Points of Integral Canonical Models of Shimura Varieties of Preabelian Type, p-divisible Groups, and Applications

Adrian Vasiu

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ABSTRACT. We prove the existence of integral canonical models of Shimura varieties of preabelian type w.r.t. primes dividing 3 (6 in the C and $D^{\mathbb{H}}$ type case), as well as w.r.t. some non-hyperspecial subgroups and primes dividing a prime $p \geq 2$. We completely classify the standard Hodge situations. We prove that, under a slight restriction, Shimura-canonical lifts of Shimura-ordinary abelian varieties over finite fields have complex multiplication. We classify Shimura p-divisible groups over closed subschemes of $\mathrm{Spec}(W(k)[[x_1, ..., x_n]])$, with $n \in \mathbb{N}$, and k a perfect field of characteristic $p \geq 3$. We prove theorems on extending reductive groups.

Key words: integral canonical models, Shimura varieties of preabelian type, (Shimura-ordinary) p-divisible groups, complex multiplication, abelian varieties.

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Contents

§1. Introduction

1.0. For how part two of this paper is organized we refer to [Va4, 1.2] and [Va2, 1.16-21]. In this part 2C we present the results announced in [Va2, 1.17] and some of their applications. It is entirely independent of part 2B, and so we do not hesitate to write it before [Va5].

1.1. Conventions and notations. In the whole of the paper we use the notations and conventions of [Va2, 2.1 and 2.3.7]. Whenever we consider (cf. [Va2, 2.3]) a SHS (standard Hodge situation) $(f, L_{(p)}, v)$, we use freely the notations introduced in [Va2, 2.3.1-3]. Always by k we denote a perfect field of characteristic a prime $p \geq 2$. $W(k)$ is the Witt ring of k, and $B(k) := W(k)\left[\frac{1}{p}\right]$. For F a field we denote by $\bar{F}$ its algebraic closure. We denote by E_v the completion of a number field E with respect to a prime of it v. The connected component of the origin of a Lie group G is denoted by G^0.

The different extension properties to be used are abbreviated as in [Va1, 3.2.3]; so the extended extension property is abbreviated as EEP, the smooth extension property is abbreviated as SEP, etc. For generalities on integral models of Shimura varieties we refer to [Va1, §3], on Shimura crystals we refer to [Va2, 2.2], on Shimura p-divisible groups we refer to [Va2, 2.2.20] and [Va4, §2], on Fontaine's categories we refer to [Va2, 2.1 and 2.2.1], on Shimura-canonical lifts (or G-canonical lifts) and Shimura-ordinary points (or G-ordinary points) we refer to [Va2, 3.1, 4.2, 4.4 and 4.9], on different types of Shimura varieties we refer to [Va1, 2.5] and [De1], and on well positioned families of tensors we refer to [Va1, 4.3].

The operation of Weil restriction of a group G over S_1 will be denoted by $\mathrm{Res}_{S_1/S}(G)$; here $S \hookrightarrow S_1$ is a finite étale homomorphism of $\mathbb{Z}$-algebras. If E is a number field, we denote by $E_{(p)}$ the normalization of the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ (with respect to p) in E. Deviating from the approach of [Va2], we denote by σ the Frobenius automorphism (of the field of fractions) (of the Witt ring) of any perfect field of characteristic p.

1.2. In §2 we recall the main tools needed in §4-5. In 2.1 we review some simple facts on the classification of classical Hermitian symmetric domains. In 2.2 we review Satake's work (see [Sa]) on different (good) embeddings between them, in the $\mathbb{Z}_{(p)}$ context involving Shimura varieties, and needed in 2.3.

While 2.1 is dealing with real (rational) aspects, in 2.2-3 we move to different $\mathbb{Z}_{(p)}$ versions of 2.1. So in 2.3 we present a list of the most useful tricks of achieving embeddings between Shimura pairs of preabelian type, in a $\mathbb{Z}_{(p)}$ context. Most of these tricks were introduced previously in (scattered parts of) [Va1-3]. So here, for use in §4-5 and for future references, we systematize them. There are nine tricks altogether to be presented. Here we will detail some of them but very briefly.

Trick one (see 2.3.1) introduces the use of special quadruples: they play the same role as the one played by special pairs for Shimura pairs. Trick three (see 2.3.3) introduces the standard procedure of proving that different properties of a Shimura quadruple of preabelian type which are defined using a $\mathbb{Z}_{(p)}$ embedding into a Siegel modular variety of another Shimura quadruple having the same adjoint quadruple, are well defined, i.e. they do not depend on the chosen $\mathbb{Z}_{(p)}$ embedding used.

Trick five (see 2.3.5) formalizes the relative standard PEL situation, generalizing the usual standard PEL situations (of [Ko] or of [Va2, 2.3.5.1]). Trick six captures the essence of a great part of [Va1] involving particular (but the most useful) tensors of degree four.

Tricks seven and eight (i.e. the unitary trick and the orthogonal trick) are essentially presented in [Va3, §3] and respectively in [Va1, 6.6.2]. One of the goals of the unitary trick is to achieve good embeddings (in the $\mathbb{Z}_{(p)}$ context) of some Shimura pairs of C_n, $D_n^{\mathbb{H}}$, B_n or $D_n^{\mathbb{R}}$ type into unitary Shimura pairs. The goal of the orthogonal trick is to achieve good embeddings (in the $\mathbb{Z}_{(p)}$ context) of some orthogonal (i.e. of B_n or $D_n^{\mathbb{R}}$ type) Shimura pairs into (most of the time non-compact) adjoint Shimura pairs of some B_m or $D_m^{\mathbb{R}}$ type, with $m > n$, whose reflex fields are $\mathbb{Q}$.

In 2.4 we review the most inspiring (to us) recent moments related to p-divisible groups, and which play a central role in §4-5. They are due to Faltings (see [Fa3]), to de Jong (see [dJ1]), and to Milne (see [Mi3] and [Va4]), and are recalled respectively in 2.4.1, 2.4.2, and 2.4.3. Along them, we recall some parts of [Va1-4] related to them.

In 2.5, for applications in §4-6, a more general variant of [Va1, 3.1.6] is mentioned.

1.3. Before plunging in the technical aspects of the existence of integral canonical models of Shimura varieties of preabelian type (see §4) and of some of their variants (see §5), we start in §3 an abstract and general theory of Shimura envelopes. It is meant to generalize the theory of (quotients of) integral canonical models of Shimura varieties (of Hodge type). A Shimura envelope is a versal Shimura p-divisible group over a regular formally smooth $W(k)$-scheme, which is maximal in some sense (see def. 3.1 3)). The idea is: many properties of integral canonical models of Shimura varieties of Hodge type (in mixed characteristic $(0,p)$) can be read out just from the p-divisible groups of (any of) the (special in the sense of [Va1, 1.2.2]) universal abelian schemes over them. This incipient theory of Shimura envelopes is meant also to provide a frame ground for the case of Shimura varieties of special type (cf. 3.4.2 3) and [Va1, 3.2.7 8)]).

§3 consists mostly of definitions and remarks. However, in two theorems (see 3.1.2 and 3.4.1 a) and b)) we capture three fundamental ideas: we capture the essence of [Va2, 4.2.1] and respectively of [Va1, the proof of 5.1], and the essence of how one can apply 2.4.2 in the context of Shimura p-divisible groups.

§3 also illustrates (to a great extend) the difficulties to be encountered in §4, as well as the reaches of the geometro-arithmetical context provided by Shimura envelopes "emerging" from (quotients of) integral canonical models of Shimura varieties of Hodge type. Chronologically, many of our ideas related to integral canonical models of Shimura varieties of preabelian type, were obtained thinking in this abstract context of Shimura envelopes.

Some generalizations of parts of 2.4 and §3 pertaining to (Shimura) p-divisible groups, involving different Fontaine's categories are also mentioned (see Theorem 3 of 2.4.1 and 3.4.2 2)). Warning: in 6.5.2 we obtain, using a more involved machinery and essentially restricting to the case $p \geq 3$, much more refined forms of 3.4.1 a) and its versions involving different Fontaine's categories; however, as 3.4.1 a) is all one needs in §4-5 and it handles to some extend even the case $p = 2$, to make as well the connection with the previous work of [Va1], we include it first.

For simplicity, in 3.1-4 we concentrate on the unramified case. However, for an application in 5.4 and for future references, assuming $p \geq 3$, the classification of Shimura p-divisible groups over complete noetherian local $W(k)$-schemes of residue field k, is presented in 3.5. Its essence is [Va3, the proof of 2.4] (cf. also [Va3, 2.4.2]); so in theorem 3.5.1 we just restate loc. cit., in a convenient language allowed by [Va4]. Many variants of 3.5.1 are mentioned in 3.5.2-3.

1.4. The first goal of §4 is to weaken the (very) technical hypothesis (of the existence of families of tensors which are $\mathbb{Z}_{(p)}$-very well positioned with respect to a symplectic form for a reductive group) we needed for the proof of the basic result 5.1 of [Va1]. To state the first two main results we use the terminology introduced in [Va2, 2.3] and the notations of [Va1, 3.2.6]. In particular we use definition [Va2, 2.3.4]. From now on, without a special reference, we assume that $p \geq 3$. Let $f: (G, X) \hookrightarrow (GSp(W, \psi), S)$ be an embedding of a Shimura pair of Hodge type into a Shimura pair defining a Siegel modular variety, and let $L_{(p)}$ be a $\mathbb{Z}_{(p)}$-lattice of W good (in the sense of [Va1, 5.8.3]) with respect to f. Let $K := GSp(W, \psi)(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p)$, and let $H := K \cap G(\mathbb{Q}_p)$.

1.4.1. We have (cf. 4.1.1-2) the following first two main theorems:

Main Theorem 1. $\mathrm{Sh}_p(G, X)$ *exists and is equal to the normalization of the Zariski closure of* $\mathrm{Sh}_H(G, X)$ *in* $\mathrm{Sh}_p(GSp(w, \psi), S)_{E(G,X)_{(p)}}$.

Main Theorem 2. $L_{(p)}$ *is automatically crystalline well positioned for the injective map* f*. In other words, for any prime* v *of* $E(G, X)$ *dividing* p*, the triple* $(f, L_{(p)}, v)$ *is a SHS.*

The above two theorems are the maximum one can say in the context of $\mathbb{Z}_{(p)}$ embeddings into Siegel modular varieties (see 2.3.2 for a precise definition of such embeddings), from the point of view of smoothness and of reductiveness (of the groups involved) in the integral de Rham cohomology context (and so, cf. [Va2, §4] and [Va4] from many other points of view). In 4.1.1 Theorem 1 is stated in terms of Shimura quadruples. Theorem 2 fulfills the promise of [Va2, 2.3.8 2)]: it classifies completely the standard Hodge situations (cf. 4.1.3).

1.4.2. Following the pattern of [Va1] we get directly (see 4.3) from Theorem 1:

Main Corollary. *Any Shimura quadruple* (G_1, X_1, H_1, v_1) *of preabelian type, with* $(v_1, 2) = 1$*, has a uniquely determined integral canonical model* $\mathcal{N}_1$. $\mathcal{N}_1$ *is a quasi-projective integral model; it is strongly smooth (in the sense of [Va1, 3.4.8]).* $\mathcal{N}_1$ *is a projective integral model iff* (G_1, X_1) *is of compact type.*

1.4.3. 4.2 is entirely dedicated to the proof of Theorems 1-2. There are six Steps involved. To explain the ideas of these Steps, we start considering a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W good with respect to p, and a prime v of $E(G, X)$ dividing p. In what follows k is a perfect field containing $\mathbb{F}_p$: if desired it can be assumed to be $\mathbb{F}$ (and so that $k = \bar{k}$). We use the standard notations of [Va2, 2.3.1-3], with the only difference: we define $\mathcal{N}$ to be the normalization of the Zariski closure of $\mathrm{Sh}_H(G, X)$ in $\mathcal{M}_{O_{(v)}}$. The main idea is to use special quadruples (see trick one of 2.3.1) and to prove (using directly Milne's conjecture recalled in 2.4.3) first that the property $(*)$ of [Va2, 2.3.4] is true for special points $z \in \mathcal{N}(W(\bar{k}))$ (i.e. for points "coming" from a $W(\bar{k})$-valued point of the integral canonical model of a special quadruple of (G, X, H, v), defined in [Va1, 3.2.10]; cf. also [Va1, 3.2.8]), and for G-canonical lifts of G-ordinary points of $\mathcal{N}_{k(v)}$. Such G-ordinary points and G-canonical points can be defined following the pattern of [Va2, 4.1-4 and 4.5.12.2], even before we know that $\mathcal{N}$ is the integral canonical model of the Shimura quadruple (G, X, H, v) (for the case $p \geq 5$ this is explained in 4.2 M), while for the case $p = 3$ it is needed only after we already know that $\mathrm{Sh}_3(G, X)$ exists, and once this holds, 4.2 M) applies as well for $p = 3$).

Then the local deformation theory of [Va1, 5.4] can be applied (cf. Step 1 of 4.2) for these particular points z, and so (as in [Va2, 2.3.13.1]) we deduce that the property $(*)$ of [Va2, 2.3.4] remains true for all points $z_1 \in \mathcal{N}(W(\bar{k}))$ lifting a point $y_1 \in \mathcal{N}(\bar{k})$ of an open neighborhood (in $\mathcal{N}_{\bar{k}}$) of a point $y \in \mathcal{N}(\bar{k})$ which is liftable to such a special point or G-canonical lift $z \in \mathcal{N}(W(\bar{k}))$. Moreover $\mathcal{N}$ is formally smooth in all these points y_1.

This is enough for the case $p \geq 5$ (cf. Step 2 of 4.2). In other words, if $p \geq 5$, we already know that $\mathcal{N}$ is a smooth integral model (cf. [Va1, 3.2.12 and 6.4.1]); so we can extra use the ideas of [Va1, 6.4.2 and the proof of 4.3.10 b)] and of [Va2, 4.2.1, 4.4.1 and 4.9], in the context of the trick three of 2.3.2, in order to conclude that $L_{(p)}$ is indeed crystalline well positioned for the map f.

1.4.4. The same ideas (of using special points coming from special quadruples) is exploited to prove Theorems 1 and 2 (cf. Steps 4 and 5 of 4.2) for the case $p = 3$, and to obtain (cf. 5.2-3) the existence of integral canonical models (having either the EEP or the SEP, depending on how much we assume a priori) of Shimura varieties of Hodge type with respect to primes dividing odd rational primes and to a (large class) of non-hyperspecial subgroups.

For proving Theorems 1 and 2 for the case $p = 3$, we need to check two things:

i) that the validity of the property $(*)$ of [Va2, 2.3.4] for the $W(\bar{k})$-valued points of an open formally smooth $O_{(v)}$-subscheme $\mathcal{U}$ of $\mathcal{N}$ having a non-empty special fibre, extends to all the $W(\bar{k})$-valued points of $\mathcal{N}$;

ii) any $\bar{k}$-valued point of $\mathcal{N}$ has a $W(\bar{k})$-valued lift.

It is useful to think of the part of i) involving $W(\bar{k})$-valued points of the maximal open subscheme $\mathcal{N}_0$ of $\mathcal{N}$ such that the special fibre of $\mathcal{U}$ is Zariski dense in its special fibre, as a property (which turns out to be, cf. 3.3 4) and 6.5.2) true for any Shimura p-divisible group over the field of fractions $\mathcal{K}$ of a DVR $\mathcal{V}$ of equal characteristic p, which extends to a p-divisible group over $\mathcal{V}$; though $\mathcal{K}$ is not perfect, we can imitate the definition of Shimura p-divisible groups over $\overline{\mathcal{K}}$ to ones over $\mathcal{K}$. So, in our context we have:

Main Property. *The p-divisible group over $\mathcal{U}_{\mathbb{F}}$ can not be extended over a normal locally noetherian $\mathbb{F}$-scheme $\mathcal{U}_1$ (having $\mathcal{U}_{\mathbb{F}}$ as an open dense subscheme) in such a way that the reductiveness property needed (cf. $(*)$ of [Va2, 2.3.4]) in the integral de Rham cohomology context is destroyed.*

The main new idea in checking this property is to use the tricks of 2.3.8-10. In other words one can check directly (using 2.4.2 in the form of 3.4.1 a)), that i) and ii) are satisfied provided we are in the context of one of the last three tricks of 2.3. Strictly speaking, only the unitary trick of 2.3.8 is really needed. The usage of the tricks has to be exploited case by case: we can assume that $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is a simple adjoint pair; let τ be its (Shimura) type. Till 1.4.4.1 we assume that we are in a particular and convenient context provided by this (to be explained) usage.

If $\tau = A_n$, then we can assume that we are in the case of a standard PEL situation, cf. [Va2, 2.3.5.1]. So [Ko] and [LR] apply.

The case (G, X) is of compact type (see [Va1, def. 4 of 2.5]), with τ arbitrarily, is much simpler: it is treated in Step 3 of 4.2, as a preparation for the harder Steps 4 and

5. It goes as follows. Using 2.4.2 (in the form of 3.4.1 a)) in the context of 2.3.8, we "get full hold" (in the sense of 1.4.3) of one connected component of $\mathcal{N}_{\mathbb{F}}$; using [Va1, 6.4.1.1 1)] we show (standard arguments) that up to $G(\mathbb{A}_f^p)$-translation, there are no other such connected components.

So in the four cases still to be mentioned below, we can assume that (G, X) is of non-compact type.

If $\tau = C_n$, then the unitary trick (see Step 4 of 4.2) allows us to treat the situation similarly to the standard PEL situation: we just have to use extra principal quasi-polarizations of the (direct factors of the) p-divisible groups involved (over complete local $W(\mathbb{F})$-schemes).

If $\tau = D_n^{\mathbb{H}}$, then the situation is entirely similar to the $\tau = C_n$ case; the only difference is: we have to consider principal symmetric quasi-polarizations (cf. the terminology of end of 4.2 T)).

If $\tau = B_n$, then the situation (cf. either the orthogonal trick or the above reduction to the non-compact type) reduces to the case when $k(v) = \mathbb{F}_p$, $E(G, X) = \mathbb{Q}$, and $G_{\mathbb{R}}^{\mathrm{ad}}$ does not have compact factors. In this situation, it can be checked directly (see Step 5 of 4.2), using a standard toroidal induction, that the ordinary $\mathbb{F}$-valued points of the special fibre $\mathcal{N}_{\mathbb{F}_p}$ of $\mathcal{N}$ are dense in $\mathcal{N}_{\mathbb{F}_p}$, and their canonical lifts are defining $W(\mathbb{F})$-points of $\mathcal{N}$ (so in particular we have $\mathcal{N}_0 = \mathcal{N}$). Once we have this, arguments similar to the ones of [Va3, 2.2.3] allow us to get (see 4.2 V)) the Main Property.

The mentioned toroidal induction is based entirely on [FC], [Har] and [Va2, P1 of 4.6], cf. 4.2 W). In other words, different suitable quotients $\mathcal{N}/H_0$ (of finite type) of $\mathcal{N}$ have (not a priori smooth) toroidal compactifications, obtained by taking the normalization $\mathcal{N}^{\mathrm{tc}}/H_0$ of the Zariski closure of the generic fibre of $\mathcal{N}/H_0$ in some smooth projective toroidal compactification (see [FC]) of a suitable quotient (of finite type) $\mathcal{M}/K_0$ of $\mathcal{M}$ (we recall that we are using the standard notations of [Va2, 2.3.1-3]). Moreover, the boundaries of $\mathcal{N}/H_0$ in $\mathcal{N}^{\mathrm{tc}}/H_0$ are related (cf. [Har] for the rational, i.e. characteristic zero, picture) to different $\mathbb{Z}_{(p)}$ embeddings (involving smaller dimensions from all points of view) into Siegel modular varieties. As, we have (see [FC]) a natural semi-abelian scheme over $\mathcal{N}^{\mathrm{tc}}/H_0$, using induction, [Va2, P1 of 4.6] and [FC, 5.11 of p. 70] apply directly. [FC, 5.11 of p. 70] allows us to conclude that any abelian variety over the field of fractions of $k[[x]]$ specializing to a semi-abelian variety over k which is the extension of an ordinary abelian variety by a torus, is ordinary.

If $\tau = D_n^{\mathbb{R}}$, the situation is entirely similar to the $\tau = B_n$ case. Of course, there is as well a standard reduction of the $D_n^{\mathbb{R}}$ case to the B_n case (see 4.2.1 4)).

1.4.4.1. So the last three tricks of 2.3 are enough to prove via (2.4.2-3), that $\mathrm{Sh}_3(G, X)$ exists. Once we have this, Step 2 of 4.2 applies entirely to give us that i) and ii) above hold without any assumption; so, implicitly, the Main Property holds without any assumption.

1.4.5. As a roundabout we point out (cf. Step 6 of 4.2) that for the proof of Theorems 1-2 (and so of [Va1, 5.1]) the use of [Fa3, cor. 9] and of [Fa1, 8.1] can be avoided to a great extent, i.e. the mentioned results are used only in the simplest, down to earth situations. This has very important pedagogical and psychological impacts, as [Fa1, 8.1] is a very technical and general theorem, and so not so suitable for being included in courses. Also the use of [dJ1] can be entirely avoided, cf. 4.2.1 6).

1.5. A natural question arises: what about the other cases, when we are not in the context of a $\mathbb{Z}_{(p)}$ embedding into a Siegel modular variety? Three such situations are detailed respectively in 5.1-3. There are no new ideas besides the ones mentioned above in connection to Theorems 1-2 (perhaps the only exceptions are 5.2.2 8) and Exercise 2 of 5.3); so the results of 5.1-3 are not stated explicitly in this introduction. Here we will just mention, that the second one (see 5.2) handles the situation (called the case of quasi-hyperspecial subgroups), when the Zariski closure of $G_{\mathbb{Q}_p}$ in $GSp(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}, \psi)$ is "formed by two pieces" in such a way that the first piece G_1 is a reductive group over $\mathbb{Z}_p$, and the deformation theory of [Va1, 5.4] can be performed using only it (so in particular G_1^{ad} contains all the "non-compact factors" of $G_{\mathbb{Q}_p}^{\mathrm{ad}}$).

5.2.2 8) represents an elementary way in showing that in some Shimura towers of Hodge type (involving a reductive group $G_{\mathbb{R}}^{\mathrm{ad}}$ which has compact factors), we do get étale covers in the mixed characteristic situation. It represents a considerable simplification to the inductive process of [Ca] (and so of [HT]). Exercise 2 of 5.3 and 5.3.2 are dealing with the non-smooth variants of [Fa3, th. 10 and the remarks after]; moreover they also show (cf. 5.3.1 1)) that working (in the context of integral models of Shimura varieties of Hodge type) just with the SEP instead of EP, the results become essentially trivial (and so very incomplete).

1.6. Theorems 1-2 (and their proof) have many applications, besides the ones mentioned in the Corollary or in 1.5. Some of the immediate ones are presented in §6. They are grouped in 5 sections. First (see 6.1) we fulfill the promise of [Va6] of handling as in loc. cit. the case $p = 3$ for (suitable) moduli schemes of pseudo-polarized K3 surfaces of degree relatively prime to 3.

Second (see 6.2.1-2) we point out explicitly some parts of [Va2] where we worked under the assumption that Theorem 2 holds. Moreover, we fulfill the promise of [Va2, 4.4.6], provided we are in a context slightly weaker than the one of the Corollary. We have (see 6.2.3.2 1)):

Main Theorem 3. *Let (G, X, H, v) be a Shimura quadruple of preabelian type, with $(v, 2) = 1$. Let $\mathcal{N}$ be its integral canonical model (cf. 1.4.2). We assume that all simple factors of $(G^{\mathrm{ad}}, X^{\mathrm{ad}}, H^{\mathrm{ad}}, v^{\mathrm{ad}})$ of some $D_n^{\mathbb{H}}$ type, are partially U-simple in the sense of 6.2.3.0 C. Then each Shimura-canonical lift Spec$(W(\mathbb{F})) \to \mathcal{N}$ of a Shimura-ordinary point Spec$(\mathbb{F}) \to \mathcal{N}$ (see [Va2, 4.9.8] and 6.2.2), gives birth to a special point (see def. [Va1, 2.10]) Spec$(B(\mathbb{F})) \to \mathcal{N}$.*

This theorem is the generalization to the context of integral canonical models of Shimura quadruples of preabelian type (whose adjoints are not having simple factors isomorphic to simple adjoint Shimura quadruples of $D_n^{\mathbb{H}}$ type which are not U-simple), of the following well known fact of the classical Serre–Tate theory: the canonical lift of an ordinary abelian variety over a finite field has complex multiplication. Warning: the restrictions in the $D_n^{\mathbb{H}}$ type encountered in Theorem 3 (i.e. to be encountered in 6.2.3), will be removed in the third part. This completes the generalization of the Serre–Tate theory started in [Va2].

Related to the notion of (partially) U-simple, we will just mention that it is inspired from the theory of U-ordinariness started in [Va2, 4.4.13].

The general philosophy of (or behind) Theorem 3 is:

GP. *The Shimura-canonical lifts of Shimura-ordinary polarized projective varieties over finite fields are of CM type (in some sense), and so are defined over a localization of the ring of integers of a number field.*

This philosophy is exemplified here in 6.2.3 (in the context of abelian varieties emerging from a SHS), in 6.2.3.5 (in the context of pseudo-polarized K3 surfaces emerging from a situation which gets reduced to a SHS), and in 6.4.1 1) (in the context of abelian varieties emerging from a $p=2$ SHS). For the case of curves of genus 4 (!), cubic surfaces, polarized Enriques surfaces, canonically polarized surfaces of general type with $p_g=1$ and $K^2=1$, cubic fourfolds, and polarized hyperkählerian varieties (the ones mentioned in [Va6]) see [Va11-12]; this list (except the case of the mentioned surfaces of general type, for which some adjustments are in order) is in conformity with [Va6, GP of 1.6].

Third, we revisit Milne's conjecture (of 2.4.3): Theorem 2 allows us to restate (see 6.3) better some parts of [Va4] and of [Va6, 4.6]. Moreover, in 6.3.1, based on 5.3.2 F (resp. on 1.4.1), we present the first variants of Milne's conjecture in a non-reductive context; they generalize [Va4, V4] and Theorem 2 (resp. [Va4, V3]).

Fourth, the left aside case $p=2$ is mentioned in 6.4. For not making this paper too long, we develop here the $p=2$ theory only as much as to provide a reach class of examples where we can prove the existence of integral canonical models, as well as some uniqueness statements on them. So in 6.4.1 (resp. 6.4.1.1) we treat the $p=2$ case just for symplectic (resp. Hermitian, i.e. of $D_n^{\mathbb{H}}$ type) Shimura varieties. Warning: we get easily the equivalent of Theorem 1 and Theorem 3, but we get just a partial form of Theorem 2 (i.e. we get the equivalent form of Theorem 2 for the compact type cases when $k(v^{\mathrm{ad}})=\mathbb{F}_2$); the reason for this is explained in 6.4.2 5) (and implicitly in the proof of 6.4.1).

Moreover, we point out (see 6.4.3.2) that the integral canonical models we have constructed in mixed characteristic $(0,2)$ for symplectic and Hermitian Shimura varieties of compact type, are in fact uniquely determined. For a full general $p=2$ theory, see [Va8].

Fifth, very abstract, practical, and generalized forms of the above Main Property are included in 6.5.1-3. It is worth pointing out, that often, in connection to [Va1, 4.3.10], or 3.4.1 a), or $p=2$, instead of Killing forms of some Lie algebras, one can work as well with some trace forms of (different representations of) them. Using this in 6.5.4 we complement [Va1, 4.3.10.1]: we obtain different results on when a reductive group over $U_{\mathcal{S}}$ extends to a reductive group over $\mathcal{S}$; here $\mathcal{S}$ is a normal, integral, locally noetherian scheme, while $U_{\mathcal{S}}$ is an open subscheme of it containing all the points of codimension at most one. The results are very refined for the classical Lie types. The main results of 6.5 are contained in 6.5.1-2, 6.5.4.2.1 and 6.5.4.3.1. Here we will just point out (cf. 6.5.4) that in case the classical purity theorem holds for $\mathcal{S}$ (for instance if $\mathcal{S}$ is regular) then the situation is somehow simpler. In 6.5.4 we restrict mostly to the case $p\geq 3$. In some sense 6.5.4 represents an overview of what we did: see 6.5.5 for an interpretation.

1.7. There is very little literature to be mentioned in connection to the above three Main Theorems and to §5-6. Related to Theorem 1, prior to [Va1], regardless of how p is, there is nothing to be added besides what was already mentioned in [Va1, third paragraph before

1.1]. In other words, [MFK], [LR], [Mo], [Ca] and [Ko] form the whole literature on the existence of integral canonical models in existence before [Va1]. Related to Theorem 2 there is nothing to be mentioned. Related to Theorem 3 there is nothing concrete to be mentioned outside of the standard PEL type situations (recalled in 2.3.4) involving a prime of the reflex field having a prime number of elements. Related to §5, we need to mention that [Ca] handles cases similar to the ones of 5.2, but for Shimura curves, while [HT] handles some simple higher dimensional unitary varieties; [Ca] and [HT] contain also situations where the primes of the residue fields are ramified (see 5.4.1 4) for an extra comparative discussion, and see 5.4 and [Va13] for our approach towards such primes).

4.1.1 and 6.4 represent progress in Milne's conjecture of [Va1, 3.2.5], while §5 extends it to a more general context.

Also we could not find any literature in connection to 6.5.

The ideas we did not have time to deal with, or we were able to sort them out just partially, are mentioned as questions and problems. Not to make this part 2C too long, 10 simple exercises are included.

1.8. We spent a lot of time, here as well as in [Va1-4], to present the ideas in such a way that they can be also used in the construction of integral canonical models of Shimura varieties of special type. The previous tools ([Va1, 3.2.7 8) and 4.1.5], [Va2, 3.4.1.6 and 3.5-6], [Va3, 2.4] and its variants -see [Va9]- in a suitable context of different categories $\mathcal{MF}^{\nabla}_{[-1,1]}(*)$) are complemented here by 4.2, 6.4 and 6.5.1. So the tools are getting more and more refined to fully attack this problem (without putting aside some small primes).

1.9. In some sense, this part 2C is formed by different parts of [Va1-4] and [Va6] essentialized, generalized and simplified to extreme (i.e. very close to the maximum one could say). All its main ideas were already clear to us in the beginning of 1996. Many parts of it strongly support the quickly unfolding general opinion that in problems pertaining to geometry, as long as one could hope so, the primes 2 and 3 are as good as the other primes.

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1.10. In the version to be submitted for publication, V5 of p. 48 and 5.2.3 will be considerable enlarged. Also a proof of 6.5.2 for $p = 2$ and the maximum one can say in connection to 6.5.1 and 5.5.4 for $p = 2$ (resulting in a $DEG4-$ theory complementing 2.3.7), will be included. The complete $p = 2$ for Shimura varieties of preabelian type which are also of compact type, will be also included.

§2. Preliminaries

We start by reviewing some well known facts on the classification of classical Hermitian symmetric domains (see 2.1), and of their (good) embeddings (see 2.2) into each other, using a $\mathbb{Z}_{(p)}$ language. Then we adapt 2.2 to the integral $\mathbb{Z}_{(p)}$ setting of embeddings of Shimura varieties of abelian type into each other (see 2.3). Most of the time, for the simplicity sake and to be in vogue with the approach of [Va2] of taking the case $p = 2$ aside, we assume $p \geq 3$. Implicitly, in 2.3-4, while presenting the most useful tricks of constructing embeddings of Shimura varieties of abelian type into (other) Shimura varieties of abelian type, and tools to exploit their "good part", we present (partially recall) the most important historical moments (used or to be used) in the proof of the existence of integral canonical models of Shimura varieties of preabelian type. The most recent such moments besides the ones emerging directly from our work, are singled out in 2.4. They do play a crucial role in §4. For many concrete situations of 2.1-3 below see [Va7].

2.1. The classical Hermitian symmetric domains. The main reference for this section is [He, §6 of Ch. X]: loc. cit. will be used without any extra comment. Let τ be a classical Lie type. Let $n \in \mathbb{N}$ be its rank. So $\tau \in \{A_n, B_n, C_n, D_n\}$.

Case one: $\tau = A_n$**.** In this case there are $[\frac{n+1}{2}]$ isomorphism classes of Hermitian symmetric domains associated to τ. They are in one to one correspondence to pairs $(a, b) \in \mathbb{N} \times \mathbb{N}$ with the properties that $a \leq b$ and $a + b = n + 1$. The Hermitian symmetric domain corresponding to such a pair (a, b) is

$$AIII_{a,b} := G(\mathbb{R})^0/K_\infty,$$

where G is the adjoint group of the semisimple $\mathbb{Q}$–group $SU(a, b)$ whose group of $\mathbb{Q}$–valued points is the group of matrices $g \in SL(a + b)(\mathbb{Q}(i))$ which leave invariant the Hermitian form

$$-z_1\overline{z_1} - ... - z_a\overline{z_a} + z_{a+1}\overline{z_{a+1}} + ... + z_{a+b}\overline{z_{a+b}}$$

in $a + b$ (complex) variables. Here $\overline{z}$ is the complex conjugate of the complex number z, while K_∞ is a maximal compact subgroup of $G(\mathbb{R})^0$. The complex dimension of $AIII_{a,b}$ is ab, while the $\mathbb{Q}$–rank of G is $\min\{a, b\}$.

We denote by $\tilde{SU}(n, n)$ the quotient of $SU(n, n)$ by the central finite subgroup $\mu_2 \tilde{\to} \mathbb{Z}/2\mathbb{Z}$ whose two elements are the scalar multiplication of all variables by 1 or -1.

Case two: $\tau = B_n$**.** In this case there is only one (up to isomorphism) Hermitian symmetric domain associated to τ. It is denoted by $BDI_{2,2n-1}$, and is isomorphic to $G(\mathbb{R})^0/K_\infty$. Here $G = SO(2, 2n - 1)$ is the absolutely simple adjoint $\mathbb{Q}$–group defined as the SO-group of the quadratic form

$$-x_1^2 - x_2^2 + x_3^2 + ... + x_{2n+1}^2$$

in $2n + 1$ variables, while K_∞ is a maximal compact subgroup of $G(\mathbb{R})^0$. The complex dimension of $BDI_{2,2n-1}$ is $2n - 1$. If $n \geq 2$ (resp. $n = 1$) then the $\mathbb{Q}$–rank of G is 2 (resp. 1).

Case three: $\tau = C_n$**.** Also in this case there is only one (up to isomorphism) Hermitian symmetric domain associated to τ. It is denoted by CI_n, and is isomorphic to $G(\mathbb{R})^0/K_\infty$. Here G is the adjoint group of the split simply connected group $Sp(n)$ of symplectic isomorphisms of a symplectic space (W, ψ) over $\mathbb{Q}$ with $\dim_{\mathbb{Q}}(W) = 2n$, while K_∞ is a maximal compact subgroup of $G(\mathbb{R})^0$. The complex dimension of CI_n is $\frac{n(n+1)}{2}$. G is a split group, and so its $\mathbb{Q}$–rank is n.

Case four: $\tau = D_n$**.** We assume $n \geq 4$. In this case there are two (resp. one) (up to isomorphism) Hermitian symmetric domains associated to τ, depending on the fact $n \geq 5$ (resp. $n = 4$). The first one is denoted by $BDI_{2,2n-2}$, has complex dimension $2n - 2$ and is constructed as in case two, but working with just $2n$ variables; the $\mathbb{Q}$–rank of G is 2. The second one (for $n \geq 5$) is denoted by $DIII_n$, has complex dimension $\frac{n(n-1)}{2}$, and is isomorphic to $G(\mathbb{R})^0/K_\infty$. Here G is the adjoint group of the semisimple $\mathbb{Q}$–group $SO^*(2n)$ whose group of $\mathbb{Q}$–valued points is the group of matrices $g \in SO(2n, \mathbb{Q}(i))$ which leave invariant the skew Hermitian form

$$-z_1\overline{z_{n+1}} + z_{n+1}\overline{z_1} - \ldots - z_n\overline{z_{2n}} + z_{2n}\overline{z_n},$$

while K_∞ is obtained as above. The $\mathbb{Q}$–rank of G is $\left[\frac{n}{2}\right]$. Of course we can define $SO^*(8)$ as well: we have $SO^*(8)^{\mathrm{ad}} = SO(2,6)^{\mathrm{ad}}$.

Let $\mathbb{S}_\mathbb{Q}$ be the Weil restriction from $\mathbb{Q}(i)$ to $\mathbb{Q}$ of the one dimensional split torus. In all the above four cases we can choose K_∞ to be the subgroup of $G(\mathbb{R})^0$ commuting with a homomorphism

$$h^G\colon \mathbb{S}_\mathbb{Q} \to G$$

with the property that the pair

$$(G, X_G),$$

with X_G as the $G(\mathbb{R})$-conjugacy class of $h^G_\mathbb{R}$, is a Shimura pair. For the rest of 2.1 we disregard completely the case of Hermitian symmetric domains $AIII_{a,b}$ with $a \neq b$. So in what follows, G is either not of some A_n Lie type, or is $SU(m,m)^{\mathrm{ad}}$ for some $m \in \mathbb{N}$. We refer to such a Shimura pair (G, X_G) obtained as above as a classical adjoint Shimura pair with trivial involution (and obtained working with the standard quaternion algebra $\mathbb{Q}(i,j)$ over $\mathbb{Q}$).

Below we encounter a reductive $\mathbb{Q}$–group $\tilde{G}$ having G as its adjoint, and $\mathbb{G}_m$ as its abelianization, and such that h^G can be lifted to a monomorphism

$$\tilde{h}^G\colon \mathbb{S}_\mathbb{Q} \hookrightarrow \tilde{G}.$$

So the connected component of the origin of the center $Z(\tilde{G})$ of $\tilde{G}$ is $\mathbb{G}_m$. In case G is of B_n or C_n Lie type, or if we have $\tilde{G}^{\mathrm{der}} = SO^*(2n)$, the natural quotient group $\tilde{G}/\mathbb{G}_m$ will be G itself. So in these cases the $\tilde{G}(\mathbb{R})$-conjugacy class $\tilde{X}_G$ of $\tilde{h}^G_\mathbb{R}$ can be identified with X_G.

If G is of A_n Lie type, with $n \geq 2$, the natural quotient group $\tilde{G}/\mathbb{G}_m$ will be $\tilde{SU}(n,n)$, and so it will not be an adjoint group. In this case we denote by $h^{\tilde{SU}(n,n)}$ the resulting homomorphism $\mathbb{S}_\mathbb{Q} \to \tilde{SU}(n,n)$ obtained from $h^{\tilde{G}}$ by passage to the quotient. Let $X_{\tilde{SU}(n,n)}$ be the $\tilde{SU}(n,n)(\mathbb{R})$-conjugacy class of $h^{\tilde{SU}(n,n)}_\mathbb{R}$; we get a Shimura pair

$$(\tilde{SU}(n,n), X_{\tilde{SU}(n,n)}).$$

As above, the $\tilde{G}(\mathbb{R})$-conjugacy class $\tilde{X}_G$ of $\tilde{h}^G_{\mathbb{R}}$ can be identified with $X_{\tilde{SU}(n,n)}$. Moreover $\tilde{G}^{\mathrm{der}} = SU(n,n)$.

If $G = SO(2,2n-2)^{\mathrm{ad}}$ and $G^{\mathrm{der}} = \mathrm{Spin}(2,2n-2)$ is simply connected ($n \geq 4$), the situation is entirely similar to the A_n case: we just have to replace everywhere $\tilde{SU}(n,n)$ by $SO(2,2n-2)$. So $h^{SO(2,2n-2)}$, $X_{SO(2,2n-2)}$ and $\tilde{X}_G$ will be defined in the same manner.

In all the above situations, $\tilde{G}$ and $\tilde{h}^G$ are uniquely determined. The Shimura pairs $(\tilde{G},\tilde{X}_G)$ we get will be referred as the classical Shimura pairs with trivial involution (and obtained working with $\mathbb{Q}(i,j)$). All of them are of non-compact type, cf. [BHC]. We have:

$$E(\tilde{G},\tilde{X}_G) = E(G,X_G) = \mathbb{Q}.$$

If (G,X_G) is of B_n, C_n or $D_n^{\mathbb{H}}$ type, then the natural map $(\tilde{G},\tilde{X}_G) \to (G,X_G)$ is a cover in the sense of [Va1, 2.4]. Similarly, if G is of A_n type with $n \geq 2$ (resp. if $\tilde{G}^{\mathrm{der}} = \mathrm{Spin}(2,2n-2)$), then we get a cover $(\tilde{G},\tilde{X}_G) \to (\tilde{SU}(n,n), X_{\tilde{SU}(n,n)})$ (resp. $(\tilde{G},\tilde{X}_G) \to (SO(2,2n-2), X_{SO(2,2n-2)})$).

2.2. A review of Satake's work. In what follows we review the part of [Sa] needed in 2.3. Satake's work can be formulated in terms of Lie algebras, in terms of semisimple (mostly adjoint) groups, or in terms of reductive groups. Here we recall it in terms of semisimple groups (this part is also recalled in [Va3, 3.0]), and in terms of reductive groups whose abelianizations are a split one dimensional torus. Moreover we will state the results over $\mathbb{Z}_{(p)}$ instead of $\mathbb{R}$. Without explicitly stated we assume $p \geq 3$. We again have to consider cases. Working with $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_{(p)}(i)$ instead of $\mathbb{Q}$ and $\mathbb{Q}(i)$, we define as in Case one of 2.1, a semisimple simply connected group $SU(m,m)_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$, for any $m \in \mathbb{N}$. Similarly we define semisimple groups $SO(2,m)_{\mathbb{Z}_{(p)}}$, $Sp(n)_{\mathbb{Z}_{(p)}}$ and $SO^*(2q)_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$, for each $m,n,q \in \mathbb{N}$, $q \geq 4$, m odd or greater than 5. Let $\mathrm{Spin}(2,m)_{\mathbb{Z}_{(p)}}$ be the simply connected group cover of $SO(2,m)_{\mathbb{Z}_{(p)}}$. Similarly we define $GSp(n)_{\mathbb{Z}_{(p)}}$ to be the group of symplectic similitude isomorphisms of a symplectic space of dimension 2n over $\mathbb{Z}_{(p)}$. Let $\mathbb{S}_{\mathbb{Z}_{(p)}} := \mathrm{Res}_{\mathbb{Z}_{(p)}(i)/\mathbb{Z}_{(p)}}(\mathbb{G}_m)$ (the notation is justified: its generic fibre is $\mathbb{S}_{\mathbb{Q}}$).

The below homomorphisms a_n, b_n, c_n, d_n and e_n are just the $\mathbb{Z}_{(p)}$ version of ones over $\mathbb{R}$ of [Sa]: they are obtained by replacing in [Sa] everywhere $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H} := \mathbb{C}(j)$ by $\mathbb{Z}_{(p)}$, $\mathbb{Z}_{(p)}(i)$, and respectively by $\mathbb{Z}_{(p)}(i,j)$.

Warning: [Sa] is worked out mostly in terms of Lie algebras. As we assumed $p > 2$, there is an automatic passage from Lie algebras to semisimple groups: for the case A below this is trivial (cf. [Sa, 1.4]; see 6.4 for its version for $p = 2$), while for the other cases this is a consequence of the existence of a_n, and of the fact that the orders of the centers of simply connected semisimple split groups of B_n, C_n and D_n Lie type are relatively prime to p. So by pointing out that these homomorphisms a_n, b_n,..., e_n can be easily read out respectively from paragraphs 1.4, 3.5, 1.4, 3.3 and 3.5 of [Sa], we will introduce them below without any extra further comment.

Case A: the case of $SU(n,n)$. We have a homomorphism

$$a_n \colon SU(n,n)_{\mathbb{Z}_{(p)}} \to Sp(2n)^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$$

of finite kernel such that $h^{\mathrm{Sp}(2n)^{\mathrm{ad}}}$ can be chosen to be defined naturally by a homomorphism from $\mathbb{S}_{\mathbb{Z}_{(p)}}$ into its image $\tilde{SU}(n,n)_{\mathbb{Z}_{(p)}}$.

Let $GSU(n,n)_{\mathbb{Z}_{(p)}}$ be the reductive subgroup of $GSp(2n)_{\mathbb{Z}_{(p)}}$ generated by the center of $GSp(2n)_{\mathbb{Z}_{(p)}}$ and by the simply connected subgroup naturally isogeneous to the image of a_n. So we get a monomorphism

$$\tilde{a}_n \colon GSU(n,n)_{\mathbb{Z}_{(p)}} \hookrightarrow GSp(2n)_{\mathbb{Z}_{(p)}},$$

giving birth to an injective map

$$\tilde{a}_n \colon (GSU(n,n), X_{\tilde{SU}(n,n)}) \hookrightarrow (GSp(2n), X_{Sp(2n)^{\mathrm{ad}}}).$$

Case B: the case of $SO(2,2n-1)$. We have a homomorphism

$$b_n \colon \mathrm{Spin}(2,2n-1)_{\mathbb{Z}_{(p)}} \to SU(2^{n-1},2^{n-1})^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$$

of finite kernel such that $h^{SU(2^{n-1},2^{n-1})^{\mathrm{ad}}}$ can be chosen to be defined naturally by a homomorphism from $\mathbb{S}_{\mathbb{Z}_{(p)}}$ into its image $SO(2,2n-1)_{\mathbb{Z}_{(p)}}$. As in Case A we define a reductive group $\mathrm{GSpin}(2,2n-1)_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$ whose abelianization is $\mathbb{G}_m$, and of whose adjoint is $SO(2,2n-1)_{\mathbb{Z}_{(p)}}$. So we get a monomorphism

$$\tilde{b}_n \colon \mathrm{GSpin}(2,2n-1)_{\mathbb{Z}_{(p)}} \hookrightarrow GSU(2^{n-1},2^{n-1})_{\mathbb{Z}_{(p)}},$$

giving birth to an injective map

$$\tilde{b}_{n\mathbb{Q}} \colon (\mathrm{GSpin}(2,2n-1), X_{\mathrm{SO}(2,2n-1)}) \hookrightarrow (GSU(2^{n-1},2^{n-1}), X_{\tilde{SU}(2^{n-1},2^{n-1})}).$$

Case C: the case of $Sp(n)$. We have a homomorphism

$$c_n \colon Sp(n)_{\mathbb{Z}_{(p)}} \to SU(n,n)^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$$

of finite kernel such that $h^{SU(n,n)^{\mathrm{ad}}}$ can be chosen to be defined naturally by a homomorphism from $\mathbb{S}_{\mathbb{Z}_{(p)}}$ into its image $PSp(n)_{\mathbb{Z}_{(p)}}$. As in Case A we obtain a monomorphism

$$\tilde{c}_n \colon GSp(n)_{\mathbb{Z}_{(p)}} \hookrightarrow GSU(n,n)_{\mathbb{Z}_{(p)}},$$

giving birth to an injective map

$$\tilde{c}_{n\mathbb{Q}} \colon (GSp(n), X_{\mathrm{Sp}(n)^{\mathrm{ad}}}) \hookrightarrow (GSU(n,n), X_{\tilde{SU}(n,n)}).$$

Case $D^{\mathbb{H}}$: the case of $SO^*(2n)$. We have a homomorphism

$$d_n \colon SO^*(2n)_{\mathbb{Z}_{(p)}} \to SU(n,n)^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$$

of finite kernel such that $h^{SU(n,n)^{\mathrm{ad}}}$ can be chosen to be defined naturally by a homomorphism from $\mathbb{S}_{\mathbb{Z}_{(p)}}$ into its image $SO^*(2n)^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$. As in Case A we define a reductive group $GSO^*(2n)_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$ having $\mathbb{G}_m$ as its abelianization and $SO^*(2n)^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$ as its adjoint, and a monomorphism

$$\tilde{d}_n\colon GSO^*(2n)_{\mathbb{Z}_{(p)}} \hookrightarrow GSU(n,n)_{\mathbb{Z}_{(p)}},$$

giving birth to an injective map

$$\tilde{d}_{n\mathbb{Q}}\colon (GSO^*(2n), X_{SO^*(2n)^{\mathrm{ad}}}) \hookrightarrow (GSU(n,n), X_{\tilde{SU(n,n)}}).$$

Case $D^{\mathbb{R}}$: the case of $SO(2,2n-2)$. We have a homomorphism

$$e_n\colon \mathrm{Spin}(2,2n-2)_{\mathbb{Z}_{(p)}} \to SU(2^{n-1},2^{n-1})^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$$

of finite kernel such that $h^{SU(2^{n-1},2^{n-1})^{\mathrm{ad}}}$ can be chosen to be defined naturally by a homomorphism from $\mathbb{S}_{\mathbb{Z}_{(p)}}$ into its image $SO(2,2n-2)_{\mathbb{Z}_{(p)}}$. The reason we choose 2^{n-1} and not 2^{n-2} is that we want to use both the half spin representations, and so we define e_n by composing similar homomorphisms

$$e_n^1, e_n^2\colon \mathrm{Spin}(2,2n-2)_{\mathbb{Z}_{(p)}} \to SU(2^{n-2},2^{n-2})_{\mathbb{Z}_{(p)}}$$

obtained using just one half spin representation, with the natural composite homomorphism

$$SU(2^{n-2},2^{n-2})_{\mathbb{Z}_{(p)}} \times SU(2^{n-2},2^{n-2})_{\mathbb{Z}_{(p)}} \hookrightarrow SU(2^{n-1},2^{n-1})_{\mathbb{Z}_{(p)}} \to SU(2^{n-1},2^{n-1})^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}.$$

As in Cases A and B we get a monomorphism

$$\tilde{e}_n\colon \mathrm{GSpin}(2,2n-2)_{\mathbb{Z}_{(p)}} \hookrightarrow GSU(2^{n-1},2^{n-1})_{\mathbb{Z}_{(p)}},$$

giving birth to an injective map

$$\tilde{e}_{n\mathbb{Q}}\colon (\mathrm{GSpin}(2,2n-2), X_{SO(2,2n-2)}) \hookrightarrow (GSU(2^{n-1},2^{n-1}), X_{\tilde{SU(2^{n-1},2^{n-1})}}).$$

The fact that we made use of both the half spin representations, guarantees that the notations are justified, i.e. the derived group of $\mathrm{GSpin}(2,2n-2)_{\mathbb{Z}_{(p)}}$ is simply connected.

2.2.1. Remark. One might wonder why we worked above with $\mathbb{Q}(i)$. Of course, instead of it we can work as well with any other imaginary quadratic extension E of $\mathbb{Q}$ which is unramified above the odd prime p of interest; in such a case $\mathbb{Q}(i,j)$ needs to be replaced as well by a quaternion algebra over $\mathbb{Q}$ containing E, and ramified at the place of infinity. The nice thing about $\mathbb{Q}(i)$ is that it works for all such primes. The bad part of $\mathbb{Q}(i)$ is: p might not split in it. Of course if p splits in $\mathbb{Q}(i)$ (resp. in E), then all the groups of 2.2 (resp. of 2.2 worked out with E instead of $\mathbb{Q}(i)$) split over $\mathbb{Q}_p$. In such situation, the passage from homomorphisms over $\mathbb{Q}$ to ones over $\mathbb{Z}_{(p)}$ can be argued as well using the following general remark.

2.2.2. Remark. We consider $p \geq 2$. Let O be a DVR of mixed characteristic $(0, p)$ and perfect residue field. Let $K_O := O[\frac{1}{p}]$. Let $i : G^1 \hookrightarrow G^2$ be a monomorphism between reductive groups over K_O. We assume that G^1 and G^2 split over the completion $\widehat{O}$ of O. Then:

Fact. *$i_{\widehat{O}}$ extends to a monomorphism of reductive groups over $\widehat{O}$.*

It was impossible to trace a reference to this fact in [SGA, Vol. 3] or in [Ja]. So here we briefly mention how the argument goes. Let $K_1 := \widehat{O}[\frac{1}{p}]$. As any split torus T^1 of $G^1_{K_1}$ is contained in a maximal split torus T^2 of $G^2_{K_1}$, and as T^2 is the generic fibre of a maximal subtorus of a reductive group over $\widehat{O}$ having $G^2_{K_1}$ as its generic fibre, we can assume that G^1 and G^2 are both semisimple groups. Moreover, using the conjugacy property of hyperspecial subgroups of [Ti1, p. 47] we can assume that $G^{2\mathrm{ad}}$ is simple. So the argument has to continue case by case, depending on the Lie type of G^2. The case G^2 is of A_n Lie type is covered by [Ja, 10.4]. The other classical Lie types can be handled similarly. As in applications to Shimura varieties, only the A_n case is usually needed, we will not detail here the rest of the argument.

But now, from the above Fact and [Va1, 3.1.3.1], we deduce that i itself extends to a monomorphism of reductive groups over O.

2.3. The most useful tricks. The essence of the below eleven tricks can be adapted to the abstract case of Shimura p-divisible groups. As we think this adaptation can be done automatically by any interested reader, to keep the requirements as low as possible, we will deal here mostly with the context of Shimura varieties. We will work with $p \geq 2$. Let (G, X, H, v) be a Shimura quadruple (cf. [Va1, 3.2.6]). So the prime v of $E(G, X)$ divides a rational prime p such that G is unramified over $\mathbb{Q}_p$; it is unramified above p, cf. [Mi2, 4.7]. Let $G_{\mathbb{Z}_{(p)}}$ be a reductive group over $\mathbb{Z}_{(p)}$ whose group of $\mathbb{Z}_p$ valued points is H (cf. [Va1, 3.1.3] and the definition of hyperspecial subgroups).

2.3.1. Trick one: special quadruples. The main reference for this trick is [Va1, 3.2.10-11]. Here we just restate loc. cit. in a form useful in applications. There are injective special maps

$$(T, \{h\}, H_T, v_T) \hookrightarrow (G, X, H, v),$$

with T a torus. Moreover we can assume that the Zariski closure $T_{\mathbb{Z}_{(p)}}$ of T in $G_{\mathbb{Z}_{(p)}}$ is a torus. We can also assume that $T_{\mathbb{Z}_p}$ contains a maximal split subtorus of $G_{\mathbb{Z}_p}$; moreover we can prescribe as well the H_l-conjugacy class of $T_{\mathbb{Q}_l}$, for l running through a finite set of primes relatively prime to p, and for H_l an arbitrarily chosen compact open subgroup of $G(\mathbb{Q}_l)$. In particular, if $G_{\mathbb{Q}_p}$ is split, we can assume that $T_{\mathbb{Z}_p}$ itself is split; in this case we have $k(v_T) = k(v) = \mathbb{F}_p$.

It is known that $(T, \{h\}, H_T, v_T)$ has a uniquely determined integral canonical model $\mathcal{T}$, cf. [Va1, 3.2.8]. We assume now that (G, X, H, v) has a normal integral model $\mathcal{N}$ over $O_{(v)}$ having the SEP (or the EP, if $p > 2$) (cf. [Va1, definitions of 3.2.3]). This assumption implies the existence of a unique morphism $\mathcal{T} \to \mathcal{N}$, extending the natural morphism $\mathrm{Sh}_{H_T}(T,\{h\}) \to \mathrm{Sh}_H(G,X)$ at the level of generic fibres. We deduce:

Fact. *$\mathcal{N}$ has $W(\mathbb{F})$-valued points, such that the resulting morphisms $Spec(B(\mathbb{F})) \to \mathcal{N}$ are special points.*

From now on, in what follows we assume that (G,X) is of preabelian type.

2.3.2. Trick two: the standard preabelian type situation. This is obtained by moving to an abstract form of a SHS. We consider an injective map

$$f\colon (G_1,X_1,H_1,v_1) \hookrightarrow (G,X,H,v).$$

So (G_1,X_1) is also of preabelian type, cf. [De1] or [Sa]. We refer to the pair (f,v_1) as a weak standard preabelian type situation, to be abbreviated as: a weak SPTS. We assume now that the Zariski closure of G_1 in $G_{\mathbb{Z}_{(p)}}$ is a reductive group $G_{1\mathbb{Z}_{(p)}}$. We refer to the triple $(f,G_{\mathbb{Z}_{(p)}},v_1)$ as a standard preabelian type situation, to be abbreviated as SPTS; sometimes we refer to such a triple as a $\mathbb{Z}_{(p)}$ embedding (of $\mathrm{Sh}(G_1,X_1)$ or of (G_1,X_1)) (into $\mathrm{Sh}(G,X)$ or into (G,X)). If (G,X) is of abelian type, then (easy exercise starting from definitions) (G_1,X_1) is also of abelian type, and we speak about the triple $(f,G_{\mathbb{Z}_{(p)}},v_1)$ as a standard abelian type situation, to be abbreviated as SATS. Obviously any SHS $(f,L_{(p)},v)$ gives birth to a SATS $(f,GSp(L_{(p)},\psi),v)$, and the same remains true for a $p=2$ SHS.

We assume now that (G,X,H,v) has a normal integral canonical model and that $(f,G_{\mathbb{Z}_{(p)}},v_1)$ is a SPTS. We recall (cf. [Va1, 3.4.1]) that:

Proposition. *The normalization $\mathcal{N}_1$ of $\mathrm{Sh}_{H_1}(G_1,X_1)$ in $\mathcal{N}_{O_{(v_1)}}$ is also a normal integral canonical model of (G_1,X_1,H_1,v_1).*

If by chance we can interpret $\mathcal{N}$ as some moduli scheme, then the formalism of a SHS introduced in [Va2, 2.3.1-3] can be extended to the present context of a SPTS. For instance, if (G,X) is of abelian type and has its weight defined over $\mathbb{Q}$, then [Mi2, 3.27] can be entirely adapted following the pattern of [Va2, 2.3.1-3]. We hope to come back to this idea in a near future. If (G,X) is not of abelian type, then we can consider the adjoint map $(G,X,H,v) \to (G^{\mathrm{ad}},X^{\mathrm{ad}},H^{\mathrm{ad}},v^{\mathrm{ad}})$ and use [Va2, 4.9.20] for reducing the initial SPTS to a SATS very close in nature to it (see [Va5] for details): by doing this, from the local point of view, we do not lose any information at all. We also hope (cf. 1.8) that this last idea can be used as well to the context of Shimura varieties which are not of preabelian type.

2.3.2.1. In Step 1 of 4.2 we need to be able to recognize very quickly the simplest type of a SHS. So let $f\colon (T,\{h\}) \hookrightarrow (GSp(W,\psi),S)$ be a special pair of a Shimura pair defining a Siegel modular variety. We consider a SATS $(f,GSp(L_{(p)},\psi),v_T)$, with $L_{(p)}$ a $\mathbb{Z}_{(p)}$-lattice of W good w.r.t. f. We have:

Exercise 1. The triple $(f,L_{(p)},v_T)$ is a SHS. Hint: use [Va1, 4.3.13].

2.3.2.2. $\mathbb{Z}_{(p)}$ maps. If by chance we are in a context in which f is not injective, but the Zariski closure in $G_{\mathbb{Z}_{(p)}}$ of the image of G_1 in G is a reductive group, we refer to the triple $(f,G_{\mathbb{Z}_{(p)}},v_1)$ as a $\mathbb{Z}_{(p)}$ map.

2.3.3. Trick three: the diagonal and the Segre embeddings. The main references for what follows are [Va1, Example 3 of 2.5, 6.2 F) and 6.5.1], [Va2, 4.9.2] and [Va3, 3.2

C)]. We start with two ($p=2$) SHS's $(f_1, L^1_{(p)}, v_1)$ and $(f_2, L^2_{(p)}, v_2)$. As usual we use the standard notations of [Va2, 2.3.1-3] but we put everywhere a lower right index 1 or 2, except for the $\mathbb{Z}_{(p)}$-lattices for which we put upper right indices. So let $L_{(p)} := L^1_{(p)} \oplus L^2_{(p)}$ and let $\psi := \psi_1 \oplus \psi_2$. Let $W := W_1 \oplus W_2$. $L_{(p)}$ is a $\mathbb{Z}_{(p)}$-lattice of W, self dual w.r.t. ψ. As in [Va1, Example 3 of 2.5] we consider a Hodge quasi product (G_3, X_3) of (G_1, X_1) and (G_2, X_2). So G_3 is a normal subgroup of $G_1 \times G_2$ such that the quotient group is $\mathbb{G}_m$. So $E(G_3, X_3)$ is the composite field of $E(G_1, X_1)$ and $E(G_2, X_2)$. Let v_3 be the prime of it dividing v_1 and v_2. Let

$$f_3 : (G_3, X_3) \hookrightarrow (GSp(W, \psi), S)$$

be the natural injective map as in loc. cit. We recall that we call it a Segre embedding. Directly from the definition of a SHS (see also [Va1, 4.3.17]) we get:

Fact. *The triple $(f_3, L_{(p)}, v_3)$ is a ($p=2$) SHS.*

We assume now that the two adjoint Shimura quadruples $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}, H_1^{\mathrm{ad}}, v_1^{\mathrm{ad}})$ and $(G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}}, H_2^{\mathrm{ad}}, v_2^{\mathrm{ad}})$ are isomorphic, under an isomorphism $i\colon G^{\mathrm{ad}}_{1\mathbb{Z}_{(p)}} \tilde{\to} G^{\mathrm{ad}}_{2\mathbb{Z}_{(p)}}$. Let G_4 be the reductive subgroup of G_3 generated by the connected component of the origin of $Z(G_3)$ and by a semisimple group naturally isogeneous to the (image of the) monomorphism

$$(1_{G_1^{\mathrm{ad}}}, i_{\mathbb{Q}}) : G_1^{\mathrm{ad}} \hookrightarrow G_1^{\mathrm{ad}} \times G_2^{\mathrm{ad}} = G_3^{\mathrm{ad}}.$$

Let X_4 be such that we get an injective map $i_4\colon (G_4, X_4) \hookrightarrow (G_3, X_3)$ with the property that the composite of the inverse of the resulting isomorphism (via i_4 and the first projection $(G_3, X_3) \to (G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}})$) $(G_4^{\mathrm{ad}}, X_4^{\mathrm{ad}}) \tilde{\to} (G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}})$ with the similarly obtained one $(G_4^{\mathrm{ad}}, X_4^{\mathrm{ad}}) \tilde{\to} (G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$ is $i_{\mathbb{Q}}$. Let

$$f_4 := f_3 \circ i_4 : (G_4, X_4) \hookrightarrow (GSp(W, \psi), S);$$

we call it a diagonal embedding associated to $i_{\mathbb{Q}}$. We obviously have $E(G_4, X_4) = E(G_3, X_3)$. We have (cf. [Va2, 4.9.2]; [Va4, V3 of 3.2] offers a simplification in [Va2, 4.9.2], cf. loc. cit.):

Proposition. *If $p \geq 3$, then the triple $(f_4, L_{(p)}, v_3)$ is a SHS.*

We do believe that this proposition holds for $p=2$ as well. Sometimes it is desirable to work with G_3' and G_4' instead of G_3 and G_4, where $G_i' \hookrightarrow G_i$, $i = \overline{3,4}$, is a monomorphism inducing an isomorphism at the level of adjoint groups. Warning: G_4^{der} might not be isomorphic to G_1^{der} or to G_2^{der} (this can happen precisely when none of G_1^{der} and G_2^{der} is a group cover of the other one). This diagonal trick is very useful in two situations:

i) when we want to show that some properties pertaining to a class of finite maps between Shimura quadruples of preabelian (abelian) type do not depend on the ($p=2$) SHS's involved in obtaining them (see [Va2, 4.9] for examples);

ii) when we expect that a property $\mathcal{PR}$ (pertaining to Shimura quadruples of preabelian type) depends only on the adjoint Shimura quadruples involved, and we want to pass accordingly (see also GP of 2.3.3.1) from a ($p=2$) SHS $(f_1, L^1_{(p)}, v_1)$ to a "simpler" one $(f_2, L^2_{(p)}, v_2)$ (for examples see [Va3, 3.2] and [Va2, 4.9.23]; see also the proofs of the existence of integral canonical models of Shimura varieties of preabelian type); here "simpler" means: for which we already know that $\mathcal{PR}$ holds.

Once we proved Theorem 2 of 1.4.1, the above two situations, for $p \geq 3$, are saying more or less the same thing.

2.3.3.1. Of course this trick can be adapted to the context of SATS's (but then we do not have to proof any proposition), as follows. We use the previous notations, with $p \geq 2$; but we just assume that $(f_i, GSp(L^i_{(p)}, \psi), v_i)$ is a $\mathbb{Z}_{(p)}$ embedding. Let $\mathcal{N}_i$ be the normalization of the Zariski closure of $\mathrm{Sh}_{H_i}(G_i, X_i)$ in $\mathcal{M}_{iO_{(v_i)}}$, $i = \overline{1,2}$ (we use as well the same standard notations as for a ($p = 2$) SHS). Let $\mathcal{N}_4$ be the normalization of the Zariski closure of $\mathrm{Sh}_{H_4}(G_4, X_4)$ in $\mathcal{M}_{O_{(v_4)}}$, with $\mathcal{M}$ as the integral canonical model of the Shimura quadruple $(GSp(L_{(p)}, \psi), S, GSp(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p), p)$. Let $\mathcal{M}_{12}$ be the maximal open closed subscheme of

$$\mathcal{M}_{1O_{(v_4)}} \times_{O_{(v_4)}} \mathcal{M}_{2O_{(v_4)}}$$

which is naturally a closed subscheme of $\mathcal{M}_{O_{(v_4)}}$, cf. [Va1, 3.2.14 and 3.2.16], and the fact that the there is an injective finite map

$$((GSp(L^1_{(p)}, \psi_1) \times GSp(L^2_{(p)}, \psi_2)) \cap GSp(L_{(p)}, \psi), S_1 \tilde{\times} S_2) \hookrightarrow (GSp(L^1_{(p)}, \psi_1), S_1) \times (GSp(L^2_{(p)}, \psi_2), S_2);$$

here $S_1 \tilde{\times} S_2$ is the $(GSp(L^1_{(p)}, \psi_1) \times GSp(L^2_{(p)}, \psi_2)) \cap GSp(L_{(p)}, \psi)(\mathbb{R})$-conjugacy class defined by any element of X_4. So $\mathcal{N}_4$ is naturally a finite scheme over $\mathcal{M}_{12}$.

It is more convenient to work over $W(\mathbb{F})$. We follow [Va2, 4.9]. We consider a connected component $\mathcal{C}_4$ of $\mathcal{N}_{4W(\mathbb{F})}$, such that we have natural morphisms $i_j : \mathcal{C}_4 \to \mathcal{C}_j$, $j = \overline{1,2}$, with $\mathcal{C}_j$ a suitable connected component of $\mathcal{N}_{jW(\mathbb{F})}$. From the proof of [Va1, 3.4.1] we get that directly that the morphism

$$i_1 \times i_2 : \mathcal{C}_4 \to \mathcal{C}_1 \times \mathcal{C}_2$$

is a finite map. We got the following situation:

$$\mathcal{M}_{2W(\mathbb{F})} \leftarrow \mathcal{C}_2 \leftarrow \mathcal{C}_4 \to \mathcal{C}_1 \to \mathcal{M}_{1W(\mathbb{F})}. \tag{1}$$

Of course, we can restate everything in terms of quotients of finite type, i.e. when $i_1 \times i_2$ is obtained by taking a pro-étale cover of a similarly constructed morphism of finite type, obtained by working with some finite level symplectic similitude structures. The general philosophy of (1) (or of 2.3.3 ii)) is:

GP. *Often we can shift properties enjoyed by $\mathcal{C}_1$ to $\mathcal{C}_2$, by "working" via $\mathcal{C}_4$.*

2.3.4. Trick four: the standard PEL situation. The main references for what follows are [Ko, §5] and [Va2, 2.3.5]. For the original rational context we refer to [Sh1-2]. Here we review this trick in a form closer to the spirit of [Va2, 2.3.5]. We assume $p \geq 3$. Roughly speaking a standard PEL situation is a SHS $(f, L_{(p)}, v)$, where we can essentially restrict to considering only tensors of degree 2. In other words, if there is a $\mathbb{Z}_{(p)}$-subalgebra $\mathcal{B}$ of $\mathrm{End}(L_{(p)})$ which over $W(\mathbb{F})$ is a product of matrix algebras, which is self dual w.r.t.

ψ, and is such that the subgroup of $GSp(L_{(p)}, \psi)$ fixing its elements has $G_{\mathbb{Z}_{(p)}}$ as its connected component of the origin, then the quadruple $(f, L_{(p)}, v, \mathcal{B})$ is called a standard PEL situation. In fact we do not need to assume a priori that $(f, L_{(p)}, v)$ is a SHS: if $(f, L_{(p)}, v)$ is a SATS, and $\mathcal{B}$ exists having the above properties, then $(f, L_{(p)}, v)$ is automatically a SHS, cf. [Ko, §5]; see also [Va1, 3.4.11 or 5.6.3] and [Va2, 2.3.6]. The same remains true in the context of a $p = 2$ standard PEL situation (defined as above starting from a $p = 2$ SHS $(f, L_{(2)}, v)$ for which a $\mathbb{Z}_{(2)}$-algebra $\mathcal{B}$ as above exists; cf. [Va2, 2.3.18]). We have (cf. [Ko, §5] and [LR] and [Va2, 2.3.18 and 4.14]):

Theorem. *If $(f, L_{(p)}, v, \mathcal{B})$ is a ($p = 2$) standard PEL situation, then $\mathcal{N}$ is an integral canonical model, and is a closed subscheme of $\mathcal{M}_{O_{(v)}}$.*

2.3.4.1. Remarks. 1) This theorem is reobtained in [Va1, 5.6.3-4]. Strictly speaking, [Va1, 5.6.3] is stated for primes $p \geq 3$, as the proof of [Va1, 5.1] deals only with such primes; however as [Va1, 4.3.11] is true for $p = 2$, one can check that the proof of [Va1, 5.1] can be adapted to remove the condition $p \geq 3$ in [Va1, 5.6.3] (cf. [Va2, 2.3.18 and 4.14.2]).

2) For a nice description of the possibilities of $\mathcal{L} := \mathcal{B} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ see [Sh1-2]. See also [Ko, §5] for nice descriptions of different properties of the (first historically constructed) integral canonical models $\mathcal{N}$ obtained starting from a ($p = 2$) standard PEL situation.

2.3.5. Trick five: the degree two trick or the relative PEL situation. The main reference for what follows is [Va1, 4.3.16]. We start capturing the essence of loc. cit.; then we particularize to the context of $\mathbb{Z}_{(p)}$ embeddings into Siegel modular varieties.

Let $p \geq 2$. Let O be a DVR of mixed characteristic $(0, p)$ having a perfect residue field. Let $K_O := O[\frac{1}{p}]$. Let M be a free O-module of finite rank. We consider reductive subgroups G^1 and G^2 of $GL(M)$.

The case of a monomorphism. We assume that G^1 is a subgroup of G^2. We denote by i^1 the inclusion $G^1 \hookrightarrow GL(M)$. We consider a family $(e_\alpha)_{\alpha \in \mathcal{J}^1}$ of endomorphisms of $M \oplus M^*$ fixed by G^1. Let O_1 be the completion of the strict henselization of O. Let R be a faithfully flat O_1-algebra. We consider free R-modules M_R which are R-lattices of $M \otimes_O R[\frac{1}{p}]$ (in the sense that we have $M_R[\frac{1}{p}] = M \otimes_O R[\frac{1}{p}]$), and such that $\forall \alpha \in \mathcal{J}^1$, e_α is an R-endomorphism of $M_R \otimes_R M_R^*$.

2.3.5.1. Definitions. We say that i^1 is a degree 2 representation well adapted w.r.t. G^2 (to be abbreviated as a $DEG2$ representation w.r.t. G^2), if such a family $(e_\alpha)_{\alpha \in \mathcal{J}^1}$ exists having the property:

(*) for any R integral and as above, and for every M_R as above and such that the Zariski closure $\tilde{G}^2_R$ of $G^2_{R[\frac{1}{p}]}$ in $GL(M_R)$ is a reductive group over R, the Zariski closure of $G^1_{R[\frac{1}{p}]}$ in $GL(M_R)$ is a reductive group over R.

If we allow R to be only normal integral (resp. we allow R to be reduced), then we define as in [Va1, paragraph before 4.3.6] the notion of weakly (resp. strongly) $DEG2$ representation w.r.t. G^2. If the trace form on $\mathrm{Lie}(G^{2\mathrm{der}})$ obtained via M is perfect, then, referring to (*), the trace form on $\mathrm{Lie}(\tilde{G}^{2\mathrm{der}}_R)$ obtained via M_R is also perfect (argument: the statement gets easily reduced to the case when R is a complete DVR, and in such a case it is obvious); in such a situation we will use the terminology: (strongly or weakly) degree 2+ representation well adapted w.r.t. G^2, and the abbreviation will be: (strongly or weakly) $DEG2+$ representation w.r.t. G^2.

A more general case. We do not assume any more that G^1 is a subgroup of G^2, but we do assume that the intersection G^3 of G^2 with the centralizer $C(G^1)$ of G^1 in $GL(M)$ is a reductive group over O.

2.3.5.2. Definitions. We say that G^1 and G^2 (warning: the order matters) are well matched as subgroups of $GL(M)$, if for any R integral and as above, and for every R-lattice M_R of $M \otimes_O R[\frac{1}{p}]$ such that the Zariski closures $\tilde{G}^1_R$ and $\tilde{G}^2_R$ of $G^1_{R[\frac{1}{p}]}$ and respectively of $G^2_{R[\frac{1}{p}]}$ in $GL(M_R)$ are reductive groups over R, the Zariski closure of $G^3_{R[\frac{1}{p}]}$ in $GL(M_R)$ is also a reductive group over R. As above we define the notions of strongly or weakly matched.

2.3.5.3. Examples. 1) Let $L \subset \mathrm{End}(M)$ be a semisimple O-algebra such that $L \otimes_O O_1$ is a product of algebras of the form $\mathrm{End}(M_1)$, with M_1 a free O_1-module. We assume that the subgroup of $C(L)$ of $GL(M)$, defined as the centralizer of L, is a reductive group over O. We also assume G^1 is the connected component of the origin of $C(L) \cap G^2$ (defined as the Zariski closure in G^2 of the connected component of the origin of the generic fibre of the intersection $C(L) \cap G^2$). We also assume that one of the following two conditions is satisfied:

a) There is a subtorus T of $GL(M)$ such that G^1 is contained in the intersection G^3 of G^2 and of the centralizer $C(T)$ of T in $GL(M)$, and T and G^2 are well matched as subgroups of $GL(M)$; we also assume that we have $G^{1\mathrm{ab}} = G^{3\mathrm{ab}}$, and that the inclusion $G^{1\mathrm{der}} \hookrightarrow G^{3\mathrm{der}}$ becomes over O_1 the diagonal embedding of $G^{1\mathrm{der}}_{O_1}$ in a product of a finite number of copies of itself, which are permuted transitively (under conjugation) by the invertible elements of $L \otimes_O O_1$;
b) The bilinear form on $\mathfrak{g} := \mathrm{Lie}(G^{2\mathrm{der}})$ induced by the trace form Tr on $\mathrm{End}(M)$ is perfect, and $\pi(\mathfrak{g} \otimes_O K_O)$ (defined as in [Va1, 4.2.1]) leaves invariant $\mathrm{Lie}(C(L))$; moreover a rational multiple of Tr restricts to a perfect form on $\mathrm{Lie}(C(L))$.

If a) holds, then i^1 is a $DEG2$ representation w.r.t. G^2, cf. [Va1, 4.3.16]: we can take as $(e_\alpha)_{\alpha \in \mathcal{J}^1}$, the family formed by all endomorphism of M fixed by G^1; it contains L, all elements of the Lie algebra of the center of G^1, and all elements of $\mathrm{Lie}(T)$. We refer to the situation when a) holds as a type I $DEG2$ representation w.r.t. G^2. If b) holds then i^1 is a $DEG2+$ representation w.r.t. G^2, cf. [Va1, 4.3.16]: we can take as $(e_\alpha)_{\alpha \in \mathcal{J}^1}$, the same family as above. We refer to the situation when b) holds as a type II $DEG2$ representation w.r.t. G^2.

2) We assume that G^1 is the centralizer in G^2 of a subtorus T of G^2. We can take $(e_\alpha)_{\alpha \in \mathcal{J}^1}$ to be the family of all endomorphisms of M fixed by T. Following [Va1, 4.3.13-14] we get directly that i^1 is a $DEG2$ representation w.r.t. G^2. We refer to this situation as a type III $DEG2$ representation w.r.t. G^2. It is the most particular case of type I: obviously T and G^2 are well matched, and we can take L such that its invertible elements are $T(O)$. Similarly, if T is such that the following properties hold:

– it is a subgroup of $GL(M)$ normalizing G^2,

– its connected component of the origin is contained in G^2,

– it is the extension of a finite étale group scheme by a torus in such a way that its O_1-valued points are semisimple elements of $GL(M \otimes_O O_1)$,

and if G^1 is the connected component (assumed to be reductive) of the origin of the centralizer of T in G^2, we will speak about a type IV $DEG2$ representation w.r.t. G^2.

3) Let $n \in \mathbb{N}$, $n \geq 2$, and let (M_i, ψ_i) be a symplectic space over O. Let

$$(M, \psi) := \oplus_{i=1}^{n} (M_i, \psi_i).$$

Let T be the maximal subtorus of $GL(M)$ such that it leaves invariant M_i, for $i = \overline{1, n}$. Let $G^2 := GSp(M, \psi)$. Then T and G^2 are strongly well matched. To see this, we use the notations of 2.3.5.2, with R a reduced faithfully flat O_1-algebra. We can assume, working in the étale topology, that $\tilde{G}^2_R$ and $\tilde{T}_R$ are split. So we can also assume the existence of a perfect symplectic form $\psi_R: M_R \otimes_R M_R \to R$, such that $G^2_R = GSp(M_R, \psi_R)$. The fact that $\tilde{T}_R$ is a split subgroup of $GL(M_R)$ implies that we have a direct sum decomposition $M_R := \oplus_{i=1} M_{iR}$ such that $M_{iR}\left[\frac{1}{p}\right] = M_i \otimes_O R\left[\frac{1}{p}\right]$. But ψ and ψ_R differ from each other, over any generic point of $\mathrm{Spec}(R)$ just by a non-zero scalar. This implies that $\psi_R(x_i, x_j) = 0$, if $x_l \in M_{lR}$, for $l \in \{i, j\} \subset \{1, ..., n\}$, with $i \neq j$. So the restriction of ψ_R to M_{iR} is perfect. The conclusion follows.

2.3.5.4. The ($p = 2$) relative PEL situation. We start by considering two injective maps $f: (G, X) \hookrightarrow (G_1, X_1)$ and $f_1: (G_1, X_1) \hookrightarrow (GSp(W, \psi), S)$. Let v be a prime of $E(G, X)$ dividing p; let v_1 be the prime of $E(G_1, X_1)$ divided by v. Let $L_{(p)}$ be a $\mathbb{Z}_{(p)}$-lattice of W such that $\psi : L_{(p)} \otimes_{\mathbb{Z}_{(p)}} L_{(p)} \to \mathbb{Z}_{(p)}$ is perfect. We assume that the following two triples

$$\mathcal{T}_1 := (f_1, GSp(L_{(p)}, \psi), v_1)$$

and

$$\mathcal{T} := (f, G_{1\mathbb{Z}_{(p)}}, v)$$

are $\mathbb{Z}_{(p)}$ embeddings. Obviously the composite of two $\mathbb{Z}_{(p)}$ embeddings is a $\mathbb{Z}_{(p)}$ embedding; so the triple $(f_1 \circ f, GSp(L_{(p)}, \psi), v)$ is as well a $\mathbb{Z}_{(p)}$ embedding. If $p \geq 3$ (resp. $p = 2$) we say that $\mathcal{T}$ is a relative (resp. $p = 2$ relative) PEL situation w.r.t. $\mathcal{T}_1$, if the representation $G_{\mathbb{Z}_{(p)}} \hookrightarrow GSp(L_{(p)}, \psi)$ is a $DEG2+$ or a $DEG2$ representation w.r.t. $G_{1\mathbb{Z}_{(p)}}$. As in 2.3.5.3 we assign (if possible) a type to it: type I, type II, type III or type IV. The type II is the closest generalization of 2.3.4, cf. [Va1, 4.3.11 and 4.3.16] (so in examples 1) and 6) of 2.3.6, we get type II). In the case of type III or IV, we can (and often do) disregard the part of the terminology: "w.r.t. $\mathcal{T}_1$".

2.3.5.5. Theorem. *We assume that $(f_1, L_{(p)}, v_1)$ is a ($p = 2$) SHS, and that $\mathcal{T}$ is a ($p = 2$) relative PEL situation w.r.t. $\mathcal{T}_1$. Then the triple $(f_1 \circ f, L_{(p)}, v)$ is a ($p = 2$) SHS.*

The proof of this theorem is entirely the same as the proof of the theorem of 2.3.4, and so left as an exercise.

2.3.6. Standard examples of $\mathbb{Z}_{(p)}$ embeddings. The terminology of 2.3.2-5 allows us to quickly present the main list of standard examples; for future references they are (arbitrarily) numbered. The first five examples are just a restatement of the five type cases of 2.2, while the sixth one is obtained by combining two such examples. The next four examples are an adequate restatement of a simplified version of [Va1, 6.6.2] (cf. also 2.3.11 13)), while the eleventh one is just meant to start the passage from the classical setting, to the more general one (involving totally real number fields), to be dealt extensively in the next tricks. The last one is just a $\mathbb{Z}_{(p)}$ version of the SSO of [Va3, 3.1.4]. We assume $p \geq 3$.

1) The triple $(\tilde{a}_{n\mathbb{Q}}, GSp(2n)_{\mathbb{Z}_{(p)}}, p)$ is a $\mathbb{Z}_{(p)}$ embedding into a Siegel modular variety. Writing $Sp(2n) = Sp(W, \psi)$, with (W, ψ) a symplectic space over $\mathbb{Q}$ of dimension $4n$, we can assume the existence of a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W such that $\psi: L_{(p)} \otimes_{\mathbb{Z}_{(p)}} L_{(p)} \to \mathbb{Z}_{(p)}$ is perfect and $GSp(2n)_{\mathbb{Z}_{(p)}} = GSp(L_{(p)}, \psi)$: if needed we can replace ψ by a non-zero rational multiple of it (cf. [Ti1, end of 2.5]). Let $\mathcal{B}$ be as in 2.3.4. The quadruple $(\tilde{a}_{n\mathbb{Q}}, L_{(p)}, p, \mathcal{B})$ is a standard PEL situation.

2) The triple $(\tilde{b}_{n\mathbb{Q}}, GSU(2^{n-1}, 2^{n-1})_{\mathbb{Z}_{(p)}}, p)$ is a $\mathbb{Z}_{(p)}$ embedding into a classical unitary variety having trivial involution.

3) The triple $(\tilde{c}_{n\mathbb{Q}}, GSU(n, n)_{\mathbb{Z}_{(p)}}, p)$ is a $\mathbb{Z}_{(p)}$ embedding into a classical unitary variety having trivial involution.

4) The triple $(\tilde{d}_{n\mathbb{Q}}, GSU(n, n)_{\mathbb{Z}_{(p)}}, p)$ is a $\mathbb{Z}_{(p)}$ embedding into a classical unitary variety having trivial involution.

5) The triple $(\tilde{e}_{n\mathbb{Q}}, GSU(2^{n-1}, 2^{n-1})_{\mathbb{Z}_{(p)}}, p)$ is a $\mathbb{Z}_{(p)}$ embedding into a classical unitary variety having trivial involution.

6) The triple $(\tilde{a}_{n\mathbb{Q}} \circ \tilde{d}_{n\mathbb{Q}}, GSp(2n)_{\mathbb{Z}_{(p)}}, p)$ is a $\mathbb{Z}_{(p)}$ embedding into a Siegel modular variety. If $L_{(p)}$ is chosen as in 1) above, and we define $\mathcal{B}$ as usual, the quadruple $(\tilde{a}_{n\mathbb{Q}} \circ \tilde{d}_{n\mathbb{Q}}, L_{(p)}, p, \mathcal{B})$ is a standard PEL situation.

7) Let $n, m \in \mathbb{N}$. Let $(L_{(p)}, \psi)$ be a symplectic space over $\mathbb{Z}_{(p)}$ of rank $2n$. We consider the natural monomorphism

$$\tilde{f}_n: GSp(L_{(p)}, \psi) \hookrightarrow GSp(L_{(p)}^{\oplus m}, \psi^{\oplus m});$$

here the upper right index $\oplus m$ means direct sum of m copies. It gives birth to an injective map $\tilde{f}_{n\mathbb{Q}}: (GSp(n), X_{\mathrm{Sp}(n)^{\mathrm{ad}}}) \hookrightarrow (GSp(nm), X_{\mathrm{Sp}(nm)^{\mathrm{ad}}})$. Obviously the triple

$$(\tilde{f}_{n\mathbb{Q}}, GSp(L_{(p)}^{\oplus m}, \psi^{\oplus m}), p)$$

is a $\mathbb{Z}_{(p)}$ embedding. If $\mathcal{B}$ is defined as usual, then the quadruple $(\tilde{f}_{n\mathbb{Q}}, L_{(p)}^{\oplus m}, p, \mathcal{B})$ is a standard PEL situation.

8) Let $n \geq 2$. We consider the standard monomorphism

$$g_n: SO(2, 2n-3)_{\mathbb{Z}_{(p)}} \hookrightarrow SO(2, 2n-1)_{\mathbb{Z}_{(p)}}$$

defined (with the notations of Case two of 2.1 and of 2.2) by: $SO(2,2n-3)_{\mathbb{Z}_{(p)}}$ acts trivially on the $\mathbb{Z}_{(p)}$-lattice generated by x_{2n} and x_{2n+1} and naturally on the $\mathbb{Z}_{(p)}$-lattice generated by the first $2n-1$ variables x_1,..., x_{2n-1}. Lifting it at the level of GSpin's groups we get a natural monomorphism

$$\tilde{g}_n\colon \mathrm{GSpin}(2,2n-3)_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GSpin}(2,2n-1)_{\mathbb{Z}_{(p)}},$$

giving birth to an injective map

$$\tilde{g}_{n\mathbb{Q}}\colon (\mathrm{GSpin}(2,2n-3), X_{SO(2,2n-3)}) \hookrightarrow (\mathrm{GSpin}(2,2n-1), X_{SO(2,2n-1)}).$$

So we get a $\mathbb{Z}_{(p)}$ embedding $(\tilde{g}_{n\mathbb{Q}}, \mathrm{GSpin}(2,2n-1)_{\mathbb{Z}_{(p)}}, p)$ into an orthogonal Shimura variety of B_n type.

9) Let $n, m \in \mathbb{N}$. We assume $n \geq 4$. As in 7) we get a natural monomorphism

$$\tilde{h}_n\colon GSO^*(2n)_{\mathbb{Z}_{(p)}} \hookrightarrow GSO^*(2nm)_{\mathbb{Z}_{(p)}}.$$

It gives birth to an injective map

$$\tilde{h}_{n\mathbb{Q}}\colon (GSO^*(2n), X_{SO^*(2n)^{\mathrm{ad}}}) \hookrightarrow (GSO^*(2nm), X_{SO^*(2nm)^{\mathrm{ad}}}).$$

We get a $\mathbb{Z}_{(p)}$ embedding $(\tilde{h}_{n\mathbb{Q}}, GSO^*(2nm)_{\mathbb{Z}_{(p)}}, p)$ into a classical Shimura variety of $D_{nm}^{\mathbb{H}}$ type with trivial involution.

10) Let $n \geq 4$. We consider the standard monomorphism

$$i_n\colon SO(2,2n-2)_{\mathbb{Z}_{(p)}} \hookrightarrow SO(2,2n)_{\mathbb{Z}_{(p)}}$$

defined in the same manner as g_n. Lifting it at the level of GSpin's groups we get a natural monomorphism

$$\tilde{i}_n\colon \mathrm{GSpin}(2,2n-2)_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GSpin}(2,2n)_{\mathbb{Z}_{(p)}},$$

giving birth to an injective map

$$\tilde{i}_{n\mathbb{Q}}\colon (\mathrm{GSpin}(2,2n-2), X_{SO(2,2n-2)}) \hookrightarrow (\mathrm{GSpin}(2,2n), X_{SO(2,2n)}).$$

So we get a $\mathbb{Z}_{(p)}$ embedding $(\tilde{i}_{n\mathbb{Q}}, \mathrm{GSpin}(2,2n)_{\mathbb{Z}_{(p)}}, p)$ into an orthogonal Shimura variety of $D_{n+1}^{\mathbb{R}}$ type.

11) Let (G,X) be a simple adjoint Shimura pair of preabelian type. From [De1, 2.3.4] we deduce the existence of a totally real number field $F := F(G,X)$ (cf. the notations of [Va3, 2.5]) and of an absolutely simple adjoint F-group G^F of classical Lie type such that

$$G = \mathrm{Res}_{F/\mathbb{Q}}(G^F).$$

Let F_1 be a totally real number field F_1 containing F. We denote by $G^{F_1} := G^F_{F_1}$. Let $G(F_1) := \mathrm{Res}_{F_1/\mathbb{Q}}(G^{F_1})$. The convention is $G = G(F)$. We assume that G is unramified over $\mathbb{Q}_p$ and that F_1 is unramified above p. As in [Va1, 6.5.1] we deduce that G^{F_1} is unramified over the completion of F_1 w.r.t. any prime of it dividing p. Let G_p^F be an adjoint group over $F_{(p)}$ having G^F as its generic fibre. Let $G_p^{F_1}$ be its pull back to $F_{1(p)}$. Let $G_{\mathbb{Z}_{(p)}} := \mathrm{Res}_{F_{(p)}/\mathbb{Z}_{(p)}}(G_p^F)$ and $G_{1\mathbb{Z}_{(p)}} := \mathrm{Res}_{F_{1(p)}/\mathbb{Z}_{(p)}}(G_p^{F_1})$. They are both adjoint groups over $\mathbb{Z}_{(p)}$; their generic fibres are respectively G and $G(F_1)$. We have a natural monomorphism

$$j(F_1)^{(G,X)}: G_{\mathbb{Z}_{(p)}} \hookrightarrow G_{1\mathbb{Z}_{(p)}}.$$

Let $X(F_1)$ be the $G(F_1)(\mathbb{R})$-conjugacy class generated by X via the extension to $\mathbb{R}$ of this monomorphism; $X(F_1)$ is a disjoint union of $[F_1 : F]$ copies of X. We get an injective map

$$j(F_1)^{(G,X)}_{\mathbb{Q}}: (G,X) \hookrightarrow (G(F_1), X(F_1)),$$

and a $\mathbb{Z}_{(p)}$ embedding $(j(F_1)^{(G,X)}_{\mathbb{Q}}, G_{1\mathbb{Z}_{(p)}}, p)$.

12) Let $n \geq 5$. We consider the inclusion

$$j_n : SO(2,n)_{\mathbb{Z}_{(p)}} \hookrightarrow SO(2,n+1)_{\mathbb{Z}_{(p)}}$$

defined in the same manner as g_n. If n is even (resp. n is odd) let $g(n) \in SO(2,n+1)_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})$ (resp. $g(n) \in O(2,n+1)_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})$) be an element of order 2 such its centralizer in $SO(2,n+1)_{\mathbb{Z}_{(p)}}$ is the extension of $\mathbb{Z}/2\mathbb{Z}$ by $SO(2,n)_{\mathbb{Z}_{(p)}}$ (resp. is $SO(2,n)_{\mathbb{Z}_{(p)}}$). Lifting j_n at the level of GSpin's groups we get a natural monomorphism

$$\tilde{j}_n : \mathrm{GSpin}(2,n)_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GSpin}(2,n+1)_{\mathbb{Z}_{(p)}},$$

giving birth to an injective map

$$\tilde{j}_{n\mathbb{Q}}: (\mathrm{GSpin}(2,n), X_{SO(2,n)}) \hookrightarrow (\mathrm{GSpin}(2,n+1), X_{SO(2,n+1)}).$$

So we get a $\mathbb{Z}_{(p)}$ embedding $(\tilde{j}_{n\mathbb{Q}}, \mathrm{GSpin}(2,n+1)_{\mathbb{Z}_{(p)}}, p)$ from an orthogonal Shimura variety into another orthogonal Shimura variety of a different Lie type.

Let $G^2\mathrm{Spin}(2,n+1)_{\mathbb{Z}_{(p)}}$ be the extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathrm{GSpin}(2,n+1)_{\mathbb{Z}_{(p)}}$, corresponding to the extension of $O(2,n+1)_{\mathbb{Z}_{(p)}}$ by $\mathbb{G}_m$. Moreover $g(n)$ lifts to a torsion element (uniquely determined up to sign)

$$\tilde{g}(n) \in G^2\mathrm{Spin}(2,n+1)_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})$$

having the same properties. If n is even, then $\tilde{g}(n) \in \mathrm{GSpin}(2,n+1)_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})$.

2.3.6.1. Remarks. 1) The main philosophy of 1) to 12) above is: all extra $\mathbb{Z}_{(p)}$ embeddings to be considered below are obtained from the above eleven ones of 1-10) and 12), via 11) and its versions (in a reductive context), through an adequate twisting process. This twisting process was first used in [Va1, last sentence of 6.6.2] and [Va3, 3.0]. It will be reviewed and detailed in 2.3.8-10 below.

2) In examples 7), 8), 10) and 12) we get relative PEL situations (cf. 2.3.5.3-4); in examples 7), 8) and 10) we get a type I relative PEL situation, while in example 12) we get a type IV relative PEL situation (for example 7) cf. 2.3.5.3 3)). Also we can get a relative

PEL situation for example 9), provided we work w.r.t. $(\tilde{a}_{nm\mathbb{Q}} \circ \tilde{d}_{nm\mathbb{Q}}, GSp(2nm)_{\mathbb{Z}_{(p)}}, p)$ (cf. 2.3.5.3; the argument is entirely the same as the one of 2.5.3.5 3)); it is also of type I.

3) Example 1) can be extended to the case of an arbitrary $SU(n,m)$, i.e. when $n \neq m$. As this is not needed here, it will not be included here.

2.3.7. Trick six: the degree 4 trick. The main reference for this section is [Va1, 4.3, 5.7.1, 5.7.4 and 6.5-6]. What follows is somehow similar to 2.3.5: we just move from a $DEG2$ theory to a $DEG4$ theory, following 2.3.5. Again we start with a general setting, and then we rephrase it in terms of $\mathbb{Z}_{(p)}$ embeddings into Siegel modular varieties. Moreover, for simplicity, as in 2.3.5 we concentrate on the mixed characteristic context (see [Va1, 4.3.10.1] for how the general context is accomplished).

2.3.7.1. Definitions. 1) Let p, O, O_1, M and K_O be as in 2.3.5. We consider a faithful representation $\rho : G^1 \hookrightarrow GL(M)$, with G^1 a reductive group. We say that ρ is a $DEG4$ representation, if the following three requirements are satisfied:

i) an integral power of 2 times the Killing form on $\mathfrak{g}_1 := \mathrm{Lie}(G^{1\mathrm{der}})$ is perfect;

ii) there is a projector π_M of $\mathrm{End}(M) \otimes_O O_1$ on $\mathfrak{g}_1 \otimes_O O_1$ fixed by G^1;

iii) p is greater or equal to the number $s(\mathfrak{g}_1 \otimes_O K_O, M \otimes_O K_O)$ introduced in [Va1, 4.3.3].

2) We consider a $\mathbb{Z}_{(p)}$ embedding $\mathcal{T} := (f, GSp(L_{(p)}, \psi), v)$ into a Siegel modular variety. Here $f\colon (G,X) \hookrightarrow (GSp(W,\psi), S)$ is as usual. We say that $\mathcal{T}$ is a $DEG4$ embedding if the representation $G_{\mathbb{Z}_{(p)}} \hookrightarrow GSp(L_{(p)}, \psi)$ is a $DEG4$ representation.

We recall that

$$s(\mathrm{Lie}(G^{\mathrm{der}}), W) = 2$$

regardless of how f is, cf. [Va1, 5.7.1]. So, as we have [De1, 2.3.10], there is no surprise that we also have the following result:

2.3.7.2. Theorem. *Let (G,X) be a simple adjoint Shimura pair of preabelian type. Let $p \geq 3$ be a prime such that G is unramified over $\mathbb{Q}_p$. Then there is an injective map $f_1\colon (G_1, X_1) \hookrightarrow (GSp(W_1, \psi_1), S_1)$ into a Shimura pair defining a Siegel modular variety such that:*

a) $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) = (G^{\mathrm{ad}}, X^{\mathrm{ad}})$;

b) for any Shimura pair (G_3, X_3) of abelian type having (G,X) as its adjoint, there is a natural isogeny $G_1^{\mathrm{der}} \to G_3^{\mathrm{der}}$;

c) f_1 factors through an injective map $f_2\colon (G_2, X_2) \hookrightarrow (GSp(W_1, \psi_1), S_1)$ such that:

- *i) there is a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W such that $\psi : L_{(p)} \otimes_{\mathbb{Z}_{(p)}} L_{(p)} \to \mathbb{Z}_{(p)}$ is perfect, and the triples $\mathcal{T}_1 := (f_1, GSp(L_{(p)}, \psi), v_1)$ and $\mathcal{T}_2 := (f_2, GSp(L_{(p)}, \psi), v_2)$ are $\mathbb{Z}_{(p)}$ embeddings, for any prime v_1 of $E(G_1, X_1)$ dividing p, and with v_2 as the prime of $E(G_2, X_2)$ divided by v_1;*
- *ii) $\mathcal{T}_2$ is a $DEG4$ embedding;*
- *iii) denoting by f_0 the factorization $(G_1, X_1) \hookrightarrow (G_2, X_2)$ of f_1, the triple $(f_0, G_{1\mathbb{Z}_{(p)}}, v_1)$ is a relative PEL situation w.r.t. $\mathcal{T}_2$;*
- *iv) G_2^{ad} is a simple $\mathbb{Q}$–group;*

v) the connected component of the origin of $Z(G_2)$ is the same as the connected component of the origin of $Z(G_1)$.

Proof: This is nothing else but [Va1, 6.6.2 and 6.6.6]: to be able to use [Va1, 6.5.1.1 vi)] we just needed to rephrase (see 2.3.7.1) the definition [Va1, 5.8.1]. We recall some of the details. If G is of A_n or C_n Lie type, with p not dividing $n+1$, or of B_n Lie type with p not dividing $2n-1$, or of D_n Lie type with p not dividing $n-1$, then we can take f_0 to be the identity (so $f_1 = f_2$), and [Va1, 6.5.1.1] applies entirely. The above excluded cases are in conformity with [Va1, 5.7.2-3 and 6.6.6]. Moreover the general case of G being of A_n Lie type is handled by [Va2, 2.3.5.1]: if we have a standard PEL situation $(f_1, L_{(p)}, v_1, \mathcal{B}_1)$, then we can take f_2 to be the identity and $f_0 = f_1$. So from now on in the proof, when we refer to the case G is of A_n Lie type, we assume that n is odd and that all simple factors of $G_{\mathbb{R}}$ are either compact or have $\frac{n+1}{2}$ as their $\mathbb{R}$-ranks: we just want to exemplify even more how 2.3.6 can be used in practice.

If G is of A_{pm-1} or of C_{pm-1} Lie type, with $m \in \mathbb{N}$, then we can take G_2^{ad} to be of C_{pm} Lie type and respectively of C_{2pm-2} Lie type. If G is of $B_{pm+\frac{p+1}{2}}$ Lie type, $m \in \mathbb{N} \cup \{0\}$, then we can take G_2^{ad} to be of $B_{1+pm+\frac{p+1}{2}}$ Lie type. If (G, X) is of $D^{\mathbb{R}}_{pm+1}$ type, with $m \in \mathbb{N}$, then we can take $(G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$ to be of $D^{\mathbb{R}}_{pm+2}$ type. If (G, X) is of $D^{\mathbb{H}}_{pm+1}$ type, with $m \in \mathbb{N}$, then we can take $(G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$ to be of $D^{\mathbb{H}}_{2pm+2}$ type. All these choices are in conformity with [Va1, 6.6.2].

We just have to recall (cf. [Va1, 6.6.5]) how we effectively construct f_1 and f_2 subject to all the requirements, in the excluded cases. Discussion has to go case by case. Here we recall fully just the B case, i.e. when G is of $B_{pm+\frac{p+1}{2}}$ Lie type, $m \in \mathbb{N} \cup \{0\}$, as all the other cases are entirely the same. We will use the notations of 2.3.6 11). The group G^F is an outer form of the group $SO(2, 2pm+p)_F$. But both these two groups are unramified over $\mathbb{Q}_p$, and over $\mathbb{R}$ are inner forms of each other (regardless of the considered embedding $F \hookrightarrow \mathbb{R}$), cf. [De1, 1.2.1]. As in [Va3, 3.0 F) and G)] we deduce the existence of a totally real number field F_1 containing F and unramified over p, and such that G^{F_1} is an inner form of $SO(2, 2pm+p)_{F_1}$. Let

$$\gamma \in H^1(\mathrm{Gal}(\overline{F_1}/F_1), SO(2, 2pm+p)_{F_1})$$

be the element achieving this twist. Of course, as we are in the B case, we can take $F_1 = F$; but we present the details in such a manner that they work out for the other cases as well. So we can take G_2^{ad} to be the Weil restriction $\tilde{G}_2^{F_1}$ from F_1 to $\mathbb{Q}$ of the twisted form of $SO(2, 2pm+p+2)_{F_1}$, obtained via γ and via the standard inclusion $g_{pm+\frac{p+3}{2}F_1} : SO(2, 2pm+p)_{F_1} \hookrightarrow SO(2, 2pm+p+2)_{F_1}$.

We immediately get that G_2^{ad} is unramified over $\mathbb{Q}_p$, and that we have a monomorphism $G \hookrightarrow G_2^{\mathrm{ad}}$ which lifts to a $\mathbb{Z}_{(p)}$ monomorphism $G_{\mathbb{Z}_{(p)}} \hookrightarrow G^{\mathrm{ad}}_{2\mathbb{Z}_{(p)}}$, and which gives birth (cf. [Va3, Fact of 3.0 G)]) to an injective map $(G, X) \hookrightarrow (G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$. So now we have to apply [Va1, 6.5.1.1] to $(G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$ and the $\mathbb{Z}_{(p)}$ reductive form of G_2^{ad} we got. We automatically get a) and b), as G_1^{der} is simply connected. iv) of c) is satisfied, cf. the construction, while v) of c) can always be achieved. i) of c) is automatically satisfied, while ii) of c) is the context of loc. cit. The fact that the triple $(f_0, G_{1\mathbb{Z}_{(p)}}, v_1)$ is a relative

PEL situation of type III (and so of type I as well) (w.r.t. $\mathcal{T}_2$), is a consequence of the fact that G is the derived subgroup of the centralizer of a subtorus of G_2^{ad}, obtained from Weil restriction from F_1 to $\mathbb{Q}$ of a one dimensional subtorus of $\tilde{G}_2^{F_1}$, and which extends to a subtorus of $G^{\mathrm{ad}}_{2\mathbb{Z}_{(p)}}$. This settles the B case.

The case (G,X) is of $D^{\mathbb{R}}_{pm+1}$ type is entirely the same, starting from 2.3.6 10). The case (G,X) is of $D^{\mathbb{H}}_{pm+1}$ type is very similar: starting from 2.3.6 9), the only difference is that we get a type I (can be viewed also as type II) relative PEL situation w.r.t. $\mathcal{T}_2$, cf. 2.3.6.1 2). The case (G,X) is of C_{pm-1} type is entirely similar to the case of $D^{\mathbb{H}}_{pm+1}$ type, starting from 2.3.6 8). In case G is of A_{pm-1} Lie type, then we can either quote directly [Va2, 2.3.5.1] (so we can take f_2 to be the identity; but this does not fit the desire of having G_2^{ad} of C_{pm} Lie type), or we can just follow the same pattern, starting from 2.3.6 1). This ends the proof of the theorem.

We come back to the context of 2.3.7.1 1). In the following proposition we recall the essence of [Va1, 4.3.10 b), 4.3.10.1 1), 4.3.13 and 4.3.6 2)]. We have:

2.3.7.3. Proposition: the fundamental proposition of the $DEG4$ **theory (in mixed characteristic).** *We assume $p\geq 3$. Then there is a family of homogeneous tensors of $End(M\otimes_O O_1)$ and of $End(End(M\otimes_O O_1))$ (i.e. of degrees 2 and 4), containing precisely three tensors of degree 4, and which is strongly O_1-very well position for $G^1_{O_1}$.*

Proof: The proof of this proposition, based on [Va1, 4.3.10 b), 4.3.10.1 1), 4.3.13 and 4.3.6 2)], is entirely the same as the part of the proof of [Va1, 5.7.1] (cf. also [Va1, 5.7.4]) involving $\mathbb{Z}_{(p)}$-well position families of tensors. So it is not repeated here. We just recall that the three tensors of degree 4 are: π_M, and two others tensors B and B^* obtained as in [Va1, 4.3.2], and "capturing" what i) of 2.3.7.1 1) is saying.

2.3.8. Trick seven: the unitary trick. This is nothing else but [Va3, 3.1 and 3.1.2 2)]: here we just recall the results. We use the notations of 2.3.6 11) (so $p\geq 3$). If (G,X) is of A_n type, then [Va2, 2.3.5.1] represents the first part of this trick. From now on we assume that (G,X) is not of A_n type. We have (cf. [Va3, 3.1]) the following second part of the unitary trick:

Proposition. *There are injective maps*

$$(G_2,X_2)\hookrightarrow(G_3,X_3)\hookrightarrow(G_1,X_1)\overset{f_1}{\hookrightarrow}(GSp(W,\psi),S)$$

such that:

a) $(G_2^{\mathrm{ad}},X_2^{\mathrm{ad}})=(G,X)$ and $(G_3^{\mathrm{ad}},X_3^{\mathrm{ad}})=(G(F_1),X(F_1))$, for some totally real number field containing F and unramified above p;

b) we get a monomorphism $G_2^{\mathrm{ad}}\hookrightarrow G_3^{\mathrm{ad}}$ which (under the identifications of a)) is nothing else but $j(F_1)^{(G,X)}_{\mathbb{Q}}$;

c) there is a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W such that $\psi\colon L_{(p)}\otimes_{\mathbb{Z}_{(p)}}L_{(p)}\to\mathbb{Z}_{(p)}$ is perfect and we get a standard PEL situation $(f_1,L_{(p)},v_1,\mathcal{B})$, for any prime v_1 of $E(G_1,X_1)$ dividing p;

d) denoting by f_j the resulting inclusion $(G_j,X_j)\hookrightarrow(GSp(W,\psi),S)$, $j\in\{2,3\}$, the triple $\mathcal{T}_j:=(f_j,GSp(L_{(p)},\psi),v_j)$ is a $\mathbb{Z}_{(p)}$ embedding; here v_j is an arbitrary prime of $E(G_j,X_j)$ dividing p;

e) if (G,X) is without involution (resp. with involution; this implies that (G,X) is of $D_n^{\mathbb{H}}$ type), then $E(G_2,X_2)=E(G_3,X_3)$ are totally real number fields containing $E(G_1,X_1)$ (resp. $E(G_2,X_2)=E(G_3,X_3)$ are totally imaginary number fields containing strictly $E(G_1,X_1)$);

f) $Z^0(G_2)\subset Z^0(G_3)\subset Z^0(G_1)$ (here $Z^0(*)$ denotes the toric part of the center of the group $*$);

g) if (G,X) is not of $D_n^{\mathbb{H}}$ type, then G_2^{der} and G_3^{der} are simply connected;

h) denoting by $G^d_{3\mathbb{Z}_{(p)}}$ the image of $G^{\mathrm{der}}_{3\mathbb{Z}_{(p)}}$ in $G^{\mathrm{ad}}_{1\mathbb{Z}_{(p)}}$, the monomorphism $G^d_{3\mathbb{Z}_{(p)}}\hookrightarrow G^{\mathrm{der}}_{1\mathbb{Z}_{(p)}}$ is obtained from Weil restriction from $F_{1(p)}$ to $\mathbb{Z}_{(p)}$ of a monomorphism $m_{F_{1(p)}}$, obtained by twisting via an inner form (in the $D_n^{\mathbb{R}}$ type case defined by a cohomological class liftable to the context of $SO(2,2n-2)_{F_{1(p)}}$) a monomorphism defined as follows:

(i) if G is of B_n Lie type, $n\geq 2$, then we have the monomorphism defined by the extension to $F_{1(p)}$ of the monomorphism $SO(2,2n-1)_{\mathbb{Z}_{(p)}}\hookrightarrow SU(2^{n-1},2^{n-1})^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$ of the Case B of 2.2;

(ii) if G is of C_n Lie type, $n\geq 2$, then we have the monomorphism defined by the extension to $F_{1(p)}$ of the monomorphism $PSp(n)_{\mathbb{Z}_{(p)}}\hookrightarrow SU(n,n)^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$ of the Case C of 2.2;

(iii) if (G,X) is of $D_n^{\mathbb{H}}$ type, $n\geq 4$, then we have the monomorphism defined by the extension to $F_{1(p)}$ of the monomorphism $SO^*(2n)^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}\hookrightarrow SU(n,n)^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$ of the Case $D^{\mathbb{H}}$ of 2.2;

(iv) if (G,X) is of $D_n^{\mathbb{R}}$ type, $n\geq 4$, then we have the monomorphism defined by the extension to $F_{1(p)}$ of the monomorphism $SO(2,2n-2)_{\mathbb{Z}_{(p)}}\hookrightarrow SU(2^{n-1},2^{n-1})^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$ of the case $D^{\mathbb{R}}$ of 2.2;

i) the representations $G^{\mathrm{der}}_{j\mathbb{C}}\hookrightarrow GL(W\otimes_{\mathbb{Q}}\mathbb{C})$, $j=\overline{1,3}$, involve no tensor products;

j) writing $G_1^{\mathrm{ad}}=Res_{F_1/\mathbb{Q}}(\tilde{G})$, then for any embedding $F_1\hookrightarrow\mathbb{R}$, $\tilde{G}_{\mathbb{R}}$ is compact iff $G_{\mathbb{R}}$ is compact;

l) the faithful representation $G^{\mathrm{der}}_{1W(\mathbb{F})}\hookrightarrow GL(L_{(p)}\otimes_{\mathbb{Z}_{(p)}}W(\mathbb{F}))$ is a direct sum of SD-standard representations (in the sense of [Va3, 3.2.3]) associated to a simple factor of the simply connected derived group $G^{\mathrm{der}}_{1W(\mathbb{F})}$;

m) the monomorphism $G^{\mathrm{der}}_{1\mathbb{Z}_{(p)}}\hookrightarrow GL(L_{(p)})$ is constructed as in [Va1, 6.6.5].

Warning: not all the above properties are contained in the statements of [Va3, 3.1]; those which are not mentioned in [Va3, 3.1] are obtained in the process [Va3, 3.0] leading to [Va3, 3.1]. For instance, c) is contained in [Va3, 3.0 I)], j) is contained in [Va3, 3.0 G)], while m) is contained in [Va3, 3.0 I)]. Related to e) we have to mention: the similar statement involving adjoint Shimura pairs is obvious from the construction; but $E(G_i,X_i)$, $i=\overline{1,3}$, is the composite field of $E(G_i^{\mathrm{ad}},X_i^{\mathrm{ad}})$ and of the reflex field E of $(G_2^{\mathrm{ab}},X_2^{\mathrm{ab}})$ (cf. f) and [De3, 3.8]), and so [De1, 2.3.10] allows us to make the appropriate choice for E so that e) holds. For a version of the above proposition, in the case (G,X) is a product of simple adjoint Shimura pairs, see [Va3, 3.1.2 2)].

2.3.9. Trick eight: the orthogonal trick. The main reference for what follows is [Va1, 6.6.2]. As the presentation in loc. cit. might be too quick for non-specialists, here we repeat it with full details. We consider $p \geq 3$. To go faster we treat the cases B_n and $D_n^{\mathbb{R}}$ simultaneously. Let $m \in \mathbb{N}$, $m \geq 6$ or odd. Let F be a totally real number field unramified above p. We consider the semisimple group $SO(2, m+2)_{\mathbb{F}_{(p)}}$ over $\mathbb{F}_{(p)}$ obtained from $SO(2, m+2)_{\mathbb{Z}_{(p)}}$ by extension of scalars. We want to consider a subtorus T of it of relative dimension one, such that the derived subgroup $C(T)^{\mathrm{der}}$ of its centralizer $C(T)$ in $SO(2, m+2)_{\mathbb{F}_{(p)}}$ has the following property:

– for each embedding $i_F: F \hookrightarrow \mathbb{R}$, $T_{\mathbb{R}}$ is compact and $C(T)_{\mathbb{R}}^{\mathrm{der}}$ is of prescribed type $\tau(I_F)$, where $\tau(i_F)$ is either the compact type $SO(m+2)_{\mathbb{R}}$ or is the type $SO(2, m)_{\mathbb{R}}$.

As usual, we allow the replacement of F by a totally real number field F_1, unramified above p.

Once we have such a torus T, the Weil restriction

$$\mathrm{Res}(i_T): \mathrm{Res}_{F_{(p)}/\mathbb{Z}_{(p)}}(C(T)) \hookrightarrow \mathrm{Res}_{F_{(p)}/\mathbb{Z}_{(p)}}(SO(2, m+2)_{F_{(p)}})$$

of the monomorphism $i_T: C(T) \hookrightarrow SO(2, m+2)_{F_{(p)}}$, gives birth naturally to $\mathbb{Z}_{(p)}$ embeddings

$$(\mathrm{Res}(i_T)_{\mathbb{Q}}, \mathrm{Res}_{F_{(p)}/\mathbb{Z}_{(p)}}(SO(2, m+2)_{F_{(p)}}), v_1),$$

where

$$i_{T\mathbb{Q}}: (G_1, X_1) := (\mathrm{Res}_{F/\mathbb{Q}}(C(T)_F), X_1) \hookrightarrow (G_2, X_2) := (\mathrm{Res}_{F/\mathbb{Q}}(SO(2, m+2)_F), X_2)$$

is the logical injective map between Shimura pairs of orthogonal type, and where v_1 is a prime of $E(G_1, X_1)$ dividing p.

The construction of such a torus is just a simple application of the approximation theory. We start considering a quadratic form

$$B(F_{(p)})(a_1, a_2, a_3, a_4) := a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2 + x_5^2 + \ldots + x_{m+4}^2,$$

where $a_i \in \mathbb{G}_m(F_{(p)})$, $i = \overline{1, 4}$, are such that:

i) they are invertible squares in the completion of $F_{(p)}$ w.r.t. any finite prime of it;

ii) if i_F is a real embedding as above, then $i_F(a_1)$ and $i_F(a_2)$ (resp. $i_F(a_3)$ and $i_F(a_4)$) are both positive or negative (resp. both negative or positive), depending on the fact that $\tau(i_F)$ is compact or not.

We consider the SO group of $B(F_{(p)})$; we denote it by $SO(B(F_{(p)})$. It is a semisimple group over $F_{(p)}$, such that for any real embedding i_F, $SO(B(F_{(p)}))_{\mathbb{R}}$ is non-compact and isomorphic to $SO(2, m+2)_{\mathbb{R}}$. Let T_1 be its subgroup acting trivially on the subspace corresponding to the quadratic form $a_3 x_3^2 + a_4 x_4^2 + x_5^2 + \ldots + x_{m+4}^2$. It is a one dimensional torus over $F_{(p)}$.

As in [Va3, 3.0 F)] we deduce the existence of a totally real number field F_1 unramified over p and such that $SO(B(F_{(p)})_{F_{1(p)}}$ and $SO(2, m+2)_{F_1(p)}$ are inner forms of each other, and in case m is even, the inner form is defined by an element of $H^1(F_1, SO(B(F_{(p)})_{F_1})$. Of course, by enlarging F_1 (via a totally real quadratic or quartic extension) we can assume as well that this inner twist is trivial. These facts are recalled as well in the proofs of C) and D) of the Proposition below.

So now, we just take T to be the subtorus of $SO(2, m+2)_{F_1(p)}$ corresponding to $T_{1F_{1(p)}}$ via this twist (this makes sense at least after enlarging F_1). Obviously it has the desired properties. We get:

2.3.9.1. Proposition. *A) Let (G,X) be a simple adjoint Shimura pair of B_n (resp. of $D_n^{\mathbb{R}}$) type. We use the notations of 2.3.6 11). Let $p \geq 3$ be a prime such that G is unramified over $\mathbb{Q}_p$. Then for any $l \in \mathbb{N}$ there is an injective map $f_1\colon (G_1, X_1) \hookrightarrow (GSp(W_1, \psi_1), S_1)$ into a Shimura pair defining a Siegel modular variety such that:*

a) $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) = (G, X)$ and G_1^{der} is simply connected;

b) f_1 factors through an injective map $f_2 \circ f_3$, where $f_2\colon (G_2, X_2) \hookrightarrow (GSp(W_1, \psi_1), S_1)$ and $f_3\colon (G_3, X_3) \hookrightarrow (G_2, X_2)$ are injective maps, with $E(G_2, X_2) = \mathbb{Q}$, $G_2^{\mathrm{ab}} = \mathbb{G}_m$ and $E(G_3^{\mathrm{ad}}, X_3^{\mathrm{ad}}) = E(G,X)$, and such that:

- *i) there is a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W such that $\psi : L_{(p)} \otimes_{\mathbb{Z}_{(p)}} L_{(p)} \to \mathbb{Z}_{(p)}$ is perfect, and the following three triples $\mathcal{T}_1 := (f_1, GSp(L_{(p)}, \psi), v_1)$, $\mathcal{T}_2 := (f_2, GSp(L_{(p)}, \psi), p)$ and $\mathcal{T}_3 := (f_2 \circ f_3, GSp(L_{(p)}, \psi), v_3)$ are $\mathbb{Z}_{(p)}$ embeddings, for any prime v_1 of $E(G_1, X_1)$ dividing p and dividing a prime v_3 of $E(G_3, X_3)$;*
- *ii) denoting by f_0 the factorization $(G_1, X_1) \hookrightarrow (G_3, X_3)$ of f_1, the triple $(f_0, G_{3\mathbb{Z}_{(p)}}, v_1)$ (resp. $(f_3, G_{2\mathbb{Z}_{(p)}}, v_3)$) is a relative PEL situation w.r.t. $\mathcal{T}_3$ (resp. w.r.t. $\mathcal{T}_2$) of type I (resp. of type III);*
- *iii) $(G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$ is a simple adjoint Shimura pair of B_{n+l} (resp. of $D_{n+l}^{\mathbb{R}}$) type;*
- *iv) we have an isomorphism $(G_3^{\mathrm{ad}}, X_3^{\mathrm{ad}}) \xrightarrow{\sim} (G(F_1), X(F_1))$, for some totally real number field F_1 containing F, and under it, the monomorphism $G_1^{\mathrm{ad}} = G \hookrightarrow G_3^{\mathrm{ad}}$ becomes nothing else but $j(F_1)_{\mathbb{Q}}^{(G,X)}$.*

B) We can choose $l \in \{1,2\}$ such that $\mathcal{T}_2$ is a DEG4 embedding.

C) If G is of D_4 Lie type, or of D_n Lie type, with $n \geq 5$, or of B_n Lie type, we can choose respectively F_1 such that $[F_1 : F]$ is a divisor of 12, or 4, or 1.

D) By enlarging F_1, we can assume as well that (G_2, X_2) is of non-compact type with $G_{2\mathbb{Z}_{(p)}}^{\mathrm{der}}$ as the extension to $F_{1(p)}$ of $Spin(2, 2n-1)_{\mathbb{Z}_{(p)}}$ (resp. of $Spin(2, 2n-2)_{\mathbb{Z}_{(p)}}$): if G is of D_4 Lie type, or of D_n Lie type, with $n \geq 5$, or of B_n Lie type, we can choose respectively F_1 such that $[F_1 : F]$ is a divisor of 48, or 16, or 2.

Proof: We need to choose $\tau(i_F)$ to be compact iff $G_{\mathbb{R}}^F$ obtained via i_F is compact. What we did above for m and $m+2$ can be easily extended to m and $m+2l$: of course the above torus T gets replaced by an l-dimensional torus. So let $F_{1(p)}$ be a totally real number field unramified over p, and such that there is an l-dimensional subtorus $\tilde{T}$ of $SO(2, m+2l)_{F_{1(p)}}$ such that its centralizer $C(\tilde{T})$ in $SO(2, m+2l)_{F_{1(p)}}$ has the property that: for any embedding $i_{F_1} : F_1 \to \mathbb{R}$, $C(\tilde{T})_{\mathbb{R}}^{\mathrm{der}}$ is either compact or (isomorphic to) $SO(2,m)_{\mathbb{R}}$, depending on $\tau(i_{F_1}|F)$ being compact or not. We take $m = 2n-1$ (resp. $m = 2n-2$), if (G,X) is of B_n (resp. of $D_n^{\mathbb{R}}$) type. As in [Va3, 3.0 G)] we can assume that $C(\tilde{T})_{F_1}^{\mathrm{ad}}$ and G^{F_1} are inner forms of each other, and that this inner form of G^{F_1} is trivial w.r.t. any embedding of F_1 into a real field or a p-adic field. Twisting the monomorphism $C(\tilde{T}) \hookrightarrow SO(2, m+2l)_{F_{1(p)}}$, we get a monomorphism

$$C(\tilde{T}') \hookrightarrow SO(2, m+2l)'_{F_{1(p)}}$$

between reductive groups over $F_{1(p)}$, where $C(\tilde{T}')^{\mathrm{ad}}_{F_1}$ is G^{F_1}; here $\tilde{T}'$ is an l-dimensional subtorus of $SO(2, m+2l)'_{F_{1(p)}}$. So the A) part follows from the second part of the unitary trick applied to the (uniquely determined) adjoint Shimura variety defined by the group

$$G_2^{\mathrm{ad}} := \mathrm{Res}_{F_1/\mathbb{Q}}(SO(2, m+2l)'_{F_1}{}^{\mathrm{ad}}).$$

The fact that we can choose $G_2^{\mathrm{ab}} = \mathbb{G}_m$, is implied as in [De1, 2.3.13], by the type of $(G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$, by the fact that all the factors of $G_{2\mathbb{R}}^{\mathrm{ad}}$ are non-compact, and by 2.3.8 i).

B) is a consequence of the proof of 2.3.7.2: we can choose $l \in \{1, 2\}$ in order the Killing form on $\mathrm{Lie}(C(T)^{\mathrm{der}})$ to be perfect.

C) is a standard consequence of the structure of the group of outer automorphisms in the mentioned cases, cf. [Va3, 3.0 J)]. Strictly speaking, A) above speaks about two inner twists; but of course, referring to the paragraph before 2.3.9.1, we can twist directly the natural monomorphism from the derived subgroup of the centralizer of T_{F_1} into $SO(B(F_{(p)})_{F_1}$.

D) is a consequence of [Ha, Hauptsatz and 5.5.3]. In other words, to lift an element of $H^1(\mathrm{Gal}(\bar{F}/F), G^F)$ to an element of $H^1(\mathrm{Gal}(\bar{F}/F), G^{F\mathrm{sc}})$, with $G^{F\mathrm{sc}}$ as the simply connected group cover of G^F, the first obstruction is encoded in an element of $H^2(\mathrm{Gal}(\bar{F}/F), \mu_2)$. Such an element corresponds to a quaternion F-algebra. Applying [Ha, 5.5.3] to the group of invertible elements of this algebra, it becomes trivial, by passage (in our case) to a totally real quadratic extension of F unramified above p. In the D_n case there will be a second such obstruction. So the numbers of C) have to be multiplied respectively by 4, 4, and 2. This ends the proof of the proposition.

2.3.9.2 Remark. There are many variants of the above proposition. For instance, we have a lot of freedom in choosing $G^{\mathrm{der}}_{2\mathbb{Z}_{(p)}}$. A first example: we can assume that F_1 is a Galois extension of $\mathbb{Q}$ or of F. A second example: let E be a totally imaginary extension of $\mathbb{Q}$ or of F unramified above p; we can assume that G^{ad} and G_2^{der} split over the composite field of E and F_1. Typical situation: we choose E to be a totally imaginary quadratic extension of $\mathbb{Q}$ in which p splits.

Moreover, from [Se] or [De1, p. 272] we deduce that for the D_4 Lie type we can divide the numbers of 2.3.9.1 C) and D) by 3 (i.e. we do not need to make distinction between D_4 and D_n with $n \geq 5$ Lie types.

2.3.10. Trick nine: the symplectic trick. This is just a variant of 2.3.8. Instead of a unitary Shimura pair (G_1, X_1) of some A_{2m-1} type, we can use a Shimura pair of C_{2m} type. This is obtained by twisting not the monomorphisms mentioned in i) to iv) of 2.3.8 h), but the composite of their (isogeny) lifts to monomorphisms into $\tilde{SU}(m, m)_{F_{1(p)}}$, with adequate monomorphisms defined by the images of the homomorphisms $a_{m F_{1(p)}}$ of the Case A of 2.2. Of course, in general we lose the standard PEL situation property of 2.3.10

c); but the situation is somehow closer to the "spirit" of $\mathbb{Z}_{(p)}$ embeddings into symplectic Shimura pairs.

2.3.11. Remarks. 1) It would be theoretically interesting to see other DEG theories, like $DEG3$, $DEG3+$, etc. For instance, [Va1, 4.3.10 a)] can be thought as an alternative to the above $DEG4$ theory. Such theories would be very useful in connection to [Va4, 4.2.1 2)].

2) It is worth reminding that 2.3.7.2 and 2.3.8-10 are much depending on Deligne's work [De1, 2.3.10], cf. also [Va1, 6.5-6].

3) Once we have 2.3.8 worked out with one F_1, it can be stated as well for any other totally real number field F_2 containing F_1 and unramified over p. The same applies to 2.3.9-10.

4) In 2.3.7.2 we can assume as well (cf. its proof) that we have a property similar to 2.3.8 j).

5) In 2.3.7.2, if $p \geq 5$, then $(f_1, L_{(p)}, v_1)$ is a SHS. This is the essence of [Va1, 6.5-6], via [Va1, 5.1 and 5.6.5]; the argument is entirely the same as the one in [Va1, 6.6.3]: we just have to replace good embeddings (terminology used in [Va1, 5.8.1]) by the terminology SHS. One of the first goals of 4.2 is to show that the same is true even for $p = 3$.

6) If in 2.3.7.2 $(f_2, L_{(p)}, v_2)$ is a SHS, then $(f_1, L_{(p)}, v_1)$ is also a SHS, cf. 2.3.5.5. The same thing remains valid in the contexts of 2.3.8-10.

7) As we will see in 4.2, in many instances the existence of a projector π_M as in 2.3.7.1 1), can be entirely ignored (for instance see S2 of 4.2 K) below). So the main point (i.e. the essence) in 2.3.7.1 1) is contained in the other two requirements.

8) The main philosophy of 2.3.9 is: from many points of view, in the study of orthogonal Shimura varieties, we can restrict to the non-compact ones.

9) We have $\tilde{e}_n = \tilde{b}_n \circ \tilde{j}_{2n-2}$.

10) There are many small things we could add to 2.3.1-10. Like estimates on $\dim_{\mathbb{Q}}(W)$ in 2.3.7.2 or in 2.3.8, like statements on the residue fields of $k(v_j)$ in terms of the residue field of the prime v of $E(G, X)$ divided by v_1 in 2.3.8, etc. All these things, are very simple, and so will be stated in the other parts of the paper, at the needed moments.

11) In 2.3.7-9 (and their applications) [De1, 2.3.10] is exploited to a great extend. See [Va7] for an intensive exploitation (through examples) of its variant [De1, 2.3.11]; in other words in [Va7] we deal also with the situations where we do have tensor products (of representations) involved (i.e., for instance, when 2.3.8 i) is no longer satisfied).

12) In 2.3.8, if G_1^{ad} is not of some A_{pk-1} Lie type, with $k \in \mathbb{N}$, then $(f_1, GSp(L_{(p)}, \psi), v_1)$ is a $DEG4$ embedding (cf. the proof of 2.3.7.2).

13) There is a difference from 2.3.7.2 and [Va1, 6.6.2] for the C_n and $D_n^{\mathbb{H}}$ type cases. To explain it we restrict to the C_n type. In loc. cit., in order to get its d) part for the C_n types, we used forms of the following type of embeddings (let's say over $\mathbb{C}$): a product of a split torus of rank n times a split simply connected group of C_n can be embedded naturally in a split simply connected group of C_{2n} Lie type.

14) It is easy to see, starting from 2.3.7.2 or 2.3.8, that [Va2, 2.3.5.1] has versions for simple adjoint Shimura pairs of C_n or $D_n^{\mathbb{H}}$ type, defined by groups whose extensions to $\mathbb{R}$ have no compact factors. This can be read out automatically from [Va1, Case 1 of 6.6.5].

2.4. Three other recent important moments. For the proof of the existence of integral canonical models of Shimura varieties of preabelian type, the above tricks of 2.3 are not entirely enough. They have to be supplemented by (motivic) statements on p-divisible groups. There are three such important ideas, which emerged recently (more or less at the same type: end of 1993, 1994, beginning of 1995). We start with a more general setting, and then we present these three important ideas.

Let p be a prime, and let V be a complete DVR of mixed characteristic $(0,p)$, and of perfect residue field k. Let $V_0 := W(k)$. Let $K := V\left[\frac{1}{p}\right]$ and $K_0 := V_0\left[\frac{1}{p}\right]$. Let $e := [V : V_0]$. Let $\Gamma := \mathrm{Gal}(\bar{K}/K)$. Let H be a free $\mathbb{Z}_p$-module of finite rank. Let

$$\rho\colon \Gamma \to GL(H)(\mathbb{Z}_p)$$

be a crystalline representation. We assume that it factors through the $\mathbb{Z}_p$-points of an algebraic subgroup G of $GL(H)$, and that there is a family $(v_\alpha)_{\alpha\in\mathcal{J}}$ of homogeneous tensors of $\mathcal{T}(H)$ such that $G_{\mathbb{Q}_p}$ is the subgroup of $GL(H^1\left[\frac{1}{p}\right])$ fixing its members. If $G_{\mathbb{Q}_p}$ is reductive, then such a family of tensors automatically exists, cf. [De2, 3.1]. Till the end of 2.4.1 we assume $p \geq 3$.

Let $B^+(V)$ be the Fontaine's ring of V as defined in [Fa3]. Let Re be the piano ring of k of convergent resonance e (see [Va3, 2.1]). Let π_V be a uniformizer of V. Using it one defines naturally (see loc. cit.) a V_0-monomorphism $Re \hookrightarrow B^+(V)$, and a V_0-epimorphism $Re \twoheadrightarrow V$. We assume the existence of a filtered F-crystal

$$\mathcal{C} := (M, (F^i)_{i\in I}, \Phi_M, \nabla_M)$$

over Re, with I a finite set, and with M a free Re-module of finite rank such that we have an isomorphism

$$\mathcal{F}\colon M \otimes_{Re} B^+(V)\left[\frac{1}{p}\right] \tilde{\to} H \otimes_{\mathbb{Z}_p} B^+(V)\left[\frac{1}{p}\right]$$

of $B^+(V)\left[\frac{1}{p}\right]$-modules, respecting the Frobenius, the Galois actions, and the filtrations. See [Fa3] for such situations, for the natural filtrations of Re and of $B^+(V)$, and for the natural Γ-action on $M \otimes_{Re} B^+(V)$. In all that follows we assume that $\mathcal{F}$ is obtained as in [Fa3, §4] (so I is included into the translate by an integer of the set $\{0, ..., p-2\}$). If $e = 1$, the "complication" with Re is not needed, as one can use directly Fontaine's theory (cf. [FI]).

Let $w_\alpha \in F^0(\mathcal{T}(M\left[\frac{1}{p}\right]))$ correspond to v_α via $\mathcal{F}$; we have $\nabla_M(w_\alpha) = 0$, as one can check easily. Three fundamental questions can be posed:

Q1. Let $\alpha \in \mathcal{J}$. When $w_\alpha \in \mathcal{T}(M)$?

Q2. We assume that G is a reductive group. When the Zariski closure $\tilde{G}_{Re}$ of the subgroup of $GL(M\left[\frac{1}{p}\right])$ fixing w_α, $\forall\alpha \in \mathcal{J}$, in $GL(M)$, is a reductive subgroup of $GL(M)$?

Q3. When there is an isomorphism

$$\rho_M\colon H \otimes_{\mathbb{Z}_p} Re \tilde{\to} M$$

taking v_α into w_α, $\forall \alpha \in \mathcal{J}$?

In what follows we are mostly concerned with these three questions in the simplified context of (Shimura) p-divisible groups. Occasionally, a wider setting is also presented.

2.4.1. M1: Faltings's work and some of its immediate developments. In [Fa3] a general (but restricted) answer to Q1 is given. We recall the answer, in a much weaker form very closely related to [Fa3, cor. 9]. We assume that ρ and (M, Φ_M, ∇_M) are attached to a p-divisible group D_V over V. So $I = \{1\}$, and we can take F^1 to be a direct summand of M such that under the natural identification $H^1_{dR}(D_V/V) = M \otimes_{Re} V$ (see [Fa3, §6]) it becomes the natural Hodge filtration $F^1(M \otimes_{Re} V)$. We have (cf. [Fa3, cor. 9]):

Theorem 1 (Faltings). *If $v_\alpha \in H^{\otimes a} \otimes (H^*)^{\otimes b}$, with a and b non-negative integers such that $a, b \leq p-2$, then $w_\alpha \in \mathcal{T}(M)$. Moreover w_α is divisible by p iff v_α is.*

In loc. cit. the result is stated for $a = b$; but the same argument applies for the case $a, b \leq p-2$; in seems to us that would be enough to assume just that $a+b \leq 2(p-2)$. Using the above theorem, in [Va1, 5.2] we have shown that in many situations the answer to Q2 is yes, i.e. $\tilde{G}_{Re}$ is reductive. We recall one such form of our work, i.e. we recall what is the abstract form of loc. cit. when supplemented by [Va4, V5 of 3.2], in the context of [Va1, 5.7.1]. We have:

Theorem 2. *We assume that:*

i) $p \geq 5$ and that G is a reductive group;

ii) there is a cocharacter $\mu_K : \mathbb{G}_m \to \tilde{G}_K$ producing a direct sum decomposition $M \otimes_{Re} K = F^1(M \otimes_{Re} V) \otimes K \oplus F^0(M \otimes_{Re} V) \otimes K$, with $\beta \in \mathbb{G}_m(K)$ acting through μ_K on $F^i(M \otimes_{Re} V) \otimes K$ as the multiplication with β^{-i};

iii) the Killing form and the trace form (obtained via H) on Lie(G^{dernc}) are both perfect (here G^{dernc} refers to the subgroup of G^{der}, naturally isogeneous to the maximal non-compact factor G^{adnc} of G^{ad}; such factor is definable as in [Va3, 5.1.3] starting from the assumption of ii)).

Then in the context of Theorem 1, $\tilde{G}_{Re}$ is a reductive group over Re, and moreover, if $k = \bar{k}$, the answer to Q3 is yes.

Proof: One just needs to combine [Va1, 5.2 and 5.7.1] (cf. 2.3.7.3) and [Va4, V5 of 3.2], once we remark the following thing. We can assume that the image of μ_K is not contained in the center of $GL(M \otimes_{Re} K)$; so as in [Va3, 5.1.3] we can define the compact simple factors of G^{ad}. Assuming that $k = \bar{k}$, we can eliminate them (i.e. we can pass to a reductive normal subgroup of G). This elimination is supported by the fact that we will show that the answer to Q3 is yes (of course there are many other ways to argue it; for instance using [Va1, 3.1.6]). So we can assume (the argument is the same as the one of [Va1, 5.7.1] based on minimal weights and the assumption of ii)) that the number $s(\mathrm{Lie}(G^{\mathrm{der}}_{\mathbb{Q}_p}), H[\frac{1}{p}])$ is 2. As in [Va1, 5.2 and 5.7.1] we get that $\tilde{G}_{Re}$ is a reductive group.

As in [Va1, 5.3.1] we get that we can assume that μ_K lifts to a cocharacter $\mu_V : \mathbb{G}_m \to \tilde{G}_V$. Using the same argument of [Va1, 5.2.1.1] we get that $\tilde{G}_{Re}$ is a split group having a maximal split torus T_{Re}. We can assume that μ_V mod p factors through $T_{Re/pRe}$; so we can assume that μ_V mod p lifts to a cocharacter $\mu_{Re}:\mathbb{G}_m\to\tilde{G}_{Re}$. Let F^1 be the maximal direct summand of M on which it acts non-trivially. To the quadruple (M,F^1,Φ_M,∇_M) it corresponds a p-divisible group D_{Re} over Re (see [BBM]), lifting $D_{V/pV}$. So we can apply [Va4, V5 of 3.2] (cf. also the proof of 3.5.1 below) to the resulting Shimura p-divisible group $(D_{Re},(w_\alpha)_{\alpha\in\mathcal{J}})$ over Re. This proves the theorem.

There are much more general variants of Theorem 2, in which we do no get restricted to the case of p-divisible groups, or where we work in the general context of an arbitrary ($\mathbb{Z}_p$- or $W(\mathbb{F})$- or $W(k)$-) well positioned family of tensors. We state here explicitly just one very simple variant, which we think (due to its simplicity) can be useful in many other situations. We assume that $I\subset\{0,1,...,a\}$, with $a\in\mathbb{N}$, $a\le p-2$, and that $\mathcal{C}$ is a p-divisible object of $\mathcal{FM}^{\nabla}_{[0,a]}(Re)$ in the sense of [Va2, 2.1] (in [Fa3] the terminology "object of $\mathcal{MF}_{[0,p-2]}(V)$" is used). We have:

Theorem 3. *We assume $2a\le p-2$ and that G is a reductive group. We also assume that the faithful representation $G\hookrightarrow GL(H)$ is a DEG4 representation. Then in the context of $\mathcal{F}$, $\tilde{G}_{Re}$ is a reductive group.*

Proof: We can assume that $k=\bar{k}$. From 2.3.7.3 we deduce the existence of a family $\mathcal{F}_1=(v_\alpha)_{\alpha\in\mathcal{J}_1}$ of homogeneous tensors of $\mathcal{T}(H\otimes_{\mathbb{Z}_p}W(k))$ of degrees 2 and 4, and which is strongly $W(k)$-very well positioned for $G_{B(k)}$. These tensors are $W(k)$-linear combinations of homogeneous tensors of $\mathcal{T}(H)$ fixed by G. For the rest of the proof, we just need to copy entirely the proof of [Va1, 5.2.12], once we remark that [Fa3, iii) of th. 5] allows us (cf. the inequality $2a\le p-2$) to assume that $w_\alpha\in\mathcal{T}(M)$, $\forall\alpha\in\mathcal{J}_1$. This ends the proof.

2.4.1.1. Remark. It is possible to estimate the number $s(\mathrm{Lie}(G^{\mathrm{der}}_{\mathbb{Q}_p}),H[\frac{1}{p}])$ in terms of a. For instance, this is very easy, as $\mathcal{C}$ satisfies an W-condition similar to the one introduced in [Va2, 3.4.1.5], cf. [Wi]. As the faithful representation $G^{\mathrm{der}}\hookrightarrow GL(H)$ can involve tensors products, these estimates can be quite weak. Moreover, if instead of a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,a]}(Re)$, we consider the corresponding one of $\mathcal{MF}^{\nabla}_{[b,a+b]}(Re)$ for some $b\in\mathbb{Z}$, then these estimates can differ considerable. So we prefer not to present an analogue of [Va1, 5.8.4-6] in this situation.

2.4.1.2. Remark. The condition ii) of Theorem 2 is automatically satisfied, cf. [Va3, 5.5.1] and [Va4, V2 of 3.2].

2.4.2. M2: de Jong's work. We restrict to what we need. Let k be a perfect field of characteristic $p\ge 2$. Let $R:=W(k)[[T]]$, with T an independent variable. Let $S:=\mathrm{Spec}(R/pR)$, and let $K_S:=\mathrm{Spec}(k((T)))$. A simpler form of [dJ1, 1.1] is:

Theorem 4 (de Jong). *The natural functor from non-degenerate F-crystals over S to non-degenerate F-crystals over K_S is fully faithful.*

For the notion non-degenerate crystal [dJ1] refers to [Saa, 3.1.1 of p. 331]. For us the only important thing is: this notion is stable under tensor products, and any p-divisible object of some Fontaine's category $\mathcal{MF}^{\nabla}_{[0,a]}(\widetilde{R})$ becomes (after forgetting the filtrations) a non-degenerate F-crystal over $\widetilde{R}/p\widetilde{R}$, for any $a\in\mathbb{N}$ and any $\widetilde{R}$ an affine regular formally smooth $W(k)$-scheme. The above Theorem 4 is very much related to Q1 and Q2. Here we state one such application of it as suited (cf. 3.4.1 a)) for 4.2. Let Y be a regular formally smooth $W(k)$-scheme, and let $\mathcal{D} := (D, (v_\alpha)_{\alpha\in\mathcal{J}})$ be a Shimura p-divisible group over Y. Let Y_k be the special fibre of Y. We can assume that it is connected (and so integral). Let Z_k be an integral separated k-scheme containing Y_k as an open subscheme and such that D extends to a p-divisible group $\tilde{D}_k$ over Z_k. Let $y\colon \mathrm{Spec}(k) \to Z_k$ be an arbitrary morphism. Let

$$m\colon \mathrm{Spec}(S) \to Z_k$$

be a morphism lifting y and such that its restriction m_{K_S} to K_S factors through Y_k. Let (M, ϕ) be the σ-crystal over k defined by the restriction of $\tilde{D}_k$ to the special fibre of m, and let

$$\mathcal{C}(R) := (M_R, \phi_R, \nabla_R, (w_\alpha)_{\alpha\in\mathcal{J}})$$

be the F-crystal with cycles over R/pR defined by $m^*(\tilde{D}_k)$ and $m^*_{K_S}(\mathcal{D})$. For each $\alpha \in \mathcal{J}$, a multiple of v_α by a non-negative power of p, when restricted to K_S (via m_{K_S}), can be viewed as a morphism between non-degenerate F-crystals over K_S. From Theorem 4 we deduce that this multiple extends to a morphism between non-degenerate F-crystals over S. We deduce:

Fact 1. *$\forall \alpha \in \mathcal{J}$, to v_α it is naturally associated (via m) an element $w^m_\alpha \in \mathcal{T}(M[\frac{1}{p}])$ fixed by ϕ.*

Fact 2. *Let $\alpha \in \mathcal{J}$. We assume the existence of a morphism $m_1 : Spec(W(\bar{k})) \to Y$ such that, the corresponding crystalline (de Rham) cycle $m_1^*(v_\alpha)$ is integral. Then $w_\alpha \in \mathcal{T}(M_R)$.*

Proof: As Y_k is connected, from [Fa3, rm. iii) after th. 10] we deduce easily (to be compared also with [Va2, 2.3.13.1]) that $m_2^*(v_\alpha)$ is integral for any morphism $\mathrm{Spec}(W(k_1)) \to Y$, with k_1 an algebraically closed field containing k; of course if $p \geq 3$ we can quote as well the variant of [Va4, 3.1] recalled in Theorem 5 below. So Theorem 4 applies directly.

We ignore here the problem of the independence of w^m_α of m, as in the cases needed in 4.2, this is obvious. In other words, if Z_k is formally smooth over k or a pro-étale cover of a k-scheme of finite type, using alterations (for the pro-étale situation) (cf. [dJ2]), we get immediately this independence.

2.4.3. M3: Milne's conjecture. We come back to Q1-3. Obviously, if an isomorphism ρ_M exists, then $w_\alpha \in \mathcal{T}(M)$ iff $v_\alpha \in \mathcal{T}(H)$. We refer to the problem of the existence of ρ_M as one variant of Milne's conjecture (cf. [Va4, beginning of §3] for the first moments of the unfolding of this conjecture). See [Va4, V5 of 3.2] for a general result (in the context of p-divisible groups) pertaining directly to Q3. Here, in two theorems, we just recall the main result [Va4, 3.1] and one of its variants, in the unramified setting. So, to conform with the needs of §4, we assume that $k = \bar{k}$ and that $V = V_0$. Let $(M, F^1, \phi, G, (t_\alpha)_{\alpha\in\mathcal{J}})$ be a Shimura filtered σ-crystal over k. As in [Va4, 2.4] we define (via Fontaine's comparison theory) its étale counterpart $(H^1, G_{\mathbb{Z}_p}, (v_\alpha)_{\alpha\in\mathcal{J}})$. We have (cf. [Va4, 3.1 and resp. V3 of 3.2]):

Theorem 5. *We assume $p \geq 3$. Then there is an isomorphism*

$$H^1 \otimes_{\mathbb{Z}_p} W(k) \tilde{\to} M$$

taking v_α to t_α, $\forall \alpha \in \mathcal{J}$.

Theorem 6. *Let $(f, L_{(p)}, v)$ be a SHS. Then for any morphism $Spec(W(k)) \to \mathcal{N}$, if $(M, F^1, \phi, G_{W(k)}, (t_\alpha)_{\alpha\in\mathcal{J}}, p_M)$ is the principally quasi-polarized Shimura filtered σ-crystal attached to it (see [Va2, 2.3.10] and [Va4, 4.4 2)]), there is an isomorphism*

$$L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(k) \tilde{\to} M$$

taking v_α into t_α, $\forall \alpha \in \mathcal{J}'$, and taking $\tilde{\psi}$ into p_M.

2.4.3.1. Remark. It is worth pointing out that in the proof of Theorems 5-6, we used the proof of Theorem 1 in the simplest case when G is a subtorus of $GL(H)$, and when $e = 1$. Moreover in many situations the condition $p \geq 3$ of Theorem 5 is not needed (see [Va4, V6] and [Va8]).

2.5. A variant of [Va1, 3.1.6]. Let $\mathcal{S}$ be an integral scheme, and let Spec($\mathcal{K}$) be its generic point. Let $\mathcal{F}$ be a locally free sheaf on $\mathcal{S}$ of finite rank. Let $\mathcal{G}$ be the $\mathcal{S}$-group scheme defined by (linear) automorphisms of $\mathcal{F}$. So $\mathcal{G} = GL(\mathcal{F})$. Let $n \in \mathbb{N}$, and let G_i be a reductive subgroup of the generic fibre $\mathcal{G}_{\mathcal{K}}$ of $\mathcal{G}$, $i = \overline{1, n}$. We assume that:

a) G_i's are commuting with each other;
b) the Zariski closure of G_i in $\mathcal{G}$ is a reductive subgroup.

The first assumption implies that the subgroup G_0 of $\mathcal{G}_{\mathcal{G}}$ generated by G_i's is a reductive group. We have the following extension of [Va1 3.1.6 and 3.1.6.1]:

2.5.1. Lemma. *The Zariski closure of G_0 in $\mathcal{G}$ is a reductive subgroup.*

Proof: We can assume that $\mathcal{S}$ is a local noetherian scheme. We can also assume that

$$\mathrm{Lie}(G_0) = \sum_{i=1}^{n} \mathrm{Lie}(G_i)$$

(otherwise we have to decompose, up to isogeny, the groups involved into products involving at most one reductive group which is not a torus). Using induction, we can assume that $n = 2$. The rest is entirely the same as in loc. cit. (i.e. it is irrelevant if G_1 is a torus or not). This proves the lemma.

2.5.2. Remark. Of course, above we can replace $GL(\mathcal{F})$ by any reductive $\mathcal{S}$-group $\mathcal{H}$. Argument: locally in the étale topology of $\mathcal{S}$, $\mathcal{H}$ has representations in such GL's $\mathcal{S}$-groups; moreover, as pointed put in the proof of loc. cit., the fact that by doing these localizations we might lose the integrality part, it is entirely irrelevant.

§3. Shimura envelopes

In this chapter we start an abstract general theory of Shimura envelopes, generalizing the theory of integral canonical models of Shimura varieties of Hodge (even preabelian) type. It consists mostly of definitions, examples, remarks and questions. Let p be an arbitrary prime, and let O be a DVR faithfully flat over $\mathbb{Z}_{(p)}$. For simplicity we assume in 3.1-4 that the morphism $\mathrm{Spec}(O) \to \mathrm{Spec}(\mathbb{Z}_{(p)})$ is of index of ramification one; one reason for this choice is: in 3.5.4 1) we motivate why the passage to the ramified situation is automatic.

In particular, this assumption implies that the residue field k of O is perfect. Let $L := O[\frac{1}{p}]$. Let Y be a regular formally smooth separated O-scheme. For simplicity and to have a reacher context, we also assume that we have the following Zariski density properties:

– the $W(\bar{k})$-valued points of Y are Zariski dense in Y, and that the $\bar{k}$-valued points of Y_k are Zariski dense in Y_k.

Let

$$\mathcal{D} := (D, (v_\alpha)_{\alpha \in \mathcal{J}})$$

be a Shimura p-divisible group over Y, cf. [Va2, 2.2.20].

3.0. Convention. In the whole of §3 we drop the part "not necessarily quasi-split" whenever we speak about Shimura (filtered) σ-crystals which are not necessarily quasi-split (see [Va2, 2.2.9 7)]).

3.1. Definitions. 1) We say that $\mathcal{D}$ is compact (or proper) if Y is a proper scheme over O.

2) We say that $\mathcal{D}$ is a Shimura quasi-envelope, if $\mathcal{D}$ is versal (in the sense of [Va2, 3.4.19 B]) at each $\bar{k}$-valued point of Y.

3) We say that $\mathcal{D}$ is a saturated Shimura envelope (resp. a Shimura envelope) if it is a Shimura quasi-envelope, and if any normal, locally noetherian, flat (resp. regular formally smooth), separated O-scheme Z containing Y as an open subscheme and with Y_L Zariski dense in Z_L (resp. and with $Z_L = Y_L$), and such that D extends to a p-divisible group over Z, is automatically Y itself.

4) By an automorphism of Y w.r.t. $\mathcal{D}$ we mean an O-automorphism f of Y such that $f^*(\mathcal{D})$ is isomorphic to $\mathcal{D}$. We denote by $\mathrm{Aut}(Y; \mathcal{D})$ the group of such automorphisms.

5) We say that $\mathcal{D}$ has the ST property (here ST stands for Serre–Tate), if there is a Zariski dense open subset U_k of Y_k such that the Newton polygons associated to $\bar{k}$-valued points of U_k and to D, are all the same. We always choose U_k to be maximal with this property.

6) Let $H \subset \mathrm{Aut}(Y; \mathcal{D})$ be a subgroup. We say that it is transitive if it permutes transitively the connected components of Y_L. We assume now that $\mathcal{D}$ has the ST property. We say that $\mathcal{D}$ has the C property w.r.t. H (here C stands for Chai), if the H-orbit of any $\bar{k}$ valued point of U_k is Zariski dense in Y_k. If $H = \mathrm{Aut}(Y; \mathcal{D})$, then we drop the expression w.r.t. H.

7) We say that $\mathcal{D}$ has the completion property, if for any algebraically closed field k_1 containing k, every $W(k_1)$-valued point z_1 of $Y_{W(k_1)}$ lifting a k_1-valued point of Y_{k_1} has the following property: denoting by $(M_1, F_1^1, \phi_1, G_1)$ the Shimura filtered σ-crystal associated to z_1 and $\mathcal{D}$ (as in [Va2, 2.3.10]), for any $g \in G_1(W(k_1))$, the Shimura filtered σ-crystal $(M_1, F_1^1, g\phi_1, G_1)$ is associated to a $W(k_1)$-valued point z_{1g} of $Y_{W(k_1)}$ lifting a k_1-valued point of the same connected component of Y_{k_1} to which y_1 belongs.

8) We say that $\mathcal{D}$ has the I property (here I stands for isogeny) if for any Shimura filtered σ-crystal $\mathcal{C}_1 := (M_1, F_1^1, \phi_1, G_1)$ associated to a $W(k_1)$-valued point z_1 of Y (with k_1 arbitrary as in 7) above), any representative $\mathcal{C}_2$ of an element of the isogeny class of $\mathcal{C}_1$ (see [Va2, 2.2.22 2)]), is isomorphic to the Shimura filtered σ-crystal associated to a $W(k_1)$-valued point of Y and to $\mathcal{D}$.

9) We say that $\mathcal{D}$ has the Z property (here Z stands for T. Zink), if it has the I property, and if for any $\mathcal{C}_1$ as in 8) above, the number $iso-SCM-d(\mathcal{C}_1)$ (defined in [Va2, 3.13.3 5-6)]) is finite.

10) We say that $\mathcal{D}$ is a stable Shimura envelope, if we have:

i) for any morphism $m\colon \mathrm{Spec}(O_1) \to \mathrm{Spec}(O)$ with O_1 a DVR, which is of index of ramification one, $\mathcal{D}_{O_1}$ is a Shimura envelope (over Y_{O_1});

ii) for any pro-étale cover $Y_1 \to Y$, $\mathcal{D}_{Y_1}$ is a Shimura envelope (over Y_1).

11) Similarly to 10) we define stable saturated Shimura envelopes.

12) We define 1) to 10) above in the principally quasi-polarized context as well, following the pattern of [Va2, 2.2.21 4)]. The only extra thing we need to add: in 7) above g has to fix (not only normalize) the resulting principal quasi-polarization of (M_1, ϕ_1). So we speak about a p.q-p. Shimura envelope, etc. Here p.q-p. stands for principally quasi-polarized.

13) We assume that $\mathcal{D}$ is a Shimura quasi-envelope having the ST property. Following the pattern of [Va2], the points of U_k with values in perfect fields will be called Shimura-ordinary (cf. [Va2, 3.12.1] and [Fa3, rm. ii) after th. 10]). From [Va2, 2.3.17.2 and 3.1] (resp. [Va2, 2.3.18, 3.14 and 4.14.3] in the case $p = 2$) we deduce that each such Shimura-ordinary point y_1 with values in a perfect field k_1, has a unique lift (resp. has a finite number of lifts), called the (resp. called) Shimura-canonical lift (resp. lifts) of y_1, $z_1\colon \mathrm{Spec}(W(k_1)) \to Y$, such that the Shimura filtered Lie σ-crystal attached to it is of parabolic type in the sense of [Va2, 2.2.12 a)].

14) We assume that Y is locally of finite type over O and that k is a subfield of $\mathbb{F}$. We assume that $\mathcal{D}$ is a Shimura quasi-envelope having the ST property. We say that $\mathcal{D}$ has the ACM property (here ACM stands for the algebraization of the quasi CM type of some Shimura-canonical lifts; see def. [Va2, 2.3.17]), if for any Shimura-canonical lift $z\colon \mathrm{Spec}(W(\mathbb{F})) \to Y$ lifting a Shimura-ordinary point $\mathrm{Spec}(\mathbb{F}) \to U_k$, factors through the normalization of O in a finite field extension of L.

15) We say that $\mathcal{D}_k$ is p-consistent (or has the p-consistency property), if for any normal locally noetherian separated k-scheme Z_k containing Y_k as a dense subscheme, and such that D_k extends to a p-divisible group $\tilde{D}_k$ over Z_k, every morphism $y\colon \mathrm{Spec}(k_1) \to Z_k$, with k_1 a perfect field, has the property:

– the σ-crystal with cycles attached to the triple $(y, \mathcal{D}, \tilde{D})$ (cf. Fact 1 of 2.4.2) is a Shimura σ-crystal.

Similarly, we say that $\mathcal{D}$ is consistent (or has the consistency property), if for any complete DVR V of perfect residue field and faithfully flat over $\mathbb{Z}_{(p)}$, for every morphism $m_V : \mathrm{Spec}(V[\frac{1}{p}]) \to Y$ such that $m_V^*(D)$ extends to a p-divisible group D_V over V, the pair $(D_V, m_V^*((v_\alpha)_{\alpha\in\mathcal{J}}))$ (it is easy to see that the mentioned pull back of cycles makes sense) is a Shimura p-divisible group over V.

16) We consider a point z_1 as in 8), but we assume that k_1 is just perfect. As in [Va4, 2.1.4] we define the étale counterpart

$$\mathcal{E}_1 = (H^1_1, G_{1\mathbb{Z}_p}, (w_\alpha)_{\alpha \in \mathcal{J}})$$

of $\mathcal{C}_1$. We say that $\mathcal{D}$ is of constant étale type, if $\mathcal{E}_1$ does not depend on k_1 or on z_1. Similarly we speak in a p.q-p. context. We say that $\mathcal{D}$ is of constant filtration type, if it is of constant étale type, and if for any algebraically closed field k_1 and point $z_1 \in Y(W(k_1))$, the $G(W(k_1))$-conjugacy class of the cocharacter $\mu : \mathbb{G}_m \to G_{1W(k_1)}$ defined (following the pattern of [Va1, 5.3.1]) via Fontaine's comparison theory (for $p \geq 3$ also via Theorem 5 of 2.4.2) and the filtration class (see [Va2, 2.2.9 3)]) of $\mathcal{C}_1$, does not depend on the choice of z_1.

17) Whatever property a Shimura quasi-envelope $\mathcal{D}$ has, we refer to Y as the support of $\mathcal{D}$. Expressions like Y supports a Shimura envelope having some property, will prevail.

18) By a Shimura FTQHT-variety over O (here FTQHT stands for finite type plus quasi-Hodge type), we mean a separated O-scheme Y of finite type which:

i) is the support of a stable saturated Shimura envelope $\mathcal{D}$;

ii) has a pro-étale cover $Y_1 \to Y$, such that the $\mathrm{Aut}(Y_1; \mathcal{D}_{Y_1})$-obit of any point of Y_{1L} with values in an algebraically closed field L_1 containing L, is Zariski dense in Y_{1L_1}.

To Y_L we refer as a Shimura FTQHT-variety over L.

3.1.1. Example. Let $(f, L_{(p)}, v)$ be a SHS. We assume that all the Hodge cycles of $\mathcal{A}$ are also Hodge cycles of $\mathcal{A}_{H_0}$. Let $O := O_{(v)}$ and let $Y := \mathcal{N}/H_0$. Let

$$\mathcal{D}_{H_0} := (D, (v_\alpha)_{\alpha \in \mathcal{J}'}, p_D)$$

be the principally quasi-polarized Shimura p-divisible group over Y defined by the principally quasi-polarized p-divisible group of $(\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ and the family of de Rham components of the family $(w_\alpha)_{\alpha \in \mathcal{J}'}$ of Hodge cycles with which $\mathcal{A}_{H_0}$ is naturally (cf. the extra assumption) endowed.

Obviously $\mathcal{D}_{H_0}$ is a stable saturated Shimura envelope. It has property ST, cf. [Va2, 4.2.1]. It is compact iff (G, X) is of compact type (cf. [Va1, 6.4.1.1 2) and 6.4.11] and [Va2, 2.3.3.1]). It is expected that the pull back $\mathcal{D}$ of $\mathcal{D}_{H_0}$ to $\mathcal{N}$ has the C property (cf. [Va2, 4.10.1]). In the third part we will see that it has the ACM property (cf. also 6.2.3.3), the I property, and the completion property, and so (cf. [Va2, 3.13.3.1 3)]) we do expect that it has the Z property.

We have: $G(\mathbb{A}_f^p)$ is naturally a subgroup of $\mathrm{Aut}(\mathcal{N}; \mathcal{D})$. It is transitive, cf. [Va1, 3.3.2]. It can happen that $\mathrm{Aut}(\mathcal{N}; \mathcal{D})$ is much bigger than $G(\mathbb{A}_f^p)$ (see [Va1, 3.2.7 9)]). For instance this is the case if $(G^{\mathrm{ad}}, X^{\mathrm{ad}}, H^{\mathrm{ad}}, v^{\mathrm{ad}})$ has two isomorphic simple factors.

We can work out the above example using Shimura triples, as in [Va1, 3.2.6]. Warning: the principally quasi-polarized p-divisible group we would get over $\mathrm{Sh}_p(G, X)$ is not necessarily of constant étale type: for instance this is the case if $k(v) = \mathbb{F}_p$, and there is another prime v_1 of $E(G, X)$ dividing p and with $k(v_1)$ having at least p^2 elements, cf. [Va2, P1 of 4.6].

$\mathcal{N}/H_0$ is a Shimura FTQHT-variety over $O_{(v)}$.

3.1.2. Theorem. *We assume $p \geq 3$. We assume that $\mathcal{D}$ is a Shimura quasi-envelope of constant filtration class. We also assume that the connected components of Y_L are permuted transitively by $Aut(Y;\mathcal{D})$. Then $\mathcal{D}$ has the ST property.*

Proof: We can assume that Y_L is connected. For each $W(\bar{k})$-valued point z of Y, we can define as in [Va4, 2.1.4] the étale counterpart $\mathcal{E}$ of (the Shimura filtered σ-crystal $\mathcal{C}_z$ associated to) $z^*(\mathcal{D})$. The fact that Y_L is connected implies that $\mathcal{E}$ does not depend on z. So from Theorem 5 of 2.4.3 and the fact that $\mathcal{D}$ is of constant filtration class, we deduce that the G-ordinary type produced by $\mathcal{C}_z$ (cf. terminology of [Va2, 3.1.1]) is independent of z. So the theorem follows from [Va2, 3.12.1] and [Fa3, rm. ii) after th. 10] (or 3.2 8) below).

3.2. Remarks. 1) We do not know if (or when) a (saturated) Shimura envelope is automatically stable.

2) The most important problem of the theory of Shimura envelopes is, in our opinion, the classification of stable saturated Shimura envelopes. One should start classifying first the compact ones.

3) A compact Shimura quasi-envelope is automatically a stable saturated Shimura envelope.

4) We will see in [Va9] and part 2D that one can abstractly construct Shimura envelopes starting from [Va2, 3.4].

5) We can weaken 3.1 1), by either just requiring that the Kodaira-Spencer map is just injective in any $\bar{k}$-valued point of Y (i.e. we have a quasi-versal deformation in the sense of [Va2, 3.4.19 B]), or by allowing it to have an image in each such point which is the same as the image of a versal situation (cf. [Va2, 3.4.14 and 3.4.19 v)]). We will not persuade this idea here.

6) The convention of [Va2, 2.2.21 7)] will apply as well in the context of Shimura (quasi-) envelopes.

7) We view 3.1.2 as the abstract form of [Va2, 4.2.1], as it represents a second (new) proof of loc. cit. It is arguable which one of the two is an easier proof; however we feel that the proof of 3.1.2 is more of a principal. In a similar manner we expect that most of the properties of the stable saturated Shimura envelopes introduced in 3.1.1 extend to all the stable saturated Shimura envelopes.

8) We assume that $\mathcal{D}$ is a Shimura quasi-envelope. Then the filtered F-crystal with cycles defined by $\mathcal{D}$ over the completion of a point $z_1\colon \mathrm{Spec}(W(k_1)) \to Y_{W(k_1)}$ (with k_1 as in 3.1 13)), is isomorphic to a Shimura filtered F-crystal defined (see [Va2, 2.2.10]) by a sextuple $(M_1^1, F_1^1, \phi_1, G_1, H_1, \tilde{f}_1)$, where $(M_1^1, F_1^1, \phi_1, G_1)$ has a similar meaning as in 3.1 7), and H_1 is a smooth subgroup of G_1 such that the morphism $H_1 \to G_1/P_1$, with P_1 as the parabolic subgroup of G_1 leaving F_1^1 invariant, is étale in the origin. All these are in conformity with the proof of [Va2, 3.12.1].

9) Our justification for the choice of the word "envelope" is based on how the Steps of 4.2 unfold, while the expressions like "Shimura envelope" are motivated by 3.1.1.

10) The notion Shimura FTQHT-variety is stable under morphisms $\mathrm{Spec}(O_1) \to \mathrm{Spec}(O)$ of index of ramification one. In [Va5] this terminology will be extended (in a

more canonical way; see also 3.4.2 3)) to the one of Shimura FTQPT-varieties (here QPT stands for quasi-preabelian type).

11) In [Va2, 2.3.18, 3.14 and 4.14.3] it is shown that 3.1.1 holds as well for a $p=2$ SHS. Moreover, using [Va4, V6 of 3.2] we deduce that in many situations 3.1.2 holds as well for $p=2$; in fact using the proof of [Va2, 4.2.1] it can be shown that in 3.1.2 the condition $p\geq 3$ is not needed at all.

3.3. Problems and questions. 1) We assume that L is a number field. Let $\mathcal{D}$ be a stable saturated Shimura envelope over a quasi-projective smooth O-scheme Y.

Q4. Is it true that an étale cover of Y has geometric fibres of general type?

Q5. Is it true (at least in the case when Y is proper over O) that for any number field E containing L, the number of E-valued points of Y_L is finite? This question is closely related to [Fa2].

Q6. When can Y be embedded as a locally closed subscheme of a finite type integral canonical model of some Shimura variety of preabelian type (see def. of [Va6, 3.1.2 2)])?

Q7. Is Y an integral S-canonical model in the sense of [Va6, 3.1.1 b)]?

2) We come back to the general situation of a stable saturated Shimura envelope $\mathcal{D}$ over some O-scheme Y. Does Y has the EP or the SEP?

3) We assume that $\mathcal{D}$ is a Shimura quasi-envelope. Let Y_1 be a regular formally smooth O-scheme, satisfying the same Zariski density properties as Y, containing Y as an open subscheme in such a way that Y_L is dense in Y_{1L}, and with the property that $\mathcal{D}$ extends to a Shimura p-divisible group $\mathcal{D}_1$ over Y_1. If $\mathcal{D}_1$ is a Shimura envelope we say that it is an envelope of $\mathcal{D}$.

Q8. Is it true that $\mathcal{D}$ always has an envelope? If yes, are such envelopes uniquely determined?

Q9. If $k\subset\mathbb{F}$ and Y is locally of finite type over O, (when) does $\mathcal{D}$ has the ACM property?

4) Is it true that always $\mathcal{D}_k$ is p-consistent? It turns out that at least for $p\geq 3$ the answer to this question is yes (see 3.4.1 a) below and 6.5.2). The main (starting) motivation we had in expecting the answer to be yes was: 4.1.2 is obtained under no restriction on the existence of a good family of étale tensors, while [Va2, 3.4] shows that the general situation can be treated very closely to the situation of 4.1.2.

3.4. We assume that Y is connected. As in the proof of 3.1.2, we consider the étale counterpart $\mathcal{E}:=(H^1,G_{\mathbb{Z}_p},(u_\alpha)_{\alpha\in\mathcal{J}})$ of an arbitrary point $z\in Y(W(\bar{k}))$. We assume the existence of a family $(t_\beta)_{\beta\in\mathcal{J}_0}$ of homogeneous tensors of $\mathcal{T}(H^1\otimes_{\mathbb{Z}_p}W(\mathbb{F}))$ which is $W(\mathbb{F})$-very well positioned w.r.t. $G_{B(\mathbb{F})}$. We have:

3.4.1. Theorem. *a) $\mathcal{D}_k$ is p-consistent.*

b) We assume that both the partial degrees (defined in [Va4, Case 2 of 3.1]) of t_β are less or equal to $\max\{1,p-2\}$, $\forall\beta\in\mathcal{J}_0$. *Then $\mathcal{D}$ is consistent.*

Proof: Part a) is an easy consequence of Fact 2 of 2.4.2, starting from the very definition [Va1, 4.3.4], and from [Va1, 4.3.15]. To see that indeed we can apply def. [Va1, 4.3.4], we use the notations of 2.4.2 (referring to 3.1 15) we can assume that $k = \bar{k}$ and that we are dealing with a morphism $y : \mathrm{Spec}(k) \to Z_k)$. Let O^1 be the local ring of the generic point of the special fibre of $\mathrm{Spec}(R)$, and let k^1 be its field of fractions. We apply Fontaine's comparison theory for the Shimura p-divisible group over $\overline{k^1}$ we get naturally. We deduce: passing to the normalization R_1 of R in a finite field extension $\mathcal{R}_1$ of the field of fractions $\mathcal{R}$ of R, we can assume that there is an isomorphism

$$i_1 : H^1 \otimes_{\mathbb{Z}_p} \mathcal{R}_1 \tilde{\to} M_R \otimes_R \mathcal{R}_1$$

taking t_β into its de Rham counterpart of Fact 2 of 2.4.2, $\forall \beta \in \mathcal{J}$. We have:

Lemma. *We can assume that* $i_1(H^1 \otimes_{\mathbb{Z}_p} R_1[\frac{1}{p}]) = M_R \otimes_R R_1[\frac{1}{p}]$.

Proof: $R_1[\frac{1}{p}]$ is a Dedekind ring, while $R[\frac{1}{p}]$ is a UFD. Moreover there is a non-zero element $f \in R \setminus pR$ such that $i_1(H^1 \otimes_{\mathbb{Z}_p} R_1[\frac{1}{pf}]) = M_R \otimes_R R_1[\frac{1}{pf}]$. From 2.4.2, as $k = \bar{k}$, we deduce that:

(*) a multiple by a non-negative power of p, of any element of $\mathcal{T}(H^1 \otimes_{\mathbb{Z}_p} B(\mathbb{F}))$ fixed by $G_{B(\mathbb{F})}$, is integral w.r.t. M_R.

In particular we can assume that we have direct sum decompositions $H^1 \otimes_{\mathbb{Z}_p} R_1[\frac{1}{p}] = \oplus_{i \in I} H(i)$ and $M_R \otimes_R R_1[\frac{1}{p}] = \oplus_{i \in I} M(i)$, whose summands when tensored with $\mathcal{R}_1$, are all absolutely irreducible $G_{\mathcal{R}_1}$-representations. Moreover, $M(i)$'s are definable over $R[\frac{1}{p}]$; as $R[\frac{1}{p}]$ is a UFD, we deduce that they are free $R_1[\frac{1}{p}]$-modules. Similarly, $H(i)$'s are definable over $B(k)$ and, so are free $R_1[\frac{1}{p}]$-modules.

But the Zariski closures of $G_{\mathcal{R}_1}$ in $GL(i_1(H^1 \otimes_{\mathbb{Z}_p} R_1[\frac{1}{p}]))$ and in $GL(M_R[\frac{1}{p}])$ are reductive groups. For the second such closure, this is an immediate consequence of $(*)$: we can use 2.5.1 (or [Va1, 3.1.6.1]), [Va1, 4.3.13], and arguments involving Killing forms as in [Va1, the proof of 4.3.10 b)], cf. [Va1, 4.3.10.1 1)]. This implies that for any local ring L_1 of $R_1[\frac{1}{p}]$ which is a DVR and in which f is not invertible, $H(i)_{L_1}$ and $M(i)_{L_1}$ differ by a scalar. So the "difference" from $H(i)$ and $M(i)$ is measured by a fractional ideal $\mathcal{I}_i$ of $R_1[\frac{1}{p}]$; $\mathcal{I}_i$ becomes trivial in $R_1[\frac{1}{pf}]$. Passing to the normalization of R_1 in a finite field extension $\mathcal{R}_2$ of $\mathcal{R}_1$, we can assume that all these ideals $\mathcal{I}_i$ are principal: as $H(i)$'s and $M(i)$'s are free $R_1[\frac{1}{p}]$-modules, we can take $\mathcal{R}_2$ such that $[\mathcal{R}_2 : \mathcal{R}_1]$ is less or equal to rank of H^1 times the number of elements of the set I. Changing i_1 accordingly (i.e. by an element of $GL(H^1)(R_1[\frac{1}{p}])$ centralizing $G_{R_1[\frac{1}{p}]}$), we get the lemma.

From the lemma we deduce that we can apply [Va1, 4.3.4]. So the subgroup of $GL(M_R \otimes R_1)$ obtained by taking the Zariski closure of $G_{\mathcal{R}_1}$ in it is reductive. As R is an excellent ring, any closed subscheme $m_{W(k)} : \mathrm{Spec}(W(k)) \hookrightarrow \mathrm{Spec}(R)$, lifts to a closed subscheme $m_V : \mathrm{Spec}(V) \hookrightarrow \mathrm{Spec}(R_1)$, with V a DVR finite faithfully flat extension of $W(k)$. Moreover, the operation of taking the Zariski closure commutes with faithfully flat morphisms. So the σ-crystal with cycles $(M, \phi, \tilde{G}_{W(k)}, (t_\alpha)_{\alpha \in \mathcal{J}})$ over k we get by pulling back $\mathcal{C}(R)$ via $m_{W(k)}$, is such that $\tilde{G}_{W(k)}$ is a reductive subgroup of $GL(M)$. The existence of an injective cocharacter $\mu : \mathbb{G}_m \hookrightarrow \tilde{G}_{W(k)}$ as required in [Va2, 2.2.8 1)] can be checked working mod p, cf. [SGA3, Vol. 2 , p. 48] (to be compared with [Va1, proof of 5.3.2]). Using a direct specialization process entirely similar to the one of [Va1, 5.3.1], based on the fact that $\tilde{G}_k$ lifts to a reductive group over $\mathrm{Spec}(R_1/pR_1)$, and that we have a natural morphism $\mathrm{Spec}(\overline{k^1}) \to \mathrm{Spec}(R_1/pR_1)$, we deduce the existence of the desired injective cocharacter $\mu_k : \mathbb{G}_m \hookrightarrow \tilde{G}_k$. We deduce that $(M, \phi, \tilde{G}_{W(k)}, (t_\alpha)_{\alpha\in\mathcal{J}})$ is a Shimura σ-crystal over k. This proves a).

Part b) just captures the essence of [Va1, the proof of 5.1]: if $p \geq 3$, there is nothing needed to be changed in loc. cit., besides the movement from $\mathbb{F}$ to an arbitrary algebraically closed field containing $\mathbb{F}$; for $p = 2$ we are dealing with a relative PEL type situation, and so the same ideas as in 2.3.4 (cf. also 2.3.5.5) apply directly. This proves the theorem.

3.4.2. Remarks. 1) A faster proof of the reductiveness of the group $\tilde{G}_{W(k)}$ introduced at the end of the proof 3.4.1 a) can be obtained by starting remarking that we can choose $m_{W(k)}$ such that its generic fibre does not factor through the closed subscheme $\mathrm{Spec}(R/fR)$ of $\mathrm{Spec}(R)$.

2) There is a natural generalization of 3.4.1 (a) as well as b)), modeled entirely on the Theorem 3 of 2.4.

For the a) part this generalization is obtained automatically: we just have to consider not a Shimura p-divisible group over Y, but a p-divisible object $\mathcal{C}(a)$ with cycles of $\mathcal{MF}^{\nabla}_{[0,a]}(Y)$, with $a \leq p-2$, such that a reductiveness assumption entirely similar to one of [Va2, 2.2.20 ii)] is satisfied. The last part of the proof of 3.4.1 a) involving cocharacters can be generalized as well, as $\mathcal{C}(a)$ satisfies a W-condition similar to the one of [Va2, 3.4.1.5], cf. [Wi].

This part a) can be entirely adapted to the purely characteristic p situation, i.e. having just Y_k and not Y, even without assuming that Y_k is formally smooth (noetherian would be enough), or that the set $Y_k(\bar{k})$ is Zariski dense in Y_k. Of course, in this context, in general, we have to disregard the part involving cocharacters, or to assume that something similar to the mentioned W-condition holds.

For the part b) this generalization has to be obtained more carefully. We start considering again the context of $\mathcal{C}(a)$ above. We need to assume that both the partial degrees of t_β are less or equal to $\frac{p-2}{a}$, $\forall \beta \in \mathcal{J}$. For simplicity, we also assume that Y is either proper over $W(k)$, or is affine $Y = \mathrm{Spec}(R_Y)$; in this last case the p-adic completion $R_Y^\wedge$ of R_Y is automatically endowed with Frobenius lifts, and we choose (just to fix the ideas) one of them. In the affine case we also assume that we are in a geometric context, in the sense that the Galois representation (with $\mathbb{Z}_p$ coefficients) corresponding to $\mathcal{C}(a)$ (cf. [Fa1, 2.6]) is defined over R_Y and not only over $R_Y^\wedge$. In the affine case we assume that we are in a reductive context similar to the one mentioned above, while in the proper case we assume that this reductiveness property holds for an open subscheme U_Y of Y such that (Y, U_Y) is an extensible pair. Moreover some precautions are in order as [Va3, 2.4] does not work out in general if $a \geq 2$ (cf. [Va2, 3.4.8.9 and 3.4.18.5.5]). So if $a \geq 2$, then we have to define a corresponding consistency property by requiring that for any morphism m_V as in 3.1 15) such that its attached Galois representation (in the proper case cf. [Fa1, 2.6*]; in the affine case conform the extra assumption) of the Galois group $\mathrm{Gal}(\overline{V[\frac{1}{p}]}/V[\frac{1}{p}])$, when viewed in the resulting context similar to the one of 2.4, is such that the answer to Q2 of

2.4 is yes.

3) As in [Va2, 2.2.8 3-4) and 3.4.1.5-6] we can define (for $p \geq 3$) generalized Shimura quasi-envelopes (over Y), where instead of versal Shimura p-divisible groups we work with generalized Shimura p-divisible objects or with Shimura adjoint Lie p-divisible objects of $\mathcal{MF}_{[-1,1]}^{\nabla}(Y)$ (defined similarly to [Va2, 2.2.8 3-4) and 2.2.11 4)]) which are versal. This will be done in detail in part 2D: for the present part 2C, it has been more appropriate to work the way we did.

3.5. The classification of Shimura p-divisible groups: the complete local case, with $p \geq 3$. The notion of Shimura p-divisible groups is not well adapted for functorial purposes. However, this notion is well adapted w.r.t. quasi-isomorphisms in the sense of [Va2, 2.2.21 7)], and w.r.t. isomorphisms. So we first classify Shimura p-divisible groups (over some complete local schemes) in terms of (quasi-) isomorphism classes, by always specifying a fixed étale counterpart. Then we draw conclusions to the category of p-divisible groups (over such schemes), where the results are of course functorial.

3.5.0. Generalities. The main references for what follows are [Fa3], [Va3, 2.4] and [Va4]. Let p, k, V_0, K_0, V, π_V, K, Γ, Re and $B^+(V)$ have the same significance as in 2.4. Let $\widetilde{Re}$ be the piano ring of k of resonance e (cf. [Va1, 5.2.1] and [Va3, 2.1]). Let Φ_{Re} be the standard Frobenius lifts of Re, $\widetilde{Re}$ and R below, cf. [Va3, 2.1] (we do hope that no confusion will arise by using the same notation). We recall from loc. cit. that we have natural $W(k)$-monomorphisms

$$R := W(k)[[t]] \hookrightarrow Re \hookrightarrow \widetilde{Re} \hookrightarrow B(k)[[t]],$$

with t an independent variable, and that Φ_{Re} is compatible with σ and we have $\Phi_{Re}(t) = t^p$. Let $Re(\bar{k})$ be the piano ring of $\bar{k}$ of convergent resonance e. It is the p-adic completion of $Re \otimes_{W(k)} W(\bar{k})$; in particular it is faithfully flat over Re.

There is another way to describe $\widetilde{Re}$. Let I_e be the ideal of Re generated by p and by its first filtration $F^1(Re)$. $\widetilde{Re}$ is the completion of Re w.r.t. the divided power ideals $I_e^{[n]}$ of Re, $n \in \mathbb{N}$. So from the very definitions, we get:

Fact A. *The natural $W(k)$-monomorphism $Re \hookrightarrow B^+(V)$ (of 2.4) factors through $\widetilde{Re}$.*

Let $\mathcal{D} := (D, (u_\alpha)_{\alpha \in \mathcal{J}})$ be a Shimura p-divisible group over $W(k)$. Let

$$\mathcal{C} := (M, F^1, \phi, G, (u_\alpha)_{\alpha \in \mathcal{J}})$$

be its attached Shimura filtered σ-crystal over k, and let

$$\mathcal{E} := (H^1, G_{\mathbb{Z}_p}, (v_\alpha)_{\alpha \in \mathcal{J}})$$

be its étale counterpart, cf. [Va4, 2.1.4].

Using Fontaine's well known result on the classification of p-divisible groups over $W(k)$, from Theorem 5 of 2.4 and from the very definitions of [Va2, 2.2.20] we deduce:

Fact B. *The set of isomorphism classes of Shimura p-divisible groups over $W(k)$ having $\mathcal{E}$ as their counterpart is in one-to-one correspondence with the set of isomorphism classes of quintuples $\mathcal{C}$ such that we have an isomorphism*

$$\rho_M : M \otimes_{W(k)} W(\bar{k}) \tilde{\to} H^1 \otimes_{\mathbb{Z}_p} W(\bar{k})$$

taking u_α into v_α, $\forall \alpha \in \mathcal{J}$, and there is a monomorphism cocharacter $\mu : \mathbb{G}_m \hookrightarrow G$ giving birth to a direct sum decomposition $M = F^1 \oplus F^0$ ($\beta \in \mathbb{G}_m(W(k))$ acting via μ as the multiplication with β^{-i} on F^i). Of course, if k is a C_1 field (for instance if it is finite), then we do not need to pass to $W(\bar{k})$ in order to get ρ_M, cf. [Va4, V7 of 3.2].

Using Fact B and [Fa3, th. 10 and the rm. after], a similar thing can be stated for Shimura p-divisible groups over $W(k)[[x_1, ..., x_m]]$, for any $m \in \mathbb{N}$ (see also [Va2, 3.4.18.5.1 and 3.4.18.7]). As this is already done in loc. cit., it will not be repeated here. In what follows we will extend this result to the ramified situation. We start with a Shimura p-divisible group $\mathcal{D} := (D, (u_\alpha)_{\alpha \in \mathcal{J}})$ over V. We still denote by $\mathcal{E} = (H^1, G_{\mathbb{Z}_p}, (v_\alpha)_{\alpha \in \mathcal{J}})$ its étale counterpart, cf. [Va4, 2.1.4]. The equivalent of $\mathcal{C}$ in this ramified setting is slightly subtler. As in [Fa3] and [Va1, 5.2], based on [BBM], to $\mathcal{D}$ it is associated a quintuple

$$\mathcal{C}(\mathcal{D}) := (M, F^1(M), \Phi_M, (t_\alpha)_{\alpha \in \mathcal{J}}, \nabla_M),$$

where:

a) M is a free Re module of rank equal to the rank of H^1;

b) $F^1(M)$ is the Re-submodule of M, obtained by pulling back to M the Hodge filtration (is a direct summand) F_V^1 of $M \otimes_{Re} V$ defined by D;

c) $\Phi_M : M \to M$ is a Φ_{Re}-linear injective map such that M is generated by $\Phi_M(M)$ and by $\frac{1}{p}\Phi_M(F^1(M))$;

d) $\nabla_M : M \to M \otimes_{Re} dtRe = dtM$ is a p-nilpotent integrable connection on M such that Φ_M is ∇_M-parallel;

e) $t_\alpha \in F^0(\mathcal{T}(M[\frac{1}{p}]))$ is fixed by Φ_M, annihilated by ∇_M, and corresponds to v_α via the integral version of Fontaine's comparison theory of [Fa3], $\forall \alpha \in \mathcal{J}$.

Such a quintuple $\mathcal{C}(\mathcal{D})$ (resp. quadruple obtained from $\mathcal{C}(\mathcal{D})$ by removing the cycles) satisfying a) to e) (resp. a) to d)) above will be referred as a p-divisible object with cycles (resp. just p-divisible object) of $\widetilde{\mathcal{MF}}_{[0,1]}^{\nabla}(Re)$; here the role of the tilde is to emphasize that we are keeping track of the $W(k)$-epimorphism $f_V : Re \twoheadrightarrow V$, and so the filtrations are not as usual defined by direct summands of M, but are defined by Re-submodules of M generated by a direct summand of M and by $\mathrm{Ker}(f_V)M$. Similarly we define (p-divisible objects) (with cycles) of the categories (of objects) $\widetilde{\mathcal{MF}}_{[0,1]}^{\nabla}(\widetilde{Re})$ and $\widetilde{\mathcal{MF}}_{[0,1]}^{\nabla}(R)$.

Let $\tilde{G}_{Re}$ be the Zariski closure in $GL(M)$ of the subgroup of $GL(M[\frac{1}{p}])$ fixing t_α, $\forall \alpha \in \mathcal{J}$. From [Va4, def. 2.1.1 b)] we know that $\mathcal{D}$ is induced from a Shimura p-divisible group over $W(k)[[x_1, ..., x_m]]$, for some $m \in \mathbb{N} \cup \{0\}$, via a $W(k)$-homomorphism $m_V : W(k)[[x_1, ..., x_m]] \to V$. From [Va4, 2.1.3 1)] we deduce that m_V lifts to a $W(k)$-homomorphism

$$m_{Re} : W(k)[[x_1, ..., x_m]] \to Re.$$

From this and [Va4, V5 of 3.2] we deduce the existence of an isomorphism

$$f) \qquad H^1 \otimes_{\mathbb{Z}_p} Re(\bar{k}) \tilde{\to} M \otimes_{Re} Re(\bar{k})$$

taking v_α into t_α, $\forall \alpha \in \mathcal{J}$.

In particular we get that $\tilde{G}_{Re}$ is a reductive subgroup of $GL(M)$. We have:

3.5.1. Theorem. *The set of isomorphism classes of Shimura p-divisible groups over V having $\mathcal{E}$ as their counterpart is in one-to-one correspondence with the set of isomorphism classes of quintuples $(M, F^1(M), \Phi_M, (t_\alpha)_{\alpha \in \mathcal{J}}, \nabla_M)$ such that properties a) to f) above hold, and there is a monomorphism cocharacter $\mu_V : \mathbb{G}_m \hookrightarrow \tilde{G}_V$, giving birth to a direct sum decomposition $M \otimes_{Re} V = F^1_V \oplus F^0_V$ ($\beta \in \mathbb{G}_m(V)$ acting via μ_V as the multiplication with β^{-i} on F^i_V).*

Proof: The above association $\mathcal{C}(\mathcal{D})$ to $\mathcal{D}$ is obviously well behaved w.r.t. isomorphisms. Moreover, if $\mathcal{D}_1$ and $\mathcal{D}_2$ are two Shimura p-divisible groups over V such that $\mathcal{C}(\mathcal{D}_1)$ and $\mathcal{C}(\mathcal{D}_2)$ are isomorphic, then from [Fa3, §4-5] we get that $\mathcal{D}_{1K}$ and $\mathcal{D}_{2K}$ are isomorphic. From a well known theorem of Tate ([Ta]), we get that $\mathcal{D}_1$ and $\mathcal{D}_2$ are isomorphic.

We just need to show that any quintuple $\mathcal{C}^1_{Re} := (M, F^1(M), \Phi_M, (t_\alpha)_{\alpha \in \mathcal{J}}, \nabla_M)$, satisfying a) to f) and the extra requirement on F^1_V is associated to a Shimura p-divisible group over V. This is a consequence of [Va3, 2.4.2].

Here, we will like to mention, another approach (somehow roundabout, and somehow copying loc. cit.), i.e. that this association is in essence the content of [Va3, the proof of 2.4]: interpreting the loc. cit. just in terms of F-crystals, we get directly that the extension $\mathcal{C}^1_{\widetilde{Re}}$ of $(M, F^1(M), \Phi_M, (t_\alpha)_{\alpha \in \mathcal{J}}, \nabla_M)$ to $\widetilde{Re}$ is obtained by extension of scalars from R to $\widetilde{Re}$ of a similar quintuple

$$\mathcal{C}^2_R := (M^2, F^1(M^2), \Phi_{M^2}, (t^2_\alpha)_{\alpha \in \mathcal{J}}, \nabla_{M^2})$$

(over R) (here $F^1(M^2)$ is the pull back of a direct summand of $M^2 \otimes_R V$); moreover, by replacing $F^1(M^2)$ by a direct summand $F^1(M^2)'$ of M^2 contained in $F^1(M^2)$, the resulting quintuple $(M^2, F^1(M^2)', \Phi_{M^2}, (t^2_\alpha)_{\alpha \in \mathcal{J}}, \nabla_{M^2})$ is associated to a Shimura p-divisible group over R.

So the conclusion follows, once we show that in fact $\mathcal{C}^1_{Re}$ and $\mathcal{C}^2_{Re}$ are themselves isomorphic. For this, using [Fa3, th. 5], it is enough to show that their corresponding Galois representations (of Γ) (with $\mathbb{Z}_p$ coefficients) are isomorphic. But this is a consequence of Fact A and of the way one can recover these Galois representations in the Fontaine's comparison theory over the $W(k)$-algebra $B^+(V)$. Of course we can as well argue this at the level of modules: any element of the F^0-filtration of $\mathrm{Hom}(M, M^2) \otimes_{Re} \widetilde{Re}$ fixed by the Frobenius, has to belong to $\mathrm{Hom}(M, M^2)$, as it can be easily checked. This proves the theorem.

3.5.2. Corollary. *The category of p-divisible groups over V is anti equivalent to the category $\widetilde{\mathcal{MF}}^{\nabla}_{[0,1]}(Re)$ of quadruples $(M, F^1(M), \Phi_M, \nabla_M)$ satisfying a) to d) above.*

Proof: This is an immediate consequence of 3.5.1. A couple of remarks are in order. First, we need to take $\mathcal{J}$ to be the empty set in 3.5.1 (cf. [Va2, 2.2.11 9)]). Second, the association to a p-divisible group over V of such a quadruple is obviously functorial (see [BBM]). Third, in 3.5.1 we have avoided the "trivial" situations (cf. [Va2, 2.2.11 11)]); so in order to show that any such quadruple $(M, F^1(M), \Phi_M, \nabla_M)$ is associated to a p-divisible group over V, even if $F^1(M) = M$ or if $F^1(M) = \ker(M \twoheadrightarrow M \otimes_{Re} V)$, we just have to modify it as follows. In the first case we replace it by its direct sum with the extension to Re of the quadruple $(W(k), 0, \sigma, 0)$ (here the two zeros mean: trivial F^1-filtration, trivial connection). In the second case we replace it by its direct sum with the extension to Re of $W(k)(1) := (W(k), W(k), p\sigma, 0)$. Using the projections of these direct sums on the direct factors, we do get that they have an extra Shimura structure, so that 3.5.1 applies. This proves the corollary.

3.5.3. Variants. 1) There are too many variants of 3.5.1-2. In what follows, we will state some of them, mostly in the context of 3.5.1 (the passage to the context of 3.5.2 is automatic). One exception: we prefer to state V5 in the context of 3.5.2.

V1. The principally quasi-polarized context. If now $\mathcal{D} := (D, (v_\alpha)_{\alpha \in \mathcal{J}}, p_D)$ is a principally quasi-polarized Shimura p-divisible group over V, then we have to use sextuples: we get a principal quasi-polarization of H and respectively of M, and we have to require that the isomorphism of f) is moreover a symplectic isomorphism.

V2. The context of quasi-isomorphisms. For this variant we need to pass from isomorphisms to quasi-isomorphisms, in the sense of [Va2, 2.2.21 7)]. In other words, we still work with quintuples, but we disregard the families of tensors involved, and in their place we keep track of the reductive groups $\tilde{G}_{Re}$. So f) above has to be modified: the isomorphism of f) has to take $G_{Re(\bar{k})}$ into $\tilde{G}_{Re(\bar{k})}$.

V3. The context of a fixed filtration class. Let $V(\bar{k})$ be the completion of the strict henselization of V. This variant is entirely similar to 3.5.1; the only difference: we require that the $G_{\mathbb{Z}_p}(V(\bar{k}))$-conjugacy class of the cocharacter of $G_{V(\bar{k})}$ obtained via f) and μ_V is fixed a priori. This makes sense: as in [Va2, 2.2.9 3)] one can check that the $\tilde{G}_V$ conjugacy class of μ_V is uniquely determined by $\mathcal{C}(\mathcal{D})$.

V4. The context of $\widetilde{Re}$ or of R. From the proof of 3.5.1 we deduce that we have an entirely similar variant of 3.5.1, where instead of Re we work with $\widetilde{Re}$. In practice, we think it is more convenient to work with $\widetilde{Re}$ instead of Re (see [Va1, 5.2-3] and [Va3, 2.4]; see also V8), though the fact that Re is "smaller" (being a $W(k)$-subalgebra of $\widetilde{Re}$) makes it to be psychologically more suited.

Moreover, we can replace as well Re by R in the following sense. The problem with R is: an isomorphism over Re between two p-divisible objects with cycles of $\widetilde{\mathcal{MF}}^{\nabla}_{[0,1]}(R)$ is not necessarily defined over R. So two p-divisible objects with cycles of $\widetilde{\mathcal{MF}}^{\nabla}_{[0,1]}(R)$ will be called pseudo-isomorphic if they are isomorphic over Re. So we get a variant of 3.5.1 (cf. its proof) by working with pseudo-isomorphism classes of quintuples over R (i.e. of p-divisible objects with cycles of $\widetilde{\mathcal{MF}}^{\nabla}_{[0,1]}(R)$) satisfying (the analogues of) a) to f) above.

Definitely, among the rings Re, $\widetilde{Re}$ and R, R "offers" (from all points of view) the simplest context. So the "complications" with Re and $\widetilde{Re}$ can be avoided from many points of view.

V5. The context of extensions of truncations mod p^n. Let $n \in \mathbb{N}$. Another variant we can obtain by working with extensions of truncations mod p^n (in the sense of [Va2, 2.2.1]) of the quintuples $\mathcal{C}(\mathcal{D})$. This is somehow easier adaptable to the context of 3.5.2, but still it makes sense to the context of 3.5.1. However, for simplicity we will refer here just to 3.5.2. We have:

Corollary. *The category of finite flat commutative group schemes of p-power order over V is anti equivalent to the category of objects $\widetilde{\mathcal{MF}}^{\nabla}_{[0,1]}(Re)$ (of extensions of truncations modulo non-negative powers of p of quadruples as in 3.5.2).*

Proof: First we need to recall [Va2, 3.4.18.5.3]. Second we need to remark that any object of $\widetilde{\mathcal{MF}}^{\nabla}_{[0,1]}(Re)$ (defined similarly to $\mathcal{MF}^{\nabla}_{[0,1]}(Re)$ but involving different F^1-filtrations not defined by direct summands, but obtained, via extensions, from taking modulo a non-negative power of p Re-modules $F^1(M)$ as above) is indeed an extension of truncation of a p-divisible object of $\widetilde{\mathcal{MF}}^{\nabla}_{[0,1]}(Re)$. Third we need to recall that any finite flat commutative group schemes of p-power order over V is a subgroup of a p-divisible group over V. Last we need to point out that the integral version of Fontaine's comparison theory (see [Fa3]) has an obvious variant modulo powers of p, as long as we restrict to objects of $\widetilde{\mathcal{MF}}^{\nabla}_{[0,1]}(Re)$. This proves the corollary.

V6. The context of different Frobenius lifts. A great part of [Va2, 3.4.18] can be adapted to the present situation. First of all we can work with a Frobenius lift of R which does not necessarily take t into t^p. Denoting by I the ideal of R generated by t, it is enough to assume that I is preserved by the Frobenius Φ_{Re} of R and that we do have a Frobenius lift of R of essentially additive type (in the sense of [Va2, 3.4.18.0.1]). The reason is: [Va2, 3.4.18.4] remains valid in the context of R being replaced by Re (cf. [Va2, 3.4.20 4)]). In particular, even if Φ_{Re} is a Frobenius lift of R which is not of essentially additive type, the connections ∇_M we get by the condition that Φ_M is ∇_M-parallel, are automatically p-nilpotent and integrable (to be compared with [Va2, 3.4.1.8.4.1]).

V7. The context of $V[[x_1,...,x_m]]$. Here we pass from V to $V(m) := V[[x_1,...,x_m]]$, with $m \in \mathbb{N}$. Accordingly, we have to replace Re by $Re(m) := Re[[x_1,...,x_m]]$, $\widetilde{R}e$ by $\widetilde{R}e(m) := \widetilde{R}e[[x_1,...,x_m]]$, R by $R(m) := W(k)[[t,x_1,...,x_m]]$, F^1_V by $F^1_{V(m)}$, and Φ_{Re} by the Frobenius lift $\Phi(m)$ of $Re(m)$ compatible with Φ_{Re} and taking x_i to x_i^p. Similarly to the case of V, to any Shimura p-divisible group $\mathcal{D}_m := (D_m,(v_\alpha)_{\alpha\in\mathcal{J}})$ over $\mathrm{Spec}(V(m))$ it is associated a quintuple

$$\mathcal{C}(\mathcal{D}_m) := (M(m),F^1(M(m)),\Phi_{M(m)},(t_\alpha)_{\alpha\in\mathcal{J}},\nabla_{M(m)}),$$

having entirely similar properties as a) to e) above. Similarly we define the étale counterpart $\mathcal{E}$, and moreover the property similar to f) above holds. We get:

Theorem. *The set of isomorphism classes of Shimura p-divisible groups over Spec$(V(m))$ having $\mathcal{E}$ as their counterpart is in one-to-one correspondence with the set of isomorphism classes of quintuples $(M(m),F^1(M(m)),\Phi_{M(m)},(t_\alpha)_{\alpha\in\mathcal{J}},\nabla_{M(m)})$ such that the properties similar to a) to f) above hold, and there is a monomorphism cocharacter $\mu_{V(m)}:\mathbb{G}_m\hookrightarrow$*

$\tilde{G}_{V(m)}$, *giving birth to a direct sum decomposition* $M(m)\otimes_{Re(m)} V(m)=F^1_{V(m)}\oplus F^0_{V(m)}$ *(*$\beta\in\mathbb{G}_m(V(m))$ *acting via* $\mu_{V(m)}$ *as the multiplication with* β^{-i} *on* $F^i_{V(m)}$*).*

Proof: The proof of this is entirely similar to the proof of 3.5.1. However some points need to be detailed, as [Fa3, th. 5*] is not stately exactly as we need it. First, as $\mathrm{Spec}(W(k)[[x_1,...,x_m]])$ has a Zariski dense set of $W(k)$-valued points, using the proof of 3.5.1, we get immediately that any two such quintuples over $Re(m)$ isomorphic over $\widetilde{Re}(m)$, are in fact isomorphic over $Re(m)$. Second, as in [Va3, the proof of 2.4], we get that any such quadruple over $\widetilde{Re}(m)$ is obtained from one over $R(m)$, and that this quintuple over $R(m)$, after modifying the F^1-filtration (as in the proof of 3.5.1), is induced by a Shimura p-divisible group over $R(m)$; in other words the presence of extra variables x_i's is irrelevant.

The last thing we need to mention: any isomorphism between two such quintuples over $Re(m)$ associated to two Shimura p-divisible groups $\mathcal{D}^1$ and $\mathcal{D}^2$ over $\mathrm{Spec}(V(m))$ is induced uniquely by an isomorphism between $\mathcal{D}^1$ and $\mathcal{D}^2$. But this is a consequence of Grothendieck–Messing theory (the part involving lifting isomorphisms is needed here), and of 3.5.1 applied to the context of the closed embedding $\mathrm{Spec}(Re)\hookrightarrow Re(m)$ defined by x_i's go to zero, once we remark: there is no element of the F^0-filtration of $\mathrm{End}(M(m))$, different from $1_{M(m)}$, fixed by the Frobenius, and congruent to $1_{M(m)}$ modulo the ideal of $Re(m)$ generated by x_i's. This proves the theorem.

V8. The context of closed subschemes of Spec($V[[x_1,...,x_m]]$). Even more generally, using entirely the same ideas, a theorem similar to the one of V7 holds for any closed subscheme $Z=\mathrm{Spec}(R^Z)$ of $\mathrm{Spec}(W(k)[[x_1,...,x_m]])$, for some $m\in\mathbb{N}$. Let I_Z be the kernel of the $W(k)$-epimorphism

$$i_Z: W(k)[[x_1,...,x_m]]\twoheadrightarrow R^Z.$$

Let I_m be the ideal of $B(k)[[x_1,...,x_m]]$ generated by x_i's. Let $\widetilde{R}_Z$ (resp. R_Z) be the PD-filtration (resp. the p-adic) completion of the PD hull of the ideal $(p)+I_Z$ of $W(k)[[x_1,...,x_m]]$. For simplicity sake, we prefer to work in the context of V4. $\widetilde{R}_Z$ is a $W(k)$-subalgebra of $B(k)[[x_1,...,x_m]]$ containing $W(k)[[x_1,...,x_m]]$. It is easy to see that:

Fact C. *$\widetilde{R}_Z$ is isomorphic to the projective limit of $\widetilde{R}_Z/I^n_m\cap\widetilde{R}_Z$, $n\in\mathbb{N}$.*

We choose the Frobenius lift of $\widetilde{R}_Z$ to be compatible with σ and to take x_i into x_i^p. It is this Fact C which guarantees that [Va3, proof of 2.4] applies. We will not restate the theorem of V7 in the present context, as this is obvious. We will just point out how we can construct the étale counterpart of a quintuple (as above) over $\widetilde{R}_Z$ (to be referred as a p-divisible object $\mathcal{C}(\mathcal{D}_Z)$ with cycles of $\widetilde{\mathcal{MF}}^{\nabla}_{[0,1]}(\widetilde{R}_Z)$), even in the case when the ring R^Z is not integral of characteristic zero. Using the same argument as in [Va1, 5.3.2], by modifying as usual the F^1-filtration, we can bring everything in the context of a p-divisible object with cycles of $\mathcal{MF}^{\nabla}_{[0,1]}(\widetilde{R}_Z)$. So we just need to take its image via the closed subscheme $\mathrm{Spec}(W(k))\hookrightarrow\widetilde{R}_Z$ defined by x_i's go to zero, and to use Fontaine's comparison theory (over $W(k)$) to get its étale counterpart.

V9. Combined context. There are too many (and to easy) ways of combining the above 8 variants, to be mentioned here explicitly. We will just mention two things. First, the combination of V6 and V7 can be very useful in practice (to be compared with [Va2, 3.4.18]). Second, the combination of V4 and of V8 produces lots of simplifications: we can substitute $\widetilde{R}_Z$ (or R_Z) by $W(k)[[x_1,...,x_m]]$ (warning: we have to use the language of pseudo-isomorphisms); this is very useful from many points of view (like deformation aspects), as the local "huge" rings $\widetilde{R}_Z$ are not even noetherian.

3.5.4. Remarks. 1) Using 3.5.1-3, 3.1-4 extends automatically to the context of a (suitable) scheme Y over V. However, presently we can not see any applications of this extension.

2) The essence of 3.5.1-2, i.e. [Va3, proof of 2.4], was already contained in the preliminary version of [Va3], released on December 19, 1997. 3.5.2 and V5 of 3.5.3 has been also announced in [Br] (most probably [Br] was inspired from the mentioned version). Moreover, the general form of 3.5.2 mentioned in 1) above is also obtained in [Zi]. However we regard the approach of 3.5.1-2 (and of their general forms of V7-8 of 3.5.3) via [Va3, proof of 2.4 and 2.4.2] as much simpler. This approach relies mainly on [Fa3, th. 5] in the simplest case when (with the notations of loc. cit.) $a = 1$, and on the simple theorem 10 of [Fa3], and so it can be adapted to a great extend even for the case $p = 2$ (for instance, for $p = 2$, in connection to 3.5.1, we can get a variant of V4 of 3.5.3 by working with isomorphism classes of Shimura 2-divisible groups over V modulo an equivalence relation; see [Va8] for details). This approach is entirely different than the ones of [Br] and [Zi]. Being simpler, it is automatically more general: 3.5.1 and their variants can not be obtained via [Br] or [Zi]. We do believe that it is correct to attribute 3.5.3 to Faltings.

3) Warning: the definition of different categories $\widetilde{\mathcal{MF}}^{\nabla}_{[0,1]}(*)$ introduced above depend on the epimorphisms chosen; but different epimorphisms produce equivalent categories. For instance the definition of $\widetilde{\mathcal{MF}}^{\nabla}_{[0,1]}(R_Z)$ depends on i_Z, but not its equivalence class.

4) Warning: we do not know if the rings Re and $\widetilde{Re}$ can be substituted by R (in the sense of V4 of 3.5.3), if by chance we are just in a non-smooth (for instance non-reductive) context involving crystalline cycles; the reason is: we can use the variant of 3.5.2 provided by V4 of 3.5.3 (i.e. to get a p-divisible object $\mathcal{C}_R$ of $\widetilde{\mathcal{MF}}^{\nabla}_{[0,1]}(R)$), but it is not clear at all that actually the cycles are "traceable" (from Re) back to R, by an appropriate choice of $\mathcal{C}_R$. It seems to us that in general, these cycles are not traceable back to R: what makes us to hesitate on the issue is that [Va1, 5.3.1-2] and [Va3, proof of 2.4] can not be (normally) performed in a non-smooth context.

5) 4) above has significant philosophical implications in connection to [Va1, 4.3]. It points out that in connection to applications to Shimura varieties of Hodge type (see 4.2 below), we can not restrict a priori in def. [Va1, 4.3.4] to the class of (complete) regular noetherian local rings (of dimension 2 and residue field $\mathbb{F}$). If by chance, due to some elementary reasons, we can restrict to such a class of regular rings, then under this restriction the greatest part of [Va1, 4.3] admits considerable simplifications. Regardless of the fact we can restrict or not, we still think it is very useful to have such strong theories as the one of loc. cit.

On the other hand, referring to the notations of [Va1, 4.3.4], it would have been more convenient to work in a larger context, where we do not assume $M\big[\frac{1}{\pi}\big]=W\otimes_K R\big[\frac{1}{\pi}\big]$, but only that $M\otimes_R \mathcal{R}=W\otimes_K \mathcal{R}$, with $\mathcal{R}$ the field (ring, in the case of strongly context) of fractions of R; this is pointed out by applications, and by (surprisingly) harder work we had to perform in the proof of 3.4.1 a) in order to bring a concrete situation to the original context of loc. cit.

6) Fact A and 3.5.2 also point out that the integral version of Fontaine's comparison theory for p-divisible groups (over V) can be stated as well in terms of $\widetilde{Re}$ or R (cf. also the proof of 3.5.1).

7) Warning: we do believe that the part of V4 involving R does not remain true if instead of $\widetilde{\mathcal{MF}}^{\nabla}_{[0,1]}(*)$ we work with $\widetilde{\mathcal{MF}}^{\nabla}_{[0,a]}(*)$, with $a\in\mathbb{N}$, $p-2\geq a\geq 2$; see the limitations of [Va2, 3.4.8.9 and 3.4.18.5.5].

§4. Integral canonical models of Shimura varieties of preabelian type revisited

Two main theorems pertaining to Shimura varieties of Hodge type are stated in 4.1 and proved in 4.2. The conclusions that result to the existence of integral canonical models of Shimura varieties of preabelian type are gathered in 4.3.1-3. They are an immediate consequence of 4.1 and of [Va1].

4.1. The first main two results. Though a couple of years have passed since the first draft of [Va1] was made available, we can not add anything to [Va1, 3.2, §4 and §6] pertaining to the general theory of integral canonical models of Shimura varieties of preabelian type (meanwhile reference [Mi4] has to be added in connection to the existence of canonical models of Shimura varieties). However we can refine (so that, in particular, the case of small primes 2 and 3 is also included), extend and simplify some parts of [Va1, §5]. This is the main goal of §4 below and of parts of the subsequent chapters §5-6.

In the whole of §4 we consider a prime $p\geq 3$. Let (W,ψ) be a symplectic space over $\mathbb{Q}$ defining a Siegel modular variety, cf. [Va1, Example 2 of 2.5]. Let $L_{(p)}$ be a $\mathbb{Z}_{(p)}$-lattice of W such that we get a perfect form $\psi\colon L_{(p)}\otimes_{\mathbb{Z}_{(p)}} L_{(p)}\to\mathbb{Z}_{(p)}$. Let $K:=GSp(L_{(p)}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Z}_p,\psi)(\mathbb{Z}_p)$; it is a hyperspecial subgroup of $GSp(W,\psi)(\mathbb{Q}_p)$. Let

$$f\colon (G,X)\to (GSp(W,\psi),S)$$

be an injective map. We assume that the Zariski closure of G in $GSp(L_{(p)}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Z}_{(p)},\psi)$ is a reductive group $G_{\mathbb{Z}_{(p)}}$. So $H:=K\cap G(\mathbb{Q}_p)$ is a hyperspecial subgroup of $G(\mathbb{Q}_p)$. Let v be a prime of $E(G,X)$ dividing p. The triple $(f,GSp(L_{(p)},\psi),v)$ is a SATS. Let $\mathcal{M}$ be the integral canonical model of the Shimura quadruple $(GSp(W,\psi),S,K,p)$. Let $\mathcal{N}$ be the normalization of the Zariski closure of $\mathrm{Sh}_H(G,X)$ in $\mathcal{M}_{O_{(v)}}$.

4.1.1. First Main Theorem. *$\mathcal{N}$ is the integral canonical model of the Shimura quadruple (G,X,H,v).*

4.1.2. Second Main Theorem. *The triple $(f,L_{(p)},v)$ is a SHS.*

4.1.3. Corollary. *To give a SHS is the same as to give a $\mathbb{Z}_{(p)}$ embedding into a Siegel modular variety, with p an odd prime.*

4.2. The proofs of 4.1.1-2. To simplify the presentation, the proofs of 4.1.1-2 are divided into six steps. Moreover, different paragraphs are indexed by capital letters. Step 1 introduces the main idea, and states the Main Claim. Step 2 proves this Main Claim and 4.1.2 for the case $p \geq 5$. We would like to point out right away that for $p \geq 5$, 4.1.1 is already known: it is just a restatement of what [Va1, 3.12 and 6.4.1] when combined, say in our present situation. So we are left to refer to the case $p = 3$. The simpler case when (G, X) is of compact type (with arbitrary $p \geq 3$) is handled in Step 3. The general non-compact type case for $p = 3$ is proved in Steps 4 and 5. In Step 6 we revisit the case $p \geq 5$: we prove 4.1.1 without using [Va1, 6.4.1]. Concluding remarks are gathered in 4.2.1.

A) Step 0. The starting setting. Let L be a $\mathbb{Z}$-lattice of W such that $L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = L_{(p)}$ and $\psi : L \otimes_{\mathbb{Z}} L \to \mathbb{Z}$ is perfect. We already know that $\mathcal{N}$ is a normal integral canonical model, cf. [Va1, 3.2.12]. This means that there is a compact open subgroup H_0 of $G(\mathbb{A}_f^p)$ such that $\mathcal{N}$ is a pro-étale cover of (the normal $O_{(v)}$-scheme of finite type)

$$Y := \mathcal{N}/H_0;$$

moreover from the proof of loc. cit. and the quasi-projectiveness (in the sense of [Va1, 3.2.3 2')]) of $\mathcal{M}$ we get that Y is in fact a quasi-projective $O_{(v)}$-scheme. We can assume that H_0 is a subgroup of the subgroup K_4 of $GSp(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}[\frac{1}{p}], \psi)$ of elements congruent to 1 mod 4. So $\mathcal{M}/K_4$ is a moduli space of principally polarized abelian schemes (over $\mathbb{Z}_{(p)}$-schemes) of dimension $g_W := \frac{\dim(W)}{2}$, having a level-4 symplectic similitude structure (cf. [Va1, 3.2.9]). Let $(\mathcal{A}(4), \mathcal{P}_{\mathcal{A}(4)})$ be the universal principally polarized abelian scheme over $\mathcal{M}/K_4$. Let $(\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ be its pull back to Y, and let $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ be its pull back to $\mathcal{N}$. Let $(v_\alpha)_{\alpha \in \mathcal{J}'}$ be as usual, a family of homogeneous tensors of $\mathcal{T}(W)$ such that G is the subgroup of $GL(W)$ fixing its members (cf. the notations of [Va2, 2.3.1]). For simplicity, we assume that

$$v_\alpha \in \mathcal{T}(L_{(p)}),\ \forall \alpha \in \mathcal{J}'.$$

We also assume that H_0 is small enough so that the family $(w_\alpha^{\mathcal{A}})_{\alpha \in \mathcal{J}}$ of Hodge cycles with which $\mathcal{A}$ is naturally endowed (see [Va1, 4.1] and [Va2, 2.3.3]), is obtained by pulling back a family $(w_\alpha^{\mathcal{A}_{H_0}})_{\alpha \in \mathcal{J}}$ of Hodge cycles of $\mathcal{A}_{H_0}$. Let

$$\mathcal{D}_{H_0} := (D_{H_0}, (t_\alpha)_{\alpha \in \mathcal{J}'}, p_{D_{H_0}})$$

be the principally quasi-polarized p-divisible group with cycles over Y, naturally attached to the triple

$$\mathcal{S} := (\mathcal{A}_{H_0}, (w_\alpha^{\mathcal{A}_{H_0}})_{\alpha \in \mathcal{J}}, \mathcal{P}_{\mathcal{A}_{H_0}}).$$

Let $\mathcal{C}$ be a connected component of $Y_{W(\mathbb{F})}$. Let $d := \dim_{\mathbb{C}}(X)$. Let

$$R := R_{d+1} := W(\mathbb{F})[[x_1, ..., x_d]],$$

with x_i's as independent variables. Let Φ_R be its Frobenius lift compatible to σ and such that $\Phi_R(x_i) = x_i^p$.

B) To prove 4.1.1-2 we need to prove two things:

a) Y is smooth over $O_{(v)}$;
b) $\mathcal{D}_{H_0}$ is a principally quasi-polarized Shimura p-divisible group over Y.

a) is equivalent to: the normal separated quasi-projective $W(\mathbb{F})$-scheme $\mathcal{C}$ is smooth over $W(\mathbb{F})$, while b) is equivalent to $\mathcal{D}_{\mathcal{C}}$ is a principally quasi-polarized Shimura p-divisible group. These two equivalences are a consequence of [Va1, 3.3.2].

C) Step 1. The use of toric quadruples. We use trick one of 2.3.1 and Exercise 1 of 2.3.2.1. We deduce the existence of a SHS $(f \circ f_T, L_{(p)}, v_T)$ factoring through $(f, L_{(p)}, v)$, with $f\colon (T, \{h\}) \hookrightarrow (G, X)$ a special pair. Let $\mathcal{T}$ be the integral canonical model of $(T, \{h\}, H_T, v_T)$, with H_T as in 2.3.1 (cf. [Va1, 3.2.8]). From [Va1, 3.2.7 4)] we deduce the existence of a natural morphism $\mathcal{T} \to \mathcal{N}$. We can assume that at least one of the $W(\mathbb{F})$-valued points of $\mathcal{N}_{W(\mathbb{F})}$ obtained from ones of $\mathcal{T}_{W(\mathbb{F})}$, maps into $\mathcal{C}$. Let

$$m_T : \mathrm{Spec}(W(\mathbb{F})) \to \mathcal{C}$$

be such a morphism factoring through $\mathcal{T}_{W(\mathbb{F})}$. From Theorem 6 of 2.4, applied to the SHS $(f \circ f_T, L_{(p)}, v_T)$, we deduce that the principally quasi-polarized filtered σ-crystal with cycles over $\mathbb{F}$ attached to $m_T^*(\mathcal{S}_{\mathcal{C}})$ is a principally quasi-polarized Shimura filtered σ-crystal $(M, F^1, \phi, G_{W(\mathbb{F})}, (u_\alpha)_{\alpha\in\mathcal{J}'}, p_M)$, and that $u_\alpha \in \mathcal{T}(M)$, $\forall \alpha \in \mathcal{J}'$. Let P be the parabolic subgroup of $G_{W(\mathbb{F})}$ leaving invariant F^1.

D) Following entirely the deformation theory of [Va2, 5.4], we deduce the existence of a formally étale morphism

$$m_R : \mathrm{Spec}(R) \to \mathcal{C}$$

lifting m_T, and such that the p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$ with cycles defined by $m_R^*(\mathcal{S}_{\mathcal{C}})$ is a Shimura filtered F-crystal $(M, F^1, \phi, G_{W(\mathbb{F})}, N, \tilde{f}, p_M)$, with N a smooth subgroup of $G_{W(\mathbb{F})}$ such that the natural quotient morphism $N \to G_{W(\mathbb{F})}/P$ is an open embedding, and with $\tilde{f}$ the ring isomorphism induced naturally by an isomorphism from the completion of N in the origin and $\mathrm{Spec}(R)$.

Using a natural algebraization procedure, as in [Va2, 2.3.13.1], we deduce the existence of an open subscheme $\mathcal{U}$ of $\mathcal{C}$ such that:

c) m_R factors through it;
d) $\mathcal{U}_{\mathbb{F}}$ is connected;
e) it is smooth over $W(\mathbb{F})$;
f) $\mathcal{D}_{\mathcal{U}}$ is a principally quasi-polarized Shimura p-divisible group.

E) So the problem is: how to extend c) and d) to the whole of $\mathcal{C}$? Let $\mathcal{U}_0$ be the Zariski closure of $\mathcal{U}_{\mathbb{F}}$ in $\mathcal{C}$. The starting idea is:

Main Claim. *$\mathcal{U}_0$ is smooth over $\mathbb{F}$, and for each point $y : Spec(\mathbb{F}) \to \mathcal{U}_0$, $y^*(\mathcal{D}_{\mathcal{U}_0})$ is a Shimura p-divisible group over $\mathbb{F}$.*

The goal of the next four Steps is to prove this Main Claim. Here we just mention one thing towards its proof. From Fact 1 of 2.4.2 we get directly that the principally quasi-polarized σ-crystal with cycles over $\mathbb{F}$ $(M_y, \phi_y, (t_\alpha^y)_{\alpha\in\mathcal{J}'}, p_{M_y})$ naturally attached to $y^*(\mathcal{D}_{\mathcal{U}_0})$ has the property:

g) $t_\alpha^y \in \mathcal{T}(M_y)$, $\forall\alpha\in\mathcal{J}'$.

G) As an application of the Main Claim we get:

Corollary. *We can assume that* $\mathcal{U}_{\mathbb{F}} = \mathcal{U}_0$.

Proof: The proof is divided into two parts. First part is presented below, and handles the corollary in many situations. The general case is easily reduced to the cases treated in this first part, cf. end of Step 2 below; it forms the second part.

H) Let V be a complete DVR having $\mathbb{F}$ as its residue field and such that y lifts to a morphism $m_V : \mathrm{Spec}(V) \to \mathcal{C}$. Let $A_V := m_V^*(\mathcal{A}_{H_0\,\mathcal{C}})$. Let $H_V := H^1_{dR}(A_V/V)$, and let $F^1(H_V)$ be its Hodge filtration. Let $K_V := V\left[\frac{1}{p}\right]$. Let $A_{\mathbb{F}}$ be the special fibre of A_V. As usual (for instance see [Fa3, §6]) we have a natural isomorphism $i_y : M_y\otimes_{W(\mathbb{F})} K_V \xrightarrow{\sim} H_V\left[\frac{1}{p}\right]$ such that $\forall\alpha\in\mathcal{J}'$ we have:

h) $i_y(t_\alpha^y) \in F^0(H_V\left[\frac{1}{p}\right])$.

It allows the identification of t_α^y with the de Rham component of the Hodge cycle (corresponding to the index $\alpha\in\mathcal{J}'$) with which A_V is naturally endowed (see [Va1, 5.2.2.1 and 5.2.10]; the loc. cit. are general results: the arguments included in loc. cit. remain true in any situation (and in particular in the situations where we are not in an integral reductive context).

Let F_y^1 be a direct summand of M_y such that

$$\mathcal{C}_y := (M_y, F_y^1, \phi_y, (t_\alpha^y)_{\alpha\in\mathcal{J}'}, p_{M_y})$$

is a principally quasi-polarized Shimura filtered σ-crystal, cf. def. [Va2, 2.2.8] and the (assumed) Main Claim. Let $P(F_y^1)$ be the parabolic subgroup of $GSp(M_y, p_{M_y})$ leaving F_y^1 invariant. From h) we get that the deformation space of $\mathcal{C}_y$ has relative dimension d; being formally smooth over $W(\mathbb{F})$ it is identifiable with $\mathrm{Spec}(R)$. Let $\mathcal{C}_R$ be the principally quasi-polarized Shimura filtered F-crystal we get over $\mathrm{Spec}(R)$, and which is a versal deformation of $\mathcal{C}_y$ (see the proof of [Va2, 3.12.1]). We get naturally (as in [Va1, 5.4.7-8]) a formally closed embedding

$$m_y^R\colon \mathrm{Spec}(R) \to \mathcal{M}_{W(\mathbb{F})}$$

lifting the composite y_0 of y with the morphism $\mathcal{U}_0 \to \mathcal{M}_{W(\mathbb{F})}$, and corresponding (as in [Va1, 5.4]) to this deformation space. We just need to show that the composite m_V' of m_V with the natural morphism $\mathcal{C} \to \mathcal{M}_{W(\mathbb{F})}$ factors through m_y^R: the argument for this is entirely the same to the proof of [Va1, 5.3.3.1]. Of course one can apply directly [Va1, 4.1.5] as well: this has the advantage that [Va1, 4.1.5] does not uses the fact that $p\ge 3$.

I) This first part ends with three solutions of the Corollary, each one handling different aspects (partial cases) of it.

S1: a sketched solution for the general case. The fact that m_V' factors through m_y^R is a direct consequence of the fact that the completion of the local ring of $\mathcal{M}_{W(\mathbb{F})}$ in y_0 can be identified (this is the same as [Va1, 5.5.1]) with the completion of $GSp(M_y, p_{M_y})/P(F_y^1)$ in the origin, and of h) above.

S2: the case $V = W(\mathbb{F})$. This case is a direct consequence of [Va1, 5.3.1] and of [Va2, 2.3.17.1-2].

S3: the case when the monomorphism $G_{\mathbb{Z}_{(p)}} \hookrightarrow GL(L_{(p)})$ is a $DEG4$ representation. This case is nothing else but a particular case of 3.4.1 a). It is worth recalling the (simplified) details. Let $R_2 := W(\mathbb{F})[[x]]$. We use Fact 2 of 2.4.2, in the context of a morphism $m_2 : \mathrm{Spec}(R_2/pR_2) \to \mathcal{U}_0$, whose generic fibre factors through $\mathcal{U}$ and whose composite with the natural closed embedding $\mathrm{Spec}(\mathbb{F}) \hookrightarrow \mathrm{Spec}(R_2/pR_2)$ is y. Let

$$\mathcal{C}_{R_2} := (M_{R_2}, \phi_{R_2}, (w_\alpha^2)_{\alpha\in\mathcal{J}'}, p_{M_{R_2}}, \nabla_{R_2})$$

be the principally quasi-polarized F-crystal with cycles over R_2/pR_2 defined by $m_2^*(\mathcal{D}_{\mathcal{U}_0})$. From Fact 2 of 2.4.2, we get that

i) $w_\alpha^2 \in \mathcal{T}(M_{R_2})$, $\forall \alpha \in \mathcal{J}'$.

From this i) and the very definition of a $DEG4$ representation, in the same manner we got 2.3.7.3, we get directly that the subgroup of $GL(M_{R_2})$ obtained as the Zariski closure of the subgroup of $GL(M_{R_2}[\frac{1}{p}])$ fixing w_α^2, $\forall \alpha \in \mathcal{J}'$, is reductive over R_2, and so isomorphic to G_{R_2}. Using the part of the proof of 3.4.1 a) involving cocharacters and [SGA3, Vol. 2 , p. 48], we deduce easily the existence of an injective cocharacter $\mu : \mathbb{G}_m \hookrightarrow G_{R_2}$ giving birth to a direct sum decomposition $M_{R_2} = F^1_{R_2} \oplus F^0_{R_2}$ ($\beta \in \mathbb{G}_m(R_2)$ acts via μ on $F^i_{R_2}$ as the multiplication with β^{-i}, $i = \overline{0,1}$) such that the sextuple obtained by adding $F^1_{R_0}$ to $\mathcal{C}_{R_2}$ is a principally quasi-polarized filtered F-crystal with cycles over R_2/pR_2.

From [Fa3, rm. ii) after th. 10], we get that this sextuple is obtained from $\mathcal{C}_R$, via a morphism $m(2) : \mathrm{Spec}(R_2) \to \mathrm{Spec}(R)$. Using this, we get immediately (for instance using S2 above, with $\mathbb{F}$ replaced by the algebraic closure k of the field of fractions of $\mathbb{F}[[x]]$, and with m_V replaced by a morphism $\mathrm{Spec}(W(k)) \to \mathrm{Spec}(R_2)$, which at the level of generic fibres is the natural dominant morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(R_2/pR_2)$) that $m_y^R \circ m(2)$ factors through $\mathcal{C}$. Using [Va1, 5.4.5] we get directly that m_y^R factors through $\mathcal{C}$. Obviously this implies the corollary, and so it ends this third (partial) solution.

This ends the first part of the proof of the corollary.

J) Step 2. The case $p \geq 5$. Here we prove the Main Claim for primes $p \geq 5$. From [Va1, 6.4.1] we already know that $\mathcal{C}$ is smooth. So the completion of the local ring of y in $\mathcal{C}$ can be identified with $\mathrm{Spec}(R)$; so we get a natural formally étale morphism $m(y)\colon \mathrm{Spec}(R) \to \mathcal{C}$ lifting y. Let $(A_R, p_{A_R}) := m(y)^*(\mathcal{A}_{H_0\,\mathcal{C}}, \mathcal{P}_{\mathcal{A}_{H_0\,\mathcal{C}}})$. Let $M_R := H^1_{dR}(A_R/R)$, and let $\psi_R(y)$ be its perfect symplectic form defined by p_{A_R}.

K) Let $\tilde{G}$ be the Zariski closure in $GL(M_R)$ of the subgroup of $GL(M_R[\frac{1}{p}])$ defined by fixing the de Rham components of the Hodge cycles with which A_R is naturally endowed. We just need to show (to be compared with [Va2, 2.3.13.1]):

Lemma 1. *$\tilde{G}$ is a reductive subgroup of $GL(M_R)$.*

Proof: We present two solutions to this lemma. The first one (see S1) handles the general context, while the second one (see S2) avoids some Lie types.

S1: the general case. From f) above and the smoothness of $\mathcal{C}$ we already know, that there is an extensible pair $(\mathrm{Spec}(R), U_R)$ (in the sense of [Va1, 3.2.1 1)]) such that $\tilde{G}_{U_R}$ is a reductive group over U_R.

Let $T_{R[\frac{1}{p}]}$ be the connected component of the origin of the center of $\tilde{G}_{R[\frac{1}{p}]}$, and let T_R be its closure in $GL(M_R)$. Let $\tilde{G}^{\mathrm{der}}_R$ be the Zariski closure in $GL(M_R)$ of $\tilde{G}^{\mathrm{der}}_{R[\frac{1}{p}]}$. From 2.5.1 we deduce that it is enough to show that T_R and $\tilde{G}^{\mathrm{der}}_R$ are reductive groups over R.

The fact that T_R is a subtorus of $GL(M_R)$ is an immediate consequence of [Va1, 4.3.13]: the $\mathbb{Z}_p$ algebra $L(p)$ of endomorphisms of $L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ fixed by $G_{\mathbb{Z}_p}$ acts naturally on M_R; so $L(p) \otimes_{\mathbb{Z}_p} R$ acts on M_R, and so using different elements of it which are projectors of M_R we achieve the desired direct sum decomposition of M_R into R-submodules, needed to get (as in loc. cit.) that T_R is a subtorus.

To show that $\tilde{G}^{\mathrm{der}}_R$ is a reductive group, we have to use trick three of 2.3.3, in the form of 2.3.3.1. We can assume that we have a situation

$$\mathcal{C}_1 \leftarrow \mathcal{C}_2 \rightarrow \mathcal{C},$$

obtained as in 2.3.3.1, by putting (the not yet known to be a SHS) the triple $(f, L_{(p)}, v)$ together with a SHS $(f_1, L_{1(p)}, v_1)$ whose adjoint Shimura quadruple is isomorphic to (G, X, H, v), and by working at finite type level. The existence of such a SHS $(f_1, L_{1(p)}, v_1)$ is guaranteed by 2.3.11 4) (cf. also [Va1, 6.4.2] and [Va2, 2.3.6]). Here $\mathcal{C}_2$ is related to a $\mathbb{Z}_{(p)}$ embedding

$$(f_2, GSp(L_{(p)} \oplus L_{1(p)}, \psi \oplus \psi_1), v_2),$$

where (G_2, X_2) is a suitable Shimura subpair of $(G, X) \times (G_1, X_1)$ such that the natural projections $(G_2, X_2) \rightarrow (G_1, X_1)$ and $(G_2, X_2) \rightarrow (G, X)$ are finite maps. But the morphisms $c_2 : \mathcal{C}_2 \rightarrow \mathcal{C}$ and $c_1 : \mathcal{C}_2 \rightarrow \mathcal{C}_1$ are étale covers, cf. [Va1, 6.4.5.1]. So we can assume that $m(y)$ factors through $\mathcal{C}_2$; let $m(y)_1 : \mathrm{Spec}(R) \rightarrow \mathcal{C}_1$ be the resulting composite morphism. Moreover over $\mathcal{C}_2$ we have a principally polarized abelian scheme $(\mathcal{A}^2, \mathcal{P}_{\mathcal{A}^2})$: it is the product of $c_2^*((\mathcal{A}_{\mathcal{C}}, \mathcal{P}_{\mathcal{A}_{\mathcal{C}}}))$ and of a similar one obtained via c_1 and $(f_1, L_{1(p)}, v_1)$. We define similarly M_{1R}, via $m(y)_1$. Similarly we define $\tilde{G}_{1R}$, as a subgroup of $GL(M_{1R})$.

As $(f_1, L_{1(p)}, v_1)$ is a SHS, we already know that $\tilde{G}_{1R}$ is a reductive subgroup (for instance see [Va2, 2.3.13]), isomorphic to G_{1R}. Let H_R be the semisimple group cover of $\tilde{G}^{\mathrm{der}}_{1R}$, corresponding to the isogeny $G_2^{\mathrm{der}} \rightarrow G_1^{\mathrm{der}}$. From the existence of $\mathcal{C}_2$, we get directly that we have a central isogeny

$$H_{U_R} \rightarrow \tilde{G}_{U_R}.$$

In other words, the part of Step 1 pertaining to the first trick and to Theorem 6 of 2.4, can be applied directly to the context of $(f_2, GSp(L_{(p)} \oplus L_{1(p)}, \psi \oplus \psi_1), v_2)$. From [BLR, theorem 1 of p. 106] we deduce that the resulting homomorphism $H_{U_R} \rightarrow GL(M_R)_{U_R}$ extends to a homomorphism $q_R : H_R \rightarrow GL(M_R)$. Let H^1_R be the quotient of H_R by the finite flat central subgroup of it corresponding to the kernel of the natural isogeny $G_2^{\mathrm{der}} \rightarrow G^{\mathrm{der}}$. It is included in the kernel of q_R, as its generic fibre is. We get a homomorphism $q^1_R : H^1_R \rightarrow GL(M_R)$, which over U_R is a closed embedding. As R is regular, as H^1_R and $GL(M)$ are affine smooth R-schemes, and as (R, U_R) is an extensible pair, we deduce that q^1_R is a finite morphism. As in the proof of [Va1, 3.1.6] (or as pointed out in [Va1, 3.1.2.2

3)]) we get that in fact it is a closed embedding. So $\tilde{G}_R^{\mathrm{der}}$ is nothing else but H_R^1. This ends the proof of this lemma 1.

S2: the case when the Killing form on Lie($G^{\mathrm{der}}_{\mathbb{Z}_{(p)}}$) is perfect. It is enough to show that there is a morphism $\mathrm{Spec}(W(\mathbb{F})) \to \mathrm{Spec}(R)$ such that the Zariski closure in $GL(M_R \otimes_R W(\mathbb{F}))$ of the reductive subgroup of $GL(M_R \otimes_R B(\mathbb{F}))$ obtained by pulling back $\tilde{G}_{R[\frac{1}{p}]}$, is a reductive group over $W(\mathbb{F})$. The argument is: [Va1, 5.3.1 and 5.4] applies entirely.

We consider an embedding $i_2 : \mathrm{Spec}(R_2) \hookrightarrow \mathrm{Spec}(R)$, such that $U_{R_2} := i_2^{-1}(U_R)$ is $\mathrm{Spec}(R)$ without the maximal point. It is enough to show that the similarly constructed integral subscheme $\tilde{G}'_{R_2}$ of $GL(M_R \otimes_R R_2)$ is a reductive group over R_2. Here we put a prime index, not to create confusion of notations: $\tilde{G}'_{R_2}$ is always a closed subscheme of $\tilde{G}_{R_2}$ (in general G'_{R_2} is different from G_{R_2}). As in S1 we can concentrate on the derived part. Let $\mathfrak{g}'$ be the locally free sheaf over U_{R_2} defined by the Lie algebra of $\tilde{G}^{\mathrm{der}}_{U_{R_2}}$ (no prime index is needed here). From [FC, 6.2 of p. 181] we deduce that it extends to a locally free sheaf over R_2, which can be identified with an R_2-module $\mathfrak{g}_2$. $\mathfrak{g}_2$ has a natural Lie algebra structure. From our hypothesis on Lie($G^{\mathrm{der}}_{\mathbb{Z}_{(p)}}$), we deduce that the Killing form of $\mathfrak{g}_2$ is perfect, being perfect in codimension one. Using [Va1, Claim 2 of the proof of 4.3.10 b)], we get immediately the existence of a semisimple group over R_2 which over U_{R_2} is $\tilde{G}_{U_{R_2}}$. The rest of the argument is as in S1 (or as in S3 of I) above).

L) Obviously the Lemma 1 implies (for $p \geq 5$) the Main Claim as well as the Corollary. The next problem is: we do not know a priori that the special fibre $\mathcal{C}_{\mathbb{F}}$ is connected (this will be proved in [Va10]); however, the Corollary allows us to assume that $\mathcal{U}$ is obtained from $\mathcal{C}$ by "throwing away" potential connected components of $\mathcal{C}_{\mathbb{F}}$ which do not intersect $\mathcal{U}_0$. The question arises:

Q10. *How we can pass the information we have on $\mathcal{U}$ to the (supposed to exist) connected components of $\mathcal{C}_{\mathbb{F}} \setminus \mathcal{U}_0$?*

M) The answer is very simple: by using Shimura-canonical lifts, and by repeating the above arguments. Here are the details. We apply [Va2, 4.2.1 and 4.4.1] to the SHS $(f_1, L_{1(p)}, v_1)$. Let $\mathcal{U}_1$ be the open subscheme of $\mathcal{C}_1$ containing its generic fibre and all the Shimura-ordinary points of the special fibre of $\mathcal{C}_{1\mathbb{F}}$. We know (cf. loc. cit.) that $(\mathcal{C}_1, \mathcal{U}_1)$ is an extensible pair. Let $\mathcal{U}' := c_2(c_1^{-1}(\mathcal{U}_1))$; so $(\mathcal{C}, \mathcal{U}')$ is also an extensible pair. We just need to show:

Lemma 2. *$\mathcal{D}_{\mathcal{U}'}$ is a principally quasi-polarized Shimura p-divisible group over $\mathcal{U}'$.*

Proof: Following the way we got c) to f) above, we just have to show that the filtered σ-crystal with cycles $(M_1, F_1^1, \phi_1, (u_{1\alpha})_{\alpha\in\mathcal{J}})$ associated to $m_1^*(\mathcal{S}_{\mathcal{U}'})$ is a Shimura filtered σ-crystal over $\mathbb{F}$. Here

$$m_1 : \mathrm{Spec}(W(\mathbb{F})) \to \mathcal{U}'$$

is an arbitrary Shimura-canonical lift. As we do not know yet that $(f, L_{(p)}, v)$ is a SHS, we can not use [Va2]; so here by Shimura-canonical lift we mean: as above we can assume that m_1 factors through $\mathcal{C}_2$, and so we get a composite morphism $m_{11} : \mathrm{Spec}(W(\mathbb{F})) \to \mathcal{C}_1$

which is assumed to be a Shimura-canonical lift (cf. [Va2, 4.4.1] applied to the SHS $(f_1, L_{1(p)}, v_1)$).

Let $A_{W(\mathbb{F})} := m_1^*(\mathcal{A}_{H_0\mathcal{U}'})$. Using [Va1, 4.3.13] and Theorem 5 of 2.4, we just have to show that the natural p-adic Galois representation attached to $A_{W(\mathbb{F})}$

$$\rho : \mathrm{Gal}(\overline{B(\mathbb{F})}/B(\mathbb{F})) \to GL(H^1_{\text{ét}}(A_{\overline{B(\mathbb{F})}}, \mathbb{Z}_p))(\mathbb{Z}_p) = GL(L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p)(\mathbb{Z}_p),$$

factors through the $\mathbb{Z}_p$ valued points of a subtorus T_1 of $GL(L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p)$, for which we have $T_{1\mathbb{Q}_p} \subset G_{\mathbb{Q}_p}$. Here the identification of the two GL's groups is obtained as in [Va1, 5.1.2].

ρ factors through the $\mathbb{Q}_p$-valued points of $G_{\mathbb{Q}_p}$. So to check the existence of T_1, we can work in the adjoint context, i.e. with the resulting representation

$$\rho^{\mathrm{ad}} : \mathrm{Gal}(\overline{B(\mathbb{F})}/B(\mathbb{F})) \to G^{\mathrm{ad}}(\mathbb{Q}_p).$$

Again we need to use c_1 and c_2. We have:

Sublemma. *The representation ρ^{ad} is the same as the adjoint counterpart ρ_1^{ad} of a representation ρ_1 obtained similarly to ρ, but via m_{11}.*

This sublemma is a consequence of the existence of the abelian scheme $\mathcal{A}^2$ over $\mathcal{C}_2$. So [Va2, b) of 4.4.1 3)] (cf. also [Va2, 4.5.12.2.1]) applied to ρ^1 applies, giving us the image of T_1 in $G^{\mathrm{ad}}_{\mathbb{Z}_p}$. From this image we can recapture the desired T_1 easily. This ends the proof of this lemma 2.

N) From Lemma 2, as in the proof of the part of the Main Claim contained in K) above, we get that $\mathcal{D}_{\mathcal{U}'}$ extends to a principally quasi-polarized Shimura p-divisible group over $\mathcal{C}$. This ends the proof of 4.1.2 for primes $p \geq 5$.

O) We are left with the case $p = 3$. Using Step 2 (and [Va2, end of 4.9.2] for the part of Step 2 related to L) and M)), we deduce that for proving the Main Claim, G) above, as well as 4.1.1-2, we can assume that we are in the context of 2.3.7.2. In other words, using 2.3.11 6), we can assume that the monomorphism $G_{\mathbb{Z}_{(p)}} \hookrightarrow GL(L_{(p)})$ is a $DEG4$ representation, and that $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is simple. In this way, the proof of G) gets completed (cf. S3 of I) above), once we complete the proof of the Main Claim in the case when the monomorphism $G_{\mathbb{Z}_{(p)}} \hookrightarrow GL(L_{(p)})$ is a $DEG4$ representation. We will not mention this again, once this last case is completed.

P) Step 3. The compact case. We assume now that (G, X) is of compact type. As in [Va1, 6.4.1.1 2)] we deduce that $\mathcal{C}$ is a proper $W(\mathbb{F})$-scheme. So $\mathcal{U}_0$ is a proper $\mathbb{F}$-scheme. We first assume that the Main Claim holds. The Corollary of Step 1 (as we are in the context of a $DEG4$ representation, S3 of I) applies) implies that $\mathcal{U}$ is a smooth $W(\mathbb{F})$-scheme having connected proper fibres, and is an open subscheme of a proper $W(\mathbb{F})$-scheme. From the theorem of formal fibres of a proper morphism, we get directly that $\mathcal{C} = \mathcal{U}$. So $\mathcal{C}$ is smooth. This implies 4.1.1; when combined to f) (of D) above) it also implies 4.1.2.

So we are left with the proof of the Main Claim. In this paragraph we somehow just repeat S3 of I). We keep using the notations of Step 1. Let $\tilde{G}_y$ be the Zariski closure in $GL(M_y)$ of the subgroup of $GL(M_y[\frac{1}{p}])$ fixing t_α^y, $\forall\alpha\in\mathcal{J}'$. Using linear combinations of such tensors of degree 4 which belong to $\mathcal{T}(M_y)$, from Theorem 4 of 2.4, and from the definition of a $DEG4$ representation, we deduce the existence of a projector of $\mathrm{End}(M_y)$ fixed by $\tilde{G}_y$ and such that:

– its image $\mathfrak{g}$, when tensored with $B(\mathbb{F})$, is $\mathrm{Lie}(\tilde{G}_y^{\mathrm{der}})[\frac{1}{p}]$;
– the Killing form on $\mathfrak{g}$ is perfect.

As in [Va1, the proof of 4.3.10 b)] (cf. 2.3.7.3) we deduce that $\tilde{G}_y^{\mathrm{der}}$ (defined as the Zariski closure of $\tilde{G}_{y\mathbb{Q}_p}^{\mathrm{der}}$ in $GL(M_y)$) is a reductive group over $W(\mathbb{F})$. As in the proof of Lemma 1, we deduce that the Zariski closure of the connected component of the origin of the center of the generic fibre of $\tilde{G}_y$ in $GL(M_y)$ is a reductive group. Again lemma 2.5.1 applies: we conclude that $\tilde{G}_y$ is a reductive group (of course it is split, and so isomorphic to $G_{W(\mathbb{F})}$). From the existence of the isomorphism i_y, as in [Va1, 5.3.1], we deduce the existence of a cocharacter $\mu:\mathbb{G}_m\to\tilde{G}_y$ as required in the definition of a Shimura σ-crystal. We deduce that $(M_y,\phi_y,\tilde{G}_y,p_{M_y})$ is a principally quasi-polarized Shimura σ-crystal over $\mathbb{F}$. This ends the proof of the Main Claim, and so (cf. the first paragraph of P)) of 4.1.1-2 in the compact case.

Q) Step 4. The use of the unitary trick. We are left with the following situation: $p=3$, $(G^{\mathrm{ad}},X^{\mathrm{ad}})$ is simple of non-compact type, and the monomorphism $G_{\mathbb{Z}_{(p)}}\hookrightarrow GL(L_{(p)})$ is a $DEG4$ representation. The case G^{ad} is of A_n Lie type is handled by [Va2, 2.3.5.1]. So from now on we assume that G^{ad} is not of A_n type. We first treat the case $(G^{\mathrm{ad}},X^{\mathrm{ad}})$ is of C_n or $D_n^{\mathbb{H}}$ type. So till the end of Step 4 we have only these two types in mind.

We can assume that we are in the context of the second part of the unitary trick, cf. the proof of 2.3.7.2. So we can assume that f factors through a standard PEL situation $(f_1,L_{(p)},v_1,\mathcal{B}_1)$, via an injective map $i_1:(G,X)\hookrightarrow(G_1,X_1)$, in such a way that a bunch of properties hold. In what follows we will mention only those properties which we need.

First, all the factors of $G_{1\mathbb{R}}^{\mathrm{ad}}$ are non-compact, cf. 2.3.8 j); in fact they are isomorphic to $SU(n,n)_{\mathbb{R}}^{\mathrm{ad}}$. So $E(G_1^{\mathrm{ad}},X_1^{\mathrm{ad}})=\mathbb{Q}$. From [De1, 2.3.13] (cf. also [Va1, 6.6]), we deduce that the monomorphisms $\mathbb{S}_{\mathbb{R}}\hookrightarrow G_{1\mathbb{R}}$ corresponding to elements of X_1 factor through the subgroup of $G_{1\mathbb{R}}$ generated by $G_{1\mathbb{R}}^{\mathrm{der}}$ and by the one dimensional split torus acting on $W\otimes_{\mathbb{Q}}\mathbb{R}$ as scalar multiplication. So we can assume that $G_1^{\mathrm{ab}}=\mathbb{G}_m$, and that $E(G_1,X_1)=\mathbb{Q}$. Warning: by assuming this, we might lose the property that $(f_1,L_{(p)},v_1,\mathcal{B}_1)$ is a standard PEL situation; however, this is irrelevant for what follows, as we will be interested just in the derived groups involved and in their representations. Moreover, if $(G^{\mathrm{ad}},X^{\mathrm{ad}})$ is of C_n type then also $E(G,X)=\mathbb{Q}$; if $(G^{\mathrm{ad}},X^{\mathrm{ad}})$ is of $D_n^{\mathbb{H}}$ type without (resp. with) involution, then $E(G,X)$ is a totally real number field (resp. a totally imaginary quadratic extension of a totally real number field).

R) Second, using 2.3.11 2), we can assume that $G_1^{\mathrm{ad}}=\mathrm{Res}_{F/\mathbb{Q}}(G_1^F)$ and $G^{\mathrm{ad}}=\mathrm{Res}_{F/\mathbb{Q}}(G^F)$, with F as usual, and with G^F and G_1^F absolutely simple F-groups. Moreover, we can assume the existence of a totally imaginary quadratic extension E of F unramified above p and such that G_{1E} splits, cf. [Va1, 6.6.5] and 2.3.11 3). Let $G_{1F_{(p)}}^{\mathrm{der}}$ be the simply connected semisimple $F_{(p)}$-group, having G_1^F as the adjoint of its generic fibre, and such

$G^{\mathrm{der}}_{1\mathbb{Z}_{(p)}} = \mathrm{Res}_{F_{(p)}/\mathbb{Z}_{(p)}}(G^{\mathrm{der}}_{1F_{(p)}})$.

We recall from [Va1, 6.6.5] how $L_{(p)}$ and the monomorphism $G_{1\mathbb{Z}_{(p)}} \to GL(L_{(p)})$ are constructed. We start identifying $G^{\mathrm{der}}_{1E_{(p)}}$ with $SL(W'_{E_{(p)}})$, with $W'_{E_{(p)}}$ a free $E_{(p)}$-module of rank $2n$. Let $W_{E_{(p)}} := W'_{E_{(p)}} \oplus W'_{E_{(p)}}$ be a disjoint union of two copies of $W'_{E_{(p)}}$. We let $SL(W'_{E_{(p)}})$ act on $W_{E_{(p)}}$ in the following manner: $A \in SL(W'_{E_{(p)}})(E_{(p)})$ acts in the standard way on the first copy, and as $(A^t)^{-1}$ on the second copy of $W'_{E_{(p)}}$. Here A^t is the transpose of A, viewed as a matrix w.r.t. a fixed $E_{(p)}$-bases of $W'_{E_{(p)}}$. Let $\tilde{K}$ be a totally imaginary quadratic extension of F unramified above p. We take

$$L_{(p)} := \tilde{K}_{(p)} \otimes_{F_{(p)}} W_{E_{(p)}},$$

and we take $\rho_1 : G^{\mathrm{der}}_{1\mathbb{Z}_{(p)}} \hookrightarrow GL(L_{(p)})$ to be defined at the level of $\mathbb{Z}_{(p)}$-valued points as follows:

$$\gamma \in G^{\mathrm{der}}_{1\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}) = G^{\mathrm{der}}_{1F_{(p)}}(F_{(p)}) \subset G^{\mathrm{der}}_{1E_{(p)}}(E_{(p)})$$

acts on $L_{(p)}$ via the natural action on $W_{E_{(p)}}$. [De1, 2.3.13] points out that in fact, in the present situation, we can avoid "the complication" with $\tilde{K}$; however, for future references we will keep track of $\tilde{K}$.

Third, we move the study of our representations over $\mathbb{Z}_p$. For simplicity we assume that $K_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ is just two copies of $F_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ (i.e. that K splits above any prime of F dividing p). We write

$$F_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p := \prod_{i \in I(F)} W(k_i),$$

with $I(F)$ a finite set, and with k_i finite fields of characteristic p. Similarly we can write $G^{\mathrm{der}}_{1\mathbb{Z}_p} = \prod_{i \in I(F)} G^i_1$, where $G^i_1 := \mathrm{Res}_{W(k_i)/\mathbb{Z}_p}(SL_n(W(k_i)))$ (it is 2.3.11 3) which allows us to consider $SL_n(W(k_i))$ and not some outer form of it). We deduce that the faithful representation

$$\rho_{1\mathbb{Z}_p} : G^{\mathrm{der}}_{1\mathbb{Z}_p} \hookrightarrow GL(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p)$$

is a disjoint union of representations ρ^i_1 indexed by $i \in I(F)$. To describe it we fix $i \in I(F)$. Let

$$E_{(p)} \otimes_{F_{(p)}} W(k_i) = \prod_{j \in I(F,i)} W(k_{ij}),$$

where $I(F,i)$ is either $\{1\}$ or $\{1,2\}$. If $I(F,i) = \{1\}$, then k_{i1} is the quadratic field extension of k_i; otherwise we have $k_i = k_{i1} = k_{i2}$. The representation ρ^i_1 is nothing else but the disjoint union of two copies of the representation of G^i_1 on

$$W_{I(F,i)} := \oplus_{j \in I(F,i)} W(k_{ij})^{2n} \oplus \oplus_{j \in I(F,i)} W(k_{ij})^{2n},$$

obtained in a similar fashion as ρ_1.

S) Fourth, we need to understand how the subgroup $G^{\mathrm{der}}_{\mathbb{Z}_p}$ of $G^{\mathrm{der}}_{1\mathbb{Z}_p}$ acts on $L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$. We will use similar notations, i.e. $G^{\mathrm{der}}_{F_{(p)}}$, $G^{\mathrm{der}}_{E_{(p)}}$ and G^i will have the same meaning as the corresponding ones with a lower right index one, but working with $G_{\mathbb{Z}_{(p)}}$ instead $G_{1\mathbb{Z}_{(p)}}$. The key point is:

Fact 0. *$G^{\mathrm{der}}_{E_{(p)}}$ is the subgroup of $SL(W'_{E_{(p)}})$ fixing a perfect bilinear form b' on the $\mathbb{Z}_{(p)}$-module $W'_{E_{(p)}}$; moreover, b' is $E_{(p)}$-linear w.r.t. the natural structure of $W'_{E_{(p)}}$ as an $E_{(p)}$-module.*

This fact is in conformity with [Va3, 3.0 D)]. Of course b' is symmetric (resp. alternating) if $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is of $D_n^{\mathbb{H}}$ (resp. of C_n) type. So the $\mathbb{Z}_p$-module $W_{I(F,i)}$ comes equipped naturally with a perfect bilinear form $b_{F,i}$; it is $W(k_i)$-linear w.r.t. the natural $W(k_i)$-module structure of $W_{I(F,i)}$. Moreover, the monomorphism $G^{\mathrm{der}}_{\mathbb{Z}_p} \hookrightarrow GL(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p)$ is itself a disjoint union of representations ρ^i indexed by $i \in I(F)$. ρ^i factors through G^i, and it is nothing else but the composite of the natural monomorphism $G^i \hookrightarrow G^i_1$, with ρ^i_1. As $b_{F,i}$ is $W(k_i)$-linear, we get directly:

Fact 1. *G^i is the subgroup of G^i_1 fixing $b_{F,i}$.*

We conclude:

Fact 2. *The monomorphism $G^{\mathrm{der}}_{\mathbb{Z}_{(p)}} \hookrightarrow GL(L_{(p)})$ is a $DEG2$ representation.*

T) Fifth, we need to start analyzing an arbitrary morphism $z_V : \mathrm{Spec}(V) \to \mathcal{N}$, with V a faithfully flat finite DVR extension of $W(\mathbb{F})$. Let $y_V : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ be the morphism it defines naturally. As usual, let $(M_{y_V}, \phi_{y_V}, (t^V_\alpha)_{\alpha\in\mathcal{J}'}, p_{M_{y_V}})$ be the σ-crystal with cycles attached to y_V. Let $G_{1W(\mathbb{F})}$ be the reductive subgroup of $GL(M_{y_V})$ obtained by keeping track only of the cycles of a subfamily $(t^V_\alpha)_{\alpha\in\mathcal{J}'_1}$ of $(t^V_\alpha)_{\alpha\in\mathcal{J}'}$ (so $\mathcal{J}'_1$ is a subset of $\mathcal{J}'$), corresponding to the morphism $y_{1V} : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}_1$, obtained by composing y_V with the natural morphism $\mathcal{N} \to \mathcal{N}_1$. Warning: the family $(t^V_\alpha)_{\alpha\in\mathcal{J}'_1}$ is not necessarily formed just by endomorphisms.

Let $\tilde{G}_{W(\mathbb{F})}$ be the Zariski closure in $GL(M_{y_V})$ of the subgroup of $GL(M_{y_V}[\frac{1}{p}])$ fixing the tensors of the family $(t^V_\alpha)_{\alpha\in\mathcal{J}'}$. We have a natural direct sum decomposition

$$M_{y_V} := \oplus_{i\in I(F)} M^i_{y_V}$$

in $G_{1W(\mathbb{F})}$-submodules, which is left invariant by ϕ_{y_V}. Here $M^i_{y_V}$ corresponds (via Fontaine's theory) to ρ^i_1. Moreover, for $i \in I(F)$, we have natural direct sum decompositions $(M^i_{y_V}, \phi_{y_V}) = (M^{i0}_{y_V}, \phi_{y_V})^2$, such that each copy of $M^{i0}_{y_V}$ is left invariant by $G_{1W(\mathbb{F})}$. Each such copy corresponds to a copy of $W_{I(F,i)}$. Corresponding to $b_{F,i}$ we get a perfect bilinear form

$$b^{\mathrm{crys}}_{F,i} : M^{i0}_{y_V} \otimes_{W(\mathbb{F})} M^{i0}_{y_V} \to W(\mathbb{F})(1),$$

fixed by $\tilde{G}_{W(\mathbb{F})}$. The reason why it is perfect: the Fontaine's comparison theory is functorial. We get immediately:

Fact 3. *$\tilde{G}_{W(\mathbb{F})}$ is a reductive group.*

Obviously we can identify $\tilde{G}_{W(\mathbb{F})}$ with $G_{W(\mathbb{F})}$.

Fact 4. *The quadruple* $(M_{y_V}, \phi_{y_V}, \tilde{G}_{W(\mathbb{F})}, p_{M_{y_V}})$ *is a principally quasi-polarized Shimura* σ*-crystal over* $\mathbb{F}$.

Proof: We just need to show the existence of a cocharacter $\mu_y : \mathbb{G}_m \to G_{W(\mathbb{F})}$, such that it gives birth to a direct sum decomposition $M_{y_V} = F^1_{y_V} \oplus F^0_{y_V}$ (with $\beta \in \mathbb{G}_m(W(\mathbb{F}))$ acting via μ_y on $F^i_{y_V}$ as the multiplication by β^{-i}), with the property that $(M_{y_V}, F^1_{y_V}, \phi_{y_V}, \tilde{G}_{W(\mathbb{F})})$ is a Shimura filtered σ-crystal. Such a cocharacter exists over $V[\frac{1}{p}]$ (cf. H)); but as in [Va1, 5.3.1] we deduce that it exists over $W(\mathbb{F})$.

As in D) above we can construct starting from $(M_{y_V}, F^1_{y_V}, \phi_{y_V}, \tilde{G}_{W(\mathbb{F})})$, a formally closed morphism $m(y_V) : \mathrm{Spec}(R) \to \mathcal{N}_1$.

Claim. $m(y_V)$ *factors through* $\mathcal{N}$.

Before proving the Claim, let us notice that it implies that $\mathcal{N}$ is formally smooth at y_V, and that property (*) of [Va2, 2.3.4] holds for any $W(\mathbb{F})$-valued point lifting y_V. Of course, we can work as well instead of $\mathbb{F}$ with any arbitrary point of $\mathcal{N}$ with values in an algebraically closed field containing $\mathbb{F}$. We conclude: 4.1.1-2, as well as the Main Claim and G) above, hold in the non-compact cases of C_n or $D_n^{\mathbb{H}}$ type.

We are left with the proof of the above Claim. In what follows it is irrelevant which copy of $M^{i0}_{y_V}$ we are dealing with, and so we will treat the situation as if it would be only one copy. In the C_n type case, $b^{\mathrm{crys}}_{F,i}$ can be interpreted as a principal quasi-polarization $p^{i0}_{y_V}$ of the p-divisible group $D^{i0}_{y_V}$ over $\mathbb{F}$ corresponding to the σ-crystal $(M^{i0}_{y_V}, \phi_{y_V})$. It lifts to a principal quasi-polarization $p^{i0}_{z_V}$ of the p-divisible group $D^{i0}_{z_V}$ over V lifting naturally $D^{i0}_{y_V}$, via the lift z_V of y_V. But the morphism $\mathrm{Spec}(R) \to \mathcal{N}_1$ can be interpreted as the deformation space factoring through $\mathcal{N}_1$ of all such principally quasi-polarized p-divisible groups. This obviously implies that the composite of z_V with the morphism $\mathcal{N} \to \mathcal{N}_1$, factors through $m(y_V)$. Using [Va1, 4.1.5] the Claim follows immediately. The situation in the $D_n^{\mathbb{H}}$ is entirely similar: we just have to change the use of principal quasi-polarizations, by isomorphisms between p-divisible groups. In other words, in case $b_{F,i}$ is symmetric, then we can still view $b^{\mathrm{crys}}_{F_i}$ as an isomorphism between $\mathcal{C}^{i0}_{y_V} := (M^{i0}_{y_V}, \phi_{y_V})$ and $\mathcal{C}^{i0*}_{y_V}(1) := (M^*_{y_V}(1), \phi_{y_V})$, to be referred as a principal symmetric quasi-polarization.

Of course, for both C_n and $D_n^{\mathbb{H}}$ types, one could as well just copy [Va1, 5.2-4] in the present situation: the degrees of $b_{F,i}$'s are $2 = 2(3-2)$.

U) Step 5. The orthogonal non-compact case. We are left with the following case: $p = 3$, $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is simple of non-compact B_n or $D_n^{\mathbb{R}}$ type, and the monomorphism $G_{\mathbb{Z}_{(p)}} \hookrightarrow GL(L_{(p)})$ is a $DEG4$ representation. As in P) above, we get that the Main Claim and G) hold. If $\mathcal{C}_{\mathbb{F}}$ has only one connected component we are done. So here we need to consider the (potential) situation when $\mathcal{C}_{\mathbb{F}}$ has extra connected components besides $\mathcal{U}_{\mathbb{F}}$. Let $\mathcal{C}_i$ be such an extra connected component.

Let $m \in \mathbb{N}$ such that all the simple factors of $G^{\mathrm{ad}}_{\mathbb{R}}$ are isomorphic to $SO(2,m)^{\mathrm{ad}}_{\mathbb{R}}$. As in Step 4, we can assume that $G^{\mathrm{ab}} = G_1^{\mathrm{ab}} = \mathbb{G}_m$, and that $E(G,X) = E(G_1, X_1) = \mathbb{Q}$. Let $y_i : \mathrm{Spec}(\mathbb{F}) \to \mathcal{C}_i$ be a point. Let $m_{V_i} : \mathrm{Spec}(V_i) \to \mathcal{C}$ be a lift of it, with V_i a faithfully flat finite DVR extension of $W(\mathbb{F})$.

V) The first key point is:

Fact 5. *If y_i is an ordinary point, then $\mathcal{C}$ is smooth at all points of $\mathcal{C}_i$. Moreover, denoting by $\mathcal{U}_i$ the open subscheme of $\mathcal{C}$ formed by the generic fibre and by $\mathcal{C}_i$, $\mathcal{D}_{\mathcal{U}_i}$ is a principally quasi-polarized Shimura p-divisible group.*

Proof: This fact is just a ramified variant of [Va3, 2.2.3]. Let $(M_{y_i}, \phi_{y_i}, (t_\alpha^{y_i})_{\alpha\in\mathcal{J}'}, p_{M_{y_i}})$, A_{V_i}, K_{V_i} and H_{V_i} have the same significance as in G) and H), but working with y_i and m_{V_i} instead of y and respectively of m_V. Let $\tilde{G}_{iW(\mathbb{F})}$ be the Zariski closure in $GL(M_{y_i})$ of the subgroup of $GL(M_{y_i}[\frac{1}{p}])$ whose generic fibre is the subgroup fixing $t_\alpha^{y_i}$, $\forall\alpha\in\mathcal{J}'$. Similarly we define a subgroup $\bar{G}_{V_i}$ of $GL(H_{V_i})$. As y_i is an ordinary point, the p-adic Galois representation attached to A_{V_i}

$$\rho_i : \mathrm{Gal}(\overline{K_{V_i}}/K_{V_i}) \to GL(H_{\text{ét}})(\mathbb{Z}_p) := GL(H^1_{\text{ét}}(A_{\overline{K_{V_i}}}, \mathbb{Z}_p))(\mathbb{Z}_p) = GL(L^*_{(p)}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Z}_p)(\mathbb{Z}_p)$$

factors through the $\mathbb{Z}_p$-valued points of a Borel subgroup B of $G_{\mathbb{Z}_p}$, for instance cf. [Va2, 4.8.2] applied to $(1_{GSp(L_{(p)},\psi)}, L_{(p)}, p)$. In fact the proof of loc. cit. shows that we can replace B by a subgroup B^1 of it which is the closure in $G_{\mathbb{Z}_p}$ of the connected component of the origin of the intersection of $G_{\mathbb{Q}_p}$ with the generic fibre of a subgroup B^1_{GSp} of $GSp(L^*_{(p)}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Z}_p)$. Here, cf. (the proof of) loc. cit., B^1_{GSp} is the extension of a one dimensional split torus by an abelian unipotent group N. Of course, to be able to pass to the mentioned connected component of the origin, we might have to pass to a DVR finite faithfully flat cover of V_i. So B^1 itself is the extension of a one dimensional split torus by an abelian unipotent group N^1, where N^1 is a subgroup of the unipotent radical N^1_{GSp} of B^1_{GSp}.

B^1_{GSp} (and so also B^1) is related to the maximal subrepresentation of ρ_i corresponding to a direct summand $H(0)$ of $H_{\text{ét}}$ on which $\mathrm{Gal}(\overline{K_{V_i}}/K_{V_i})$ acts trivially. Moreover $\mathrm{Gal}(\overline{K_{V_i}}/K_{V_i})$ acts via the inverse of its cyclotomic character on the quotient $\mathbb{Z}_p$-module $H_{\text{ét}}/H(0)$. To B (resp. B^1) corresponds (via Fontaine's comparison theory) a subgroup $\tilde{B}$ (resp. $\tilde{B}^1$) of $\tilde{G}_{W(\mathbb{F})}$, and a subgroup $\bar{B}$ (resp. $\bar{B}^1$) of $\bar{G}_{V_i}$. So $\tilde{B}$ and $\bar{B}$ are also extensions of unipotent groups by a torus. Similarly we define $\tilde{B}^1_{GSp}$ and $\bar{B}^1_{GSp}$.

We need to show that $\tilde{G}_{W(\mathbb{F})}$ and $\bar{G}_{V_i}$ are reductive groups, while $\tilde{B}$ and $\bar{B}$ are respectively Borel subgroups of them. The easiest way to check this is to use [Fa3, §6] (cf. also 3.5). To apply loc. cit., let $e_i := [V_i : W(\mathbb{F})]$, and let π_i be an arbitrary uniformizer of V_i. Let Re_i be the piano ring of $\mathbb{F}$ of convergent resonance e_i. Let $M(A_{V_i})$ be the filtered F-crystal over Re_i defined by the dual of the universal vector extension of the (p-divisible group) of A_{V_i}; here, to simplify the notations, as in loc. cit., we identify it with a free Re_i-module. Let $B^+(V_i)$ be as in loc. cit. The choice of π_i makes it naturally to be an Re_i-algebra, and so a $W(\mathbb{F})$-algebra.

In loc. cit. it is shown the existence of a $B^+(V_i)$-monomorphism (functorial w.r.t. the category of p-divisible groups over V)

$$\rho(A_{V_i}) : M(A_{V_i})\otimes_{Re_i} B^+(V_i) \to H_{\text{ét}}\otimes_{\mathbb{Z}_p} B^+(V_i)$$

compatible with Frobenius and the Galois actions. To $H(0)$ corresponds a direct summand $M(0)$ of $M(A_{V_i})$, and to $H_{\text{ét}}/H(0)$ corresponds $M(A_{V_i})/M(0)$.

Let

$$\mu : \mathbb{G}_m \hookrightarrow B^1$$

be an injective cocharacter such that it produces a direct sum decomposition $H_{\text{ét}} = H(-1) \oplus H(0)$, with $\gamma \in \mathbb{G}_m(\mathbb{Z}_p)$ acting via it as the multiplication with γ^{-i} on $H(i)$. Such a cocharacter can be chosen to induce an isomorphism between $\mathbb{G}_m$ and any torus of B^1; as B^1 is smooth it is trivial to see that such tori do exist (they are all of dimension one): we can either make use of semisimple elements, or use [SGA3, Vol. 2, p. 48], starting from the fact $B^1_{\mathbb{F}_p}$ does have such non-trivial tori.

Let $\bar{B}_{Re_i}$ be the closed subscheme of $GL(M(A_{V_i}))$ corresponding to B via $\rho(A_{V_i})$; we will see below that the notation is justified, i.e. $\bar{B}_{V_i}$ is obtained from $\bar{B}_{Re_i}$ via the $W(\mathbb{F})$-epimorphism $Re_i \twoheadrightarrow V_i$ defined naturally by the above choice of π_i (cf. [Fa3, §4], and [Va3, 2.1]). Let $\beta \in B^+(V_i)$ be as in [Fa3, §4]. Similarly to $\bar{B}_{Re_i}$, we construct $\bar{G}_{Re_i}$ as a closed subscheme of $GL(M(A_{V_i})$. We have:

Subfact. *There is an element*

$$n \in N(B^+(V_i)\left[\frac{1}{p\beta}\right])$$

such that we have:

$$\rho(A_{V_i})(M(A_{V_i}) \otimes_{Re_i} B^+(V_i)) = \mu(\beta^{-1}) n_{Re_i}\big(H_{\text{ét}} \otimes_{\mathbb{Z}_p} B^+(V_i)\big). \tag{2}$$

Proof: As $\rho(A_{V_i})(M(A_{V_i}) \otimes_{Re_i} B^+(V_i)) \supset \beta H_{\text{ét}} \otimes_{\mathbb{Z}_p} B^+(V_i)$ (see [Fa3, §6]), we deduce that

$$\mu(\beta^{-1})(H(0) \otimes_{\mathbb{Z}_p} B^+(V_i)) = M(0) \otimes_{Re_i} B^+(V_i)$$

and that

$$\mu(\beta^{-1})(H_{\text{ét}}/H(0) \otimes_{\mathbb{Z}_p} B^+(V_i)) = M(A_{V_i})/M(0) \otimes_{Re_i} B^+(V_i).$$

So such an element n_{Re_i} for which (2) holds does exist in $N_{GSp^1}(B^+(V_i)\left[\frac{1}{p\beta}\right])$. Remarking that we have a direct product $N^1_{GSp} = N^1 \times N^2$ of abelian unipotent groups, we deduce that $n_{Re_i} = n_1 n_2$, with $n_i \in N^i(B^+(V_i)\left[\frac{1}{p\beta}\right])$. But B^1 acts on the Lie algebra of N^2 via its quotient B^1/N^1. This implies that $n_2 \in N(B^+(V_i))$. So we can take $n = n_1$. This proves the subfact.

Obviously (2) implies that $\bar{G}_{Re_i}$ is a reductive subgroup of $GL(M(A_{V_i}))$, that $\bar{B}_{Re_i}$ is a Borel subgroup of it, and that $\tilde{B}_{W(\mathbb{F})}$ is a Borel subgroup of $\tilde{G}_{iW(\mathbb{F})}$.

To end the proof of Fact 5 we can just follow (copy) either [Va1, 5.2-4] or [Va3, the proof of 2.4 and 2.4.1-2].

W) The second key point is:

Proposition. *The abelian variety we get naturally (by pulling back $\mathcal{A}_{H_0}$) over the generic point of $\mathcal{C}_i$ is ordinary.*

Proof: We will use toroidal induction on $m \in \mathbb{N}$: the case of small values $m \in \{1,3,5,6\}$ is already settled by Step 4. For a full formalism (and extra applications) of such type of induction see [Va10]. Here we will restrict just to what we need. Let $l \in 4\mathbb{N}$ be an integer relatively prime to p. We can assume that $H_0 \times H = G(\mathbb{A}_f) \cap K(l)$, where $K(l) := \{g \in GSp(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}) | g \text{ is congruent to 1 mod l}\}$. Let

$$\mathcal{M}^{\mathrm{stc}}/K(l)$$

be a smooth projective toroidal compactification of $\mathcal{M}/K(l)$, cf. [FC, ch. IV]. From [Har, 3.4] we deduce that we can assume that the normalization of the Zariski closure of $\mathrm{Sh}_{H_0 \times H}(G,X)$ in $\mathcal{M}^{\mathrm{stc}}_{\mathbb{Q}}/K(l)$ is a smooth projective toroidal compactification of the quasi-projective $\mathbb{Q}$–scheme $\mathrm{Sh}_{H_0 \times H}(G,X)$; loc. cit. states only the functorial aspect of this statement, but the finite part is an easy consequence of the fact that in [Har, 3.3.4] we get finite morphisms. Let $\mathcal{N}^{\mathrm{tc}}/H_0$ be the normalization of the Zariski closure of $\mathrm{Sh}_{H_0 \times H}(G,X)$ in $\mathcal{M}^{\mathrm{stc}}/K(l)$. From [FC, ch. IV] we deduce that we can assume the existence of a semi-abelian variety $\tilde{A}(l)$ over $\mathcal{M}^{\mathrm{stc}}/K(l)$ extending the universal abelian scheme $\mathcal{A}(l)$ over $\mathcal{M}/K(l)$. Here $\mathcal{A}(l)$ is the pull back of $\mathcal{A}(4)$ to $\mathcal{M}/K(l)$.

We recall that the boundaries of $\mathrm{Sh}_{H_0 \times H}$ w.r.t. $\mathcal{N}^{\mathrm{tc}}_{\mathbb{Q}}/H_0$ (resp. the boundaries of $\mathcal{M}/K(l)$ w.r.t. $\mathcal{M}^{\mathrm{stc}}/K(l)$) are related to parabolic subgroups of G (resp. of $GSp(W,\psi)$). But any parabolic subgroup $P(G)$ of G extends to a parabolic subgroup of $G_{\mathbb{Z}_{(p)}}$. Moreover, as G^{ad} is absolutely simple and obtained using the Weil restriction from F to $\mathbb{Q}$ (with F as usual) of an F-group $\tilde{G}^F$, the image of $P(G)$ in G^{ad} is obtained using the same process from a parabolic subgroup $\tilde{P}^F$ of $\tilde{G}^F$. We deduce immediately that these boundaries are related to $\mathbb{Z}_{(p)}$ embeddings of Shimura varieties of dimension smaller than d into Siegel modular varieties. Moreover these Shimura varieties either do not have factors of some B_q or $D_q^{\mathbb{R}}$ type, or they do have such factors but with $q < m$. So these $\mathbb{Z}_{(p)}$ embeddings related to boundaries are SHS, by the induction argument and by what we already proved in the previous Steps.

Moreover, the reflex fields of these Shimura varieties related to the boundaries are all equal to $\mathbb{Q}$: this is a consequence of the fact that the construction of these toroidal compactifications is over $\mathbb{Q}$ (of course we can as well argue this starting from [Mi1, 2.5 and 2.7 of Ch. V] and 2.3.9.1 D)). However for what follows a weaker form of this statement is enough: using 2.3.9.2 we deduce that we can assume that F is a Galois cover of $\mathbb{Q}$, and that $\tilde{G}^F$ is obtained from a group $\bar{G}^F$ over $\mathbb{Q}$ by extension of scalars; moreover we can assume that $\bar{G}^F$ splits over a totally imaginary quadratic extension of $\mathbb{Q}$ in which p splits. So all these reflex fields are either $\mathbb{Q}$ or a totally imaginary quadratic extension of it in which p splits. We deduce:

Fact 6. *All the $\mathbb{Z}_{(p)}$ embeddings related to the boundaries are involving primes (of some reflex fields) having $\mathbb{F}_p$ as their residue fields.*

From Fact 6, [Va2, P1 of 4.6] and [FC, §2 of ch. V and 5.8 of p. 175] we get immediately:

Fact 7. *The semi-abelian varieties over the generic points of the special fibre of the complement of $\mathcal{N}/H_0$ in $\mathcal{N}^{\mathrm{tc}}/H_0$ are all extensions of an ordinary abelian variety by a torus.*

As $\mathcal{U}$ is smooth, we deduce from the theorem of formal functions, that the special fibre $\overline{\mathcal{S}}$ of $\mathcal{N}^{\text{tc}}/H_0$ is connected. Using Fact 5, we deduce that we can assume that the Zariski closure $\mathcal{C}_i^{\text{tc}}$ of $\mathcal{C}_i$ in $\overline{\mathcal{S}}$ intersects the Zariski closure of $\mathcal{U}_0$ in $\overline{\mathcal{S}}$; from the intersection theory we deduce that the resulting intersection $\mathcal{S}^1$ is of codimension 1 in $\overline{\mathcal{S}}$. Let

$$z_k : \operatorname{Spec}(k[[x]]) \to \mathcal{C}_i^{\text{tc}}$$

be a morphism whose generic point sits over the generic point of $\mathcal{C}_i^{\text{tc}}$, and whose special point sits over a generic point of $\mathcal{S}^1$; here k is a suitable algebraically closed field containing $\mathbb{F}$. Let $\tilde{A}$ be the semi-abelian scheme over $k[[x]]$ we get naturally. From the above fact and from [FC, 5.11 of p. 70] we get immediately that the generic fibre of $\tilde{A}$ is an ordinary abelian variety. This ends the proof of the proposition.

X) V) and W) together handle the last (left) case mentioned in U). So this ends the proof of 4.1.1-2.

Y) Step 6. The case $p \geq 5$ revisited. Everything we did for $p = 3$ works out as well for $p \geq 5$. So we obtain a second proof of 4.1.1-2 which does not rely on [Va1, 5.1 or 6.4.1]. This second proof is, in our opinion, somehow simpler. First we recall that for the proof of [Va1, 5.1] we used Theorem 1 of 2.4 for homogeneous tensors of degree 4, for $G_{\mathbb{Z}_p}$ a reductive group whose adjoint has all the simple factors of classical Lie type, and for $e \in \mathbb{N}$ arbitrary. Moreover we used the de Rham conjecture (see [Fa1, 8.1]) in the context of abelian varieties over an arbitrarily ramified DVR extension of $W(\mathbb{F})$, in the form of its application of [Bl, (0.3)]. For the second proof, we used [Fa3, §4-6] only as far as it is mentioned in 2.4.3.1 and in V) above, and we used the de Rham conjecture only for abelian varieties over a Witt ring, in the form of the same application of it [Bl, (0.3)].

However, in situations where we can not be very flexible in applying the tricks of 2.3 (without perhaps quite a lot of work), the first proof can be still more convenient (for instance see 5.2.1 and 5.2.2 1) below; see also 5.2.2 6)).

4.2.1. Remarks. 1) The method of Step 4 can be entirely adapted to the general case (i.e. compact or non-compact) C_n and $D_n^{\mathbb{H}}$ type. The only difference is: the perfect bilinear forms $b_{F,i}^{\text{crys}}$ might be taking values in $W(\mathbb{F})$ as well.

2) The method of Step 5 can be used as well in the following list of non-compact types: C_n type, or $D_n^{\mathbb{H}}$ type with trivial involution, or $D_n^{\mathbb{H}}$ type with non-trivial involution but working with a prime of the reflex field whose residue field has p elements.

3) 2.3.9.1 together with Step 5 of 4.2 represents another way to handle the case of B_n or $D_n^{\mathbb{R}}$ compact type.

4) 2.3.6 12) together with 3) above (and 2.3.6.1 2)) show that we can often restrict our attention (for instance in connection with problems related to 4.1) to orthogonal Shimura varieties which are (depending on our test) either of B_n type or of $D_n^{\mathbb{R}}$ type.

5) 4.2 W) can be performed as well in the context of minimal compactifications (cf. [Va10] and [FC, §2 of ch. V]). Moreover, in [Va10] we will see, using an entirely similar toroidal induction, that $\mathcal{N}^{\text{tc}}/H_0$ of 4.2 W) is actually smooth, and so in fact $\mathcal{C}_i$ of 4.2 U) does not exist.

6) The remarks 1) and 3) above imply that the use of 2.4.2 is in fact entirely avoidable: just the proof of the Proposition of 4.2 W) has to be slightly modified; once this is

performed, we can argue the integrality of some de Rham cycles in the de Rham cohomology of $\mathcal{A}_{\mathbb{C}}$, using arguments in codimension one. However we do believe that 2.4.2 brings more light to the situation.

7) Warning: Step 4 of 4.2 is very close to the case of a standard PEL situation; however in general we do not have such a situation.

8) It would be very much desirable to have a more abstract and of principal approach to 4.2 W). We think such an approach is possible (see 5.4.1) by a refined analysis of the 3-divisible objects of $\mathcal{MF}_{[-1,1]}(Re)$ (with Re as in 2.4) endowed with a (non a priori perfect) symmetric quasi-polarization, corresponding to a perfect symmetric form in the étale $\mathbb{Z}_3$ context.

4.3. Conclusions. Before moving to extensions of 4.1.1-2 and to varied applications of them, some conclusions are in order. First let us point out that in the third part we will see that in 4.1.1 we do not need to take the normalization (cf. [Va1, 5.6.4]).

Second we recall briefly how 1.4.2 follows from 4.1.1. We follow very closely [Va1]. Following [Va1, 6.4.1, 6.4.1.1 2), 6.8.8] and [Va2, 4.6.11] we just need to show that [Va1, 6.4.2] has a variant even for $p=3$. We have:

4.3.1. Theorem. *For any adjoint Shimura quadruple (G_1, X_1, H_1, v_1) of preabelian type, there is an injective map $f\colon (G,X) \hookrightarrow (GSp(W,\psi), S)$ such that:*

i) there is a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W good w.r.t. f;

ii) denoting as usual $H := G(\mathbb{Q}_p) \cap GSp(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p, \psi)(\mathbb{Z}_p)$, there is a prime v of $E(G,X)$ dividing p and such that the adjoint quadruple of (G,X,H,v) is (isomorphic to) (G_1, X_1, H_1, v_1);

iii) for any Shimura pair (G_2, X_2) of abelian type having (G_1, X_1) as its adjoint, we have a natural isogeny $G^{\mathrm{der}} \to G_2^{\mathrm{der}}$.

Proof: The case $p \geq 5$ is handled in [Va1, 6.4.2]. We would like to point out that for such primes, 4.3.1 is weaker than loc. cit. (cf. also [Va2, 2.3.8 1)]). Moreover, using 2.3.7.2 and trick three (as in [Va1, 6.5.1]) we get a proof of 4.3.1 for all odd primes at once.

We think it is instructive to include a much simpler second proof of 4.3.1 (than the above one relying on 2.3.7.2) which works for all primes $p \geq 3$ at once.

First [De1, 2.3.10] and [Va1, Example 3 of 2.5]) guarantees the existence of an injective map

$$\tilde{f}\colon (\tilde{G}, \tilde{X}) \hookrightarrow (GSp(\tilde{W}, \tilde{\psi}), \tilde{S})$$

such that $(\tilde{G}^{\mathrm{ad}}, \tilde{X}^{\mathrm{ad}}) = (G_1, X_1)$, and $\tilde{G}^{\mathrm{der}}$ is maximal for the abelian case in the sense of iii) above. As $\tilde{G}^{\mathrm{der}}$ and $GSp(\tilde{W}, \tilde{\psi})$ are unramified over $\mathbb{Q}_p$, we deduce that the centralizer C of $\tilde{G}^{\mathrm{der}}$ in $GSp(\tilde{W}, \tilde{\psi})$ splits over $B(\mathbb{F})$. From [Ti1, 1.10] we deduce that $C_{\mathbb{Q}_p}$ has a maximal torus unramified over $\mathbb{Q}_p$. From the proof of [Va1, 3.1.4], we deduce that we can assume that $\tilde{G}$ is unramified over $\mathbb{Q}_p$. So let $(\tilde{G}, \tilde{X}, \tilde{H}, \tilde{v})$ be a Shimura quadruple having (G_1, X_1, H_1, v_1) as its adjoint. So now [Va2, 2.3.8.1] applies (via [Va1, 6.7.2]) directly. This proves the theorem.

Third we state a new corollary. We consider a weak SPTS (f, v_1). We use the notations of 2.3.2. From the Main Corollary of 1.4.2 and from [Va1, 3.2.12] we deduce that

$\mathcal{N}$ and $\mathcal{N}_1$ are integral canonical models. We consider also a finite map $(G_2, X_2, H_2, v_2) \to (G_1, X_1, H_1, v_1)$. Let $\mathcal{N}_2$ be the integral canonical model of (G_2, X_2, H_2, v_2). From [Va1, 3.2.7 4)] we deduce the existence of a natural morphism

$$m_2 \colon \mathcal{N}_2 \to \mathcal{N}_{O_{(v_2)}}.$$

We have:

4.3.2. Corollary. *m_2 is a pro-finite morphism.*

Proof: We already know that the natural morphism $m_1 \colon \mathcal{N}_1 \to \mathcal{N}_{O_{(v_1)}}$ is a pro-finite morphism. In fact the proof of [Va1, 3.4.1] shows that m_1 is finite. We just need to show that the natural morphism $\mathcal{N}_2 \to \mathcal{N}_{1O_{(v_2)}}$, through which m_2 factors, is pro-finite. But [Va1, 6.4.5 b)] and [Va2, 4.6.11] can be entirely adapted to this situation, even for $p = 3$. This proves the corollary.

4.3.3. Remarks. 1) 4.3.2 has obviously a variant for Shimura triples; it will not be stated here.

2) As mentioned in [Va2, 2.3.11], all the local deformation theory presented in [Va1, 5.4] applies for the SHS $(f, L_{(p)}, v)$ of 4.1.2. So [Va1, 5.5.1] applies as well.

3) Of course, one can be more precise in getting 2.3.7.2 or 4.3.1. For instance, the $\mathbb{Z}_{(p)}$ version (see [Va1, 6.5.1.1]) of [De2, 2.3.10] remains true for $p = 3$ as well: using 4.1.2, we just need in [Va1, 6.5.1.1] to substitute its d) part accordingly: there is a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W such that $\psi : L_{(p)} \otimes_{\mathbb{Z}_{(p)}} L_{(p)} \to \mathbb{Z}_{(p)}$ is perfect, and the triple $\mathcal{T} := (f, GSp(L_{(p)}, \psi), v)$ is a $\mathbb{Z}_{(p)}$ embedding, for any prime v of $E(G, X)$ dividing p.

4) Using 4.1.1-2 and the Galois characterization of toric points (see [Va2, 4.5.12.2.1, 4.9.17.2-3]), we get immediately the following supplement to [Va2, 4.11]:

Fact. *The toric points have a functorial aspect with respect to $\mathbb{Z}_{(p)}$ embeddings (or maps) in the context of Shimura varieties of preabelian type.*

5) 4.1.2 has many logical variants in the larger context of $\mathbb{Z}_{(p)}$ embeddings (see [Va5]).

§5. The case of non-hyperspecial subgroups

The Second Main Theorem is the maximum one can say in the context of $\mathbb{Z}_{(p)}$ embeddings (with p odd) into Siegel modular varieties. A natural question can be put: what about other situations? These other situations could mean:

A) the case when G is unramified but we are not in the context of a $\mathbb{Z}_{(p)}$ embedding;

B) the case when G is ramified over $\mathbb{Q}_p$, but the prime v is still unramified over p;

C) the case when v is ramified over p;

D) we move from Shimura varieties of preabelian type to mixed Shimura varieties (of preabelian type) (see [Mi1, §6]);

E) we move from Shimura varieties of preabelian type to Shimura varieties of special type.

In what follows we treat three situations pertaining to A) and B). They form an incipient general theory of the integral models of Shimura varieties w.r.t. some (not necessarily hyperspecial) subgroups, and we hope they will be usable to get full control in connection to C) as well. This hope is supported as well by 3.5: see 5.4 for an application and guiding general ideas. For the case D) in the contexts which can be reduced to one of a $\mathbb{Z}_{(p)}$ embedding see [Va5] and [Va10] (cf. also [Va2, 1.16] and [Va1, 1.8]). For E) see 1.8. In the whole of §5, without explicitly stated, we assume $p \geq 3$.

5.1. The quasi-standard Hodge situation. Let p, (W,ψ), $L_{(p)}$, K, f, H, v, $\mathcal{M}$, $\mathcal{N}$ and $(v_\alpha)_{\alpha\in\mathcal{J}'}$ have the same significance as in 4.2. We assume that G is unramified over $\mathbb{Q}_p$, and that H is a hyperspecial subgroup of $G(\mathbb{Q}_p)$. But we do not assume that the Zariski closure $G_{\mathbb{Z}_{(p)}}$ of G in $GSp(L_{(p)},\psi)$ is a reductive group. Plenty of such situations can be constructed (for instance, starting from [Va1, example of the proof of 3.1.2.1 b)], we can choose G to be torus). We refer to the triple $(f, L_{(p)}, v)$ as a quasi-standard Hodge situation, to be abbreviated as QSHS.

As in [Va2, 2.3.4], to each morphism $z\colon \mathrm{Spec}(W(k)) \to \mathcal{N}$, with k a perfect field, we attach a quintuple $\mathcal{C}_z = (M, F^1, \phi, \tilde{G}_{W(k)}, (t_\alpha)_{\alpha\in\mathcal{J}'}, p_M)$; it is a principally quasi-polarized filtered σ-crystal with (crystalline) cycles. We have:

5.1.1. Fact. *$\tilde{G}_{W(k)}$ is not a reductive group.*

Proof: If it would be, as in [Va2, 2.3.10] we get that $\mathcal{C}_z$ is a principally quasi-polarized Shimura filtered σ-crystal. From [Va4, V3 of 3.2 or 3.1.1] we deduce that $G_{\mathbb{Z}_p}$ is a reductive subgroup of $GSp(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p)$. Contradiction.

This fact does not stop us from asking:

Q11. Is it true that, in case $k = \bar{k}$, there is an isomorphism

$$L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(k) \tilde{\to} M$$

taking v_α to t_α, for any $\alpha \in \mathcal{J}'$, and taking $\tilde{\psi}$ (defined as in [Va2, 2.3.1]) into p_M?

In 6.3.1 we will see that the answer to Q11 is always yes.

5.2. The quasi-hyperspecial case. Let p, (W,ψ), $L_{(p)}$, K, f, H, v, $\mathcal{M}$, $\mathcal{N}$ and $(v_\alpha)_{\alpha\in\mathcal{J}'}$ have the same significance as in 4.2. We assume that v is unramified over p. So the Witt ring $W(k(v))$ is the ring of integers of E_v. As before we define $G_{\mathbb{Z}_{(p)}}$. We do not assume $G_{\mathbb{Z}_{(p)}}$ to be a reductive group; but we do assume that there is a reductive normal subgroup $\tilde{G}_{\mathbb{Q}_p}$ of $G_{\mathbb{Q}_p}$ such that we have:

i) the Zariski closure $\tilde{G}_{\mathbb{Z}_p}$ of $\tilde{G}_{\mathbb{Q}_p}$ in $G_{\mathbb{Z}_p}$ is a reductive group over $\mathbb{Z}_p$;

ii) there is an injective cocharacter $\mu\colon \mathbb{G}_m \hookrightarrow \tilde{G}_{W(k(v))}$ such that, under an inclusion of $W(k(v))$ in $\mathbb{C}$ extending the natural inclusion $O_{(v)} \hookrightarrow \mathbb{C}$, when viewed as a cocharacter of $G_{\mathbb{C}}$, is $G(\mathbb{C})$-conjugate to one of the cocharacters μ_x^* of [Va2, 2.3.1] (here $x \in X$).

We refer to H as a quasi-hyperspecial subgroup of $G(\mathbb{Q}_p)$ w.r.t. (G,X). We also assume that:

iii) there is an injective map $(T,\{h\}) \hookrightarrow (G,X)$, with T a torus, such that the Zariski closure $T_{\mathbb{Z}_p}$ of $T_{\mathbb{Q}_p}$ in $G_{\mathbb{Z}_p}$ is a torus over $\mathbb{Z}_p$;

iv) (G,X) is of compact type;

v) there is a family of homogeneous tensors $(s_\beta)_{\beta\in\tilde{\mathcal{J}}}$ of $\mathcal{T}(L^*_{(p)}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Z}_p)$ fixed by $G_{\mathbb{Q}_p}$ and which is $\mathbb{Z}_p$-well positioned w.r.t. $\tilde{G}_{\mathbb{Q}_p}$.

In many situations iii) is automatically implied by i) and ii) (to be compared with [Va1, 3.1.4]). Warning: we do not assume any restriction on the degrees of s_β's.

If G is unramified over $\mathbb{Q}_p$, then we have

$$G(\mathbb{Q}_p) = G(\mathbb{Q})H, \tag{3}$$

cf. [Mi2, 4.10]. But in general (3) does not hold (if $G^{\mathrm{ad}}_{B(\mathbb{F})}$ is not a product of absolutely simple groups, then it is easy to see that $G(\mathbb{Q})$ is not dense in $G(\mathbb{Q}_p)$). The equality (3) is equivalent to the following fact: the connected components of $\mathrm{Sh}_H(G,X)_{\mathbb{C}}$ are permuted transitively by $G(\mathbb{A}_f^p)$; the argument for this is entirely the same as the one of [Va1, 3.3], as [Mi2, 4.11] is true as well for the case when H is not necessarily hyperspecial, provided (3) holds. Moreover, if (3) holds, then the last part of [Va1, 5.1.2] involving different identifications of some $\mathbb{Z}_p$-modules holds in the present context.

So if (3) does not hold, the identification (similar to the one of [Va1, 5.1.2]) of some H^1-étale groups with coefficients in $\mathbb{Z}_p$ with $L^*_{(p)}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Z}_p$ is not always possible. However, from the way (see [Va1, p. 454]) $G(\mathbb{A}_f^p)$ acts on the set $\mathcal{A}(G,X,W,\psi)$ introduced in loc. cit., we deduce that still we can identify such H^1-étale groups with coefficients in $\mathbb{Z}_p$ with other suitable $\mathbb{Z}_p$-lattices $L_p'^*$ of $W^*\otimes_{\mathbb{Q}}\mathbb{Q}_p$, of the form $g(L^*_{(p)}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Z}_p)$, for some $g\in G(\mathbb{Q}_p)$. Here by "suitable" we mean the following fact: as $g\in G(\mathbb{Q}_p)$, the closure of $\tilde{G}_{\mathbb{Q}_p}$ in $GL(L_p'^*)$ is still a reductive group over $\mathbb{Z}_p$ (isomorphic to $G_{\mathbb{Z}_p}$). By passing from $L_{(p)}$ to the dual $L'_{(p)}$ of $L'^*_{(p)} := L_p'^*\cap W$, ψ gets replaced by a $\mathbb{G}_m(\mathbb{Z}_p)$-multiple of $g(\psi)$ (cf. loc. cit.). But, as in [Va2, 2.3.14], from the point of view of Q11 and of well positioned families of tensors this is irrelevant. So in 5.2.1 below we will not stop to emphasize that we are dealing with $L_{(p)}$ or with some $L'_{(p)}$.

Ideally would be to have $L'_p=\tilde{g}(L^*_{(p)}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Z}_p)$, with $\tilde{g}\in G(\mathbb{Q}_p)$ centralizing $\tilde{G}_{\mathbb{Z}_p}$. The possibility of choosing such an element $\tilde{g}$ is closely related to Q12 below. To state it, we first consider the natural homomorphism

$$q : G(\mathbb{Q}_p)\to \tilde{G}^{\mathrm{ad}}_{\mathbb{Z}_p}(\mathbb{Q}_p).$$

Q12. Is it true that $\mathrm{Im}(q) = q(G(\mathbb{Q}))G^{\mathrm{ad}}_{\mathbb{Z}_p}(\mathbb{Z}_p)$?

It seems to us that Q12 always holds (to be compared with the proof of [Mi2, 4.10]). Here we will just mention that: the answer to Q12 is obviously yes if G is unramified over $\mathbb{Q}_p$.

We have:

5.2.1. Theorem. *$\mathcal{N}$ is a smooth integral model having the EEP.*

Proof: As in the proof of [Va1, 5.1], we just need to prove that $\mathcal{N}$ is formally smooth over $O_{(v)}$. We follow the pattern of 4.2. As in 4.2 A) we move to a finite level over $W(\mathbb{F})$, w.r.t. a suitable compact open subgroup H_0 of $G(\mathbb{A}_f^p)$. In other words we prove that $\mathcal{Q} := \mathcal{N}/H_{0W(\mathbb{F})}$ is smooth over $W(\mathbb{F})$. As in [Va1, 6.4.1 2)] we deduce that $\mathcal{Q}$ is a projective normal scheme. Let $(v_\alpha)_{\alpha\in\mathcal{J}^1}$, with $\mathcal{J}^1$ a set containing $\mathcal{J}'\cup\tilde{\mathcal{J}}$, be an enlarged family of homogeneous tensors of $\mathcal{T}(L_{(p)}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Q}_p)$ such that $\tilde{G}_{\mathbb{Q}_p}$ is the subgroup of $GL(L_{(p)}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Q}_p)$ fixing its members; of course we assume that for $\beta\in\tilde{\mathcal{J}}$ we have $v_\beta = s_\beta$. Let T_1 be the maximal subtorus of $T_{\mathbb{Z}_p}$ contained in $\tilde{G}_{\mathbb{Z}_p}$.

We follow very closely the first three Steps of 4.2. Let H_T and v_T be as in 4.2 C). Compared to the mentioned part of 4.2, only two changes need to be made. Let $m_T : \mathrm{Spec}(W(\mathbb{F})) \to \mathcal{C}$ be as in Step 1 of 4.2. Let $(M, F^1, \phi, T_{W(\mathbb{F})}, p_M)$ be the principally quasi-polarized Shimura filtered σ-crystal attached to a morphism $\mathrm{Spec}(W(\mathbb{F})) \to \mathcal{T}$ used in defining m_T. From ii) we get that $(M, F^1, \phi, T_{1W(\mathbb{F})}, p_M)$ is still a principally quasi-polarized Shimura filtered σ-crystal. Using Theorem 5 of 2.4, we deduce that $\mathcal{C}_{\mathcal{F}} := (M, F^1, \psi, \tilde{G}_{W(\mathbb{F})}, (u_\alpha)_{\alpha\in\mathcal{J}^1}, p_M)$ is also a principally quasi-polarized Shimura filtered σ-crystal. Here u_α corresponds via Fontaine's theory to v_α, $\alpha\in\mathcal{J}^1$. Now we need to deform $\mathcal{C}_{\mathcal{F}}$ as in 4.2 D). The rest applies entirely: we get $\mathcal{U}$, a Main Claim, etc. It is iv) and v) above which show that Step 3 of 2.4 applies. This ends the proof of the theorem.

5.2.2. Comments. 1) No doubt conditions iii), iv) and v) can be considerably weaken (cf. also 3.3 4)). For instance, if $\deg(v_\alpha) \le 2(p-2)$ $\forall\alpha\in\tilde{\mathcal{J}}$, the proof of [Va1, 5.1] can be entirely adapted to get a proof of 5.2.1, without assuming iii) and iv) (strictly speaking, the non-compact case is not interesting). Moreover, in such a situation, the condition $p\ge 3$ is not needed here: the extra restriction on the degrees of v_α's implies that we are in a situation similar to the ones of 2.3.4 (or of 2.3.5.5).

We included the above proof of 5.2.1 using the above conditions iii) to v), as it is closer to the spirit of this paper, and as it handles (to a much greater extend) the case $p=3$ as well. However, as the proof of 5.2.1 reveals, we can substitute condition iii) by

iii') the Zariski closure T_1 of the intersection of $T_{\mathbb{Q}_p}$ with $\tilde{G}_{\mathbb{Q}_p}$ in $\tilde{G}_{\mathbb{Z}_p}$ is a maximal torus of $\tilde{G}_{\mathbb{Z}_p}$, and the normalization of $\mathrm{Sh}_{H_T}(T,\{h\})_{B(\mathbb{F})}$ in $\mathcal{N}_{W(\mathbb{F})}$ has a $W(\mathbb{F})$-valued point.

2) It is an easy exercise (starting from [Va1, 5.8.6]) to see that v) is automatically satisfied if $p\ge \dim_{\mathbb{Q}}(W)$.

3) The answer to Q11, in the situation described in 5.2.1 (or in 5.2.2 1)), is yes (cf. its proof and [Va4, V1 of 3.2]).

4) $\mathcal{N}$ above is the integral canonical model (over $O_{(v)}$) of $\mathrm{Sh}_H(G,X)$ (cf. def. [Va1, 3.2.3 6)]).

5) Of course 5.2.1 and 5.2.2 1) as well as [Va1, 6.4] can be extended along the usual lines (see [Va1, §6]) to situations where (G,X) is not necessarily of Hodge type. We will not state these extensions here. We will just mention that $\mathcal{N}$ is strongly smooth, and projective (resp. quasi-projective) in 5.2.1 (resp. in 5.2.2 1)). Moreover the whole of [Va2, 4.1-8] can be easily adapted to this situation. In particular, we get Shimura-ordinary types

and points, Shimura-canonical lifts, etc. (the density part of [Va2, 4.2.1] being preserved as well).

6) A great part of the tricks of 2.3 can be adapted to the situation of 5.2.1 or of 5.2.2 1): also this will not be detailed here. Just one detail: we need to work with totally real number fields F_1 containing F and unramified above some fixed set of primes of F dividing p; for instance in the context of 9) below we always need to take F_1 unramified above v.

7) v) can be stated as well in a $W(\mathbb{F})$-context, instead of a $\mathbb{Z}_p$ context, cf. [Va1, 5.6.10]. Also, it is worth pointing out that in v) above we requested the s_β's to be fixed by $G_{\mathbb{Q}_p}$. Of course this hypothesis can be removed: we just have to give an argument why in a situation as in the theorem, we can always specialize the de Rham counterparts of all these tensors (corresponding to $\alpha \in \mathcal{J}^1$), as in Fact 2 of 2.4.2. The reason is: locally in the pro-étale topology of $\mathcal{N}$, we can assume that the p-adic Galois representations involved factor through the group of $\mathbb{Z}_p$-valued points of $\tilde{G}_{\mathbb{Q}_p}$. However, we can not see any use of this: usually we get that v) holds in the $W(\mathbb{F})$ context by assuming that the faithful representation $\tilde{G}_{\mathbb{Z}_p} \hookrightarrow GL(L_{(p)})$ is a $DEG4$ representation, which can be "encoded" using a family of homogeneous tensors (as in 2.3.7.3) fixed by $G_{\mathbb{Q}_p}$.

8) Often we can pass from H to an open subgroup of it: this can be achieved by changing the pair $(f, L_{(p)})$ accordingly; however we always have to take

$$H := G(\mathbb{Q}_p) \cap GSp(L_{(p)}, \psi).$$

The process of changing the pair $(f, L_{(p)})$ goes as follows. Let $\tilde{\tilde{G}}_{\mathbb{Q}_p}$ be the semisimple subgroup of $G^{\mathrm{der}}_{\mathbb{Q}_p}$ such that we have a natural isogeny

$$\tilde{\tilde{G}}_{\mathbb{Q}_p} \times \tilde{G}^{\mathrm{der}}_{\mathbb{Q}_p} \to G^{\mathrm{ad}}_{\mathbb{Q}_p}.$$

Let $g \in \tilde{\tilde{G}}_{\mathbb{Q}_p}(\mathbb{Q}_p)$. Let $L'_{(p)}$ be as in the part of 5.2 preceding 5.2.1, and obtained via it. Let $\psi' := g(\psi)$, and let

$$\tilde{H} := H \cap gHg^{-1}.$$

Let $L_{1(p)} := L_{(p)} \oplus L'_{(p)}$ and $\psi_1 := \psi \oplus \psi'$. As in trick three of 2.3, we get a diagonal embedding $f_1 : (G, X) \hookrightarrow (GSp(L_{1(p)}, \psi_1), S_1)$. The conditions needed to apply 5.2.1 (or 5.2.2 1)) are still satisfied for the triple $(f_1, L_{1(p)}, v)$: so we can replace H by its open subgroup $\tilde{H}$. A natural limit process can be applied: we can replace H by its closed subgroup

$$\bar{H} := \cap_{g \in \tilde{\tilde{G}}_{\mathbb{Q}_p}(\mathbb{Q}_p)} gHg^{-1}.$$

where G_1 is the kernel of the natural quotient map $G_{\mathbb{Q}_p} \to \tilde{\tilde{G}}^{\mathrm{ad}}_{\mathbb{Q}_p}$. So we get:

Fact. *The normalization of* $\mathcal{N}$ *in the field of fractions of* $\mathrm{Sh}_{\bar{H}}(G, X)$ *is a regular formally smooth integral model over* $O_{(v)}$ *having the EEP.*

Let G_1 be the kernel of the natural quotient map $G_{\mathbb{Q}_p} \to \tilde{\tilde{G}}^{\mathrm{ad}}_{\mathbb{Q}_p}$. If $\tilde{\tilde{G}}_{\mathbb{Q}_p}(\mathbb{Q}_p)$ has no normal subgroups besides the subgroups of $Z(\tilde{\tilde{G}}_{\mathbb{Q}_p})(\mathbb{Q}_p)$, then we get immediately

$$\bar{H} \subset G_1(\mathbb{Q}_p).$$

9) The situation described in the theorem 5.2.1 is very common. We can obtain plenty of examples with G unramified over $\mathbb{Q}_p$. More interesting examples can be obtained in the following way. We take F to be a totally real number field, and we take a prime p such that:

– F has a prime v unramified over p and such that $k(v) = \mathbb{F}_p$;
– F has ramified primes over p.

The simplest situations are obtained when $[F : \mathbb{Q}] = 3$. We choose $\tilde{G}^F$ to be an absolutely simple group of some B_m Lie type, with $m \geq 2$, and such that: for precisely one embedding $F \hookrightarrow \mathbb{R}$ we have $\tilde{G}_{\mathbb{R}}$ non-compact, and moreover for this specific embedding we have $\tilde{G}_{\mathbb{R}} \tilde{\to} SO(2, 2m-1)_{\mathbb{R}}$. Let $\tilde{G} := \mathrm{Res}_{F/\mathbb{Q}}(\tilde{G}^F)$, and let $(\tilde{G}, \tilde{X})$ be the unique adjoint Shimura pair defined by $\tilde{G}$; from [BHC] we get that it is of compact type. We have $E(\tilde{G}, \tilde{X}) = F$ (cf. [De3, 3.8]). We can assume that v is such that it corresponds to the non-compact factor, i.e. the composite inclusion $F \hookrightarrow \mathbb{R} \hookrightarrow \mathbb{C}$ producing the non-compact factor $\tilde{G}^F_{\mathbb{R}}$ of $\tilde{G}_{\mathbb{R}}$, factors through the embedding $F \hookrightarrow \mathbb{Q}_p$ defined by v. Let K^F be a totally imaginary quadratic extension of F, such that $K^F_{(p)}$ is an étale $F_{(p)}$-algebra. Now it is easy to see that [De1, 2.3.10] can be adapted (as in [Va1, 6.5.1.1]) to a $\mathbb{Z}_{(p)}$ context so that i-iii) and v) above hold as well. For v) above we just need p to be big enough in comparison to $[F : \mathbb{Q}]$ (cf. 2) above). Of course, if we work with iii') instead of iii), then we can have the B_1 Lie type as well.

Moreover, the situation described in 1) above is even more common; in particular in the above discussion, as we do not impose iii), we can also allow $\tilde{G}^F$ to be of B_1 Lie type as well.

5.2.3. The adjoint type case. One can go beyond 5.2.2 5) and 6). The ideas of 4.2 can be also extended to the situations of 5.2.1 and 5.2.2 1). In particular, these situations (putting aside v), which is not needed for what we refer here) are stable under the process of standard toroidal induction of 4.2 W). So one can get easily the analogues of 4.1.1-2 as well. However there is a complication in stating them, as the subgroups H involved are of particular shape. So here we will state with no extra comment just an adjoint version of 4.1.1; the reason is: 5.2.1, 5.2.2 1), 5) and 6), and [Va1, end of 6.8.1] make the passage to other situations very smooth.

Let (G, X) be an adjoint Shimura pair of abelian type. Let v be a prime of $E(G, X)$ dividing a rational prime $p \geq 3$ and such that:

– $G_{\mathbb{Q}_p} = G^1_{\mathbb{Q}_p} \times G^2_{\mathbb{Q}_p}$, with $G^1_{\mathbb{Q}_p}$ unramified over $\mathbb{Q}_p$;
– the cocharacters $\mu : \mathbb{G}_m \to G_{E(G,X)_v}$, which over $\mathbb{C}$, under an inclusion $E(G, X)_v \subset \mathbb{C}$ extending the natural one $E(G, X) \subset \mathbb{C}$, are $G(\mathbb{C})$-conjugate with the cocharacters μ^*_x $(x \in X)$ defined in [Va2, 2.3.1], factor through $G^1_{E(G,X)_v}$.

Let H^1 be a hyperspecial subgroup of $G^1_{\mathbb{Q}_p}(\mathbb{Q}_p)$. We have:

5.2.3.1. Theorem. *There is a smooth integral model $\mathcal{N}^1$ of $\mathrm{Sh}_{H^1}(G, X)$ over $O_{(v)}$ having the EP, and which is equipped with a continuous right action by $G(\mathbb{A}^p_f) \times G^2(\mathbb{Q}_p)$. It is quasi-projective and strongly smooth. It is projective iff (G, X) is of compact type.*

5.2.3.2. Remark. As any two hyperspecial subgroups H^1 and H^2 of $G^1_{\mathbb{Q}_p}(\mathbb{Q}_p)$ are conjugate by an element of $G^1_{\mathbb{Q}_p}(\mathbb{Q}_p)$ ([Ti1, p. 47]) and so by an element g of $G(\mathbb{Q}_p)$, we get that $\mathcal{N}^1$ of 5.2.3.1, as a $O_{(v)}$-scheme, does not depend on H^1. In other words, the standard isomorphism $\mathrm{Sh}_{H^1}(G,X)\tilde{\to}\mathrm{Sh}_{H^2}(G,X)$ defined by g (see [Va1, 2.3]), extends uniquely, due to the EP of $\mathcal{N}^1$ and of $\mathcal{N}^2$ (obtained working with H^2), to an $O_{(v)}$-isomorphism $\mathcal{N}^1\tilde{\to}\mathcal{N}^2$ respecting the $G(\mathbb{A}^p_f)$-actions.

5.3. A more general situation. A naturally question arises: how much more we can go beyond 4.1.1-2, 5.2.1, and 5.2.2 1) and 3)? Here we start approaching this question. We start following 5.2. We assume 5.2 ii), but we weaken 5.2 i); we just assume:

i') the normalization $\tilde{G}_{\mathbb{Z}_p}$ of the Zariski closure of $\tilde{G}_{\mathbb{Q}_p}$ in $GL(L_{(p)}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Z}_p)$ is a smooth group over $\mathbb{Z}_p$.

We also assume that iii') holds. Let $\mathcal{N}^{\mathrm{sm}}$ be the formally smooth locus of $\mathcal{N}$. The condition iii') implies, as in 4.2 C) and D), that $\mathcal{N}^{\mathrm{sm}}$ is a faithfully flat $O_{(v)}$-scheme. Obviously $G(\mathbb{A}^p_f)$ acts on $\mathcal{N}^{\mathrm{sm}}$. So $\mathcal{N}^{\mathrm{sm}}$ is a smooth integral model of $\mathrm{Sh}_H(G,X)$ over $O_{(v)}$. As in the proof of [Va1, 6.4.1] we get that it is quasi-projective. We do regard i') and ii) as very natural conditions: we can not hope too much without them. Also iii') is a natural condition; moreover often it is implied by i') and ii).

The problem we would like to solve can be formulated nicely as four questions. They are formulated here in their (decreasing) order of importance.

Q13. What extra conditions assure that $\mathcal{N}=\mathcal{N}^{\mathrm{sm}}$?

Q14. What extra conditions assure that $\mathcal{N}^{\mathrm{sm}}_{\mathbb{F}}$ is a closed subscheme of $\mathcal{N}_{\mathbb{F}}$?

Q15. What extra conditions assure that $\mathcal{N}^{\mathrm{sm}}$ has the SEP?

Q16. What extra conditions assure that the answer to Q11 is true in the context of $W(\bar{k})$-valued points of $\mathcal{N}^{\mathrm{sm}}$?

We do not know how to handle Q13-4: as soon as we move out from a reductive context, it becomes very hard to deal with the questions Q17-8 below, which are very closely related to Q13-4. To formulate them, let R be an integral, normal, faithfully flat $\mathbb{Z}_{(p)}$-algebra. Let M be a free R-module of finite rank, and let G_1 be a reductive subgroup of $GL(M[\frac{1}{p}])$. Let $\mathcal{G}_1$ be the normalization of the Zariski closure of G_1 in $GL(M)$. We assume the existence of an extensible pair $(\mathrm{Spec}(R),U_R)$ such that $\mathcal{G}_{1U_R}$ is smooth.

Q17. What extra conditions assure that $\mathcal{G}_1$ is a smooth group over R?

Q18. We assume that R is a complete DVR having an algebraically closed residue field (so $U_R=R$). Let M_1 be another R-lattice of $M[\frac{1}{p}]$ such that $\mathcal{G}_1$ is also naturally a subgroup of $GL(M_1)$. Knowing that some homogeneous tensors of $\mathcal{T}(M)$ are also tensors of $\mathcal{T}(M_1)$, what extra conditions guarantee the existence of an element $g\in GL(M\otimes_R R_1[\frac{1}{p}])$ fixed by $\mathcal{G}_{1R_1}$, and such that $g(M\otimes_R R_1)=M_1\otimes_R R_1$ (here R_1 is a faithfully flat DVR extension of R)?

Question Q18 is very intimately related to [Va3, 2.3]. So, if $\mathcal{G}_1$ is a reductive group, Q18 gets a fairly good answer by [Va3, 4.3.4.8 b) and 5.5.2]. We do not know how to sort out Q18, in the case $\mathcal{G}_1$ is not reductive.

Also, we do not know how to handle Q16 in such a generality, though [Va4, 4.2 e)] offers many hopes (for instance see Exercise 3 below). However, the question Q15 is much more easier. We have:

Exercise 2. We do not assume i'), ii) or iii'). We just assume that $\mathcal{N}^{\mathrm{sm}}$ has a non-empty special fibre, and that $p \geq 2$. Then $\mathcal{N}^{\mathrm{sm}}$ has the SEP, and so, if $p \geq 3$, it is uniquely determined by its special fibre. Hint: what is needed to show is that any $W(k)$-valued point of $\mathcal{N}$, with k a perfect fields, factors through $\mathcal{N}^{\mathrm{sm}}$; using [Wi] (as in [Va3, 4.4.4]), the fact that any faithfully flat closed subgroup of $\mathbb{G}_a^m$ over $W(k)$, with $m \in \mathbb{N}$, is smooth, and a deformation theory very closely imitating [Va1, 5.4] (see the construction of $\mathcal{C}_{N_0}$ in 5.3.2 E below), this factorization can be checked very easily.

Exercise 3. We assume that iii') holds, and that the connected components of $\mathcal{N}_{\mathbb{F}}^{\mathrm{sm}}$ are permuted transitively by $G(\mathbb{A}_f^p)$. Then the answer to Q11 in this context is yes. Hint: use [Va4, 4.2 e)] and a variant of [Va2, 2.3.13.1] obtained starting from Exercise 2.

5.3.1. Remarks. 1) The moral of Exercise 2 is: to use the SEP instead of the EP in the context of Shimura varieties (of Hodge type) is meaningless: despite the fact that for $p \geq 3$ the SEP itself gives us the uniqueness part (see [Va1, 3.2.3.1 7)]), working with SEP in the mentioned context we lose a lot of information, and moreover the problem of existence of smooth integral models having SEP is a trivial one.

2) It is worth pointing out that we can not sort out Q14 and Q11 even for a connected component of $\mathcal{N}_{\mathbb{F}}$ for which we know that Q11 holds generically. To explain this, we situate ourselves in the context of 3.4.1 a). Following its proof, we start remarking that we can not guarantee at all that $\mathrm{Spec}(R_1/pR_1)$ has only one generic point. So, if $O_1, ..., O_n$ ($n \in \mathbb{N}$) are the local rings of R_1 which are DVR's of mixed characteristic, we can assume that $i_1(H \otimes_{\mathbb{Z}_p} O_i) = M_R \otimes_R O_i$ for some fixed $i \in \{1, ..., n\}$. But we can not say anything about the (potential) O_j's, with $j \in \{1, ..., n\} \setminus \{i\}$. This motivates Q18.

5.3.2. A digression: two non-smooth variants. A. We start considering the context of 2.4, but with $V = V_0$ and $K = K_0$. We take $p \geq 2$. We assume just that $G_{\mathbb{Q}_p}$ is connected, and that ρ is attached to a p-divisible group D_{V_0} over V_0. Let $(M_0, F^1, \phi_0, (w_\alpha)_{\alpha \in \mathcal{J}})$ be the filtered σ-crystal with cycles corresponding to D_{V_0} and the family $(v_\alpha)_{\alpha \in \mathcal{J}}$, via Fontaine's comparison theory. Let G_{V_0} be the subgroup of $GL(M_0)$ which is the Zariski closure of the subgroup of $GL(M_0[\frac{1}{p}])$ fixing w_α, $\forall \alpha \in \mathcal{J}$. It is a connected group. We do not assume that it is smooth.

Let $\mu : \mathbb{G}_m \to G_{V_0}$ be the canonical split of $(M_0, F^1, \phi_0, (w_\alpha)_{\alpha \in \mathcal{J}})$, cf. [Wi]. Let $M_0 = F^1 \oplus F^0$ be the usual direct sum decomposition it produces (so $\beta \in \mathbb{G}_m$ acts via μ as the multiplication with β^{-i} on F^i). Let N be the connected unipotent abelian subgroup of $GL(M_0)$ whose Lie algebra is formed by elements on which $\beta \in \mathbb{G}_m$ acts via μ as the multiplication with β. Let N_0 be the subgroup of N of whose Lie algebra is $\mathrm{Lie}(N) \cap \mathrm{Lie}(G_{K_0})$. It is a connected smooth subgroup; its extension to $W(\bar{k})$ is isomorphic to $\mathbb{G}_a^m$, for some $m \in \mathbb{N} \cup \{0\}$. So the completion of N_0 in the origin is isomorphic with

$\mathrm{Spec}(R_m) := \mathrm{Spec}(V_0[[x_1, ..., x_m]])$. We choose the Frobenius of R_m to be compatible with σ and to take x_i into x_i^p.

B. Let R be a local faithfully flat $W(k)$-algebra such that:

– we have a decreasing filtration of R by ideals I_n, $n \in \mathbb{N}$;
– R is complete with this filtration;
– $R/I_1 = W(k)$;
– I_n/I_{n+1} is a free $W(k)$-module of finite rank;
– it is equipped with a Frobenius lift Φ_R which takes I_n into I_{n+1}, for any $n \in \mathbb{N}$;
– $I_n^2 \subset I_{n+1}$.

We refer to such a local $W(k)$-algebra R as a *GD*-ring of k (here *GD* stands for good deformation). As examples of such rings R we have the rings of formal power series in a finite number of variables with coefficients in $W(k)$, or in a piano ring of k of some resonance $e \in \mathbb{N}$, or in one of the rings to be introduced in the following definition.

C. Definition. Let $m \in \mathbb{N}$. Let $(e_1, ..., e_m)$ be an m-tuple of positive integers. The subring of $B(k)[[x_1, x_2, ..., x_m]]$ formed by formal power series

$$\sum_{n_1,\dots,n_m=0}^{\infty} a_{n_1,\dots,n_m} \prod_{j=1}^{m} x_j^{n_j},$$

with all the numbers

$$a_{n_1,\dots,n_m} \prod_{i=1}^{m} [\frac{n_i}{e_i}]! \in W(k),$$

will be called a piano ring of k of resonance $(e_1, ..., e_m)$, and will be denoted by $R_{(e_1,\dots,e_m)}(k)$. Its standard Frobenius is compatible with σ and takes x_i to x_i^p.

D. All of [Va3, 2.1] pertaining to the case $m = 1$ can be transposed to the context of an arbitrary $m \in \mathbb{N}$. First, we mention that if $(e'_1, ..., e'_m)$ is another m-tuple of positive integers such that $e'_i \le e_i$, $i = \overline{1, m}$, then we have a natural $W(k)$-monomorphism

$$R_{(e'_1,\dots,e'_m)}(k) \hookrightarrow R_{(e_1,\dots,e_m)}(k).$$

Second, we recall the motivation for the word “piano”. We need to assume $p \ge 3$. Let $l \in \mathbb{N}$ such that

$$l \le \max\{e_1, ..., e_m\}.$$

Let V_l be a totally ramified extension of $W(k)$ of degree l. Let I_l be a non-empty subset of $\in \{1, ..., m\}$. We consider a map f_{I_l} from I_l into the maximal ideal of V_l. We assume that:

– for any $i \in I_l$, $f_{I_l}(i)^{e_i}$ is divisible by p, and that the image of f_{I_l} contains a uniformizer of V_l.

We get a natural $W(k)$-epimorphism (still to be denoted by)

$$f_{I_l} : R_{(e_1,\dots,e_m)}(k) \twoheadrightarrow V_l$$

taking x_i into $f_{I_l}(i)$ or zero depending on the fact that i belongs or not to I_l.

It is also worth pointing out that $R_{(e_1,\ldots,e_m)}(k)$ is obtained from the tensor product of $R_{e_1}(k)$,..., $R_{e_m}(k)$ by a suitable completion process.

E. We consider now as in [Va2, 2.2.10] a filtered F-crystal $\mathcal{C}_{N_0}$ over R_m

$$(M_0 \otimes_{V_0} R_m, F^1 \otimes_{V_0} R_m, n_0(\phi_0 \otimes 1), \nabla_0),$$

with n_0 as the universal element of N_0 defined by the natural V_0-morphism $\mathrm{Spec}(R_m) \to N_0$.

The essence of [Fa3, rm. iii) after th. 10] and of [Va3, 2.4] can be formulated as:

F. Proposition. *We consider a deformation $\mathcal{C}_R$ of $(M_0, F^1, \phi_0, (w_\alpha)_{\alpha\in\mathcal{J}})$ over R, such that $(M_0, F^1, \phi_0, (w_\alpha)_{\alpha\in\mathcal{J}})$ is obtained via the closed V_0-morphism m_0 : Spec$(R/I_1) \hookrightarrow$ Spec(R). Then there is a unique V_0-morphism*

$$m_R : Spec(R) \to Spec(R_m)$$

such that:

a) $m_R \circ m_0$ is the closed embedding Spec$(V_0) \hookrightarrow R_m$, defined by x_i's go to zero;

b) $\mathcal{C}_R$ is isomorphic to $m_R^*(\mathcal{C}_{N_0})$, under an isomorphism inducing the logical one (identification) modulo I_1.

The proof of this is entirely the same as the proofs in the above two references.

5.3.2.1. Remark. We assume now that $p \geq 3$, $k = \bar{k}$ and that $G_{\mathbb{Q}_p}$ is a torus. Then N_0 is the trivial group. Let $T^1_{V_0}$ be the smallest connected subgroup of G_{V_0} through which μ factors, and whose Lie algebra is invariant under ϕ_0. As in the proof of [Va2, 2.2.19] we get that $T^1_{V_0}$ is a torus. From [Va2, 2.2.9 1)] we deduce that $(M_0, F^1, \phi_0, T^1_{V_0})$ is a Shimura filtered σ-crystal. From Theorem 5 of 2.4.3 we deduce:

Fact. *There is an isomorphism $H^1 \otimes_{\mathbb{Z}_p} V_0 \tilde{\to} M_0$ taking v_α into w_α, $\forall \alpha \in \mathcal{J}$.*

The same can be stated in a principally quasi-polarized context.

5.4. The case v is ramified. A natural question arises: how much of 5.3 can be adapted to the situation of C) of the beginning paragraph of §5? Here we include just a small application of 3.5 in this direction, modeled on Exercise 2. Let p, (W, ψ), $L_{(p)}$, K, f, H, v, d, $\mathcal{M}$, $\mathcal{N}$, $(v_\alpha)_{\alpha\in\mathcal{J}'}$ and $\mathcal{A}$ have the same significance as in 4.2. We assume that v is ramified over p. Let $e > 1$ be its ramification index. Let V be the completion of the strict henselization of the ring of integers O_v of $E(G, X)_v$. We assume the existence of a morphism $m_V : \mathrm{Spec}(V) \to \mathcal{N}$. Let Re and R be as in 2.4. Let $(M, F^1(M), \Phi_M, \nabla_M)$ be the quadruple associated (cf. 3.5.2) to the p-divisible group D_V (over V) of $m_V^*(\mathcal{A})$. Corresponding to $(v_\alpha)_{\alpha\in\mathcal{J}'}$, we get a family of homogeneous tensors $(t_\alpha)_{\alpha\in\mathcal{J}'}$ of $F^0(\mathcal{T}(M[\frac{1}{p}]))$ fixed by Φ_M and annihilated by ∇_M. As in 3.5, we define a closed subscheme $\tilde{G}_{Re}$ of $GL(M)$. We also assume that there is a subgroup N of $GL(M)$, which is a closed subscheme of $\tilde{G}_{Re}$, is isomorphic to $\mathbb{G}_a^d$, and is such that $\mathrm{Lie}(N_{\mathbb{F}})$ does not intersect the F^0-filtration of $M \otimes_{Re} \mathbb{F}$.

Exercise 4. Show that m_V factors through the formally smooth locus $\mathcal{N}^{\mathrm{sm}}$ of $\mathcal{N}$. So $\mathrm{Sh}_H(G, X)$ has a smooth integral model $\mathcal{N}^{\mathrm{sm}}$ over $O_{(v)}$. Hint: use V7 of 3.5.3, in a manner following very closely the one of Exercise 2.

5.4.1. Remarks. 1) We do not know if $\tilde{G}_{Re}$ is automatically a subgroup of $GL(M)$. One of the difficulties encountered is: the ring Re is not regular. So one usually needs to move, if possible, from Re to R (cf. 3.5.4 4)). What we gain (in case it is possible) by doing this: R is a local regular ring of dimension 2, to which [FC, 6.2. of p. 181] applies. See [Va13] for how this can be exploited.

2) We do not know if $\mathcal{N}^{\mathrm{sm}}$ has the SEP, even if $e < p-1$.

3) We do not know how (in practice) to assure very easily that the assumptions made above are indeed satisfied. Of course, one can try to use something similar to 2.3.1 to get that m_V exists; still the situation is complicated in general. However, there are "two ways out", but not so simple.

In the first one, one usually needs to assume that the special fibre of D_V is ordinary: in this case the existence of N can be very quickly checked, as in 4.2 V). So using variants of Step 5 of 4.2, one can handle in this way (see [Va13]) many situations involving Shimura varieties of Hodge type which are not of compact type.

In the second one, one needs "to adjust the lattices". To explain this idea, let O_v be the ring of integers of $E_v := E(G,X)_v$. $G_{\mathbb{Z}_p}$ is not a reductive group; but the closure of G_{E_v} in a suitable O_v-lattice L_v of $L_{(p)} \otimes_{\mathbb{Z}_{(p)}} E_v$ is often reductive (sometimes we might have to pass from E_v to a finite field extension of it). Usually the situation gets reduced to one, where there is a clear cut logic in choosing such a lattice (for instance, if F_1 is a Galois extension of $\mathbb{Q}$ ramified above p, and we are in a context similar to the one of Step 4 of 4.2, then we can replace $F_{1(p)} \otimes_{\mathbb{Z}_{(p)}} F_{1(p)}$ by a bigger $\mathbb{Z}_{(p)}$-lattice of $F_1 \otimes_{\mathbb{Q}} F_1$ which as an $\mathbb{Z}_{(p)}$-algebra is a disjoint union of copies of $F_{1(p)}$). Moreover, we can often pass from L_v to similar lattices in the de Rham context via Fontaine's comparison theory (and Q3), and so we can "gain more information" on different Re-lattices as M above. Once these new lattices (in the de Rham context) "are obtained" at the generic points of $\mathcal{N}_{\mathbb{F}}$, we can "prolong" them to all the points of $\mathcal{N}$, using the ideas of S2 of 4.2 K). See [Va13] for details.

4) 5.2.1 and 5.2.2 1) and 8) (and 6.4.2 2) below for the case $p=2$) represent the generalization of the cases of [Ca] which can not covered by 4.1.1, and involving a prime v of the reflex reflex field of an adjoint Shimura curve unramified over a rational prime p. We do expect that the cases of good reduction, with v ramified above p, obtained in [Ca] and [HT] for some compact Shimura varieties of some A_n type, can be generalized in the same manner as well (even to non-compact Shimura varieties). It is worth pointing out that the approach in [Ca] (and so in [HT]) is based on Drinfeld' s elliptic modules and involves only simple adjoint Shimura varieties whose $\mathbb{R}$-ranks are 1; it might be possible to adapt it to a larger class of Shimura varieties (we have in mind all the Shimura varieties of some A_n, C_n or $D_n^{\mathbb{H}}$ type, cf. 4.2.1 1) and [Va2, 2.3.5.1]). However in the orthogonal case, as well as for a more general context, we view the above (second) approach of 3) (via adjusting lattices), as simpler, more natural, and more direct (for instance 5.2.2 8) represents a very elementary process of preserving the regular formal smoothness property in some Shimura tower). See [Va13] for details.

§6. Applications

6.1. The case $p=3$ in [Va6]. In this section we refer to [Va6]: we fully use the notations of different parts of loc. cit. The goal of the section is to fulfill the promise of [Va6, 1.3.5

or 3.2.1 2)]: the things are "fine" in [Va6] for the prime $p = 3$ as well. So here we want to show, based on 1.4.2, that:

6.1.1. Supplementary claim. *Everywhere in [Va6] we can replace* $6d$ *by* $2d$.

Proof: A) First in [Va6, Exercise 1 of 2.5.4] we can consider $p = 3$ as well, cf. 4.1.2. Moreover in the whole of [Va6, 2.5.4] we can replace $6d$ by $2d$; [Va1, 6.4.4] is stated in such a way that we can include the prime $p = 3$ (as soon as we have 1.4.2).

B) Second we need to refer to the proof of [Va6, 3.2]: we assume that $(2d, 3) = 1$, and that we replaced in loc. cit. $6d$ by $2d$. We just need to check that Claim 1 of the proof of loc. cit. is still valid for $p = 3$. The reference to [Va1, 5.5.1] can be substituted by 4.3.3 2). So the only part of the proof of Claim 1 which needs to be modified is the argument that the morphism m'_{1Y} is the identity. This can be proved in the same way, but in a somehow more complicated (roundabout) manner. We need two subclaims.

Subclaim 1. *Any polarized K3 surface of degree* d *relatively prime to* 3 *over* k*, lifts to an ordinary polarized K3 surface of degree* d.

Proof: This is a direct consequence of [Va2, 3.1.0]: we can perform the same shifting operation of passing from a $SO(21)_{W(k)}$ context to a $\mathrm{GSpin}(21)_{W(k)}$ context as in the proof of [Va6, 4.7. D)], for having everything in the context of Shimura σ-crystals.

Subclaim 2. *The Shimura filtered Lie* σ*-crystal associated to a polarized K3 surface over* $W(k)$ *obtained from* (Z, L) *via a* $W(k)$*-valued point* z_1 *of* Y *is the same as the Shimura adjoint filtered Lie* σ*-crystal associated (via the SHS* $(f_1, L_W \otimes_{\mathbb{Z}} \mathbb{Z}_{(3)}, 3)$*) to* $z_1 \circ m'_{1Y}$.

Proof: We first prove this in the case the $K3$ surface over k we get via y is ordinary. But in this case (cf. the theory of canonical coordinates, see [De4]) we get filtered Lie σ-subcrystals of the mentioned two Shimura (adjoint) filtered Lie σ-crystals, defined by Borel subalgebras, and which are elements of $\mathcal{MF}_{[-1,0]}(W(k))$. So Fontaine's comparison theory and [Va6, 2.7.2.1 A and B] apply again in this situation (cf. [Fa1, 2.6] or [Fa3, §4]) even for $p = 3$.

Second we deduce that under this ordinary hypothesis, m'_Y is formally étale. Let now $\mathcal{K}3_{d,p,K_{W(k)}}$ be the moduli scheme of level-K marked polarized K3 surfaces of degree d over $W(k)$ having $\mathcal{K}3_{d,p,K_{B(k)}}$ as its generic fibre. For instance it can be constructed using Artin's method, starting from [Va6, 2.6 D)]. As in the proof of [Va6] we get (by using the extension property enjoyed by $\mathcal{N}_{W(k)}$) that we have a natural morphism $m_{\mathrm{nat}} : \mathcal{K}3_{d,p,K_{W(k)}} \to \mathcal{N}_{W(k)}$. Though in the proof of [Va6, 3.2] we allowed K to be very small, this morphism has a counterpart in the finite type context. From the above ordinary part, we get that m_{nat} is formally étale at the ordinary points. From the classical purity theorem and Subclaim 1, we get that m_{nat} is formally étale (everywhere). This ends the proof of Claim 1, and so of the $p = 3$ part of [Va6, 3.2].

Third, we need to end the proof of Subclaim 2. For simplicity we assume that K is an open subgroup of $G(\mathbb{A}_f^p)$. Let $Z = \mathrm{Spec}(R)$ be an affine open subscheme of $\mathcal{A}_{\mathcal{K}3,d,p,K_{W(k)}}$ containing y, and having connected fibres. We choose arbitrarily a Frobenius lift of the 3-adic completion $R^\wedge$ of R compatible with σ. Let $n \in \mathbb{N}$. Let $\mathcal{D}_1$ (resp. $\mathcal{D}_2$) be the 3-divisible object of $\mathcal{MF}^\nabla_{[-1,1]}(R)$ defined by the Shimura filtered Lie F-crystal associated to the versal polarized $K3$ surface over Z (resp. defined by the Shimura adjoint filtered Lie σ crystal attached to the principally polarized abelian scheme over Z obtained from $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ by natural pull back, and to the family of Hodge cycles with which $\mathcal{A}$ is naturally endowed via the SHS $(f_1, L_W \otimes_{\mathbb{Z}} \mathbb{Z}_{(3)}, 3)$). Let $Z_o = \mathrm{Spec}(R_o)$ be the open subscheme of Z having the same generic fibre and of whose special fibre is the ordinary locus of Z_k. Working with n big enough and with the pull backs of $\mathcal{D}_1/3^n\mathcal{D}_1$ and $\mathcal{D}_2/3^n\mathcal{D}_2$ to R_o, from [Va6, 2.7.3.2 A and B] and Fontaine's comparison theory with $\mathbb{Z}_3$ coefficients we get (based on the ordinary part paragraph above) that:

Lemma. *$\mathcal{D}_1/3^n\mathcal{D}_{1R_o}$ and $\mathcal{D}_2/3^n\mathcal{D}_{2R_o}$ can be identified canonically.*

Proof: Two urgent warnings are in order. First warning: in [Fa1, p. 35-36] it is explained how to associate a Galois representation to an object of $\mathcal{MF}^{\nabla}_{[-1,1]}(R_o)$ (equivalently to an object of $\mathcal{MF}^{\nabla}_{[0,2]}(R_o)$. But some statements are made (see [Fa1, p. 37-43]) only in the case of objects of $\mathcal{MF}^{\nabla}_{[-1,0]}(R_o)$ (equivalently of $\mathcal{MF}^{\nabla}_{[0,1]}(R_o)$). In our case (of $\mathcal{D}_1$ and $\mathcal{D}_2$) we are saved by the fact that, after passage to a pro-étale Galois cover R_1 of the 3-adic completion $R_o^{\wedge}$ of R_o, our two 3-divisible objects $\mathcal{D}_{1R_1}$ and $\mathcal{D}_{2R_1}$ are 3-divisible objects of $\mathcal{MF}^{\nabla}_{[-1,0]}(R_1)$ in disguise, i.e. in the above sense of being determined by 3-divisible subobjects corresponding to Borel Lie subalgebras. Second warning: to apply Fontaine's comparison theory, we first multiply the period morphism by 3 in order to get a $\mathbb{Z}_3$ context, and then we can take things mod 3^n. So first we get that $\mathcal{D}_{1R_1}$ and $\mathcal{D}_{2R_1}$ can be identified canonically.

To get that $\mathcal{D}_{1R_o}$ and $\mathcal{D}_{2R_o}$ themselves are identifiable canonically, the situation is a little bit complicated, as it is hard to trace in the literature how exactly the Galois representations associated to them (as in [Fa1, p. 35-36]) relate to the similar ones derived directly from the étale context with $\mathbb{Z}_3$ coefficients of the geometric context we started with. Of course in the case of $\mathcal{D}_{2R_o}$, the corresponding Galois representations coincide, but we do not know a priori this for $\mathcal{D}_{1R_o}$ (see the limitations in [Fa1, p. 37-43]). However, from the fact that we have an identification over R_1, we can still deduce easily (standard descent) that $\mathcal{D}_{1R_o}$ and $\mathcal{D}_{2R_o}$ can be identified canonically; we just need to remark:

Exercise 5. Show that $\mathcal{D}_{2R_1}$ has no automorphisms. Hint: use the fact that such an automorphism lifts to an automorphism a_1 of the Shimura 3-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R_1)$ producing it; then evaluate a_1 at any canonical lift $\mathrm{Spec}(W(\mathbb{F})) \to \mathrm{Spec}(R_1)$). A full proof of this exercise, in a much wider setting, will be given in part 2D.

This proves the lemma.

But now it is a trivial matter to see that this lemma implies that $\mathcal{D}_1$ and $\mathcal{D}_2$ can be identified canonically (one can directly argue at the level of projective $R^{\wedge}$-submodules associated to these 3-divisible objects, or apply Theorem 4 of 2.4.2). Of course this implies Subclaim 2.

C) Third it is trivial to see that the functorial aspects of [Va6, 3.2.3] are valid with $6d$ replaced by $2d$. Moreover, having [Va6, 3.2] sorted out for $p = 3$, 1.4.2 allows us to replace in [Va6, 3.2.5] $6d$ by $2d$. Fourth, the same replacement in [Va6, 3.3 and 4.5] is obviously possible. Fifth, in [Va10] we present the case $p = 3$ as well; so in [Va6, 4.2] we can replace $6d$ by $2d$. Sixth, the replacement in [Va6, 4.7] is possible, cf. §4 and the fact that we stated the things in [Va2] for $p = 3$ as well (see also 6.2 below). Seventh, the problems in [Va6, 5.3.1, 5.5 and 5.6.2] can be posed as well for $p = 3$, and our expectations are the same.

6.2. Completions to [Va2]. In [Va2] we worked the case $p = 3$ as well: however we had to assume that 4.1.1-2 and 4.3.1-2 hold, cf. [Va2, 1.15]. So we feel appropriate to point out the different parts of [Va2] where §4 allows us to remove all these assumptions.

6.2.1. Part involving integral canonical models. First, 1.4.2 guarantees that in [Va2, 2.3.3 or 4.9.21] $\mathcal{N}$ does exist and is uniquely determined. Second, the part needed in the proof of [Va2, 2.3.9 b)] is nothing else but 1.4.2. Third, once we have 1.4.1, [Va1, 5.4] applies directly even for $p = 3$ (cf. also Step 1 of 4.2); this handles [Va2, 2.3.11].

6.2.2. Part involving stratifications. Here we have to refer to [Va2, Theorem 9 of 1.11 and 4.9]. First, the statement at the end of [Va2, 1.11] and the assumption of [Va2, 4.9.1] are covered by 4.1.2. Second, 4.3.2 covers [Va2, 4.9.1.1]. Third, it is easy to see that once we have §4, the whole of the proof of [Va2, 4.9.8] applies for $p = 3$.

6.2.3. G-canonical lifts which are special points. Here we deal with the promise of [Va2, 4.4.6]: first, we prove it in most of the cases, and second, we prove Theorem 3 of 1.6. We use extensively the unitary trick and [Va2, 4.11.2]. We also use the notations of 2.3.8.

The following definitions are complementing [Va3, 2.5].

6.2.3.0. Definitions and notations. A. Let (G, X) be a simple adjoint Shimura pair with non-trivial involution. So (cf. [De3, 3.8]), $E(G, X)$ is a totally imaginary quadratic extension of a totally real number field $E_1(G, X)$ (cf. the notations of [Va3, 2.5]). A prime v of $E(G, X)$ is said to be moved by the involution, if it divides a prime v_1 of $E_1(G, X)$ which splits in $E(G, X)$. Similarly, if (G, X, H, v) is a Shimura quadruple, we say that it is with visible non-trivial involution, if v is not moved by the involution.

The problem of non-trivial involution in the case of simple adjoint Shimura quadruples is slightly subtler than in the case of simple adjoint Shimura pairs. So to explain the above word "visible" and to introduce some extra terminology, we need some preliminaries. For the notion of Shimura group pairs, we refer to [Va2, 2.2.5]. Let $(G_{\mathbb{Z}_p}, [\mu])$ be the Shimura group pair over $\mathbb{Z}_p$ defined by (G, X, H, v), as in [Va2, 2.2.6 5)]. Here $G_{\mathbb{Z}_p}$ is the reductive group over $\mathbb{Z}_p$ having $G_{\mathbb{Q}_p}$ as its generic fibre and having H as its group of $\mathbb{Z}_p$-valued points, while $\mu : \mathbb{G}_m \hookrightarrow G_{W(k(v))}$ is the injective cocharacter, which under an embedding $W(k(v)) \hookrightarrow \mathbb{C}$ extending the standard one $O_{(v)} \hookrightarrow \mathbb{C}$, is $G(\mathbb{C})$-conjugate to the cocharacters μ_x^* defined by $x \in X$ as in [Va2, 2.3.1]. We write it as a product of simple Shimura group pairs and of pairs of the same type but involving trivial cocharacters

$$(G_{\mathbb{Z}_p}, [\mu]) = \prod_{l \in I(p)} (G_l, [\mu_l]),$$

with G_l's as simple adjoint groups over $\mathbb{Z}_p$. As usual, it makes sense to say that $(G_l, [\mu_l])$ has or has not trivial involution (if μ_l is trivial, then we have trivial involution).

If there is $l \in I(p)$ such that $(G_l, [\mu_l])$ is with non-trivial involution, then we say that (G, X, H, v) is with non-trivial involution. For instance this is the case if v is not moved by the involution. Warning: if v is moved by the involution, it can still happen that (G, X, H, v) is with non-trivial involution.

We say that (G, X, H, v) is with total non-trivial involution, if $(G_l, [\mu_l])$ is with non-trivial involution, $\forall l \in I(p)$ (so in particular none of μ_l's are zero).

B. We consider now the case when (G, X) is a simple adjoint Shimura variety of $D_n^{\mathbb{H}}$ type, with or without involution. We apply the second part of the unitary trick of 2.3.8; with its notations, let

$$E_2(G, X) := E(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}})$$

(the notations are in accordance to the ones of [Va3, 2.5]). We have natural inclusions

$$(1). \qquad E_2(G, X) \subset E_1(G, X) \subset E(G, X)$$

It is easy to see that $E_2(G, X)$ can be defined directly in an abstract way (cf. [De1, 2.3.12]), and so it can be defined as well for simple adjoint Shimura varieties of E_6 type, or of some A_n type. Following the pattern of [Va3, 2.5], for all these types, we define similarly a totally real number field $EF_2(G, X)$, and in case (G, X) is with trivial involution (resp. with non-trivial involution) a totally real (resp. a CM) Galois extension $G_2(G, X)$ of $\mathbb{Q}$. We have similar to (1) above, natural inclusions involving theese $EF(G, X)$'s and $G(G, X)$'s fields.

We refer to $E_1(G, X)$ (resp. $E_2(G, X)$) as the real (resp. theoretically real) reflex field of (G, X) or of $\mathrm{Sh}(G, X)$. We refer to $EF(G, X)$ (resp. $EF_1(G, X)$ or $E_2(G, X)$) as the RT (resp. real RT or the theoretically real RT) field of (G, X) or of $\mathrm{Sh}(G, X)$ (here RT stands for reflex plus type). We refer to $G(G, X)$ as the Galois RT field of (G, X) or of $\mathrm{Sh}(G, X)$, and accordingly we refer to $G_i(G, X)$, $i = \overline{1, 2}$. We also refer to all these fields, as the key fields for (F, X) or for $\mathrm{Sh}(G, X)$.

We say that a prime v of $E(G, X)$ is of simple type, if the residue fields of the primes v_i of $E_i(G, X)$ divided by v are the same ($i = \overline{1, 2}$). Otherwise we say that v is of non-simple type.

C. We come back to the context of A, with (G, X) of $D_n^{\mathbb{H}}$ type. Let $l \in I(p)$ such that μ_l is non-trivial. Let B_l be a Borel subgroup of G_l. Let $d_l \in \mathbb{N}$ such that the field of definition of the special fibre of μ_l is $k(v_l) := \mathbb{F}_{p^{d_l}}$. So (cf. [Mi2, 4.6-7]), we can assume that μ_l is defined over $W(k(v_l))$. So let

$$\mu_l : \mathbb{G}_m \to G_{lW(k(v_l))}.$$

It acts on $\mathrm{Lie}(G_{lW(k(v_l))})$ via the identity, trivial, and the inverse character of $\mathbb{G}_m$. We can assume that $\mathrm{Lie}(B_l)$ is included in the F^0-filtration of $\mathrm{Lie}(G_{lW(k(v_l))})$ corresponding to the trivial and the inverse of the identity characters of $\mathbb{G}_m$. We get a Shimura adjoint Lie σ-crystal

$$\mathfrak{L}_l := (\mathrm{Lie}(G_{lW(k(v_l))}), \sigma \circ \mu_l(\frac{1}{p})).$$

Here σ is the Frobenius automorphism of $W(k(v_l))$; it acts on $\mathrm{Lie}(G_{lW(k(v_l))})$ logically, i.e. it fixes $\mathrm{Lie}(G_l)$.

Let I_0 be the set of simply factors of $G_{lW(\mathbb{F})}$; we can assume that $I_0 = \{1, ..., m\}$, for some $m \in \mathbb{N}$. Following the pattern of [Va2, 3.5.3], we define:

– a new set $\tilde{I}_0$ which is either I_0 or $\{1, ..., 2m\}$ (see [Va2, 3.5.3.1]),

– and for each $i \in \tilde{I}_0$ we define a number $\varepsilon_i \in \{0, n-1, n\}$ (see [Va2, 3.5.3.4]; cf also the fact that we are dealing with the $D_n^{\mathbb{H}}$ type).

Let a_l (resp. b_l) be the number of $i \in \tilde{I}_0$ such that $\varepsilon_i = n-1$ (resp. $\varepsilon_i = n$).

Obviously, $(G_l, [\mu_l])$ is with non-trivial involution iff $\tilde{I}_0 \neq I_0$, and in such a case we have $a_l = b_l$.

We say that $(G_l, [\mu_l])$ is U-simple if $a_l \neq b_l$. We say that (G, X, H, v) is U-simple (resp. partially U-simple) if $\forall l \in I(p)$ (resp. if there is $l \in I(p)$) either μ_l is trivial or $(G_l, [\mu_l])$ is U-simple (resp. $(G_l, [\mu_l])$ is U-simple). Here the letter U is in accordance with [Va2, 4.4.13] (cf. also the Case 2 of the proof of Theorem below).

Obviously, if (G, X, H, v) is U-simple (resp. partially U-simple), then it is totally (resp. is) with non-trivial involution. Warning: not always it is possible to capture the notion of (partially) U-simple using the key fields of (G, X). Accordingly, we will not elaborate further more on this notion.

6.2.3.1. Theorem. *Let $(f, L_{(p)}, v)$ be a SHS. We assume that all simple factors of the adjoint Shimura quadruple $(G^{\mathrm{ad}}, H^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$ which are of some $D_n^{\mathbb{H}}$ type, are U-simple in the above sense. Then all the abelian varieties over $W(\mathbb{F})$ obtained by pulling back $\mathcal{A}_{H_0}$ via an arbitrary G-canonical lift Spec$(W(\mathbb{F})) \to \mathcal{N}/H_0$, have complex multiplication.*

Proof: The statement of the theorem can be rephrased: the generic fibres of the G^{ad}-canonical lifts $\mathrm{Spec}(W(\mathbb{F})) \to \mathcal{N}^{\mathrm{ad}}$, with $\mathcal{N}^{\mathrm{ad}}$ as the integral canonical model of the Shimura quadruple $(G^{\mathrm{ad}}, H^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$, are special points. This is a consequence of the proof of [Va2, 4.9.8], and so of the way we defined Shimura-canonical lifts. So we can choose the SHS $(f, L_{(p)}, v)$, subject to the conditions that the Shimura quadruple $(G^{\mathrm{ad}}, H^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$ is unchanged. So we can assume that $(G^{\mathrm{ad}}, H^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$ is simple.

If G^{ad} is of A_n Lie type, then we can use the first part of the unitary trick, i.e. we are in the context of a standard PEL situation $(f, L_{(p)}, v, \mathcal{B})$. In this case [Va2, 4.4.5] applies directly.

We assume now that G^{ad} is not of some A_n Lie type. We use the second part of the unitary trick. So we can assume that f factors as a sequence of injective maps $f_1 : (G, X) \hookrightarrow (G_1, X_1)$, $f_2: (G_1, X_1) \hookrightarrow (G_2, X_2)$, and $f_3 : (G_2, X_2) \hookrightarrow (GSp(W, \psi, S)$, such that a long list of properties hold. We summarize part of these properties, in the language allowed by 4.1.2. Let v_j be the prime of $E(G_j, X_j)$ divided by v, $j = \overline{1, 2}$. Let $\mathcal{N}_j$ be the integral canonical model of the Shimura quadruple (G_j, X_j, H_j, v_j), where $H_j := G_j(\mathbb{Q}_p) \cap GSp(L_{(p)}, \psi)$. The triples $(f_3, L_{(p)}, v_2)$ and $(f_3 \circ f_2, L_{(p)}, v_1)$ are SHS; moreover we have a standard PEL situation $(f_3, L_{(p)}, v_2, \mathcal{B}_2)$. We consider two cases.

Case 1: $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is of B_n, C_n or $D_n^{\mathbb{R}}$ type. From 2.3.8 e) and our hypotheses we deduce that $k(v) = k(v_1) = k(v_2)$. So [Va2, 4.11.2 a)] is satisfied in our context

$$i_2: \mathcal{N}_1 \to \mathcal{N}_2$$

(here i_2 is the natural morphism, cf. [Va1, 3.2.7 4)]).

Let $F_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p := \prod_{i\in I(F)} W(k_i)$, and $F_{1(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p := \prod_{m\in I(F_1)} W(k_m)$, with $I(F)$ and $I(F_1)$ as finite sets, and with k_i's and k_m's as finite fields. As in [Va1, 6.5.1] (or in 4.2 R)) we deduce that

$$G^{\mathrm{ad}}_{\mathbb{Z}_p} := \prod_{i\in I(F)} \mathrm{Res}_{W(k_i)/\mathbb{Z}_p}(G^i),$$

with G^i an absolutely simple adjoint group over $W(k_i)$. Similarly we can write

$$G^{\mathrm{ad}}_{j\mathbb{Z}_p} := \prod_{m\in I(F_1)} \mathrm{Res}_{W(k_m)/\mathbb{Z}_p}(G^m_j),$$

with G^m_j's as absolutely simple adjoint groups over $W(k_m)$, $j = \overline{1,2}$.

Using 2.3.11 3) we deduce that we can assume that all these absolutely simple adjoint groups G^m_j are split; so $k(v)$ is a subfield of the composite of all these k_m's finite fields. Using this splitting property and 2.3.8 h), it is trivial to see that [Va2, 4.11.2 b)] holds in the context of $i_2\colon \mathcal{N}_1 \to \mathcal{N}_2$. The argument goes as follows. It is enough to work with just one index $m \in I(F_1)$. Let T^m_j (resp. B^m_j) be a maximal split torus (respectively a Borel subgroup) of G^m_j.

We can assume that T^m_1 (resp. B^m_1) is a subgroup of T^m_2 (resp. of B^m_2). $T_j(m) := \mathrm{Res}_{W(k_m)/\mathbb{Z}_p}(T^m_j)$ (resp. $B_j(m) := \mathrm{Res}_{W(k_m)/\mathbb{Z}_p}(B^m_j)$) is a maximal torus (resp. a Borel subgroup) of $\mathrm{Res}_{W(k_m)/\mathbb{Z}_p}(G^m_j)$. Moreover $T_j(m)$ is such that for any subfield $\tilde{k}$ of k_m, $T_j(m)_{W(\tilde{k})}$ contains a maximal split subtorus of $\mathrm{Res}_{W(k_m)/\mathbb{Z}_p}(G^m_j)_{W(\tilde{k})}$. Let

$$k(v)^m := k(v) \cap k_m.$$

Let

$$\mu^m_j : \mathbb{G}_m \hookrightarrow \mathrm{Res}_{W(k_m)/\mathbb{Z}_p}(T^m_j)_{W(k(v)^m)}$$

be a faithful cocharacter such that under an embedding $W(k(v)) \hookrightarrow \mathbb{C}$, extending the inclusion $E(G_j, X_j) \hookrightarrow \mathbb{C}$, it is $\mathrm{Res}_{W(k_m)/\mathbb{Z}_p}(G^m_j)(\mathbb{C})$-conjugate to the corresponding projection (on the factor $\mathrm{Res}_{W(k_m)/\mathbb{Z}_p}(G^m_j)_{\mathbb{C}}$ of $G^{\mathrm{ad}}_{j\mathbb{C}}$) of a cocharacter μ^*_x, defined by some point $x \in X_j$ (see [Va2, 2.3.1]). Such a cocharacter obviously exists over $W(k_m)$, as well as over $W(k(v))$ (cf. the very definition of the reflex field, see also [Mi2, 4.7]); so it exists over $W(k(v)^m)$, again cf. loc. cit. We can assume that $\mathrm{Lie}(\mathrm{Res}_{W(k_m)/\mathbb{Z}_p}(B^m_j))$ is contained in the parabolic Lie subalgebra of $\mathrm{Lie}(\mathrm{Res}_{W(k_m)/\mathbb{Z}_p}(B^m_1))$ formed by $W(k(v)^m)$-submodules on which $\mathbb{G}_m$ acts via μ^m_j either trivially or as the inverse of the identity character. We just need to show that the composite of μ^m_1 with the natural monomorphism $\mathrm{Res}_{W(k_m)/\mathbb{Z}_p}(T^m_1) \hookrightarrow \mathrm{Res}_{W(k_m)/\mathbb{Z}_p}(T^m_2)$ is nothing else but μ^m_2. But this is an immediate consequence of 2.3.8 j) (cf. also [De1, 2.3.12]).

So [Va2, 4.11.2] applies. Combining this with [Va2, 4.11.3] we deduce that the natural morphism

$$i\colon \mathcal{N} \to \mathcal{N}_2$$

takes Shimura-canonical lifts into Shimura-canonical lifts. But we know (cf. [Va2, 4.4.5]) that the abelian varieties over $W(\mathbb{F})$ obtained from Shimura-canonical lifts $\mathrm{Spec}(W(\mathbb{F})) \to \mathcal{N}_2$ have complex multiplication. This ends the proof in Case 1.

Case 2: $(G^{\mathrm{ad}}, X^{\mathrm{ad}}, H^{\mathrm{ad}}, v^{\mathrm{ad}})$ **is of** U**-simple** $D_n^{\mathbb{H}}$ **type.** In this case there is only one modification needed to be made to Case 1. It is rooted in the fact that we do not necessarily have $k(v) = k(v_2)$, and so [Va2, 4.11.2] can not be applied. However, it is trivial to check that:

a) $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}, H_1^{\mathrm{ad}}, v_1^{\mathrm{ad}})$ is also U-simple;

b) i_2 (and so also i, cf. [Va2, 4.11.3]) takes Shimura-ordinary points into U-ordinary points, and takes Shimura-canonical lifts into U-canonical lifts (see def. [Va2, 4.4.13.1]).

For instance, to see a), let $i \in I(F)$ and let $i_1 \in I(F_1)$ such that $G^i_{W(k_{i_1})}$ is naturally a subgroup of $G_1^{i_1}$; imitating the notations of 6.2.3.0 C, the numbers a_{i_1} and b_{i_1} corresponding to i_1 are (respectively) $[k_{i_1} : k_i]$ or $\frac{[k_{i_1}:k_i]}{2}$ times the numbers a_i and b_i corresponding to i, depending on the fact that the Shimura group pair naturally defined (via $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$) by $\mathrm{Res}_{W(k_i)/\mathbb{Z}_p}(G^i)$ is or it is not with trivial involution (we recall that $G_1^{i_1}$ is split).

So instead of [Va2, 4.4.5] we have to use [Va2, 4.4.13.4 2)]. This ends the proof in the Case 2, and the proof of the theorem.

Exercise 6. Show that in 6.2.3.1 we can replace the expression "U-simple" by the stronger one "partially U-simple". Hint: use an argument similar to the one which gave us [Va3, 3.2.2] via [Va3, 3.2.1].

6.2.3.2. Remarks. 1) From Exercise 6 and [Va2, 4.9.8] we deduce directly Theorem 3 of 1.6.

2) The above proof of 6.2.3.1 details a more general form of [Va3, 3.1.1].

3) For a complete solution of Exercise 6, and how we can remove entirely (using the same ideas) its restrictions, see part three.

4) 6.2.3.1 and Exercise 6 above hold as well in the context of 5.2, cf. 5.2.2 6).

5) There are many variant of 6.2.3.1 and Exercise 6 involving U-canonical lifts (i.e. when i_2 of the proof of 6.2.3.1 takes U-canonical lifts, which are not G-canonical, into U-canonical lifts) (see [Va14-15] for complete lists).

6.2.3.3. Corollary. *We use the notations of 3.1.1. We assume that the hypothesis of 6.2.3.1 is satisfied. Then* $\mathcal{D}_{H_0}$ *has the ACM property.*

6.2.3.4. Problem. Study the CM types of the abelian varieties of 6.2.3.1 having complex multiplication: explain how they depend on $(f, L_{(p)}, v)$, and how they depend on the choice of the Shimura-canonical lift $\mathrm{Spec}(W(\mathbb{F})) \to \mathcal{N}/H_0$.

6.2.3.5. Example. Let $\mathcal{K}$ be a field of characteristic zero. A pseudo-polarized K3 surface Z over $\mathcal{K}$ is said to be of CM type, if the K3 surface over $\mathcal{K}$ birationally isomorphic to Z (obtained by blowing up the singularities of Z) is of CM type in the sense of [Va6, 5.2.1]. From 6.2.3.1 we get directly: all the Shimura-canonical (pseudo-) polarized K3 surfaces of Shimura-ordinary (pseudo-) polarized K3 surfaces over finite fields, introduced in [Va6, 4.7 C)] and implicitly introduced in the mixed characteristic $(0, 3)$ case in 6.1.1, are of

CM type. This is so as the Shimura pair $(SO(2,19), X_{SO(2,19)})$ introduced in 2.2 does not (easy exercise) have Shimura subpairs of some $D_n^{\mathbb{H}}$ type.

6.2.3.6. Definition. Let $(f, L_{(p)}, v)$ be a SHS. The abelian varieties obtained from $\mathcal{A}_{H_0}$ by pull back via Shimura-ordinary points (resp. Shimura-canonical lifts) of $\mathcal{N}$, will be called Shimura-ordinary (resp. Shimura-canonical) abelian varieties.

6.2.3.7. Remark. Referring to 6.2.3.1, the mentioned G-canonical lifts, are not always associated to $\mathbb{Z}_{(p)}$ embeddings of special quadruples. However, this does happen if the G-ordinary points of $\mathcal{N}_{k(v)}$ are T-ordinary in the sense of [Va2, 4.4.13] (cf. [Va2, P7 of 4.6]).

6.3. Milne's conjecture revisited. First, we would like to point out that 4.1.2 implies that all the assumptions of [Va6, 4.7] pertaining to the existence of some SHS, are automatically satisfied. This makes the results of loc. cit. pertaining to Milne's conjecture very refined.

Second, 4.1.2 allows very sharp upper estimates of the numbers $M(A)$ and $M(f, L)$ introduced in [Va3, 2.2.1.1 and 2.2.2.1]. For instance in [Va4, 4.3.1.1], indeed we can take $M(f, L) = 3$. In general, referring to the context of [Va3, 2.2.2], let $S(f, L)$ be the finite set formed by 2 and by the set of primes p such that the Zariski closure of G in $GL(L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$ is not a reductive group. We can take:

$$M(f, L) := 1 + \max(S(f, L)).$$

A similar thing can be stated for the numbers $M(A)$: it will not be done here.

Third, a natural question, directly related to Q2, arises. For the rest of 6.3 we consider the context and notations of 2.4 and of the first paragraph of 2.4.1, with $p \geq 3$, and with G a reductive group.

Q19. *Is it true that $\tilde{G}_{Re}$ is automatically reductive?*

[Va2, 4.12.12], [Va4, 3.1] and 4.1.2 point to a positive answer yes, at least in the case $e = 1$. In part 2D, we will analyze this question very carefully. Most cases can be reduced to the cases described in the Example 1 below, using an argument similar to the one of S1 of 4.2 K).

Moreover, for many other cases (not necessarily in the context of p-divisible groups), the same idea of S1 of 4.2 K) can be applied, at least for primes $p \geq 5$ and $e = 1$.

Example 1. We assume the existence of a polarized abelian variety (A, p_A) over V such that D_V is the p-divisible group of A, p_A is of degree relatively prime to p, and p_A as a perfect form on H is fixed by G. We also assume that $\forall \alpha \in \mathcal{J}$, v_α is the p-component of an étale component of a Hodge cycle of A. Then the answer to questions Q2-3 of 2.4 is yes (so also the answer to Q19 is yes). The argument is entirely the same as the one of [Va1, 5.8.8]: one reduces the problem to the context of a SHS (cf. 4.1.2), to which [Va4, V3 of 3.2] applies directly.

Example 2. We assume that the special fibre of D_V is an ordinary p-divisible group. As in the Subfact of 4.2 V) we get that the answer to questions Q2-3 of 2.4 is yes (so also the answer to Q19 is yes).

Exercise 7. We assume that G^{ad} does not have compact factors. Using Theorem 2 of 2.4 and [Va1, 5.8.4-7], give good estimates of how p has to be in comparison to the rank of the p-divisible group over $W(k)$ we started with, in order to get an answer yes to the question, regardless of how e is.

6.3.1. Milne's conjecture in the non-reductive context. Here we answer Q11, as a particular case of a complement to [Va4, V4 of 3.2]. We start recalling the context of loc. cit. Let Y be a regular formally smooth $W(k)$-scheme having a connected special fibre. Let $\mathcal{D} := (D_Y, (u_\alpha)_{\alpha\in\mathcal{J}})$ be a p-divisible group over Y with crystalline cycles. We assume that Y is the (p-adic completion) of a pro-étale scheme over a smooth $W(k)$-scheme. We assume that there is a morphism $z : \mathrm{Spec}(W(\bar{k})) \to Y$ such that:

(*) the Galois representation of Gal($B(\bar{k})$) attached to z (via $\mathcal{D}$) factors through the $\mathbb{Q}_p$-valued points of a torus of $G_{\mathbb{Q}_p}$.

Here $G_{\mathbb{Q}_p}$ is the generic fibre of the $\mathbb{Z}_p$-group $G_{\mathbb{Z}_p}$ of the étale counterpart (defined as in [Va4, 2.4.1]) of $z^*(\mathcal{D})$. Let now $z_0 : \mathrm{Spec}(W(\bar{k})) \to Y$ be an arbitrary morphism. Let $(M_0, F^1, \phi_0, (w_\alpha)_{\alpha\in\mathcal{J}})$ be the filtered σ-crystal with cycles over $\bar{k}$ attached to $z_1^*(\mathcal{D})$, and let $(H^1, G_{\mathbb{Z}_p}, (v_\alpha)_{\alpha\in\mathcal{J}})$ be its étale counterpart. The fact that we do have the same group $G_{\mathbb{Z}_p}$, is a consequence of [Va2, Fact of 3.4.19 D] and of the assumption on Y involving pro-étaleness. We have:

A. Proposition (the non-reductive general variant of [Va4, V4 of 3.2]). *There is an isomorphism $H^1 \otimes_{\mathbb{Z}_p} W(\bar{k}) \tilde{\to} M_0$ taking v_α into w_α, $\forall\alpha \in \mathcal{J}$.*

Proof: This is a consequence of 5.2.3 and 5.2.3.1, as Y has a connected special fibre.

Obviously, we have a principally quasi-polarized variant of this proposition.

B. Corollary. *The answer to Q11 is yes.*

Proof: We need to apply A above to an arbitrary open subscheme Y of $\mathcal{N}_{W(\mathbb{F})}$ whose special fibre is a connected component of $\mathcal{N}_{\mathbb{F}}$. Using the density of G-ordinary points (see [Va2, 4.2.1]) and the existence of G-canonical lifts (see [Va2, 4.4.1 and 4.9.8]), in a way similar to 4.2 K), we deduce that condition (*) above is satisfied for the standard principally quasi-polarized p-divisible group with cycles we naturally have (as in [Va2, 2.3.11]) over Y. The corollary follows.

Of course B represents a second proof of 5.1.1.

6.4. The case $p = 2$. Following the policy of [Va2] the case $p = 2$ is treated as asides. A great part of [Va8] is dedicated to this case, and so here we just mention some weaker (i.e. partial) results, just for pointing out that despite the general opinion that this case is harder, it can be treated entirely in the same manner, and that the main two theorems of 4.1 remain (at least in the majority of cases) true, provided we formulate the uniqueness part slightly differently.

We start pointing out that though we worked out 2.2 for $p \geq 3$, it can be extended to the case $p = 2$ as well. To support this statement, we recall the shape of classical split groups over $\mathbb{Z}_{(2)}$, cf. [Bo2, §23] for the characteristic two picture.

Let $n \in \mathbb{N}$. Let M be a free $\mathbb{Z}_{(2)}$-module of finite rank $n+1$. Then $SL(M)$ is a semisimple simply connected split group of A_n Lie type. We assume now that n is odd. So there is a perfect symplectic form $\psi_M : M \otimes_{\mathbb{Z}_{(2)}} M \to \mathbb{Z}_{(2)}$. Then $Sp(M, \psi_M)$ is a semisimple simply connected split group of $C_{\frac{n+1}{2}}$ Lie type. We assume now that $n = 2m+1$ is odd. We consider the bilinear form

$$b_M(2m+1) := x_1x_2 + x_3x_4 + ... + x_{2m-1}x_{2m} + x_{2m+1}^2$$

w.r.t. an arbitrary fixed basis $\{e_1, ..., e_{2m+1}\}$ of M. Then $SO(M, b_M(2m+1))$ is a semisimple adjoint split group of B_m Lie type. We assume now that $n = 2m$ is even. We consider the bilinear form

$$b_M(2m) := x_1x_2 + x_3x_4 + ... + x_{2m-1}x_{2m}$$

w.r.t. an arbitrary fixed basis $\{e_1, ..., e_{2m}\}$ of M. Then $SO(M, b_M(2m))$ is a semisimple split group of D_m Lie type. Based on these facts, it is easy to adapt 2.2-3 to the context of a $\mathbb{Z}_{(2)}$ setting. See [Va8] for details.

Here we concentrate just on the case of Shimura varieties of C_n type, $n \in \mathbb{N}$. For the A_n types see [Va2, 2.3.18] (in other words it is easy to pass from the theorem below and from loc. cit. to a theorem entirely similar to the one below, which combines the C_n Lie types with the A_m Lie types). For the $D_n^{\mathbb{H}}$ type case see 6.4.1.1. For the B_n and $D_n^{\mathbb{R}}$ types, as well as for a combined version of all the types, see [Va8] (see also 6.4.2 4) for a sketch).

Right away we need to mention: it is not good to work with $\mathbb{Q}(i)$, as $\mathbb{Z}_2(i)$ is ramified over $\mathbb{Z}_2$. So we need to work with any other totally imaginary quadratic extension E of $\mathbb{Q}$ in which 2 splits (is unramified would be in fact enough). Let $d \in \mathbb{N}$ square free and congruent to 7 (congruent to 3 would be enough) mod 8 such that $E = \mathbb{Q}(\sqrt{-d})$. Let (M, ψ_M) be a symplectic space over $\mathbb{Z}_{(2)}$, with M of rank $2n$. Let

$$N := M \otimes_{\mathbb{Z}_{(2)}} E_{(2)}.$$

As in [Sa, 1.4] we get that

$$F(x, y) := i\sqrt{d}\psi_M(\overline{x}, y)$$

is a Hermitian perfect form on the $E_{(2)}$-module N, with values in $E_{(2)}$, and of signature (n, n). As in Case C of 2.2 we get a monomorphism

$$\tilde{c}_n : GSp(n)_{\mathbb{Z}_{(2)}} = GSp(M, \psi_M) \hookrightarrow GSU(N, F)$$

between reductive groups over $\mathbb{Z}_{(2)}$. Using this we deduce immediately that 2.3.8 remains valid for $p = 2$ in the C_n type case.

In what follows all the integral canonical models of Shimura quadruples of a Siegel modular variety to be considered, are supposed to be the standard ones (of Mumford; see [MFK]) mentioned in [Va1, 3.2.9].

6.4.1. Theorem. *Let (G, X, H, v) be a Shimura quadruple of preabelian type, with v dividing 2, and such that all the simple factors of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ are of some C_m type. We have:*

1) It does have an integral canonical model over $O_{(v)}$. All the integral canonical models to be constructed below are quasi-projective and strongly smooth; they are projective iff (G,X) is of compact type. Moreover, [Va2, 4.2.1 and 4.4.1] and 6.2.3.1 hold as well for all these integral canonical models.

2) We assume the existence of an injective map $f:(G,X)\hookrightarrow(GSp(W,\psi),S)$ such that we do have a $\mathbb{Z}_{(2)}$ embedding $(f,GSp(L_{(2)},\psi),v)$, and $H:=G(\mathbb{Q}_p)\cap GSp(L_{(2)},\psi)(\mathbb{Z}_2)$. We also assume the existence of a special subpair $f_T:(T,\{h\})\hookrightarrow(G,X)$ such that the triple $\mathcal{T}:=(f\circ f_T,L_{(2)},v_T)$ is a $p=2$ SHS, for any prime v_T of $E(T,\{h\})$ dividing v, and (the equivalent of) Theorem 6 of 2.4.3 holds for $\mathcal{T}$. We also assume that $(G^{\mathrm{ad}},X^{\mathrm{ad}})$ is of compact type. Then an integral canonical model of (G,X,H,v) can be obtained by taking the normalization of $\mathrm{Sh}_H(G,X)$ in the extension to $O_{(v)}$ of the standard integral canonical model $\mathcal{M}$ of $(GSp(W,\psi),S,GSp(L_{(2)},\psi)(\mathbb{Z}_2),2)$. Moreover, $(f,L_{(2)},v)$ is a SHS.

3) We assume that $k(v^{\mathrm{ad}})=\mathbb{F}_2$ and that $(G^{\mathrm{ad}},X^{\mathrm{ad}})$ is of compact type. Then there is a unique integral canonical model $\mathcal{N}$ of it subject to the following functorial property:

– *if (G_1,X_1,H_1,v_1) is a Shimura quadruple which either satisfies all the requirements imposed on (G,X,H,v), or is of a Siegel modular variety, and if we have an injective map (resp. a finite map) $(G,X,H,v)\to(G_1,X_1,H_1,v_1)$, then there is a finite (resp. a profinite pro-étale) morphism (of course uniquely determined) $\mathcal{N}\to\mathcal{N}_{1O_{(v)}}$ which at the level of generic fibres is the natural morphism $\mathrm{Sh}_H(G,X)\to\mathrm{Sh}_{H_1}(G_1,X_1)_{E(G,X)}$ (here $\mathcal{N}_1$ is the integral canonical model of (G_1,X_1,H_1,v_1)).*

Proof: The proof of this theorem is very much the same as of 4.2 and of 6.2.3. But there are three complications showing up. First, we do not know the uniqueness of integral canonical models w.r.t. primes dividing 2 (cf. [Va1, 3.2.4]); however we do believe in the uniqueness part (see [Va1, 3.2.1.4 5)]). That is why we can not apply Step 2 of 4.2 without very careful considerations. Second, Theorem 6 of 2.4.3 is stated for $p\geq 3$; however, in [Va4, V6 of 3.2] some cases of it in the $p=2$ setting are also proved. Third, we can not use [Va1, 4.3.10 b)] or S3 of 4.2 I) directly; so we have to use slightly different $W(\mathbb{F})$-well positioned families of tensors, then in the case of S3 of 4.2 I).

The key points are as follows. [Va2, 6.2.2] covers the case $p=2$, while the fact that 2.3.8 remains true for $p=2$ in the C_n type case guarantees that Step 4 (i.e. the non-compact type case) of 4.2 applies entirely. Some precautions are in order in the compact case in connection to Steps 1 to 3 of 4.2 (see below). In our C Lie type case, [Va1, 6.2.3] remains valid for $p=2$, cf. [Va1, 6.2.6], while [Va1, 6.1.2] is not needed.

We are left to sort out the mentioned three complications. We can get rid (in connection to the 1) part) of the second complication as follows. We just need to point out that the following exercise makes the things work.

Exercise 8. We consider the $p=2$ version of 2.3.8, in the C_n type case. We use the notations of 2.3.8. Let T_3 be a maximal torus of G_3, such that we have a special subpair $i_{T_3}:(T_3,\{h_3\})\hookrightarrow(G_3,X_3)$, and the triple $\mathcal{T}^3:=(f_3\circ i_{T_3},L_{(2)},v_{T_3})$ is a $p=2$ SHS, for any prime v_{T_3} of $E(T_3,\{h_3\})$ dividing 2. Show that Theorem 6 of 2.4.3 remains true in the context of $\mathcal{T}$. Hint: use Fact 1 of 4.2 S).

We mentioned above that S3 of 4.2 I) does not apply directly (cf. 2.3.7.1 i)). So to get rid of the third complication we have to proceed as follows. Instead of S3 of 4.2 I) we can use 4.2.1 1) directly; another way (for the compact case) is to use the following exercise (i.e. to use an analogue of S3 of 4.2 I), where instead of the Killing form on $\mathrm{Lie}(G^{\mathrm{der}}_{W(\mathbb{F})}) = \mathrm{Lie}(Sp(n)_{W(\mathbb{F})})$ we use the standard trace form on it; we recall that this standard trace form is $\frac{1}{2(n+1)}$ times the Killing form, it is not perfect, but its natural extension to $\mathrm{Lie}(GSp(n)_{W(\mathbb{F})})$ is perfect; this last fact can be easily read out from [KMRT, p. 358]).

Exercise 9. Let O be a complete DVR of mixed characteristic $(0,2)$, and of perfect residue field. Let $K_O := O\big[\frac{1}{2}\big]$. Let $(\tilde{M},\tilde{\psi})$ be a symplectic space over K_O. Let $\tilde{M}_O$ be an O-lattice of $\tilde{M}$. Let $\mathfrak{g}_O$ be a Lie subalgebra of $\mathfrak{gl}(\tilde{M})$, which as an O-submodule is a direct summand, and whose generic fibre is $GSp(\tilde{M},\tilde{\psi})$. We assume that the trace form on $\mathfrak{g}_O$ is perfect, and that $\tilde{\psi}$ takes $\tilde{M}_O \otimes_O \tilde{M}_O$ into O. Then $\mathfrak{g}_O$ is the Lie algebra of a split reductive subgroup of $GL(\tilde{M}_O)$. The same remains true in the mixed characteristic $(0,q)$ case, with q an odd prime not dividing the $\dim_O(\tilde{M}_O)$; moreover, for an arbitrary odd prime q the same remains true if we replace GSp groups by Sp groups. Hint: show that all the elementary invariants of $(\tilde{M}_O,\tilde{\psi})$ (defined using a standard symplectic basis) must be equal.

All the above facts take care of 2) and of the first part of 1). For the second part of 1) we just need to refer to [Va2, 4.14.3] and to point out that the arguments of [Va1, 6.4.1.1 1)] apply as well to give us the part of 1) involving projectiveness. To see 3), we just need to show that in the situation described in 2), if $k(v^{\mathrm{ad}}) = \mathbb{F}_2$, then we do not need to assume that such a special subpair $f_T : (T,\{h\}) \hookrightarrow (G,X)$ does exist.

We follow a little bit the Step 2 of 4.2, including the notations. In other words, using the trick three of 2.3.3, we reach a situation which can be summarized as in the proof of 4.2 K), as morphisms

$$\mathcal{C}_1 \overset{c_1}{\leftarrow} \mathcal{C}_2 \overset{c_2}{\rightarrow} \mathcal{C}$$

between $W(\mathbb{F})$-schemes of finite type. Moreover, we can assume that $G_2^{\mathrm{der}} = G_1^{\mathrm{der}}$ are simply connected. From [Va1, 6.2.3] we get immediately that the generic fibre of c_1 is an isomorphism. We need to show that c_1 is an isomorphism, that c_2 is an étale cover, and that $\mathcal{D}_{\mathcal{C}}$ is a Shimura 2-divisible group. From the second part of 1) we deduce that there is a special point $\mathrm{Spec}(B(\mathbb{F})) \to \mathcal{C}_1$ which extends to a G_1-canonical lift $z_1 : \mathrm{Spec}(W(\mathbb{F})) \to \mathcal{C}_1$.

But $\mathcal{C}_2$ is projective over $W(\mathbb{F})$: the arguments of [Va1, 6.4.1.1 1)] apply. As a pro-étale cover of it has the EP, from the classical purity theorem we get immediately that $\mathcal{C}_2$ has the EP. We also get that z_1 lifts to a morphism $z_2 : \mathrm{Spec}(W(\mathbb{F})) \to \mathcal{C}_2$. Let $z := c_2 \circ z_2$. Let $(M, F^1, \phi_M, \tilde{G}_{W(\mathbb{F})}, p_M)$ be the principally quasi-polarized σ-crystal associated to $z^*(\mathcal{D}_{\mathcal{C}})$. As in Step 2 of 4.2, we just need to show that:

Lemma. *$\tilde{G}_{W(\mathbb{F})}$ is reductive (isomorphic to $G_{W(\mathbb{F})}$), and a similarly defined group $\tilde{G}_{2W(\mathbb{F})} \subset GL(M \oplus M_1)$ obtained by working with $(f_2, L_{(2)} \oplus L_{1(2)}, v_2)$, is as well reductive.*

To check this we have to "split" $G_{\mathbb{Z}_p}$ and $G_{B(\mathbb{F})}$ into three pieces, such that one piece corresponds to the connected component of the origin of the center of the groups involved, while the other two pieces decompose their derived groups, as in [Va3, 5.1.3]. In other words, we can define a simple factor of $G^{\mathrm{ad}}_{\mathbb{Z}_p}$ or of $G^{\mathrm{ad}}_{B(\mathbb{F})}$ of being compact or non-compact. There are two ways in doing this. First, we can use Galois representations: a simple factor $\mathcal{F}$ of $G^{\mathrm{ad}}_{\mathbb{Z}_p}$ is compact iff the 2-adic Galois representation associated to $z^*(\mathcal{A}_{\mathcal{C}_2})$ has a trivial image into it. Second, we can use an injective cocharacter $\mu : \mathbb{G}_m \hookrightarrow \tilde{G}_{W(\mathbb{F})}$ as in [Va3, 5.3.1] and [Va2, 3.10.1] to define the trivial and the non-trivial part of the filtered Lie isocrystal

$$(\mathrm{Lie}(G^{\mathrm{ad}}_{B(\mathbb{F})}), F^0(\mathrm{Lie}(G^{\mathrm{ad}}_{B(\mathbb{F})})), F^1(\mathrm{Lie}(G^{\mathrm{ad}}_{B(\mathbb{F})})), \phi_M).$$

Let $\tilde{G}^{t\mathrm{der}}_{W(\mathbb{F})}$ (resp. $\tilde{G}^{\mathrm{ntder}}_{W(\mathbb{F})}$) be the Zariski closures in $GL(M)$ of the semisimple subgroup of $G_{B(\mathbb{F})}$ naturally isogeneous to the trivial factor -defined as the product of the compact factors- (resp. of the non-trivial factor defined similarly) of $G^{\mathrm{ad}}_{B(\mathbb{F})}$. Similarly we define $G^{t\mathrm{der}}_{\mathbb{Z}_p}$ and $G^{\mathrm{ntder}}_{\mathbb{Z}_p}$.

In the usual way, using [Va1, 4.3.13], we get that the Zariski closure in $GL(M)$ of the connected component of the center of $\tilde{G}_{B(\mathbb{F})}$, is a reductive subgroup $T_{W(\mathbb{F})}$.

From [Va2, P5 of 4.6] and [Va2, 4.7.30 1)] (the second part of 1) allows us to use the mentioned results in the present context) we deduce: there is a direct sum decomposition

$$(M, F^1, \phi_M) = (M_1, F^1_1, \phi_M) \oplus (M_2, F^1_2, \phi_M),$$

with (M_1, F^1_1, ϕ_M) a usual canonical-lift, with (M_2, ϕ_M) not having the slopes 0 or 1, and with $\mathrm{Lie}(\tilde{G}^{\mathrm{ntder}}_{W(\mathbb{F})}) \subset \mathrm{End}(M_1)$.

This implies directly (the argument is entirely similar to the one of [Va3, 2.2.3], cf. also [Va4, B) of 3.2 V6] and 4.2 V)) that $\tilde{G}^{\mathrm{ntder}}_{W(\mathbb{F})}$ is a semisimple subgroup of $GL(M)$.

Using the definition of the compact factors of $G^{\mathrm{ad}}_{\mathbb{Z}_p}$ based on Galois representations, we deduce that the elements of $\mathrm{Lie}(G^{t\mathrm{der}}_{\mathbb{Z}_p})$ are fixed by this Galois representation. So $\mathrm{Lie}(G^{t\mathrm{der}}_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} W(\mathbb{F})$ is naturally a Lie subalgebra of $\mathrm{Lie}(GL(M))$. We deduce easily that $G^{t\mathrm{der}}_{W(\mathbb{F})}(W(\mathbb{F})) \subset GL(M)(W(\mathbb{F}))$. There are many ways to argue this. For instance, we can start from the fact that the simply connected group cover $\mathcal{G}^{\mathrm{der}}$ of any simple factor of $G^{\mathrm{ad}}_{W(\mathbb{F})}$ is naturally a subgroup of $G^{t\mathrm{der}}_{W(\mathbb{F})}$. So [Ti1, end of 3.1.1] applies: we deduce that $\mathcal{G}(W(\mathbb{F}))$ is a subgroup of $GL(M)(W(\mathbb{F}))$. So [Va1, 3.1.2.1 a) and c)] implies that $\mathcal{G}$ is naturally a subgroup of $GL(M)$. From 2.5.1 we deduce that $G^{t\mathrm{der}}_{W(\mathbb{F})}(W(\mathbb{F}))$ itself is a subgroup of $GL(M)(W(\mathbb{F}))$. So $G^{t\mathrm{der}}_{W(\mathbb{F})}$ can be identified with $\tilde{G}^{t\mathrm{der}}_{W(\mathbb{F})}$.

Using again 2.5.1, for the three "pieces" $T_{W(\mathbb{F})}$, $\tilde{G}^{\mathrm{ntder}}_{W(\mathbb{F})}$ and $\tilde{G}^{t\mathrm{der}}_{W(\mathbb{F})}$, we get that $\tilde{G}_{W(\mathbb{F})}$ is a reductive group.

This ends the proof of the lemma, and so of the theorem.

6.4.1.1. Exercise 10. Prove a $D_n^{\mathbb{H}}$ version of the second paragraph before 6.4.1, of Exercises 8 and 9, of theorem 6.4.1, of 2), 3) and 5) of 6.4.2 and of 6.4.3.2 below.

6.4.2. Remarks. 1) It is easy to see, using the same type of toroidal induction as in Step 5 of 4.2, that in 6.4.1 3) the hypothesis $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is of compact type can be removed, provided we know that the connected components of $\mathcal{N}_{\mathbb{F}}$ are permuted transitively by $G(\mathbb{A}_f^2)$, or that, the equivalent of [Va4, B of V6 of 3.2] holds in the present context.

It is well known that through such an induction, we do not leave the class of symplectic Shimura varieties we have considered (for instance, cf. [Mi1, Example 2.8 of p. 382]).

2) The whole of §5 can be adapted to the context of 6.4.1 (the problem of 5.2 involving a $\mathbb{G}_m(\mathbb{Z}_p)$-multiple of $g(\psi)$ can be sorted out as in the proof of [Va3, 4.3.3]).

3) [Va1, 6.4.11] can be entirely adapted to the compact type context of 6.4.1. The only difference: for the uniqueness part of [Va1, 6.4.11 C)] we presently can not use the EP; so we have to use [MM, Theorem 2] in order to get the uniqueness part. Loc. cit. applies, as for any compact type Shimura pair (G,X) and for every H_0 a small enough compact open subgroup of $G(\mathbb{A}_f^p)$, the connected components of $\mathrm{Sh}_{H_0\times H}(G,X)_{\mathbb{C}}$ are of general type.

4) Using 2.2.1-2 we will see in [Va10] that 6.4.1 extends to all other abelian types. Here is a quick sketch. Using 4.2.1 4) and 2.3.9.1, we can restrict to Shimura varieties of $D_n^{\mathbb{R}}$ types, with n odd, and whose reflex fields are $\mathbb{Q}$. As n is odd we can benefit from the orthogonal variant of Exercise 8. For the rest one needs to combine Step 5 of 4.2 with the proof of 6.4.1. Also it is worth pointing out that in most of the compact type situations, we get significant simplifications.

5) We presently can not say something very precise about the extension of 4.1.2 to the $p=2$ case. The reason is: we do not have yet a good feeling about the Galois representations of Shimura-canonical lifts of Shimura-ordinary 2-divisible groups over $\mathbb{F}$. In other words we do not have a good feeling about the integral version of Fontaine's comparison theory for 2-divisible groups over $W(\mathbb{F})$. In most cases (for instance the cases of C_n, B_n or $D_n^{\mathbb{R}}$ types), based on 2.4.2 and [Va2, 3.10.6], the situation gets easily reduced to the case of Shimura varieties of A_1 type. However, even the general situation of the A_1 type case looks complicated in connection to 4.1.2 for $p=2$.

6.4.3. More on the uniqueness part. We have seen above in the proof of 6.4.1 that one of the thorny issues in mixed characteristic $(0,2)$ is related to the uniqueness of the integral canonical model of an arbitrary Shimura quadruple (G,X,H,v) (i.e. not necessarily of preabelian type), with v a prime of $E(G,X)$ dividing 2. The following result is a natural extension of 6.4.2 3):

6.4.3.1. Proposition. *We assume that (G,X) is of compact type and that (G,X,H,v) has a proper integral canonical model over $O_{(v)}$. Then $\mathcal{N}$ is the only (up to isomorphism) integral canonical model of (G,X,H,v).*

Proof: Let $\mathcal{N}_1$ be another integral canonical model of (G,X,H,v). Let H_0 be a compact open subgroup of $G(\mathbb{A}_f^p)$ such that $\mathcal{N}_1$ (resp. $\mathcal{N}$) is a pro-étale cover of the $O_{(v)}$-scheme $\mathcal{N}_1/H_0$ (resp. of $\mathcal{N}/H_0$) of finite type. We can assume that H_0 is small enough so that the connected components of $\mathrm{Sh}_{H_0\times H}(G,X)$ are of general type. Using [MM, Theorem 2] we deduce easily that it is enough to show that $\mathcal{N}_1/H_0$ is proper over $O_{(v)}$. Using Nagata's embedding theorem (see [Vo]) we deduce that each connected component $\mathcal{C}_0$ of $\mathcal{N}_1/H_0$ is embeddable as an open subscheme into an integral proper $O_{(v)}$-scheme $\mathcal{C}$. We assume that $\mathcal{C}_0$ is not $\mathcal{C}$. So let $\mathrm{Spec}(V)\to\mathcal{C}$ be a morphism, with V a strictly henselian complete DVR of mixed characteristic $(0,2)$, such that the special point of $\mathrm{Spec}(V)$ maps into a point of $\mathcal{C}\setminus\mathcal{C}_0$. Let $K_V:=V\left[\frac{1}{2}\right]$. Let $m_V:=\mathrm{Spec}(K_V)\to\mathrm{Sh}_{H_0\times H}(G,X)$ be the resulting morphism. It extends to a morphism $\mathrm{Spec}(V) \to \mathcal{N}/H_0$ (as $\mathcal{N}/H_0$ is proper over $O_{(v)}$), and so to a morphism $\mathrm{Spec}(V) \to \mathcal{N}$. So m_V lifts to a morphism into $\mathrm{Sh}_H(G,X)$. But this contradicts the fact that $\mathcal{N}_1$ has the EP. Contradiction. This proves the proposition.

We now point out two applications of 6.4.3.1 and its proof. First, by combining it with 6.4.1 1) we get:

6.4.3.2. Corollary. *Let (G_1, X_1, H_1, v_1) be a Shimura quadruple of abelian type such that v_1 divides 2 and all the simple factors of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ are of some C_n Lie type. Then it has a uniquely determined integral canonical model.*

Second, the proof of 6.4.3.1 allows us to go beyond [Va1, 6.4.11 C]. In other words, in loc. cit. we do not need to assume that any smooth D-scheme is healthy. For not being too long we restate (see 6.4.3.3 a)) this fact here just for quadruples.

6.4.3.3. Corollary. *Let (G_2, X_2, H_2, v_2) be an arbitrary Shimura quadruple having an integral canonical model $\mathcal{N}_2$ which is proper (so (G_2, X_2) is of compact type). Let p be the rational prime divided by v_2. Let H_0 be a compact open subgroup of $G_2(\mathbb{A}_f^p)$ such that $\mathcal{N}_2$ is a pro-étale cover of $\mathcal{N}_2/H_0$. Let $\mathcal{D}$ be a Dedekind faithfully flat $O_{(v_2)}$ scheme. We have:*

a) $\mathcal{N}_2/H_{0\mathcal{D}}$ is the only proper smooth scheme over $\mathcal{D}$ having $\mathrm{Sh}_{H_0\times H}(G,X)_{\mathcal{D}[\frac{1}{p}]}$ as its generic fibre.

b) $\mathcal{N}_{2\mathcal{D}}$ is the only smooth integral model of $\mathrm{Sh}_H(G,X)$ over $\mathcal{D}$ having the EP (the def. of [Va1, 3.2.3] extend easily from DVR's to such schemes $\mathcal{D}$).

6.4.3.4. A natural question arises: what about the case of Shimura quadruples in mixed characteristic $(0,2)$ which are not of compact type? There are (presently) two ways to get the uniqueness of integral canonical models of such Shimura quadruples of preabelian type. One way is to follow the pattern of 6.4.1 3). The other way is to use toroidal compactifications and to imitate 6.4.3.1-3 above. See [Va8] for details.

6.5. The p-consistency property revisited. In what follows we include an application of the ideas of (S1 and) S2 of 4.2 K) in connection to the p-consistency property (i.e. to 3.4.1 a) and its extensions mentioned in 3.4.2 2)) and to the simplest cases pertaining to Q17. Also what follows represents a complement to [Va1, 4.3.10.1]. For not being too long, we leave it to the reader to restate 6.5.1-3 below in the more general context (than the one to be considered and) allowed by [dJ1, §1].

A. Let p be an arbitrary prime, and let k be a perfect field of characteristic p. Let $R := W(k)[[T]]$, and let $\mathcal{R}$ be the field of fractions of R. Let Φ be the Frobenius lift of R compatible with σ and taking T to T^p. Let O be the local ring of R which is a DVR of mixed characteristic $(0,p)$. Let $U_R := \mathrm{Spec}(R) \setminus \mathrm{Spec}(k)$ be the open subscheme of $\mathrm{Spec}(R)$ obtained by removing the maximal point. Let $\mathcal{C}$ be a non-degenerate F-crystal over R/pR; let M be the corresponding free R-module of finite rank, and let Φ_M be the Φ-linear map of M. Let

$$\nabla_M : M \to M \otimes_R dTR = dTM$$

be the p-nilpotent integrable connection on M making Φ_M to be ∇_M-parallel. Let $m \in \mathbb{N}$. Let $\mathcal{F} := (v_\alpha)_{\alpha\in\mathcal{J}}$ be a family of homogeneous endomorphisms of

$$\mathcal{C}(m) := \oplus_{i=0}^{m} \mathcal{C}^{\otimes i}$$

(or of $\mathcal{C}(m)_{O/pO}$ as [dJ1, §1] points out). We view it as a family of homogeneous tensors of $\mathcal{T}(M)$ fixed by Φ_M and annihilated by ∇_M. We assume that the Zariski closure in $GL(M \otimes_R O)$ of the subgroup of $GL(M \otimes_R \mathcal{R})$ fixing v_α, $\forall \alpha \in \mathcal{J}$, is a reductive group G_O over O. Let G_R be the Zariski closure of G_O in $GL(M_R)$. The most general form of the p-consistency property can be formulated as the following problem.

B. Problem. Show that G_R is a reductive group over R.

C. If G_R is a reductive group we say that $\mathcal{C}$ is p-consistent (or has the p-consistency property) w.r.t. $\mathcal{F}$. We can always assume that a p-power multiple of the 3 tensors of degree 4 of the proof of 2.3.7.3 but obtained working with the faithful representation $G_{\mathcal{R}} \hookrightarrow GL(M \otimes_R \mathcal{R})$, are members of our family $\mathcal{F}$. The same arguments of S1 and S2 4.2 K) show that:

a) G_{U_R} is a reductive group;
b) to solve the above problem, is enough to show that the Zariski closure of $G_{U_R}^{\mathrm{der}}$ in $GL(M)$ is a semisimple subgroup.

Let $\mathfrak{g}$ be the Lie algebra over R formed by the global sections of the (Lie) sheaf on U_R defined by the Lie algebra of $G_{U_R}^{\mathrm{ad}}$. As an R-module, $\mathfrak{g}$ is free. Moreover, still as in S2 of 4.2 K) we get:

c) to get b), it is in fact enough to show that $\mathfrak{g}$ is the Lie algebra of an adjoint group over R.

So we can assume that $k = \bar{k}$; this implies that R is a strictly henselian local ring. Using Tits' classification (see [Ti2]) of adjoint groups over $\mathcal{R}$, from the classical purity theorem we get easily that each simple factor of $G_{\mathcal{R}}^{\mathrm{ad}}$ is absolutely simple. So let

$$G_{U_R}^{\mathrm{ad}} = \prod_{i \in I} H_i,$$

with H_i as absolutely simple adjoint groups over U_R. Let:

$$\mathfrak{g} = \oplus_{i \in I} \mathfrak{g}_i$$

be the direct sum decomposition of $\mathfrak{g}$ in absolutely simple Lie algebras, corresponding to the above group decomposition. To sort out c) it is enough to concentrate on one such factor $\mathfrak{g}_i$ of $\mathfrak{g}$.

Part 1 of the proof of [Va1, 4.3.10 b)] can be restated (cf. [Va1, 4.3.10.1]):

D. Criterion 1. *If the Killing form on $\mathfrak{g}_i$ is perfect, then there is an adjoint group over R having $\mathfrak{g}_i$ as its Lie algebra.*

E. The above criterion can be improved if instead of Killing forms we use trace forms. This goes as follows. Let G_i be the affine group scheme over R of (Lie) automorphisms of $\mathfrak{g}_i$. Let $\tilde{G}_i$ be the closure of H_i in G_i. Following Part 1 of [Va1, proof of 4.3.10 b)], to show that $\tilde{G}_i$ is an adjoint group over R, it is enough to show that G_{ik} is the extension of a finite étale group over k by an adjoint group over k of the same Lie type as H_i. So we get:

d) to check this last thing for $p \geq 3$ in case G_i is not of G_2 Lie type and for $p \geq 5$ otherwise, it is enough to show that $\mathfrak{g}_i \otimes_R k$ is the Lie algebra of an adjoint group G_{ik}^0 over k.

The argument for d) goes as follows. The hypotheses of d) imply that $\mathfrak{g}_i \otimes_R k$ has regular semisimple elements. As any Lie automorphism $\mathcal{A}$ of it, takes semisimple regular elements into semisimple regular elements, we can assume that $\mathcal{A}$ leaves invariant the Lie algebra of a maximal torus T_{ik} of G_{ik}^0. Using outer automorphisms, we can assume as well that $\mathcal{A}$ normalizes the Lie algebra of a Borel subgroup of G_{ik}, inducing the trivial automorphism of the Dynkin diagram of G_{ik}^0. Now it is easy to check (for instance starting from [BT, 4.2-3]; the case $p \geq 5$ is entirely trivial), that $\mathcal{A}$ has to be an inner automorphism defined by an element of $T_{ik}(k)$ (we recall that $k = \bar{k}$).

d) above has many applications, as in practice it can be easily checked that it is satisfied. For instance, we have:

F. Criterion 2. *We assume that $H_{i\mathcal{R}}$ is either of some B_n Lie type and p is odd, or it is of some A_n or C_n Lie type and the simply connected group cover $\tilde{H}_i$ of H_i is naturally a subgroup of $G_{U_R}^{\mathrm{der}}$, or that it is of some D_n Lie type such that an isogeny cover of the isogeny cover of H_i corresponding to $SO(2n)_R$ is naturally a subgroup of $G_{U_R}^{\mathrm{der}}$. Then $\mathfrak{g}_i \otimes_R k$ is the Lie algebra of an adjoint group over k.*

Proof: We just present here the A_n Lie type case: the other cases are handled entirely in the same manner, starting from Exercise 9 of the proof of 6.4.1, and of its analogues in the orthogonal cases.

So let H_i be of A_n Lie type, for some $n \in \mathbb{N}$. Let Z be the finite flat commutative group scheme over R whose restriction to U_R is the center of $\tilde{H}_i$ (cf. [FC, 6.2 of p. 181]). It is easy to see, starting from the very particular shape of R (local regular strictly henselian ring), that it is isomorphic to μ_{n+1}, even if $p = 2$. So let $\bar{H}_i$ be the central extension of H_i by $\mathbb{G}_m$ and of whose derived group is $\tilde{H}_i$. Let $\overline{\mathfrak{h}}_i$ be the Lie algebra over R formed by the global sections of the (Lie) sheaf over U_R of its Lie algebra. The standard trace from on $\overline{\mathfrak{h}}_i$ is prefect, being perfect in codimension one. As $\tilde{H}_i$ is simply connected, we deduce the existence of a complete DVR V of mixed characteristic $(0, p)$ having k as its residue field, of a morphism $\mathrm{Spec}(V) \to \mathrm{Spec}(R)$ such that $\bar{H}_{V[\frac{1}{p}]}$ is split, and of a subrepresentation W_i of the representation of the split group $\bar{H}_{V[\frac{1}{p}]}$ on $\mathcal{T}(M \otimes_R V[\frac{1}{p}])$, which is a standard $n+1$ dimensional irreducible representation. We can assume that $\overline{\mathfrak{h}}_i \otimes_R V$ acts on the closure $\tilde{W}_i$ of W_i in $M \otimes_R V$. As the trace form on $\overline{\mathfrak{h}}_i \otimes_R V$ is perfect, we deduce that $\overline{\mathfrak{h}}_i \otimes_R V = \mathrm{End}(\tilde{W}_i)$. This proves the second criterion.

G. Criterion 3. *Let $p \geq 3$. We assume that $\mathfrak{g}_i$ is of classical Lie type. Then $\mathfrak{g}_i \otimes_R k$ is the Lie algebra of an adjoint group over k.*

Proof: We have to analyze the situation case by case. If $\mathfrak{g}_i$ is of some A_n Lie type, then we use the notations of F above. We have a canonical (non-commutative) R-algebra structure on $\overline{\mathfrak{h}}_i$ and a faithful representation

$$\bar{H}_i \hookrightarrow GL(\overline{\mathfrak{h}}_i)_{U_R}.$$

So F above applies as well to this representation.

If $\mathfrak{g}_i$ is of B_n Lie type then we can apply F above to the adjoint faithful representation $H_i \hookrightarrow GL(\mathfrak{g}_i)_{U_R}$.

If $\mathfrak{g}_i$ is of D_n Lie type, then using a trick similar to the one of 3.2.6 12), the situation gets reduced to the B_n Lie type. In other words, the isogeny cover of H_i corresponding to $SO(2n)_R$ is embeddable into a semisimple adjoint group over U_R of B_n Lie type; this is so as $H_{i\mathcal{R}}$ is an inner form of its split form $H_{i\mathcal{R}}^{\mathrm{sp}}$ definable (as $k = \bar{k}$ and as the case of A_1 Lie type has been already handled) by an element of $H^1(\mathcal{R}, H_{i\mathcal{R}}^{\mathrm{sp}})$, and the inner form of the split adjoint group over $\mathcal{R}$ of B_n Lie type we get, extends to an adjoint group H_i^B over U_R. As H_i^B extends to an adjoint group H_{iR}^B over R, the derived subgroup of the centralizer of an element of order 2 of $H_{iR}^B(R)$ is the extension of H_i to R.

Similarly, if $\mathfrak{g}_i$ is of C_n Lie type then we have to use a trick similar to the one of 2.3.6 7) with m of the mentioned place belonging to the set $\{1, 2\}$. So as in the proof of 2.3.7.2, the situation gets reduced to the case when the Killing form on $\mathfrak{g}_i$ is perfect. So D above applies. This proves the third criterion.

H. Using [Bo1, 6.4 and 7.3] one can get many criteria similar to the one of criterion 2 above. In what follows we will refer only to the above three criteria. We have:

6.5.1. Theorem. *Let $p \geq 3$. We assume that for any $i \in I$, $\mathfrak{g}_i$ is of classical Lie type, or it is of exceptional Lie type and D above holds for it. Then $\mathcal{C}$ is p-consistent w.r.t. $\mathcal{F}$.*

I. We will now mention applications of the above three criteria to p-divisible groups. We come back to the case of an arbitrary perfect field k. We assume that $\mathcal{C}$ is the F-crystal associated to a p-divisible group D over R/pR. Let $\mathcal{K}$ be the algebraic closure of O/pO. As in [Va2, 3.10]] we can define the compact and the non-compact factors of the adjoint Lie F-crystal $(\mathfrak{g}, \Phi_M)$. So the compact factors are related to direct summands

$$\mathfrak{g}_{I_1} := \oplus_{i \in I_1} \mathfrak{g}_i,$$

with I_1 a subset of I, such that Φ_M induces a Φ-linear isomorphism of them, and Φ_M permutes the factors $\mathfrak{g}_i$, $i \in I_1$, transitively. We have:

6.5.2. Corollary. *If $p \geq 3$, then $\mathcal{D}_k$ is automatically p-consistent.*

Proof: We can assume $k = \bar{k}$. We just have to mention: if $\mathfrak{g}_{I_1}$ is a compact factor of $(\mathfrak{g}, \Phi_M)$ (here I_1 is as above), then we have $\mathfrak{g}_{I_1} = \mathfrak{g}^{I_1} \otimes_{\mathbb{Z}_p} R$, with $\mathfrak{g}^{I_1}$ as the $\mathbb{Z}_p$-subalgebra of $\mathfrak{g}_{I_1}$ formed by elements fixed by Φ_M. The fact that $\mathfrak{g}^{I_1}$ is the Lie algebra of an adjoint group over $\mathbb{Z}_p$ can be read out by looking at $D_{\mathcal{K}}$. This implies that $\tilde{G}_i$, with $i \in I_1$, is automatically an adjoint group (so for $p = 5$ we do not have to put aside the G_2 Lie type).

So the corollary follows from 6.5.1 and from the fact that all the non-compact factors of $(\mathfrak{g}, \Phi_M)$ are of classical Lie type (cf. [Sa, p. 460-1] or [Se, ch. 3]).

6.5.3. Remarks. 1) 6.5.2 is the second way (besides 6.3.1 A) of fulfilling 4.2.1 10).

2) We do not know if the restrictions of criteria 1 to 3 (and so of 6.5.1-2) are really needed.

6.5.4. A digression on Q17. What follows is very closely related to question Q17 of 5.3. However, we start with a much wider context. Let S be a normal integral scheme, and let U_S be an open subscheme of S containing all the points of codimension at most 1. Let G_{U_S} be a reductive group over U_S. Here we will deal with the following question.

Q20. When G_{U_S} extends to a reductive group over S?

As any reductive group over S, as an S scheme is affine, such an extension if exists is uniquely determined. So the problem gets reduced easily to the case when G_{U_S} is a torus and to the case when G_{U_S} is an adjoint group. In other words, if $G_{U_S}^{\mathrm{ab}}$ (resp. $G_{U_S}^{\mathrm{ad}}$) extends to a torus G_S^{ab} (resp. adjoint group G_S^{ad}) over S, then the normalization of $G_S^{\mathrm{ab}} \times G_S^{\mathrm{ad}}$ in the field of fractions of G_{U_S}, is a reductive group G_S over S extending G_{U_S}. Sometimes it is desirable to work with semisimple groups instead of adjoint groups.

In most geometric applications we have S normal (for instance, $\mathcal{N}$ of 4.2 was initially known to be just normal). However it turns out that it is better to treat first the case when S is regular.

6.5.4.1. Case 1: G_{U_S} is a torus T_{U_S}. Using standard properties (see [BLR, Prop. 5 of p. 195]) of the process of Weil restriction from U_S^1 to U_S, with U_S^1 an étale connected cover of U_S, and [FC, 2.9 of p. 11] we get immediately:

Fact 1. *Any connected étale cover of U_S extends to an étale cover of S, iff any torus over U_S extends to a torus over S.*

From this fact and the classical purity theorem we get:

Fact 2. *We assume that S is regular. Then any torus over U_S extends to a torus over S.*

To express our results faster, as Q20 is a local question in the Zariski topology, we assume in Case 2 below that S is a $\mathbb{Z}_{(p)}$-scheme, for some prime p.

6.5.4.2. Case 2: G_{U_S} is a semisimple group and S is regular. We now treat the case when S is regular. Let $\mathcal{G}$ be the (Lie) sheaf over U_S defined by the Lie algebra of G_{U_S}. Passing to an étale connected cover of S we can assume that all the simple factors of G_{U_S} are absolutely simple, and so we can assume that G_{U_S} itself is absolutely simple. Let τ be its Lie type. We can also assume that:

INN. *(The generic fibre of) G_{U_S} is an inner form of a split group.*

Definition. If locally in the Zariski topology of S, there is a locally free sheaf $\mathcal{F}$ on S of finite rank, and there is a closed subgroup embedding

$$q : G_{U_S} \hookrightarrow GL(\mathcal{F})_{U_S}$$

(resp. such that the resulting trace form on $\mathcal{G}$ defined by q is perfect), then we say $\mathcal{G}$ or G_{U_S} (resp. that $\mathcal{G}$) has local representations (resp. that $\mathfrak{g}$ has a perfect trace form).

Using standard arguments on codimension one we get immediately:

Fact 3. *If there is a perfect trace form on $\mathcal{G}$, then $\mathcal{G}$ extends to a locally free sheaf over S.*

In practice, we go the other way round of Fact 3. For instance we have:

Fact 3'. *The existence of a perfect trace form on $\mathcal{G}$ is automatic, if G_{U_S} is adjoint, if $\mathcal{G}$ extends to a locally free sheaf over S, and if any one of the following two conditions is satisfied:*

i) S has no points of characteristic p;

ii) the Killing form on the Lie algebra of a split simple adjoint group over $\mathbb{F}_p$ of τ Lie type is non-degenerate.

Proof: We just need to apply Fact 3 to the standard adjoint representation.

We state here, with no extra comments, the following abstract extension of 6.5 D:

Fact 4. *If we are in the context of Fact 3', then G_{U_S} extends to an adjoint group over S.*

So, if $\mathcal{G}$ extends to a locally free sheaf over S, then we can assume that U_S contains all the points of characteristic zero; if moreover the characteristic of S is zero, then we can assume that (S, U_S) is an extensible pair, and we are getting closer to the context of Q17.

In general, we have:

6.5.4.2.1. Theorem. *Let $p \geq 3$. We assume that one of the following three conditions is satisfied:*

1) τ is a classical Lie type and $\mathcal{G}$ extends to a locally free sheaf over S;

2) Fact 4) above applies;

3) τ is a classical Lie type and the local rings of S are of dimension at most two.

Then G_{U_S} extends to a semisimple group over S.

Proof: The proof of 3) is entirely the same as the proof of 6.5.1. There is nothing to be added to 2). The proof of 1) is entirely the same as the corresponding part of the proof of 6.5 C and E, starting from INN (i.e. one easily reduces 1) to 3)). This proves the theorem.

6.5.4.2.2. Remarks. 1) We do not how to get extra criteria on the extension of $\mathcal{G}$ to a locally free sheaf over S, which do not assume that G_{U_S} has some (good) local representations. In particular, we do know when Fact 3 (or variants of it involving reductive group) applies, if not to $\mathcal{G}$ itself to a similar one $\mathcal{G}'$, obtained from a central isogeny of G_{U_S}.

2) See [Va9] for refinements of the above results for the E_6 and E_7 Lie types.

3) If by chance we know that G_{U_S} has a maximal torus T_{U_S}, then it extends to a torus T_S over S, and we get immediately that G_{U_S} extends to an adjoint group over S (passing to a connected étale cover of S, we can assume that T_S is split; the rest is trivial even for $p = 2$). However, in practice it is very hard to check the existence of such a torus.

4) Part of what we did above can be worked out for $p = 2$ as well (cf. also [Va8]).

5) In the above Theorem 3) is just a particular case of 1): it is included just for the sake of future references.

6.5.4.3. Case 3: G_{U_S} is a semisimple group and S is normal. The regularity of S in case 2 above was needed only in two parts: first to get that (under some conditions) $\mathcal{G}$ extends to a locally free sheaf over S, and second in the part involving (in the étale topology of S) absolutely simple factors.

So if S is the spectrum of a normal local ring for which the classical purity theorem fails, then using Weil restriction of scalars, we can easily build up examples of adjoint groups G_{U_S} over some U_S for which $\mathcal{G}$ does not extend to a locally free sheaf over S, and so they do not extend to smooth groups over S. It is not clear to us if we can build up such examples so that we do have a monomorphism as q of the def. of Case 2. However, we some extra work, we still have the following extension of 6.5.4.2.1:

6.5.4.3.1. Theorem. *We assume that G_{U_S} is an adjoint group, that $\mathcal{G}$ extends to a locally free sheaf $\tilde{\mathcal{G}}$ over S and that the Lie types of G_{U_S} are such that 6.5.1 applies to them. Then G_{U_S} extends to an adjoint group over S.*

Proof: It is an easy exercise to see that locally in the étale topology of S we get absolutely simple factors (due to the locally freeness of $\tilde{\mathcal{G}}$, the statement gets reduced to the following trivial one: a finite locally free S-algebra, which as an S-scheme is étale above points of U_S, is an étale S-algebra). But 6.5 E handles all integral schemes, and not only the regular ones (as Part 1 of [Va1, proof of 4.3.10 b)] does so). So this and loc. cit. allow us to assume that we are dealing with a classical Lie type, and to replace S by its normalization in a finite field extension of its field of fractions .

Accordingly, we can assume that S is local, noetherian and of dimension two (so one could be tempted to use the resolution of singularities, but this is not possible here), and that (the generic fibre of) G_{U_S} is split. Enlarging S even more we can assume that all the fibres of G_{U_S} are split. Now we have to use Exercise 8 of the proof of 6.4.1. We will detail this for the symplectic space (i.e. for the C_n Lie type), as the other cases are entirely the same (for the orthogonal types, one just needs to replace below symplectic by symmetric, while, for the A_n Lie types, one can use entirely the ideas of 6.5 F and G).

So let $\mathcal{M}$ be a locally free sheaf on U_S of rank $2n$ such that its generic fibre is endowed with a perfect symplectic form ψ, and the simply connected group cover of G_{U_S} is a subgroup of $GL(\mathcal{M})$ in such a way that its generic fibre centralizes ψ. We do not know a priori that (we can assume that) $\mathcal{M}$ extends to a locally free sheaf on S; but it does extend to a quasi-coherent sheaf on S. This is enough in order to apply Exercise 8. This proves the theorem.

6.5.4.3.2. Remarks. 1) We can not say anything about a potential variant of the above theorem for tori.

2) When $\mathcal{G}$ does not extend to a locally free sheaf on S, one could try to see if G_{U_S} extends at least to some S-group. Also, we can not say anything in this direction.

For future references we state here a combined version of Case 1 and Case 2. We have:

6.5.4.4. Theorem. *We assume that S is regular. Then any reductive group H over U_S such that 6.5.4.2.1 applies (after a replacement of S by an étale connected cover of it) to $H^{\mathrm{ad}}_{U_S}$ extends to a reductive group over S.*

6.5.5. An interpretation. 6.5.4.2-3 lead to the following question. Referring to the context of Q17 of 5.3, how do we check in practice the existence of an open subscheme U_R of R such that $(\mathrm{Spec}(R), U_R)$ is an extensible pair, and $\mathcal{G}_{1U_R}$ is a reductive group?

In [Va1] and in this part 2C, in a suitable context of Shimura varieties of Hodge type, we developed three methods. In the first method (cf. [Va1, 4.3]), one needs to check that some tensors are integral; it is a general method. The other two methods are strictly particular to Shimura varieties of Hodge type. In the second method, one needs to check that (the equivalent of) Milne's conjecture (dealt with in [Va4]) holds, and that we have some good closed subschemes of Spec(R), and that we have some good "deformation theory". The third method, refers to the possibility of constructing good compactifications of Spec(R) so that using them, we can show that in fact Spec(R/pR) has only one generic point.

Also it is worth pointing out, that for many applications (for instance cf. S3 of 4.2 I)), it is enough to be able to get 6.5.4.3.1, after passage to a finite surjective morphism $S_1 \to S$, with S_1 a normal scheme.

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