Shimura Varieties and the Mumford–Tate conjecture

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ABSTRACT. We prove the Mumford -Tate conjecture for those abelian varieties over number fields, whose simple factors of their adjoint Mumford–Tate groups have (over $\mathbb{R}$) certain (products of) non-compact factors. In particular we prove this conjecture for the orthogonal case. We construct embeddings of Shimura varieties of (whose adjoints are of prescribed abelian type) into unitary Shimura varieties. We also prove two general theorems involving Frobenius tori of such abelian varieties.

Key words: abelian and Shimura varieties, reductive and p-divisible groups, Galois representations and Hodge cycles.

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Contents

§1. Introduction

1.1. Let A be an abelian variety over a number field E, and let p be a rational prime. Let H_A (resp. $G_{\mathbb{Q}_p}$) be the Mumford–Tate group of A (resp. be the connected component of the origin of the algebraic envelope of the p-adic representation naturally attached to A), cf. 4.0. $H_{A\mathbb{Q}_p}$ can be viewed naturally as a subgroup of $GL(H^1_{\text{ét}}(A_{\bar{E}},\mathbb{Q}_p))$ (see 4.0). The Mumford–Tate conjecture (see 4.1) asserts: as subgroups of $GL(H^1_{\text{ét}}(A_{\bar{E}},\mathbb{Q}_p))$, we have $H_{A\mathbb{Q}_p} = G_{\mathbb{Q}_p}$. Our approach to the proof of this conjecture is via the following five main steps.

S1. A well known result (see 4.2.5) allows us to assume that p is as big as desired. In particular we can assume the existence of a prime v of E unramified over p, and such that A has good reduction with respect to it.

S2. Standard arguments on Frobenius tori (for their construction reviewed in 4.2.7.1, see [Chi] and [Pi]) allow us to assume that $G_{\mathbb{Q}_p}$ is a split reductive group (see 4.2.8).

S3. Using the main result of [Wi] we can define each simple factor $\mathcal{F}$ of $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ or of $H^{\mathrm{ad}}_{A\mathbb{Q}_p}$ to be compact or non-compact, depending on the fact that a suitable injective cocharacter

$$\mu\colon \mathbb{G}_m \hookrightarrow G_{\mathbb{Q}_p}$$

(see 4.3.3 and 4.4.4), has a trivial or a non-trivial image in $\mathcal{F}$ (for details see 5.1.3). The third step is to try to show that the product of the non-compact factors of $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ is isomorphic to the product of the non-compact factors of $H^{\mathrm{ad}}_{A\mathbb{Q}_p}$.

S4. Assuming S3, we show that in fact $G^{\mathrm{der}}_{\mathbb{Q}_p} = H^{\mathrm{der}}_{A\mathbb{Q}_p}$.

S5. Assuming S4, by standard arguments we show that $G_{\mathbb{Q}_p} = H_{A\mathbb{Q}_p}$.

Combining S1-5, we stated in [Va2, 1.19] the following expectation (see also 4.1.1):

1.2. Expectation (the split criterion). *There is $N(A) \in \mathbb{N}$, such that if $p \geq N(A)$ and if $G_{\mathbb{Q}_p}$ is a split group, then $G_{\mathbb{Q}_p} = H_{A\mathbb{Q}_p}$.*

It is well known (see 4.2.9) that this split criterion implies that $G_{\mathbb{Q}_p} = H_{A\mathbb{Q}_p}$ for any prime p.

1.3. Conventions and notations. For F a field we denote by $\bar{F}$ its algebraic closure. We denote by E_v the completion of a number field E with respect to a finite prime of it v. If G is a group acting on a set S, we denote by S^G the subset of S formed by elements fixed by G.

The expression "standard Hodge situation" is abbreviated as SHS (cf. [Va2, 2.3.5]). We use freely the notations and conventions of [Va2, 2.1 and 2.3.7]. Whenever we consider a SHS $(f, L_{(p)}, v)$ (cf. [Va2, 2.3]), we use freely the notations introduced in [Va2, 2.3.1-3]. Always by k we denote a perfect field of characteristic $p \geq 3$. $W(k)$ is the Witt ring of k, $B(k) := W(k)\left[\frac{1}{p}\right]$, while σ is the Frobenius automorphism of $W(k)$ or of $B(k)$.

For the notions of (principally quasi-polarized) Shimura (filtered) σ-crystals or p-divisible groups we refer to [Va2, 2.2]; for the case of these notions in the context of a SHS, we refer to [Va2, 2.3.9-11]. The notion of a Shimura p-divisible group (over a regular formally smooth $W(k)$-scheme) being a versal deformation (at some maximal point which is algebraic over k), is understood in the sense of [Va2, 3.4.19 B)].

If X_S is a scheme over a scheme S, then for any other S-scheme T, we often denote, without any other extra comment, by X_T the S-product of X_S and T. We use freely the standard notations for the fundamental weights of a split simple Lie algebra of A_n Lie type, as in [Bou, p. 188].

1.3.1. In [Va1-2] we developed some techniques pertaining to Shimura varieties and to the deformation theory of abelian varieties endowed with de Rham (crystalline) cycles. Here (to be continued in [Va5]) we start exploiting them for achieving progress in the proof of the Mumford–Tate conjecture. In this paper, we concentrate mostly in using techniques from reductive groups (see [Va1, 4.3]), from local deformation theory (see [Va1, 5.2-4]), and from Shimura varieties (see [Va1, 6.5-6] and §3). In [Va5] we will enlarge these techniques by the one involved in global deformations (see [Va2, 3.4]), and the ones involved in our work on the Langlands–Rapoport conjecture (see the announcements of [Va1, 1.7.1] and [Va5]). We do hope that all these techniques and ideas will match to a complete proof of the conjecture.

For samples of what the techniques presented here can achieve see 1.5. The techniques are elaborated in §3 (the ones involving Shimura varieties) and in §4 (the ones involving reductive groups and the local deformation theory). For the details of how the paper is structured see 1.6-9.

1.4. The context. Let us consider the situation of 1.2. Let $(v_\alpha)_{\alpha\in\mathcal{J}}$ be a family of homogeneous tensors of the essential tensor algebra $\mathcal{T}(H^1_{\text{ét}}(A_{\bar{E}},\mathbb{Q}_p))$ of $H^1_{\text{ét}}(A_{\bar{E}},\mathbb{Q}_p)\oplus H^1_{\text{ét}}(A_{\bar{E}},\mathbb{Q}_p)^*$ such that $G_{\mathbb{Q}_p}$ is the subgroup of $GL(H^1_{\text{ét}}(A_{\bar{E}},\mathbb{Q}_p))$ fixing v_α, $\forall\alpha\in\mathcal{J}$. Such a family exists (cf. [De2, 3.1 c)]). Let $G_{\mathbb{Z}_p}$ be the split reductive group over $\mathbb{Z}_p$ having $G_{\mathbb{Q}_p}$ as its generic fibre. For our proposed number $N(A)\in\mathbb{N}$ see 4.3.0; the main new ideas are based on the philosophy of 4.3.4.1. Here we just mention that $N(A)$ depends only on A, and that it is effectively computable. In what follows we refer to this context, but we do not mention $N(A)$: we just take p to be big enough and odd, and such that we are as well in the context of S1. Also it is irrelevant which complex embeddings we choose for different number fields, and so we do not mention them.

1.5. The main results. To state some of the results of this paper, let (H_A,X_A) be the Shimura pair defined by A (cf. [Va1, 2.12 3)]). We write the adjoint Shimura pair

$$(H_A^{\text{ad}},X_A^{\text{ad}})=\prod_{i\in I}(H_i,X_i)$$

as a finite product of simple adjoint Shimura pairs. We have (see 5.1.2):

Theorem 1. *The split criterion holds if $\forall i\in I$, the adjoint Shimura pair (H_i,X_i) is one of the following types:*

a) non-special A_n type;
b) B_n type;
c) C_n type, with n odd;
c') C_2 type;
d) $D_n^{\mathbb{H}}$ type, with n odd and such that $2n$ is not of the form $C_{2^{m+1}}^{2^m}$, with $m\in\mathbb{N}$;
d') $D_n^{\mathbb{H}}$ type with non-trivial involution, with $n\in\{2q,6q|q\text{ a prime greter than }3\}\cup\{4,6,12\}$ or with n odd;
e) $D_n^{\mathbb{R}}$ type.

Here we do not describe in details what means non-special A_n type (see 5.1.2.1). We just mention four large families of cases of such types.

Examples (see 5.1.2.3). *(H_i,X_i) is of non-special A_n type, if one of the following four conditions is satisfied:*

a) $H_{i\mathbb{R}}$ *has two simple non-compact factors* $SU(a_i,b_i)^{\mathrm{ad}}_{\mathbb{R}}$, $i=\overline{1,2}$, *such that* $a_1 < a_2 \le b_2 < b_1$, *and either* $(a_1,b_1)=1$ *or* $(a_2,b_2)=1$;

b) $H_{i\mathbb{R}}$ *has a simple non-compact factor* $SU(a,b)^{\mathrm{ad}}_{\mathbb{R}}$, *with* a *and* b *relatively prime and such that the pair* (a,b) *is not of the form* (C_r^s, C_r^{s-1}), *for* $s,r\in\mathbb{N}$, $2\le s\le r-1$;

c) $H_{i\mathbb{R}}$ *has as a factor a product* $SU(a,a)^{\mathrm{ad}}_{\mathbb{R}}\times SU(b,2a-b)^{\mathrm{ad}}_{\mathbb{R}}$ *with* b *different from* a *and odd;*

d) H_i *is of* A_{q-1} *Lie type, with* q *equal to* 4 *or to a prime.*

Let A_v be the reduction of A with respect to v. We consider the Frobenius torus T_v defined by A_v. We have:

Theorem 2 (see 4.4.8). *We assume that* A_v *is an ordinary abelian variety. Then* T_v *is isomorphic to a subtorus of* H_A, *and* $T_{v\mathbb{Q}_p}$ *is isomorphic to a subtorus of* $G_{\mathbb{Q}_p}$.

Theorem 3 (see 3.2.2 and 3.2.2.1). *Let* $i\in I$. *Let* $l\ge p+2$ *be a prime. As above, for any prime* w *of* E *dividing* p, *we construct a Frobenius torus* T_w. *We view* $T_{w\mathbb{Q}_l}$ *as a subgroup of* $G_{\mathbb{Q}_l}$. *We have: the image of* $T_{w\overline{\mathbb{Q}_l}}$ *in any simple factor of* $H_{i\overline{\mathbb{Q}_l}}$ *has the same dimension.*

1.6. Our work on the Mumford–Tate conjecture grew out (see [Va2, 3.4.20 2)]) of Milne's conjecture ([Mi1, 0.1]; see also [Va1, 5.6.6]). A great part of [Va5] is dedicated to this conjecture; so here (in §2) we first state it as a question in a more general context, and then prove it in two generic situations (see 2.2.1-2) as well as in a special case (see 2.2.3), just to pay due homage to it and to be able to apply it in §3. Some extra complements to [Va1-2] are included in 2.3-5. In particular we draw attention on 2.4: it is a complement to the induced property of [Va2, 3.4.18.5], in the most simple ramified context.

1.7. To get hold of the compact factors (i.e. to achieve S4 above and to prove Theorem 3) we need to use a proposition on adjoint Shimura varieties of abelian type. So in the first part of §3 we prove a rational version of the result stated in [Va2, 1.23 1) to 4)]. It is just the rational version of the results (over $\mathbb{R}$) of [Sa], obtained following the pattern of [Va1, 6.6.2]. In other words, starting from an adjoint Shimura pair (G,X) of abelian type, we construct (see 3.1 and 3.1.2 2)) useful injective maps

$$(G_0,X_0)\hookrightarrow(G_1,X_1)\xhookrightarrow{f_1}(GSp(W,\psi),S),$$

with $(G_0^{\mathrm{ad}},X_0^{\mathrm{ad}})=(G,X)$, with f_1 a PEL type embedding (see [Ko] for the meaning of this), and with G_1^{der} a simply connected group whose adjoint group has all its simple factors of A_n Lie type, for some $n\in\mathbb{N}$. This rational version of [Va2, 1.23] is good enough to get as well an effective generic version of it (see 3.1.1 and 3.1.2 3)). The proof of 3.1 relies on [De1, 2.3.10], being as well the unitary version of loc. cit. Instead of [De1, 2.3.10] one can use as well the PEL type embeddings of Shimura (see [Sh]); this approach leads to a new proof of the essence of [De1, 2.3.10], cf. 3.1.2 5). 3.1.1 has many applications (see 1.7.1 and 3.1.2 4)): as part of its proof is slightly sketched, they are not stated here.

1.7.1. In the second part of §3 we prove Theorem 3. It is the main new ingredient needed to accomplish (via Zarhin's lemma stated in 4.2.5.1, and 3.2.3 ADV) and 3.2.3.1) S4 of

1.1 (see Step 2 of 5.1.3). The proof of Theorem 3 relies on 3.1, on standard arguments involving Frobenius tori, and on the fact that the Langlands–Rapoport conjecture is known to be true (see [Mi1]) for Shimura varieties of A_n type, $n \in \mathbb{N}$. In fact, for what we need from this last conjecture, the main result of [Zi] suffices. We do not stop here to detail 3.2.3 ADV) and 3.2.3.1 in the general case; however, as a sample, see 1.8 for how they work in the case $(H_A^{\mathrm{ad}}, X^{\mathrm{ad}})$ is a simple adjoint Shimura variety of $D_n^{\mathbb{R}}$ type.

This application (i.e. Theorem 3) of 3.1, though it is used just in 5.1.3-4 beginning with the Step 2 of 5.1.3, it is stated and proved in 3.2, for not interfering with the flow of ideas of §4-5. For it, we assume that the reader is familiar with the results reviewed in the parts running from 4.0 to 4.2.14, as well with the beginning paragraph of 5.1.3. So the reader less familiar with these reviewed results, should read 3.2 only after reading 4.0-2 and the mentioned part of 5.1.3.

The same ideas of 3.1-2 can be used in many other situations; in particular they reobtain (see 3.2.2.2) the results of [Pi, 5.10-11] (recalled in 4.2.14), provided we know that the ordinary reduction conjecture (for instance see [Oo, p. 11]) is true for A.

1.8. The proof of Theorem 1 is standard (to be compared with [Pi, 5.14-15]). There are three main new ideas. First, in the A_n type case, besides Serre's result (see [Pi, 3.5]) and Pink's results (see [Pi, 5.10-11]), we rely on ideas of [Zi]. In other words, if all the Shimura pairs (H_i, X_i) are of some A_n type, then using the main result of [Zi] we can check directly that T_v is isomorphic to a subtorus of an inner form of H_A (we recall that p is big enough).

Second, we do not need requirements on (the dimension or the structure of) H_A^{ab}, as the local deformation theory presented below together with the simple Fact 4.2.7.2, make them unnecessary. This achieves S5 of 1.1 (cf. Step 3 of 5.1.3).

Third, we use more intensively the general properties of Shimura varieties (like [De1, 2.3.10] and [Va1, 6.5-6]; see 3.2.3 for samples). In particular 3.2.3 ADV) and 3.2.3.1 are the only new ingredients needed to get e) of Theorem 1 (see 5.1.4 E)). As a sample of how these new ingredients work, let us assume that I has just one element and that $(H_i, X_i) = (H_A^{\mathrm{ad}}, X_A^{\mathrm{ad}})$ is as in the e) part of Theorem 1.

It is known (cf. [Pi, 4.3 and 5.10]) that the image $\mathcal{F}_1$ of $G_{\overline{\mathbb{Q}_p}}$ in a simple factor $\mathcal{F}_0$ of $H^{\mathrm{ad}}_{A\overline{\mathbb{Q}_p}}$ is either $\mathcal{F}_0$ itself or a suitable product PB of two simple adjoint groups of some B_n Lie type (here the B_0 Lie type corresponding to the trivial group is allowed); but 3.2.3 ADV) allows us to assume that the representation of $\mathrm{Lie}(\mathcal{F}_0)$ on $H^1 := H^1_{\text{ét}}(A_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p) \otimes \overline{\mathbb{Q}_p}$ involves both the half-spin representations, and that any irreducible (non-trivial) subrepresentation of $\mathrm{Lie}(H^{\mathrm{der}}_{A\overline{\mathbb{Q}_p}})$ on H^1 factors through a simple factor of it (i.e. no tensor products are involved). So if $\mathcal{F}_1$ is not $\mathcal{F}_0$, we reach a contradiction to the well known theorem of Faltings recalled in 4.2.3, as the representation of $\mathrm{Lie}(PB)$ on a simple $\mathrm{Lie}(\mathcal{F}_0)$-submodule $H^1_{\prime}$ of H^1, left invariant by $\mathrm{Lie}(G_{\overline{\mathbb{Q}_p}})$, is the same, regardless of which half-spin representation is involved in defining $H^1_{\prime}$.

It is 3.2.3.1 which shows that we can assume that, for any arbitrarily chosen such $\mathrm{Lie}(\mathcal{F}_0)$-submodule $H^1_{\prime}$, the representation on it of the connected component of the center of $H_{A\overline{\mathbb{Q}_p}}$ is the same scalar representation as for a similar simple $\mathrm{Lie}(\mathcal{F}_0)$-submodule $H^1_{\prime\prime}$ but involving the other half-spin representation; so indeed in the above paragraph we can apply 4.2.3. Moreover, an entirely similar argument (based on 3.2.3.1) shows that $\mathrm{Lie}(\mathcal{F}_0)$ can not project onto two or more distinct simple factors of $\mathrm{Lie}(H^{\mathrm{ad}}_{A\overline{\mathbb{Q}_p}})$.

In the case (H_i, X_i) is of non-special A_n type with non-trivial involution, the above first and third ideas have to be combined in order to get that a simple factor of $\mathrm{Lie}(G^{\mathrm{ad}}_{\overline{\mathbb{Q}_p}})$ does not project onto two or more simple factors of $\mathrm{Lie}(H^{\mathrm{ad}}_{A\overline{\mathbb{Q}_p}})$ (cf. Step 2 of 5.1.3).

1.8.1. To our knowledge, e) and a) of Theorem 1 are completely new general instances for the validity of the conjecture. For instance the e) part was not known not even in the case when $\mathrm{End}(A_{\mathbb{C}}) = \mathbb{Z}$. Very particular cases of a), b), c), c'), d) and d') of Theorem 1, but involving Shimura varieties of arbitrary dimension, were known previously (see the references of 4.2.16). From the point of view of possibilities of types of $(H_A^{\mathrm{ad}}, X_A^{\mathrm{ad}})$, Theorem 1 handles about 77 percent. Referring to parts of Theorem 1, this percentage is counted as follows: 20 percent for each b) and e), 16 percent for a), just 10 percent for c) and c'), and 11 percent for d) and d'). For the simplest cases not settled by Theorem 1 see 5.1.2.4.

1.9. To explain the ideas behind Theorem 2, we need some notations. Let $W(\mathbb{F})$ be the completion of the maximal unramified cover of the ring of integers O_v of E_v, and let σ be its Frobenius automorphism. It is quite convenient (from the notations point of view) to work what follows over $W(\mathbb{F})$ instead of over O_v (however see 4.4.7.2).

After some nice arrangements (cf. 4.3) we can put everything in the context of a Shimura filtered σ-crystal

$$\mathcal{C} := (M, F^1, \phi, G_{W(\mathbb{F})}, (t_\alpha)_{\alpha \in \mathcal{J}})$$

over $\mathbb{F}$. Here M, F^1 and ϕ are as explained below, while t_α, $\alpha \in \mathcal{J}$, is the homogeneous tensor of the essential tensor algebra of $M \oplus M^*[\frac{1}{p}]$ corresponding under the Fontaine's comparison theory to v_α; moreover $G_{W(\mathbb{F})}$ is the subgroup of $GL(M)$ whose generic fibre is defined as the subgroup fixing t_α, $\forall \alpha \in \mathcal{J}$. So here is the place where we need p to be big enough: for these arrangements we use (see 4.3.4) the theory of $\mathbb{Z}_p$-well positioned family of tensors (cf. [Va1, 4.3]; we mainly need [Va1, 4.3.10 b)]).

To get to this context of $\mathcal{C}$ we need to pass to a $\mathbb{Z}[\frac{1}{p}]$-isogeny A' of $A_{E'}$ (with E' a finite field extension of E) and to a prime v' of E' dividing v, cf. 4.3.5-6. What we achieve by doing this: the closure of $G_{\mathbb{Q}_p}$ in $GL(H^1_{\acute{e}t}(A'_{\overline{E'}}, \mathbb{Z}_p))$ is (isomorphic to) $G_{\mathbb{Z}_p}$.

If v' is still unramified over p, then $M := H^1_{\mathrm{crys}}(A'_{\mathbb{F}}/W(\mathbb{F}))$, with $A'_{\mathbb{F}}$ as the special fibre of the (abelian scheme which is the) Néron model $A'_{W(\mathbb{F})}$ of the extension of A' to $B(\mathbb{F})$, ϕ is the resulting Frobenius endomorphism of M, and F^1 is the Hodge filtration of M defined by $A'_{W(\mathbb{F})}$.

If v' is ramified over p, then the filtered σ-crystal (M, F^1, ϕ) still comes naturally from an abelian scheme A^0 over $W(\mathbb{F})$; its special fibre $A^0_{\mathbb{F}}$ is the extension to $\mathbb{F}$ of the reduction of A' with respect to v'. To explain how we get A^0, let L be the completion of the maximal unramified algebraic field extension of $E'_{v'}$, and let O_L be its ring of integers. Let $e_A := [O_L : W(\mathbb{F})]$ and let $\widetilde{Re}_A$ be the ring used in [Va1, 5.2], but working with e_A instead of e of loc. cit. (cf. also 2.1 where such a ring is called a piano ring). We consider the abelian scheme A'_{O_L} over O_L defined by the Néron model (it is an abelian scheme) of A'_L. As $G_{\mathbb{Z}_p}$ is a reductive group over $\mathbb{Z}_p$ and as p is big enough, we can assume (see 4.3.4) that the family of tensors $(v_\alpha)_{\alpha\in\mathcal{J}}$ has a subfamily which is $\mathbb{Z}_p$-very well position with respect to $G_{\mathbb{Q}_p}$ and is enveloped by $H^1_{\text{ét}}(A'_{\bar{L}},\mathbb{Z}_p)$. So the machinery of [Va1, 5.2-3] applies to A'_{O_L}. We get (see 4.4.5; cf. also 1.9.1 below which explains that we can assume that we are in a polarized context) an abelian scheme over $\mathrm{Spec}(\widetilde{Re}_A)$ such that:

– in the natural O_L-valued point of $\mathrm{Spec}(\widetilde{Re}_A)$ (see loc. cit.) it is A'_{O_L};

– in the natural $W(\mathbb{F})$-valued point of $\mathrm{Spec}(\widetilde{Re}_A)$ (see loc. cit.) it becomes A^0;

– it is endowed naturally with a family (indexed by $\alpha\in\mathcal{J}$) of de Rham (crystalline) cycles, which gives birth by pull back to the family $(t_\alpha)_{\alpha\in\mathcal{J}}$.

So in this case it also makes sense to speak about the family of tensors $(t_\alpha)_{\alpha\in\mathcal{J}}$ of being of the essential tensor algebra of $M\oplus M\left[\frac{1}{p}\right]$ (see [Va1, 5.3.4-12]). As A' is $\mathbb{Z}\left[\frac{1}{p}\right]$-isogeneous to $A_{E'}$ it makes sense to also view t_α as an element of $\mathcal{T}(H^1_{dR}(A'_L/L))$.

1.9.1. Regardless of how is v' over p, we can assume (this is standard, see 4.3.2 and 4.3.6) that all the abelian schemes considered above (i.e. A, A' and A^0) are principally polarized and have some level-N symplectic similitude structure (for $N\in\mathbb{N}$ greater than 2 and relatively prime to p; we take $N=4$, as $p\geq 3$); in the case when v' is ramified over p, we can also assume that the $\mathbb{Z}\left[\frac{1}{p}\right]$-isogeny between $A_{E'}$ and A' is compatible with the chosen polarizations, cf. 4.3.5-7.

In the case when v' is unramified over p, we denote $A'_{W(\mathbb{F})}$ from now on by A^0. So A^0 is well defined regardless of how v' is over p. Let $\psi_M\colon M\otimes M\to W(\mathbb{F})(1)$ be the perfect alternating form defined by the principal polarization of A^0. So we get a principally quasi-polarized Shimura filtered σ-crystal

$$\mathcal{CP}:=(M,F^1,\phi,G_{W(k)},(t_\alpha)_{\alpha\in\mathcal{J}},\psi_M).$$

1.9.2. Next we just apply the machinery of [Va1, 5.4]. We get (see 4.4.7) a versal deformation of $\mathcal{CP}$ over a regular ring R of formal power series with coefficients in $W(\mathbb{F})$; it corresponds to a principally polarized abelian scheme (A_R,p_{A_R}) over $\mathrm{Spec}(R)$, endowed with de Rham cycles. 2.4.1 shows that A'_{O_L} and its principal polarization are obtained from (A_R,p_{A_R}) via an O_L-valued point of $\mathrm{Spec}(R)$. If $A^0_{\mathbb{F}}$ has ordinary reduction, then [Va1, 5.5.1] implies that its canonical lift of $A'_{\mathbb{F}}$ is obtained from A_R through a $W(\mathbb{F})$-valued point of $\mathrm{Spec}(R)$. In this way we get Theorem 2 (via the fact that all the Hodge cycles of A'_L are "preserved" under the local deformation we achieved; see [Va1, 5.4.5 and 5.2.10]).

1.9.3. The ideas of 1.9.1-2 can be used (see 4.4.10) to reobtain (via [Va2]) the results of [Pi, 5.10-11]. We also hope that they will be useful in other situations as well (in the part of [Va5] related to the Mumford–Tate conjecture, we start from where the ideas of 4.4.7-9 and 3.2 stop; see 4.4.11 for a sample).

1.10. In §5 we also present (besides Theorem 1 and the Examples of 1.5) briefly some applications (see the definition of Serre's volumes in 5.3, and the part 5.4 2) on the ordinary reduction conjecture) and some other conclusions (like a method of slightly extending the main result of [Wi] in the case of some p-divisible groups with étale cycles: the ramified case can be handled in many cases; see 5.5.1). For a complement to [Va1, 4.3.10.1] see 5.5.2.

1.11. We would like to thank University of California at Berkeley (resp. University of Utah) for providing us with good conditions for the writing of this manuscript (resp. for improving it). We would like to thank Prof. J. S. Milne for providing us with the manuscript [Mi1] at the right time for being able to build up properly on [Mi1, 0.1]. We would like to thank Prof. S. G. Tankeev for making us aware of 4.2.5, and Prof. G. Faltings for pointing out an inexactitude in 2.2.2.1. Also we are obliged to [Pi]: though this work was done independently of [Pi], as afterthoughts, we did not hesitate to simplify the things using [Pi] in 5.1; moreover the definition 5.1.2.1 (based on [Pi, 5.10]) and the numerical analysis of 5.1.2.2, are slightly more general than the ones we initially thought of. This research was also supported by the NSF grant DMF 97-05376.

§2. Some complements

In this chapter we include some simple complements to [Va1-2]. Let p be an odd rational prime and let k be a perfect field of characteristic p.

2.1. The piano rings. Let $e \in \mathbb{N}$. Let T be an independent variable. We denote by $\widetilde{Re}$ (resp. by Re) the subring of $B(k)[[T]]$ formed by formal power series $\sum_{n=0}^{\infty} a_n T^n$ for which $a_n \left[\frac{n}{e}\right]! \in W(k)$ for all n (resp. for which the sequence $b_n := a_n \left[\frac{n}{e}\right]!$ is formed by elements of $W(k)$ and converges to 0). Sometimes, when we want to emphasize k, we write $\widetilde{Re}(k)$ (resp. $Re(k)$) instead of $\widetilde{Re}$ (resp. of Re). Their Frobenius (see [Fa2] or [Va1, 5.2]) are compatible with the Frobenius automorphism of $W(k)$, and are defined by: T goes to T^p. Obviously we have a natural $W(k)$-monomorphism $Re \hookrightarrow \widetilde{Re}$. The rings Re were used for the first time with significant impact in [Fa2], while the rings $\widetilde{Re}$ were used for the first time with significant impact in [Va1, 5.2]. We refer to $\widetilde{Re}$ (resp. to Re) as the piano ring of k of resonance e (resp. of convergent resonance e). This terminology is inspired from their construction and from the following important fact (see [Fa2, §2] and [Va1, 5.2.1]):

Fact. *For any DVR V which is a finite flat totally ramified extension of $W(k)$ of degree e, there are $W(k)$-epimorphisms $Re \twoheadrightarrow V$ and $\widetilde{Re} \twoheadrightarrow V$; they are defined by: T is mapped into an arbitrarily chosen uniformizer of V. Moreover, mapping T to 0, we get $W(k)$-epimorphisms $Re \twoheadrightarrow W(k)$ and $\widetilde{Re} \twoheadrightarrow W(k)$.*

We view the $W(k)$-epimorphisms of the above Fact as the keys of a piano tuned on all (convergent or not) resonances of length e. If $e' \in \mathbb{N}$ is divisible by e, then we have natural $W(k)$-monomorphisms (inclusions)

$$Re' \hookrightarrow Re$$

and

$$\widetilde{Re}' \hookrightarrow \widetilde{Re}.$$

So in the above Fact, we can substitute the degree e by: the degree a divisor of e.

2.2. Digression on a conjecture of Milne. For the original conjecture of Milne we have in mind we refer to [Mi1, 0.1] and to [Va1, 5.6.6]. Let A be an abelian variety over a DVR V of mixed characteristic having a residue field of characteristic p. Let $K := V[\frac{1}{p}]$. Let $(s_\alpha)_{\alpha\in\mathcal{J}}$ be a suitable family of Hodge cycles of A_K. In the context of abelian varieties, the conjecture asserts that under some mild conditions the below question has a positive answer.

Q. Is there a faithfully flat p-adically complete V-algebra R such that there is an R-linear isomorphism

$$\rho: H^1_{\text{ét}}(A_{\bar{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} R \tilde{\to} H^1_{dR}(A/V) \otimes_V R$$

taking the p-component of the étale component of s_α into the de Rham component of s_α, for any α an element of the set $\mathcal{J}$?

A similar question can be put for other classes of varieties or objects (like p-divisible groups). For an intensive study of this question in the broader context of p-divisible groups see [Va5]; for the analogue study in the case of K3 surfaces we refer to [Va7]. Here we "touch" this question just as far as it brings more light to parts of §3.

Warning: Tate-twists are of course allowed; i.e. $\mathbb{Z}_p(m)$, with $m \in \mathbb{Z}$, is always naturally identified, as a $\mathbb{Z}_p$-module, with $\mathbb{Z}_p$. For abelian varieties the situation is very much simplified: they are always polarized and so, as in [Va2, 2.3.1], in practice we just need to keep track of such polarizations and of Hodge cycles which involve no Tate-twists and of whose different realizations (like Betti, de Rham, etc.) are homogeneous tensors of corresponding essential tensor algebras.

To study question **Q** we can assume that V is a complete DVR having an algebraically closed residue field. If this is so, and if the subgroup G of $GL := GL(H^1_{dR}(A/V))$ defined as the closure of the subgroup of GL_K fixing the de Rham component of s_α, $\forall\alpha \in \mathcal{J}$, is a reductive group over V, then (this is standard) the answer to the above question is yes, iff such an isomorphism ρ exists for $R = V$.

2.2.0. Definition. We say that A has the MC property, if the answer to the above question is true, provided we work with the family of all Hodge cycles of A (here MC stands for Milne's conjecture).

2.2.1. The generic situation of an abelian variety over a number field. Let A be an abelian variety over a number field E. We take $(s_\alpha)_{\alpha\in\mathcal{J}}$ to be the family of all Hodge cycles of A. We assume that it is equal to the family of all Hodge cycles of $A_{\bar{E}}$ (i.e. we assume that by enlarging E we do not gain extra Hodge cycles, cf. [De2]). We choose an embedding $E \hookrightarrow \mathbb{C}$; in what follows $A_{\mathbb{C}}$ refers with respect to it. Let

$$L := H^1_B(A_{\mathbb{C}}, \mathbb{Z})$$

be the first group of the Betti cohomology of $A_{\mathbb{C}}$ with coefficients in $\mathbb{Z}$. Let $W := L \otimes \mathbb{Q}$. Let s^B_α be the Betti realization of s_α, $\alpha \in \mathcal{J}$. So s^B_α is an element of the tensor algebra of $W \oplus W^*$.

Let O_E be the ring of integers of E, and let $N \in \mathbb{N}$ such that A extends to an abelian scheme A_N over $\mathrm{Spec}(O_E[\frac{1}{N}])$. Let $D := H^1_{dR}(A_N/O_E[\frac{1}{N}])$. We have:

Fact. *There is a finite field extension E_1 of E such that we have an isomorphism*

$$W \otimes_{\mathbb{Q}} E_1 \tilde{\to} D \otimes_E E_1$$

taking s_α^B into the de Rham realization s_α^{dR} of s_α, $\forall \alpha \in \mathcal{J}$. Moreover, there is $N_1 \in \mathbb{N}$ a multiple of N such that under this isomorphism $L \otimes O_{E_1}[\frac{1}{N_1}]$ is mapped onto $D \otimes O_{E_1}[\frac{1}{N_1}]$ (here O_{E_1} is the ring of integers of E_1).

This Fact is an immediate (standard) consequence of the fact that we have an isomorphism (cf. [De2])

$$W \otimes \mathbb{C} \tilde{\to} D \otimes \mathbb{C}$$

taking s_α^B into s_α^{dR}, $\forall \alpha \in \mathcal{J}$. We deduce:

2.2.1.1. Corollary. *There is a multiple $M(A) \in \mathbb{N}$ of N such that the fibre of A_N over any localization of Spec$(O_E[\frac{1}{M(A)}])$ with respect to a finite prime of it has the MC property.*

In [Va5] we will see that the number $M(A)$ is effectively computable.

2.2.2. The generic situation in the context of Shimura varieties of Hodge type. Let $f\colon (G, X) \hookrightarrow (G\mathrm{Sp}(W, \psi), S)$ be an embedding of the Shimura pair (G, X) of Hodge type into a Shimura pair defining a Siegel modular variety. Let L be a $\mathbb{Z}$-lattice of W such that we get a perfect alternating form $\psi\colon L \otimes L \to \mathbb{Z}$. Let

$$K_4 := \{\mathrm{g} \in G\mathrm{Sp}(L \otimes \widehat{\mathbb{Z}}, \psi) | \mathrm{g} \text{ mod } 4 \text{ is the identity element}\}.$$

Let $\mathcal{M}$ be the $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$-scheme parameterizing isomorphism classes of principally polarized abelian schemes of relative dimension equal to $\dim(W)/2$, and having level-4 symplectic similitude structure; so we have a natural identification

$$\mathcal{M}_{\mathbb{Q}} = \mathrm{Sh}_{K_4}(G\mathrm{Sp}(W, \psi), S),$$

cf. [Va1, 3.2.9 and 4.1.0]. Let H be a compact open subgroup of $G(\mathbb{A}_f)$ contained in K_4. We get naturally a principally polarized abelian scheme $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ over the normalization $\mathcal{N}$ of $\mathcal{M}$ in the ring of fractions of $\mathrm{Sh}_H(G, X)$. For the sake of simplicity, we assume that all the Hodge cycles of $\mathcal{A}_{\mathcal{K}^{\mathrm{al}}}$, with $\mathcal{K}^{\mathrm{al}}$ as the algebraic closure of the field of fractions $\mathcal{K}$ of an arbitrary connected component of $\mathcal{N}$, are also Hodge cycles of $\mathcal{A}_{\mathcal{K}}$. In other words we assume that H is small enough. We have:

2.2.2.1. Lemma. *There is a number $M(f, L) \in \mathbb{N}$, such that for every DVR V of mixed characteristic having a residue field of characteristic $p \geq M(f, L)$, any abelian scheme over V obtained from $\mathcal{A}$ by pull back through a morphism Spec$(V) \to \mathcal{N}$, and of whose Mumford–Tate group is G, has the MC property.*

Proof: This is entirely the same as 2.2.1, once we remark that $\mathcal{N}$ is of finite type over $\mathbb{Z}$. So instead of number fields and their rings of integers we have to work with the fields of fractions of different connected components of $\mathcal{N}_{\mathbb{Q}}$, and with the rings of global functions of different affine integral open subschemes of $\mathcal{N}$. We deduce the existence of a dense open subscheme U of $\mathcal{N}$ and of a number $N(U) \in \mathbb{N}$ such that the statement of the lemma is true for morphisms $\mathrm{Spec}(V) \to \mathcal{N}_{\mathbb{Z}[\frac{1}{(N(U)-1)!}]}$ factoring through U.

Let $N \in \mathbb{N}$, with $2N \geq \dim(W)$, such that the closure of G in $GL(L \otimes \mathbb{Z}_{(q)})$ is a reductive group over $\mathbb{Z}_{(q)}$, for any prime $q \geq N$. From [Va1, 5.8.6] and [Va2, 2.3.6] we deduce that for any prime v of $E(G,X)$ dividing a prime $q \geq N$, the triple $(f, L \otimes \mathbb{Z}_{(q)}, v)$ is a SHS.

It is known (see [Har]) that $\mathcal{N}_{\mathbb{Q}}$ has smooth toroidal compactifications. Let $N_1 \in \mathbb{N}$ such that $\mathcal{N}_{\mathbb{Z}[\frac{1}{(N_1-1)!}]}$ has smooth compactifications. So we can assume that for any prime $q \geq N_1$, the special fibre of U over $\mathrm{Spec}(\mathbb{F}_q)$ is dense in $\mathcal{N}_{\mathbb{F}_q}$.

Now we can take

$$M(f,L) := \max\{N, N_1, N(U)\}.$$

To see this let $m_V: \mathrm{Spec}(V) \to \mathcal{N}_{\mathbb{Z}[\frac{1}{M(f,L)!}]}$ be a morphism, such that the resulting abelian scheme A_V over $\mathrm{Spec}(V)$ has G as its Mumford–Tate group. From the definition of $M(f,L)$ and [Va2, 2.3.13.1] we deduce that we can assume that m_V factors through U. So a_V has the MC property. This ends the proof of the lemma.

In [Va5] we will see that the number $M(f,L)$ is effectively computable.

2.2.3. The case of canonical lifts. We assume that $k = \bar{k}$ and that $V = W(k)$. We assume that A is the canonical lift of an ordinary abelian variety over k. Let (M, F^1, ϕ) be the filtered σ-crystal over k defined by A. The slopes of (M, ϕ) are just 0 and 1. Let $\mu: \mathbb{G}_m \to GL(M)$ be the canonical split of (M, F^1, ϕ) (cf. [Wi]). We get a direct sum decomposition $M = F^1 \oplus F^0$, with $\gamma \in \mathbb{G}_m(W(k))$ acting through μ as the multiplication with γ^{-i} on F^i, $i = \overline{0,1}$. Moreover ϕ leaves invariant F^1 and F^0. Let $B^+(W(k))$ be the Fontaine's ring (as defined in [Fa2]; see also [Va1, 5.2]). We consider the $B^+(W(k))$-monomorphism of the integral version of Fontaine's comparison map (see [Fa2])

$$\rho: M \otimes_{W(k)} B^+(W(k)) \hookrightarrow H^1 \otimes_{\mathbb{Z}_p} B^+(W(k));$$

here $H^1 := H^1_{\text{ét}}(A_{\overline{B(k)}}, \mathbb{Z}_p)$. Let $\beta \in B^+(W(k))$ be defined as in [Va1, 5.2.3], but working with $B^+(W(k))$. We have:

$$\rho\big(\mu(\beta)(M \otimes_{W(k)} B^+(W(k)))\big) = H^1 \otimes_{\mathbb{Z}_p} B^+(W(k)).$$

This can be checked immediately starting from a basis $\{f_i | i \in \{1, ..., \dim(A)\}\}$ of F^1 such that for all i we have $\phi(f_i) = pf_i$, and from a basis $\{e_i | i \in \{1, ..., \dim(A)\}\}$ of F^0 such that for all i we have $\phi(e_i) = e_i$. But any Hodge cycle of A is a de Rham cycle in the sense of [Bl]: the proof in loc. cit. works in the general case of an arbitrary variety over a complete DVR of mixed characteristic, cf. [Va1, 5.2.16]. We deduce that the $\mu(\beta)$ takes the de Rham component of any Hodge cycle of A into the p-component of its étale component. We conclude:

Fact. *A has the MC property.*

2.3. Representation well positioned families of tensors. Let O be a discrete valuation ring, let π be a uniformizer of it, and let K be its field of fractions. Let M_O be an O-lattice of a finite dimensional K-vector space W. Let G_K be a reductive subgroup of $GL(W)$. For the notion (resp. theory) of O-well positioned families of tensors with respect to G_K we refer to [Va1, 4.3.4 and 4.3.7 1)] (resp. to [Va1, 4.3]). Here we introduce some extra definitions, in order to complement [Va1, 4.3.10.1 2)] later on (see 5.5.2).

2.3.1. Definition. A family $(v_\alpha)_{\alpha\in\mathcal{J}}$ of G_K-invariant tensors of the tensor algebra of $M_O \oplus M_O^*$ is said to be M_O-representation very well positioned for the reductive group G_K if:

a) it is O-very well positioned for the group G_K;
b) for every local integral faithfully flat O-algebra R, and for any free R-module M, with $M[\frac{1}{\pi}] = W \otimes_K R[\frac{1}{\pi}]$, enveloping it, and such that $\forall\alpha \in \mathcal{J}$, v_α is a tensor of the tensor algebra of $M_O \oplus M_O^*$ having a non-zero image modulo π iff it is a tensor of the tensor algebra of $M \oplus M^*$ having a non-zero image modulo m (with m the maximal ideal of R), there is an isomorphism

$$\rho: M_O \otimes_O R^{\text{sh}} \tilde{\to} M \otimes_R R^{\text{sh}},$$

taking v_α into v_α, $\forall\alpha \in \mathcal{J}$;
c) G_K is the subgroup of $GL(M[\frac{1}{\pi}])$ fixing v_α, $\forall\alpha \in \mathcal{J}$ (so ρ of b) is defined by an element of $G_K(W \otimes_K R^{\text{sh}}[\frac{1}{p}])$).

As in [Va1, 4.3.4] we have variants: we speak about a weakly (resp. strongly) M_O-representation very well positioned family of tensors; for this we keep c), we request R in b) to be normal (resp. we request R to be just reduced), and in a) we request the family to be weakly (resp. strongly) O-very well positioned for G_K.

2.3.2. Definition. A family $(v_\alpha)_{\alpha\in\mathcal{J}}$ of G_K-invariant tensors of the tensor algebra of $M_O \oplus M_O^*$ is said to be M_O-representation well positioned for the group G_K if 2.3.1 a) is true, while 2.3.1 b) is true only for DVR's which are faithfully flat O-algebras (warning: for the sake of generality and flexibilty, 2.3.1 c) is ignored here).

2.4. An inducing property of p-divisible groups in the ramified case. What follows is a complement to the induced property of [Va2, 3.4.18.5]. It is desirable to have a form of loc. cit. for complete regular $W(k)$-algebras which are not formally smooth. Here we touch this problem in the simplest case of a totally ramified finite DVR extension of $W(k)$.

Let V be a DVR which is a finite flat totally ramified extension of $W(k)$. Let K be its field of fractions. Let $e := [V : W(k)]$. Let H_V be a p-divisible group over V. Let

$$H^1_{\text{ét}}(H_V, \mathbb{Z}_p) := T_p(H_V)^*$$

be the dual of the Tate module of H_V. Let $(w_\alpha)_{\alpha\in\mathcal{J}}$ be a family of tensors of $\mathcal{T}(H^1_{\text{ét}}(H_V, \mathbb{Z}_p))$, fixed by $\text{Gal}(\bar{K}/K)$. We assume the existence of a reductive subgroup $\tilde{G}$ of the general linear group $GL(H^1_{\text{ét}}(H_V, \mathbb{Z}_p))$ with the properties:

i) its generic fibre is the subgroup of $GL(H^1_{\acute{e}t}(H_V,\mathbb{Z}_p))_{\mathbb{Q}_p}$ fixing w_α, $\forall\alpha\in\mathcal{J}$;

ii) there is a subset $\mathcal{J}_0$ of $\mathcal{J}$ such that the family of homogeneous tensors $(w_\alpha)_{\alpha\in\mathcal{J}_0}$ is $\mathbb{Z}_p$-well positioned for $\tilde{G}_{\mathbb{Q}_p}$.

We also assume that $\forall\alpha\in\mathcal{J}_0$ we have: the partial degrees of w_α are smaller or equal to $p-2$. Here by partial degrees of w_α we mean the numbers $a_\alpha, b_\alpha\in\mathbb{N}\cup\{0\}$ such that

$$w_\alpha\in(H^1_{\acute{e}t}(H_V,\mathbb{Z}_p))^{\otimes a_\alpha}\otimes(H^1_{\acute{e}t}(H_V,\mathbb{Z}_p)^*)^{\otimes b_\alpha}.$$

So in particular we have:

$$\deg(w_\alpha):=a_\alpha+b_\alpha\le 2(p-2).$$

Let $\widetilde{Re}$ (resp. Re) be the piano ring of k of resonance e (resp. of convergent resonance e). Let

$$(M,F^1(M),\Phi_M,\nabla)$$

be the filtered Frobenius crystal over Re obtained by taking the dual of the Lie algebra of the universal vector extension of the p-divisible group associated to H_V (cf. [Me], [BBM] and [Fa2, ch. 6]). Let $(t_\alpha)_{\alpha\in\mathcal{J}}$ be the family of tensors of $\mathcal{T}(M[\frac{1}{p}])$, corresponding to $(w_\alpha)_{\alpha\in\mathcal{J}}$, under the integral version of Fontaine's comparison map, cf. [Fa2, ch. 6].

We recall that we have natural ring inclusions

$$R:=W(k)[[t]]\subset Re\subset\widetilde{Re},$$

for t an independent variable, and that the Frobenius of these rings are compatible with the Frobenius of $W(k)$, and take t into t^p (see 2.1); moreover $\widetilde{Re}$ is the subring of $B(k)[[t]]$ formed by formal power series $\sum_{n=0}^\infty a_nt^n$ such that $a_n\left[\frac{n}{e}\right]!\in W(k)$, $\forall n\in\mathbb{N}\cup\{0\}$ (cf. 2.1). We fix arbitrarily a uniformizer π_V of V. It defines naturally $W(k)$-epimorphisms $R\twoheadrightarrow V$, $Re\twoheadrightarrow V$ and $\widetilde{Re}\twoheadrightarrow V$ (cf. the Fact of 2.1); moreover we have $W(k)$-epimorphisms $R\twoheadrightarrow W(k)$, $Re\twoheadrightarrow W(k)$ and $\widetilde{Re}\twoheadrightarrow W(k)$ defined by: t goes to 0. We refer to these six $W(k)$-epimorphisms as the natural ones (defined by π_V).

We assume the existence of a cocharacter

$$\mu_K:\mathbb{G}_m\to GL(M\otimes_{Re}K)$$

fixing the image t_α^V of t_α in $\mathcal{T}(M\otimes_{Re}K)$, $\forall\alpha\in\mathcal{J}$, and such that it produces a direct sum decomposition

$$M\otimes K=F^1(M)\otimes_{Re}K\oplus F^0_K;$$

here $\beta\in\mathbb{G}_m(K)$ acts through μ_K as the multiplication with β^{-1} on the summand $F^1(M)\otimes_{Re}K$, and trivially on the summand F^0_K.

Let $G_{\widetilde{Re}}$ be the reductive group over $\widetilde{Re}$, and let $H_{\widetilde{Re}}$ be a p-divisible group over $\widetilde{Re}$, constructed as in [Va1, 5.2-3]; so $G_{\widetilde{Re}}$ is the extension to $\widetilde{Re}$ of a reductive group G_{Re} over Re. Loc. cit. is worked out in the context of abelian schemes; but the same construction (finalized in [Va1, 5.3.3]) applies to the context of p-divisible groups. We can apply loc. cit. as the family of tensors $(w_\alpha)_{\alpha\in\mathcal{J}_0}$ is $\mathbb{Z}_p$-well positioned for $\tilde{G}_{\mathbb{Q}_p}$, is enveloped by $H^1_{\text{ét}}(H_V, \mathbb{Z}_p)$, and is of partial degree not bigger than $p-2$. The only difference from loc. cit.: as we did not assume k to be algebraically closed it might happen that the groups $\tilde{G}_{\widetilde{Re}}$ and $G_{\widetilde{Re}}$ are not isomorphic.

Let $(M_0, F^1_0, \phi, G_{W(k)}, (t^0_\alpha)_{\alpha\in\mathcal{J}})$ be the Shimura filtered σ-crystal over k defined by pulling back $H_{\widetilde{Re}}$ and the family of tensors $(t_\alpha)_{\alpha\in\mathcal{J}}$ through the natural $W(k)$-epimorphism $\widetilde{Re} \twoheadrightarrow W(k)$. It corresponds to a Shimura p-divisible group $(H_{W(k)}, (t^0_\alpha)_{\alpha\in\mathcal{J}})$ over $W(k)$. Let

$$\mathfrak{C} := (M_0, F^1_0, \phi, G_{W(k)}, G^0_{W(k)}, \tilde{f})$$

be a Shimura filtered F-crystal defined by a smooth subgroup $G^0_{W(k)}$ of $G_{W(k)}$. We assume that the morphism $G^0_{W(k)} \to G_{W(k)}/P$, with P as the parabolic subgroup of $G_{W(k)}$ leaving F^1_0 invariant, is smooth. Let $(H, (t^0_\alpha)_{\alpha\in\mathcal{J}})$ be the Shimura p-divisible group over the completion $\widehat{G}^0$ of $G^0_{W(k)}$ in its origin producing $\mathfrak{C}$, cf. [Va2, 3.4.18.5.1 and 2.2.21 1)]. We have:

Proposition. *H_V together with its de Rham tensors $(t^V_\alpha)_{\alpha\in\mathcal{J}}$ is induced from H and its crystalline tensors $(t^0_\alpha)_{\alpha\in\mathcal{J}}$, through a V-valued point of $\widehat{G}^0$.*

Proof: For $n \in \mathbb{N}$, let I_n (resp. J_n) be the ideal of $\widetilde{Re}$ (resp. of R) formed by formal power series as above but with $a_0 = a_1 = ... = a_{n-1} = 0$ (resp. generated by t^n).

As in [Va1, 5.3.1] we get that we can assume that μ_K lifts to a cocharacter $\mu_V\colon \mathbb{G}_m \to GL(M \otimes_{Re} V)$ factoring through G_V. The construction of $H_{\widetilde{Re}}$ corresponds (see [Va1, 5.2-3]) to a lift of μ_V to a cocharacter of $G_{\widetilde{Re}}$. From [Va2, 2.3.17.2] we get immediately that it is irrelevant which lift we choose. In other words we are allowed to choose such a lift in order of making the situation as simple as possible.

We assume first the existence of a filtered Frobenius crystal

$$\mathcal{C}_R := (M_R, F^1(M_R), \Phi_{M_R}, \nabla_R)$$

over $k[[t]]$ (so M_R is a free R-module of finite rank, $F^1(M_R)$ is a direct summand of M_R, etc.) such that we have an isomorphism

$$\rho\colon M_R \otimes \widetilde{Re} \xrightarrow{\sim} M \otimes \widetilde{Re}$$

of non-filtered F-crystals, with the property that, viewed just as a morphism, it respects the filtrations. We view it as an identification. We also assume that under this identification (isomorphism) ρ we have:

a) $t_\alpha \in \mathcal{T}(M_R[\frac{1}{p}])$, $\forall\alpha \in \mathcal{J}$;
b) $t_\alpha \in \mathcal{T}(M_R)$, $\forall\alpha \in \mathcal{J}_0$;
c) there is a reductive subgroup G_R of $GL(M_R)$ which over $\widetilde{Re}$ becomes $G_{\widetilde{Re}}$.

We consider now $\mu\colon \mathbb{G}_m \to G_R$ a cocharacter such that: under the natural $W(k)$-epimorphisms $R \twoheadrightarrow V$ and $R \twoheadrightarrow W(k)$, it becomes the cocharacters defined by pulling back, under the natural $W(k)$-epimorphisms $\widetilde{Re} \twoheadrightarrow V$ and respectively $\widetilde{Re} \twoheadrightarrow W(k)$, the cocharacter of $G_{\widetilde{Re}}$ chosen to define H. The existence of such a cocharacter is proved in entirely the same manner as [Va1, 5.3.2]. It gives birth to a direct summand F^1_R of M_R. We have $F^1_R \otimes_R W(k) = F^1_0$. We choose the cocharacter of $G_{\widetilde{Re}}$ lifting μ_V and defining $H_{\widetilde{Re}}$ to be the extension of μ to $\mathrm{Spec}(\widetilde{Re})$. So H_V together with its tensors $(t^V_\alpha)_{\alpha\in\mathcal{J}}$ is induced from the Shimura p-divisible group $(H^R, (t_\alpha)_{\alpha\in\mathcal{J}})$ over $\mathrm{Spec}(R)$ defined (cf. [Va2, 3.4.18.5.1] and [Va2, 2.2.21 1)]) by $\mathcal{C}_R$ and its de Rham tensors $(t_\alpha)_{\alpha\in\mathcal{J}}$, through an V-valued point of $\mathrm{Spec}(R)$.

But the p-divisible object (M_R, F^1_R, Φ_{M_R}) of $\mathcal{MF}_{[0,1]}(R)$ we get (by forgetting ∇_R), is in the desired shape needed for applying [Fa2, rm. iii) after th. 10]. So from loc. cit. and [Va2, 3.4.18.5.1] we deduce that the Shimura p-divisible group $(H^R, (t_\alpha)_{\alpha\in\mathcal{J}})$ is induced from $(H, (t^0_\alpha)_{\alpha\in\mathcal{J}})$ through an R-valued point of $\widehat{G}^0$. We recall that the first part of [Fa2, rm. iii) after th. 10] is not quite correct, cf. [Va2, 2.2.9 1) and 1')]. However, as we have i) above, [Fa2, rm. iii) after th. 10] does apply.

So we are left with the proof of the existence of a filtered F-crystal $\mathcal{C}_R$ over $k[[T]]$ for which an isomorphism ρ of the desired form and properties exists. As $\widetilde{Re}$ is the projective limit of $\widetilde{Re}/I_n$, $n \in \mathbb{N}$, and as $J_n = R \cap I_n$, $\forall n \in \mathbb{N}$, to show the existence of a $\mathcal{C}_R$ with the desired properties we can proceed by induction on $n \in \mathbb{N}$. So, inductively we show the existence of a filtered Frobenius crystal

$$\mathcal{C}_n = (M_n, F^1(M_n), \Phi_{M_n}, \nabla_n)$$

over $k[[t]]$ such that for an adequate lift of μ_V to a cocharacter of $G_{\widetilde{Re}}$, we have a desired isomorphism of non-filtered F-crystals modulo I_n:

$$\rho_n \colon M_n \otimes_R \widetilde{Re}/I_n \xrightarrow{\sim} M \otimes_{Re} \widetilde{Re}/I_n,$$

and an isomorphism of filtered F-crystals

$$r_n \colon M_{n-1} \otimes_R \widetilde{Re}/I_{n-1} \xrightarrow{\sim} M_n \otimes_R \widetilde{Re}/I_{n-1}$$

such that ρ_{n-1} is the composite of r_n with ρ_n modulo I_{n-1}/I_n, and the above requirements a) to c) are satisfied modulo I_n. This is easily accomplished following the pattern of the proof of [Va1, 5.3.2], due to the following three reasons.

A) We already know that this is possible for $1 \le n \le ep$, as for such an n we have $\widetilde{Re}/I_n = R/J_n$.

B) By induction it makes sense to speak about such a reductive group (i.e. of G_{R/J_n}) over R/J_n, if we know it over R/J_{n-1}, cf. the Grothendieck–Messing theory: the ideal J_{n-1}/J_n of R/J_n and the ideal I_{n-1}/I_n of $\widetilde{Re}/I_n$ have a natural divided power structure, and so [Fa2, proof of th. 10 and rm. iii) after th. 10] applies. In other words, as I_{n-1}/I_n is a free V_0-module of rank one, we can apply loc. cit. to get that $H_{\widetilde{Re}}$ modulo $\widetilde{Re}/I_n$ and its family (still denoted by) $(t_\alpha)_{\alpha\in\mathcal{J}}$ of crystalline cycles, is induced from the Shimura p-divisible group $(H, (t^0_\alpha)_{\alpha\in\mathcal{J}})$ through a morphism $m_n : \mathrm{Spec}(\widetilde{Re}/I_n) \to \widehat{G}^0$ (by induction we know that m_n modulo I_{n-1}/I_n factors through $\mathrm{Spec}(R/J_{n-1})$).

C) We always choose the lift of μ_V (to a cocharacter of $G_{\widetilde{Re}}$) so that modulo I_n is the pull back of a cocharacter of G_{R/J_n} through the morphism $\mathrm{Spec}(\widetilde{Re}/I_n) \to \mathrm{Spec}(R/J_n)$

defined by the natural $W(k)$-monomorphism $R/J_n \hookrightarrow \widetilde{Re}/I_n$. So, under this choice we get that m_n factors through $\mathrm{Spec}(R/J_n)$.

So we get a compatible system of filtered Frobenius crystals over $k[[t]]$ and of truncated isomorphisms (i.e. what we get modulo J_n and modulo I_n lifts what we got modulo J_{n-1} and modulo I_{n-1}; here $n \in \mathbb{N}$, $n \geq 1$). Their limits are the desired filtered Frobenius crystal $\mathcal{C}_R$ and respectively the desired isomorphism ρ: obviously a) to c) above hold. This ends the proof of the proposition.

2.4.1. Variant. The same result can be stated in the relative context when we have a principal quasi-polarization. The same proof applies (cf. [Fa2, rm. iii) after th. 10], and [Va2, 2.2.9 12)]).

2.4.2. Remark. The above proof of 2.4 gives us much more information. It shows that any sextuple as above $\mathcal{C}_V := (M, F^1(M), \Phi_M, \nabla, G_{Re}, (t_\alpha)_{\alpha\in\mathcal{J}})$ over Re, such that G_{Re} is a reductive subgroup of $GL(M)$ and μ_K exists, is induced from a similar sextuple over R by extension of scalars, without assuming that 2.4 ii) holds or that the partial degrees of w_α are smaller or equal to $p-2$, $\forall \alpha \in \mathcal{J}_0$. This is so as (cf. the proof of 2.4):

i) $\mathcal{C}_V$ depends only on the pair $(H_V, (t_\alpha^V)_{\alpha\in\mathcal{J}})$;

ii) The pair of i) above is induced from the Shimura p-divisible group $(H^R, (t_\alpha)_{\alpha\in\mathcal{J}})$ through a composite morphism : $\mathrm{Spec}(V) \twoheadrightarrow \mathrm{Spec}(Re) \to \mathrm{Spec}(R)$.

The same remains true in a principally quasi-polarized context.

In [Va5] we will see that this remark is the main ingredient needed for the classification of Shimura p-divisible groups over such DVR's.

2.4.3. Remark. Of course, for the proof of Proposition of 2.4 we could have just quoted [Fa2, rm. iii) after th. 10] (and [Va2, 2.2.9 1) and 1')]). But the proof we have included is more detailed, and it can be used as well in other situations, when instead of $\mathrm{Spec}(V)$ we have an arbitrary closed subscheme of $\mathrm{Spec}(W(k)[[x_1, ..., x_m]])$, with $m \in \mathbb{N}$. For such variants of 2.4 and 2.4.1 see [Va5].

2.5. Some notations involving reflex fields. Here we just introduce some notations pertaining to reflex fields, in order to have a better understanding later on of the d') part of Theorem 1 of 1.5, and in order to facilitate future references.

Let (G, X) be a simple adjoint Shimura variety. It is well known (see [De3, 3.8]) that there is a totally real number field $E_1(G, X)$ such that $E(G, X)$ is either it or a totally imaginary quadratic extension of it, depending on the fact that (G, X) is not or is with non-trivial involution. Let $F(G, X)$ be the totally real number field such that G is the Weil restriction from $F(G, X)$ to $\mathbb{Q}$ of an absolutely simple adjoint $F(G, X)$-group (cf. [De1, 2.3.4]). Let $EF_1(G, X)$ (resp. $EF(G, X)$) be the composite field of $F(G, X)$ with $E_1(G, X)$ (resp. with $E(G, X)$). Let $G_1(G, X)$ (resp. $G(G, X)$) be the Galois extension of $\mathbb{Q}$ generated by $EF_1(G, X)$ (resp. by $EF(G, X)$). $G_1(G, X)$ is a totally real number field, while in case (G, X) is with non-trivial involution, then $G(G, X)$ is a totally imaginary quadratic extension of $G_1(G, X)$. Using standard facts on number fields (for instance, see [La, p. 168]), we get:

Fact. *If (G,X) is with non-trivial involution, then the set of primes p which split in $G_1(G,X)$ but do not split in $G(G,X)$, and are such that G is unramified over $\mathbb{Q}_p$, is of positive Dirichlet density.*

If p is such a prime, then $G_{\mathbb{Q}_p}$ is a product of absolutely simple groups, and any prime of $E_1(G,X)$ (resp. of $E(G,X)$) dividing p has residue field $\mathbb{F}_p$ (resp. $\mathbb{F}_{p^2}$). So (a simple factor of) the Shimura group pair (see [Va2, 2.2.5 1) and 2.2.6 5)]) $(G_{\mathbb{Q}_p},[\mu_p])$ defined naturally by (G,X), is, in the logical sense, with non-trivial involution.

§3. Good embeddings into unitary Shimura varieties

This chapter is divided into two parts. In 3.1 (resp. 3.1.1-2) we prove a rational version (resp. an effective generic version) of the result stated in [Va2, 1.23], while in 3.2 we apply it to the understanding of Frobenius tori in the context of a principally polarized abelian variety over a number field. As part of the proof of 3.1.1 is slightly sketched, 3.1.1 is starred (cf. 3.1.2 4)).

3.0. The setting. A) What follows is very close in spirit to [Va1, 6.5-6]. We start with a simple adjoint Shimura variety $\mathrm{Sh}(G,X)$ of abelian type. So G is an adjoint simple $\mathbb{Q}$–group. From [De1, 2.3.4] we deduce the existence of a totally real number field F and of an absolutely simple F-group $G(F)$ (of classical Lie type) such that

$$G=\mathrm{Res}_{F/\mathbb{Q}}G(F).$$

For any totally real number field F_1 containing F we denote by

$$G^{F_1}:=\mathrm{Res}_{F_1/\mathbb{Q}}G(F)_{F_1}.$$

As in [Va1, 6.6.1] we get naturally a Shimura pair (G^{F_1},X^{F_1}) and an injective map

$$(G,X)\hookrightarrow(G^{F_1},X^{F_1}),$$

with X^{F_1} as in loc. cit. Let $p\geq 3$ be a prime such that G is unramified over $\mathbb{Q}_p$; this forces F to be unramified above p. In what follows we always choose F_1 to be unramified over p: this implies that G^{F_1} is also unramified over $\mathbb{Q}_p$.

B) In practice, it is convenient (though not really needed) to choose whenever possible F_1 to be big enough so that $G(F)_{F_1}$ splits over the completion of F_1 with respect to any finite prime of it. If G is of B_n or C_n Lie type, then such a field F_1 always exists as $G(F)$ itself splits over all but finite completions of F with respect to its finite primes; argument: a simple adjoint group of B_n or C_n Lie type over the ring of integers of a local field, is a split group.

However, for the sake of generality, in what follows, even when possible, we do not assume that F_1 is such that $G(F)_{F_1}$ splits over the completion of F_1 with respect to any finite prime of it. This generality is hinted at in E) below, but in fact is not really used here.

C) From now on till the end of Proposition 3.1 we assume that G is not of A_n Lie type, $n \in \mathbb{N}$. We now introduce the "simplest" Shimura pair $(\tilde{G}, \tilde{X})$ with the property that $\tilde{G}_{\mathbb{R}}$ is isomorphic to a simple non-compact factor of $G_{\mathbb{R}}$. So $\tilde{G}$ is an absolutely simple $\mathbb{Q}$–group of adjoint type, of the "simplest" type. Depending on the type of (G, X) we have the following choices for $\tilde{G}$:

– if (G, X) is of B_n type ($n \in \mathbb{N} \setminus \{1\}$), then $\tilde{G} = SO(2, 2n-1)$ is the group of $2n+1$ by $2n+1$ matrices of determinant one leaving invariant the quadratic form $-x_1^2 - x_2^2 + x_3^2 + ... + x_{2n+1}^2$ on $\mathbb{Q}^{2n+1}$;

– if (G, X) is of C_n type ($n \in \mathbb{N} \setminus \{1, 2\}$), then $\tilde{G}$ is the split simple $\mathbb{Q}$–group of C_n Lie type;

– if (G, X) is of $D_n^{\mathbb{R}}$ type ($n \in \mathbb{N}$, $n \geq 4$), then $\tilde{G} = SO(2, 2n-2)^{\mathrm{ad}}$ ($SO(2, 2n-2)$ is defined in the same manner as $SO(2, 2n-1)$);

– if (G, X) is of $D_n^{\mathbb{H}}$ type ($n \in \mathbb{N}$, $n \geq 4$), then $\tilde{G} = SO^*(2n)^{\mathrm{ad}}$ is the $\mathbb{Q}$–version of the $\mathbb{R}$-version described in [He, p. 445]; so $SO^*(2n)(\mathbb{Q})$ is the subgroup of $SO(2n)(\mathbb{Q}(i))$ formed by matrices leaving invariant the skew hermitian form $-z_1\overline{z}_{n+1} + z_{n+1}\overline{z}_1 - .... - z_n\overline{z}_{2n} + z_{2n}\overline{z}_n$ over $\mathbb{Q}(i)$ (here $\overline{z}_i$ means the complex conjugate of the complex number z_i, while $SO(m)$, $m \in \mathbb{N}$, is the semisimple $\mathbb{Q}$–group leaving invariant the quadratic form $x_1^2 + ... + x_m^2$ on $\mathbb{Q}^m$).

The existence of such a group $\tilde{G}$ is a consequence of the fact that the Shimura varieties of D_n^{mixt} type are not of preabelian type (see [De1]). The values of $n \in \mathbb{N}$ are chosen so that we do not get repetitions among the types (so for the $D_4^{\mathbb{H}}$ and $D_4^{\mathbb{R}}$ types, the convention of [De1, end of 2.3.8] applies). If $\tilde{G}$ is not $SO^*(2n)^{\mathrm{ad}}$ for some $n \in \mathbb{N}$, $n \geq 4$, then the $\tilde{G}(\mathbb{R})$-conjugacy class $\tilde{X}$ of homomorphisms $SO(2)_{\mathbb{R}} \to \tilde{G}_{\mathbb{R}}$ (given birth to a Shimura pair $(\tilde{G}, \tilde{X})$) is uniquely determined; in the excluded cases we have two choices for such a conjugacy class, cf. [De1, 1.2.7-8]. In what follows we are interested just in $\tilde{G}$: the role of $\tilde{X}$ is irrelevant; $\tilde{X}$ is mentioned just to emphasize that $\tilde{G}$ can be used to define a simple adjoint Shimura pair.

D) Similarly, in what follows $SU(m, m)$ ($m \in \mathbb{N}$) will be the $\mathbb{Q}$–version of the $\mathbb{R}$–version described in [He, p. 444]. We denote by $G(A)$ the following simply connected semisimple $\mathbb{Q}$–group (till the end of 3.1.1 we have the same restrictions on n as in C) above):

– if (G, X) is of B_n or $D_n^{\mathbb{R}}$ type, then $G(A) = SU(2^{n-1}, 2^{n-1})$;
– if (G, X) is of C_n or $D_n^{\mathbb{H}}$ type, then $G(A) = SU(n, n)$.

The above choices for $G(A)$ are in accordance with [Sa, 3.3-7 or p. 461]: we have naturally a homomorphism

$$h_{\mathbb{R}}: \tilde{G}_{\mathbb{R}}^{\mathrm{sc}} \to G(A)_{\mathbb{R}}$$

of finite kernel, and which, at the level of Lie algebras satisfies the condition (H_2) of [Sa, p. 427]. Here $\tilde{G}^{\mathrm{sc}}$ is the simply connected group cover of $\tilde{G}$. This finite kernel is trivial if (G, X) is of B_n, C_n or $D_n^{\mathbb{R}}$ type.

The only difference in comparison with [Sa, p. 461] is: in the case when (G, X) is of $D_n^{\mathbb{R}}$ type we consider the spin representation and not just a half-spin representation; so we have $G(A) = SU(2^{n-1}, 2^{n-1})$ and not $SU(2^{n-2}, 2^{n-2})$, i.e. we work with a product of two copies of $SU(2^{n-2},2^{n-2})$ embedded naturally in $SU(2^{n-1},2^{n-1})$. Of course, if (G,X) is of $D_n^{\mathbb{R}}$ type we can work as well with just one half-spin representation (i.e. we can work with $SU(2^{n-2},2^{n-2})$), but then for n even the above finite kernel is non-trivial and we can not get 3.1 f) below.

Moreover we can assume that $h_{\mathbb{R}}$ is the extension to $\mathbb{R}$ of a homomorphism

$$h\colon \tilde G^{\mathrm{sc}} \to G(A)$$

(so the notation $h_{\mathbb{R}}$ is justified). This is a consequence of the constructions in [Sa, 3.3-7]: instead of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, in loc. cit. we can work as well with $\mathbb{Q}$, $\mathbb{Q}(i)$ and the standard quaternion algebra $\mathbb{Q}(i,j)$ over $\mathbb{Q}$. Let $\tilde G^d$ be the image of h in $G(A)$.

E) Of course instead of $\mathbb{Q}(i)$, in C) and D) above, we can work as well with a totally imaginary quadratic extension K of $\mathbb{Q}$ unramified over p, and instead of $\mathbb{Q}(i,j)$ of D) above we can work as well with a quaternion algebra over $\mathbb{Q}$ containing K and ramified at the infinite place. What we mean by this: we have variants, where instead of $\tilde G$ or $G(A)$, we work with some inner forms of them, which split over K. However, as $\mathbb{Q}(i)$ is unramified above any odd prime, it is all we need in what follows; so we do not detail the case of an arbitrary K as above. Just one example: in some cases we can work with the SO-group of the quadratic form $-x_1^2-x_2^2+lx_3^2...+lx_5^2$ instead of $SO(2,3)$, and with $K=\mathbb{Q}(\sqrt{-l})$ instead of $\mathbb{Q}(i)$, etc. (here $l\in\mathbb{N}$ is such that K is unramified above p).

F) We divide this part in three subparts.

i) Taking F_1 big enough (subject to the requirements of A) above), $G(F)_{F_1}$ becomes an inner form of $\tilde G_{F_1}$. We recall the arguments for this well known fact.

If G is of B_n or C_n Lie type, then this fact is obvious: we can take $F_1=F$, as there are no outer automorphisms involved.

If G is of D_n Lie type, $n\geq 4$, then this fact is a consequence of the following remarks:

– $G(F)$ is a form (not a priori inner) of $\tilde G_F$;

– the group scheme of outer automorphisms of $\tilde G_F$ is $\mathbb{Z}/2\mathbb{Z}$ if $n\geq 5$, and is the symmetric group S_3 if $n=4$;

– we have a short exact sequence $\mathbb{Z}/3\mathbb{Z}\hookrightarrow S_3\twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$;

– working in the étale topology of $\mathrm{Spec}(F)$, the non-trivial torsors of $\mathbb{Z}/2\mathbb{Z}$ (resp. of $\mathbb{Z}/3\mathbb{Z}$) correspond to quadratic extensions of F (resp. to Galois field extensions of F of degree 3);

– for any embedding $i_F\colon F\hookrightarrow\mathbb{R}$, $G(F)_{\mathbb{R}}$ is an inner form of $\tilde G_{\mathbb{R}}$.

So if $n\geq 5$ (resp. $n=4$), moving from F to a totally real field extension F_1 of it of degree at most 2 (resp. of degree at most 6), $G(F)_{F_1}$ becomes an inner form of $\tilde G_{F_1}$. Moreover, as G and $\tilde G$ are unramified over $\mathbb{Q}_p$, we deduce that F_1 is unramified over p.

ii) It is worth making slightly more explicit the situation when (G,X) is of $D_n^{\mathbb{R}}$ type. We first consider an inner form G'_F of $\tilde G_F$ with the property that with respect to any embedding $i_F\colon F\hookrightarrow\mathbb{R}$, we have: $G'_{\mathbb{R}}$ is isomorphic to $G(F)_{\mathbb{R}}$. For instance, we can take G'_F to be the adjoint group of the SO-group of the quadratic form $a_1x_1^2+a_2x_2^2+x_3^2+...+x_{2n}^2$, where $a_1,a_2\in F$ are such that (cf. approximation theorem):

– under an embedding i_F as above, they are both negative or positive, depending on the fact that the resulting group $G(F)_{\mathbb{R}}$ is non-compact or compact;

– G'_F is unramified over $\mathbb{Q}_p$.

Enlarging F to F_1 in the way allowed, we can assume (see end of i) above) that $G(F)_{F_1}$ is an inner form of G'_{F_1}, and so of $\tilde{G}_{F_1}$.

iii) We come back to the general situation of i) above. Using this form to twist the homomorphism h_{F_1} (or more precisely the resulting monomorphism $\tilde{G}^d_{F_1} \hookrightarrow G(A)_{F_1}$) defined by the extension of h to F_1, we get a monomorphism

$$h^t: G(F)^d_{F_1} \hookrightarrow G(A)^t,$$

where $G(A)^t$ is an inner form of $G(A)_{F_1}$ and $G(F)^d_{F_1}$ is the central isogeny cover of $G(F)_{F_1}$ corresponding to the group cover $\tilde{G}^d$ of $\tilde{G}$. After potentially having to pass to a totally real quadratic extension of F_1, this twisting is possible as:

– if (G,X) is not of $D_n^{\mathbb{R}}$ type with n even, then the center of $\tilde{G}^d$ is contained in the center of $G(A)$;

– if (G,X) is of $D_n^{\mathbb{R}}$ type with n even, then we have a monomorphism

$$SO(2,2n-2) \hookrightarrow G(A)^{\mathrm{ad}}$$

defined by taking the image of $\tilde{G}^{\mathrm{sc}}$ in $G(A)^{\mathrm{ad}}$ (this is a consequence of the fact that we are working with the spin representation, cf. D) above; see [Bou, (VIII) of planche IV] why this image is $SO(2,2n-2)$);

– the compact inner form $SO(2n)$ of $SO(2,2n-2)$ is defined by twisting via an element in $H^1(\mathbb{Q},SO(2,2n-2))$ (and not only in $H^1(\mathbb{Q},SO(2,2n-2)^{\mathrm{ad}})$); so, as the kernel of the natural isogeny $SO(2,2n-2) \to SO(2,2n-2)^{\mathrm{ad}}$ is $\mathbb{Z}/2\mathbb{Z}$, we can choose (cf. the standard Galois interpretation of $H^2(F,\mu_2)$ and [Ha, 5.5.1]) F_1 as above such that the inner form $G(F)_{F_1}$ of $\tilde{G}_{F_1}$ is defined by an element in $H^1(F_1,SO(2,2n-2))$.

So in fact we first twist the resulting monomorphism $\tilde{G}^e_{F_1} \hookrightarrow G(A)^{\mathrm{ad}}_{F_1}$ (with $\tilde{G}^e$ as the image of $\tilde{G}^d$ in $G(A)^{\mathrm{ad}}$), and then we "lift" the things to the simply connected group cover of the resulting twist of $G(A)^{\mathrm{ad}}_{F_1}$. However, in what follows, by abuse of language, we say that we twist h_{F_1}; similarly, in other situations (like over $\mathbb{R}$), we use the same language.

G) We have the following fact:

Fact. *If for an embedding $i_{F_1}: F_1 \hookrightarrow \mathbb{R}$, $G(F)_{F_1}$ becomes a compact group (resp. it becomes $\tilde{G}_{\mathbb{R}}$), then, under the same embedding, $G(A)^t$ becomes a compact group (resp. it becomes $G(A)_{\mathbb{R}}$).*

Proof: What we need to prove is: if we twist the homomorphism $h_{\mathbb{R}}$, by the compact inner form of $\tilde{G}^{\mathrm{sc}}_{\mathbb{R}}$, then the inner form IF of $G(A)_{\mathbb{R}}$ we get, is the compact form. This is a general fact on symmetric domains, due to the fact that $h_{\mathbb{R}}$ satisfies the condition (H_2) of [Sa]: it can be checked easily using the Cartan decompositions (referred to in [Sa, p. 427]; see also [He, §7 of Ch. 3]). Here we would like to give an elementary proof of this fact, in a form needed for future references.

Case 1: (G, X) is of C_n or $D_n^{\mathbb{H}}$ type. This is easy, and it can be proved in many ways. We choose two such ways. For both we use the classical fact on the conjugacy of maximal compact subgroups of a semisimple group over $\mathbb{R}$ (for instance see [He, 2.2 (ii) of p. 256]). First, looking at the inner forms of $SU(n,n)_{\mathbb{R}}$, the only one which can contain a compact subgroup of C_n or D_n Lie type, is the compact inner form. Second, Zarhin's lemma (recalled in 4.2.5.1) applies (to the extension to $\mathbb{C}$ of the inclusion of a maximal compact subgroup of IF into IF).

Case 2: (G, X) is of B_n or $D_n^{\mathbb{R}}$ type. Let us first remark that we have an infinite chain of injective maps between Shimura pairs:

$$(SO(2,1), X(1)) \hookrightarrow (SO(2,2), X(2)) \hookrightarrow \ldots \hookrightarrow (SO(2,n), X(n)) \hookrightarrow \ldots \tag{1}$$

Here $X(n)$, $n \in \mathbb{N}$, are the logical Hermitian symmetric domains. Using the fact that $\tilde{G}_{\mathbb{R}}^{\mathrm{sc}}$ is the simply connected group cover of $SO(2,m)_{\mathbb{R}}$ for some $m \in \mathbb{N}$, $m \geq 3$, we just need to show: twisting the resulting homomorphism (obtained via $h_{\mathbb{R}}$ and the inclusions of (1))

$$SO(2,1) \hookrightarrow SU(n,n)_{\mathbb{R}}^{\mathrm{ad}},$$

through the compact inner form of $SO(2,1)$, we get the compact inner form of $G(A)_{\mathbb{R}}^{\mathrm{ad}}$. So we need to show: under the natural diagonal embedding of $SU(1,1) = SO(2,1)^{\mathrm{sc}}$ in $SU(q,q)$ (i.e. under the composite of the diagonal monomorphism $SU(1,1) \hookrightarrow SU(1,1)^q$, with the natural monomorphism $SU(1,1)^q \hookrightarrow SU(q,q)$; here $q \in \mathbb{N}$), through the twist corresponding to the compact inner form of $SU(1,1)$, we get the compact inner form of $SU(q,q)$. This is an easy exercise, that can be checked directly (for instance by direct calculation, or by reduction to the situation of C_q of Case 1 above, or by using induction and the same lemma of Zarhin). This ends the proof of the fact.

H) Let $G_1^{\mathrm{sc}} := \mathrm{Res}_{F_1/\mathbb{Q}} G(A)^t$. From G) above we deduce the existence of a Shimura pair of adjoint type of the form $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}})$, with G_1^{ad} as the adjoint group of G_1^{sc}; [De1, 1.2.8 (ii)] guarantees that such a Shimura pair is uniquely determined.

I) We consider (cf. [Va2, 2.3.5.1]) an embedding

$$f_1 \colon (G_1, X_1) \hookrightarrow (G\mathrm{Sp}(W, \psi), S)$$

such that:

– $G_1^{\mathrm{der}} = G_1^{\mathrm{sc}}$ (so the notations are justified);

– there is a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W for which we get a perfect form $\psi \colon L_{(p)} \otimes L_{(p)} \to \mathbb{Z}_{(p)}$;

– denoting by $\mathcal{B}_{(p)}$ the $\mathbb{Z}_{(p)}$-subalgebra of $\mathrm{End}(L_{(p)})$ fixed by G_1, we get a standard PEL situation $(f, L_{(p)}, v, \mathcal{B}_{(p)})$ (see [Va2, 2.3.5] for the meaning of this), for any prime v of $E(G_1, X_1)$ dividing p.

We recall that [Va2, 2.3.5.1] is just a $p \geq 3$ version of parts of [Va1, 6.5.1.1 and 6.6.5] where we had $p \geq 5$, and that loc. cit. is rooted in [De1, 2.3.10].

J) Moreover, there is $M(G, X) \in \mathbb{N}$, effectively computable, such that

$$\dim_{\mathbb{Q}}(W) \leq M(G, X),$$

regardless of the prime $p \geq 3$ we keep track of: this is obvious, cf. the construction in [Va1, 6.6.5] and the fact that we can choose (cf. i) and iii) of F) above) F_1 so that $[F_1 : F] \leq 12$.

3.1. Proposition. *There are injective maps*

$$(G_2, X_2) \hookrightarrow (G_3, X_3) \hookrightarrow (G_1, X_1)$$

such that:

a) $(G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}}) = (G, X)$ *and* $(G_3^{\mathrm{ad}}, X_3^{\mathrm{ad}}) = (G^{F_1}, X^{F_1})$*;*

b) we get a monomorphism $G_2^{\mathrm{ad}} \hookrightarrow G_3^{\mathrm{ad}}$ *which (under the identifications of a)) is nothing else but the natural monomorphism* $G \hookrightarrow G^{F_1}$*;*

c) $G_3^{\mathrm{der}} = Res_{F_1/\mathbb{Q}} G_{F_1}^d$*, and under this identification, the monomorphism* $G_3^{\mathrm{der}} \hookrightarrow G_1^{\mathrm{der}}$ *becomes the Weil restriction of* h^t*;*

d) if (G, X) *is without involution (resp. is with involution; this implies that* (G, X) *is of* $D_n^{\mathbb{H}}$ *type), then we have* $E(G_2, X_2) = E(G_3, X_3) = E(G_1, X_1)$ *(resp.* $E(G_2, X_2) = E(G_3, X_3)$ *is a quadratic extension of* $E(G_1, X_1)$*);*

e) $Z^0(G_2) \subset Z^0(G_3) \subset Z^0(G_1)$ *(here* $Z^0(*)$ *denotes the toric part of the center of the group* $*$*);*

f) if (G, X) *is not of* $D_n^{\mathbb{H}}$ *type, then* G_2^{der} *and* G_3^{der} *are simply connected.*

Proof: We take G_3 to be generated by $Z^0(G_1)$ and by the subgroup $\mathrm{Res}_{F_1/\mathbb{Q}}(G(F)_{F_1}^d)$ of G_1^{der}. This takes care of c). We take G_2 to be the natural subgroup of G_3 generated by $Z^0(G_1)$ and such that b) holds. So e) holds as well. The fact that $h_{\mathbb{R}}$ at the level of Lie algebras satisfies the condition (H_2) of [Sa, p. 427] gets translated (cf. G) and H) above) in: there is $x_1 \in X_1$ such that its attached monomorphism $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{1\mathbb{R}}$ factors through $G_{2\mathbb{R}}$, resulting in a monomorphism $SO(2)_{\mathbb{R}} \to G_{\mathbb{R}}$ corresponding to an element of X. This takes care of a). d) is an easy consequence of [De3, 3.8] and [De1, 2.3.10 (ii)]: we can always choose the totally imaginary quadratic extension of F_1 needed to build up $Z(G_1)$ such that in the case (G, X) is with non-trivial involution we do have that $E(G_3, X_3)$ is a quadratic extension of $E(G_1, X_1)$. f) is a direct consequence of the construction of h^t. This proves the proposition.

It is now an easy exercise to see (as in [Va1, 6.5-6]) that h^t has a version over the normalization $F_{1(p)}$ of $\mathbb{Z}_{(p)}$ in F_1. In other words, referring to E) above, instead of $\mathbb{Q}$, a totally imaginary quadratic extension K of $\mathbb{Q}$ unramified over p, and a quaternion algebra $H_{\mathbb{Q}}$ over $\mathbb{Q}$ containing K and ramified at the infinite place, we need to work respectively with: $\mathbb{Z}_{(p)}$, the normalization $K_{(p)}$ of $\mathbb{Z}_{(p)}$ in K, and with a maximal order of $H_{\mathbb{Q}}$ containing $K_{(p)}$. Typical situation: we work with $\mathbb{Z}_{(p)}$, with $\mathbb{Z}_{(p)}(i)$ and with $\mathbb{Z}_{(p)}(i, j)$. We immediately get:

3.1.1*. Corollary. *The result stated in [Va2, 1.23] is true for* (G, X) *provided*

$$p \geq \max\{5, M(G, X)/2\}.$$

Proof: The above part of 3.1 takes care of 1) to 4) of [Va2, 1.23]. [Va2, 1.23 5)] (i.e. the part involving "G-ordinary" things) is an immediate consequence of [Va2, 4.11.2-3]: the condition b) of [Va2, 4.11.2] can be easily checked (see [Va5] for details). This proves the corollary.

The inequality $p \geq \max\{5, M(G, X)/2)\}$ is in conformity with [Va1, 5.8.6]. If we assume [Va2, 2.3.8 2)] (see [Va5]), then the condition $p \geq \max\{5, M(G, X)\}$ gets replaced by $p \geq 3$.

3.1.2. Remarks. 1) If (G, X) is of A_n type ($n \in \mathbb{N}$), then 3.1 gets substituted by [Va2, 2.3.5.1].

2) Using standard arguments we get that 3.1 has a version for the case when (G, X) is a product of simple adjoint Shimura pairs of abelian type. To state it let $(G, X) = \prod_{i\in I}(G_i, X_i)$ be written as a product of such simple Shimura pairs. Let F^i be the totally real number field such that G_i is the Weil restriction from F^i to $\mathbb{Q}$ of an absolutely simple F^i-group $\widetilde{G}_i$. Then there is a product $(G^1, X^1) = \prod_{i\in I}(G_i^1, X_i^1)$ of simple Shimura adjoint pairs of some A_n type ($n \in \mathbb{N}$), and there are injective maps

$$(G_2, X_2) \hookrightarrow (G_3, X_3) \hookrightarrow (G_1, X_1) \overset{f_1}{\hookrightarrow} (GSp(W_1, \psi_1), S_1)$$

such that:

a) $(G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}}) = (G, X)$ and $(G_3^{\mathrm{ad}}, X_3^{\mathrm{ad}}) = \prod_{i\in I}(G_i^{F_1^i}, X_i^{F_1^i})$, with F_1^i a totally real number field containing F^i;

b) we get a monomorphism $G_2^{\mathrm{ad}} \hookrightarrow G_3^{\mathrm{ad}}$ which is a product of the natural monomorphisms $G_i \hookrightarrow G_i^{F_1^i}$;

c) $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) = (G^1, X^1)$, G_1^{der} is a simply connected semisimple group, and f_1 is a PEL type embedding;

d) the monomorphism $G_3^{\mathrm{der}} \to G_1^{\mathrm{der}}$ is a product indexed by $i \in I$ of monomorphisms which are either isomorphisms or are obtained as in 3.1 c), depending on the fact that G_i is or it is not of A_n Lie type for some $n \in \mathbb{N}$;

e) if none of the Shimura pairs (G_i, X_i) is of $D_n^{\mathbb{H}}$ type with non-trivial involution (resp. if there is such a Shimura pair of $D_n^{\mathbb{H}}$ type with non-trivial involution), then we have: $E(G_2, X_2) = E(G_3, X_3) = E(G_1, X_1)$ (resp. we have: $E(G_2, X_2) = E(G_3, X_3)$ is an extension of $E(G_1, X_1)$ of degree 2^m, for some $m \in \mathbb{N} \cup 0$; moreover m is not bigger than the number of such Shimura pairs);

f) 3.1 e) remains true;

g) if none of the Shimura pairs (G_i, X_i) is of $D_n^{\mathbb{H}}$ type, then G_2^{der} and G_3^{der} are simply connected semisimple groups.

3) Using standard arguments (as in [Va1, 6.5.1]), from 3.1.1 we get that the result stated in [Va2, 1.23] remains true for any adjoint Shimura quadruple $(\bar{G}, \bar{X}, \bar{H}, \bar{v})$ of abelian type, with $\bar{v}$ dividing a prime $p \geq \max\{5, M(\bar{G}, \bar{X})\}$; here $M(\bar{G}, \bar{X}) \in \mathbb{N}$ is effectively computable (for instance we can take it to be the product of the numbers $M(*)$, with $*$ running through the simple factors of $(\bar{G}, \bar{X})$).

4) 3.1.1 has many consequences (cf. [Va1, 6.8], [Va2, 1.23, 1.23.1 and 4.4.6]). We will come back to 3.1.1 and its consequences in [Va5]. In what follows we do not need 3.1.1: we need just 3.1 and its variant of 2) above.

5) We also view 3.1 and 3.1.2 2) as the unitary version of [De1, 2.3.10]. Moreover, combining 3.1 and 3.1.2 2) with the PEL type embeddings of Shimura (see [Sh]), one reobtains the essence of [De1, 2.3.10] (in particular reobtains the fact that any Shimura variety of A_n, B_n, C_n, $D_n^{\mathbb{H}}$ or $D_n^{\mathbb{R}}$ is of preabelian type).

3.1.3. Definition. We refer to (1) of 3.0 G) as the standard SO-sequence of injective maps between Shimura pairs (to be abbreviated shortly as SSO).

3.1.4. Point of view. It is a widely spread opinion that the closest generalization of the elliptic modular curve is provided by Siegel or Hilbert–Blumenthal modular varieties. In our opinion this is just partially correct: the Shimura varieties defined by Shimura pairs of SSO are equally well a very close generalization of the elliptic modular curve. We refer to these Shimura varieties as classical spin modular varieties.

3.1.5. Remark. Here we refer to [Va1, 6.6.2] and its proof. With the notations of loc. cit., if (G,X) is of C_l (resp. of $D_l^{\mathbb{R}}$) type, then we can choose as well G^1 to be of A_{2l} type (cf. 3.1); but in this case [Va1, 6.6.2 d)] has to be modified: there is a perfect alternating form $\psi_L\colon L\otimes L\otimes W(\mathbb{F})\to W(\mathbb{F})$ (resp. a perfect symmetric form $B_L\colon L\otimes L\otimes W(\mathbb{F})\to W(\mathbb{F})$) such that $G^0_{W(\mathbb{F})}$ is the subgroup of $G^1_{W(\mathbb{F})}$ normalizing ψ_L (resp. B_L). So [Va1, 5.6.9] still applies to get the existence of $\mathrm{Sh}_p(G_4,X_4)$ (see [Va1, 3.2.6] for this notation).

3.2. An application of 3.1 to Frobenius tori. A) We start by introducing a new setting. We use the notations of 2.2.2. We still denote by $(\mathcal{A},\mathcal{P}_{\mathcal{A}})$ its restriction to $\mathrm{Sh}_H(G,X)$. Let

$$m\colon \mathrm{Spec}(E)\to\mathcal{N}$$

be a morphism, and let A be the abelian variety over E obtained from $\mathcal{A}$ by pull back. We assume that the Mumford–Tate group of A is G itself.

B) Let

$$(G_2,X_2)\hookrightarrow(G_3,X_3)\hookrightarrow(G_1,X_1)\overset{f_1}{\hookrightarrow}(GSp(W_1,\psi_1),S_1)$$

be injective maps constructed following the pattern of 3.1, but starting from the adjoint Shimura pair $(G_2^{\mathrm{ad}},X_2^{\mathrm{ad}}):=(G^{\mathrm{ad}},X^{\mathrm{ad}})$ (cf. 3.1.2 2)). So G_1^{ad} has all the simple factors of A_n Lie type (for potentially different values of n). Moreover, the embedding $(G_1,X_1)\hookrightarrow(GSp(W_1,\psi_1),S_1)$ is a PEL type embedding (see [Va1, 4.3.11] and [Ko]). We assume in what follows familiarity with 4.2.0-11.

We assume that the intersection X_{20} of X_2 and X inside $X_2^{\mathrm{ad}}=X^{\mathrm{ad}}$ is non-empty, cf. [Va1, 2.4.0 and rm. 3) of 3.2.7] and [Va2, 4.9.1]. We can assume that m can be lifted to a point $\mathrm{Spec}(\mathbb{C})\to\mathrm{Sh}(G,X)$ defined by a pair $[x_m,g_m]$ (see [Va1, 4.1]), with $x_m\in X_{20}$ (here $g_m\in G(\mathbb{A}_f)$). Let L_1 be a $\mathbb{Z}$-lattice of W_1 such that we get a perfect form $\psi_1\colon L_1\otimes L_1\to\mathbb{Z}$. Let $H_2:=\{\mathrm{g}\in G_2(L_1\otimes\widehat{\mathbb{Z}})|\mathrm{g}\bmod 4 \text{ is the identity}\}$. As in 2.2.2, we get a principally polarized abelian scheme $(\mathcal{A}_2,\mathcal{P}_{\mathcal{A}_2})$ over $\mathrm{Sh}_{H_2}(G_2,X_2)$.

C) Let $(W_0,\psi_0):=(W\oplus W_1,\psi\oplus\psi_1)$. We consider as in [Va1, Example 3 of 2.5] and [Va2, 4.9.2] the diagonal Hodge embedding of the embeddings $f\colon(G,X)\hookrightarrow(GSp(W,\psi),S)$ and $f_2\colon(G_2,X_2)\hookrightarrow(GSp(W_1,\psi_1),S_1)$. We get an injective map:

$$f_3\colon(G_4,X_4)\hookrightarrow(GSp(W_0,\psi_0),S_0)$$

where (G_4,X_4) embeds into a Hodge quasi product of (G,X) and (G_2,X_2), via an injective map

$$f_4\colon(G_4,X_4)\hookrightarrow(G,X)\times(G_2,X_2),$$

which gives birth to identifications (to be denoted by I) $(G_4^{\mathrm{ad}},X_4^{\mathrm{ad}})=(G^{\mathrm{ad}},X^{\mathrm{ad}})=(G_2^{\mathrm{ad}},X_2^{\mathrm{ad}})$. These identifications I are defined via the projections of $(G,X)\times(G_2,X_2)$ onto its factors.

Let $H_4 := G_4(\mathbb{Q}_p) \cap H \times H_2$. Let $L_0 := L \oplus L_1$; it is a $\mathbb{Z}$-lattice of W_0, and we have a perfect form $\psi_0 \colon L_0 \otimes L_0 \to \mathbb{Z}$. We get naturally (starting from f_3), as in 2.2.2, a universal principally polarized abelian scheme $(\mathcal{A}_4, \mathcal{P}_{\mathcal{A}_4})$ over $\mathrm{Sh}_{H_4}(G_4, X_4)$.

D) We assume that E is big enough so that we have a morphism

$$m_4 \colon \mathrm{Spec}(E) \to \mathrm{Sh}_{H_4}(G_4, X_4),$$

whose projection on $\mathrm{Sh}_H(G, X)$ (via f_4) is m. The assumption $x_m \in X_{20}$ guarantees the existence of such a morphism m_4, with E big enough, provided we pass to isogenies and change a little bit the level structures (to be compared with the definition of $\mathfrak{C}_0$ in [Va2, 4.9.1.1]). Though we do not need this, we would like to point out that [Va1, 3.3.1] guarantees that this passage and these changes of level structures can be performed so that we still keep track of any a priori given rational prime q for which the closure of G (resp. of G_2) in $GL(L \otimes \mathbb{Z}_{(q)})$ (resp. in $GL(L_1 \otimes \mathbb{Z}_{(q)})$) is a reductive group over $\mathbb{Z}_{(q)}$. What we do need here: these two operations are irrelevant to the proof of Mumford–Tate conjecture (see 4.2.11); so, not to complicate the notations, we can assume that the performance of these two operations are not needed.

We also assume that E contains $E(G_4, X_4)$. Let A_2 be the abelian variety over E obtained from $\mathcal{A}_2$ via m_4, f_4 and the embedding $(G_2, X_2) \hookrightarrow (GSp(W_1, \psi_1), S_1)$. So the abelian variety over E obtained via m_4 and f_4 is $A \times A_2$. We also assume that E is big enough so that all the Hodge cycles of $A_{\bar{E}}$, $A_{2\bar{E}}$ and $A_{\bar{E}} \times A_{2\bar{E}}$ are defined over E.

E) Let v be a prime of E such that:

i) A and A_2 have good reduction with respect to v;

ii) the Frobenius torus T_v constructed as in 4.2.7.1 (for A) has maximal rank, equal to the rank of the algebraic envelope of the p-adic representation attached to A, for any rational prime p (cf. 4.2.4); moreover this property also holds for A_2 and respectively $A \times A_2$ (we get Frobenius tori T_v^2 and respectively T_v^4);

iii) it divides an odd rational prime $l \geq M(f_1, L_1)$;

iv) for any prime $p \geq l$ we get standard Hodge situations $(f_3, L_0 \otimes \mathbb{Z}_{(p)}, v_4(p))$, $(f_2, L_1 \otimes \mathbb{Z}_{(p)}, v_2(p))$ and $(f_1, L_1 \otimes \mathbb{Z}_{(p)}, v_1(p))$ (here for $i \in \{1, 2, 4\}$, $v_i(p)$ is an arbitrary prime of $E(G_i, X_i)$ dividing p; we choose $v_i(l)$ to be the prime of $E(G_i, X_i)$ divided by v, $i \in \{1, 2, 4\}$).

The fact that such a prime v does exist is implied by [Va1, 5.8.7] and [Va2, 2.3.6] (they take care of iv)), and by 4.2.7-8 (they take care of ii) and iii)).

F) Let $p \geq l+2$ be a prime such that G^{ad} splits over $\mathbb{Q}_p$. The essential condition is $p \geq l+2$; the other one is inserted just to simplify the language (cf. 3.2.2.1). So G_2^{ad} and G_4^{ad} also split over $\mathbb{Q}_p$. In what follows we work with sufficiently high positive integral powers of the Frobenius automorphism of $\overline{k(v)}$ fixing $k(v)$, so that all the Frobenius elements to be introduced below are defining valued points of the Frobenius tori involved (i.e. we prefer to work not with groups of multiplicative type, but with tori). Let $Fr^i \in T_v^i(\mathbb{Q}_p)$, $i \in \{2, 4\}$ be a Frobenius element, as in 4.2.7.2. Similarly, we get a Frobenius element $Fr \in T_v(\mathbb{Q}_p)$. We have natural monomorphisms $T_{v\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}_p}$ and $T^i_{v\mathbb{Q}_p} \hookrightarrow G_{i\mathbb{Q}_p}$. We can assume that under the identifications I, the images of Fr and Fr^2 in $G^{\mathrm{ad}}_{\mathbb{Q}_p}(\mathbb{Q}_p)$ are the same.

3.2.1. The key point is: as the embedding f_1 is a PEL type embedding, with G_1^{der} a simply connected group which is a product of semisimple groups of A_n Lie type having as their adjoints simply $\mathbb{Q}$–groups, the Langlands–Rapoport conjecture is true for the SHS $(f_1, L_0 \otimes \mathbb{Z}_{(l)}, v_1(l))$, (cf. [Mi1, 6.12] and 2.2.2.1; in fact, the reference to 2.2.2.1 can be substituted by [Mi1, (0.2) c)], see also the third part). What we need from this key point in our context is:

Fact. *There is a subtorus T_1 of G_1, and an element $Fr_{\mathbb{Q}} \in T_1(\mathbb{Q})$, such that there is $g \in G_1(\mathbb{Q}_p)$ taking (under conjugation) $T_{1\mathbb{Q}_p}$ into $T^2_{v\mathbb{Q}_p}$, and $Fr_{\mathbb{Q}}$ into Fr^2.*

As the manuscript [Mi1] is not yet published, we would like to point out that for what follows, instead of [Mi1] we can use as well the main result of [Zi] as follows. A suitable isogeny $A'_{2\bar{v}}$ of the extension to the algebraic closure of $k(v)$ of the reduction of A_2 with respect to v (see i) of 3.2 E)) has a CM lift A'_2 to characteristic zero, in such a way that the $\mathbb{Q}$–endomorphisms of $A'_{2\bar{v}}$ corresponding to the elements of $\mathcal{B} := \mathcal{B}_{(l)} \otimes \mathbb{Q}$, and the $\mathbb{Q}$–polarization of $A'_{2\bar{v}}$ (naturally induced from $\mathcal{P}_{\mathcal{A}_2}$) lift to $\mathbb{Q}$–endomorphisms and respectively to a $\mathbb{Q}$–polarization of A'_2 (see [Zi]). Using the Betti homology of an extension of A'_2 to $\mathbb{C}$, we get a symplectic space (W'_1, ψ'_1) over $\mathbb{Q}$, and a $\mathbb{Q}$–monomorphism $\mathcal{B} \hookrightarrow \mathrm{End}(W'_1)$.

Let G'_1 be the subgroup of $GSp(W'_1, \psi'_1)$ fixing all the elements of $\mathcal{B}$. It is an inner form of G_1 defined by an element $\gamma \in H^1(\mathbb{Q}, G_1)$. We have (see the first three paragraphs of [Ko, §8]):

HW. *The images of γ in $H^1(\mathbb{Q}_q, G_{1\mathbb{Q}_q})$ and in $H^1(\mathbb{R}, G_{1\mathbb{R}})$ are zero (here q is an arbitrary rational prime).*

In particular we have

$$G_{1\mathbb{Q}_p} = G'_{1\mathbb{Q}_p}$$

and

$$Z(G_1) = Z(G'_1).$$

In fact we can assume that $G'_1 = G_1$ (cf. [Mi1, 6.12]; see also the first five paragraphs of [Ko, §7] for the cases when G^{ad} has no simple factors of A_{2m+1} Lie type, with $m \in \mathbb{N}$); in what follows we will not use this identification at all, due to two reasons. The first reason is: the ideas to be introduced below can be used as well in the case of other PEL type embeddings (see [Ko] and [Zi]) (we recall that [Va1, 1,7.1] is not handling fully the D case of the PEL type embeddings). The second reason is: for applications to §5, what we need is just the part of HW involving ($\mathbb{Q}_p$ and) $\mathbb{R}$. To explain this, let (G'_1, X'_1) be the Shimura pair defined by A'_2. From [Ko, 4.2 and the first three paragraphs of §8] we deduce the existence of an isomorphism

$$W_1 \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} W'_1 \otimes_{\mathbb{Q}} \mathbb{R}$$

such that:

– it takes ψ into ψ', and $b \in \mathcal{B}$ into b, $\forall b \in \mathcal{B}$ (so it takes $G_{1\mathbb{R}}$ into $G'_{1\mathbb{R}}$);
– it takes X_1 into X'_1.

In particular we deduce that the zero dimensional Shimura varieties $\mathrm{Sh}(G_1^{\mathrm{ab}}, X_1^{\mathrm{ab}})$ and $\mathrm{Sh}(G_1'^{\mathrm{ab}}, X_1'^{\mathrm{ab}})$ are isomorphic. We formulate this as follows:

AB. *The smallest subtorus $Z_{\min}$ of $Z(G_1)$ such that all homomorphisms $Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{1\mathbb{R}}$ defined by elements of X_1, factor through the extension to $\mathbb{R}$ of the subgroup of G_1 generated by G_1^{der} and by $Z_{\min}$, is the same as the smallest subtorus of $Z(G_1')$ with the property that all homomorphisms $Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G'_{1\mathbb{R}}$ defined by elements of X_1', factor through the extension to $\mathbb{R}$ of the subgroup of G_1' generated by $G_1'^{\mathrm{der}}$ and by $Z_{\min}$.*

Above, HW points out to a Hasse–Witt type property, while AB stands for the abelian counterpart picture.

Coming back to our Frobenius tori, as A_2' has complex multiplication, we deduce the existence of a subtorus T_1' of G_1', and of an element $Fr'_{\mathbb{Q}}$ such that:

Fact'. *There is $g' \in G_1(\mathbb{Q}_p)$ taking (under conjugation) $T'_{1\mathbb{Q}_p}$ into $T^2_{v\mathbb{Q}_p}$, and $Fr'_{\mathbb{Q}}$ into Fr^2.*

3.2.1.0. Directly from the form of the monomorphism $G^{\mathrm{der}}_{2\mathbb{R}} \hookrightarrow G^{\mathrm{der}}_{1\mathbb{R}}$ obtained in 3.1 we get: each simple factor of $\mathrm{Lie}(G^{\mathrm{der}}_{3\mathbb{R}})$ is absolutely simple and contained in an absolutely simple factor of $\mathrm{Lie}(G^{\mathrm{der}}_{1\mathbb{R}})$; moreover each simple factor $\mathrm{Lie}(\mathcal{F})$ of $\mathrm{Lie}(G^{\mathrm{der}}_{2\mathbb{R}})$ is embedded diagonally in a product of simple factors of $\mathrm{Lie}(G^{\mathrm{der}}_{3\mathbb{R}})$, all isomorphic to $\mathrm{Lie}(\mathcal{F})$.

3.2.1.1. As usual (see 5.1.3), we can speak about the compact or non-compact factors of the adjoint group of $G_{1\mathbb{Q}_p} = G'_{1\mathbb{Q}_p}$. The important thing is: any compact factor $\mathcal{F}_c$ of $G'^{\mathrm{ad}}_{1\mathbb{Q}_p}$ is "connected over $\mathbb{Q}$ through $G_1'^{\mathrm{ad}}$" with a non-compact factor of $G'^{\mathrm{ad}}_{\mathbb{Q}_p}$ (as G_1' is an inner form of G_1). What we mean by this: there is a $\mathbb{Q}$–simple factor of G'^{ad} whose extension to $\mathbb{Q}_p$ contains $\mathcal{F}_c$ and a non-compact factor of $G'^{\mathrm{ad}}_{\mathbb{Q}_p}$.

We also have: the images of the integral powers of $Fr'_{\mathbb{Q}}$ in $G_1'^{\mathrm{ad}}$ generate the image $T_1'^{\mathrm{ad}}$ of T_1 in $G_1'^{\mathrm{ad}}$. So these images, viewed as $\mathbb{Q}_p$-valued points of $T'^{\mathrm{ad}}_{1\mathbb{Q}_p}$ generate $T'^{\mathrm{ad}}_{1\mathbb{Q}_p}$. We deduce immediately: the image of $T_{v\mathbb{Q}_p}$ in $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ is equal to the conjugate of $T'^{\mathrm{ad}}_{1\mathbb{Q}_p}$ through g'. Summing up the picture, we conclude (cf. 3.2.1.0):

3.2.2. Theorem. *Let $T^{\mathrm{ad}}_{v\mathbb{Q}_p}$ be the image of $T_{v\mathbb{Q}_p}$ in $G^{\mathrm{ad}}_{\mathbb{Q}_p}$. If two factors of $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ are "connected over $\mathbb{Q}$" (in the same sense as in 3.2.1.1), then the images of $T^{\mathrm{ad}}_{v\mathbb{Q}_p}$ in these two factors have the same dimension.*

3.2.2.1. Remark. The condition ii) of 3.2 E) is not needed at all for getting 3.2.2. Moreover the condition that $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ is a split adjoint group can be entirely removed: we have to state 3.2.2 over $\overline{\mathbb{Q}_p}$ instead of over $\mathbb{Q}_p$.

3.2.2.2. Remark. A natural question arises: can we reobtain the results [Pi, 5.10-11] (see 4.2.14), using the above ideas of 3.2? We do not know the answer, as there is one obstruction. It consists in the fact that we do not know (working under the assumption ii) of 3.2 E)), if we can assume that the Mumford–Tate group $T(A_2')$ of A_2' is (isomorphic to) T^2_v. Obviously we always have a natural monomorphism $T^2_v \hookrightarrow T(A_2')$.

However, we can assume that this monomorphism is an isomorphism, provided the reduction of A_2 with respect to v is an ordinary abelian variety (cf. [Va2, 4.4.1 2) a) and P2 of 4.6]). So we can reobtain [Pi, 5.10-11] provided the ordinary reduction conjecture (see

[Oo, p. 11]) holds for A_2; this is equivalent (cf. [Va2, 4.9.23]) to: the ordinary reduction conjecture holds for A.

3.2.3. The shifting process. Here we assume familiarity with 4.0-1. Let G_p (resp. $G_p(2)$) be the connected component of the p-adic representation (see 4.0) naturally attached to A (resp. to A_2). Then the images of G_p^{der} and $G_p(2)^{\mathrm{der}}$ in $G_{\mathbb{Q}_p}^{\mathrm{ad}} = G_{2\mathbb{Q}_p}^{\mathrm{ad}}$ (cf. the identifications I) are the same. So to show that $G_p^{\mathrm{der}} = G_{\mathbb{Q}_p}^{\mathrm{der}}$ is the same as showing that $G_p(2)^{\mathrm{der}} = G_{2\mathbb{Q}_p}^{\mathrm{der}}$. In other words we have the following useful fact:

Fact. *To prove the part of the Mumford–Tate conjecture involving derived groups, for the pair (A,p), is the same as proving it for the pair (A_2,p).*

The supplementary advantage (besides getting 3.2.2) obtained by replacing A by A_2 is:

ADV) *The irreducible subrepresentations of the representation of Lie$(G_{2\mathbb{C}})$ on $W_1 \otimes \mathbb{C}$ factor through a simple factor of Lie$(G_{2\mathbb{C}})$ (i.e. we do not have to deal with tensor products of irreducible representations of simple factors of Lie$(G_{2\mathbb{C}})$), and are SD-standard representations of these simple factors. Moreover, if $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ has a simple factor (G_f, X_f) of $D_n^{\mathbb{R}}$ type, then for a simple factor of Lie$(G_{f\mathbb{C}})$ we get as irreducible subrepresentations of it on $W_1 \otimes \mathbb{C}$, both the half-spin representations.*

Here an irreducible representation of a split simple Lie algebra of classical Lie type $\mathcal{T}$ over a field K of characteristic 0 is said to be an SD-standard representation, if it is associated to a minimal weight, and if, in case $\mathcal{T} = A_n$, $n \in \mathbb{N}\setminus\{1,2\}$, it is of dimension $n+1$ (i.e. it is associated to the minimal weight $\overline{w}_1$ or to $\overline{w}_n$). Here SD stands for Shimura-Deligne (cf. [De1, 2.3.10] and [Va1, 6.5-6]). Similarly, we speak about SD-standard representations of semisimple split groups over K, whose adjoint groups are simple. This ADV) property is a consequence of 3.2.1.0 and of 3.0 I).

The disadvantage we get by replacing A by A_2 is:

DADV) *We lose the control on $Z(G)$, while, in general, the dimension of A_2 is much bigger than the dimension of A.*

3.2.3.1. Extra advantages. Warning: A and A_2 are by now means, in the general case, related through some isogeny property. However we can call them adjoint-isogeneous. Of course, instead of A_2 we can work out 3.2.3 ADV) using any other abelian variety over E adjoint-isogeneous to A, by starting from another injective map

$$f_2'\colon (G_2', X_2') \hookrightarrow (GSp(W_1', \psi_1'), S_1'),$$

with the property that $(G_2'^{\mathrm{ad}}, X_2'^{\mathrm{ad}}) = (G^{\mathrm{ad}}, X^{\mathrm{ad}})$. So for getting 3.2.3 ADV), we could have just quoted the proof of [De1, 2.3.10]; we worked 3.2 the way we did, to get besides 3.2.3 ADV), 3.2.2 as well.

However, using the proof of [De1, 2.3.10] we can get as well extra advantages. This goes as follows. We start by assuming that $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is a simple adjoint Shimura pair which is not of some A_n type with non-trivial involution ($n \geq 2$). From loc. cit. and [Va1, 6.5.1.1 iv) and Case 1 of 6.6.5] we deduce that we can assume as well that the injective map f_2' is such that:

EADV) *For any irreducible subrepresentation $\mathcal{W}$ of the connected component of the origin of $Z(G'_{2\mathbb{C}})$ on $W'_1 \otimes \mathbb{C}$, the representation of $G'^{\mathrm{der}}_{2\mathbb{C}}$ on $\mathcal{W}$ is faithful, involving no tensor products, and in the case $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is of $D_n^{\mathbb{R}}$ type (resp. of A_n type with trivial involution), for each simple factor of Lie$(G'_2) \otimes_{\mathbb{Q}} \mathbb{C}$, both the half spin representations (resp. both the representations associated to the weights $\overline{w}_1$ and $\overline{w}_n$) of it are among the irreducible subrepresentations of Lie$(G'^{\mathrm{ad}}_{2\mathbb{C}})$ on $\mathcal{W}$.*

This extra advantage can be enjoyed in the general case as follows. To explain this, we write (as in 3.1.2 2)) $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) = \prod_{i\in I}(G_i, X_i)$ as a product of simple adjoint Shimura pairs. As above we assume that none of the pairs (G_i, X_i) is of A_n type with non-trivial involution. Let F_i be the totally real number field such that G_i is the restriction from F_i to $\mathbb{Q}$ of an absolutely simple adjoint F_i-group. Let F be the totally real number field generated by F_i, $i \in I$ (so F is unramified above any prime over which all F_i's are unramified). Let

$$(G_5, X_5) := \prod_{i\in I}(G_i^F, X_i^F).$$

We just have to repeat now the constructions in the proof of [De1, 2.3.10] and in [Va1, 6.5.1.1 iv)] for this product at once. We get injective maps of Shimura pairs

$$(G'_2, X'_2) \hookrightarrow (G_6, X_6) \hookrightarrow (GSp(W'_1, \psi'_1), S'_1),$$

such that $(G_6^{\mathrm{ad}}, X_6^{\mathrm{ad}}) = (G_5, X_5)$, the resulting injective map $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \hookrightarrow (G_5, X_5)$ is a product (indexed by I) of the standard ones $(G_i, X_i) \hookrightarrow (G_i^F, X_i^F)$, and EADV) holds.

The case when there is $i \in I$ such that (G_i, X_i) is of A_n type with non-trivial involution is subtler, and so it will be treated just in Step 2 of 5.1.3.

3.2.2 is used just in Step 2 of 5.1.3 (and so implicitly in 5.1.4), while 3.2.3 is used just in 5.1.3-4, beginning with Step 2 of 5.1.3.

§4. The basic techniques

4.0. Let E be a number field, and let A be an abelian variety over E. Let p be a rational prime, and let $G_{\mathbb{Q}_p}$ be the algebraic group over $\mathbb{Q}_p$ defined (see [Se]) as the connected component of the origin of the algebraic envelope of the natural p-adic representation

$$\rho\colon \mathrm{Gal}(\bar{E}/E) \to GL(H^1_{\acute{e}t}(A_{\bar{E}}, \mathbb{Q}_p))(\mathbb{Q}_p).$$

Let

$$H^1 := H^1_{\acute{e}t}(A_{\bar{E}}, \mathbb{Z}_p).$$

So $G_{\mathbb{Q}_p}$ is a connected algebraic subgroup of $GL(H^1[\frac{1}{p}])$, and there is a finite field extension E_1 of E such that $\rho(\mathrm{Gal}(\bar{E}/E_1))$ is a compact open subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$.

Let H_A be the Mumford–Tate group of A. We recall that H_A is a reductive group over $\mathbb{Q}$, and that any embedding $E \hookrightarrow \mathbb{C}$ allows the interpretation of H_A as the smallest subgroup of $GL(H^1_B(A_{\mathbb{C}}, \mathbb{Q}))$ with the property that the cocharacter

$$\mu(\mathbb{C})\colon \mathbb{G}_m \to GL(H^1_B(A_{\mathbb{C}}, \mathbb{Q}) \otimes \mathbb{C})$$

(defined by: if $H^1_B(A_{\mathbb{C}},\mathbb{Q}) \otimes \mathbb{C} = F^{1,0} \oplus F^{0,1}$ is the usual Hodge decomposition, then $\beta \in \mathbb{G}_m(\mathbb{C})$ acts through $\mu(\mathbb{C})$ as the multiplication with β^{-1} on $F^{1,0}$ and as identity on $F^{0,1}$) factors through H_A (cf. [De2, 3.4]). Here the left lower index B refers to the Betti cohomology. From now on we fix (arbitrarily) an embedding $i_E\colon E \hookrightarrow \mathbb{C}$. Let $H^1_B := H^1_B(A_{\mathbb{C}},\mathbb{Z})$ be obtained using it.

The connexion between the Betti cohomology of $A_{\mathbb{C}}$ and its étale cohomology with coefficients in $\mathbb{Q}_p$ allows us to view $H_{A\mathbb{Q}_p}$ as a subgroup of $GL(H^1[\frac{1}{p}])$ (it does not matter which embedding $i_E\colon E \hookrightarrow \mathbb{C}$ we fixed). We have the following remarkable prediction (cf. [Mu1] and [Se]):

4.1. The Mumford–Tate conjecture (for A with respect to p). *As subgroups of $GL(H^1[\frac{1}{p}])$, $G_{\mathbb{Q}_p} = H_{A\mathbb{Q}_p}$.*

Our main goal in this paper is to approach (and prove in many cases) this conjecture, via the following (expected to work) strategy:

4.1.1. Expectation (the split criterion). *There is $N(A) \in \mathbb{N}$, effectively computable and easy expressible in terms of A, such that if $p \geq N(A)$ and if $G_{\mathbb{Q}_p}$ is a split reductive group, then 4.1 is true (i.e. $G_{\mathbb{Q}_p} = H_{A\mathbb{Q}_p}$).*

4.1.1.1. There is a simpler variant of the above strategy, where we assume that both $G_{\mathbb{Q}_p}$ and $H_{A\mathbb{Q}_p}$ are split reductive groups. References to this variant are made in: 4.2.8.1, 4.2.9-10, 4.3.3.1, 4.3.6.1.1 and Step 3 of 5.1.3.

4.1.2. In 4.2 we review the well known facts about the conjecture 4.1, and then we show how it is implied by these facts and by the (assumed validity of the) split criterion 4.1.1, or by its variant 4.1.1.1. Then in 4.3-4 and 5.1 we prove the split criterion in many situations, and so (implicitly) we prove the Mumford–Tate conjecture in many situations.

4.2. The previously used tools.

4.2.1. Faltings' theorem. *$G_{\mathbb{Q}_p}$ is a reductive group (cf. [Fa1]).*

4.2.2. Pyatetskii-Shapiro–Deligne–Borovoi's theorem. *As a subgroup of $GL(H^1[\frac{1}{p}])$, $G_{\mathbb{Q}_p}$ is a subgroup of $H_{A\mathbb{Q}_p}$ (for instance, cf. [De2]).*

4.2.3. Faltings' theorem. *We have*

$$End(A_{\bar{E}})^{\mathrm{op}} \otimes \mathbb{Q}_p \xrightarrow{\sim} End(H^1 \otimes \mathbb{Q}_p)^{G_{\mathbb{Q}_p}(\mathbb{Q}_p)} = End(H^1 \otimes \mathbb{Q}_p)^{H_{A\mathbb{Q}_p}(\mathbb{Q}_p)};$$

the first isomorphism is defined by the action of endomorphisms of $A_{\bar{E}}$ on its étale cohomology (with coefficients in $\mathbb{Q}_p$) (cf. [Fa1]).

4.2.4. Serre's theorem. *The rank r_p of $G_{\mathbb{Q}_p}$ does not depend on p (cf. [Chi]).*

4.2.5. Theorem. *If the Mumford–Tate conjecture for the abelian variety A is true for a prime p, then it is true for all rational primes (cf. [LP, 4.3] or [Ta1, 3.4]).*

The proof of this result (cf. [Ta1, 3.4]) relies on 4.2.1-4 and on:

4.2.5.1. Zarhin's lemma. *Let $\mathcal{V}$ be a finite dimensional vector space over a field of characteristic zero. Let $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset End(\mathcal{V})$ be reductive algebraic Lie subalgebras. We assume that the centralizer of $\mathfrak{g}_i$ in $End(\mathcal{V})$ is independent on $i \in \{1,2\}$, and that the rank of $\mathfrak{g}_1$ is equal to the rank of $\mathfrak{g}_2$. Then $\mathfrak{g}_1 = \mathfrak{g}_2$ (cf. [Za, key lemma of §5]).*

4.2.6. The Mumford–Tate conjecture is true for abelian varieties (over number fields) having complex multiplication over $\overline{\mathbb{Q}}$ (this is well known).

4.2.7. The Frobenius torus. *There is a number field E^T and a torus T of H_{AE^T} (called a Frobenius torus of A) such that for any prime v of E^T dividing a rational prime p, $T_{E^T_v}$ is isomorphic to a maximal torus of $G_{E^T_v}$.*

This is a trivial consequence of [Chi, ch. 3]: we get 4.2.7 for p a prime big enough; passing to a finite field extension, we can control the left aside small values of p (cf. 4.2.4).

4.2.7.1. The construction of Frobenius tori. For this construction, briefly reviewed here, we refer to [Chi] and [Pi]. Let q be a rational prime relatively prime to p. Let v_q be a prime of E dividing q and such A has good reduction with respect to v_q. Let $\overline{v}_q$ be any prime of $\bar{E}$ dividing v_q. Let $F(\overline{v}_q) \in \mathrm{Gal}(\bar{E}/E)$ be a Frobenius element of $\overline{v}_q$. Since the restriction of ρ to the inertia group of $\overline{v}_q$ is trivial, $\rho(F(\overline{v}_q))$ is well defined. Let T_{v_q} be the $\mathbb{Q}$–model of the connected component of the origin of the Zariski closure of the set $\{\rho(F(\overline{v}_q)^n | n \in \mathbb{Z}\}$ in $GL(H^1 \otimes \mathbb{Q}_p)$. So T_{v_q} is a torus over $\mathbb{Q}$ and $T_{v_q\mathbb{Q}_p}$ is a subtorus of $G_{\mathbb{Q}_p}$.

It is well known (see [Chi, 3.b] and [Pi, 3.6-7]) that for an infinite set of primes v_q of E (subject to the above conditions, i.e. $(v_q, p) = 1$ and v_q is a prime of good reduction for A), $T_{v_q\mathbb{Q}_p}$ is a maximal torus of $G_{\mathbb{Q}_p}$.

As $\mathrm{Lie}(Z(H_A))$, viewed as a $\mathbb{Q}$–Lie subalgebra of $\mathrm{End}(H^1 \otimes \mathbb{Q}_p)$, is generated by $\mathbb{Q}$–linear combinations of $\mathbb{Q}_p$-components of endomorphisms of the reduction of A with respect to v_q, from the very construction of T_{v_q} we get:

4.2.7.2. Fact. *If $T_{v_q\mathbb{Q}_p}$ is a maximal torus of $G_{\mathbb{Q}_p}$ containing a maximal torus $T_{v_q}(ss)$ of $H^{\mathrm{der}}_{A\mathbb{Q}_p}$, then there is a subtorus $\widetilde{T}_{v_q}$ of $Z(H_A)$ such that $T_{v_q\mathbb{Q}_p}$ is generated by $T_{v_q}(ss)$ and by $\widetilde{T}_{v_q\mathbb{Q}_p}$.*

4.2.8. The set of primes p such that $G_{\mathbb{Q}_p}$ is a split reductive group is of positive Dirichlet density, and so infinite. This is a direct consequence of 4.2.7 (T splits over a finite field extension of E^T; the set of rational primes which split completely in this extension is of positive Dirichlet density, for instance see [La, p. 168]).

4.2.8.1. We have a variant of 4.2.8: the set of primes p such that $G_{\mathbb{Q}_p}$ and $H_{A\mathbb{Q}_p}$ are both split reductive groups is of positive Dirichlet density, and so infinite (in the above argument we need to replace E^T by its composite with a number field over which an arbitrarily chosen torus of H_A splits).

4.2.9. Corollary. *The conjecture 4.1 is implied by 4.1.1.1, and so by 4.1.1.*

Proof: This is a direct consequence of 4.2.7-8, via 4.2.5.

4.2.10. The conjecture 4.1 (or the expectation 4.1.1, or its variant 4.1.1.1) is true for A iff it is true for $A^n \times (A^t)^m$, with n and m non-negative integers such that $n+m>0$.

4.2.11. The conjecture 4.1 is true for A iff it is true for an isogeny A' of A (defined over a finite field extension E' of E).

The statements 4.2.10-11 are obvious.

4.2.12. The following assertions are equivalent:

a) $A_{\bar{E}}$ has complex multiplication.
b) H_A is a torus.
c) For a (any) prime p, $G_{\mathbb{Q}_p}$ is a torus.

The equivalence of a) and b) results directly from definitions. The equivalence of b) and c) results from 4.2.3.

4.2.13. $G_{\mathbb{Q}_p}$ commutes (cf. 4.2.3) with the center of $H_{A\mathbb{Q}_p}$. Let $\tilde{G}_{\mathbb{Q}_p}$ be the reductive subgroup of $H_{A\mathbb{Q}_p}$ generated by the connected component of the origin of $Z(H_A)_{\mathbb{Q}_p}$ and by $G_{\mathbb{Q}_p}$. We have $G^{\mathrm{der}}_{\mathbb{Q}_p} = \tilde{G}^{\mathrm{der}}_{\mathbb{Q}_p}$.

From our results on the Langlands–Rapoport conjecture (cf. the announcement in [Va1, 1.7.1]) we deduce easily, based on the "smallest" interpretation of H_A (see 4.0), that $\tilde{G}_{\mathbb{Q}_p} = G_{\mathbb{Q}_p}$, and that in 4.2.7, the isomorphism of $T_{E_v^T}$ with a maximal torus of $G_{E_v^T}$ is achieved by conjugation with an element of $H_A(E_v^T)$. In what follows we do not make use of these facts, though it would have simplified the things to some extend.

Of course we could have worked out 4.1.1, assuming that $\tilde{G}_{\mathbb{Q}_p}$ is a split group. We preferred to keep the original form of the split criterion, while we simultaneously work with its simpler variant 4.1.1.1.

4.2.14. Pink's theorem. *All simple factors of $G^{\mathrm{ad}}_{\overline{\mathbb{Q}_p}}$ are of A_l, B_l, C_l or D_l Lie type, $l \in \mathbb{N} \cup \{0\}$; moreover, the irreducible subrepresentations of the natural representation of the Lie algebra of any such simple factor on $H^1 \otimes \overline{\mathbb{Q}_p}$, are associated to minimal weights. (cf. [Pi, 5.11]; see also [Ta2] for a slightly less general result, and [Va3] for a completely different approach).*

This theorem is obtained as a corollary to:

4.2.14.1. Pink's theorem. *$G_{\overline{\mathbb{Q}_p}}$ is generated by cocharacters, whose resulting representations on $H^1 \otimes \overline{\mathbb{Q}_p}$, are associated to the trivial and the inverse of the identity character of $\mathbb{G}_m$, the multiplicities being the same (cf. [Pi, 5.10]).*

The proof of 4.2.14.1 (cf. [Pi, 3.15]) is obtained by combining a first result of Serre (see [Pi, 3.5]) with a second result (see [Pi, 2.3]) on isogeny classes of p-divisible groups. This second result is obtained also in [RR]. For two other approaches (one particular and the other one general) see 3.2.2.2 and 4.4.10 below.

4.2.15. Bogomolov's theorem. *$G_{\mathbb{Q}_p}$ contains the one dimensional split torus acting as the scalar multiplication on $H^1\left[\frac{1}{p}\right]$ (cf. [Bog]).*

4.2.16. For the cases for which the conjecture 4.1 was previously known to be true, see [Pi, the paragraph after the proof of 5.8] and [Pi, 5.13-15].

4.2.17. In 4.3 we start the proof of 4.1.1 in the general context. In 5.1 we accomplish its end in many situations. The basic techniques are presented in 4.3-4, while most of the conclusions of §3-4 are collected in §5.

4.3. Nice arrangements.

4.3.0. Let p be a prime such that $G_{\mathbb{Q}_p}$ is a split reductive group. The natural numbers $N_1(A)$, $N_2(A)$ and $N_3(A)$ are defined below in 4.3.1, 4.3.4, and respectively in 5.1.0. As their notation points out, they depend only on A. We take

$$N(A) := \max(N_1(A), N_2(A), N_3(A)).$$

The condition $p \geq N_1(A)$ guarantees that E is unramified above p and A has good reduction with respect to primes of E dividing p. The condition $p \geq N_2(A)$ says that we can obtain out of the faithful representation $G_{\mathbb{Q}_p} \hookrightarrow GL(H^1[\frac{1}{p}])$ "whatever we desire"; i.e. it guarantees that the mentioned representation is "strongly unramified", in some sense that will not be made explicitly here. The condition $p \geq N_3(A)$ says (cf. 3.2 F); see Step 2 of 5.1.3) that in some context we get a SHS enjoying nice properties. We are not at all interested here in a tight bound for $N(A)$ (cf. 4.2.5); so we treat the natural numbers $N_2(A)$ and $N_3(A)$ in a loose way, i.e. we do not stop here to get tight bounds for them. However, we do think that it is of some theoretical interest to have tight bounds for $N_2(A)$ (or similarly constructed numbers); accordingly we present the ideas in such a way, that in any concrete situation, the reader can trace out such tight bounds.

We deal with the problem: if $p \geq N(A)$ show that $G_{\mathbb{Q}_p} = H_{A\mathbb{Q}_p}$. An afterthought of the approach that follows towards the solution of this problem is included in 5.5.2. Those who are interested just in this approach (to the proof 4.1.1), can look just at 4.3.1-3, at 4.3.4 but just working with a $\mathbb{Z}_p$-very well positioned family of tensors (so at 4.3.4.1, at ii), iv) and vi) of 4.3.4.2, at 4.3.4.6 with $\max\{d(A), 4\}$ being replaced by 4, and at 4.3.4.8 a)), at 4.3.5-8, and at the whole of 4.4 . For any prime q, $G_{\mathbb{Q}_q}$ (resp. $\tilde{G}_{\mathbb{Q}_q}$) will have the same significance as $G_{\mathbb{Q}_p}$ (resp. as $\tilde{G}_{\mathbb{Q}_p}$), but working in the $\mathbb{Q}_q$-context. We view naturally $G_{\mathbb{Q}_q}$ and $\tilde{G}_{\mathbb{Q}_q}$ as subgroups of $GL(H^1_B \otimes_{\mathbb{Z}} \mathbb{Q}_q)$.

4.3.1. There is a smallest integer $N_1(A) \geq 3$ such that any prime v of E dividing a rational prime $l \geq N_1(A)$, is unramified over l, and is a prime of good reduction for A. Obviously $N_1(A)$ depends only on A (and on E). From now on we assume that $p \geq N_1(A)$.

4.3.2. We can assume that A has a principal polarization p_A: the Zarhin's trick says that $B := A^4 \times (A^t)^4$ has a principal polarization, and so we can replace if needed A by B (cf. 4.2.10). Another way is via [Mu2, cor. 1, p. 234] and 4.2.11.

This second approach is more convenient: we can assume (cf. 4.2.11) that A is a product of absolutely simple abelian varieties and that p_A is a product of principal polarizations of these simple factors. We do not need this assumption in what follows: we include it just for the sake of future references. So let

$$A = \prod_{i \in I_f} A_i,$$

and

$$p_A = \prod_{i \in I_f} p_{A_i},$$

with I_f a finite set, and with (A_i, p_{A_i}) a principally polarized abelian variety such that A_i is absolutely simple, $\forall i \in I_f$. Similarly we get a direct sum decomposition

$$H^1 = \oplus_{i \in I_f} H^1(i),$$

with $H^1(i) := H^1_{\text{ét}}(A_{i\bar{E}}, \mathbb{Z}_p)$.

For $i \in I_f$ let $H_A(i)$ be the Mumford–Tate group of A_i. As A_i is an absolutely simple abelian variety we get:

Fact. *$H_A(i)^{\text{ad}}$ is either a simple $\mathbb{Q}$–group or the $\mathbb{Q}$–rank of each simple factor of it is zero. So, in this last case, the Shimura variety $(H_A(i), X_A(i))$ defined by (A_i, p_{A_i}) as in [Va1, 2.12 3)] is of compact type (see Definition 4 of [Va1, 2.5]), cf. [BHC].*

The argument of this well known fact is simple and short, and so we include it. The irreducible representations of a product of Lie algebras over $\mathbb{C}$ are obtained by taking tensor products of irreducible representations of its simple factors. But in the case of $\text{Lie}(H_A(i)^{\text{der}}_{\mathbb{C}})$ we can not have such tensors products involving two non-compact factors of $\text{Lie}(H_A(i)^{\text{der}}_{\mathbb{R}})$ (we recall that the simple factors of $H_A(i)^{\text{ad}}_{\mathbb{R}}$ are absolutely simple, cf. [De1, 1.2.1]).

We can assume that $(H_A(i)^{\text{ad}}, X_A(i)^{\text{ad}})$ has a simple factor $(H(i_0), X(i_0))$ whose extension to $\mathbb{R}$ does not have compact factors. For instance, this happens if we have a simple factor whose $\mathbb{Q}$–rank is positive. This assumption implies that the $\mathbb{Q}$–vector subspace of $H_{1B}(A_{i\mathbb{C}}, \mathbb{Q})$ generated by irreducible subrepresentations of the Lie algebra $\text{Lie}(H(i_0))$ is $H_A(i)$-invariant and fixed by the other simple factors of $\text{Lie}(H_A(i)^{\text{der}})$ different from $\text{Lie}(H(i_0))$; here we view $\text{Lie}(H(i_0))$ naturally as a Lie subalgebra of $\text{Lie}(H_A(i)^{\text{der}})$. So from Weyl's complete reductibility theorem, as A_i is an absolutely simple abelian variety, we get that $H_A(i)^{\text{ad}} = H(i_0)$. This proves the fact.

We view $H_A(i)$ naturally as a quotient of H_A. We get naturally a monomorphism

$$H_A^{\text{ad}} \hookrightarrow \prod_{i \in I_f} H_A(i)^{\text{ad}}.$$

Warning: H_A^{ad} is isomorphic to a product of simple factors of this product, but it is not necessarily equal to the whole product; if it is not equal to the whole product, then two of the adjoint Shimura varieties $\text{Sh}(H_A(i)^{\text{ad}}, X_A(i)^{\text{ad}})$, $i \in I_f$, have isomorphic simple factors.

p_A induces a perfect form (still denoted by p_A)

$$p_A: H^1 \otimes H^1 \to \mathbb{Z}_p(-1)$$

(fixed by $\text{Gal}(\bar{E}/E)$). We still denote by (H^1, p_A) the symplectic space over $\mathbb{Z}_p$ we get.

4.3.3. Lemma. *There is a $\mathbb{Z}_p$-lattice $H^1_\prime$ of $H^1[\frac{1}{p}]$ such that we still get a perfect form $p_A: H^1_\prime \otimes H^1_\prime \to \mathbb{Z}_p$, and the closure of $G_{\mathbb{Q}_p}$ in $GL(H^1_\prime)$ is a (split) reductive group $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$.*

Proof: We get this by combining the following three facts:

– the inclusion $G_{\mathbb{Q}_p} \hookrightarrow G\mathrm{Sp}(H^1\left[\frac{1}{p}\right], p_A)$ of split reductive groups over $\mathbb{Q}_p$ can be extended to an inclusion $G_{\mathbb{Z}_p} \hookrightarrow G\mathrm{Sp}_{\mathbb{Z}_p}$ of split reductive groups over $\mathbb{Z}_p$;

– any $\mathbb{Z}_p$-reductive form $G\mathrm{Sp}_{\mathbb{Z}_p}$ of $G\mathrm{Sp}(H^1\left[\frac{1}{p}\right], p_A)$ is associated to a $\mathbb{Z}_p$-lattice $H^1_\prime$ of $H^1\left[\frac{1}{p}\right]$ such that $\beta p_A \colon H^1_\prime \otimes H^1_\prime \to \mathbb{Z}_p$ is a perfect form, with $\beta \in \mathbb{G}_m(\mathbb{Q}_p)$ (i.e. $G\mathrm{Sp}_{\mathbb{Z}_p} = G\mathrm{Sp}(H^1_\prime, \beta p_A)$ and $G\mathrm{Sp}_{\mathbb{Z}_p}(\mathbb{Z}_p) = G\mathrm{Sp}(H^1_\prime, \beta p_A)(H^1_\prime)$) (cf. [Ti, end of 2.5]);

– there is a cocharacter $\mu_p \colon \mathbb{G}_m \to G_{\mathbb{Q}_p}$ producing a direct sum decomposition $H^1\left[\frac{1}{p}\right] = F^1_p \oplus F^0_p$, with μ_p acting on F^0_p trivially and on F^1_p as the inverse of the identity character of $\mathbb{G}_m$; it allows us to get rid of the above factor β.

Such a cocharacter exists over $W(\mathbb{F})\left[\frac{1}{p}\right]$, cf. Fontaine's comparison theory and the main result of [Wi]. The details are recalled in 4.4.4; here we just point out that the condition $p \geq N_1(A)$ is essential here. As $G_{\mathbb{Q}_p}$ is split, such a cocharacter exists over $\mathbb{Q}_p$. This ends the proof of the lemma.

Warning: we do not require $H^1_\prime$ to be stabilized by $\mathrm{Gal}(\bar{E}/E)$. Of course it is stabilized by an open subgroup of $\mathrm{Gal}(\bar{E}/E)$.

4.3.3.0. We can assume that $H^1_\prime = \oplus_{i \in I_f} H^1_\prime \cap H^1(i)\left[\frac{1}{p}\right]$: we perform 4.3.3 above for each factor A_i of A, $i \in I_f$.

4.3.3.1. We have a variant of 4.3.3 and 4.3.3.0: if $H_{A\mathbb{Q}_p}$ is split, then we can assume as well that its closure in $G\mathrm{Sp}(H^1_\prime, p_A)$ is a split reductive group over $\mathbb{Z}_p$. To see this for 4.3.3, in the above first fact we have to start with inclusions of split reductive groups $G_{\mathbb{Z}_p} \hookrightarrow H_{A\mathbb{Z}_p} \hookrightarrow G\mathrm{Sp}_{\mathbb{Z}_p}$; the rest is the same. As $H_{A\mathbb{Q}_p}$ is split iff $H_A(i)_{\mathbb{Q}_p}$ is split $\forall i \in I_f$, using entirely the same method we get the variant of 4.3.3.0.

4.3.4. Proposition. *There is $N_2(A) \in \mathbb{N}$ such that if also $p \geq N_2(A)$, then there is a family $(v_\alpha)_{\alpha \in \mathcal{J}}$ of tensors of the tensor algebra of $H^1_\prime \oplus H^{1*}_\prime$ which is $\mathbb{Z}_p$-very well positioned for the group $G_{\mathbb{Q}_p}$.*

Almost the whole of 4.3.4 is dedicated to the proof of this proposition: its proof ends with the proof of 4.3.4.8 a).

4.3.4.1. We need first to define $N_2(A)$. In what follows $\mathcal{K}$ is an arbitrary field of characteristic zero. It is a well known fact (for instance see [Hu]) that the numbers introduced below do not depend on the choice of $\mathcal{K}$: we need such a field $\mathcal{K}$ just to simplify the presentation. We start remarking that:

a) for any $d \in \mathbb{N}$, there is only a finite number of isomorphism classes of semisimple split groups over $\mathcal{K}$ of dimension not bigger than d;
b) for any semisimple split group over $\mathcal{K}$ there is only a finite number of isomorphism classes of representations of it of dimension smaller or equal to a given $r \in \mathbb{N}$.

For a), cf. the classification of semisimple split groups over $\mathcal{K}$. For b), cf. the classification of the irreducible representations -all of them are absolutely irreducible- of a split semisimple Lie algebra over $\mathcal{K}$, the Weyl's complete reductibility theorem, and the dimension formula -in the case of a simple split semisimple Lie algebra over $\mathcal{K}$- of these representations. For all these facts see [Hu]. [Hu] is worked out over an algebraically closed field of arbitrary characteristic; as we are restricting our statements to split semisimple groups, the same remains true for any such field $\mathcal{K}$.

4.3.4.2. Some invariants. Let g_A be the dimension of A. Let $d_A = 4g_A^2$, and let $r_A = 2g_A$. Of course, in our case we can work with smaller expressions of d_A and r_A; for instance, the dimension of $G_{\mathbb{Q}_p}$ is smaller or equal to 1 plus the dimension of a split simple adjoint group of C_{g_A} Lie type. But we recall (cf. 4.3.0) that we are not interested here for the best bound of $N_2(A)$. We need $N_2(A)$ to be greater than:

i) all orders of the centers of semisimple groups over $\mathcal{K}$ of dimension not bigger than d_A;
ii) all the numbers $B(G)$ (defined in [Va1, 5.7.2]), with G running through the set of all simple adjoint groups over $\mathcal{K}$ of dimension not bigger than d_A;
iii) $\max\{2d_A, r_A\}$;
iv) the numbers $s(\mathfrak{g}, W)$ (defined in [Va1, 4.3.3]), for any split semisimple Lie algebra $\mathfrak{g}$ over $\mathcal{K}$ of dimension not bigger than d_A, and for any representation W of $\mathfrak{g}$ over $\mathcal{K}$ of dimension not bigger than r_A;
v) the numbers $k(\mathfrak{g}, W_0)$ (defined in 4.3.4.2.4 below), for any split semisimple Lie algebra over $\mathcal{K}$ of dimension not bigger than d_A, and for any irreducible representation W_0 of it of dimension not bigger than r_A;
vi) the natural numbers $d(A)$ and $\Pi(A)$ defined in 4.3.4.2.1 and respectively in 4.3.4.2.3 below.

4.3.4.2.1. Here $d(A) \in \mathbb{N}$ is defined as the smallest number with the property that for any prime q, $\tilde{G}_{\mathbb{Q}_q}$ can be defined as the subgroup of $GL(H_B^1 \otimes \mathbb{Q}_q)$ fixing some homogeneous tensors of the tensor algebra of $(H_B^1 \oplus H_B^{1*}) \otimes \mathbb{Q}_q$ of degree not bigger than $d(A)$.

4.3.4.2.2. To see why such a number $d(A)$ does exist, let us first remark that, for a prime q, the group $\tilde{G}_{\mathbb{Q}_q}$ can be defined as the subgroup of $GL(H_B^1 \otimes \mathbb{Q}_q)$ fixing some homogeneous tensors of the tensor algebra of $(H^1 \oplus H_B^{1*}) \otimes \mathbb{Q}_q$ of degree not bigger than a given $d(A, q) \in \mathbb{N}$, iff the group $\tilde{G}_{\overline{\mathbb{Q}_q}}$ can be defined as the subgroup of $GL(H_B^1 \otimes \overline{\mathbb{Q}_q})$ fixing some homogeneous tensors of the tensor algebra of $(H_B^1 \oplus H_B^{1*}) \otimes \overline{\mathbb{Q}_q}$ of degree not bigger than $d(A, q)$.

Let now T^0 be the connected component of the origin of the center $Z(H_A)$ of H_A. For any rational prime q, $T^0_{\mathbb{Q}_q}$ is the connected component of the origin of the center of $\tilde{G}_{\mathbb{Q}_q}$ (cf. 4.2.13). We can pass to a finite field extension E^{T^0} of E^T over which T^0 becomes a split torus.

The essential point is: there is a finite set $\{\tilde{G}_j | j \in S(A)\}$ of split reductive groups over E^{T^0}, which are subgroups of $GL(H_B^1(A_{\mathbb{C}}, \mathbb{Q}) \otimes E^{T^0})$ having $T^0_{E^{T^0}}$ as the connected component of the origin of their centers, and such that for any prime q, $\tilde{G}_{\overline{\mathbb{Q}_q}}$ is $\widetilde{GL}(q)$-conjugate, where $\widetilde{GL}(q) := GL(H_B^1 \otimes \mathbb{Q}_q)(\overline{\mathbb{Q}_q})$, with one of the groups $\tilde{G}_{j\overline{\mathbb{Q}_q}}$ (it does not matter which embedding $E^{T^0} \hookrightarrow \overline{\mathbb{Q}_q}$ we choose), $j \in S(A)$. This is an easy consequence of 4.3.4.1.

Now the existence of $d(A)$ results from the above first remark, and from the fact that the set $S(A)$ is finite.

4.3.4.2.3. $\Pi(A) \in \mathbb{N}$ is defined as the smallest number such that for any representation $W_{\mathbb{Z}}$ over $\mathbb{Z}$, of dimension not bigger than r_A, of a split semisimple Lie algebra $\mathfrak{g}_{\mathbb{Z}}$ of a semisimple group (over $\mathbb{Z}$)) of dimension not bigger than d_A, which is a direct sum of irreducible representations of $\mathfrak{g}_{\mathbb{Z}}$, the bilinear form on $\mathfrak{g}_{\mathbb{Z}}\left[\frac{1}{\Pi(A)}\right]$ induced by the trace form on $\mathrm{End}(W_{\mathbb{Z}}\left[\frac{1}{\Pi(A)}\right])$ is perfect. 4.3.4.1 guarantees that the smallest such number $\Pi(A) \in \mathbb{N}$ is well defined.

4.3.4.2.4. Notation. For an irreducible representation W_0 of a split semisimple Lie algebra $\mathfrak{g}$ over an arbitrary field $\mathfrak{L}$ of characteristic zero we denote by $k(\mathfrak{g}, W_0)$ the natural number defined by the property: it is the smallest number such that the Casimir element $C_{\mathbb{Z}}$ of the irreducible representation $W_{0\mathbb{Z}}$ over $\mathbb{Z}$ of the Lie algebra $\mathfrak{g}_{\mathbb{Z}}$ of a split simply connected semisimple group over $\mathbb{Z}$, which over $\mathfrak{L}$ becomes the representation W_0 of $\mathfrak{g}$ (so $\mathfrak{g} = \mathfrak{g}_{\mathbb{Z}} \otimes \mathfrak{L}$ and $W_0 = W_{0\mathbb{Z}} \otimes \mathfrak{L}$), produces an isomorphism

$$C_{\mathbb{Z}}: W_{0\mathbb{Z}}\left[\frac{1}{k(\mathfrak{g}, W_0)}\right] \tilde{\to} W_{0\mathbb{Z}}\left[\frac{1}{k(\mathfrak{g}, W_0)}\right].$$

4.3.4.3. So 4.3.4.1 assures the existence of a smallest number $N_2(A) \in \mathbb{N}$ for which the needed properties i)-vi) of 4.3.4.2 are satisfied (cf. 4.3.4.2.1-4). Obviously $N_2(A)$ depends only on the dimension of A. As $d_A \geq 4$, we deduce from 4.3.4.1 iii) that $N_2(A) \geq 5$. From now on we also assume that $p \geq N_2(A)$. So we have $p \geq \max(N_1(A), N_2(A))$.

4.3.4.4. Let T_p^0 be the connected component of the origin of $Z(G_{\mathbb{Z}_p})$. It is a split torus (as $G_{\mathbb{Z}_p}$ is a split reductive group over $\mathbb{Z}_p$). Let $H_{\prime}^1 = \oplus_{\gamma \in \Gamma} H_{\prime}^{\gamma}$ be the direct sum decomposition associated to the representation $T_p^0 \hookrightarrow GL(H_{\prime}^1)$: Γ is a set of characters of T_p^0, and $\beta \in T_p^0(\mathbb{Z}_p)$ acts on $H_{\prime}^{\gamma}$ by the multiplication with $\gamma(\beta) \in \mathbb{G}_m(\mathbb{Z}_p)$.

For any $\gamma \in \Gamma$, let $H_{\prime}^{\gamma}\left[\frac{1}{p}\right] = \oplus_{i \in I_{\gamma}} H^i\left[\frac{1}{p}\right]$ be a direct sum decomposition in irreducible $G_{\mathbb{Q}_p}^{\mathrm{der}}$-representations. Let

$$I := \cup_{\gamma \in \Gamma} I_{\gamma}$$

(disjoint union). We get:

$$H_{\prime}^1\left[\frac{1}{p}\right] = \oplus_{i \in I} H^i\left[\frac{1}{p}\right].$$

Let $H^i = H^i\left[\frac{1}{p}\right] \cap H_{\prime}^1$, $i \in I$.

4.3.4.5. As $p \geq N_2(A)$, we deduce easily from 4.3.4.2 v), by using the same argument involving Casimir elements as in the case of a field of characteristic zero, that we have a direct sum decomposition:

$$H_{\prime}^1 = \oplus_{i \in I} H^i.$$

4.3.4.6. Let now $(v_{\alpha})_{\alpha \in \mathfrak{J}_0}$ be the family of tensors formed by all homogeneous tensors of the tensor algebra of $H_{\prime}^1 \oplus H_{\prime}^{1*}$ fixed by $G_{\mathbb{Q}_p}$, and of degree at most $\max\{d(A), 4\}$; so the set $\mathfrak{J}_0$ is infinite. In particular it contains the following tensors:

1) The projection p_{γ} of $H_{\prime}^1$ on $H_{\prime}^{\gamma}$ associated to the direct sum decomposition $H_{\prime}^1 = \oplus_{\gamma \in \Gamma} H^{\gamma}$ ($\gamma \in \Gamma$). So $p_{\gamma} \in \mathrm{End}(H_{\prime}^1) \tilde{\to} H_{\prime}^1 \otimes (H_{\prime}^1)^*$.
2) The tensor $\pi(\mathrm{Lie}(G_{\mathbb{Q}_p}^{\mathrm{der}}), H_{\prime}^1\left[\frac{1}{p}\right])$ defined in [Va1, 4.2.1] (the property iv) of 4.3.4.2 and 4.3.4.5 imply that it is enveloped by -i.e. it is integral with respect to- $H_{\prime}^1$).

3) The tensors B and B^* defined in [Va1, 4.3.2] but for the Lie subalgebra $\mathrm{Lie}(G^{\mathrm{der}}_{\mathbb{Q}_p})$ of $\mathfrak{gl}(H^1_{\prime}[\frac{1}{p}])$ (cf. 2) above).
4) The projection p^i of $H^1_{\prime}$ on H^i associated to the direct sum decomposition of 4.3.4.5 ($i \in I$).

4.3.4.6.1. Remark. The tensors $\pi(\mathrm{Lie}(G^{\mathrm{der}}_{\mathbb{Q}_p}), H^1_{\prime}[\frac{1}{p}])$, B and B^* are fixed by any automorphism of $H^1_{\prime}[\frac{1}{p}]$ normalizing $G_{\mathbb{Q}_p}$.

4.3.4.7. The subgroup of $GL(H^1_{\prime}[\frac{1}{p}])$ fixing v_α, $\forall \alpha \in \mathfrak{J}_0$, is a subgroup of $\tilde{G}_{\mathbb{Q}_p}$ containing $G_{\mathbb{Q}_p}$, cf. the property vi) of 4.3.4.6 and the definition of the family $(v_\alpha)_{\alpha\in\mathfrak{J}_0}$. Let $(v_\alpha)_{\alpha\in\mathfrak{J}}$, with $\mathfrak{J}$ a set containing $\mathfrak{J}_0$, be an enlarged family of tensors of the tensor algebra of $H^1_{\prime}\oplus H^1_{\prime}$ such that $G_{\mathbb{Q}_p}$ is the subgroup of $GL(H^1_{\prime}[\frac{1}{p}])$ fixing its members. We can assume that all the p-components of the étale components of the Hodge cycles of $A_{\mathbb{C}}$ are elements of this family of tensors, indexed by $\mathfrak{J}$. This forces the set $\mathfrak{J}\setminus\mathfrak{J}_0$ to be infinite.

We have $\deg(v_\alpha) < 2p-2$, $\forall \alpha \in \mathfrak{J}_0$ (cf. the inequalities $p \geq N_2(A) > 2d_A \geq 8$).

For simplifying the presentation, we prefer to assume that (up to isomorphism, see [Va2, 2.3.1]) $v_\alpha \in \mathcal{T}(H^1_{\prime}[\frac{1}{p}])$, $\forall \alpha \in \mathfrak{J}$.

4.3.4.8. Claim. a) *The family of tensors $(v_\alpha)_{\alpha\in\mathfrak{J}_0}$ of the tensor algebra of $H^1_{\prime}\oplus H^{1*}_{\prime}$ is $\mathbb{Z}_p$-very well positioned for the group $G_{\mathbb{Q}_p}$.*

b) *If $\tilde{G}_{\mathbb{Q}_p} = G_{\mathbb{Q}_p}$, then this family is $H^1_{\prime}$-representation well positioned for $G_{\mathbb{Q}_p}$.*

Proof: a) Let R be an integral faithfully flat $\mathbb{Z}_p$-algebra, and let $M \subset R\otimes_{\mathbb{Z}_p} H^1_{\prime}[\frac{1}{p}]$ be a free R-module, with $M[\frac{1}{p}] = R\otimes_{\mathbb{Z}_p} H^1_{\prime}[\frac{1}{p}]$, and such that it envelopes the family of tensors $(v_\alpha)_{\alpha\in\mathfrak{J}_0}$ (cf. the definitions of [Va1, 4.3]). Let $S := R[\frac{1}{p}]$.

The subfamily of tensors $(p_\gamma)_{\gamma\in\Gamma}$ is obviously $\mathbb{Z}_p$-well positioned for T^0_p. So the closure of T^0_{pS} in $GL(M)$ is a torus over R.

The subfamily of tensors formed by B, B^* and $\pi(\mathrm{Lie}(G^{\mathrm{der}}_{\mathbb{Q}_p}), H^1_{\prime}[\frac{1}{p}])$ is $\mathbb{Z}_p$-well positioned for the semisimple group $G^{\mathrm{der}}_{\mathbb{Q}_p}$ (cf. [Va1, 4.3.10 b) and 4.3.10.1] and the property iv) of 4.3.4.2). So the closure of G^{der}_S in $GL(M)$ is a semisimple group over R.

From [Va1, 3.1.6] we deduce that the closure of G_S in $GL(M)$ is a reductive group over R. We deduce that the family of tensors $(v_\alpha)_{\alpha\in\mathfrak{J}_0}$ is $\mathbb{Z}_p$-very well positioned for the group $G_{\mathbb{Q}_p}$. This ends the proof of 4.3.4.

b) Let us now assume that R is a strictly henselian DVR V, that $\tilde{G}_{\mathbb{Q}_p} = G_{\mathbb{Q}_p}$, and that for any $\alpha \in \mathfrak{J}_0$, v_α has a non-zero image in $H^1_{\prime}/pH^1_{\prime}$ iff it has a non-zero image in $M/\pi_V M$, with π_V an uniformizer of V (cf. 2.3.1). We need to show that there is an isomorphism $M \tilde{\to} H^1_{\prime}\otimes_{\mathbb{Z}_p} V$ taking v_α into v_α, $\forall \alpha \in \mathfrak{J}_0$. 4.3.4.7 allows us to express the existence of such an isomorphism in terms of a reduced V-scheme (which is sort of a torsor of G_V) not being a scheme over $\mathrm{Spec}(V/\pi_V \times V[\frac{1}{p}])$. So we can assume that V is a complete DVR, and we can replace V by its normalization V_1 in a finite field extension of $V[\frac{1}{p}]$.

We have a direct sum decomposition of G_V-modules $M = \oplus_{i\in I} M^i$, with $M^i = p^i(M)$ (cf. 4.3.4.6 4)).

We have two hyperspecial subgroups of $G_{\mathbb{Q}_p}(V[\frac{1}{p}])$: $G_{V[\frac{1}{p}]}(M)$ and $G_{V[\frac{1}{p}]}(H^1_{\prime}\otimes_{\mathbb{Z}_p} V)$. They are conjugate by an element of $G^{\mathrm{ad}}_{\mathbb{Q}_p}(V[\frac{1}{p}])$ (cf. [Ti, p. 47]), and so, over a normal finite flat extension V_1 of V properly chosen, by an element $g_1 \in G_{\mathbb{Q}_p}(V_1\left[\frac{1}{p}\right])$. Replacing V with V_1, and replacing $M \otimes V_1$ with $g_1(M \otimes V_1)g_1^{-1}$, we can assume that they are the same.

On the other hand, the representation of the special fibre of G_V on $M^i/\pi_V M^i$ is absolutely irreducible. This is a consequence of [Bo1, 6.4 and 7.3] and of the fact that $p \geq 2d_A \geq \dim_V(M^i)$ (the situation gets reduced to the case of finite fields). We deduce that for any $i \in I$, M^i is of the form $\delta_i H^i \otimes_{\mathbb{Z}_p} V$, for some $\delta_i \in \mathbb{G}_m(V\left[\frac{1}{p}\right])$. For any $i \in I$, we write $\delta_i = \pi_V^{n_i} u_i$, with $n_i \in \mathbb{Z}$ and $u_i \in \mathbb{G}_m(V)$. Let

$$\delta \in GL(H^1_{\prime} \otimes_{\mathbb{Z}_p} V\left[\frac{1}{p}\right])$$

be such that it takes $m_i \in H^i$ into $\pi_V^{n_i} m_i$, $\forall i \in I$. It commutes with G_V. We have

$$\delta(H^1_{\prime} \otimes_{\mathbb{Z}_p} V) = M.$$

As the family of tensors $(v_\alpha)_{\alpha \in \mathcal{J}_0}$ is enveloped by M and by $H^1_{\prime} \otimes_{\mathbb{Z}_p} V$, we deduce by induction on $n \in \mathbb{N}$ that the family of tensors $(\delta^{-n}(v_\alpha))_{\alpha \in \mathcal{J}_0}$ is enveloped by $H^1_{\prime} \otimes_{\mathbb{Z}_p} V$, $\forall n \in \mathbb{N}$: $\delta^{-n}(v_\alpha)$ is a V-linear combination of the members of the family $(v_\alpha)_{\alpha \in \mathcal{J}_0}$, cf. 4.3.4.6.

This easily implies (cf. the fact that $\forall \alpha \in \mathcal{J}_0$, the image of v_α in $\mathcal{T}(H^1_{\prime}/pH^1_{\prime})$ is non-zero iff its image in $\mathcal{T}(M/\pi_V M)$ is non-zero) that $\delta(v_\alpha) = v_\alpha$, $\forall \alpha \in \mathcal{J}_0$. The argument goes as follows. Let $\alpha \in \mathcal{J}_0$. We can assume that the image of v_α in $\mathcal{T}(H^1_{\prime}/pH^1_{\prime})$ is non-zero. We start writing

$$v_\alpha = \sum_{s \in \mathbb{Z}} v_\alpha^s,$$

with v_α^s a homogeneous tensor of $\mathcal{T}(H^1_{\prime})$ of the same degree as v_α and such that

$$\delta^{-1}(v_\alpha^s) = \pi_V^s v_\alpha^s.$$

We immediately get that $v_\alpha^s = 0$ if $s < 0$, and that the image of v_α^0 in $\mathcal{T}(H^1_{\prime}/pH^1_{\prime})$ is non-zero. Using a suitable quotient of the difference $v_\alpha - \delta^{-1}(v_\alpha)$ by a non-negative integral power of π_V, and the fact the tensor we get is a V-linear combination of similar tensors w_α^s (of $\mathcal{T}((H^1_{\prime}))$ with $s > 0$, we get immediately that $v_\alpha^s = 0$ for any $s > 0$. So $v_\alpha = v_\alpha^0$.

So 2.3.1 b) is satisfied for such a ring R. This, together with part a) above, ends the proof of the part b) of the claim, and so of the claim.

4.3.5. Let $A'_{\mathbb{C}}$ be an abelian variety over $\mathbb{C}$ which is $\mathbb{Z}\left[\frac{1}{p}\right]$-isogeneous to $A_{\mathbb{C}}$, and such that

$$H^1_{\text{ét}}(A_{\mathbb{C}}, \mathbb{Z}_p) = H^1_{\prime}.$$

It is defined over a subfield E' of $\mathbb{C}$ which is a finite field extension of E. We denote by $i_{E'}\colon E' \hookrightarrow \mathbb{C}$ this inclusion. We denote by A' the (abelian variety) model of $A'_{\mathbb{C}}$ over E'. A' is obtained from $A_{E'}$ by taking its quotient through a finite flat group scheme C over E' of p-power order, which is a subgroup of $A_{E'}$: let $n \in \mathbb{N}$ such that

$p^n H^1_{\prime} \subset H^1$; the finite (abstract) group $C(\mathbb{C})$ corresponds to the quotient group $H^1/p^n H^1_{\prime}$ (i.e. $H^1_{\text{ét}}(A_{\mathbb{C}}/C_{\mathbb{C}}, \mathbb{Z}_p) \tilde{\to} p^n H^1_{\prime}$). The pull back of the polarization p_A of A under the endomorphism $p^n: A \to A$ (multiplication with p^n) descends, to give birth to a principal polarization $p_{A'}$ of A'. The perfect form $p_{A'}: H^1_{\prime} \otimes H^1_{\prime} \to \mathbb{Z}_p(-1)$ (of $\mathrm{Gal}(\bar{E}/E')$-modules) induced by the polarization $p_{A'}$ of A' is precisely p_A.

4.3.5.1. From 4.3.3.0 we get that A' is a product of absolutely simple abelian varieties and that $p_{A'}$ is a product of principal polarizations of these simple factors.

4.3.6. Passing to a finite field extension of E' (unramified above primes not dividing $(N_1(A) - 1)!$, cf. 4.3.1), we can assume that both A' and A have a level-4 symplectic similitude structure. By this we mean similitude isomorphisms $k'_4: (L/4L, \psi) \tilde{\to} (A'[4], p_{A'})$ and $k_4: (L/4L, \psi) \tilde{\to} (A[4], p_A)$ of symplectic spaces over $\mathbb{Z}/4\mathbb{Z}$. Here ψ is the symplectic form on

$$W := H_1(A_{\mathbb{C}}, \mathbb{Q}) = H_1(A'_{\mathbb{C}}, \mathbb{Q})$$

defined by the polarization p_A of A, while L is a $\mathbb{Z}$-lattice of W such that $\psi: L \otimes L \to \mathbb{Z}$ is a perfect form. We take L to be the first group of the Betti homology of $A'_{\mathbb{C}}$ with coefficients in $\mathbb{Z}$, and we take k'_4 to be the identity isomorphism of $L/4L$. Let $K_p := G\mathrm{Sp}(L, \psi)(L \otimes \mathbb{Z}_p)$. It is a hyperspecial subgroup of $G\mathrm{Sp}(W, \psi)(\mathbb{Q}_p)$.

4.3.6.1. We get a Siegel modular variety $\mathrm{Sh}(G\mathrm{Sp}(W, \psi), S)$, and an injective map

$$f_A: (H_A, X_A) \hookrightarrow (G\mathrm{Sp}(W, \psi), S)$$

of Shimura pairs; here X_A is the Hermitian symmetric domain defined by H_A and by the Hodge structure of W corresponding to $A_{\mathbb{C}}$ or to $A'_{\mathbb{C}}$ (see also [Va1, 2.12 3)]). We also get a Shimura quadruple $(G\mathrm{Sp}(W, \psi), S, K_p, p)$. Let $\mathcal{M}$ be its integral canonical model (cf. [Va1, 3.2.9]). So $GSp(L, \psi)(L \otimes_{\mathbb{Z}} \mathbb{A}^p_f)$ acts on $\mathcal{M}$.

We have $\dim_{\mathbb{Q}}(W) = 2g_A$. Let $\mathcal{A}_{g_A,1,4}$ be the moduli scheme over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$ parameterizing isomorphism classes of principally polarized abelian schemes of dimension g_A (over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$-schemes) having level-4 symplectic similitude structure. Its fibre over $\mathbb{Z}_{(p)}$ is the quotient of $\mathcal{M}$ by the compact open subgroup of $G\mathrm{Sp}(W, \psi)(L \otimes \widehat{\mathbb{Z}}^p)$ formed by elements acting trivially on $L/4L$. Here $\widehat{\mathbb{Z}}^p$ is $\widehat{\mathbb{Z}}$ with the p-component omitted. Corresponding to the two triples $(A', p_{A'}, k'_4)$ and (A, p_A, k_4) (of elements defined above) we get two morphisms

$$l': \mathrm{Spec}(E') \to \mathcal{A}_{g_A,1,4}$$

and respectively

$$l: \mathrm{Spec}(E) \to \mathcal{A}_{g_A,1,4}.$$

4.3.6.1.1. If $H_{A\mathbb{Q}_p}$ is a split group, then, cf. 4.3.3.1, we get another Shimura quadruple $(H_A, X_A, H_A(H^1_{\prime}), v_A)$, with v_A a fixed prime of the reflex field $E(H_A, X_A)$ dividing p. Let $\mathcal{N}(A)$ be its integral canonical model, cf. [Va1, 6.4.1]. We get an injective map

$$(H_A, X_A, H_A(H^1_{\prime}), v_A) \hookrightarrow (G\mathrm{Sp}(W, \psi), S, K_p, p),$$

and so, cf. [Va1, rm. 4) of 3.2.7] a morphism $\mathcal{N}(A) \to \mathcal{M}$. 4.3.3.1 implies that in fact we get a SHS $(f_A, L_{(p)}, v_A)$ (as $p \geq 2d_A$, [Va1, 5.8.6] applies).

4.3.6.2. Passing, if needed, to a finite field extension of E', we can assume that

$$\rho(\mathrm{Gal}(\overline{E'}/E')) \subset G_{\mathbb{Q}_p}(\mathbb{Q}_p),$$

i.e. we can assume that the tensors of the family $(v_\alpha)_{\alpha\in\mathcal{J}}$ are fixed by the Galois group $\mathrm{Gal}(\overline{E'}/E')$.

4.3.7. Passing to a finite field extension of E unramified above primes relatively prime to $(N_1(A)-1)!$ we can assume that we have natural inclusions $E(H_A, X_A) \subset E \subset E' \subset \mathbb{C}$. Let v' be a prime of E' dividing v_A and let v be the prime of E divided by it. Warning: v' might be ramified over p. Let e be the index of ramification of v' over p. Let $\mathbb{F}$ be the algebraic closure of $k(v')$. Let $V_0 := W(\mathbb{F})$, and let V be the completion of the maximal unramified cover of the ring of integers $O_{v'}$ of $E'_{v'}$. V is a V_0-algebra, with $e = [V : V_0]$. Let $K_0 := V_0\left[\frac{1}{p}\right]$ and $K := V\left[\frac{1}{p}\right]$.

Let A_{V_0} be the abelian scheme over V_0 having A_{K_0} as its generic fibre, and let A'_V be the abelian scheme over V having A'_K as its generic fibre (cf. the inequality $p \geq N_1(A)$, the definition of $N_1(A)$, and the fact that the good reduction property is preserved by passage to an isogeny). We still denote by p_A and $p_{A'}$ the principal polarizations of A_{V_0} and respectively of A'_V, whose restrictions to generic fibres are obtained from p_A and respectively $p_{A'}$, by extension of scalars.

4.3.7.1. We get morphisms $l_{V_0}: \mathrm{Spec}(V_0) \to \mathcal{A}_{g_A,1,4}$ and $l_V: \mathrm{Spec}(V) \to \mathcal{A}_{g_A,1,4}$ corresponding respectively to the triples (A_{V_0}, p_A, k_4) and $(A'_V, p_{A'}, k'_4)$. Their generic fibres are the extension to K_0 and respectively to K of l and respectively of l'.

From the definition of A', we get a $\mathbb{Z}\left[\frac{1}{p}\right]$-isogeny

$$i_V: A'_V \to A_V$$

inducing the isomorphism $H^1_{\text{ét}}(A_{\bar{K}}, \mathbb{Q}_p) \xrightarrow{\sim} H^1_{\text{ét}}(A'_{\bar{K}}, \mathbb{Q}_p)$, defined by the inclusion $H^1 \subset H^1_{\prime}\left[\frac{1}{p}\right]$. Also it induces a $\mathbb{Z}\left[\frac{1}{p}\right]$-isogeny $i_{\mathbb{F}}: A'_{\mathbb{F}} \to A_{\mathbb{F}}$.

4.3.8. From now on we assume (cf. 4.2.12) that $G_{\mathbb{Q}_p}$ is not a torus. So $G^{\mathrm{der}}_{\mathbb{Q}_p}$ is a non-trivial semisimple group.

4.4. Local deformation of A'_V and its de Rham cycles. For what follows we rely heavily on [Va1, 5.2-3], [Va2, 3.4] and 4.3.

4.4.1. Let π_V be a uniformizer of V. Let Re, $\widetilde{Re}$ and $B^+(V)$ have the same significance as in [Va1, 5.2.1]. So $\widetilde{Re}$ (resp. Re) is the piano ring of $\mathbb{F}$ of resonance e (resp. of convergent resonance e) (cf. 2.1 and loc. cit.), while $B^+(V)$ is the Fontaine's ring (of V).

Let $(M', F^1(M'), \Phi_{M'}, \nabla')$ be the filtered Frobenius crystal over Re obtained by taking the dual of the Lie algebra of the universal vector extension of the abelian variety A'_V (or of the p-divisible group associated to A'_V) (cf. [Me], [BBM]; see also [Fa2] and [Va1, 5.2.2]).

Let $M'_V := M' \otimes_{Re} V$; here the natural V_0-epimorphism $Re \twoheadrightarrow V$ is defined by π_V, cf. 2.1 and [Va1, 5.2.1]. M'_V is canonically identified with $H^1_{dR}(A'_V/V)$, cf. [Va1, 5.2.2]. Let

$$M_0 := H^1_{\mathrm{crys}}(A_{\mathbb{F}}/W(\mathbb{F}))$$

and

$$M'_0 := H^1_{\mathrm{crys}}(A'_{\mathbb{F}}/W(\mathbb{F})).$$

Let F^1_0 be the Hodge filtration of M_0 defined by A_{V_0}. Let ϕ_0 and ϕ' be the σ-linear endomorphism of M_0 and respectively of M'_0. The isocrystals $(M_0[\frac{1}{p}], \phi_0)$ and $(M'_0[\frac{1}{p}], \phi'_0)$ are naturally isomorphic (cf. the existence of the isogeny $i_{\mathbb{F}}$).

4.4.2. Using the integral version of Fontaine's comparison theorem (cf. [Fa2, th. 7]) for the p-divisible group of A'_V we get (cf. also [Va1, 5.2.5]) an inclusion of filtered $B^+(V)$-modules

$$\rho' \colon M' \otimes_{Re} B^+(V) \hookrightarrow H^1_{\prime} \otimes_{\mathbb{Z}_p} B^+(V),$$

which becomes an isomorphism ρ'_1 by inverting p. Similarly, cf. [Va1, 5.2.2.1 and 5.2.4], for A_V we get an isomorphism of filtered $B^+(V)[\frac{1}{p}]$-modules

$$\rho_1 \colon M_0 \otimes_{V_0} B^+(V)\left[\frac{1}{p}\right] \xrightarrow{\sim} H^1 \otimes_{\mathbb{Z}_p} B^+(V)\left[\frac{1}{p}\right].$$

In fact ρ_1 is obtained (by making p invertible) from a monomorphism of filtered $B^+(V)$-modules

$$\rho \colon M_0 \otimes_{V_0} B^+(V) \hookrightarrow H^1 \otimes_{\mathbb{Z}_p} B^+(V),$$

as it can be checked easily.

4.4.2.1. The $\mathbb{Z}[\frac{1}{p}]$-isogeny i_V (of 4.3.7.1) gives birth to two isomorphisms of $B^+(V)[\frac{1}{p}]$-modules

$$i_V^{dR} \colon M_0 \otimes_{\mathbb{Z}_p} B^+(V)\left[\frac{1}{p}\right] \xrightarrow{\sim} M' \otimes_{Re} B^+(V)\left[\frac{1}{p}\right]$$

and

$$i_V^{\text{ét}} \colon H^1 \otimes_{\mathbb{Z}_p} B^+(V)\left[\frac{1}{p}\right] \xrightarrow{\sim} H^1_{\prime} \otimes_{\mathbb{Z}_p} B^+(V)\left[\frac{1}{p}\right]$$

such that $\rho'_1 \circ i_V^{dR} = i_V^{\text{ét}} \circ \rho_1$ (cf. the functoriality of the integral version of Fontaine's comparison map).

4.4.3. Let $(w'_\alpha)_{\alpha \in \mathcal{J}}$ be the family of tensors of $\mathcal{T}(M'[\frac{1}{p}])$ obtained from $(v_\alpha)_{\alpha \in \mathcal{J}}$ through ρ' (cf. Fontaine's comparison theory and 4.3.6.2). They are elements of the F^0-filtration of $\mathcal{T}(M'[\frac{1}{p}])$, fixed by $\Phi_{M'}$, and annihilated by ∇'. [Fa2, cor. 9] and 4.3.4.7 guarantee that $w'_\alpha \in \mathcal{T}(M')$, $\forall \alpha \in \mathcal{J}_0$. All these facts are as in [Va1, 5.2.8-9].

4.4.3.1. From the fact that $(v_\alpha)_{\alpha \in \mathcal{J}_0}$ is a $\mathbb{Z}_p$-well positioned family of tensors for the group $G_{\mathbb{Q}_p}$ and from 4.4.3, we deduce (as in [Va1, 5.2.12]), that the closure G_{Re} in $GL(M')$ of the subgroup of $GL(M'[\frac{1}{p}])$ fixing w'_α, $\forall\alpha\in\mathcal{J}$, is a reductive group over Re. [Va1, 5.2.1.1] shows that the notation G_{Re} is justified.

4.4.3.2. Under $(i_V^{dR})^{-1}$, w'_α, $\alpha\in\mathcal{J}$, is mapped into an element w_α of the F^0-filtration of $\mathcal{T}(M_0[\frac{1}{p}])$ fixed by ϕ_0. For instance, this can be read out from the fact that i_V^{dR} is the extension to $B^+(V)[\frac{1}{p}]$ of a similar isomorphism over $Re[\frac{1}{p}]$ and from [Va1, 5.2.2.1].

4.4.4. From the main result of [Wi] we deduce the existence of a cocharacter $\mu\colon\mathbb{G}_m\to GL(M_0)$ fixing w_α, $\forall\alpha\in\mathcal{J}$. It produces a direct sum decomposition $M_0=F_0^1\oplus F_0^0$, with $\beta\in\mathbb{G}_m(V_0)$ acting through μ on F_0^i as the multiplication with β^{-i}, $i=\overline{0,1}$.

4.4.4.1. The $\mathbb{Z}[\frac{1}{p}]$-isogeny i_V transforms the cocharacter μ into a cocharacter $\mu'\colon\mathbb{G}_m\to GL(M'_V[\frac{1}{p}])$ fixing ($\forall\alpha\in\mathcal{J}$) the image t_α of w'_α under the epimorphism $M'[\frac{1}{p}]\twoheadrightarrow M'_V[\frac{1}{p}]$ of V_0-modules. So μ' factors through G_K, the reductive subgroup of $GL(M'_V[\frac{1}{p}])$ fixing t_α, $\forall\alpha\in\mathcal{J}$. It produces a direct sum decomposition $M'_V[\frac{1}{p}]=F_{\prime}^1\oplus F_{\prime}^0$, with $\beta\in\mathbb{G}_m(K)$ acting through μ' as the multiplication with β^{-i} on $F_{\prime}^i$, $i=\overline{0,1}$, and with $F_{\prime}^1$ as the Hodge filtration of $M'_V[\frac{1}{p}]=H^1_{dR}(A'_K/K)$ defined by A'_K.

4.4.4.2. Repeating the arguments of [Va1, 5.3.1] we deduce the existence of a cocharacter $\mu'_V\colon\mathbb{G}_m\to G_V$, producing similarly a direct sum decomposition $M'_V=F^1_{\prime V}\oplus F^0_{\prime V}$, with $F^1_{\prime V}:=F^1_{\prime}\cap M'_V$.

4.4.5. Repeating the arguments of [Va, 5.3.2-12], we deduce the existence of a principally polarized abelian scheme $(A'_{\widetilde{Re}},p_{A'_{\widetilde{Re}}})$ over $\mathrm{Spec}(\widetilde{Re})$ such that:

1) Its principally quasi-polarized filtered Frobenius crystal is
$$(M'\otimes_{Re}\widetilde{Re},F^1_{\widetilde{Re}},\Phi_{M'},\nabla',p_{A'_{\widetilde{Re}}}),$$
with $F^1_{\widetilde{Re}}$ a direct summand of $M'\otimes_{Re}\widetilde{Re}$ of dimension g_A (we still denote by $\Phi_{M'}$ and ∇' their extensions to $\widetilde{Re}$).
2) There is a family of tensors $(w'_\alpha)_{\alpha\in\mathcal{J}}$ of $\mathcal{T}(M'\otimes\widetilde{Re}[\frac{1}{p}])$, belonging to the F^0-filtration of $\mathcal{T}(M'\otimes\widetilde{Re}[\frac{1}{p}])$ defined by $F^1_{\widetilde{Re}}$, fixed by $\Phi_{M'}$, and annihilated by ∇'. Moreover $w'_\alpha\in\mathcal{T}(M'\otimes\widetilde{Re})$, $\forall\alpha\in\mathcal{J}_0$.
3) Under the pull back through the canonical V_0-epimorphism $\widetilde{Re}\twoheadrightarrow V$ defined by the uniformizer π_V (see 2.2.1), it becomes the principally polarized abelian variety $(A'_V,p_{A'})$, together with the family of tensors $(t_\alpha)_{\alpha\in\mathcal{J}}$ of $\mathcal{T}(M'_V[\frac{1}{p}])$.
4) Under the pull back through the canonical V_0-epimorphism $\widetilde{Re}\twoheadrightarrow V_0$ (see 2.2.1), it becomes a principally polarized abelian variety (A^0,p_{A^0}) over V_0, whose special fibre $(A^0_{\mathbb{F}},p_{A^0_{\mathbb{F}}})$ is the special fibre of $(A'_V,p_{A'})$.
5) The principally quasi-polarized filtered σ-crystal over $\mathbb{F}$ associated to (A^0,p_{A_0}) is $(M'_0,F_0^{1\prime},\phi',p_{A^0})$. Here $F_0^{1\prime}:=F^1_{\widetilde{Re}}\otimes_{\widetilde{Re}}V_0\subset M'_0=M'\otimes_{Re}V_0$.
6) There is a family $(t^0_\alpha)_{\alpha\in\mathcal{J}}$ of elements of the F^0-filtration of $\mathcal{T}(M'_0[\frac{1}{p}])$ (defined by $F_0^{1\prime}$), fixed by ϕ', and such that we get a principally quasi-polarized Shimura filtered

σ-crystal $\mathcal{C}_0 := (M'_0, F_0^{1\prime}, \phi', G_{V_0}, (t^0_\alpha)_{\alpha\in\mathcal{J}}, p_{A^0})$. It is obtained naturally from the family $(w'_\alpha)_{\alpha\in\mathcal{J}}$, via the V_0-epimorphism $\widetilde{Re} \twoheadrightarrow V_0$ of 4) above. Moreover $t^0_\alpha \in \mathcal{T}(M'_0)$, $\forall\alpha\in\mathcal{J}_0$.

$A'_{\widetilde{Re}}$ is defined by a lift of the cocharacter μ'_V of 4.4.4.2 to a cocharacter $\mu'_{\widetilde{Re}}\colon\mathbb{G}_m \to G_{\widetilde{Re}}$. It gives birth to a cocharacter $\mu'_{V_0}\colon\mathbb{G}_m \to G_{V_0}$. It produces a direct sum decomposition $M'_0 = F_0^{1\prime}\oplus F_0^{0\prime}$, with $\beta\in\mathbb{G}_m(V_0)$ acting through μ'_{V_0} as the multiplication with β^{-i} on $F_0^{i\prime}$, $i=\overline{0,1}$.

One could identify t^0_α with t_α; however, in what follows, not to create any unclarity, this will not be done.

4.4.5.1. So $A^0_{K_0}$ gets naturally a family of étale cycles indexed by $\mathcal{J}$; it is obtained from $(t^0_\alpha)_{\alpha\in\mathcal{J}}$ via Fontaine's comparison theory.

4.4.5.2. As $\widetilde{Re}$ is a projective limit of artinian rings having $\mathbb{F}$ as their residue fields, we deduce that the level-4 symplectic similitude structure of A'_V lifts to give us a level-4 symplectic similitude structure of $(A'_{\widetilde{Re}}, p_{A'_{\widetilde{Re}}})$. So we get naturally a morphism

$$\mathrm{Spec}(\widetilde{Re}) \to \mathcal{A}_{g_A,1,4}$$

through which the morphism l_V (of 4.3.7.1) factors; it naturally gives birth to a V_0-morphism

$$l^1_{V_0}\colon \mathrm{Spec}(V_0)\to\mathcal{A}_{g_A,1,4}.$$

4.4.6. Let d be the relative dimension of the quotient G_{V_0}/P_{V_0}, where P_{V_0} is the parabolic subgroup of G_{V_0} leaving invariant the direct summand $F_0^{1\prime}$ of M'_0. Let $R := V_0[[x_1,..,x_d]]$, with x_i as independent variables. Let Φ_R be its Frobenius lift, compatible with σ and taking x_i to x_i^p. Obviously we have $d\le\dim(X_A)$.

4.4.6.1. Lemma. *We have* $d\ge 1$.

Proof: If v_α is the p-component of the étale component of a Hodge cycle of $A_{\mathbb{C}}$ (i.e. of A'_K, cf. 4.3.6.2), for some $\alpha\in\mathcal{J}$ (see 4.3.4.7), then t^0_α is the de Rham component of the same Hodge cycle (of A'_K), cf. [Va1, 5.2.10]. So the group $H_{A\overline{K_0}}$ canonically can be viewed as a subgroup of $GL(M'_0\otimes\overline{K_0})$. The inclusion we get $G_{\overline{K_0}}\hookrightarrow H_{A\overline{K_0}}$ is nothing else but the extension to $\overline{K_0}$ of the initial inclusion $G_{\mathbb{Q}_p}\hookrightarrow H_{A\mathbb{Q}_p}$. This is entirely similar to [Va1, 5.2.17.2] (being a property of Fontaine's comparison theory). If $d=0$, then $G^{\mathrm{der}}_{\overline{K_0}}$ is contained in a parabolic subgroup of $H^{\mathrm{der}}_{A\overline{K_0}}$, different from $H^{\mathrm{der}}_{A\overline{K_0}}$, and so is included in a Levi subgroup of $H^{\mathrm{der}}_{A\overline{K_0}}$. This contradicts 4.2.3. So $d\ge1$.

4.4.7. Theorem. *There is a morphism* $l_R\colon Spec(R)\to\mathcal{A}_{g_A,1,4}$ *such that:*

a) l_V *factors through it;*

b) $l^1_{V_0}$ *is obtain from it by "taking"* x_i *to be* 0*;*

c) The principally quasi-polarized p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$ *defined by the principally polarized abelian scheme* (A_R, p_{A_R}) *over* R *it gives birth to, is isomorphic to one of the form*

$$\mathcal{C}_R := (M'_0\otimes R, F_0^{1\prime}\otimes R, g_{\mathrm{univ}}(\phi'\otimes 1), \nabla^0, p_{A^0}),$$

with $g_{\mathrm{univ}} \in G_{V_0}(R)$ such that $\mathcal{C}_R$ is a versal deformation (in the sense of [Va2, 3.4.19 B)]) of $\mathcal{C}_0 = (M_0', F_0^{1\prime}, \phi', G_{V_0}, (t_\alpha^0)_{\alpha\in\mathcal{J}}, p_{A_0})$.

Proof: Parts b) and c) are just the abstract form of [Va1, 5.4.4-8], while part a) is nothing else but 2.4.1.

4.4.7.1. Remark. [Fa2, th. 10 and the remarks after] implies that the connection ∇^0 on $M_0' \otimes R$ is G_{V_0}-invariant. So, if v_α is the p-component of the étale component of a Hodge cycle of $A_{\mathbb{C}}$ (cf. 4.3.4.7), then t_α^0, viewed as a homogeneous tensor of $\mathcal{T}(M_0' \otimes R[\frac{1}{p}])$, is the de Rham component of a Hodge cycle of the generic fibre of A_R. This is a consequence of [Va1, 5.2.10] (cf. the above proof of 4.4.6.1) and of [Va1, 5.4.4-8]. We conclude (this is entirely the same as [Va1, 5.4.5]):

Fact. *The morphism l_R factors through the normalization $\mathcal{N}(4)$ of the Zariski closure in $\mathcal{A}_{g_A,1,4_{O_{(v)}}}$ of the natural image of $\mathrm{Sh}(H_A, X_A)_E$ in $\mathcal{A}_{g_A,1,4_E}$.*

4.4.7.2. Remark. In all the above part 4.3.7 to 4.4.7 we could have worked as well with the ring of integers of E_v and $E'_{v'}$, as [Fa2, th. 10 and the remarks after] are valid over any perfect field. The only difference: we would get reductive groups over different integral rings involved, which are not necessarily split (however their generic fibres are inner forms of the logical extensions of $G_{\mathbb{Q}_p}$).

4.4.8. Corollary. *If $A_{\mathbb{F}}^0$ is an ordinary abelian variety, then*

a) its canonical lift A^1 is obtained from A_R via a V_0-valued point of Spec(R);
b) the Frobenius torus T_v (of 4.2.7) is isomorphic to a subtorus of H_A;
c) $T_{v\mathbb{Q}_p}$ is isomorphic to a subtorus of $G_{\mathbb{Q}_p}$.

Proof: a) is a consequence of [Va2, 2.3.17.2 and 3.1.0 b)] and of [Va2, 3.11.1], once we remark that the G_{V_0}-ordinary type produced by $\mathcal{C}_0$ (see [Va2, 3.1.1] for a definition) is a usual ordinary type, and the degree of definition of $\mathcal{C}_0$ (see [Va2, 3.11.3] for a definition) is 1. The first stated fact is a consequence of the assumption of the corollary, while the second one is obtained in the same manner as [Va2, P2 of 4.6].

It is also worth pointing out that these two facts are also a consequence of [Va2, 4.4.1 2)] and of the fact that $G_{\mathbb{Q}_p}$ is a split group, even without the assumption of the corollary. [Va2, 4.4.1 2)] is worked out in the context of a SHS, but the same argument works in the more general context of Shimura σ-crystals. In fact the degree of definition of the G_{V_0}-canonical lift $\mathcal{C}_1$ produced by (M_0', ϕ', G_{V_0}) (see [Va2, 3.1.1] for a definition) can be computed using the étale cohomology with $\mathbb{Z}_p$ coefficients, cf. [Va2, c) of 4.4.1 2) and 4.1].

b) is a consequence of the fact that A^1 is the extension to $W(\mathbb{F})$ of an abelian variety A_f^1 over the Witt ring of a finite field k_1 containing $k(v')$ (cf. 4.4.7.2 or standard arguments based on the fact that $\mathcal{A}_{g_A,1,4}$ is a Spec$(\mathbb{Z}[\frac{1}{2}])$-scheme of finite type), of the Fact of 4.4.7.1, and of the fact that the Frobenius endomorphism π_1 of the special fibre of A_f^1 lifts to an endomorphism of A^1. So the Mumford–Tate group T_{A^1} of A^1 (or of A_f^1) is a subtorus of H_A; moreover π_1 defines a $\mathbb{Q}$–valued point of T_{A_1}, and allows us to identify T_v with T_{A_1} (cf. also [Pi, 3.5]).

c) is a consequence of the fact that we can identify $H^1_{\text{ét}}(A^1_{\overline{B(\mathbb{F})}}, \mathbb{Z}_p)$ and its étale cycles $(v^1_\alpha)_{\alpha\in\mathcal{J}}$ obtained from $(t^0_\alpha)_{\alpha\in\mathcal{J}}$ via Fontaine's comparison theory, with $H^1_\prime$ and its étale cycles $(v_\alpha)_{\alpha\in\mathcal{J}}$. This is a consequence of the global deformations of [Va2, 3.15], and of [Fa3, 2.6] and [Fa2, th. 10 and the remarks after]; [Fa3, 2.6] remains valid in a pro-étale context, cf. [Va2, Fact of 3.4.19], while the condition $p \geq N_2(A)$ allows us to use it, cf. 4.3.4.7. Of course this identification is not canonical. Under this identification, the connected component of the algebraic envelope of the p-adic representation

$$\mathrm{Gal}(B(k_1)) \to GL(H^1_\prime)(\mathbb{Q}_p)$$

attached to A^1_f, is a subtorus of $G_{\mathbb{Q}_p}$ isomorphic (for instance cf. Katz–Messing's result recalled in [Pi, 3.10]; see also [Pi, 3.13]) to $T_{v\mathbb{Q}_p}$. This proves the corollary.

4.4.9. Corollary. *We do not assume anymore that $A^0_{\mathbb{F}}$ is ordinary. The abelian variety we get over the generic point of Spec(R/pR) is an ordinary abelian variety. Moreover, for any point y_k of Spec(R/pR) with values in a perfect field k and such that we get (from A_R) an ordinary abelian variety $A(k)$ over k, the canonical lift of $A(k)$ is obtained from A_R through a $W(k)$-valued point of Spec(R) lifting y_k.*

Proof: This is a consequence of [Va2, 3.12.1] and of the above proof of 4.4.8 a) and b).

4.4.10. Remark. Combining 4.4.5 6) with [Va2, 3.1.0 and the Lie Criterion of the Appendix], we regain (in a different manner) the second result (on isogeny classes), needed (cf. 4.2.14.1), to obtain Pink's results mentioned in 4.2.14.

4.4.11. An approach. We assume now that there is an infinite number of primes v of E dividing a prime $p \geq N(A)$ and such that A has ordinary good reduction with respect to them. From 4.4.8 b) we deduce the existence of an infinite number of such tori T_v of H_A. So if 4.1 does not hold for A, from 4.4.8 c) we deduce that for any prime q there is a proper reductive subgroup $G_{\mathbb{Q}_q}$ of $H_{A\mathbb{Q}_q}$ containing as a maximal torus an $H_A(\mathbb{Q}_q)$-conjugate of any one of these tori, and such that 4.2.3 (with p replaced by q) holds. It should not be too difficult to show (at least in most of the cases) that there is a proper reductive subgroup $G_{\mathbb{Q}}$ of H_A containing as a maximal torus an $H_A(\mathbb{Q})$-conjugate of any one of these tori, and such that the equivalent of 4.2.3 holds for it; for this an interpolation similar to the one of [Pi, §6] is needed.

This should lead to a contradiction (perhaps using 5.2.1 below). This is one of the approaches to be taken in [Va5]. Based on [Va1, 1.7.1] in [Va5] we will see that an entirely similar approach is possible, without assuming the existence of an infinite number of primes v of E dividing a prime $p \geq N(A)$ and such that A has ordinary good reduction with respect to them; however, to reach a contradiction as above looks much more difficult.

§5. Some conclusions

In this chapter we include some conclusions of what the techniques of §3-4 can achieve.

5.1. Applications to 4.1. For the proof of 5.1.1 (resp. 5.1.2) see 5.1.3 (resp. 5.1.4).

5.1.0. Let $N_3(A) := 2 + l(A)$, where the prime $l(A)$ is obtained in the same manner as the prime l of 3.2 E). For the meaning of $N(A)$ see 5.3.0.

5.1.1. Proposition. *If we also have $p \geq N_3(A)$, i.e. if $p \geq N(A)$, then the equality $\dim(X_A) = d$ implies that $G_{\mathbb{Q}_p} = H_{A\mathbb{Q}_p}$.*

We write now the adjoint Shimura pair

$$(H_A^{\mathrm{ad}}, X_A^{\mathrm{ad}}) = \prod_{j \in \mathfrak{I}} (H_j, X_j),$$

as a finite product of simple adjoint Shimura pairs.

5.1.2. Theorem. *If $p \geq N(A)$ then the split criterion is true (and so 4.1 is true for A, cf. 4.2.9) provided $\forall j \in \mathfrak{I}$ we have:*

A) (H_j, X_j) is of non-special A_n type (see definition 5.1.2.1);
B) (H_j, X_j) is of B_n type;
C) (H_j, X_j) is of C_n type, with n odd;
C') (H_j, X_j) is of C_2 type;
D) (H_j, X_j) is of $D_n^{\mathbb{H}}$ type, with n odd and such that $2n$ is not of the form $C_{2m+1}^{2^m}$, with $m \in \mathbb{N}$;
D') (H_j, X_j) is of $D_n^{\mathbb{H}}$ type with non-trivial involution, with n odd or $n \in \{4, 6, 12\} \cup \{2q, 6q | q \text{ a prime greater than } 3\}$;
E) (H_j, X_j) is of $D_n^{\mathbb{R}}$ type, $n \in \mathbb{N}$, $n \geq 4$.

5.1.2.1. Definition. Let (H_0, X_0) be a simple adjoint Shimura pair of A_n type, $n \in \mathbb{N}$. We say that it is of non-special A_n type if there is an injective map $f_1 \colon (H_1, X_1) \hookrightarrow (GSp(W_1, \psi_1), S_1)$ such that:

a) $(H_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) = (H_0, X_0)$;

b) for any $N \in \mathbb{N}$, there is a prime $q \geq N$, such that for every subtorus T_1 of H_1 with the property that T_1 is the smallest subtorus of H_1 with the property that a set of cocharacters $\mathbb{G}_m \to H_{1\mathbb{C}}$ which are $H_1(\mathbb{C})$-conjugate to the cocharacters $\mu_{x_1}^*$ defined in [Va2, 2.3.1] factor through $T_{1\mathbb{C}}$ (here $x_1 \in X_1$), any reductive subgroup G_1 of $H_{1\mathbb{Q}_q}$ containing $T_{1\mathbb{Q}_q}$ as a maximal torus, whose adjoint has all the simple factors of classical Lie type, and having the property that $\mathrm{End}(W_1 \otimes \mathbb{Q}_q)^{G_1(\mathbb{Q}_q)} = \mathrm{End}(W_1 \otimes \mathbb{Q}_q)^{H_{1\mathbb{Q}_q}(\mathbb{Q}_q)}$, has automatically the property that there is a simple factor of G_1^{ad} of A_n Lie type.

5.1.2.2. Numerical analysis. Let (H_0, X_0) be as above. Here we present numerical conditions needed to be satisfied in order that (H_0, X_0) is or is not of non-special A_n type. There is nothing deep in obtaining them, so we just state the results. The results are based on the standard table (see [Pi, 4.2]) of minimal weights of the split simple Lie algebras of classical Lie type and on 4.2.5.1 and 4.2.14 (to be compared with 5.1.3-4 below). For these numerical conditions we just need to take arbitrarily the embedding f_1, subject to the following two conditions:

i) the irreducible subrepresentations of the representation of Lie($H_{1\mathbb{C}}^{\mathrm{der}}$) on $W_1 \otimes \mathbb{C}$ factor through a simple factor of Lie($H_{1\mathbb{C}}^{\mathrm{der}}$) and are SD-standard representations of these factors (cf. 3.2.3 ADV));

ii) in case (H_0, X_0) is of trivial involution, then 3.2.3.1 EADV) holds as well.

We write

$$H_{0\mathbb{R}} = \prod_{j \in I_1} SU(a_j, b_j)_{\mathbb{R}}^{\mathrm{ad}} \times \prod_{j \in I_0} SU(n+1, 0)_{\mathbb{R}}^{\mathrm{ad}},$$

with $a_j, b_j \in \mathbb{N}$, $a_j \leq b_j$, $a_j + b_j = n+1$, $\forall \alpha \in I_1$. Here I_0 and I_1 are finite sets, keeping track of the non-compact and respectively compact factors of $H_{0\mathbb{R}}$. I_1 is not empty, cf. the axiom [Va1, SV3]. For $j \in I_1$, let

$$c_j := \frac{a_j}{b_j}.$$

Let

$$C := \{c_j | j \in I_1\} \subset \{\frac{1}{n}, \frac{2}{n-1}, ..., \frac{[\frac{n+1}{2}]}{n+1-[\frac{n+1}{2}]}\}.$$

So C has at least one element and at most $[\frac{n+1}{2}]$ elements. Let c (resp. d) be the minimum (resp. the maximum) of the numbers of C. If $1 \in C$, then n is odd. In what follows we consider pairs $(r, s) \in \mathbb{N} \times \mathbb{N}$ such that $2s - 1 \leq r$. For such a pair we denote by

$$c_{(r,s)} := \frac{C_r^{s-1}}{C_r^s},$$

and

$$e_{(r,s)} := C_{r+1}^s.$$

Let us first point out one elementary thing. If (H_0, X_0) is not of non-special A_n type and if $d < 1$, then the "smallest property" of T_1 (see 5.1.2.2) forces any subgroup G_1 of $H_{1\mathbb{Q}_q}$ (with q a prime and with G_1 as in 5.1.2.1 above) to have the property:

AT) The simple factors of Lie(G_1^{ad}) are of (different) A_m Lie types.

Case 1: $c = d < 1$. If (H_0, X_0) is not of non-special type, then from AT) we deduce that for any $j \in I_1$ there are sequences of pairs $(r_{j,m}, s_{j,m})$, $m = \overline{1, t_j}$, with $t_j \in \mathbb{N}$, such that:

1) $\sum_{m=1}^{t_j} r_{j,m}$ does not depend on $j \in I_1$;
2) $c_{(r_{j,m}, s_{j,m})} = c$, $\forall m \in \{1, ..., t_j\}$;
3) $a_j = C_{r_{j,m}}^{s_{j,m}-1} \times \prod_{l=1, l \neq m}^{t_j} e_{(r_{j,l}, s_{j,l})}$;
4) $b_j = C_{r_{j,m}}^{s_{j,m}} \times \prod_{l=1, l \neq m}^{t_j} e_{(r_{j,l}, s_{j,l})}$;
5) If $t_j = 1$, then $s_{j,1} \geq 2$.

Case 2: $c < d < 1$. From AT) we deduce the existence, for any $j \in I_1$, of a sequence of numbers $n_{j,m} \in \mathbb{N}$, with m running as in the Case 1 above, such that:

6) the sum $\sum_{m=1}^{t_j} n_{j,m}$ does not depend on $j \in I_1$;
7) if $t_j = 1$, then $n_{j,1} < n$;

8) there is a map $f(j)\colon \{1,...,t_j\} \to \{0,1\}$ with the property that

– if $f(j)(m) = 0$, then we can write $n_{j,m} + 1 = a_{j,m} + b_{j,m}$, with $a_{j,m}, b_{j,m} \in \mathbb{N}$, such that $u_{j,m} := \frac{a_{j,m}}{b_{j,m}} \in C$;

– if $f(j)(m) = 1$, then there is $s_{j,m} \in \mathbb{N}$, $2s_{j,m} - 1 \leq n_{j,m}$, such that $u_{j,m} := c_{(n_{j,m}, s_{j,m})} \in C$;

– the map $q_j\colon \{1,...,t_j\} \to C$, taking m to $u_{j,m}$ is surjective;

– $n + 1 = \prod_{m\in\{1,\ldots,t_j\}, f(j)(m)=0} (n_{j,m} + 1) \times \prod_{m\in\{1,\ldots,t_j\}, f(j)(m)=1} e_{(n_{j,m}, s_{j,m})}$.

5.1.2.2.1. Remarks. 1) The conditions 5.1.2.2 1) and 6) just express 3.2.2 (via the fact that T_1 of 5.1.2.1 is defined over $\mathbb{Q}$). In both above cases, t_j counts the number of simply factors of the image of $G^{\mathrm{der}}_{1\overline{\mathbb{Q}_p}}$ in the simple factor $\mathcal{F}(j)$ of $H_{0\overline{\mathbb{Q}_p}}$ corresponding to j. In Case 2 above, the function $f(j)$ keeps track of which of these factors are embedded in $\mathcal{F}(j)$ via SD-representations.

2) Of course in the case $c < d = 1$ some similar numerical conditions as in Case 2 of 5.1.2.2 can be obtained. We do not stop to list them all here. Just one sample: we similarly get surjective maps q_j onto C. Warning: the property AT) does not a priori hold.

5.1.2.3. Examples. We present now some examples to which (the ideas of) the two cases of 5.1.2.2 apply.

Example 1. (H_0, X_0) is of non-special A_n type if there is $j \in I_1$ such that a_j and b_j are relatively prime (so $c_j < 1$ if $n > 1$) and the pair (a_j, b_j) is not of the form (C_r^{s-1}, C_r^s), for (r, s) a pair as above, with $s \geq 2$. In particular (H_0, X_0) is of non-special type if there is $j \in I_1$ with $a_j = 1$ (i.e. if $\frac{1}{n} \in C$).

Example 2. We assume that $c < d$ and that there is $j \in I_1$ such that a_j and b_j are relatively prime. Then (H_0, X_0) is of non-special A_n type.

To see this, we first remark that the fact that a_j and b_j are relatively prime implies that for any prime q, under the situation of 5.1.2.1, the image of $\mathrm{Lie}(G_{1\overline{\mathbb{Q}_q}})$ in the simple factor $\mathcal{L}_j$ of $\mathrm{Lie}(H^{\mathrm{ad}}_{1\overline{\mathbb{Q}_q}})$ corresponding to this j, is a simple Lie algebra $\mathcal{L}(0)$. Now, the surjectivity property mentioned in 5.1.2.2 and in 5.1.2.2.1 2), implies that $\mathcal{L}(0) = \mathcal{L}_j$. In the same way we argue Example 1.

Example 3. We assume that $c < d < 1$ and that there is $j \in I_1$ such that the greatest common divisor of a_j and b_j is 2. Then (H_0, X_0) is of non-special A_n type.

Example 4. We assume that $c < d < 1$ and that there is $j \in I_1$ such that the greatest common divisor d_j of a_j and b_j does not have factors which can be written as a product of numbers of the form $\tilde{a}_j + \tilde{b}_j$, with $\tilde{a}_j, \tilde{b}_j \in \mathbb{N}$ such that $\frac{\tilde{a}_j}{\tilde{b}_j} \in C$, or of the form $e_{(\tilde{r}_j, \tilde{s}_j)}$, with the pair $(\tilde{r}_j, \tilde{s}_j)$ subject to the above conditions and such that $c_{(\tilde{r}_j, \tilde{s}_j)} \in C$. Then (H_0, X_0) is of non-special A_n type, cf. 8) above. This example generalizes the previous one.

Example 5. If $c = d = 1$ then n is odd, and (H_0, X_0) is of non-special type iff $n = 3$ or 4 does not divide $n + 1$. The necessity of these conditions can be seen by just using a central quotient by μ_2 of a product of two simply connected semisimple split group of A_1 and respectively of A_m Lie type ($m \in 2\mathbb{N} + 1$), embedded logically (using tensor products of representations) in a simply connected semisimple split group of A_{2m+1} Lie type.

To see the sufficiency part, let us first remark that $c = d = 1$ implies that (H_0, X_0) is of trivial involution, and so we can use 5.1.2.2 ii). As 4 does not divide $n+1$ we deduce that the image of $G_{1\overline{\mathbb{Q}_p}}$ in a simple factor of $H_{0\overline{\mathbb{Q}_p}}$ is a simple adjoint group. Using the part of 3.2.3.1 EADV) involving the weights $\overline{w}_1$ and $\overline{w}_n$, we get immediately that this image must be of A_n Lie type.

Example 6. If $c < d = 1$, and if (H_0, X_0) is not of non-special type, then for any $j \in I + 1$ such that $c_j < 1$, 2 divides a_j and b_j. This is an immediate consequence of the surjectivity part of 5.1.2.2.1 2). If moreover $n \neq 3$, then the converse of this is also theoretically true, as we can use a similar product as in Example 5. What we mean here by "theoretically": referring to 5.1.2.1, the possibility of choosing for any prime q big enough a torus T_1 such that a subgroup G_1 whose adjoint is a product of the type mentioned does exist, has to be checked.

Example 7. If $n+1$ is 4 or a prime, than (H_0, X_0) is of non-special type, regardless of how C is this. This is an elementary exercise.

We do not stop here to restate 5.1.2.2-3 in terms of the values of n, similar to the statements of Examples 5 and 7.

5.1.2.4. The simplest cases not settled by 5.1.2. Here, for the sake of future references, we include the simplest (in the sense of involving adjoint groups of small rank) cases not settled by 5.1.2. For all of them we assume that H_A^{ad} is a simple $\mathbb{Q}$–group, but we do not assume any restriction on p or on $G_{\mathbb{Q}_p}$. We denote by IM the image of $G_{\overline{\mathbb{Q}_p}}$ in an arbitrarily chosen simple factor of $H^{\mathrm{ad}}_{A\overline{\mathbb{Q}_p}}$. It is an adjoint group, not necessarily simple. We will use the sign $+$ to denote the Lie type of products of adjoint simple $\overline{\mathbb{Q}_p}$ groups.

A5. H_A^{ad} is of A_5 Lie type and $C = \{1, \frac{1}{2}\}$. IM could theoretically be of $A_1 + A_2$ Lie type. So among the 7 possibilities for the set C, only one is not settled.

A7. H_A^{ad} is of A_7 Lie type, and $C = \{1\}$ or $C = \{1, \frac{1}{3}\}$. IM could theoretically be of $A_1 + A_3$ Lie type. So among the 15 possibilities for the set C, only two are not settled.

A8. H_A^{ad} is of A_8 Lie type and $C = \{\frac{1}{2}\}$. IM could theoretically be of $A_2 + A_2$ Lie type. So among the 15 possibilities for the set C, only one is not settled

A9. H_A^{ad} is of A_9 Lie type. IM could theoretically be of A_4 Lie type, and then $C = \{\frac{2}{3}\}$, or IM could theoretically be of $A_1 + A_4$, and then $C = \{1, \frac{1}{4}\}$ or $C = \{1, \frac{2}{3}\}$. So among the 31 possibilities for the set C, only three are not settled.

A11. H_A^{ad} is of A_{11} Lie type. IM could theoretically be of $A_1 + A_5$, or $A_1 + A_1 + A_2$, or $A_2 + A_3$, or of $A_2 + C_2$ Lie type. It is worth pointing out that $A_1 + A_1 + A_2$ and $A_2 + C_2$ have the same rank, but in these both situations we have $C = \{1, \frac{1}{2}\}$. This is the first situation where a C_m Lie type, with $m \geq 2$, could theoretically show up in the case H_A^{ad} is of some A_n Lie type.

C4. H_A^{ad} is of C_4 Lie type. IM could theoretically be of $A_1 + A_1 + A_1$ Lie type.

C6. H_A^{ad} is of C_6 Lie type. IM could theoretically be of $A_1 + A_3$ Lie type.

D4. $(H_A^{\mathrm{ad}}, X_A^{\mathrm{ad}})$ is of $D_4^{\mathbb{H}}$ type with trivial involution. IM could theoretically be of B_3 or of $B_2 + B_1$ Lie type.

D6. $(H_A^{\mathrm{ad}}, X_A^{\mathrm{ad}})$ is of $D_6^{\mathbb{H}}$ type with trivial involution. IM could theoretically be of $C_1 + C_3$ Lie type.

D8. $(H_A^{\mathrm{ad}}, X_A^{\mathrm{ad}})$ is of $D_8^{\mathbb{H}}$ type. IM could theoretically be of B_4, or D_5, or of $A_1 + A_1 + A_1 + A_1$ Lie type. If $(H_A^{\mathrm{ad}}, X_A^{\mathrm{ad}})$ is with non-trivial involution, then IM has to be of $A_1 + A_1 + A_1 + A_1$ or of B_4 Lie type; moreover, for an infinite number of primes p and for an adequate choice of a simply factor of $H^{\mathrm{ad}}_{\mathbb{Q}_p}$, IM is of $A_1 + A_1 + A_1 + A_1$ (to be compared with D') of 5.1.4).

D35. $(H_A^{\mathrm{ad}}, X_A^{\mathrm{ad}})$ is of $D_{35}^{\mathbb{H}}$ type with trivial involution. IM could theoretically be of A_7 Lie type.

Also we would like to mention that the Lie types A_6, A_8, A_{10} and A_{12} are settled by Example 7.

5.1.2.5. Remark. Let F_0 be the totally real number field such that H_0 is the Weil restriction from F_0 to $\mathbb{Q}$ of an absolutely simple adjoint F_0-group. If by chance we know that there is an infinite number of primes p above which F_0 has only one prime, then one can reobtain easily most of the numerical conditions of 5.1.2-3, without being "bothered" about mentioning T_1 in 5.1.2.1 or referring to 3.2.3 or to 3.2.3.1. But still this is not enough to get them all (we have in mind Example 5 above) or to get 5.1.2 A). So even if we have a good understanding of the (assumed to be very simple) arithmetics of F_0, still S1 of 5.1.3 below (and so implicitly 3.2.3 and 3.2.3.1) looks to us unavoidable. However, in practice it can be often very easily checked that the numbers t_j of the two cases of 5.1.2.2 do not depend on $j \in I_1$, and accordingly, that the numbers $s_{j,m}$ and $r_{j,m}$ do not depend on $j \in I_1$. Of course, as in A9 of 5.1.2.4, we might have not a unique choice for such a t_j.

5.1.3. The proof of 5.1.1. It is achieved in three steps. We refer to the equality $d = \dim(X_A)$ as the DD assumption (property).

Let $G_{V_0}^{\mathrm{ad}} = \prod_{i \in J} G_{iV_0}$ be the decomposition of the adjoint group of G_{V_0} into (absolutely) simple V_0-factors. Let I^{nc} be the subset of J formed by those elements $i \in J$ with the property that the composition of the cocharacter μ'_{V_0} of 4.4.5 with the natural quotient homomorphism $G_{V_0} \to G_{iV_0}$, is non-zero. Let $I^c := J \setminus I^{nc}$. We can write

$$G^{\mathrm{ad}}_{\mathbb{Z}_p} = G^{\mathrm{adnc}}_{\mathbb{Z}_p} \times G^{\mathrm{adc}}_{\mathbb{Z}_p}$$

as a product of two adjoint groups over $\mathbb{Z}_p$, where

$$G^{\mathrm{adnc}}_{\mathbb{Z}_p} = \prod_{i \in I^{nc}} G_i,$$

and where

$$G^{\mathrm{adc}}_{\mathbb{Z}_p} = \prod_{i \in I^{c}} G_i$$

are decomposition in product of simple split adjoint factors, with the extension of G_i to V_0, being the factor G_{iV_0} of $G_{V_0}^{\mathrm{ad}}$ (so the notation G_{iV_0} is justified). Let $\mu_{V_0}^{nc}$ be the composition of the cocharacter μ'_{V_0} of 4.4.5 with the natural quotient homomorphism $G_{V_0} \twoheadrightarrow G_{V_0}^{\mathrm{adnc}}$.

We get a Shimura group pair $(G_{V_0}^{\mathrm{adnc}}, [\mu_{V_0}^{nc}])$ over V_0 (cf. 4.4.6.1 and def. 1) of [Va2, 2.2.5]) which is a product of (absolutely) simple Shimura group pairs over V_0.

Similarly we can write $H_{AK_0}^{\mathrm{ad}} = H_{AK_0}^{\mathrm{adnc}} \times H_{AK_0}^{\mathrm{adc}}$. Warning: we do not know a priori that the simple factors of $H_{AK_0}^{\mathrm{adnc}}$ or of $H_{AK_0}^{\mathrm{adc}}$ are absolutely simple.

Step 1. DD implies that $G_{K_0}^{\mathrm{adnc}} = H_{AK_0}^{\mathrm{adnc}}$. This can be seen moving over $\mathbb{C}$ (using an embedding $i_K\colon K \hookrightarrow \mathbb{C}$ extending the embedding $E' \hookrightarrow \mathbb{C}$ of 4.3.7): DD says that the compact Hermitian symmetric spaces defined by $G_{\mathbb{C}}^{\mathrm{adnc}}$ and $H_{A\mathbb{C}}^{\mathrm{adnc}}$ and their parabolic subgroups naturally defined by the F^1-filtration $F_0^{1\prime}$ of M_0' (see [He]) are the same. But these compact symmetric spaces determine these complex adjoint groups (this well known fact can be grasped from [He, ch. VIII, 6.1]: the situation gets easily reduced to the case when both our compact Hermitian spaces are simple). This is the first step.

Step 2. Step 1 implies that $G_{\mathbb{Q}_p}^{\mathrm{adnc}}$ is a direct factor of $H_{A\mathbb{Q}_p}^{\mathrm{ad}}$. The second step is to get, using this and the tools 4.2.1-7, that $G_{\mathbb{Q}_p}^{\mathrm{der}} = H_{A\mathbb{Q}_p}^{\mathrm{der}}$. We present three solutions to this statement: the first one, self contained, treating the general case, the second one treating the general case but based on the results stated in [Va1, 1.7], while the third one refers to the particular case when $\mathrm{End}(A_{\bar{E}}) = \mathbb{Z}$. As the results stated in [Va1, 1.7] will be proved only in [Va5], the Fact of the second solution below has a star, meaning its proof is not included here.

S1. Solution 1 (the general case). As we have $p \geq N(A) \geq N_3(A)$, 3.2.2.1 applies. Warning: the notations of 3.2.2.1 and of §4-5 do not correspond (in 3.2.2.1 G is defined over $\mathbb{Q}$ and is related to a Shimura pair (G, X)). From 3.2.2.1 and the above Step 1 we get: for any simple factor $\mathcal{F}_A^1$ of $H_{A\overline{\mathbb{Q}_p}}^{\mathrm{ad}}$, the image $\mathcal{F}_B^1$ of $\tilde{G}_{\overline{\mathbb{Q}_p}}$ in it, has the same rank as $\mathcal{F}_A^1$. From 3.2.3 ADV) and 4.2.3 we get (via 4.2.5.1) directly that $\mathcal{F}_A^1 = \mathcal{F}_B^1$.

But, due to the same reasons, a simple factor of $\mathrm{Lie}(G_{\overline{\mathbb{Q}_p}}^{\mathrm{ad}})$ can not project surjectively into two or more simple factors of $\mathrm{Lie}(H_{A\overline{\mathbb{Q}_p}}^{\mathrm{der}})$. The argument for this goes as follows. We can assume that all the simple factors of $H_{A\mathbb{C}}^{\mathrm{ad}}$ have the same Lie type LT. If it is not an A_n Lie type ($n \geq 2$), then we can assume (cf. 3.2.3) that the 3.2.3.1 EADV) holds; so if a simple factor of $\mathrm{Lie}(G_{\overline{\mathbb{Q}_p}}^{\mathrm{ad}})$ projects surjectively into two or more simple factors of $\mathrm{Lie}(H_{A\overline{\mathbb{Q}_p}}^{\mathrm{der}})$, then we reach a contradiction to 4.2.3.

We are left with the case when LT is A_n, $n \in \mathbb{N}$, $n \geq 2$. We can also assume (cf. 3.2.3.1) that at least one simple factor of $(H_A^{\mathrm{ad}}, X_A^{\mathrm{ad}})$ has non-trivial involution. To handle this case we follow closely [De2, 2.3.7-10] and [Va1, 6.5-6]; warning: as in the general case of 3.2.3.1 we work with all the factors of (H_A, X_A) at once. In other words, we want to show that something very similar to 3.2.3.1 EADV) allows us (via 4.2.3) to handle the present situation. We already know that all the simple factors of $G_{\overline{\mathbb{Q}_p}}^{\mathrm{ad}}$ are of some A_n Lie type. Let (as in the proof of [De2, 2.3.10]) $\mathcal{S}$ be the set of extremal points of the Dynkin diagram $\mathcal{D}$ of $\mathrm{Lie}(H_{A\mathbb{C}}^{\mathrm{der}})$. $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathcal{S}$, and so we can identify $\mathcal{S}$ with the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-set of $\mathbb{Q}$–homomorphisms from $K_{\mathcal{S}}$ to $\mathbb{C}$, for $K_{\mathcal{S}}$ a convenient product of finite field extensions of $\mathbb{Q}$.

We can assume that the injective map f_A (of 4.3.6.1) is "very close" to one obtained following the pattern of the proof of [De2, 2.3.10] (cf. 3.2.3); to be compared with [Va1, 6.5.1.1 iv) and Case 2 of 6.6.5]. Here by "very close" we mean: we can assume that f_A factors through an injective map

$$f_A^1\colon (H_A^1, X_A^1) \hookrightarrow (GSp(W,\psi), S),$$

with $H_A^{1\mathrm{ad}} = H_A^{\mathrm{ad}}$ (so $(H_A^{\mathrm{ad}}, X_A^{\mathrm{ad}}) = (H_A^{1\mathrm{ad}}, X_A^{1\mathrm{ad}})$), and with f_A^1 obtained as in [De2, 2.3.10] (or as in [Va1, Case 2 of 6.6.5]).

In what follows, for an étale $\mathbb{Q}$–algebra $\tilde{\mathcal{L}}$, we denote by $\mathrm{Res}_{\tilde{\mathcal{L}}/\mathbb{Q}}$ the torus over $\mathbb{Q}$ defined by the invertible elements of $\tilde{\mathcal{L}}$. We recall from loc. cit. that the connected component of the origin of $Z(H_A^1)$ is generated by two tori:

– one is $T_{\mathrm{good}} := \mathrm{Res}_{K/\mathbb{Q}}$, where K is a suitable totally imaginary quadratic extension of the totally real number field F_A such that H_A^{ad} is the restriction from F_A to $\mathbb{Q}$ of an absolutely simple F_A-group;

– the second one is the compact part T_c (it is a $\mathbb{Q}$–subtorus) of the torus $\mathrm{Res}_{K_{\mathcal{S}}/\mathbb{Q}}$.

As in the way we got 3.2.3.1 EADV, we do not have to bother about T_{good}; in other words, T_{good} does not create problems in connection to 3.2.3.1 EADV). So the reason we can not use directly the same arguments as in 3.2.3.1 is: denoting by $\mathcal{K}$ the $\mathbb{Q}$–torus obtained by intersecting T_c with the subtorus of $Z(H_A^1)$ generated by the connected component of the origin of $Z(H_A)$ and by T_{good}, the irreducible $\mathcal{K}_{\overline{\mathbb{Q}_p}}$-submodules of $W \otimes \overline{\mathbb{Q}_p}$ not always contain a faithful representation of $H^{\mathrm{der}}_{A\overline{\mathbb{Q}_p}}$. So for our purposes it is enough to show:

Fact. *Such irreducible $\mathcal{K}_{\overline{\mathbb{Q}_p}}$-submodules do contain faithful representations of the Lie algebra of the product of all those simple factors of Lie$(H^{\mathrm{der}}_{A\overline{\mathbb{Q}_p}})$ on which a given simple factor of Lie$(G_{\overline{\mathbb{Q}_p}})$ surjects.*

As in the previous cases of other Lie types, we get that this Fact implies that $G^{\mathrm{der}}_{\mathbb{Q}_p} = H^{\mathrm{der}}_{A\mathbb{Q}_p}$. So to prove it, we can assume (cf. 4.2.4-7) that the Frobenius torus T_v has the same rank as $G_{\mathbb{Q}_p}$. Moreover, using the last paragraph of [Va1, Case 2 of 6.6.5], we deduce based on 3.2.1, that we can assume that T_v is (isomorphic to) a $\mathbb{Q}$–subtorus of a very particular inner form of H_A. In what follows, as we are interested in "minimizing" the toric part of the center of H_A^1, based on 3.2.1 AB, not to complicate the notations we will assume that this inner form is trivial. So using the fact that T_v is a $\mathbb{Q}$–subtorus of H_A, we achieve easily a Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$-permuting partition of (equivalence relation on) $\mathcal{S}$ as follows. We say that two elements of $\mathcal{S}$ are in relation if:

– they are points of Dynkin diagrams of two simple factors of Lie$(H^{\mathrm{der}}_{A\overline{\mathbb{Q}_p}})$ on which the same simple factor Lie$(\mathcal{F})$ of Lie$(G^{\mathrm{der}}_{\overline{\mathbb{Q}_p}})$ projects surjectively;

– with respect to the torus T_v, they are either both original points or are both ending points of the Dynkin diagrams.

Let $\mathcal{S}^1$ be the set of the equivalence classes of this partition. As Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathcal{S}^1$, as above we define $K_{\mathcal{S}^1}$; we identify it with a $\mathbb{Q}$–subalgebra of $K_{\mathcal{S}}$. Due to the existence of the cocharacter μ'_{V_0}, we deduce that $\mathcal{K}$ is contained in the compact part of $\mathrm{Res}_{K_{\mathcal{S}^1}/\mathbb{Q}}$.

Recalling how $\mathrm{Res}_{K_{\mathbb{S}}/\mathbb{Q}}$ (and so how $\mathrm{Res}_{K_{\mathbb{S}1}/\mathbb{Q}}$) acts on W, we deduce immediately the above Fact.

We deduce (in the general case) that $G^{\mathrm{der}}_{\mathbb{Q}_p} = H^{\mathrm{der}}_{A\mathbb{Q}_p}$. So $\tilde{G}_{\mathbb{Q}_p} = H_{A\mathbb{Q}_p}$. This represents the first solution.

S2. Solution 2 (the general case). We include this second solution, just to emphasize that the machinery of 3.1 (needed in 3.2) can be avoided, provided we are eager to assume another (in fact much more technical) machinery of Langlands–Rapoport conjecture. It is an easy exercise to see that our results on Langlands–Rapoport conjecture referred to in [Va2, 1.24] (the infinite set of primes referred to in loc. cit. is of positive Dirichlet density, cf. also [Va1, end of 1.7.1]) when combined with 4.2.7.1 do imply that $G^{\mathrm{der}}_{\mathbb{Q}_p} = H^{\mathrm{der}}_{A\mathbb{Q}_p}$. In other words the mentioned results imply (to be compared with 3.2.1):

Fact*. *There is a maximal torus of a conjugate of $G_{\mathbb{Q}_p}$ by an element of $H_A(\mathbb{Q}_p)$ which is the extension to $\mathbb{Q}_p$ of a subtorus of H_A.*

So we get a subtorus of H_A which projects into a maximal subtorus of each simple non-compact factor of $H_{A\mathbb{Q}_p}$. As there is no compact factor of $H^{\mathrm{ad}}_{A\mathbb{R}}$ defined over $\mathbb{Q}$ this implies that the extension to $\mathbb{C}$ of this subtorus projects into a maximal torus of each simple factor of $H^{\mathrm{ad}}_{A\mathbb{C}}$. Applying again 4.2.3 and 4.2.5.1 we get (as in S1 above, i.e. via 3.2.3.1 and the extra arguments in the A_n case of S1) $G^{\mathrm{der}}_{\mathbb{Q}_p} = H^{\mathrm{der}}_{A\mathbb{Q}_p}$. This is the second solution.

S3. Solution 3 (the case $\mathrm{End}(A_{\bar{E}}) = \mathbb{Z}$). In many cases we do not need to refer to the Langlands–Rapoport conjecture. In all these cases we know a priori that the Fact of S2 is true. For instance if $\mathrm{End}(A_{\bar{E}}) = \mathbb{Z}$ and if $H^{\mathrm{ad}}_{A\mathbb{R}}$ has no compact factors of C_r Lie type, with $r \in \mathbb{N}$ greater than 2, than the Fact of S2 is implied by [Pi, 7.2] and [Va2, P2 of 4.6]. In fact the set of r's for which we have troublesome C_r compact factors is much smaller, cf. [Pi]. The first problem occurs if we have compact factors of C_4, C_{10}, etc., Lie type.

In fact to treat the case $\mathrm{End}(A_{\bar{E}}) = \mathbb{Z}$ in full generality, we can skip completely the reference to the Fact of S2 or to the Langlands–Rapoport conjecture. [Pi, 5.13 (c) and 5.12] together with the Step 1, imply that we can assume that all the simple factors of $H^{\mathrm{ad}}_{A\overline{\mathbb{Q}_p}}$ and of $G^{\mathrm{ad}}_{\overline{\mathbb{Q}_p}}$ have the same Lie type. From this and 4.2.3 we get (by simple reasons of counting dimensions) that the numbers of such such simple factors must be the same. We immediately get that $\tilde{G}_{\mathbb{Q}_p} = H_{A\mathbb{Q}_p}$. This is the third solution.

Step 3. The third step is to get $G_{\mathbb{Q}_p} = H_{A\mathbb{Q}_p}$ knowing that their derived groups are equal. This is achieved using 4.2.7 and the main result of [Wi]. We present the details. From 4.2.7.2 we deduce the existence of a subgroup H'_A of H_A such that $G_{\mathbb{Q}_p} = H'_{A\mathbb{Q}_p}$. We have:

Fact. *Using the embedding i_K of Step 1, the cocharacter μ of 4.4.4 and the cocharacter μ'_{V_0} of 4.4.5, over $\mathbb{C}$, become $H_A(\mathbb{C})$-conjugate to the cocharacter $\mu(\mathbb{C})$ of 4.0.*

This is a trivial exercise (to be compared with [Va2, 2.2.9 3)]).

There is a more fancy way to argue this Fact. To present it we first remark: if A^0 is a canonical lift this is obvious and a well known fact. We recall quickly how we get this only in the cases we need: 4.2.5 allows us (cf. 4.3.3.1) to shift to another prime for which we can apply directly [Va2, a) of 4.4.1 3)] for a SHS $(f_A, L_{(p)}, v_A)$, obtained as in 4.3.6.1.1. Of course, if we work under the variant 4.1.1.1, then we already can apply loc. cit., and so we do not have to depend on 4.2.5.

Second, we remark: 4.4.7-9 and [Va2, 3.1.4-5] allow us to shift our attention to canonical lifts over $W(k)$, with k the perfection of the field of fractions of R/pR.

So, from the very definition of H_A and the above Fact, we get that $H'_A = H_A$. So $G_{\mathbb{Q}_p} = H_{A\mathbb{Q}_p}$. This is the third step; it ends the proof of 5.1.1.

5.1.3.1. Remark. It is worth remarking that if $H^{\mathrm{ad}}_{A\mathbb{R}}$ has no compact factors, then we have nothing to prove in Step 2. Similarly, if $\mathrm{End}(A_{\bar{E}}) = \mathbb{Z}$, then Step 3 is nothing else but 4.2.15.

5.1.3.2. Exercise. Using [An, 1.5.1-2] reobtain (in a different way) the result of [An, 1.6.1 3)]) pertaining to Mumford–Tate groups. Hint: use 5.1.2.

5.1.4. The proof of 5.1.2. We follow closely the proof (see 5.1.3) of 5.1.1. From Step 3 of 5.1.3 we deduce that it is enough to concentrate on the adjoint groups of $G_{\mathbb{Q}_p}$ and of $H_{A\mathbb{Q}_p}$. From 3.2.3.1 EADV) and S1 of Step 2 of 5.1.3 we deduce that we can assume that H^{ad}_A is a simple $\mathbb{Q}$–group. From 3.2.2.1 and S1 of Step 2 of 5.1.3 we deduce that it is enough to show that there is a simple factor of $H^{\mathrm{ad}}_{A\overline{\mathbb{Q}_p}}$ on which $G_{\overline{\mathbb{Q}_p}}$ surjects naturally. We show this by treating the cases one by one.

A) This is a consequence of 3.2.3 and of [Va2, 2.3.5.1]. In other words we can assume that the embedding f_A is a PEL type embedding (see [Ko]) and that 3.2.3 ADV) holds. As in 3.2.1 we can assume (cf. 4.2.7.1) that $G_{\mathbb{Q}_p}$ contains a maximal torus which is the extension to $\mathbb{Q}_p$ of a subtorus T_A of H_A. Strictly speaking, in 3.2.1, we did not bother to explain in full generality why in fact $G_1 = G'_1$; accordingly, we should rigorously use an inner form H'_A of H_A of whose adjoint is still a simple $\mathbb{Q}$–group. But from the point of view of the Fact below this is irrelevant: we have $G_{1\mathbb{Q}_p} = G'_{1\mathbb{Q}_p}$ and their extensions to $\mathbb{C}$ give birth to the same conjugacy classes of characters (cf. 3.2.1 HW); so we will not complicate the notations with H'_A. So we can assume that:

Fact. *T_A is the smallest subtorus of H_A such that a set of cocharacters of $H_{A\mathbb{C}}$ which are $H_A(\mathbb{C})$-conjugate to the cocharacters of $\mu^*_{x_A}$ of [Va2, 2.3.1] (here $x_A \in X_A$) factor through $T_{A\mathbb{C}}$.*

This is just a restatement of [Pi, 3.16 and 5.10] for our situation. So case A) results from the definition 5.1.2.1 and from the arbitrary choice made for f_1 in 5.1.2.2.

For the rest of the cases we need to remind about the existence of the cocharacter μ'_{V_0} of G_{V_0} (so we can always choose non-compact factors of $H^{\mathrm{ad}}_{A\overline{\mathbb{Q}_p}}$), and of 4.2.14.

B) This case is nothing else but [Pi, 4.3].

C) This case is the well known case of Serre, cf. [Chi, p. 330].

C') This case is trivial.

D) This case is just the orthogonal (versus symplectic) variant of C). It can be easily read out from the table [Pi, 4.2].

D') To treat this case, we have to "move out" from the split criterion. So for this case, we do not need any reference to $N(A)$ or to A'. We just assume that we are just in the context of 4.0 and of 4.3.1, that E contains the number field $G(H_A^{\mathrm{ad}}, X_A^{\mathrm{ad}})$ defined in 2.5, and that p is such that the Fact of 2.5 applies. From the mentioned fact and the paragraph after it we deduce that there is an absolutely simple factor AS of $H^{\mathrm{ad}}_{A\mathbb{Q}_p}$, such that the Shimura group pair we naturally get (as in 5.1.3) $(AS, [\mu_{AS}])$ is with non-trivial involution. The numerical conditions we have imposed are such that the image IM of $G_{\mathbb{Q}_p}$ in AS is automatically an adjoint group which is a product of absolutely simple groups of different Lie types, which can not have non-trivial involution. We assume that $AS \neq IM$. So, if n is odd, then IM is absolutely simple of a suitable A_m Lie type; if $n = 2q$ with q a prime, then IM is of $C_1 + C_q$ Lie type, and if $n = 6q$ with q a prime different from 3, then IM is of $C_1 + C_{3q}$ or of $C_3 + C_q$ Lie type. Using 4.4.4 and Step 3 of 5.1.3, we deduce that we can assume that we have a Shimura subgroup pair $(IM, [\mu_{AS}])$ of $(AS, [\mu_{AS}])$. But the Lie types of IM can not have non-trivial involution; in other words, we can assume that μ_{AS} is a cocharacter of IM itself (and not of its extension to V_0). So from [Mi2, 4.6-7] we deduce that the Shimura group pair $(AS, [\mu_{AS}])$ is with trivial involution. Contradiction. So we do have $IM = AS$.

E) We can assume that 3.2.3 ADV) and 3.2.3.1 EADV) hold (cf. 3.2.3). It is known (see [Pi, 4.3]) that the existence of the cocharacter μ'_{V_0} (of 4.4.5) implies that the image $\mathcal{F}_1$ of $G_{\overline{K_0}}$ in a non-compact simple factor $\mathcal{F}_0$ of $H^{\mathrm{ad}}_{A\overline{K_0}}$ is either $\mathcal{F}_0$ itself or a product PB of two simple adjoint groups of $B_{n_1} + B_{n_2}$ type, with $n_1 + n_2 = n - 1$; here the so called B_0 Lie type corresponding to the trivial group is allowed. But the representation of $\mathrm{Lie}(\mathcal{F}_0)$ on $H^1 \otimes \overline{K_0}$ involves both the half-spin representations, and moreover any irreducible subrepresentation of $\mathrm{Lie}(H^{\mathrm{der}}_{A\overline{K_0}})$ on $H^1 \otimes \overline{K_0}$ factors through a simple factor of it (i.e. no tensor products are involved) (cf. 3.2.3 ADV)). So if $\mathcal{F}_1$ is not $\mathcal{F}_0$, we reach a contradiction to the well known result 4.2.3 (cf. 3.2.3.1 EADV)), as the representation of $\mathrm{Lie}(PB)$ on a simple non-trivial $\mathrm{Lie}(\mathcal{F}_0)$-submodule H^2 of $H^1 \otimes \overline{K_0}$ is the same (it is the product of the spin representations of the two simple factors of PB, cf. [Pi, 4.3]), regardless of which half-spin representation is involved in defining H^2. We conclude that $\mathcal{F}_0 = \mathcal{F}_1$. Of course we can replace above $\overline{K_0}$ by $\overline{\mathbb{Q}_p}$. This ends the proof of the case E), and so of the theorem 5.1.2.

In what follows the abelian motives over a field of characteristic zero are understood in the sense of absolute Hodge cycles, see [DM].

5.1.5. Corollary. *If the Mumford–Tate conjecture is true for the members of a finite set of motives over E, then it is true for their direct sum DS.*

Proof: Let A_{DS} be an abelian variety over E such that summands of DS are objects of the Tannakian category of abelian motives over E generated by its H^1-motive and by the Tate-twists (see [DM]). Using the same trick as in [Va2, 2.3.5.2] we can assume that each simple factor of the adjoint group of the Mumford–Tate group $H_{A_{DS}}$ of A_{DS} surjects onto a simple factor of the adjoint group of the Mumford–Tate group of DS. Using Step 2 of 5.1.3 and our hypothesis we deduce that the Mumford–Tate conjecture holds for the derived part of $H_{A_{DS}}$. From Step 3 of 5.1.3 we deduce that the Mumford–Tate conjecture holds for A_{DS}. The corollary follows.

5.2. A criterion. It is desirable to have an approach in checking DD. In [Va5] we approach the proof of the equality $d = \dim(X_A)$ via [Va2, 3.4] and the following lemma and its versions of 5.2.1.

5.2.0. Lemma. *We assume that 4.4.7 remains true if we replace R by an integral smooth V_0-algebra R_1, of relative dimension d. In other words we assume that l_R factors through a morphism $l_{R_1}\colon Spec(R_1) \to \mathcal{A}_{g_A,1,4}$, with R_1 as mentioned, and for which we have:*

– the Frobenius Φ_R is obtained from a Frobenius Φ_{R_1} of the p-adic completion $R_1^\wedge$ of R_1 (via the formally étale factorization $Spec(R) \to Spec(R_1)$);

– the principally quasi-polarized p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(R_1)$ defined by the principally polarized abelian scheme $(A_{R_1}, p_{A_{R_1}})$ over R_1 we get via l_{R_1}, is isomorphic to one of the form

$$\mathcal{C}_{R_1} := (M_0' \otimes R_1, F_0' \otimes R_1, g^1_{\text{univ}}(\phi' \otimes 1), \nabla^1, p_{A^0}),$$

with $g^1_{\text{univ}} \in G_{V_0}(R_1^\wedge)$ such that $\mathcal{C}_{R_1}$ is a versal deformation in the sense of [Va2, 3.4.19 B)] of $\mathcal{C}_0$.

We also assume that the étale cycle of $A_{R_1^\wedge[\frac{1}{p}]}$ corresponding via [Fa1, 6.2] to $t_\alpha^0 \in \mathcal{T}(M_0' \otimes R_1^\wedge[\frac{1}{p}])$ is in fact defined over $R_1[\frac{1}{p}]$; so we get a family (still denoted by) $(v_\alpha)_{\alpha\in\mathcal{J}}$ of étale cycles of $A_{R_1[\frac{1}{p}]}$. Then $d = \dim(X_A)$.

Proof: From 4.4.7.1 we deduce that l_{R_1} factors through $\mathcal{N}(4)$. As in the proof of 4.4.8, we get (via 4.4.9) that:

a) there is a Zariski dense set of $\mathbb{F}$-valued points of $\mathrm{Spec}(R_1/pR_1)$ over which we get (from A_{R_1}) ordinary abelian varieties;

b) the canonical lifts of such varieties are obtained from A_{R_1} through V_0-valued points of $\mathrm{Spec}(R_1)$.

From this and [Mo, 3.11 and 5.2] we deduce that l_{R_1} dominates a Shimura subvariety of $\mathcal{A}_{g_A,1,4_{O_{(v)}}}$. From 4.4.7 a) we deduce that this subvariety is a connected component Y of $\mathcal{N}(4)$.

Dropping the assumption that R_1 is smooth over V_0, we can assume that the affine Y_{V_0}-scheme $Y^1_{V_0} := \mathrm{Spec}(R_1)$ is obtained from an integral, finite type $Y_{O_{(w)}}$-scheme Y^1, with $O_{(w)}$ a local étale $\mathbb{Z}_{(p)}$-algebra containing $O_{(v)}$, by extension of scalars. This is a general fact: it can always be achieved by passing, if needed, to a finite field extension F of E, included in K_0, and replacing $Y^1_{V_0}$ by a scheme of finite type over it (so $O_{(w)} = F \cap V_0$).

From 4.4.7 a) and from 4.3.6.2, we deduce that the family of étale cycles $(v_\alpha)_{\alpha\in\mathcal{J}}$ is defined over the generic fibre Y^1_F of Y^1. There is a quasi-finite dominant morphism $Y^2_F \to Y_F$, factoring through the morphism $Y^1_F \to Y_F$ (argument: we can assume that Y^1_F is smooth over Y_F; so we can use [EGA, 17.16.2]).

We get a family, still to be denoted by $(v_\alpha)_{\alpha\in\mathcal{J}}$, of étale cycles defined over Y^2_F. With this we are essentially done. There are many directions to be persuaded to conclude the proof. We choose one.

We can assume that Y^2_F is geometrically connected: this is achieved by replacing F with a finite field extension of it (we are not anymore interested to have F as a subfield of K_0). Let H^1_A be the biggest connected subgroup of H_A with the property that $H^1_{A\mathbb{Q}_p}$ is included in $G_{\mathbb{Q}_p}$. We move over $\mathbb{C}$, i.e we consider the extension $Y^2_{\mathbb{C}} \to Y_{\mathbb{C}}$ of the morphism $Y^2_F \to Y_F$ to $\mathbb{C}$ (via the inclusions $F \subset K_0 \subset \mathbb{C}$). The key point is: $Y^2_{\mathbb{C}}$ has an analytically dense set $\mathcal{SP}$ of complex points, over which we get abelian varieties having complex multiplication; argument: for $Y_{\mathbb{C}}$ this is a property of Shimura varieties, while the morphism $Y^2_{\mathbb{C}} \to Y_{\mathbb{C}}$ is quasi-finite and dominant. In all these points, the étale cycles v_α, $\forall \alpha \in \mathcal{J}$, become $\mathbb{Q}_p$-linear combination of p-components of étale components of Hodge cycles, cf. 4.2.6. This implies that all the Hodge structures on $W \otimes_{\mathbb{Q}} \mathbb{C}$ defined by complex abelian varieties obtained from points of $\mathcal{SP}$, correspond to cocharacters of $G\mathrm{Sp}(W,\psi)_{\mathbb{C}}$ factoring through $H^1_{A\mathbb{C}}$. As the set $\mathcal{SP}$ of complex points of $Y^2_{\mathbb{C}}$ is analytically dense in $Y^2_{\mathbb{C}}$, we deduce that this remains true for all the complex points of $Y^2_{\mathbb{C}}$. This obviously implies (cf. the very definition of H_A) that $H^1_A = H_A$. So $H^1_{A\mathbb{Q}_p} = G_{\mathbb{Q}_p} = H_{A\mathbb{Q}_p}$, and so also $d = \dim(X_A)$. This ends the proof of the lemma.

5.2.1. Remark. If by chance we know only that the subfamily of $(v_\alpha)_{\alpha\in\mathcal{J}}$ formed by tensors fixed by the centralizer of $G^{\mathrm{dernc}}_{\mathbb{Q}_p}$ in $H_{A\mathbb{Q}_p}$ is defined over a scheme $Y^1_{V_0}$ as above, then the same argument as above, shows that $G^{\mathrm{adnc}}_{\mathbb{Q}_p}$ is a direct factor of $H^{\mathrm{ad}}_{A\mathbb{Q}_p}$. So $G^{\mathrm{adnc}}_{K_0} = H^{\mathrm{adnc}}_{AK_0}$.

Similarly, if we know only that the subfamily of $(v_\alpha)_{\alpha\in\mathcal{J}}$ formed by tensors fixed by $\tilde{G}_{\mathbb{Q}_p}$ is defined over a scheme $Y^1_{V_0}$ as above, then we get that $\tilde{G}_{\mathbb{Q}_p} = H_{A\mathbb{Q}_p}$, i.e. that $G^{\mathrm{der}}_{\mathbb{Q}_p} = H^{\mathrm{der}}_{A\mathbb{Q}_p}$.

In both the above situations we do get as well $d = \dim(X_A)$.

5.2.2. An example. We assume that H^{ad}_A is a simple $\mathbb{Q}$–group of C_{2q} type, with q a prime different from 2 and 5. We also assume that $H^{\mathrm{ad}}_{A\mathbb{R}}$ has only one non-compact factor. So $\dim(X_A) = q(2q+1)$. On the other hand $d \in \{1, \frac{q(q-1)}{2}, q(2q+1)\}$. This is in conformity with [He, table V of p. 518] and the fact that the image of $G_{\overline{\mathbb{Q}_p}}$ in a simple factor of $H^{\mathrm{ad}}_{A\overline{\mathbb{Q}_p}}$ is either of $C_1 + D_q$ Lie type or of C_{2q} Lie type. If $d = 1$, as $G_{\mathbb{Q}_p}$ is a split group, then the abelian variety A_v over $k(v)$ is either ordinary or supersingular; of course, if the Frobenius torus T_v has rank $q+1$, then from Serre's result (see [Pi, 3.5]) we get immediately that A_v must be ordinary, and so 4.4.8 applies. The case $d = \frac{q(q-1)}{2}$ is theoretically subtler.

On the other, it seems to us that the ideas of [Ta3, 0.7] can be entirely adapted to show that in fact the Lie type $C_1 + D_q$ does not show up, even if we do not assume that $H^{\mathrm{ad}}_{\mathbb{R}}$ has only one compact factor.

5.3. Serre's volumes. We come back to the general situation described in 4.0: A is an arbitrary abelian variety over a number field E and p is a an arbitrary prime; it is convenient to denote ρ of 4.0 by ρ_p, to emphasize p. We denote by

$$\prod_{p \text{ a prime}} \rho_p : \mathrm{Gal}(\bar{E}/E) \to GL(H^1_{\text{ét}}(A_{\bar{E}}, \mathbb{A}_f))(\mathbb{A}_f)$$

the product $\mathbb{A}_f$-representation of all ρ_p. We assume that the Mumford–Tate conjecture is true for A.

Let E^{conn} be the smallest finite field extension of E such that all the Hodge cycles of $A_{\bar{E}}$ are defined over E^{conn}. It is a Galois extension of E: our assumption implies that it is the same field extension of E as the one defined (see [Pi, 3.6]) by Serre. Two of the most important global invariants of the isogeny class of A are the Shimura pair (H_A, X_A) (see 4.3.6.1) and E^{conn} (or just $[E^{\mathrm{conn}} : E]$). We consider the Haar measure $\mathcal{M}(p)$ on $H_A(\mathbb{Q}_p)$ normalized by the fact that a maximal compact (open) subgroup $H(p)$ of $H_A(\mathbb{Q}_p)$ of maximum volume, has volume 1. So $H(p)$ is a hyperspecial subgroup of $H_A(\mathbb{Q}_p)$ if H_A is unramified over $\mathbb{Q}_p$, cf. [Ti, p. 55]. We define Serre's (local) p-volume $\mathcal{V}_p(A)$ of A by:

$$\mathcal{V}_p(A) := \mathcal{M}(p)(\rho(\mathrm{Gal}(\bar{E}/E^{\mathrm{conn}}))).$$

Similarly we define the total Serre's volume $\mathcal{V}(A)$ of A by:

$$\mathcal{V}(A) := \prod_{p \text{ a prime}} \mathcal{V}_p(A).$$

Obviously $\mathcal{V}_p(A)$ is a rational number of the interval (0,1]. So $\mathcal{V}(A)$ is a real number of the interval [0,1]. It is expected that $\mathcal{V}(A)$ is a rational number of (0,1] (see [Se, 11.4?]). Working with a Haar measure $\mathcal{M}(\mathbb{A}_f)$ of $H_A(\mathbb{A}_f)$, normalized in the same manner, we can define the global Serre's volume of A by:

$$\mathcal{V}_g(A) := \mathcal{M}(\mathbb{A}_f)(\prod_{p \text{ a prime}} \rho_p(\mathrm{Gal}(\bar{E}/E^{\mathrm{conn}}))).$$

We have

$$0 \le \mathcal{V}_g(A) \le \mathcal{V}(A).$$

It is expected (see [Se, 11.4?]) that $0 < \mathcal{V}_g(A)$ and that $\mathcal{V}(A)$ and $\mathcal{V}_g(A)$ are rational numbers. $\mathcal{V}_g(A)$, and $\mathcal{V}_p(A)$, with p a prime, are as well, in our opinion, very important global invariants of the isogeny class of A.

5.3.1. Remarks. 1) The study of Serre's volumes is very closely related to the study of finite quotients of the Galois group $\mathrm{Gal}(\bar{E}/E)$. We expect the possibility of proving that any finite group (resp. any simple finite group) C is a quotient of the Galois group $\mathrm{Gal}(\mathbb{Q})$ of $\mathbb{Q}$ by just using the computations of local Serre's volumes of particularly chosen abelian varieties (resp. absolutely simple abelian varieties) over $\mathbb{Q}$. At least in the case when C is a simple finite group associated to a Chevalley group (over a finite field) of classical Lie type, it should be possible to construct an epimorphism $\mathrm{Gal}(\mathbb{Q}) \twoheadrightarrow C$, starting from 5.1.2 and [Va1, 6.5.1.1]. We will come back to these ideas in [Va4].

2) We do believe that everything one would like to know about the finite quotients of $\mathrm{Gal}(\mathbb{Q})$ is codified in SSO. This represents our second approach to the understanding of such quotients (see [Va2, 3.4.18.4.6 H] for the first approach).

3) In the case when the closure $H_{A\mathbb{Z}_p}$ of $H_{A\mathbb{Q}_p}$ in $GL(H^1)$ is a reductive group over $\mathbb{Z}_p$, the study of the fact that $\mathcal{V}_p(A)$ is or it is not 1, is closely related to the fact that the natural adjoint representation over $\mathbb{F}_p$ of the finite group $H^{\mathrm{der}}_{A\mathbb{Z}_p}(\mathbb{F}_p)$ on $\mathrm{Lie}(H^{\mathrm{der}}_{A\mathbb{Z}_p}) \otimes \mathbb{F}_p$ is or it is not irreducible. For a quite elementary solution to this problem of the theory of representations of finite groups of Chevalley type, see [Va6].

4) We think that the terminology Serre's volumes is justified, as J.-P. Serre was the first one to study the numbers (representations) involved; for instance see the report [Se] and its references.

5.4. Extra remarks. 1) 5.1.2 implies (this is standard) a similar result for finitely generated fields of characteristic zero. Of course, we can replace abelian varieties by abelian motives over such fields. Similarly, the same is true in the context of 5.1.5.

2) Many previous approaches (for instance see [Pi]) to the proof of 4.1 were intermingled with approaches to the proof of the ordinary reduction conjecture (for instance see [Oo, p. 11]) for abelian varieties over number fields. So a natural question arises: what the above proof of 5.1.2 and the techniques of §3-4 bring new in this direction? Our approach to the proof of 5.1.2 was mainly based on Shimura varieties, and much less on the "inner" study of Frobenius tori (we needed mainly just their construction and 4.2.4-5). So here we do not bring anything new, besides more information on Frobenius tori (see 3.2.2.1 and 4.4.8). However, we invite the reader to solve the following two exercises (their complete proofs will be presented in [Va5]).

Exercise 1. We consider the situation of 4.0. We assume that the Shimura pair (H_A, X_A) is such that all the simple symmetric domain factors of a connected component of X_A are (see [He, p. 518]) either of type $AIII(q=1)$ or of type $BDI(q=2)$ with arbitrary $p \in \mathbb{N}$, $p \geq 3$ (i.e. we assume that all the simple non-compact factors of $H^{\mathrm{ad}}_{A\mathbb{R}}$ are either of B_n Lie type, or are either of the form $SU(p,1)^{\mathrm{ad}}_{\mathbb{R}}$, with $p \in \mathbb{N}$ arbitrary, or of the form $SO(p,2)^{\mathrm{ad}}_{\mathbb{R}}$, with $p \geq 3$ an arbitrary even integer). Using 5.1.2, [Pi], and the results on the Langlands–Rapoport conjecture stated in [Va1, 1.7.1], show that the ordinary reduction conjecture is true for A. Hint: use the same idea as in S2 of 5.1.3, [Va2, 4.9.23] and 3.2.3.

Exercise 2. Using Exercise 1 and 5.1.3.2, show that the ordinary reduction conjecture is true for all the motives over number fields mentioned in [An, 1.5]. Hint: use [An, 1.5.2].

3) The family of tensors of 4.3.4.8 a) is strongly $\mathbb{Z}_p$-very well position for $G_{\mathbb{Q}_p}$. This is a consequence of its proof and of [Va1, 4.3.6 2)].

4) In [Va5] we will see that instead of $N_3(A)$ we can work as well with $N_3'(A) \in \mathbb{N} \setminus \{1,2\}$ defined by: for any prime $q \geq N_3'(A)$, H_A is unramified over $\mathbb{Q}_q$.

5) We assume now that $\mathrm{End}(A_{\bar{E}}) = \mathbb{Z}$ and that the simple adjoint Shimura pair $(H^{\mathrm{ad}}_A, X^{\mathrm{ad}}_A)$ is of $D^{\mathbb{R}}_n$ type. Here $n \geq 4$ has to be an odd number (see the autoduality part in [Pi, 4.6]). 5.1.2 allows another proof of [Pi, p. 230-236]. In other words, 3.2.3 ADV) represents another way (based on 5.1.2) to get directly the lifting property of the l-adic representations mentioned in the first paragraph of [Pi, p. 230].

6) From the point of view of the possibilities of types of adjoint Shimura varieties of abelian type, 5.1.2 covers about 77 percents of the possible cases (cf. 1.8.1). 5.1.2 does not cover [Pi, 5.14] or the part of [Ta3, 0.7] involving C_{2p}; to our knowledge it does encompass all other previous results pertaining to 4.1, besides the mentioned ones. Warning: presently one can not obtain a general form of [Pi, 5.14] in the pattern of 5.1.2 C); however, restricted to the case when $\mathrm{End}(A_{\mathbb{C}}) = \mathbb{Z}$, then loc. cit. does have such a general form (the proof of this is the same as the proof of it). On the other hand, we can not see any serious dificulty to get such a general form for the C_{2q} types, with q a prime different from 2 and 5 (cf. 5.2.2).

7) Shimura varieties of A_n (resp. of either B_n or of $D^{\mathbb{R}}_n$) type are also called unitary (resp. orthogonal or spin Shimura varieties). 5.1.2 supports the point of view of [Va1, 1.2.6.2] that the study of unitary and orthogonal Shimura varieties is easier than the one of Shimura varieties of C_n or $D_n^{\mathbb{H}}$ type. Based on this point of view, on [Va1, 1.7.1], on 3.1.4 and on 5.3.1 2), we do expect that the mathematical role played by Siegel modular forms in the twentieth century will be substituted in the next century by the one played by orthogonal (spin) modular forms.

8) For the remaining cases of the Mumford–Tate conjecture, we can assume that the adjoint group H_A^{ad} is $\mathbb{Q}$–simple (and if it is helpful that 3.2.3 ADV) or 3.2.3.1 EADV) holds), cf. 5.1.3-4.

9) For any A_n, B_n or D_n Lie type τ, there is a simple adjoint Shimura pair (H_0, X_0), with H_0 an absolutely simple group of τ Lie type, such that, with the notations of 4.0, the Mumford–Tate conjecture is true for A, provided $H_{A\mathbb{R}}^{\mathrm{ad}}$ is isomorphic with $H_{0\mathbb{R}}$. Argument: for instance we can take H_0 such that H_0 is $SU(1, n-1)^{\mathrm{ad}}$ or $SO(2,m)^{\mathrm{ad}}$ with $m \in \{2n-2, 2n-1\}$, so that 5.1.2 applies. But this is not yet known to be true for arbitrary C_{2n} Lie types (cf. 6) above and 5.1.2.4).

10) From the Fact of Step 3 of 5.1.3, as $G_{\mathbb{Q}_p}$ is split, we get immediately that $k(v_A) = \mathbb{F}_p$ (for v_A see 4.3.7), cf. also [Mi2, 4.6-7]. This is as well in conformity with 4.4.9, and with [Va2, P1 of 4.6].

11) The approaches used in this paper, can be used as well for other classes of motives, provided (at least some of) the equivalent results of [De1, 2.3.10-13], 4.2.3, [Sa] and [Zi] are known to be true. Also we would like to point out, that surprisingly, in connection to 4.1, the results [De1, 2.3.10-13] were neither used nor quoted before.

5.5. Final complements.

5.5.1. A method of extending the main result of [Wi] to some ramified cases. We use the notations of 2.4. We do not assume a priori the existence of the cocharacter μ_K. It is desirable to have easily checkable conditions on the family of tensors $(w_\alpha)_{\alpha\in\mathcal{J}}$ which guarantee the existence of such a cocharacter. Some time ago we proposed one type of such a condition:

– the family of tensors $(w_\alpha)_{\alpha\in\mathcal{J}}$ is moreover $H^1_{\text{ét}}(H_V, \mathbb{Z}_p)$-representation well positioned for the group $\tilde{G}_{\mathbb{Q}_p}$.

However, it turns out that this extra condition is not needed, as we have:

Lemma. *Under the assumptions on the family $(w_\alpha)_{\alpha\in\mathcal{J}}$ of étale cycles of 2.4, the cocharacter μ_K exists automatically.*

Proof: Using Tate's classical result [Tat, cor. 2 of p. 180] and standard arguments of algebraic geometry, we deduce immediately that such a cocharacter exists over a finite field extension K_1 of K. We deduce that the quadruple $(M_0, \phi, G_{W(k)}, (t_\alpha)_{\alpha\in\mathcal{J}})$ (it is the same for H_V as for H_{V_1}, where V_1 is the ring of integers of K_1) becomes a Shimura σ-crystal over a finite Galois extension k_1 of k; so, with the terminology of [Va2, 2.2.9 1')], it is a quasi Shimura σ-crystal over k.

But the Galois group $\mathrm{Gal}(k_1/k)$ acts on the F^1-filtrations $F^1(M_0 \otimes_{W(k)} W(k_1))$ of $M_0 \otimes_{W(k)} W(k_1)$ such that the quadruple $(M_0 \otimes_{W(k)} W(k_1), F^1(M_0 \otimes_{W(k)} W(k_1)), \phi, G_{W(k_1)})$ is a Shimura filtered σ-crystal over k_1. As $F^1(M_0 \otimes_{W(k)} W(k_1))$ mod p is defined over k, by induction on n we can assume that $F^1(M_0 \otimes_{W(k)} W(k_1))$ mod p^n is defined pver

$W(k)/p^nW(k)$. For instance, the space of such filtrations mod p^2 is a k_1-vector space of the extension to k_1 of a vector space over k; so [Bo2, 14.2 of p. 30] applies. The induction follows.

We deduce that we can assume that $F^1(M_0 \otimes_{W(k)} W(k_1))$ is the extension to $W(k_1)$ of a direct summand F^1 of M. So we can construct a Shimura p-divisible group $(H, (t^0_\alpha)_{\alpha\in\mathcal{J}})$ as in 2.4; moreover we can assume that the Kodaira–Spencer map of H is an injective map. As $(H_{V_1}, (t^V_\alpha)_{\alpha\in\mathcal{J}})$ is induced from $(H, (t^0_\alpha)_{\alpha\in\mathcal{J}})$, it follows that $(H_V, (t^V_\alpha)_{\alpha\in\mathcal{J}})$ is induced from $(H, (t^0_\alpha)_{\alpha\in\mathcal{J}})$. The proposition follows.

5.5.2. 4.2.14 and [Bou, ch. VIII, §7.3] imply that the condition 4.3.4.2 iv) can be discarded in the context of 4.3: the numbers of 4.3.4.2 iv) obtained in the case of the representation $G_{\mathbb{Q}_p} \hookrightarrow GL(H^1[\frac{1}{p}])$ are all 2 (to be compared with [Va1, 5.7.1]). However the conditions i) to vi) of 4.3.4.2 are such that they handle any arbitrary situation. We have:

Proposition. *Let D be an integral Dedekind ring of characteristic zero having an infinite number of maximal ideals. Let $K(D)$ be its field of fractions. We assume that for any $N \in \mathbb{N}$, there is only a finite number of prime ideals of D of characteristic smaller than N. Let M_D be a projective D-module of finite rank, and let $G_{K(D)}$ be a reductive subgroup of $GL(M_D \otimes K(D))$. Then there is a non-zero ideal I_D of D, and there is a family of tensors $(v_\alpha)_{\alpha\in\mathcal{J}_D}$ of $\mathcal{T}(M_D \otimes K(D))$ fixed by $G_{K(D)}$, such that for any maximal ideal m of D not containing I_D, it is enveloped by $M_D \otimes D_{(m)}$ and is strongly $M_D \otimes D_{(m)}$-representation well positioned for the group $G_{K(D)}$. Moreover, the ideal I_D does not depend on $G^{\mathrm{der}}_{K(D)}$: it depends only on the representation of the connected component Z^0 of the origin of $Z(G_{K(D)})$ on $M_D \otimes K(D)$.*

Proof: Localizing and passing to a Galois extension, we can assume that Z^0 is a split torus. For the definition of $N_2(A)$ in 4.3.4.2 it was irrelevant that $G_{\mathbb{Q}_p}$ was split. In the proof of 4.3.4.8 b) the only place where it was relevant that $G_{\mathbb{Q}_p}$ is split was that the representation of its derived group on $H^1[\frac{1}{p}]$ is absolutely irreducible. But using an argument similar to the one of 4.3.4.2.2, we deduce that instead of $D_{(m)}$ we can work as well with its strict henselization. So the proposition follows from 4.3.4.8 b) and 5.4 3).

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