.16]). Show that the refined canonical Lie stratification of $\mathcal{M}_{k(w)}$ is the product stratification of the refined canonical Lie stratifications of $\mathcal{M}_{ik(w)}$.

4.9.20. Exercise. With the notations of 4.9.17, let $H_0 \subset G(\mathbb{A}_f^p)$ be a subgroup such that $H_0 \times H$ is a subgroup of $G(\mathbb{A}_f)$ smooth for (G, X). Show that for any formally smooth morphism $f_s : \mathrm{Spec}(R) \to \mathcal{N}/H_0$, we can define uniquely the Shimura adjoint filtered Lie F-crystal (over $\mathrm{Spec}(R/pR)$) attached to f_s.

A proof of this exercise will be given in §5. In what follows we need only the case when the first fundamental group of $\mathrm{Spec}(R)$ is trivial: this case can be treated entirely as in 4.9.2-4 (where we have $R = \mathbb{F}$).

4.9.21. The relative Oort-Moonen problem. Let (G, X, H, v) be a Shimura quadruple of preabelian type, with v dividing a prime $p > 2$. Let $\mathcal{N}$ be its integral canonical model (cf. 6.4.1 for $p > 3$, and cf. §6 for $p = 3$). Let F be a number field containing $E(G, X)$. Let $H_0 \subset G(\mathbb{A}_f^p)$ be a compact open subgroup such that the subgroup $H_0 \times H$ of $G(\mathbb{A}_f)$ is smooth for (G, X). Let Z be an irreducible subvariety of $\mathrm{Sh}_{H_0 \times H}(G, X)_F$. Let $Z_{(v)}$ be the Zariski closure of Z in $(\mathcal{N}/H_0)_{F_{(v)}}$ (here $F_{(v)}$ is the normalization of $O_{(v)}$ in F). We assume that there is a prime w of F dividing v and a G-ordinary point $y : \mathrm{Spec}(\overline{k(v)}) \to \mathcal{N}_{k(w)}/H_0$ factoring through $Z_{(v)}$, such that $Z_{(v)}$ has formally quasi-linear components at y. Then Z is of preabelian type, i.e. any irreducible component of $Z \times_F \mathbb{C}$ is a subvariety of $\mathcal{N}_{\mathbb{C}}/H_0$ of preabelian type (cf. def. [Mo, 3.7]).

Due to the existence of G-multiplicative coordinates (cf. 4.9.16) the notion of having formally quasi-linear components of y can be defined as in [Mo]. It seems to us that we can use 4.7.30 and 4.7.36, to prove that this notion is independent of all choices referred to in 4.7.30 8); if not, we have to define the notion of formally quasi-linearity by working with respect to all -allowed, i.e. logical- choices mentioned in 4.7.30 8)).

For not being too long, we simplify the things by just giving an example (without being particular about the precise choices defining G-multiplicative coordinates) in the case when w is unramified over v. If the closed subscheme (defined naturally by $Z_{(v)}$) of the spectrum of the completion of the local ring of the point $y_1 : \mathrm{Spec}(\overline{k(v)}) \to \mathcal{N}_{W(\overline{k(v)})}/H_0$ defined by y, is a formal subtorus of the formal torus (describing G-multiplicative coordinates) defined (cf. 4.9.16) by the G-ordinary point y_1, then (by the very definition) $Z_{(v)}$ has formally quasi-linear components at y.

4.9.21.1. Exercise. Using [Mo, 5.2] show that the relative Oort-Moonen problem is true if $k(v^{\mathrm{ad}}) = \mathbb{F}_p$ (here v^{ad} is the prime of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ divided by v). Hint: make use of 4.7.30 4), and of [Va2, 6.5.1.1].

4.9.22. Definition. A simple adjoint Shimura variety $\mathrm{Sh}(G, X)$ is said to be of $\mathbb{Z}$-End type if there is a principally polarized abelian variety over $\mathbb{C}$ with $\mathrm{End}(A) = \mathbb{Z}$, whose attached (see [Va2, 2.12 3)]) Shimura variety has $\mathrm{Sh}(G, X)$ as its adjoint variety.

We refer to the situation of 4.9.2. Let E be a number field. Let H_{00} be the subgroup of $G_0(\mathbb{A}_f^p)$ generated by the images of H_{01} and H_{02} (we recall that H_{0i} is a subgroup of $G_i(\mathbb{A}_f^p)$ subject to some conditions; $i \in \{1, 2\}$). We consider two morphisms $a(i)$: $\mathrm{Spec}(E) \to \mathrm{Sh}_{H_{0i}\times H_i}(G_i, X_i)$, $i \in \{1, 2\}$, giving birth to the same morphism $\mathrm{Spec}(E) \to \mathrm{Sh}_{H_{00}\times H_0}(G_0, X_0)$. Let A_i be the abelian variety over E naturally associated to $a(i)$ and the embedding f_i of 4.9.1. We have:

4.9.23. Corollary. *The ordinary conjecture (see [Oo, p. 11]) is true for A_1 iff it is true for A_2.*

Proof: We can assume that E is big enough such that G_{iE}^{ab} are split tori. We consider only those primes of E such that:

– they are unramified above rational primes;

– their residue fields have a prime number of elements;

– $A_1 \times A_2$ has good reduction with respect to them.

From 4.9.4 and P1 of 4.6 we get immediately: A_1 has an ordinary reduction with respect to such a prime iff A_2 has. The corollary follows.

From 4.9.23 and [Pi, 7.2] we deduce:

4.9.23.1. Corollary. *Let A be an abelian variety over a number field E. We assume that the Shimura variety attached to it is such that all the factors of its adjoint variety are of $\mathbb{Z}$-End type and are not of C_n type with $n \geq 3$. Then the ordinary conjecture is true for A.*

4.9.23.2. Exercise. Show that there are situations as in 4.9.23.1 with A an absolutely simple variety such that $\mathrm{End}(A_{\bar{E}}) \neq \mathbb{Z}$. Hint: use Mumford curves and 2.3.5.1.

4.10. Hecke orbits of G-ordinary points. Let (G, H, H, v) be a Shimura quadruple of preabelian type, with $(v, 2) = 1$, and let $\mathcal{N}$ be its integral canonical model over $O_{(v)}$. Let $\mathcal{U} \hookrightarrow \mathcal{N}_{k(v)}$ be the open (Zariski dense) subscheme of G-ordinary points (cf. 4.9.8).

From the proof of 2.3.9 we deduce that the field of fractions of any connected component of $\mathcal{N}_{k(v)}$ contains $\mathbb{F}$.

4.10.1. Conjecture. *$\mathcal{U}$ is the smallest non-empty open subscheme of $\mathcal{N}_{k(v)}$ which is $G(\mathbb{A}_f^p)$-invariant.*

4.10.1.1. Standard arguments show that another way to formulate 4.10.1 is: the Hecke orbit of a G-ordinary point y (of $\mathcal{U}$) with values in $k = \bar{k}$, i.e. the set of points obtained from y through $G(\mathbb{A}_f^p)$-translations, is dense in $\mathcal{N}_k$, i.e. it does not factor through a closed subscheme of $\mathcal{N}_k$ which is not $\mathcal{N}_k$ itself.

4.10.2. The expectations of the validity of the statement of 4.10.2 are based on:

a) 4.2.8.1;

b) exercise 8) of 4.5.6;

c) the part d) (or c)) of 4.4.1 3), and b) and c) of 4.4.1 2);

d) the validity of 4.10.1 in the case when $\mathrm{Sh}(G,X)$ is a Siegel modular variety [Ch];

e) 4.10.3 below;

f) 4.10.5 below.

4.10.3. Proposition. *The above conjecture is true if* $\dim(X) \leq 1$.

Proof: If $\dim(X) = 0$, i.e. if G is a torus, this is obvious (cf. [Va2, 3.2.8 and 3.3.1]). If $\dim(X) = 1$, then everything results from 4.2.8.1, from 4.9.12, and from the fact the $G(\mathbb{A}_f^p)$-orbit of any point of $\mathrm{Sh}(G,X)(\mathbb{C})$ is Zariski dense in $\mathrm{Sh}(G,X)_{\mathbb{C}}$. As we did not include a proof of 4.2.8.1, we would like to remark that for Shimura curves 4.2.8.1 is a consequence of [Va2, 6.4.11] (the case of non-compact Shimura curves, i.e. of Shimura curves having the same adjoint curve as the elliptic modular curve, is well known; see [FC, ch. IV, 5.10 and 6.12]).

4.10.4. Variants. There are variants of 4.10.1. For instance by working not with the full group $G(\mathbb{A}_f^p)$ of Hecke operators, but just with $G(\mathbb{Q}_q)$, where q is a rational prime, different from p, and such that G is unramified over $\mathbb{Q}_q$ (cf. [Ch]), or more generally with an adequate (i.e. of interest) subgroup of $G(\mathbb{A}_f^p)$. However in such a variant we have to be careful: $G(\mathbb{Q}_q)$ might not permute transitively the connected components of $\mathrm{Sh}_H(G,X)_{\mathbb{C}}$ (for instance this is the case when $G_{\mathbb{Q}_p}$ is anisotropic). Another possibility is of replacing G-ordinary points and $\mathcal{U}$ respectively by $G(\omega)$-ordinary points and the $\mathfrak{s}_\omega$ stratum (cf. 4.5.12). Again some precautions have to be taken: we might not get dense Hecke orbits. So in some of these variants 4.10.1 needs to be reformulated.

4.10.5. Remark. The gluing principle of 3.4.19 D can be interpreted as a local Hecke correspondence. Using this we prove (see [Va7]) the following IOTA (if one then all) criterion:

4.10.5.1. Criterion. *Let (G,X,H,v) be a quadruple of preabelian type, with $k(v^{\mathrm{ad}}) = \mathbb{F}_p$. We assume that either $p \geq 3$ or $p = 2$ and (G,X,H,v) is associated to a $p = 2$ SHS. Let $\mathcal{U}$ be the G-ordinary locus of the integral canonical model $\mathcal{N}$ of (G,X,H,v). If the*

$G(\mathbb{A}_f^p)$*-orbit of a* G*-ordinary point of* $\mathcal{U}$ *is dense in* $\mathcal{N}_{k(v)}$ *(i.e. it is dense in* $\mathcal{U}$*), then the* $G(\mathbb{A}_f^p)$*-orbit of any* G*-ordinary point of* $\mathcal{U}$ *is dense in* $\mathcal{N}_{k(v)}$.

This criterion recovers all the previous known cases of density of Hecke orbits in positive characteristic and provides plenty of new examples of such densities (like the case of classical Spin modular varieties, see [Va2, 5.7.5] and [Va4]).

4.11. The functorial behaviour of G-ordinary points.

4.11.1. We consider now a SHS $(f, L_{(p)}, v)$ defined by an injective map $f : (G, X) \hookrightarrow \big(\mathrm{GSp}(W, \psi), S\big)$ which factors as the composite $f_1 \circ f_2$ of two injective maps, with $f_2 : (G, X) \hookrightarrow (G_1, X_1)$, and with $f_1 : (G_1, X_1) \hookrightarrow \big(\mathrm{GSp}(W, \psi), S\big)$. Let v_1 be the prime of $E(G_1, X_1)$ divided by v. We assume that $(f_1, L_{(p)}, v_1)$ is also a SHS. We still use for it the standard notations of 2.3.1-3, but everything having a right lower index 1, except $L_{(p)}$. We have a morphism $i_2 : \mathcal{N} \to \mathcal{N}_1$ (cf. [Va2, rm. 4) of 3.2.7]). Two natural questions arises:

Q1. When i_2 takes G-ordinary points of $\mathcal{N}_{k(v)}$ into G_1-ordinary points of $\mathcal{N}_{1k(v_1)}$?

Q2. If i_2 takes G-ordinary points of $\mathcal{N}_{k(v)}$ into G_1-ordinary points of $\mathcal{N}_{1k(v_1)}$, is it true that a G-canonical lift of a G-ordinary point y of $\mathcal{N}_{k(v)}$ is mapped by i_2 into a G-canonical lift of $i_2 \circ y$?

$Q1$ can be reformulated (cf. 4.2.1): when the dense open stratum of $\mathcal{N}_{k(v)}$ is the pull back through i_2 of the dense open stratum of $\mathcal{N}_{1k(v_1)}$?

The case $k(v_1) = \mathbb{F}_p$ is completely handled by properties P1 and P2 of 4.6. The answer to $Q1$ is yes when $k(v) = \mathbb{F}_p$, and then $Q2$ also has a positive answer. What about $k(v_1) = \mathbb{F}_{p^{q_1}}$ with $q_1 \in \mathbb{N}$, $q_1 \geq 2$? We have:

4.11.2. Theorem. *The morphism* i_2 *maps* G*-ordinary points of* $\mathcal{N}_{k(v)}$ *into* G_1*-ordinary points of* $\mathcal{N}_{1k(v_1)}$*, and* G*-canonical lifts of* $\mathcal{N}$ *into* G_1*-canonical lifts of* $\mathcal{N}_1$ *if and only if the following two conditions are satisfied:*

a) $k(v) = k(v_1)$*;*

b) there is a torus T *of* $G_{\mathbb{Z}_p}$*, a cocharacter* $\mu : \mathbb{G}_m \hookrightarrow T_{W(k(v))}$*, and a Borel subgroup* B_1 *of* $G_{1\mathbb{Z}_p}$ *such that:*

i) Over an embedding of $W(k(v))$ *in* $\mathbb{C}$*, extending the composite inclusion* $O_{(v)} \subset E(G, X) \subset \mathbb{C}$*,* μ *becomes* $G(\mathbb{C})$*-conjugate to the cocharacters* μ_x^**,* $x \in X$*;*

ii) Under the direct sum decomposition $L_p^* \otimes_{\mathbb{Z}_p} W(k(v)) = F^1 \oplus F^0$ *produced by* μ *(*$\beta \in \mathbb{G}_m\big(W(k(v))\big)$ *acts through* μ *on* F^i *as the multiplication with* β^{-i}*),* F^1 *is left invariant by* B_1 *(i.e* $\mathrm{Lie}(B_1) \subset F^0(\mathrm{Lie}(G_{W(k(v))})$*);*

iii) $B_1 \cap G_{\mathbb{Z}_p}$ *is a Borel subgroup* B *of* $G_{\mathbb{Z}_p}$.

Proof: The fact that the conditions a) and b) imply that the i_2 takes G-ordinary points of $\mathcal{N}_{k(v)}$ into G_1-ordinary points of $\mathcal{N}_{1k(v_1)}$ results from 4.1-2: the G-ordinary type of the SHS $(f, L_{(p)}, v)$ and the G_1-ordinary type of the SHS $(f_1, L_{(p)}, v_1)$ can both be computed (cf. 4.1) starting from the data of T, B and μ (resp. of T, B_1 and μ), and so they are equal. The fact that a) and b) imply also that i_2 takes G-canonical lifts into

G_1-canonical lifts is a direct consequence of c) of 4.4.1 3) and of 4.4.1 1): there is $n \in \mathbb{N}$ such that the filtered σ^n-isocrystals attached to G-canonical lifts $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$, with $k = \bar{k}$, depend only on the data of b).

Let us now assume that i_2 takes G-ordinary points into G_1-ordinary points and G-canonical lifts into G_1-canonical lifts. Let $z : \mathrm{Spec}\big(W(\mathbb{F})\big) \to \mathcal{N}$ be a G-canonical lift of a G-ordinary point $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}_{k(v)}$, and let $\mathfrak{C} := (M_y, F_z^1, \varphi_y, G_{W(\mathbb{F})})$ be the Shimura filtered σ-crystal attached to z. We get another Shimura filtered σ-crystal $\mathfrak{C}_1 = (M_y, F_z^1, \varphi_y, G_{1W(\mathbb{F})})$; it is the one attached to $y_2 \circ z$.

Now the canonical split $\mu_y : \mathbb{G}_m \hookrightarrow G_{W(k)}$ of $\mathfrak{C}$ when composed with $G_{W(k)} \hookrightarrow G_{1W(k)}$, produces the canonical split of $\mathfrak{C}_1$ (cf. 3.1.5). This implies that the degree of definition of $\mathfrak{C}$ is equal to the degree of definition of $\mathfrak{C}_1$ and so $k(v) = k(v_1)$ (cf. 4.4.7). This proves a). Writing $\varphi_y = a\mu_y(\frac{1}{p})$, a becomes a σ-linear automorphism of M_y. Then $M_y^a := \big\{x \in M_y | a(x) = x\big\}$ is a $\mathbb{Z}_p$-structure on M_y. We get groups $\tilde{G}_{\mathbb{Z}_p} \overset{\tilde{f}_2}{\hookrightarrow} \tilde{G}_{1\,\mathbb{Z}_p} \hookrightarrow GL(M_y^a)$ and a cocharacter $\mu_y : \mathbb{G}_m \hookrightarrow \tilde{G}_{W(k(v))}$. Let $\tilde{B}$ be a Borel subgroup of $\tilde{G}_{\mathbb{Z}_p}$ leaving invariant F_z^1 and such that μ_y factors through $\tilde{B}_{W(k(v))}$, and let $\tilde{B}_1$ be a Borel subgroup of $\tilde{G}_{1\,\mathbb{Z}_p}$ leaving invariant F_z^1 and containing $\tilde{B}$ (the existence of $\tilde{B}$ and of $\tilde{B}_1$ is a direct consequence of b) of 4.4.1 2)). Let $\tilde{T}$ be a maximal torus of $\tilde{G}_{\mathbb{Z}_p}$ contained in $\tilde{B}$ and such that μ_y factors through $\tilde{T}_{W(k(v))}$.

Now b) results from the fact that we have isomorphisms $\tilde{G}_{\mathbb{Z}_p} \tilde{\to} G_{\mathbb{Z}_p}$ and $\tilde{G}_{1\,\mathbb{Z}_p} \tilde{\to} G_{1\mathbb{Z}_p}$, through which $\tilde{f}_2$ becomes the natural inclusion $\bar{f}_2 : G_{\mathbb{Z}_p} \hookrightarrow G_{1\mathbb{Z}_p}$ as subgroups of $GL(L_p^*)$. This is implied by d) of 4.4.1 3). As we postponed the proof of d) of 4.4.1 3), we present another argument of why b) is satisfied. To show the existence of an isomorphism $\tilde{G}_{\mathbb{Z}_p} \tilde{\to} G_{\mathbb{Z}_p}$, it is enough to show the existence of an isomorphism $\tilde{G}^{\mathrm{ad}}_{\mathbb{Z}_p} \tilde{\to} \tilde{G}^{\mathrm{ad}}_{\mathbb{Z}_p}$, as obviously $Z(\tilde{G})_{\mathbb{Z}_p} = Z(G)_{\mathbb{Z}_p})$ (cf. the definition of $\tilde{G}_{\mathbb{Z}_p}$) and as $\tilde{G}_{W(\mathbb{F})} = G_{W(\mathbb{F})}$. But this is implied by b) of 4.4.1 2). Similarly we get an isomorphism $\tilde{G}_{1\,\mathbb{Z}_p} \tilde{\to} G_{1\,\mathbb{Z}_p}$.

The fact that in c) of 4.4.1 3) we can take $n = 1$ or not for the SHS $(f, L_{(p)}, v)$ and for the point y, is expressed in terms of a torsor of $\tilde{G}_{\mathbb{Q}_p}$ being zero or not. But the resulting torsor of $\tilde{G}^{\mathrm{ad}}_{\mathbb{Q}_p} = G^{\mathrm{ad}}_{\mathbb{Q}_p}$ is trivial, cf. b) of 4.4.1 2). So it corresponds to a torsor of $Z(\tilde{G}_{\mathbb{Q}_p}) = Z(G)_{\mathbb{Q}_p}$. As this group is included in $\tilde{T}$, $\tilde{B}$ and $\tilde{B}_1$, we get that the condition b) is satisfied. This ends the proof of the theorem.

4.11.3. Example. We consider now the case when f_2 satisfies the conditions:

a) The toric part of $Z(G)$ is the toric part of $Z(G_1)$;

b) $G^{\mathrm{ad}} = \mathrm{Res}_{F/\mathbb{Q}} G^s$, with F a totally real number field, and with G^s an absolutely simple adjoint F-group;

c) f_2 induces an injective map $f_2^{\mathrm{ad}} : (G^{\mathrm{ad}}, X^{\mathrm{ad}}) \hookrightarrow (G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}})$ which is the injective map $f_{2\,F_1} : \mathrm{Sh}(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \hookrightarrow \mathrm{Sh}^{F_1}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ defined in [Va2, 6.6.1] (with F_1 a totally real number field containing F) (i.e. $G_1^{\mathrm{ad}} = \mathrm{Res}_{F_1/\mathbb{Q}} G^s_{F_1}$, and $f_2^{\mathrm{ad}} : G^{\mathrm{ad}} \hookrightarrow G_1^{\mathrm{ad}}$ is the canonical inclusion).

Then $E(G_1, X_1) = E(G, X)$, and a) and b) of 4.11.2 hold. We get:

Fact. *i_2 takes G-ordinary points of $\mathcal{N}_{k(v)}$ into G_1-ordinary points of $\mathcal{N}_{1k(v_1)}$ and G-canonical lifts of $\mathcal{N}$ into G_1-canonical lifts of $\mathcal{N}_1$.*

This result is going to be used in §12.

There are two other natural questions inspired by 4.11.2:

Q3. If i_2 takes G-ordinary points into G_1-ordinary points, does it automatically take G-canonical lifts into G_1-canonical lifts?

Q4. Does i_2 take G-ordinary points into G_1-ordinary points if we know only that $k(v) = k(v_1)$?

4.11.4. Exercise. Show that the answer to Q3 is always yes (Hint: use 4.4.8 2)).

4.11.5. Remark. The answer to Q4 is no if $k(v) = k(v_1) \neq \mathbb{F}_p$. There are many examples with G a torus which provide a negative answer to Q4. If $k(v) = k(v_1) = \mathbb{F}_p$ then the condition b) of 4.11.2 is automatically satisfied, and this motivates the form of properties P1 and P2 of 4.6.

4.11.6. Example. Here we concentrate just on Shimura σ-crystals. We assume that $G^{\mathrm{ab}}_{1\,\mathbb{Z}_p}$ is a split torus, and that $G^{\mathrm{ad}}_{1\,\mathbb{Z}_p}$ is an absolutely simple group of A_ℓ Lie type with non-trivial involution (so $\ell \geq 2$), and which splits over $W(\mathbb{F}_{p^2})$. Let T be a maximal torus of $G_{1\,\mathbb{Z}_p}$ which splits over $W(\mathbb{F}_{p^2})$, cf. [Ti, 1.10]. Let B a Borel subgroup of $G_{1\,\mathbb{Z}_p}$ containing T. Let $\mu : \mathbb{G}_m \to T_{W(\mathbb{F}_{p^2})}$ such that the conditions of 4.1 are satisfied. Let $\omega \in G_{\mathbb{Z}_p}\big(W(\mathbb{F}_{p^2})\big)$ normalizing $T_{W(\mathbb{F}_{p^2})}$ and taking $B_{W(\mathbb{F}_{p^2})}$ into its opposite (with respect to $T_{W(\mathbb{F}_{p^2})}$) Borel subgroup of $G_{1\mathbb{Z}_p}$. Let $\sigma_1 := \omega\sigma$ and let $T_1 \hookrightarrow \tilde{G}_{1\,\mathbb{Z}_p}$ be an inclusion of reductive groups which over $W(\mathbb{F}_{p^2})$ becomes the inclusion $T_{W(\mathbb{F}_{p^2})} \hookrightarrow G_{1\,W(\mathbb{F}_{p^2})}$ and which is defined at the level of Lie algebras by

$$\mathrm{Lie}(T_1) := \mathrm{Lie}\big(T_{W(\mathbb{F}_{p^2})}\big)^{\sigma_1} \hookrightarrow \mathrm{Lie}\big(G_{1\,W(\mathbb{F})}\big)^{\sigma_1} =: \mathrm{Lie}(\tilde{G}_{1\,\mathbb{Z}_p})$$

(the upper index σ_1 means we take the elements fixed by σ_1). We get:

– the Shimura σ-crystals $\big(L_p^*\otimes_{\mathbb{Z}_p}W(\mathbb{F}), \sigma_1\mu(\frac{1}{p}), T_{1\,W(\mathbb{F})}\big)$ and $\big(L_p^*\otimes_{\mathbb{Z}_p}W(\mathbb{F}), \sigma\mu(\frac{1}{p}), G_{1\,W(\mathbb{F})}\big)$ are both Shimura-ordinary, with the degree of definition equal to 2.

Obviously the $T_{1\,W(\mathbb{F})}$-ordinary type we get is different from the $G_{1\,W(\mathbb{F})}$-ordinary type we get.

4.11.7. Example. We assume that, as in 4.3.1, we have a product decomposition $G^{\mathrm{ad}}_{1\mathbb{Z}_p} = \times_{i\in\mathcal{H}} G_{1i}$ indexed by the same set as for G. We also assume that the image of $\mathrm{Lie}(G_i)$ in $\mathrm{Lie}(G^{\mathrm{ad}}_{1\mathbb{Z}_p})$ lies in $\mathrm{Lie}(G_{1i})$. As in 4.3.8 4) we can define the i-th A-degree $d_{1A}(i)$ of definition of $(f_1, L_{(p)}, v_1)$. We can have $k(v^{\mathrm{ad}}) = k(v_1^{\mathrm{ad}})$, without having $d_A(i) = d_{1A}(i)$ for all $i \in \mathcal{H}$. Easy examples can be obtained with $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ simple of $D_n^{\mathbb{H}}$ type with non-trivial involution, and with $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}})$ simple of A_{2n} type. Obviously, if there is $i \in \mathcal{H}$ such that $d_A(i) \neq d_{1A}(i)$, then a G-ordinary point is not mapped by i_2 into a G_1-ordinary point.

4.11.8. Exercise. Under the assumptions a) and b) of 4.11.2, if the A-degree of definition of a G-canonical lift $\mathrm{Spec}(W(\mathbb{F})) \to \mathcal{N}$ is 1, prove that: if $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$, with $k = \bar{k}$, is a G-ordinary point, then the formal torus (of the formal moduli scheme) of G-deformations of A_y is a subtorus of the formal torus of G_1-deformation of A_y (Hint: use 4.7.36 and 4.7.30 4)).

This is a generalization in the case of an arbitrary SHS with the A-degree of definition 1 of the main result of [No1]. There are variants (cf. 4.7.30 8)) of this exercise for an arbitrary A-degree of definition.

4.12. The integral Manin problem. Let $e \in \mathbb{N}$, and let p be a rational prime. Let k be an algebraically closed field of characteristic p. Let σ be the Frobenius automorphism of $W(k)$ and of $B(k)$.

The (original) Manin problem asked for the proof of the predicted classification of isocrystals over k attached to (principally polarized) abelian varieties over k, of dimension e. This problem was positively solved by the Honda–Serre–Tate theory (cf. [Ta, p. 98]). For a second, more recent, proof, see [Oo2].

4.12.0. The goal of this section is to formulate (and solve in some cases; see §11 for the general case) the integral Manin problem for (special fibres of) integral canonical models of Shimura varieties of preabelian type with respect to primes dividing a rational prime greater than 2. For the solution of this integral problem in the cases in which we know the completion property, we rely heavily on 3.4.15 and 4.4.1, and so we first treat the case $p \geq 3$. As a roundabout (cf. 3.14 and 4.12.12.1) we gain the case $p = 2$ as well. From now on, except for a special reference, $p \geq 3$.

4.12.1. The integral Manin problem for Siegel modular varieties. Determine all isomorphism classes of principally quasi-polarized σ-crystals over k attached to principally polarized abelian varieties over k of dimension e.

4.12.2. The integral Manin problem for a SHS $(f, L_{(p)}, v)$. Determine all isomorphism classes of principally quasi-polarized Shimura σ-crystals and of Shimura Lie σ-crystals over k attached to k-valued points of $\mathcal{N}_{k(v)}$.

4.12.3. The integral Manin problem for a Shimura quadruple (G, X, H, v) of preabelian type, with v dividing a rational prime $p \geq 3$. Determine all isomorphism classes of Shimura adjoint Lie σ-crystals attached (cf. 4.9.13) to k-valued points of the special fibre $\mathcal{N}_{k(v)}$ of the integral canonical model $\mathcal{N}$ of the quadruple (G, X, H, v).

The answer to all these three versions of the integral Manin problem is provided by 4.4.1, provided 3.4.15 holds.

4.12.4. Solution of 4.12.1. Let $(\mathrm{GSp}(W, \psi), S)$ define a Siegel modular variety, with $\dim_{\mathbb{Q}}(W) = 2e$. Let $L_{(p)}$ be a $\mathbb{Z}_{(p)}$-lattice of W such that $\psi : L_{(p)} \otimes_{\mathbb{Z}_{(p)}} L_{(p)} \to \mathbb{Z}_{(p)}$ is a perfect form. We get a SHS $(1_{(\mathrm{GSp}(W,\psi),S)}, L_{(p)}, p)$. Let $I_{2e} := \{1, 2, ..., 2e\}$. Let $\{w_i | i \in I_{2e}\}$ be a $\mathbb{Z}_{(p)}$-basis of $L^*_{(p)}$ such that for $i, j \in I_{2e}$ satisfying $j \geq i$, $\tilde{\psi}(w_i, w_j)$ is equal to 1 if $j - i = e$, and is equal to 0 otherwise. Let $M_0 := L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(k)$, and let $F^1 \subset M_0$, be the $W(k)$-submodule of M_0 generated by w_i, $i \in \{1, ..., e\}$. Let φ be the

σ-linear endomorphism of M_0 such that $\varphi(w_i) = pw_i$, if $i \in \{1, ..., e\}$, and $\varphi(w_i) = w_i$, if $i \in \{e+1, ..., 2e\}$. We have:

Answer: *The set $\mathcal{S}_e(k)$ of isomorphism classes of σ-crystals attached to principally polarized abelian varieties over k of dimension e is precisely the set of isomorphism classes of principally quasi-polarized σ-crystals defined by elements of the set*

$$\{(M_0, g\varphi, \tilde{\psi}) | g \in Sp(M_0, \tilde{\psi})(W(k))\}.$$

This is an easy consequence of [Ta, p. 98] and of the following isogeny property:

The isogeny property. *If $(M_0', g\varphi, \tilde{\psi})$, with M_0' a $W(k)$-lattice of $M_0[\frac{1}{p}]$, is the principally quasi-polarized σ-crystal attached to a principally polarized abelian variety $(A', p_{A'})$ over k, then there is a principally polarized abelian variety (A, p_A) over k, $\mathbb{Z}[\frac{1}{p}]$-isogeneous to $(A', p_{A'})$ and of whose principally quasi-polarized σ-crystal, under this $\mathbb{Z}[\frac{1}{p}]$-isogeny, is identifiable with $(M_0, g\varphi, \tilde{\psi})$.*

4.12.5. Solution of 4.12.2 in the case $\mathcal{N}$ has the completion property. Let $M_0 := M \otimes_{W(k(v))} W(k)$, $F_0^1 := F^1 \otimes_{W(k(v))} W(k)$, and let $\varphi_0 := \varphi \otimes 1$ be the Frobenius endomorphism of M_0, where M, F^1 and φ have the same significance as in 4.1. We have:

Answer*: *The set $\mathcal{S}_{(f, L_{(p)}, v)}(k)$ of isomorphism classes of principally quasi-polarized Shimura σ-crystals attached to k-valued points of $\mathcal{N}$ is precisely the set of isomorphism classes of principally quasi-polarized Shimura σ-crystals defined by elements of the set*

$$\{(M_0, g\varphi_0, G_{W(k)}, (v_\alpha)_{\alpha \in \mathcal{J}}, \tilde{\psi}) | g \in G^0_{W(k)}(W(k))\}.$$

Answer: *The set $\mathrm{Lie}(\mathcal{S}_{(f, L_{(p)}, v)})(k)$ of isomorphism classes of Shimura Lie σ-crystals attached to k-valued points of $\mathcal{N}$ is precisely the set of isomorphism classes of Shimura Lie σ-crystals defined by elements of the set $\{(\mathrm{Lie}(G_{W(k)}), g\varphi_0) | g \in G^0_{W(k)}(W(k))\}$.*

The proof of these two statements is a direct consequence of 3.4.15, and of d) of 4.4.1 3) (resp. of c) of 4.4.1 2)) for the first statement (resp. for the second one), cf. also 4.4.12. As we postponed the proof of d) of 4.4.1 3), we invite the reader to formulate a version of the first statement, inspired from 3.4.15, for the k-valued points of any connected component $\mathcal{C}$ of $\mathcal{N}_k$ (in fact 4.2.8.1 takes care of all connected components of $\mathcal{N}_k$ at once). We can use 4.4.7 and 2.3.13.1: we get the above starred answer (for $\mathcal{C}$) without knowing that (M, F^1, φ) is the one introduced in 4.1.

4.12.6. Solution of 4.12.3 in the case the integral canonical model $\mathcal{N}^{\mathrm{ad}}$ of the adjoint quadruple of (G, X, H, v) has the completion property. We refer to 4.9.17.4 for the terminology. Let $G^{\mathrm{ad}}_{W(k)}$ be the reductive group whose special fibre is $G^{\mathrm{ad}}_{B(k)}$. Let v^{ad} be the prime of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ divided by v. Let $\mu^{\mathrm{ad}} : \mathbb{G}_m \to G^{\mathrm{ad}}_{W(k(v^{\mathrm{ad}}))}$ be a cocharacter which over an embedding of $W(k(v^{\mathrm{ad}}))$ in $\mathbb{C}$, extending the composite inclusion

$O_{(v^{\mathrm{ad}})} \subset E(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \subset \mathbb{C}$, is $G^{\mathrm{ad}}(\mathbb{C})$-conjugate to the cocharacters $\mu_x : \mathbb{G}_m \hookrightarrow G^{\mathrm{ad}}_{\mathbb{C}}$, with $x \in X^{\mathrm{ad}}$ defined in [Va2, 2.2]. We have:

Answer: *The set $\mathcal{S}_{(G,X,H,v)}(k)$ of isomorphism classes of Shimura adjoint Lie σ-crystals attached to k-valued points of $\mathcal{N}$ is precisely the set of isomorphism classes of Shimura adjoint Lie σ-crystals defined by elements of the set*

$$\{(\mathrm{Lie}(G^{\mathrm{ad}}_{W(k)}), g\sigma\mu^{\mathrm{ad}}(\frac{1}{p}))|g \in G^{\mathrm{ad}}_{W(k)}(W(k))\}.$$

This is a consequence of 4.9.17.4, [Va2, 6.4.2] and of 4.4.1 2).

4.12.6.1. We would like to point out that $\mathcal{N}^{\mathrm{ad}}$ has the completion property, provided 3.4.15 is true for a certain standard Hodge situation, involving a Shimura quadruple having the same adjoint as (G, X, H, v). To see this, we can assume that G^{ad} is a $\mathbb{Q}$–simple adjoint group. Let $o(G^{\mathrm{ad}})$ be the order of the center of the simply connected group cover of $G^{\mathrm{ad}}_{\mathbb{C}}$. The proof of the above statement in the case when p does not divide $o(G^{\mathrm{ad}})$ is a direct consequence of 4.9.17 and of b) of 4.4.1 2) (if p does not divide $o(G^{\mathrm{ad}})$ then the natural homomorphism $G^{\mathrm{sc}}_{W(k)}(W(k)) \to G^{\mathrm{ad}}_{W(k)}(W(k))$ is surjective).

As we assumed $p \geq 3$, p divides $o(G^{\mathrm{ad}})$ only in the case when $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is of A_l Lie type, with p dividing $l + 1$. But in such a case there is an injective map $f : (G_1, X_1) \hookrightarrow (\mathrm{GSp}(W, \psi), S)$, with $(G^{\mathrm{ad}}_1, X^{\mathrm{ad}}_1) = (G^{\mathrm{ad}}, X^{\mathrm{ad}})$, and with G^{der}_1 simply connected, such that we have a standard PEL situation $(f, L_{(p)}, v_1, \mathcal{B})$, with v_1 dividing v^{ad}, and with the homomorphism $G_{1W(k)}(W(k)) \to G^{\mathrm{ad}}_{W(k)}(W(k))$ surjective, cf. 2.3.5.1. The surjectivity parts results from the fact that for any simple factor of $G^{\mathrm{ad}}_{W(k)}$ there is a general linear group over $W(k)$ which is included in $G_{1W(k)}$ and which has the given factor as its adjoint group, cf. 2.3.5.1 and [Ko1, (A) of p. 395]. Again, directly from def. 4.9.17 we get the validity of the above statement in this case, provided $\mathcal{N}_1$ has the completion property.

4.12.7. The Manin problem for a SHS $(f, L_{(p)}, v)$. Classify all principally quasi-polarized Shimura isocrystals (resp. all Shimura adjoint Lie isocrystals) over k defined by the principally quasi-polarized Shimura σ-crystals (resp. by the Shimura adjoint Lie σ-crystals) attached to k-valued points of $\mathcal{N}$.

4.12.8. The Manin problem for a Shimura quadruple (G, X, H, v) of pre-abelian type, with v dividing a rational prime $p \geq 3$. Classify all adjoint Lie isocrystals over k defined by the Shimura adjoint Lie σ-crystals attached to k-valued points of the integral canonical model of (G, X, H, v).

4.12.9. Remarks. 1) To our knowledge the result (answer) of 4.12.4 has not been stated as such for Siegel modular varieties of dimension greater than 1. It says: any principally quasi-polarized σ-crystal (i.e. any principally quasi-polarized p-divisible group) over k, comes from a principally polarized abelian variety over k, and any principally quasi-polarized filtered σ-crystal (i.e. any principally quasi-polarized p-divisible group over $W(k)$), comes from a principally polarized abelian scheme over $W(k)$.

2) Of course, 4.12.5-6 provide a solution of the problems 4.12.7-8 (provided we assume the completion property). But it is very much desirable, to have a more concrete (and finite) answer to 4.12.7-8, similar to the answer of the (original) Manin problems, in terms of formal isogeny types. We will deal with this in §10.

4.12.10. Example. Let $n \geq m$ be natural numbers, and let $l := m + n$. Let $(f, L_{(p)}, v)$ be a SHS such that $G^{\text{der}}_{\mathbb{R}} = SU(n,m)_{\mathbb{R}}$, $G^{\text{der}}_{\mathbb{Z}_p} = SL(N)$, with N a free $\mathbb{Z}_p$-module of rank l, and $\mathcal{N}$ has the completion property. Then the set of isomorphism classes of Shimura adjoint Lie isocrystals over k defined by the Shimura adjoint Lie σ-crystals attached to k-valued points of $\mathcal{N}$, is the set obtained by taking $\mathfrak{sl}(*)$ of isomorphism classes of isocrystals $(\mathcal{I}, \varphi_{\mathcal{I}})$ over k having only slopes which are rational numbers of the interval $[0,1]$, whose total sum is m, with $\mathcal{I}$ a free $B(k)$-vector space of dimension l.

For more examples see §10 and [Va4].

4.12.11. Problems. 1) Let (G, X, H_p, v) be a Shimura quadruple of preabelian type, with v dividing a rational prime $p \geq 3$. Let $k_1(v)$ be a finite field extension of $k(v)$, and let H_0 be a (compact open) subgroup of $G(\mathbb{A}^p_f)$, such that the subgroup $H_0 \times H_p$ of $G(\mathbb{A}_f)$ is smooth for the pair (G, X). Let σ_1 be the Frobenius automorphism of $W(k_1(v))$. Determine all Shimura adjoint Lie σ_1-crystals over $k_1(v)$ attached to $k_1(v)$-valued points of $\mathcal{N}/H_0$, where $\mathcal{N}$ is the integral canonical model of the quadruple (G, X, H_p, v). In particular, determine conditions on H_0 and $k_1(v)$, which assures that $\mathcal{N}/H_0(k_1(v))$ is not empty.

2) In the above situation, for a fixed subgroup H_0 of $G(\mathbb{A}^p_f)$, determine the smallest algebraic (resp. finite) field extension $k_1(v)$ of $k(v)$ with the property that any isomorphism class of Shimura adjoint Lie σ-crystals (resp. isocrystals) over $\overline{k(v)}$ attached to $\overline{k(v)}$-valued points of $\mathcal{N}$, is defined (by passage to $\overline{k(v)}$) by the Shimura adjoint Lie σ-crystal (resp. isocrystal) attached to a $k_1(v)$-valued point of $\mathcal{N}/H_0$.

3) The same problems 1)-2) above, but for a SHS $(f, L_{(p)}, v)$.

4.12.11.1. Remark. For the problem 1) of 4.12.11, if there is a SHS $(f, L_{(p)}, v)$, with $H_p = G(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p)$, then for $k_1(v)$ big enough, the answer to the problem is of the same form as the one given in 4.12.6, but with k replaced by $k_1(v)$, and with g running through a subset $\mathcal{G}\big(k_1(v), \mathcal{N}/H_0\big)$ of $G_{W(k_1(v))}(W(k_1(v)))$. In §14 we will deal with the problem of determining this subset and how "big" $k_1(v)$ has to be.

4.12.12. Theorem. *Any p-divisible group D over k has a versal deformation to a p-divisible group over a smooth affine $W(k)$-scheme $\mathcal{M}_D$.*

Proof: Let (M, φ) be the σ-crystal defined by D, and let (M, F^1, φ) be the filtered σ-crystal defined by an arbitrary lift of D to a p-divisible group over $W(k)$. If F^1 is not a proper summand of M we have nothing to prove. So we can assume that we have a Shimura filtered σ-crystal $(M, F^1, \varphi, GL(M))$. Let $a := \dim_{W(k)}(F^1)$ and $b := \dim_{W(k)}(M/F^1)$. We have $ab \neq 0$. We can assume that $\varphi = g\varphi_0$, with $g \in SL(M)(W(k))$, and with φ_0 a Frobenius endomorphism of M such that $(M, F^1, \varphi_0, GL(M))$ is a $GL(M)$-canonical lift. So F^1 is left invariant by φ_0, and the slopes of the isocrystal defined by (M, φ_0) are 0 and

1 with multiplicities b and respectively a. If $ab = 1$, then D is the p-divisible group of an elliptic curve, and so the theorem is well known. We assume now that $ab \geq 2$.

We consider a SHS $(f, L_{(p)}, v)$ such that:

a) $G_{\mathbb{R}}^{\mathrm{der}}$ is $SU(a,b)_{\mathbb{R}}$;

b) G splits over $\mathbb{Q}_p$; so $k(v) = \mathbb{F}_p$ and $G_{\mathbb{Z}_p}^{\mathrm{der}} = SL(\mathbb{Z}_p^{a+b})$;

c) G^{ab} is a one or two dimensional torus depending on the fact that a is equal to be b or not;

d) $E(G,X)$ is $\mathbb{Q}$ or a totally imaginary quadratic extension of $\mathbb{Q}$ depending on the fact that a is equal to be b or not;

e) the representation $G_{\mathbb{Z}_p} \to GL(L_{(p)}^* \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p)$ is a direct sum of two irreducible representations ρ_1 and ρ_2 which restricted to $G_{\mathbb{Z}_p}^{\mathrm{der}}$ are respectively the representations associated to the two minimal fundamental weights ω_1 and ω_{a+b-1} (we use the standard notations for weights); in other words we have a direct sum decomposition as $G_{\mathbb{Z}_p}$-modules $L_{(p)}^* \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p = \mathbb{Z}_p^{a+b} \oplus \mathbb{Z}_p^{a+b}$ such that $A \in SL(\mathbb{Z}_p^{a+b})(\mathbb{Z}_p)$ acts as A on the first summand and as the inverse of the transpose of A on the second direct summand.

It is well known that the conditions a) to e) imply (for an elementary solution see [Ko, p. 392]) that (G,X) is of compact type.

The existence of such a SHS is an easy thing, being a simple property of unitary Shimura varieties (for the case $a = 1$ and $b = 2$ -or vice-verse- see [Go]). In fact we get a standard PEL situation $(f, L_{(p)}, v, \mathcal{B})$, with $\mathcal{B}$ as the subalgebra of $\mathrm{End}(L_{(p)})$ formed by endomorphisms fixed by G. If $a \neq b$ then $\mathcal{B}$ is the normalization of $\mathbb{Z}_{(p)}$ in $E(G,X)$. So for $a \neq b$ we need a totally imaginary quadratic extension of $\mathbb{Q}$ which splits over p, together with a symplectic representation $\mathrm{Res}_{\mathcal{B}/\mathbb{Z}_{(p)}}\mathbb{G}_m \hookrightarrow \mathrm{GSp}(L_{(p)}, \psi)$ modelled on $SU(a,b)$. The case $a = b$ is even simpler, and so left as an exercise.

We assume that H_0 is small enough so that all the Hodge cycles of $\mathcal{A}$ are Hodge cycles of $\mathcal{A}/H_0$ as well. So $\mathcal{B} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ acts on the p-divisible group over $\mathcal{N}_{W(k)}/H_0$ defined by $\mathcal{A}_{H_0}$. We deduce (cf. e)) that this p-divisible group is a direct sum of two p-divisible groups $\mathcal{D}_1$ and $\mathcal{D}_2$. In fact $\mathcal{D}_1 = \mathcal{D}_2^t$. Moreover $\mathcal{D}_1$ and $\mathcal{D}_2$ are versal deformations in each k-valued point of $\mathcal{N}_{W(k)}/H_0$. We can assume that $\mathcal{D}_1$ has dimension a in each k-valued point of $\mathcal{N}_{W(k)}/H_0$.

We check now that $\mathcal{N}$ has the completion property. We follow the idea of 4.12.4. The similar isogeny property can be easily checked in this context: with the notations of 4.12.4, if any invertible element of $\mathcal{B}$, when viewed naturally (via de Rham components) as an automorphism of M_0', is also an automorphism of M_0, then all these elements (assumed to define automorphisms of A') do define automorphisms of A', and so [Ko, ch. 5] applies. So we just need to check that for any cyclic diagonalizable Shimura σ-crystal $\mathcal{C}_N = (N, F_N^1, \varphi_N, GL(N))$, with N a free $W(k)$-module of rank $a+b$, and with F_N^1 a direct summand of it of rank b, there is a k-valued y_N point of $\mathcal{N}_{W(k)}/H_0$ such that the Shimura σ-crystal we get over k by pulling back $\mathcal{D}_1$ (see 2.2.9 9)), is isomorphic to $\mathcal{C}_N$.

To see this let T_N be a maximal torus of $GL(N)$ such that $(N, F_N^1, \varphi_N, T_N)$ is a Shimura σ-crystal (cf. 2.2.22). We consider the Galois representation associated to it; it can be viewed as a homomorphism

$$\rho_N : \mathrm{Gal}(B(k)) \to \tilde{T}_N(\mathbb{Z}_p),$$

where $\tilde{T}_N$ is a maximal subtorus of the image $GL(1)$ of $G_{\mathbb{Z}_p}$ in the GL-group $GL(1)$ of the first $\mathbb{Z}_p^{a+b}$-summand of $L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$. Its extension to $W(\mathbb{F})$ is naturally equipped with a cocharacter μ_N. Let (cf. [Har]) T_0 be a maximal $\mathbb{Z}_{(p)}$-subtorus of $G_{\mathbb{Z}_{(p)}}$ such that:

– over $\mathbb{R}$ it is the extension of a compact torus by a one dimensional torus;

– its extension to $\mathbb{Z}_p$ has an image in $GL(1)$ which is $GL(1)(\mathbb{Z}_p)$-conjugate to $\tilde{T}_N$.

We get naturally (from μ_N) a cocharacter

$$\mu(T_0) : \mathbb{G}_m \to T_{0\mathbb{C}}.$$

As T_0 is the extension of a split torus by a torus which is compact over $\mathbb{R}$, we deduce that the image of $\mu(T_0)$ is defined over $\mathbb{R}$. We get immediately a special quadruple $(T_{0\mathbb{Q}}, \{h_0\}, T_0(\mathbb{Z}_p), v_p)$, with $h_0 : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \to T_{0\mathbb{R}}$ defined by $\mu(T_0)$, and with v_p that prime of $E(T_0, \{h_0\})$ dividing v which has the property that the cocharacter of $T_{0W(k(v_p))}$ we naturally get (see 4.1) is modelled on μ_N. Now one can easily check that any k-valued point of $\mathcal{N}_{W(k)}/H_0$ factoring through the integral canonical model of this special quadruple, can be chosen as the desired y_N point.

Now the theorem is a consequence of 4.12.5: there is $y \in \mathcal{N}_{W(k)}/H_0$ such that $\mathcal{D}_1$ over y is isomorphic to D; as $k(v) = \mathbb{F}_p$, to get 4.12.5 for our SHS $(f, L_{(p)}, v)$ we can use 4.6.12 instead of d) of 4.4.1 3. This ends the proof of the theorem.

4.12.12.0. Remark. In the above proof of 4.12.12 we can substitute arguments on unitary Shimura varieties by 2.3.5.1: what is important that over $\mathbb{Z}_p$ we get a right direct sum decomposition of $L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ into irreducible representations so that "we can single" out a p-divisible group $\mathcal{D}_1$ as above which is "related" to D. In fact we do not need a direct sum decomposition: we just need an adequate irreducible subrepresentation. So e) above can be replaced by more general versions, inspired from [Va2, 6.5-6].

We use this idea to show that 4.12.12 remains true for any Shimura p-divisible group (M, φ, G): for the case when G together with its filtration class is related to a Shimura variety of preabelian type see §7; for the case when the situation is related to Shimura varieties of special type see [Va6]. Moreover it seems to us that the condition $k = \bar{k}$ is not (really) needed: in other words, (in many situations) it is enough to work with a perfect field (as the combination of 3.15 and 4.12.12 suggests).

4.12.12.1. Remark. The natural question arises: what about the case $p = 2$? Defining the completion property for $p = 2$ as in 3.4.16.1 and 4.9.17.4, we have:

Answer: *4.12.5-6 remain true for the case of a $p = 2$ SHS $(f, L_{(2)}, v)$ whose integral canonical model $\mathcal{N}$ has the completion property.*

Moreover, for 4.12.6 it is enough to assume that $\mathcal{N}$ has the completion property. With the notations of 4.12.6, we just need to check that in 4.12.6 we can still use any element of

$G^{\mathrm{ad}}_{W(k)}(W(k))$ for our $p=2$ SHS $(f, L_{(2)}, v)$. This goes as follows. For any adjoint simple factor $\tilde{G}$ of $G^{\mathrm{ad}}_{\mathbb{Z}_p}$, there is a semisimple subgroup $\tilde{G}^1$ of $G_{\mathbb{Z}_p}$ such that we have a central isogeny $q_1 : \tilde{G}^1 \to \tilde{G}$. If $\tilde{T}$ is a maximal torus of $\tilde{G}$, then

$$\tilde{G}(W(k)) = q_1(\tilde{G}^1(W(k)))\tilde{T}(W(k));$$

so the natural variant of 3.11.1 (see also 3.6.5) involving adjoint Lie σ-crystals applies to give us (with the notations of 4.12.6):

Fact. *Any Shimura adjoint Lie σ-crystal* $(\mathrm{Lie}(G^{\mathrm{ad}}_{W(k)}, g\sigma\mu^{\mathrm{ad}}(\frac{1}{p}))$, *with $g \in G^{\mathrm{ad}}_{W(k)}(W(k))$ is isomorphic to one defined similarly, but for some $g^1 \in G^{\mathrm{der}}_{W(k)}(W(k))$.*

From 4.12.12 and 2.3.18 we get directly:

4.12.12.2. Corollary. *4.12.12 remains true for $p=2$: any 2-divisible group over an algebraically closed field $\tilde{k}$ of characteristic 2 has a versal deformation (to a 2-divisible group) over a smooth affine $W(\tilde{k})$-scheme.*

4.12.12.3. Tate deformation algebras and Shimura p-divisible groups. Let $\tilde{k}$ be a perfect field of characteristic $p \geq 2$. Let $\tilde{\sigma}$ be the Frobenius automorphism of $W(\tilde{k})$ and $\tilde{k}$. Let $(\tilde{M}, \tilde{\varphi}, \tilde{G})$ be the not necessary quasi-split Shimura $\tilde{\sigma}$-crystal associated to a Shimura p-divisible group $\tilde{D}$ over $\tilde{k}$. Let $\tilde{d} := dd(\tilde{M}, \tilde{\varphi}, \tilde{G}))$. Let $\tilde{R}$ be the p-adic completion of the henselization $\tilde{R}(h)$ of $W(\tilde{k})[x_1, ..., x_{\tilde{d}}]$. There is (cf. Fact $1'$ of 3.15) a versal Shimura p-divisible group $D_{\mathrm{def}}(\tilde{D})$ over $\mathrm{Spec}(\tilde{R})$, which in the $\tilde{k}$-valued point $\tilde{y}$ of $\mathrm{Spec}(\tilde{R})$ it becomes $\tilde{D}$. Warning: $D_{\mathrm{def}}(\tilde{D})$ is not uniquely determined; here unicity is up to isomorphism of $D_{\mathrm{def}}(\tilde{D})$ and of $\tilde{R}$.

We refer to $D_{\mathrm{def}}(\tilde{D})$ as a Tate deformation Shimura p-divisible group of $\tilde{D}$. Writing $D_{\mathrm{def}}[p^n] = \mathrm{Spec}(R_n(\tilde{D}))$, we refer to $R_n(\tilde{D})$ as a level-n Tate deformation algebra of $\tilde{D}$ or of $\tilde{D}[p^n]$. We think it is an interesting (and extremely important) problem to study (classify) these finite flat $\tilde{R}$-algebras. We have a similar problem (much harder) to understand (classify) the similarly defined $\tilde{R}(h)$-algebras, in the case $D_{\mathrm{def}}(\tilde{D})$ is definable over $\tilde{R}(h)$ itself.

4.12.12.4. Exercise. Use the proof of 4.12.12 to get a new proof of the original Manin problem, and so of 4.12.4. Hint: using products of principally quasi-polarized abelian varieties, reduce to the case when we have only the slopes α and $1-\alpha$, with $\alpha \in (0, \frac{1}{2})$; writing $\alpha = \frac{a}{a+b}$, with $(a,b)=1$, $a, b \in \mathbb{N}$, $b \geq a$, make use of the SHS mentioned in the proof of 4.12.12.

4.12.13. Remarks. 1) Let $p \geq 2$. 3.15 (or 3.4.7.1 and 3.11.1 for $p \geq 3$) can be interpreted as the solution of an integral Manin problem: Any Shimura filtered σ-crystal $(M, F^1, \varphi, \tilde{G})$ over k can be connected (through the p-adic completion of an $\mathbb{N}$-pro-étale scheme over a smooth scheme over $W(k)$, having a connected special fibre) with a G-canonical lift $(M, F^1, \varphi_1, \tilde{G})$. Moreover, 3.15 and 3.4.18.8.3 point out that we have a version of this for any perfect field of characteristic p.

We have as well a version for quasi (even pseudo) Shimura filtered σ-crystals over perfect fields (cf. 3.4.1.6).

2) The proof of 4.12.12 and 4.12.12.0 show that, in 4.12.12 and 4.12.12.2, if D and its dual are not étale, then instead of an affine $W(k)$-scheme $\mathcal{M}_D$ we can work with a connected projective smooth scheme $\mathcal{P}_D$ over $W(k)$, such that the p-divisible group over it is versal in each k-valued point of $\mathcal{P}_D$. In other words, in case D is the p-divisible group of an elliptic curve, we can use 4.12.12.0 in the context of a $(p=2)$ SHS $(f, L_{(p)}, v)$, with G^{ad} an absolutely simple $\mathbb{Q}$-group of A_1 Lie type whose $\mathbb{Q}$-rank is zero and such that it splits over $\mathbb{Q}_p$ (cf. 2.3.5.1, 2.3.18 and 4.3.6).

Moreover, choosing H_0 small enough, we can assume that $\mathcal{P}_D$ is of general type.

4.12.14. The general local integral Manin problem for Siegel modular varieties. We consider a SHS of the form $(1_{(\mathrm{GSp}(W,\psi),S)}, L_{(p)}, p)$. So, with the standard notations, $\mathcal{M} = \mathcal{N}$. Let $m \in \mathbb{N} \cup \{0\}$. Let $R := W(k)[[x_1, ..., x_m]]$ and let Φ_R be a Frobenius of it compatible with σ. The problem is to determine all principally quasi-polarized p-divisible objects of $\mathcal{MF}_{[0,1]}(R)$ defined by principally quasi-polarized p-divisible groups over R, obtained from $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ through morphisms $\mathrm{Spec}(R) \to \mathcal{N}$.

4.12.15. The general local integral Manin problem for a SHS $(f, L_{(p)}, v)$. Let $(f, L_{(p)}, v)$ be a SHS. The problem is the same: which principally quasi-polarized p-divisible objects with cycles of $\mathcal{MF}_{[0,1]}(R)$ are obtained (from $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ and the Hodge cycles $(w_\alpha^{\mathcal{A}})_{\alpha \in \mathcal{J}'}$) through morphisms $\mathrm{Spec}(R) \to \mathcal{N}$?

4.12.16. The general local integral Manin problem for a Shimura quadruple (G, X, H, v) of preabelian type, with $(v, 2) = 1$. The problem is to determine which p-divisible objects of $\mathcal{MF}_{[-1,1]}(R)$ are isomorphic to Shimura adjoint filtered Lie F-crystals over R/pR attached to (cf. 4.9.20) morphisms $\mathrm{Spec}(R) \to \mathcal{N}$.

4.12.17. The solution of 4.12.14. The answer is: any principally quasi-polarized p-divisible object of $\mathcal{MF}_{[0,1]}(R)$. This is an easy consequence of 4.12.4, [Va2, 5.4] and 3.4.18.7.

4.12.18. The solution of 4.12.15 in the case $\mathcal{N}$ has the completion property. We use the notations of 4.12.5. The answer is: any such p-divisible object of the form $(M_0 \otimes_{W(k)} R, F_0^1 \otimes_{W(k)} R, g_R(\varphi_0 \otimes 1), (v_\alpha)_{\alpha \in \mathcal{J}}, \tilde{\psi})$, with g_R an arbitrary element of $G^0_{\mathbb{Z}_{(p)}}(R)$. This is an easy consequence of 4.12.5, [Va2, 5.4] and 3.4.18.7.

4.12.19. The solution of 4.12.16 in the context of 4.12.6. We use the notations of 4.12.6. The answer is: any such p-divisible object of the form

$$(\mathrm{Lie}(G_R^{\mathrm{ad}}), g_R^{\mathrm{ad}}(\sigma\mu^{\mathrm{ad}}(\frac{1}{p}) \otimes 1), F^0(\mathrm{Lie}(G_R^{\mathrm{ad}})), F^1(\mathrm{Lie}(G_R^{\mathrm{ad}}))),$$

with $g_R^{\mathrm{ad}} \in G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}(R)$. Here the filtrations F^0 and F^1 are obtained as usual from the cocharacter μ^{ad} of 4.12.6. This is an easy consequence of 4.12.6, of 3.4.18.7, of def. 4.9.17, of [Va2, 5.4] and of 2.3.17.

4.12.16-18 admit a non-filtered version. Not to be too long this is not going to be presented here.

4.12.20. Exercise. State and prove the analogues of 4.12.14-15 and 4.12.17-18 for the case when $p = 2$. Hint: use 2.3.18 and 4.12.12.1.

4.12.21. The general (integral) Manin problems. Given an arbitrary $\mathbb{Z}_{(p)}$-scheme Y we would like to understand which principally quasi-polarized p-divisible groups (resp. which isogeny classes of principally quasi-polarized p-divisible groups) over Y are coming from principally polarized abelian schemes (resp. from isogeny classes of principally polarized abelian schemes) over Y of relative dimension e. This forms the general integral Manin problem (resp. the general Manin problem) for the Siegel modular variety of 4.12.4. Similarly we can define the general (integral) Manin problem for an arbitrary SHS $(f, L_{(p)}, v)$: we need to replace p-divisible groups by Shimura p-divisible groups (see 2.2.20; the definition there can be adapted to any $O_{(v)}$-scheme). We can also define the general (integral) Manin problem in the context of 4.12.3: again 4.9.20 can be extended to any $O_{(v)}$-scheme (see §5 for details).

The general (integral) Manin problems look very hard in such a generality. In the cases when we are dealing with (the p-adic completion of) a (regular formally) smooth separated scheme Y (over DVR's of index of ramification 1 over $\mathbb{Z}_{(p)}$ or $O_{(v)}$) we hope that some partial solutions can be obtained based on 3.4.

4.13. The local form of the invariance principle. Let $p \geq 3$ be a rational prime, and let k be a perfect field of characteristic p. Let (G_i, X_i, H_i, v_i), $i = \overline{1,2}$, be two Shimura quadruples of preabelian type, with v_i dividing p. Let $\mathcal{N}_i$ be the integral canonical model of (G_i, X_i, H_i, v_i), $i = \overline{1,2}$. Let $H_{0i} \subset G_i(\mathbb{A}_f^p)$ be a subgroup such that $H_{0i} \times H_i$ is a subgroup of $G_i(\mathbb{A}_f)$ smooth for (G_i, X_i), $i = \overline{1,2}$. Let $y_i : \mathrm{Spec}(k) \to \mathcal{N}_{ik}/H_{0i}$, $i = \overline{1,2}$, such that the non-trivial part of the Shimura adjoint Lie σ-crystal attached to y_1 is isomorphic to the non-trivial part of the Shimura adjoint Lie σ-crystal attached to y_2. We fix such an isomorphism $\mathcal{I}$. Let O_{y_i} be the local ring of y_i (in $\mathcal{N}_{iW(k)}/H_{0i}$), and let $O_{y_i}^{h\wedge}$ be the p-adic completion of its henselization, $i = \overline{1,2}$. For simplicity we assume that $k = \bar{k}$. From 4.9.20 we deduce the existence of a Shimura adjoint filtered Lie F-crystal $\mathfrak{C}_{y_i}^{\mathrm{ad}}$ over $O(p)_{y_i}^h := O_{y_i}^{h\wedge}/pO_{y_i}^{h\wedge}$. Let $\mathfrak{C}_{y_i}^{\mathrm{nt}}$ be its non-trivial part (it is defined as in 3.10.1). The refined canonical Lie stratification of $\mathcal{N}_{ik(v_i)}$ makes $\mathrm{Spec}(O(p)_{y_i}^h)$ to be a stratified scheme.

Let $n \in \mathbb{N}$.

4.13.1. Theorem. *There is an isomorphism*

$$\rho : \mathrm{Spec}(O_{y_1}^{h\wedge}) \tilde{\to} \mathrm{Spec}(O_{y_2}^{h\wedge})$$

inducing an isomorphism

$$\rho(p) : \mathrm{Spec}(O(p)_{y_1}^h) \tilde{\to} \mathrm{Spec}(O(p)_{y_2}^h)$$

of stratified schemes, and such that $\rho^*(\mathfrak{C}_{y_2}^{\mathrm{nt}}/p^n\mathfrak{C}_{y_2}^{\mathrm{nt}})$ is isomorphic to $\mathfrak{C}_{y_1}^{\mathrm{nt}}/p^n\mathfrak{C}_{y_1}^{\mathrm{nt}}$, through an isomorphism lifting $\mathcal{I}$ mod p^n.

4.13.2. Theorem. *Let $(f^i, L_{(p)}^i, v^i)$, $i = \overline{1,2}$, be two standard Hodge situations. Using the standard notations of 2.3.1-3 for the SHS $(f^i, L_{(p)}^i, v^i)$, but putting everywhere* *an upper right index i, let $y_i : \mathrm{Spec}(k) \to \mathcal{N}_k^i/H_0^i$ be a morphism, $i = \overline{1,2}$. Here k is an arbitrary perfect field containing $k(v^1)$ and $k(v^2)$. We assume that the principally quasi-polarized Shimura σ-crystal attached to y_1 is isomorphic to the principally quasi-polarized Shimura σ-crystal attached to y_2. We fix such an isomorphism $\mathcal{I}$. Then there is an isomorphism*

$$\rho : \mathrm{Spec}(O_{y_1}^{h\wedge})\tilde{\to}\mathrm{Spec}(O_{y_2}^{h\wedge})$$

inducing an isomorphism

$$\rho(p) : \mathrm{Spec}(O(p)_{y_1}^{h})\tilde{\to}\mathrm{Spec}(O(p)_{y_2}^{h})$$

of stratified schemes, and such that the pull back under ρ, of the kernel of the multiplication by p^n of the principally quasi-polarized Shimura p-divisible group $\mathcal{D}_{y_2}$ over $\mathrm{Spec}(O_{y_2}^h)$ *obtained from the principally quasi-polarized Shimura p-divisible group over $\mathcal{N}^2/H_0^2$ of the principally polarized abelian scheme $(\mathcal{A}_{H_0^2}^2, \mathcal{P}_{\mathcal{A}_{H_0^2}^2})$ by pull back under the canonical morphism* $\mathrm{Spec}(O_{y_2}^h) \to \mathcal{N}^2/H_0^2$, *is isomorphic to the kernel of the multiplication by p^n of the principally quasi-polarized Shimura p-divisible group $\mathcal{D}_{y_1}$ over* $\mathrm{Spec}(O_{y_1}^h)$, *defined similarly as $\mathcal{D}_{y_2}$ (here $O_{y_i}^h$, $O(p)_{y_i}^h$ have the same meaning as in 4.13.1), through an isomorphism lifting $\mathcal{I}$ mod p^n.*

4.13.1 is a consequence of 3.4.14.4; for the stratification part, we need to take n big enough and to apply [Ka2, 1.4.4]. 4.13.2 is a consequence of 3.4.14.4 (or of [Va2, 5.4], via Artin's approximation theorem); for both theorems cf. definitions 4.9.17 and 4.9.20 (in the case of a SHS, 4.9.20 is trivial, just introducing some terminology) (in the case of 4.13.2 cf. 2.2.21 9) as well).

4.13.3. Remarks. 1) The form of 4.13.2, with $n = \infty$, where instead of taking henselization we take the completion, was obtained previously: the result of Serre–Tate that (in many situations) deforming (locally) an abelian variety is the same as deforming its p-divisible group implies this form for a standard PEL situation; for the general case cf. [Va2, 5.4].

2) The condition $k = \bar{k}$ in 4.13.1 can be weaken: see §5 (cf. 4.9.20).

4.14. Final remarks. Here we include some remarks which are sort of overview of great parts of [Va2] and of this paper.

4.14.1. The generic situation. Many of the results presented here or in [Va2] are trivial or very easy, from the generic point of view, i.e. working with a prime p big enough. But the main inconvenience of this simple approach is that we do not have at all effective bounds (of how big p has to be), and so, from the point of view of concreteness, is void. We list now some of the results which generically "are easy", and motivate why this is so.

a) [Va2, 3.4.1 and 3.4.7] is all we need to get generically the existence of integral canonical models of Shimura varieties of Hodge type.

b) [Va2, 6.2.2 a), 6.2.3 and 6.8] is all we need to extend this to Shimura varieties of preabelian type.

c) [Va2, 6.4.4] generically is obvious.

d) The density part of 4.2.1 is obvious generically (due to the existence of smooth toroidal compactifications of canonical models of Shimura varieties and of c) above).

e) Using [Va2, 3.2.11] and d) above we get immediately the generic form of 4.2.1 itself.

f) By the same reason as in e) above, P1 of 4.6 is obvious generically.

For e) and f) cf. also the proof of 4.12.12. The list can keep going on.

4.14.2. The special type case. There are plenty of Shimura σ-crystals of (so called) special type. So, even before the proof of the existence of integral canonical models of Shimura varieties of special type and of the validity of the expectation of [Va2, a) of 3.2.7 8)], we can speak about stratifications (as in 4.5) of schemes of moduli of Shimura p-divisible groups obtained from Shimura σ-crystals of special type, cf. 3.4.1.3 (and §7). For the study of these stratifications see [Va6].

4.14.3. The case of a $p=2$ SHS. From 2.3.18 and 3.14 we get immediately that most of the results of 4.1-13 remain true for a $p=2$ SHS $(f, L_{(2)}, v)$ (see also [Va5]). However some of them need some reformulation. Below we refer to such a $p=2$ SHS.

A. All the results of 4.1-3 remain: no extra reformulation is needed.

B. 4.4.1 1) and 3) as well as b), c) and d) of 4.4.1 2) remain true without any modification. We do expect (at least in the majority of cases) that d) of 4.4.1 3) and 4.4.12 also hold. But b) of 4.4.1 2) and its version for U-ordinary points (see 4.4.13.1 d)) need substantial reformulation.

As the G-canonical lifts for $p \geq 3$ were defined at the level of filtrations in the crystalline cohomology context (cf. a) of 4.4.1 2)), for $p=2$, for their (desired) uniqueness (and the parts involving it) as well as for 4.4.5-6 we have to proceed very carefully, cf. 2.3.18.1. As a first warning we have: making use of the existence of (to be mentioned in E below) G-crystalline (additive or multiplicative) coordinates (the argument for this is the same as in the ordinary case) we get that for any G-ordinary point of $\mathcal{N}_{k(v)}$ with values in k, denoting by $(M_0, \varphi_0, G_{W(k)})$ its attached Shimura σ-crystal, any lift $\mathrm{Spec}(W(k)) \to \mathcal{M}_{W(k(v)}$ such that the resulting quadruple $(M_0, F_0^1, \varphi_0, G_{W(k)})$ is a Shimura filtered σ-crystal, does not necessarily factor through $\mathcal{N}$. So even if $k(v) = \mathbb{F}_2$ (i.e. when we can use P1 of 4.6, cf. D below), we can not get (at least theoretically) the unicity part by specifying that we choose the usual canonical lift.

We explain the situation, in the slightly more general case when $k(v^{\mathrm{ad}}) = \mathbb{F}_2$. We use the notations and terminology of 4.7.21 and of 4.7.30 1). The formal moduli scheme of G-deformations of (A_y, p_{A_y}), inside the formal torus of deformations of D_y^1 (of 4.7.30 1)) is the translation of a formal subtorus by a 2-torsion point. This subtorus is uniquely determined, but not the 2-torsion point, provided it is non-zero. So if the referred 2-torsion point is zero, then we can define the G-canonical lift uniquely, but not otherwise. We do not know when such a 2-torsion point is zero.

So, in general, we have to allow either a finite number of such G-canonical lifts, or to try to apply something else. As a second warning we have: some of G-canonical lifts might be defined only up to passage from k to an abelian extension whose Galois group is a subgroup of $(\mathbb{Z}/2\mathbb{Z})^d$. Presently we do not know what in general would be the best "something else". For instance, one could try to choose one such G-canonical lift to which a "bigger" (or specified) part of the $\mathbb{Z}$-Lie algebra of G-endomorphisms (or of the $\mathbb{Z}$-algebra of endomorphisms, or of the group of G-automorphisms or of automorphisms) of A_y, lifts; or in case k is not algebraically closed, one could try to specify that the G-canonical lift has to be defined over $W(k)$, etc. However we do have:

Fact. *In the case the Lie p-rank $p - \mathrm{Lie}_G(y)$ is zero, then we can define a unique G-canonical lift.*

Proof: We refer again to the decomposition $M_y = M_y^1 \oplus M_y^2$ of 4.7.30 1). The hypothesis $p - \mathrm{Lie}_G(y) = 0$ gets translated in $\mathrm{Lie}(N) \subset \mathrm{End}(M_y^2)$. So we are deforming D_y^2, while fixing D_y^1. The fact follows, cf. 2.3.18.1.

Similarly, for U-canonical lifts, the above part on uniqueness of G-canonical lifts applies. So 4.4.8 has to be modified as well. Moreover, the above fact points out the following thing: for $p = 2$ it is easier to understand the U-canonical lifts of U-ordinary points which in some sense are far from being ordinary.

Also 4.4.4 has to be modified, in case a G-canonical lift is not uniquely determined by its filtration in the crystalline cohomology context. The easiest way to do it is as follows.

Corollary. *Referring to the notations of 4.4.4 we have:*

$$4\mathrm{End}_G(A_y) \subset \mathrm{End}_G(A_z)$$

and the elements of $\mathrm{Aut}_G(A_y)$ *which are congruent to one mod* 4 *are also elements of* $\mathrm{Aut}_G(A_z)$.

Proof: This is a consequence of Serre–Tate's theory, see [Me].

Accordingly, 4.4.5-6 remain true. The rest, i.e. 4.4.7 and 4.4.9-12, does not need modifications.

C. 4.5 involves just Newton polygons, and so it needs no modification at all.

D. The whole of 4.6 except the parts of P2 and 4.6.10 A referring to canonical lifts remains true without modifications (of course 4.6.11 B makes no sense). Moreover, P2 and 4.6.10 A can be modified as follows: any G-canonical lift is isogeneous to the (usual) canonical lift.

E. Related to 4.7.1 there are only two things to be modified. First, in 4.7.21 the unit element can corresponds to any one of the G-canonical lifts of y; accordingly, in 4.7.36 we have to speak not about formal subtori, but about the translate of a formal subtorus by a 2-torsion element. Second, the unpublished work of A. Ogus on ordinary elliptic curves implies that in 4.7.30 8), with $p = 2$, the Frobenius of R_n which makes g_{R_n} to be the identity element, is not unique (but their number is a power of 2, and so finite).

F. The whole of 4.8 remains true for $p = 2$, as Fontaine's comparison theory with rational coefficients is still true in the case of good reduction and mixt characteristic $(0, 2)$, see [Fa1].

G. Also part of 4.9 can be adapted to some extend to this situation: as we do not know if $\mathcal{N}$ is uniquely determined, we have to work out 4.9.8 using just $\mathcal{N}$ (and so we presently lose the functorial aspect of 4.9.8). In particular, we keep quiet regarding the fact that $(f_4, L_{1(p)} \oplus L_{2(p)}, v_4)$ is or is not a SHS.

H. 4.10 remains true, provided we assume that $\mathcal{N}$ exists and is such that 4.9.8 in its non-functorial form applies to it.

I. No modifications are needed to 4.11.

J. Related to 4.12 see 4.12.12.1 and 4.12.20 1).

K. Moreover the whole of 4.13 remains true (cf. 3.14 and the proof in 3.15).

4.14.4. Some extensions. The greatest part of the terminology used before can be extended to (filtered) (Lie) σ-$\mathcal{S}$-crystals. In particular we get σ-$\mathcal{S}$-crystals which are U-ordinary, T-ordinary, Shimura-ordinary, (potentially or essentially) cyclic diagonalizable, etc. Similarly, in a geometric context we get U-ordinary, T-ordinary and G-ordinary points, etc; see [Va9] for an extensive study. This extension from categories $\mathcal{MF}_{[0,1]}(W(k))$ to other categories $\mathcal{MF}_{[a,b]}(W(k))$ is automatic.

If we extend the notion of cyclic diagonalizability to σ-$\mathcal{S}$-crystals, then in definition 2.2.12 c) we should require as well the $W(k)$-basis $\{e_i | i \in A\}$ to be compatible with the Lie structure in the sense that $[e_i, e_j] \in e_{k(i,j)} W(k)$, with $k(i,j) \in A$, $\forall i, j \in A$ (warning: it is unreasonable to require $[e_i, e_j] \in e_{k(i,j)}\mathbb{Z}$). It can be easily checked that in all the previous cases of cyclic diagonalizable Shimura filtered Lie σ-crystals we do get such bases.

Appendix

It is desirable to make the connection between the present work and [RR]. There are three results here related to [RR]. They were obtained independently, and their main part precedes [RR], cf. [Va1] and 1.25. These results are:

A) The part of 3.1.0 expressed in rm. 2) of 3.4.6.1. This is related to [RR, ch. 4]. The loc. cit. obtains rm. 2) of 3.4.6.1 in a more general context. However our ideas can be adapted to this more general context (see below).

B) The proof of b) of 4.4.1 2). Its idea is the same as in the general result [RR, 1.16].

C) 4.5.4 and 4.5.6.1. The idea of 4.5.4 is the same underlying [RR, def. 3.3]. The proposition 4.5.6.1 is related to the main result 3.6 of [RR]. It can be used to restate this main result in a simpler way, provided we restrict the class of groups. We use the notations of [RR, ch. 3]. We have:

The Lie criterion. *If all the factors of G^{ad} are of A_n, B_n, C_n or D_n Lie type ($n \in \mathbb{N}$), then for two points s_1, s_2 of S, we have $\bar{b}(s_1) < \bar{b}(s_2)$, iff the Newton polygon of the Lie isocrystal over $\bar{s}_1$ (defined by* $\mathrm{Lie}(G)$*, or in the sense of [RR, 3.3], corresponding to the adjoint representation of G on its Lie algebra) is above the Lie isocrystal over $\bar{s}_2$.*

The proof of this is entirely the same as of 4.5.6.1. We do not know if (or when) this criterion is true without the assumption on the Lie type of the simple factors of G^{ad}. In practice it is easier to work with the above criterion than with the partial ordering on $B(G)$ defined in [RR, 2.3]. However [RR, 2.4] is a very powerful theoretical tool (for instance, the proof of [RR, 4.2] is very elegant).

As a conclusion: the present work is integral, and so more refined (handling the deformation aspects and subtler things mod p, like 3.9) and more restricted (we work in a context of good reduction and of p-divisible groups), while [RR] works rationally, and so less refined, but more general.

The greatest part of the ideas and results presented in 3.4 can be used (i.e extended) to more general contexts. Let us start with an arbitrary filtered σ-$\mathcal{S}$-crystal

$$\mathcal{C} = (\tilde{M}, (\tilde{F}^i_{i\in S(\tilde{a},\tilde{b})}), \tilde{\varphi}, \tilde{G});$$

we do not assume that it satisfies the W-condition, or that $\tilde{G}$ is reductive group (see 3.4.1.5). There are four steps in achieving this extension.

1) If $\mathcal{C}$ satisfies the W-condition and $\tilde{G}$ is a quasi-split reductive group, then 3.2.3 and 3.4.6 remain valid, with no essential change. So the obstructions of proving 3.1, 3.4.1.2-3, and 3.5-6 in this context come just from 3.8.9 (cf. the discussion in 3.4.1.6) and 3.6.6. The obstruction coming from 3.6.6 is not a very serious one (as we expect 3.6.6 to be a known result), as it can be substituted (cf. 3.4.6.0) by the expectation of 3.4.1.6. So the serious obstruction is in achieving global deformations of $\mathcal{C}$. We hope to come back to this in a future paper.

2) If $\mathcal{C}$ satisfies the W-condition and $\tilde{G}$ is a reductive group which is not quasi-split, then (provided we achieved the global deformations in 1) above) 3.8 applies.

3) If $\mathcal{C}$ satisfies the W-condition, but $\tilde{G}$ is not a reductive group, we have to proceed as follows. For the problems of Newton polygons (cf. [Ka2, 1.4.4]) it matters only how $\tilde{G}^{\wedge}$ looks like. This reduces easily the problem to the case when $\tilde{G}$ is the extension of a reductive group by a nilpotent group. So to work out 3.5-6 and 3.4.6 in this situation we have to combine the case of a reductive with the case of a nilpotent group. How this is accomplished is explained in [Va5] (cf. 4.6.8 2)).

4) The case when $\mathcal{C}$ does not satisfy the W-condition (we recall that the main result of [Wi] is obtained under some conditions). This is an obstacle in getting 3.4.6 and 3.5-6 in this context. Still 3.1.0 should be provable in this context, using global deformations. We hope to come back to this step in a future paper.

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present address:

Adrian Vasiu
University of Utah
Department of Mathematics
Salt Lake City, UT-84112-0090, U.S.A.

e-mail: adrian@math.utah.edu