Points of integral canonical models of Shimura varieties of preabelian type, p-divisible groups, and applications

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§1 *Introduction*

1.0. Recently we managed to prove the existence of integral canonical models of Shimura varieties of preabelian type with respect to primes $p \geq 3$ (cf. [Va2, 6.4.1 and the definitions of 3.2.6]; see [Va2, 6.8], 4.6.11 and [Va3] for the case of Shimura varieties of preabelian type which are not of abelian type, and see §6 and [Va5] for the case $p = 3$). For the general theory of such integral models we refer to [Va2, §3]. With this paper, which is a natural continuation of [Va2], we start an extensive programme of proving that many well-known results for integral canonical models of Siegel modular varieties remain true (under proper formulation) for integral canonical models of Shimura varieties of preabelian type. Here we are concerned with points of these models with values in regular formally smooth schemes over rings of Witt vectors of a given length (finite or infinite) of a perfect field. Special attention is paid to points with values in (the Witt ring of) a perfect field.

The results mentioned in [Va2, 1.6-7] are proved in this paper. For a faster publication of this paper, we divide it in three parts, to be published separately. The first part contains §1–4, the second part will contain §5–10, and the third part will contain §11–14. In this first part we concentrate (this was the initial goal) in reobtaining in a new manner:

- the theory (due to Serre–Tate; see [Me]) of ordinary abelian varieties (over perfect fields) and of their canonical lifts,
- the theory (due to Dwork; see [Ka1]) of canonical coordinates of the formal moduli scheme of deformations of an ordinary abelian variety over an algebraically closed field,

and in extending these theories to the context of integral canonical models of Shimura varieties of preabelian type, and even more generally to the context (see 1.1 below) of Shimura σ-crystals. On the way we obtain many other results pertaining to p-divisible groups, to abelian varieties, and to the above mentioned integral canonical models. In particular we obtain versal global deformations, i.e. versal deformations over the p-adic completion of pro-étale schemes over smooth affine schemes over (Witt rings of) perfect fields (of characteristic $p > 0$), of p-divisible groups over such fields, cf. 3.15. In 4.12.12, we show that in the case when we are over an algebraically closed field we can drop the words "p-adic completion" and "pro-étale".

1.1. In what follows k is a perfect field of positive characteristic p, $W(k)$ is the ring of (infinite) Witt vectors of k, and σ is the Frobenius automorphism of $W(k)$. We denote by $\bar{E}$ the algebraic closure of a field E. The proof of the basic result 5.1 of [Va2] leads to the study of σ-crystals endowed with extra structure (cf. [Va2, 5.6.5]). These are σ-crystals (M, φ) over k (so M is a free $W(k)$-module of finite rank, and φ is a σ-linear endomorphism of M producing a σ-linear automorphism of $M[\frac{1}{p}]$) such that:

i) There is a family $(v_\alpha)_{\alpha\in\mathcal{J}}$ of tensors in spaces of the form $M^{\otimes n}\otimes M^{*\otimes m}[\frac{1}{p}]$ (with M^* the dual of M, and with $n,m\in\mathbb{N}\cup\{0\}$), fixed by φ, and such that the closure G of the subgroup of $GL(M[\frac{1}{p}])$ fixing v_α, $\forall\alpha\in\mathcal{J}$, in $GL(M)$, is a quasi-split reductive group over $W(k)$;

ii) There is a cocharacter $\mu:\mathbb{G}_m\to G$ producing a direct sum decomposition $M=F^1\oplus F^0$ such that $\beta\in\mathbb{G}_m(W(k))$ acts through μ on F^i as the multiplication with β^{-i}, $i=\overline{0,1}$. Moreover F^1 is a proper direct summand of M;

iii) The triple (M,F^1,φ) is a filtered σ-crystal.

The resulting triple (M,φ,G) (resp. quadruple $(M,F^1,\varphi,G,)$) is called a Shimura σ-crystal (resp. a Shimura filtered σ-crystal), while the pair $(\mathrm{Lie}(G),\varphi)$ is called the Shimura Lie σ-crystal attached to the Shimura σ-crystal (M,φ,G) (we still denote by φ the resulting σ-linear automorphism of $\mathrm{Lie}(G)[\frac{1}{p}]$; here $\mathrm{Lie}(G)$ is the Lie algebra of G).

Fixing the group G, the family of tensors $(v_\alpha)_{\alpha\in\mathcal{J}}$, and the $G(W(k))$-conjugacy class of the cocharacter μ, a σ-linear endomorphism φ_1 of M corresponding to any other σ-crystal (M,φ_1) satisfying the above properties i) to iii), up to isomorphism, can be put in the form $\varphi_1=g\varphi$, with $g\in G(W(k))$.

1.2. The case $p=2$ is treated as asides. So, in all that follows, without a special reference we assume that $p>2$. Our basic results are grouped in 3.1 and 3.4. In 3.1 we present the first group of basic results:

Theorem 1. *a) Among all Shimura σ-crystals $(M,g\varphi,G)$ (with $g\in G(W(k))$), the ones which have the smallest Newton polygon (in the sense that all others have a Newton polygon strictly above it) are precisely the ones $(M,g_1\varphi,G)$ (called G-ordinary σ-crystals) for which there is a cocharacter $\mu:\mathbb{G}_m\to G$ as in ii) above, such that the parabolic Lie subalgebra $\mathfrak{p}_1$ of $\mathrm{Lie}(G)$ corresponding to non-negative slopes of the Shimura Lie σ-crystal $(\mathrm{Lie}(G),g_1\varphi)$ is contained in the parabolic Lie subalgebra of $\mathrm{Lie}(G)$ leaving invariant the F^1-filtration of M defined (as in ii) of 1.1) by μ.*

b) If $(M,g_1\varphi,G)$ is a G-ordinary σ-crystal, then there is a unique Shimura filtered σ-crystal $(M,F^1,g_1\varphi,G)$ (called the G-canonical lift of the G-ordinary σ-crystal $(M,g_1\varphi,G)$) such that $\mathfrak{p}_1$ leaves invariant F^1.

c) Among all Shimura σ-crystals $(M,g\varphi,G)$ (with $g\in G(W(k))$), the ones which have the smallest Newton polygon of their attached Shimura Lie σ-crystals $(\mathrm{Lie}(G),g\varphi)$, are precisely the G-ordinary σ-crystals.

d) Any Shimura σ-crystal $(M,g\varphi,G)$ is the specialization of a $G_{W(k_1)}$-ordinary σ_1-crystal (here σ_1 is the Frobenius automorphism of the Witt ring of a perfect field k_1 containing k; we can take k_1 to be the field of fractions of the perfection of $k[[X]]$, where X is an independent variable).

1.2.0. The proof of Theorem 1 is achieved by first (an idea suggested by G. Faltings) proving it for Shimura Lie σ-crystals $(\mathrm{Lie}(G),g\varphi)$. To achieve this and to complete the proof of Theorem 1 (cf. 3.4.7) we construct global deformations of such (as in iii) of 1.1) filtered σ-crystals; their goal is: to connect any Shimura σ-crystal $(M,g\varphi,G)$ (with $g\in G(W(k))$) with (the extension to $\bar{k}$ of) a G-ordinary σ-crystal $(M,g_1\varphi,G)$. Here by

global we mean: deformations over the p-adic completion of pro-étale schemes over smooth affine schemes over $W(k)$, whose special fibres are connected and have a Zariski dense set of $\bar{k}$-valued points. This takes care of d) of Theorem 1, as it can be checked easily starting from 3.5.11-12.

1.2.1. The proof of Theorem 1 for Shimura Lie σ-crystals $(\mathrm{Lie}(G), g\varphi)$ (with $g \in G(W(k))$), means three things:

1) there is an element $g_1 \in G(W(k))$ such that the parabolic Lie subalgebra $\mathfrak{p}_1$ of $\mathrm{Lie}(G)$ (defined in a) of Theorem 1) leaves invariant F^1 (cf. 3.2.3);
2) for any $g \in G(W(k))$, the Newton polygon of the Lie σ-crystal $(\mathrm{Lie}(G), g\varphi)$ is strictly above or equal to the Newton polygon of the Lie σ-crystal $(\mathrm{Lie}(G), g_1\varphi)$ (cf. 3.5.12);
3) if in 2) above we have equality of Newton polygons, then an explicit expression of g can be determined, and the σ-crystal $(M, g\varphi)$ has the same Newton polygon as the σ-crystal $(M, g_1\varphi)$ (cf. 3.5.11).

Part 1) is easy (as G is a quasi-split reductive group), while 3) is obtained on the way of proving 2) (cf. 3.5.8-11 and 3.7.5). From these facts and 1.2.0 we get easily a), b) and c) of Theorem 1.

For the proof of 2) we first show that for any $g \in G(W(k))$, the Newton polygon of $(\mathrm{Lie}(G), g\varphi)$ can not be below the Newton polygon of $(\mathrm{Lie}(G), g_1\varphi)$ (cf. 3.5.12). This is achieved (cf. the proof of 3.5.6 needed to get 3.5.8, and so needed for geting 3.5.12) by making use of the Lie stable p-rank of a Shimura σ-crystal (M, φ, G) (it is defined in 3.9.1; see also 3.5.5), via Theorem 2 below. The proof of 3.5.6 is carried out in 3.6. It involves standard techniques pertaining to parabolic subgroups of $G_{\bar{k}}$. To conclude the proof of 2) we use the above mentioned global deformations.

1.2.1.1. For a simpler way of proving Theorem 1, see 3.5.14.

1.2.2. We have (cf. 3.9.2) the following important supplement of the first group of basic results:

Theorem 2. *A Shimura σ-crystal (M, φ, G) is a G-ordinary σ-crystal iff for any $g \in G(W(k))$, the Lie stable p-rank of $(M, g\varphi, G)$ is smaller than or equal to the Lie stable p-rank of (M, φ, G).*

In particular any Shimura σ-crystal $(M, g\varphi, G)$ having the same Lie stable p-rank as a G-ordinary σ-crystal $(M, g_1\varphi, G)$, is a G-ordinary σ-crystal. Theorem 2 represents the generalization of the characterization of ordinary σ-crystals in terms of the Hasse-Witt matrices: we get Shimura-Hasse-Witt matrices (cf. 4.3.7 and 4.3.10). The essence of its proof is 3.6.

1.2.3. In 2.2 we present the basic facts pertaining to Shimura crystals. The notion of Shimura p-divisible groups (these are p-divisible groups endowed with crystalline (de Rham) cycles and satisfying some axioms inspired from the ones satisfied by Shimura crystals) is introduced in 2.2.20. The first group of basic results (and their proves) is presented in 3.1-7, while 3.8 shows that Theorem 1 remains true if G is just a reductive group (i.e. it is not necessarily quasi-split). The construction of the global deformations mentioned in 1.2.0, is carried out in 3.4, being part of the second group of basic results.

In section 3.4 we start the presentation of ten fundamental principles of the crystalline theory. These principles are:

– the unicity principle (cf. 3.4.1.2, 3.4.18.8 and 3.4.18.8.1);

– the ∇ principle (cf. 3.4.1.3, 3.4.18.4 and 3.4.18.6 a));

– the deformation principle (to be described in 1.3.2) (cf. 3.4.14 and 3.4.14.1);

– the (very) (weak) gluing principle (cf. 3.4.2.2 and 3.4.19);

– the surjectivity principle (cf. 3.4.8.1.2 a), 3.4.8.2, 3.4.1.3 5) and 3.4.18.6 b));

– the integrability principle (cf. 3.4.18.4.1);

– the moduli principle (cf. 3.4.18.4.2 and 3.4.1.3);

– the invariance principle (cf. 4.13, §9-10 and Theorem 11 below);

– the slice principle (cf. §7);

– the integral Manin problem (cf. 1.14 and 4.12).

Seven other useful properties (the constancy property, the constructibility property, the inducing property, the liftability property, the touching property, the purity property and the rigidity property) are presented respectively in 3.4.8.9, 3.4.18.4.5, 3.4.18.5 (see also its abstract form 3.4.8.1.2 b)), 3.4.18.5.2, 3.4.18.4.3, 3.4.8.1.4 and §7. 3.4.18.4-6 can be extended to a relative situation (i.e. to a situation where a smooth group is involved, as in the context of 1.1): see 3.4.18.7. It would be too long to fully describe these principles (or properties) here. So we concentrate in outlying the main features.

1.2.4. For the definition of the Fontaine's categories $\mathcal{MF}_{[0,1]}(X)$ and $\mathcal{MF}^{\nabla}_{[0,1]}(X)$, of their p-divisible objects, and how such objects are pulled back under pro-étale (affine) morphisms see 2.1, 2.2.1 c), and respectively 3.4.1.1.1. Here X stands for an affine (in some cases even for a non-affine) regular formally smooth scheme over $W(k)$, endowed with an arbitrary (but fixed) Frobenius lift compatible with σ.

Let $U = \mathrm{Spec}(R)$ be an open subscheme of G such that the identity element a_0 of $G(W(k))$ belongs to $U(W(k))$. We assume the existence of an étale morphism $U \to \mathrm{Spec}(W(k)[x_1, ..., x_{d(G)}])$, where $d(G)$ is the relative dimension of G, such that a_0 factors through the closed subscheme of U defined by $x_i = 0$, $i = \overline{1, d(G)}$. Let Φ_R be the Frobenius of the p-adic completion $U^\wedge := \mathrm{Spec}(R^\wedge)$ of U which at the level of rings takes x_i into x_i^p, $\forall i \in \{1, ..., d(G)\}$, and is compatible with the Frobenius σ of $W(k)$ (for the case of a general type of Frobenius Φ_R see 3.4.18.8.3). Let M_R be the p-divisible object of $\mathcal{MF}_{[0,1]}(U)$ defined by the triple

$$(M \otimes R^\wedge, F^1 \otimes R^\wedge, \Phi_{M_U}).$$

Here Φ_{M_U} is the Φ_R-linear (Frobenius) endomorphism of $M \otimes R^\wedge$ which takes an element $m \in M$ into $u(\varphi(m))$, with $u \in G(R^\wedge)$ as the universal element naturally defined by the inclusion $U \hookrightarrow G$. So $\Phi_{M_U}(v_\alpha) = v_\alpha$, $\forall \alpha \in \mathcal{J}$. In their simplified form, the ∇, the moduli, and the surjectivity principles for Shimura σ-crystals can be formulated (cf. 3.4.1.3):

Theorem 3. *There is a pro-étale affine morphism $\ell : U_1 = \mathrm{Spec}(R_1) \to U$, with the special fibre of U_1 a geometrically connected scheme over k, such that:*

a) *The pull back M_{U_1} of M_U to the p-adic completion of U_1, is a Shimura p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(U_1)$. In other words there is a unique connection*

$$\nabla : M \otimes R_1^{\wedge} \to M \otimes_{W(k)} \Omega_{R/W(k)} \otimes_R R_1^{\wedge}$$

for which the Frobenius $\Phi_{M_{U_1}}$ of $M \otimes R_1^{\wedge}$ (defined by Φ_{M_U}; the Frobenius of $R_1^{\wedge}$ being the one naturally induced from Φ_R through ℓ) is ∇-parallel (in the usual sense; see [Fa2, ch. 7]). ∇ is integrable and nilpotent mod p. Moreover it respects the G-action;

b) *There is a closed subscheme $\mathcal{B}$ of the special fibre U_k of U not containing the identity element of G_k, with the property that the fibres of ℓ above points of $U_k \setminus \mathcal{B}$ are non-empty.*

From Theorem 3 and [Fa1, 7.1] we deduce the existence of a p-divisible group over $\mathrm{Spec}(R_1^{\wedge})$ corresponding to this p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(U_1)$. As ∇ annihilates all v_α, it gets a natural structure of a Shimura p-divisible group.

In their full form, the above principles for Shimura σ-crystals describe how we can choose ℓ to get an universal property and what extra properties it enjoys (subject to this special choice).

1.2.5. Let X be as in 1.2.4. The ∇ principle for an arbitrary object $\mathcal{C}$ of $\mathcal{MF}_{[0,1]}(X)$ shows the existence, in the étale topology of X (i.e. when we pass to a surjective étale affine morphism $X' \to X$), of connections satisfying the equations (see 3.4.1.1.1) needed for $\mathcal{C}$ to be an object of $\mathcal{MF}^{\nabla}_{[0,1]}(X')$. It also computes the number $\nabla(\mathcal{C})$ of such connections (in the case when the special fibre of X is a local strictly henselian scheme), cf. 3.4.18.4.4, and expresses upper bounds for this number (in the general case), cf. 3.4.18.9. The ∇ principle for a p-divisible object $\mathcal{C}$ of $\mathcal{MF}_{[0,1]}(X)$ shows the existence of such connections in the pro-étale topology of X (i.e. when we pass to a surjective pro-étale affine morphism $X' \to X$), and asserts the existence of a p-divisible group over the p-adic completion of X' whose filtered F-crystal, viewed just as a p-divisible object of $\mathcal{MF}_{[0,1]}(X')$, is the pull back of $\mathcal{C}$ to X'. It also describes how such an X-scheme X' can be obtained: the special fibre of X' is obtained from the special fibre of X by using quasi Artin-Schreier systems, while X' is obtained by lifting what we get mod p. These systems (cf. terminology of 3.4.18.4.6 A) are systems of n equations in n variables x_1,..., x_n over an algebra $R(p)$ over the field $\mathbb{F}_p$ with p elements, of the form

$$x_i = L_i(x_1^p, ..., x_n^p),$$

$i \in \{1, ..., n\}$, where L_i are linear forms (not necessarily homogeneous) in n variables with coefficients in $R(p)$. Here $n \in \mathbb{N}$.

The unicity principle tell us when the above number $\nabla(\mathcal{C})$ is 1. There are two forms of it: one in terms only of the Frobenius of X, and another one in terms only of the Frobenius of (the module underlying) $\mathcal{C}$. The integrability principle asserts that all the connections, whose existence is asserted by the ∇ principle, are integrable. The ideas involved in obtaining it can be easily grasped from 3.4.18.4.1. These two principles together with the inducing property of 3.4.18.5 form a natural extension of [Fa2, th. 10].

The moduli principle shows the existence of moduli schemes of connections (satisfying the equations mentioned above) for an arbitrary (p-divisible) object $\mathcal{C}$ of $\mathcal{MF}_{[0,1]}(X)$. The surjectivity principle asserts that, in the case of a p-divisible object, any connected component of the special fibre of a resulting moduli scheme, maps into a dense constructible subset of a connected component of the special fibre of X. The gluing principle has many variants (cf. 3.4.19). One such variant shows that in many cases two (Shimura) filtered F-crystals over the special fibre of a smooth scheme Y over $W(k)$, under some conditions which make them to look similar, are isomorphic over the special fibre of Y_1, where $Y_1 \to Y$ is a pro-étale affine morphism, whose imagine contains an open dense subscheme of the special fibre of Y.

1.2.5.1. The guiding main new idea for obtaining these principles: we work inductively mod p^n, $n \in \mathbb{N}$ (even in the case of a regular ring of formal power series over $W(k)$). What we gain by doing this: the systems of equations we get are of a very simple and practical form (see 3.4.8; in particular see the simple independent lemma 3.4.8.1, whose proof is based on intersection theory). We use these systems do define Artin-Schreier systems and Artin-Schreier fundamental groups of $\mathbb{F}_p$-algebras (see 3.4.18.4.6 B). See 3.4.18.4.6 C for the fundamental lemma of the π_1-theory in positive characteristic. The idea behind the surjectivity principle is the constancy property: the quasi Artin-Schreier systems we get working with different powers of p are very close (in shape) one to another, i.e. they differ, cf. 3.4.8.9, only by adding some constants (so the abstract surjectivity principle, cf. 3.4.1.8.2 a), applies). For an application to difference varieties see 3.4.8.1.1.

1.2.6. In 3.9.4-5 we present the notion of the refined Lie stable p-rank of a Shimura σ-crystal, and a theorem for it similar to the Theorem 2 above. In 3.10 we introduce the terminology (and notion) of cyclic Shimura Lie σ-crystals to be used in the future, as well as a formulae for them. In 3.12 we show that for many Shimura filtered F-crystals $(M, F^1, \varphi, G, H, \tilde{f})$ (defined in 2.2.10), the Shimura σ_ν-crystal over the algebraic closure ν of the field of fractions of the completion of the special fibre of H in its origin, is a $G_{W(\nu)}$-ordinary σ_ν-crystal (here σ_ν is the Frobenius automorphism of $W(\nu)$). Our hopes for a good (understandable and useful) theory of lifts of Shimura σ-crystals (M, φ, G) which are not G-ordinary (or cyclic diagonalizable; see def. b) of 2.2.1 and 2.2.22) are gathered in 3.13: we define different deviations of Shimura (filtered) (Lie) σ-crystals. The case $p = 2$ is treated in 3.14. Some conclusions, obtained by combining the basic results of 3.1 and 3.4, are included in 3.15 (see also 3.1.8). These conclusions are refined in the case of an algebraically closed field in 4.12.12.

1.2.7. The special features of the case $k = \bar{k}$ are presented in 3.11. Using the definitions 2.2.1 b) and 2.2.12, we have (cf. rm. 5-6) of 2.2.9):

Theorem 4. *If $k = \bar{k}$, then all G-canonical lifts $(M, F^1, g\varphi, G)$ ($g \in G(W(k))$) are all isomorphic and cyclic diagonalizable, and their Shimura filtered Lie σ-crystals are cyclic diagonalizable and of Borel and parabolic type.*

In particular there is a basis $\{e_i | i \in B(M)\}$ of M, and a permutation π of the set $B(M)$, such that $g\varphi(e_i) = p^{\varepsilon_i} e_{\pi(i)}$, where ε_i is 1 or 0 depending on the fact that e_i is or is not in F^1. The proof of Theorem 4 is a simple consequence of 1.2.1 3).

1.3. For a better presentation of the applications (see §4) of the basic results, we introduce in 2.3 the standard Hodge situation.

We start with an injective map $f : \mathrm{Sh}(G,X) \hookrightarrow \mathrm{Sh}\big(\mathrm{GSp}(W,\psi),S\big)$ of Shimura pairs. Here the pair $(\mathrm{GSp}(W,\psi),S)$ defines a Siegel modular variety, cf. [Va1, 2.5]. Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ with respect to the rational prime $p \geq 3$. We assume the existence of a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W which is crystalline well positioned for the map f (cf. def. 2.3.4). Let $K_p := \{g \in \mathrm{GSp}(W,\psi)(\mathbb{Q}_p) \mid g(L_{(p)} \otimes \mathbb{Z}_p) = L_{(p)} \otimes \mathbb{Z}_p\}$. The hypothesis on $L_{(p)}$ implies that $L_{(p)}$ is a good $\mathbb{Z}_{(p)}$-lattice with respect to f. This means that ψ induces a perfect form $L_{(p)} \otimes L_{(p)} \to \mathbb{Z}_{(p)}$ (i.e. the induced map from $L_{(p)}$ into its dual $L^*_{(p)}$ is an isomorphism), and that the closure $G_{\mathbb{Z}_{(p)}}$ of G in $\mathrm{GSp}(L_{(p)},\psi)$ is a reductive group over $\mathbb{Z}_{(p)}$. So K_p is a hyperspecial subgroup of $\mathrm{GSp}(W,\psi)(\mathbb{Q}_p)$ and the intersection $H := \mathrm{GSp}(W,\psi)(\mathbb{Q}_p) \cap K_p$ is a hyperspecial subgroup of $G(\mathbb{Q}_p)$.

We choose a $\mathbb{Z}$-lattice $L_{\mathbb{Z}}$ of W such that ψ induces a perfect form $\psi : L_{\mathbb{Z}} \otimes L_{\mathbb{Z}} \to \mathbb{Z}$ and $L_{(p)} = L_{\mathbb{Z}} \otimes \mathbb{Z}_{(p)}$. Let $(v_\alpha)_{\alpha\in\mathcal{J}}$ be a family of tensors in spaces of the form $(W \oplus W^*)^n$, $n \in \mathbb{N}$, such that G is the subgroup of $\mathrm{GSp}(W,\psi)$ fixing the tensors of this family. The choice of the lattice $L_{\mathbb{Z}}$ and of the family $(v_\alpha)_{\alpha\in\mathcal{J}}$ allows the interpretation of $\mathrm{Sh}(G,X)(\mathbb{C})$ as the set of isomorphism classes of principally polarized abelian varieties over $\mathbb{C}$ of dimension e (with $2e = \dim_{\mathbb{Q}}(W)$), carrying a family of Hodge cycles $(w_\alpha)_{\alpha\in\mathcal{J}}$, and satisfying some additional conditions and level structures (cf. [Va2, 4.1]). Let v be a prime of the reflex field $E(G,X)$ dividing p and let $O_{(v)}$ be the localization of the ring of integers of $E(G,X)$ with respect to v.

The triple $\big(f, L_{(p)}, v\big)$ is called a standard Hodge situation.

It is known that the scheme $\mathcal{M}$ over $\mathbb{Z}_{(p)}$ parametrizing isomorphism classes of principally polarized abelian schemes of dimension e over $\mathbb{Z}_{(p)}$-schemes, having level-N symplectic similitude structure for any $N \in \mathbb{N}$ relatively prime to p, together with the canonical action of $\mathrm{GSp}(W,\psi)(\mathbb{A}_f^p)$ on it (via L), is an integral canonical model of $\mathrm{Sh}\big(\mathrm{GSp}(W,\psi),S\big)/K_p$ (see [Va2, 3.2.9]).

The normalization $\mathcal{N}$ of the closure of $\mathrm{Sh}(G,X)/H$ in $\mathcal{M}_{O_{(v)}}$ is (cf. 2.3.3) an integral canonical model of $\mathrm{Sh}(G,X)/H$ (or of the Shimura quadruple (G,X,H,v); see [Va2, 3.2.6] for the definition of these quadruples). Let $k(v)$ be the residue field of v and let $\mathcal{N}_{k(v)}$ be the special fibre of $\mathcal{N}$. The universal principally polarized abelian scheme over $\mathcal{M}$ gives birth, by pull back, to a principally polarized abelian scheme $(\mathcal{A},\mathcal{P}_{\mathcal{A}})$ over $\mathcal{N}$, which we call special.

1.3.1. The crystalline well positioned property of $L_{(p)}$ for the map f also says: any point $z \in \mathcal{N}(W(k))$ gives birth to a principally polarized abelian scheme (A,p_A) over $W(k)$ (obtained from $(\mathcal{A},\mathcal{P}_{\mathcal{A}})$ by pull back) having a family $(w_\alpha)_{\alpha\in\mathcal{J}}$ of Hodge cycles such that the filtered σ-crystal (M,F^1,φ) (with $M := H^1_{\mathrm{crys}}(A/W(k))$, with φ as the σ-linear endomorphism of M, and with F^1 as the Hodge filtration of M defined by A) gets a natural structure of a Shimura filtered σ-crystal $(M,F^1,\varphi,G_{W(k)})$. In other words:

i) The subgroup of $GL(M)$ obtained by taking the closure of the subgroup of $GL\big(M[\frac{1}{p}]\big)$ fixing the perfect alternating form $p_A : M \otimes M \to W(k)(1)$ (induced by the polarization p_A of the abelian variety A), and the family $(t_\alpha)_{\alpha\in\mathcal{J}}$ of (crystalline) de Rham

components of $(w_\alpha)_{\alpha\in\mathcal{J}}$ (these are tensors in the tensor algebra of $(M\oplus M^*)[\frac{1}{p}]$), is a reductive group $\tilde{G}_{W(k)}$ over $W(k)$.

ii) There is a cocharacter $\mu:\mathbb{G}_m\to\tilde{G}_{W(k)}$ producing a direct sum decomposition $M=F^1\oplus F^0$, with $\beta\in\mathbb{G}_m(W(k))$ acting through μ on F^i as the multiplication with β^{-i}, $i=\overline{0,1}$. Moreover F^1 is a proper direct summand of M.

There is a finite nilpotent Galois cover $\mathrm{Spec}(W(k_1))$ of $\mathrm{Spec}(W(k))$ of index of nilpotency at most two and of a very particular type (see 2.3.9 b) and d)) such that $\tilde{G}_{W(k_1)}$ is isomorphic to $G_{W(k_1)}$. Moreover, for many standard Hodge situations we can show (see 2.3.9 c)) that it is in fact isomorphic to $G_{W(k)}$. So, for the sake of convenience, we make the convention (see 2.3.9.2) that, without any extra reference, we consider from now on only points $\mathrm{Spec}(k)\to\mathcal{N}$ with the property that for any lift of it $\mathrm{Spec}(W(k))\to\mathcal{N}$, the group we get, as in i) above, is $G_{W(k)}$.

Using this convention, the triple $(M,\varphi,G_{W(k)})$ (resp. the quadruple $(M,F^1,\varphi,G_{W(k)})$) is called the Shimura σ-crystal (resp. the Shimura filtered σ-crystal) attached to the point $y\in\mathcal{N}(k)$ defined by z (resp. attached to z). When we want to emphasize the principal polarization we speak also about the principally quasi-polarized Shimura (resp. Shimura filtered) σ-crystal attached to y (resp. to z). Notation: $(M,\varphi,G_{W(k)},p_A)$ (resp. $(M,F^1,\varphi,G_{W(k)},p_A)$).

From this we get easily that for any regular formally smooth scheme $\tilde{U}$ over $W(k)$, and for every morphism $\tilde{U}\to\mathcal{N}$, we get naturally a principally quasi-polarized Shimura p-divisible group over $\tilde{U}$, cf. 3.2.11.

1.3.2. The deformation principle in its essence says:

Theorem 5. *We assume that a starting assumption (see SA1-2 of 3.4.14) is satisfied. For any point $y:\mathrm{Spec}(k)\to\mathcal{N}$, there is a smooth morphism $\mathcal{N}(y)\to\mathcal{N}$ to which y lifts, and there is a pro-étale morphism $\mathcal{N}(y)^1\to\mathcal{N}(y)$, surjective mod p, with the special fibre of $\mathcal{N}(y)^1$ a geometrically connected scheme over k, and such that the principally quasi-polarized Shimura p-divisible group over the p-adic completion of $\mathcal{N}(y)^1$, obtained by pull back, has a very nice (down to earth) expression. What we mean by this: it is obtained by pulling back (under the p-adic completion of a pro-étale morphism) the principally quasi-polarized Shimura p-divisible group over $U_1^\wedge$ defined by (cf. [Fa1, 7.1]) a principally quasi-polarized Shimura p-divisible object M_{U_1}, produced as in Theorem 3, but starting from the principally quasi-polarized Shimura σ-crystal attached to y (so here U is an open subscheme of the maximal connected subgroup of $G_{W(k)}$ fixing the bilinear form p_A). Moreover y lifts to a morphism $\mathrm{Spec}(k)\to\mathcal{N}(y)^1$.*

We get this principle by combining standard arguments with our strategy of working inductively modulo powers of p. The starting assumption refers to the fact that, working mod p, the variant of Theorem 5, obtained by working with the henselization of the local ring of the k-valued point of $\mathcal{N}_k$ defined by y, is apriori satisfied. In practice is hard to check if a starting condition is satisfied or not. However the proof of Theorem 5 involves some very important techniques, and its variants modulo powers of p (see 3.4.14.4), involving no assumption, are also very useful. Moreover, the fact that a starting condition is satisfied or not, is at the root of many important problems (see §5-8).

Of course there are variants of this principle: we work with a point $y_{H_0} : \mathrm{Spec}(k) \to \mathcal{N}/H_0$ (here H_0 is a compact open subgroup of $G(\mathbb{A}_f^p)$ such that the quotient scheme $\mathcal{N}/H_0$ is smooth over $O_{(v)}$ and $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ descends to $\mathcal{N}/H_0$), or with an étale morphism $\mathcal{N}(y) \to \mathcal{N}$, cf. 3.4.14 and 3.4.14.1-2.

1.4. As a straightforward consequence of Theorem 1, we are able to extend and reobtain the well known results (pertaining to integral canonical models of Siegel modular varieties and universal principally polarized abelian schemes over them) concerning the existence of an ordinary type and the existence of the canonical lift of a (principally polarized) abelian variety of ordinary type, to any integral canonical model $\mathcal{N}$ of a Shimura variety $\mathrm{Sh}(G, X)$ of Hodge type and to any special principally polarized abelian scheme $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ over $\mathcal{N}$, arising (as in 1.3) from a standard Hodge situation $(f, L_{(p)}, v)$. With the notations of 1.3, we get three things.

a) First we get a G-ordinary type (cf. 4.1). It depends only on f and v, but not on $L_{(p)}$. It is the formal isogeny type of abelian varieties (obtained from $\mathcal{A}$ by pull back) over geometric points of a Zariski dense open subscheme of $\mathcal{N}_{k(v)}$. It can be described as follows.

The isomorphism $L_{(p)} \tilde{\to} L_{(p)}^*$ induced by ψ allows us to identify $G_{\mathbb{Z}_{(p)}}$ with a subgroup of $GL(L_{(p)}^*)$. Let $L_p^* := L_{(p)}^* \otimes \mathbb{Z}_p$; so $G_{\mathbb{Z}_p}$ is naturally a subgroup of $GL(L_p^*)$. Let T be a maximal torus of $G_{\mathbb{Z}_p}$ such that:

1) there is a cocharacter $\mu : \mathbb{G}_m \to T_{W(k(v))}$ which over an embedding of $W(k(v))$ in $\mathbb{C}$ (extending the inclusion $E(G, X) \subset \mathbb{C}$) is $G(\mathbb{C})$-conjugate to the cocharacters $\mu_x^* : \mathbb{G}_m \to G_{\mathbb{C}}$, $x \in X$, which are the cocharacters μ_x mentioned in [Va2, 2.2], but in the dual context (so they are the cocharacters of $G_{\mathbb{C}}$ defining the Hodge $\mathbb{Q}$–structures on W^* associated to points $x \in X$);
2) there is a Borel subgroup B of $G_{\mathbb{Z}_p}$ whose Lie algebra leaves invariant the F^1-filtration of $L_p^* \otimes W(k(v))$ defined by μ, i.e. $\mathrm{Lie}(B) \subset F^0(\mathrm{Lie}(G_{W(k(v))}))$.

We recall that μ gives birth to a direct sum decomposition $L_p^* \otimes W(k(v)) = F^1 \oplus F^0$, and that $\beta \in \mathbb{G}_m(W(k(v)))$ acts through μ on F^i as the multiplication with β^{-i}, $i = \overline{0,1}$.

The G-ordinary type we get is the formal isogeny type associated to the σ-crystal

$$(L_p^* \otimes W(k(v)), \varphi)$$

defined by $\varphi := \sigma \circ \mu(\frac{1}{p})$. Here σ acts as identity on L_p^* and as Frobenius on $W((k(v))$, and $\frac{1}{p} \in \mathbb{G}_m\big(W\big(k(v)\big[\frac{1}{p}\big]\big)\big)$.

b) Second we get G-ordinary points of $\mathcal{N}_{k(v)}$ (cf. 4.2). These are the points, with values in a field, with the property that the abelian varieties over them, obtained from $\mathcal{A}$ by pull back, have the above G-ordinary type as their formal isogeny type.

c) Third we get G-canonical lifts of G-ordinary points of $\mathcal{N}_{k(v)}$ with values in perfect fields (cf. 4.4). These G-canonical lifts are points of $\mathcal{N}$ with values in Witt rings over perfect fields.

The G-ordinary type we obtain is a usual ordinary type (i.e. it is the formal isogeny type $e(1,0)+e(0,1)$) if and only if $k(v)$ is $\mathbb{F}_p$ (cf. the property P1 of 4.6). If $k(v)=\mathbb{F}_p$ then the abelian variety over $W(k)$ obtained from $\mathcal{A}$ by pull back through a G-canonical lift of a G-ordinary point (with values in the perfect field k) of $\mathcal{N}_{k(v)}$, is the canonical lift of an ordinary abelian variety (cf. the property P2 of 4.6).

1.5. To any point $y:\mathrm{Spec}(k)\to\mathcal{N}_{k(v)}$ we attach (cf. 2.3.10) a Shimura Lie σ-crystal $(\mathfrak{g},\varphi)$, with $\mathfrak{g}$ the Lie algebra of $G_{W(k)}$ (cf. 1.3.1). Any lift $z:\mathrm{Spec}(W(k))\to\mathcal{N}$ of y, makes $(\mathfrak{g},\varphi)$ to be a filtered Lie σ-crystal, i.e. it makes it to be a p-divisible object of $\mathcal{MF}_{[-1,1]}(W(k))$. This means that we get a filtration $0=F^2(\mathfrak{g})\subset F^1(\mathfrak{g})\subset F^0(\mathfrak{g})\subset F^{-1}(\mathfrak{g})=\mathfrak{g}$ such that $\varphi\big(\frac{1}{p}F^1(\mathfrak{g})+F^0(\mathfrak{g})+p\mathfrak{g}\big)=\mathfrak{g}$. $F^0(\mathfrak{g})$ is a parabolic Lie subalgebra of $\mathfrak{g}$ and $F^1(\mathfrak{g})$ is its nilpotent radical. b) of Theorem 1 implies that the point y is a G-ordinary point iff there is a lift z of it to $W(k)$ which makes the Lie σ-subcrystal of $(\mathfrak{g},\varphi)$ corresponding to non-negative slopes to be a p-divisible object of $\mathcal{MF}_{[0,1]}(W(k))$. Such a lift z, when exists, is unique and defines (via 2.3.17) the G-canonical lift of y.

The G-ordinary points of $\mathcal{N}_{k(v)}$ and the G-canonical lifts of the G-ordinary points have all the usual properties of ordinary points (and of their canonical lifts) of an integral canonical model of a Siegel modular variety with respect to p, which can be read out from the Shimura Lie σ-crystals attached to points (with values in perfect fields). As examples of this philosophy see 1.7-9. Of course these properties have to be reformulated accordingly.

1.6. These attached Shimura Lie σ-crystals allow us (cf. 4.5) to achieve a stratification (called the canonical Lie stratification) of $\mathcal{N}_{k(v)}$ into $G(\mathbb{A}_f^p)$-invariant locally closed subschemes, indexed by Newton polygons of the attached Shimura Lie σ-crystals $(\mathfrak{g},\varphi)$. It is similar (in nature and properties) to the well-known stratification of the special fibres of integral canonical models of Siegel modular varieties.

For any representation $\rho:G_{\mathbb{Z}_p}\to GL(N)$, with N a free $\mathbb{Z}_p$ of finite rank, we define (cf. 4.5.4) a ρ-stratification of $\mathcal{N}_{k(v)}$ in $G(\mathbb{A}_f^p)$-invariant locally closed subschemes. The intersection of all these ρ-stratifications defines the absolute stratification. This absolute stratification is the most refined stratification which can be obtained naturally (i.e. in an adequate Shimura context, involving representations of $G_{\mathbb{Z}_p}$) with the help of Newton polygons.

We also introduce (4.5.2) the refined canonical Lie stratification and show that it coincides with the absolute stratification.

The G-ordinary points of $\mathcal{N}_{k(v)}$ are points of the generic, Zariski dense open stratum (it does not matter which stratification we use, cf. 4.5.6 8)). In the case of the special fibre of an integral canonical model of a Siegel modular variety, all non-trivial stratifications (i.e. stratifications which do not consist just from one stratum) introduced above coincide with the usual Newton polygon stratification; but this is not true for an arbitrary Shimura variety of Hodge type (see 4.5.4-6).

1.7. In 4.3 we restate 1.2.2 in the case of a standard Hodge situation.

1.8. In 4.6 we include some examples to illustrate the general theory presented in 4.1-5 and to point out the complexities which are arising. We also list the main properties

of the general theory. As a simple application of 1.4 b) we present (cf. 4.6.9) a general principle concerning the non-ordinary reduction of an abelian variety A over a number field E, with respect to primes of E:

Theorem 6 (the non-ordinary criterion). *If the reflex field of the Shimura variety defined (see [Va2, 2.12 3)]) by the Mumford–Tate group of A, is not $\mathbb{Q}$, then A has a non-ordinary reduction with respect to an infinite number of primes of E.*

4.6.9.4 says very precisely how "to cook" such primes.

1.9. We consider a standard Hodge situation $\big(f, L_{(p)}, v\big)$ as in 1.3. Let v^{ad} be the prime of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ divided by v. In 1.9-10 we assume that $k = \bar{k}$. For any point $y : \mathrm{Spec}(k) \hookrightarrow \mathcal{N} \times_{O_{(v)}} W(k)$ defined by a G-ordinary point $y_0 \colon \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$, the principally quasi-polarized filtered F-crystal (defined by $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$) over the completion $k[[X_1, \ldots, X_d]]$ of the local ring of y in $\mathcal{N}_k$ (with $d := \dim \mathrm{Sh}(G, X)$), can be put in a very special form (cf. 4.7.2). Using this we get (see 4.7.30 6)):

Theorem 7. *The formal moduli scheme of G-deformations (i.e. deformations factoring through $\mathcal{N}$) of the principal polarized abelian variety (A_y, p_{A_y}) over k (obtained from $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ through the point y_0) over Artinian local $W(k)$-algebras, having k as their residue field, is naturally isomorphic to the formal multiplicative group $\mathbb{G}_m^{d\wedge}/W(k)$. Under such an isomorphism, the G-canonical lift becomes the origin.*

This is part of the general theory of crystalline coordinates for p-divisible groups over k, developed in 4.7.30. This is a generalization of the main result of [No1] (in the less general case of a standard Hodge situation) and of [De2, 2.1.3]. We obtain G-multiplicative (crystalline) coordinates. When $k(v^{\mathrm{ad}}) = \mathbb{F}_p$, the situation is very close to the one in [De2, 1.4] (cf. 4.7.5-29): we obtain G-canonical (crystalline) coordinates.

As a further generalization we have (cf. 4.7.33.1):

Proposition. *If A_k is an abelian variety over k of dimension d, and if it has a lift A (over $W(k)$) of quasi CM type (cf. def. 2.2.17), then the formal moduli scheme of deformations of A_k is naturally isomorphic to the formal torus $\mathbb{G}_m^{d^2\wedge}/W(k)$, with the abelian variety A corresponding to its origin.*

This is the most general context in which we can (presently) prove the existence of good (useful) crystalline (natural) coordinates. For variants of this Proposition, including the additive case, see 4.7.33.2 and 4.7.30 5).

1.10. In 4.8 we present the following Galois property of G-ordinary points of $\mathcal{N}_{k(v)}$; it answers a question of C. -L. Chai. We consider a standard Hodge situation $(f, L_{(p)}, v)$. With the notations of 1.3, we consider a G-ordinary point $\mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ and a lift of it $z : \mathrm{Spec}(V) \to \mathcal{N}$, with V a discrete valuation ring which is a finite flat extension of $W(k)$. Let $K := V\big[\frac{1}{p}\big]$. Let A_V be the abelian variety over V, obtained from $\mathcal{A}$ by pull back through the point z. We have:

Theorem 8. *There is a finite field extension K_1 of K such that the Galois representation $\rho : \mathrm{Gal}(\bar{K}/K_1) \to GL(H^1_{\acute{e}t}(A_{\bar{K}}, \mathbb{Z}_p))(\mathbb{Z}_p) = GL(L^*_{(p)} \otimes \mathbb{Z}_p)(\mathbb{Z}_p)$ factors through the group of $\mathbb{Z}_p$-valued points of a solvable subgroup of $G_{\mathbb{Z}_p}$.*

The identification $H^1_{\acute{e}t}(A_{\bar{K}}, \mathbb{Z}_p) = L^*_{(p)} \otimes \mathbb{Z}_p$ used here is not canonical, cf. [Va2, 5.1.2]. The proof of Theorem 8, as R. Pink has pointed out to us, is a direct consequence of 4.4.7 and of 4.7.2 (and so of Theorem 7). For variants of the above Galois property (including the case of a finite field) see 4.8.3 d) and e).

1.11. The phenomena of 1.4-6 are an intrinsic property of $\mathcal{N}$ (and of $\mathcal{N}_{k(v)}$), not depending on the special principally polarized abelian scheme $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ we got over $\mathcal{N}$ (only the notion of a G-ordinary type does depend on it), and so they can be extended to special fibres of integral canonical models of Shimura varieties of preabelian type. We have (cf. 4.9.8):

Theorem 9. *Let $\mathcal{N}^1_{k(v^1)}$ be the special fibre of an integral canonical model $\mathcal{N}^1$ of a Shimura variety $\mathrm{Sh}(G^1, X^1)$ of preabelian type with respect to a prime v^1 of $E(G^1, X^1)$ dividing a rational prime $p > 2$. Then $\mathcal{N}^1_{k(v^1)}$ has a refined canonical Lie stratification in $G^1(\mathbb{A}^p_f)$-invariant locally closed subschemes. It is functorial with respect to finite maps between Shimura quadruples; moreover in the case of a standard Hodge situation we regain the refined canonical Lie stratification mentioned in 1.6. The points (with values in fields) of the Zariski dense open stratum are called G^1-ordinary points of $\mathcal{N}^1_{k(v^1)}$, and any such point with values in k, has a uniquely defined G^1-canonical lift $\mathrm{Spec}(W(k)) \to \mathcal{N}^1$.*

In 4.9 we present a proof of this for primes $p \geq 5$ (for the case $p = 3$ see §6). The proof relies on the fact that for any Shimura quadruple (G_0, X_0, H_0, v_0) (with (G_0, X_0) defining an adjoint Shimura variety of abelian type, with v_0 a prime of $E(G_0, X_0)$ dividing a rational prime $p \geq 3$ such that G_0 is unramified over $\mathbb{Q}_p$, and with H_0 a hyperspecial subgroup of $G_0(\mathbb{Q}_p)$), there is a standard Hodge situation $(f^0, L^0_{(p)}, v^0)$, with f^0: $\mathrm{Sh}(G^0, X^0) \hookrightarrow \mathrm{Sh}(\mathrm{GSp}(W, \psi), S)$ an injective map such that $(G^{0\mathrm{ad}}, X^{0\mathrm{ad}}) = (G_0, X_0)$ and v^0 divides v_0 (for $p \geq 5$ cf. [Va2, 6.4.2] and 2.3.6; for $p = 3$ cf. §6).

1.12. We do expect (cf. 4.10.1) that the open stratum of $\mathcal{N}^1_{k(v^1)}$ is the smallest $G^1(\mathbb{A}^p_f)$-invariant open subscheme of $\mathcal{N}^1_{k(v^1)}$ (cf. also [Ch, p. 441]). This is proved for Siegel modular varieties in [Ch]. Using Theorem 9, we prove it for Shimura curves (cf. 4.10.3). For a more substantial approach to the general case see 4.10.5.

1.13. In 4.11 we prove (this is standard) the functorial behaviour of G-ordinary points and of G-canonical lifts of them. Let $f_0 : (G, X, H, v) \hookrightarrow (G_1, X_1, H_1, v_1)$ be an injective map between two Shimura quadruples of Hodge type, having integral canonical models $\mathcal{N}$, and respectively $\mathcal{N}_1$. We assume that v is relatively prime to 2. So we have a morphism (still denoted by) $f_0 : \mathcal{N} \to \mathcal{N}_1$ (cf. [Va2, rm. 4) of 3.2.7]). We also assume the existence of an injective map $f_1 : (G_1, X_1, H_1, v_1) \hookrightarrow \big(\mathrm{GSp}(W, \psi), S, \mathrm{GSp}(W, \psi)(L_{(p)} \otimes \mathbb{Z}_p), p\big)$ (with $L_{(p)}$ a $\mathbb{Z}_{(p)}$-lattice of W for which $\psi : L_{(p)} \otimes L_{(p)} \to \mathbb{Z}_{(p)}$ is a perfect form) such that $(f_1, L_{(p)}, v_1)$ and $(f_1 \circ f_0, L_{(p)}, v)$ are standard Hodge situations. So the closure of G (resp. G_1) in $\mathrm{GSp}(L_{(p)}, \psi)$ is a reductive groups $G_{\mathbb{Z}_{(p)}}$ (resp. $G_{1\mathbb{Z}_{(p)}}$). Let $k(v)$ and $k(v_1)$ be the residue field of v and respectively of v_1. We have:

Theorem 10. *A G-ordinary point of $\mathcal{N}_{k(v)}$ is mapped by f_0 into a G_1-ordinary point of $\mathcal{N}_{1k(v_1)}$, and a G-canonical lift of $\mathcal{N}$ is mapped by f_0 into a G_1-canonical lift of $\mathcal{N}_1$ iff $k(v) = k(v_1)$ and there is a torus T of $G_{\mathbb{Z}_p}$ such that:*

1) there is a cocharacter $\mu : \mathbb{G}_m \hookrightarrow T_{W(k(v))}$, which over an embedding of $W(k(v))$ in $\mathbb{C}$, extending the inclusion $E(G,X) \subset \mathbb{C}$, is $G(\mathbb{C})$-conjugate to the cocharacters μ_x^*, $x \in X$, defined as in 1.4 (see also 2.3.1).

2) there is a Borel subgroup B_1 of $G_{1\mathbb{Z}_p}$ containing a Borel subgroup of $G_{\mathbb{Z}_p}$ and whose Lie algebra leaves invariant the F^1-filtration of $L^*_{(p)} \otimes W(k(v))$ defined by μ (as in 1.4 a)).

1.14. In 4.12-13 we present two important applications inspired from the deformation principle of 3.4.14. First, in 4.12 we formulate the integral Manin problem for a standard Hodge situation, as well as for integral canonical models of Shimura varieties of preabelian type with respect to primes relatively prime to 2. For a standard Hodge situation $(f, L_{(p)}, v)$, with the notations of 1.3, the problem is to determine all the principally quasi-polarized Shimura σ-crystals attached to k-valued points of (a connected component of) $\mathcal{N}_k$; here $k = \bar{k}$. A solution (cf. §11; see 4.12 for some particular cases) to this problem is provided by the Milne's conjecture (see d) of 4.4.1 3) and 1.16) and by the completion property. This property is stated in 3.4.15 as an expectation, and is proved in §11.

For different local (resp. global) integral Manin problems see 4.12.14-19 (resp. 4.12.21).

Second, in 4.13 we present (partially; to be continued in §9-10, cf. 1.21) the invariance principle. Let $p \geq 3$ be a rational prime. Let (G_i, X_i, H_i, v_i), $i = \overline{1,2}$, be two Shimura quadruples of preabelian type, with v_i dividing p. Let $\mathcal{N}_i$ be the integral canonical model of (G_i, X_i, H_i, v_i), $i = \overline{1,2}$. Let $H_{0i} \subset G_i(\mathbb{A}_f^p)$ be a subgroup such that the subgroup $H_{0i} \times H_i$ of $G_i(\mathbb{A}_f)$ is smooth for the pair (G_i, X_i) (see def. [Va2, 2.11]), $i = \overline{1,2}$. Let k be a perfect field containing $k(v_1)$ and $k(v_2)$. Let $y_i : \mathrm{Spec}(k) \to \mathcal{N}_{ik}/H_{0i}$, $i = \overline{1,2}$, be points such that the non-trivial part of the Shimura adjoint Lie σ-crystal attached to y_1 is isomorphic to the non-trivial part of the Shimura adjoint Lie σ-crystal attached to y_2 (for definitions cf. 2.3.10 and 3.10.1). Let O_{y_i} be the local ring of y_i (in $\mathcal{N}_{iW(k)}$), and let $O_{y_i}^{h\wedge}$ be the p-adic completion of its henselization, $i = \overline{1,2}$. The refined canonical Lie stratification of $\mathcal{N}_{ik(v_i)}$ makes $\mathrm{Spec}(O(p)^h_{y_i}) := \mathrm{Spec}(O_{y_i}^{h\wedge}/pO_{y_i}^{h\wedge})$ to be a stratified scheme. One variant of the local form of the invariance principle is:

Theorem 11. *We assume $k = \bar{k}$. Let $n \in \mathbb{N}$. There is an isomorphism*

$$\rho : \mathrm{Spec}(O_{y_1}^{h\wedge}) \tilde{\to} \mathrm{Spec}(O_{y_2}^{h\wedge})$$

inducing an isomorphism

$$\rho(p) : \mathrm{Spec}(O(p)^h_{y_1}) \tilde{\to} \mathrm{Spec}(O(p)^h_{y_2})$$

of stratified schemes, and such that the pull back under ρ of the non-trivial part of the Shimura adjoint filtered Lie F-crystal over $\mathrm{Spec}(O(p)^h_{y_2})$ taken mod p^n, defined by the natural morphism $\mathrm{Spec}(O_{y_2}^{h\wedge}) \to \mathcal{N}_2$, is isomorphic to the non-trivial part of the Shimura adjoint filtered Lie F-crystal over $\mathrm{Spec}(O(p)^h_{y_1})$ taken mod p^n, defined by the natural morphism $\mathrm{Spec}(O_{y_1}^{h\wedge}) \to \mathcal{N}_1$ (see 4.9.20 and 4.13 for the definition of these Lie F-crystals).

For an application of what the completion property says in some partical cases in which G^{ad} is of A_n type, see 4.12.12.

1.15. Some afterthoughts are presented in 4.14 and in the appendix. For the case of a $p = 2$ standard Hodge situation introduced in 2.3.18, see 4.12.12.1 and 4.14.3. In the appendix we make the connexion and comparison with [RR]. Two proves of Theorem 1 are presented. The first one starts in 3.1 and ends in 3.5.14, via 3.6; it completely avoids the use of 3.4. The second one is just a more elaborated form of the first one, in the sense that it leads to a much more stronger form of d) of Theorem 1 (cf. 3.15). It starts in 3.1 and ends in 3.7.5; it uses from 3.4 just the part till 3.4.11 inclusive. As 3.4 grew much more than expected initially, we recommend that after the reading of 3.3, the reader should skip to 3.5-6, and only after that to proceed with 3.4, 3.7, 3.8, etc. Also for 4.1-8 and 4.11 which form a sequence, continuing 2.3, we need from 3.4 just 3.4.6: 3.4.6 is independent of the other part of 3.4 except of the first paragraph of 3.4.0. The parts 3.4.13-16 are needed just in 3.4.19, in 4.3.6 (as an exemplification) and in 4.12-13. They are independent of what follows after them in 3.5-15; so we do not hesitate to use for them results stated in 3.5-12, and so their reading should procede only after 3.12; we felt that it is appropriate to include them as part of 3.4, for having the deformation part of the paper presented fluently. The sequence 3.4.17-20 is unrelated to what follows after it; so it can be read at any time after 3.4.1-9, as well as it can be skipped. For the present work we were motivated by very concrete problems, imerging from the theory of Shimura varieties. Whenever we were aware from the very beginning that the things work out in a more general context, we worked in such general context; when not, we stated (or presented) the things in the more general context as remarks following the "simpler" context. We tried to present the ideas in the most natural order. But we did not hesitate to point out different connexions between them, which somehow were going against this order.

There are two facts stated (and referred) here whose proves are postponed to future papers: 4.2.8.1 and d) of 4.4.1 3); they are labeled with a $*$. Whenever we use them we present a second proof entirely independent of them: the only exception is the starred answer of 4.12.5, as it is easy to restate it in a way which does not rely on d) of 4.4.1 3)) (see the paragraph following it). Also we would like to point out that for the majority of cases, 4.4.7 is equally good as d) of 4.4.1 3). So the paper is self contained modulo the limitations of [Va2, 6.8.6] (they are considerable reduced here (cf. 4.6.11)), and the existence of integral canonical models of Shimura quadruples (G, X, H, v) of preabelian type with v dividing $p = 3$. So the reader, if desires, can everywhere we are dealing with a standard Hodge situation $(f, L_{(p)}, v)$ (resp. with a Shimura quadruple (G, X, H, v)), with the Lie factors of G of B_n, C_n or D_n Lie type and with v dividing 3 (cf. 2.3.5.1 of why we do not mention factors of A_n Lie type), or with factors of (G, X) of $D_n^{\mathbb{H}}$ of the type described in [Va2, 6.8.6] and not covered by 4.6.11, can add the condition $\mathcal{N}$ exists (resp. the condition that there is a standard Hodge situation $(f^1, L^1_{(p)}, v^1)$ such that $(G^{1\mathrm{ad}}, X^{1\mathrm{ad}}, H^{1\mathrm{ad}}, v^{1\mathrm{ad}}) = (G^{\mathrm{ad}}, X^{\mathrm{ad}}, H^{\mathrm{ad}}, v^{\mathrm{ad}})$, and in the case we are dealing with a case not covered by [Va2, §6] or 4.6.11, the condition that one of the conditions of [Va2, 6.8.2] is satisfied).

Due to the fact that the results presented in this first part, have been largely detailed in this first chapter, we do no stop at the beginning of the other chapters §2-4, to outline their contents.

1.16. We present now in a very brief manner the contents of the other two parts.

In §5 we elaborate on the notion of Shimura p-divisible groups introduced in 2.2.20, and prove the conjecture of Milne stated in [Va2, 5.6.6], in the wider context of Shimura p-divisible groups. We also present the general theory of representation well behaved family of tensors (to be compared with [Va2, 4.3]).

As a corollary of Milne's conjecture we get the integrality theorem of Shimura p-divisible groups, and so also the integrality theorem of Shimura varieties of Hodge type. Here we formulate it only for a $W(k)$-valued point (it can be stated also for a V-valued point, with V a discrete valuation ring which is a finite flat extension of $W(k)$, or for the rings $Re(V)(k)$ and $\tilde{Re}(V)(k)$ defined similarly as in [Va2, 5.2.1], but for k instead of $\mathbb{F} := \overline{\mathbb{F}_p}$; here $e(V) := [V : W(k)]$) of an integral canonical model of a Shimura variety of Hodge type.

If $(f, L_{(p)}, v)$ is a standard Hodge situation, then, with the notation of 1.3, the p-component of the étale component of w_α is integral (i.e. an element of the tensor algebra of $H^1_{\acute{e}t}\big(A_{\bar{K}_0}, \mathbb{Z}_p\big) \oplus \big(H^1_{\acute{e}t}\big(A_{\bar{K}_0}, \mathbb{Z}_p\big)\big)^*$ (with $\bar{K}_0$ the algebraic closure of $W(k)[\frac{1}{p}]$) iff the de Rham component t_α of w_α is integral (i.e. it is an element of the tensor algebra of $M \oplus M^*$). What is worth remarking: there is no restriction on the degree of w_α, $\alpha \in \mathcal{J}$, and it also holds for $p = 3$.

We apply the conjecture of Milne to show how, for a standard Hodge situation $(f, L_{(p)}, v)$, and for a representation $\rho : G_{\mathbb{Z}_p} \to GL(N)$ (with N a finitely generated $\mathbb{Z}_p$-module, not necessarily free) there is naturally associated and object (if N as a set is finite) or a quasi p-divisible object (otherwise) $\mathfrak{C}_\rho$ of $\mathcal{MF}^\nabla(\mathcal{N})$ (with $\mathcal{MF}^\nabla(\mathcal{N})$ as in [Fa1, §2]; here by quasi p-divisible object we mean the extension of an object by a non-trivial p-divisible object). If N is a free $\mathbb{Z}_p$-module of finite rank then $\mathfrak{C}_\rho$ is a filtered crystal on $\mathcal{N}_{k(v)}$ (cf. the conjecturally expected result mentioned in [Ch, end of $(Q1)_D$]). The association $\rho \to \mathfrak{C}_\rho$ is tannakian and functorial. It is uniquely determined by requesting two other natural properties to be satisfied.

We introduce the notion of automorphic vector bundles on an integral canonical model of a Shimura variety. We use the existence of $\mathfrak{C}_\rho$ to prove the existence and unicity of automorphic vector bundles on $\mathcal{N}$ (associated to representations of $G_{\mathbb{Z}_{(p)}}$ on $\mathbb{Z}_{(p)}$-free modules of finite rank, or to representations of $G_{O_{(v)}}$ on $O_{(v)}$-free modules of finite rank) or on $\mathcal{N} \times O_{(v_1)}$ (with $O_{(v_1)}$ a local ring which is an étale $O_{(v)}$-algebra) (associated to representations of $G_{O_{(v_1)}}$ on $O_{(v_1)}$-free modules of finite rank). We also extend this to Shimura varieties of preabelian type, under slight conditions on the representations involved.

1.17. In §6 we prove that for an injective map $f : (G, X) \hookrightarrow \big(\mathrm{GSp}(W, \psi), S\big)$, any $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ (of W) good with respect to f, is automatically crystalline well positioned for the map f. We apply this for proving the existence of integral canonical models of Shimura varieties of preabelian type with respect to primes dividing 3, and the existence of integral canonical models (see §6 for their definition) of Shimura varieties of Hodge type with respect to primes dividing $p \geq 3$ and non-hyperspecial subgroups.

1.18. In §7 we introduce a method by which, from particular Shimura σ-crystals, we construct connected normal schemes over local rings which are formally pro-étale over $\mathbb{Z}_{(p)}$,

with p a prime greater than 2, which either have an open (dense) set of formally smooth points, or have the extension property. By this method we recover all the connected components of integral canonical models of Shimura varieties of Hodge type. We (conjecturally) expect to get some other interesting integral models, even smooth ones. The motivation of this expectation is based on the existence of Shimura σ-crystals (M, φ, G) for which the derived group G^{der} of G (cf. the notations of 1.1) is of a special type (of E_7 Lie type, etc.). We call these normal schemes twisted connected Shimura varieties. These varieties are supposed to lead to the construction of integral canonical models of Shimura varieties of special type (see Definitions 2 of [Va2, 2.5]). They either provide a negative answer to Oort conjecture [Oo1, p. 6], or lead to very beautiful mathematics.

1.19. In §8, we apply some of the ideas of §3, §5 and §7 to prove the Mumford–Tate conjecture for certain abelian varieties over number fields. We also present a simple strategy meant to prove the following general criterion (it is known that it implies the Mumford–Tate conjecture):

Expectation (the split criterion). *Let A be an abelian variety over a number field E. There is $N(A) \in \mathbb{N}$ (very easily expressable in terms of A) such that if ℓ is a prime greater or equal to $N(A)$, and if the reductive group $G_{\mathbb{Q}_\ell}$ over $\mathbb{Q}_\ell$, defined as the connected algebraic envelope of the Galois representation $\rho : \mathrm{Gal}(\bar{E}/E) \to GL(H^1_{\acute{e}t}(A_{\bar{E}}, \mathbb{Q}_\ell))(\mathbb{Q}_\ell)$, is a split reductive group, then the Mumford–Tate conjecture is true for A with respect to ℓ, i.e. we have $G_{\mathbb{Q}_\ell} = H_{A\mathbb{Q}_\ell}$. Here H_A is the Mumford–Tate group of A.*

1.20. In §9 we introduce the notion of the null stratum of the special fibre of an integral canonical model of a Shimura variety of preabelian type. This is the stratum of the refined canonical Lie stratification, which is "opposite" to the dense open stratum, and which is the natural generalization of the supersingular stratum of the special fibre of an integral canonical model of a Siegel modular variety. With the notations of 1.3-5, a point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ is a point of the null stratum, iff the Shimura Lie σ-crystal attached to it has all the slopes zero. We prove that the null stratum is always nonvoid. Of course it is closed. We formulate our expectations about the properties enjoyed by the null strata, inspired by the properties (see [LO, 0.2]) of supersingular strata of the special fibres of integral canonical models of Siegel modular varieties, and prove (partially) some of these expectations. In particular we obtain a non-supersingular criterion similar to the non-ordinary one of Theorem 6.

1.21. The goal of §10 is to show that the number $N(G, X, v)$ of strata of the refined canonical Lie stratification of the special fibre of an integral canonical model of a Shimura variety $\mathrm{Sh}(G, X)$ of preabelian type with respect to a prime v of $E(G, X)$ dividing $p > 2$, and the dimensions of the geometrically connected components of these strata are easily effectively (algorithmically) computable. The philosophy is:

The number $N(G, X, v)$, the above mentioned dimensions, as well as the local geometry of strata (like the type of singularities in the étale topology of a given stratum, like which stratum specializes to which stratum, etc.) depend only on the pair $(G^{\mathrm{adnc}}_{\mathbb{Q}_p}, \mu^{\mathrm{ad}}_g)$, where $G^{\mathrm{adnc}}_{\mathbb{Q}_p}$ is the smallest direct product factor of $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ with the property that the generic fibre

μ_g^{ad} *of the cocharacter* $\mu^{\mathrm{ad}} : \mathbb{G}_m \to G^{\mathrm{ad}}_{W(k(v^{\mathrm{ad}}))}$ *(with* $k(v^{\mathrm{ad}})$ *the residue field of the prime* v^{ad} *of* $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ *divided by* v*) factors through the extension of* $G^{\mathrm{adnc}}_{\mathbb{Q}_p}$ *to* $W(k(v^{\mathrm{ad}}))[\frac{1}{p}]$*. Here* μ^{ad} *is a cocharacter such that it becomes (under an embedding of* $W(k(v^{\mathrm{ad}}))[\frac{1}{p}]$ *in* $\mathbb{C}$ *extending the inclusion* $E(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \subset \mathbb{C}$*)* $G^{\mathrm{ad}}(\mathbb{C})$*-conjugate to the cocharacters* μ_x*,* $x \in X^{\mathrm{ad}}$ *of [Va2, 2.2].*

This forms the global form of the invariance principle.

1.22. The purpose of §11 is to prove the isogeny property mentioned in [Va2, 1.7]. The idea is: we follow entirely the pattern of [FC, 4.3, p. 264], but in the context of crystalline cohomology provided by Fontaine's categories (as the isogeny property is trivial for G-ordinary points). The hard part is when we deal with points of the null stratum (defined in 1.20); for such points we need to combine loc. cit. with ideas of [Mi1].

1.23. In §12 we prove the following result: Let (G, X, H, v) be a Shimura quadruple with (G, X) an adjoint Shimura variety of abelian type, and with $(v, 2) = 1$. Then there is an injective map $f : (G_0, X_0, H_0, v_0) \hookrightarrow (G_1, X_1, H_1, v_1)$ such that:

1) $(G_0^{\mathrm{ad}}, X_0^{\mathrm{ad}}) = (G, X)$ and (under the natural inclusion $E(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \subset E(G_0, X_0)$) v_0 divides v;

2) (G_1, X_1, H_1, v_1) is the quadruple of a Shimura variety of PEL-type, with all the factors of G^{ad} of A_ℓ type, for some $\ell \in \mathbb{N}$;

3) the residue field $k(v_0)$ of v_0 is the same as the residue field $k(v_1)$ of v_1 or $[k(v_0) : k(v_1)]$ is a power of 2, depending on the fact that (G, X) has not or has simple factors of $D_n^{\mathbb{H}}$ type with involution ($n \in \mathbb{N}$, $n > 3$);

4) there is an injective map $f_1 : (G_1, X_1) \hookrightarrow (\mathrm{GSp}(W, \psi), S)$ and a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W (with p the rational prime divided by v), with the property that the triples $(f_1, L_{(p)}, v_1)$ and $(f_1 \circ f, L_{(p)}, v_0)$ are standard Hodge situations;

5) if we have $k(v_0) = k(v_1)$, then the G-ordinary type of the standard Hodge situation $(f_1 \circ f, L_{(p)}, v_0)$ is the same as the G_1-ordinary type of the standard Hodge situation $(f_1, L_{(p)}, v)$, and a G-canonical lift of the integral model $\mathcal{N}_0$ of the Shimura quadruple (G_0, X_0, H_0, v_0) is mapped under the natural morphism $\mathcal{N}_0 \to \mathcal{N}_1$ (cf. [Va2, rm. 4) of 3.2.7]) into a G_1-canonical lift of the integral canonical model $\mathcal{N}_1$ of the Shimura quadruple (G_1, X_1, H_1, v_1)).

As a consequence of this (cf. 1.13) we prove the result concerning special points mentioned in [Va2, 1.6.1] as well as the following result.

1.23.1. Let A be an abelian scheme over the ring $W(k)$ of Witt vectors of a finite field k of characteristic $p \geq 3$. Let $B(k) := W(k)[\frac{1}{p}]$. Let $(s_\beta)_{\beta \in I}$ be a family of Hodge cycles of A, including a polarization. We assume that the closure of the subgroup G_p of $GL(L_p \otimes \mathbb{Q}_p)$ fixing the p-components of the étale components of the Hodge cycles of the family $(s_\beta)_{\beta \in I}$ in $GL(L_p)$, with $L_p := H^1_{\acute{e}t}(A_{\overline{B(k)}}, \mathbb{Z}_p)$, is a reductive group over $\mathbb{Z}_p$. We also assume the existence of a family $(v_\alpha)_{\alpha \in \mathcal{J}}$ of tensors in spaces of the form $(L_p \otimes L_p^*)^{\otimes n}$, with $n \leq p - 2$, fixed by G_p, and $\mathbb{Z}_p$-very well positioned for G_p (see def. [Va2, 4.3.4]).

Let π be the Frobenius endomorphism of A_k. Then a power of π, as an endomorphism of $H^1_{\mathrm{crys}}(A_k/W(k))$, fixes the de Rham components of the Hodge cycles of the family $(s_\beta)_{\beta\in I}$.

1.24. In §13-14, we prove the results partaining to the Langlands–Rapoport conjecture stated in [Va2, 1.7].

1.25. A small part of the results presented in this first part (essentially Theorems 1 and 2 in the context of a standard Hodge situation and their applications, presented in a) and b) of 1.4), is a revised and improved version of the second part of our thesis [Va1]. Strictly speaking b) of Theorem 1 was not stated in [Va1], but all the ingredients needed for getting it can be found in loc. cit. Besides the results mentioned in 1.2-22, this paper contains many other ideas, exercises and examples. The ideas we did not have time to deal with, are presented in the form of problems and questions, or as remarks.

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§2 *Preliminaries*

2.1. Notations and conventions. Reductive groups over fields are always assumed to be connected. So the fibres of a reductive group over a scheme are connected. For a reductive group G over a scheme we denote by G^{der}, $Z(G)$, G^{ab} and G^{ad}, respectively, the derived group of G, the center of G, the maximal abelian quotient of G and the adjoint group of G. We denote by $\mathrm{Lie}(G)$ or $\mathfrak{g}$ the Lie algebra of a reductive group scheme G over an affine scheme $X = \mathrm{Spec}(R)$. For any cocharacter $\mu : \mathbb{G}_m \to G$ and for any affine X-scheme $X^1 = \mathrm{Spec}(R^1)$ we denote by $[\mu_{R^1}]$ or $[\mu_{X^1}]$ the $G(R_1)$-conjugacy class of cocharacters defined by the extension of μ to X^1. We say that a reductive group G over $\mathbb{Q}$ is unramified over $\mathbb{Q}_p$ (p being a rational prime) if $G_{\mathbb{Q}_p}$ is unramified over $\mathbb{Q}_p$, i.e. if $G_{\mathbb{Q}_p}$ is quasi-split and splits over an unramified finite field extension of $\mathbb{Q}_p$.

If X is a set endowed with an equivalence relation $R \subset X \times X$, then for any element $x \in X$, we denote by $[x] \in X/R$ the equivalence class of x. We denote by $|B|$ the number of elements of a finite set B.

The expression (G, X) always denotes a pair defining a Shimura variety, $E(G, X)$ denotes its attached reflex field, and $\mathrm{Sh}(G, X)$ denotes the Shimura variety defined by it, identified with the canonical model of the complex variety. For an arbitrary compact subgroup K of $G(\mathbb{A}_f)$, we denote by $\mathrm{Sh}_K(G, X)$ or $\mathrm{Sh}(G, X)/K$, the quotient of $\mathrm{Sh}(G, X)$ by K. Any $x \in X$ and any $\alpha \in G(\mathbb{A}_f)$ define a complex point $[x, \alpha]$ of $\mathrm{Sh}_K(G, X)$. As in this paper we encounter several quadruples, the quadruples introduced in [Va2, 3.2.6] will be called Shimura quadruples. We are using freely the notations of [Va2, 2.4 and 3.2.6].

If k is a field we denote by $\bar{k}$ its algebraic closure. For a perfect field k, $W(k)$ is the ring of Witt vectors of k, while $B(k)$ is the field of fractions of $W(k)$. For any $n \in \mathbb{N}$ $W_n(k) := W(k)/p^n W(k)$. We denote by σ (resp. $\overline{\sigma}$) the Frobenius automorphism of $W(k)$ and $B(k)$ (resp. of $W(\bar{k})$ and $B(\bar{k})$). By a σ-crystal (over k) we mean a pair (M, φ), with M a free $W(k)$-module of finite rank, and with φ a σ-linear endomorphism of M which is a σ-linear automorphism of $M \otimes B(k)$. Warning: by a filtered σ-crystal (over k) we mean a triple (M, F^1, φ), with (M, φ) a σ-crystal, and with F^1 a $W(k)$-submodule of M which is a direct summand of M such that $\varphi(\frac{1}{p}F^1 + M) = M$. By a principal quasi-polarization of a σ-crystal (M, φ) we mean a symplectic perfect form $\psi : M \otimes M \to W(k)$ such that $\psi(\varphi(x), \varphi(y)) = p\sigma(\psi(x, y))$, $\forall x, y \in M$. We often write $\psi : M \otimes M \to W(k)(1)$. Let $n \in \mathbb{N}$; by a σ^n-isocrystal (over k) we mean a pair (M, φ), with M a finite dimensional $B(k)$-vector space, and with φ a σ^n-linear automorphism of M. When $n = 1$ we speak simply about isocrystals (over k). We still denote by φ the Frobenius of any free $W(k)$-submodule or $B(k)$-vector subspace of different tensor products (taken in any order) of a given number of copies of M with a given number of copies of its dual M^*, which is invariant under φ.

We denote by $\mathcal{MF}_{[a,b]}(R)$, $\mathcal{MF}(R)$, $\mathcal{MF}^{\nabla}_{\mathrm{big}}(R)$, $\mathcal{MF}^{\nabla}_{[a,b]}(R)$ and $\mathcal{MF}_{[a,b]}(R)$ the categories introduced in [Fa2]. We allow R to be (the p-adic completion of) a formally étale algebra (resp. affine scheme; when allowed, just scheme) over a smooth $W(k)$-algebra (resp. $W(k)$-scheme). If R and S are two $W(k)$-algebras having Frobenius lifts Φ_R and respectively Φ_S, if M is a free R-module endowed with a Φ_R-linear map $\varphi : M \to M$, and if $b : R \to S$ is a ring homomorphism compatible with the mentioned Frobenius lifts, we denote by $\varphi \otimes 1$ the Φ_S-linear map on $M \otimes_R S$ induced from φ through b.

Always V_0 denotes a Witt ring over an algebraically closed field of positive characteristic, and then K_0 automatically denotes its field of fractions. If v is a prime of a global field E, we denote by $k(v)$ the residue field of v and by $O_{(v)}$ the localization of the ring of integers of E with respect to the prime v. If O is a local ring, we denote by O^h, O^{sh} and $\widehat{O}$ respectively its henselization, its strict henselization and its completion.

Let p be a rational prime. We usually write $\mathbb{Z}_{(p)}$ instead of $O_{(p)}$. The ring of finite adèles $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is denoted by $\mathbb{A}_f$, and the ring of finite adèles with the p-component omitted is denoted by $\mathbb{A}_f^p$. We use freely different Tate-twists: $\mathbb{Q}(1), \mathbb{Q}_p(1), \mathbb{Z}_p(1), \mathbb{A}_f(1)$ etc. We denote by $\mathbb{F}_{p^d}$ the field with p^d elements ($d \in \mathbb{N}$), and by $\mathbb{F}$ its algebraic closure. If R is a faithfully flat $\mathbb{Z}_{(p)}$-algebra, we denote by $R^\wedge$ its p-adic completion. Similarly, if $X = \mathrm{Spec}(R)$, with R as before, we denote $X^\wedge := \mathrm{Spec}(R^\wedge)$.

If A is a set and $f : A \to A$ is an endomorphism of it, we denote by f^m the m-times composite of f with itself ($m \in \mathbb{N}$). If $f : A \to B$ and $g : B \to C$ are morphisms in some category we refer to $g \circ f$ as the composite of f with g.

For every free module M of finite rank over a commutative ring R, we denote by M^* its dual. We denote by $M^{\otimes n} \otimes M^{*\otimes m}$, with $m, n \in \mathbb{N} \cup \{0\}$, the tensor product of n-copies of M with m-copies of M^*. If $v_\alpha \in M^{\otimes n} \otimes M^{*\otimes m}$, we denote by $\deg(v_\alpha) = n + m$ its degree, and we call it a homogeneous tensor of degree $n + m$. By the essential tensor algebra of $M \oplus M^*$ we mean

$$\mathcal{T}(M) := \oplus_{n,m \in \mathbb{N} \cup \{0\}} M^{\otimes n} \otimes M^{*\otimes m}.$$

We do not make use of the natural structure of $\mathcal{T}(M)$ as an R-algebra (it is non-commutative if the rank of M is at least 2). Similarly we define the essential tensor algebra of $\mathcal{F} \oplus \mathcal{F}^*$ for a locally free $\mathcal{O}_X$-sheaf $\mathcal{F}$ of finite rank on a ringed space $(X, \mathcal{O}_X)$. A bilinear form on M is called perfect if it induces an isomorphism from M into its dual M^*. A pair (M, ψ) with M as above, and with $\psi : M \otimes M \to R$ a perfect alternating form on it, is called a symplectic space over R. We usually use the same notation for two perfect alternating forms if they are obtained one from another by extension of scalars. If (W, ψ) is a symplectic space over $\mathbb{Q}$, we denote by $\mathrm{Sh}\big(\mathrm{GSp}(W, \psi), S\big)$ the Siegel modular variety associated to it.

For a finite field extension $k_0 \hookrightarrow k$ and for a reductive group G over k, $\mathrm{Res}_{k/k_0} G$ denotes the group over k_0 obtained from G by restriction of scalars. If $f : G \to G_1$ is a homomorphisms of group schemes which is a closed embedding, then we identify G with its image (through f) in G_1.

We use freely the terminology of Hodge cycles of abelian varieties over a field of characteristic zero used in [De3], and by natural extension, this terminology is used also

for abelian schemes over a reduced $\mathbb{Q}$–scheme S. Even more: if S is a reduced flat $\mathbb{Z}_{(p)}$-scheme, and if A is an abelian scheme over S, we speak about Hodge cycles of A, though strictly speaking, we should speak of the Hodge cycles of the generic fibre of A. Any polarization of an arbitrary abelian scheme A is denoted by p_A (sometimes $\mathcal{P}_A$), and by abuse of notation we also denote by p_A the different maps on the cohomologies (homologies) of A induced by it. Pairs of the form (A, p_A) always denote a polarized abelian scheme.

We consider an abelian variety or a p-divisible group A over a field k of positive characteristic p. We denote by A^t its dual. If k is perfect, then by the σ-crystal of (or attached to) A (resp. by the isocrystal of A) we mean the σ-crystal defined by the first crystalline cohomology group of A with coefficients in $W(k)$ (resp. we mean the isocrystal over k defined by such a σ-crystal). If k is arbitrary, by the formal isogeny type of A, we mean the formal isogeny type (see [Og]) defined by the isocrystal of $A_{\bar{k}}$.

For Artin's approximation theorem we refer to [BLR, p. 91]. All the limits and colimits are assumed to be filtered. The projective limits between schemes have affine transition morphisms. If a projective limit is indexed in the standard manner by elements of $\mathbb{N}$, then we refer to it as an $\mathbb{N}$-projective limit; similarly we speak about an $\mathbb{N}$-pro-étale scheme over another scheme.

2.2. Shimura crystals. Let k be an arbitrary perfect field of characteristic $p > 0$.

2.2.1. Definitions. a) Let $r \in \mathbb{Q}$. An isocrystal (M, φ) over k is said to be r-symmetric, if $\forall \alpha \in \mathbb{Q}$ such that $r - \alpha$ is a slope of it, $r + \alpha$ is also a slope of it of the same multiplicity.

b) A filtered σ-crystal (M, F^1, φ) is said to be cyclic diagonalizable if there is a $W(k)$-basis $\{e_i | i \in A\}$ of M (with $A = \{1, 2, \ldots, \dim_{W(k)}(M)\}$) such that $\{e_i | i \in B\}$ (with B a subset of A) is a $W(k)$-basis of F^1, and $\varphi(e_i) = p^{\varepsilon_i} e_{\pi(i)}$, with $\varepsilon_i = 1$ if $i \in B$ and zero otherwise, and with π a permutation of A. When $\pi = 1_A$ we drop the word cyclic. If π is an $|A|$-cycle, then we replace the word cyclic by circular.

c) Let R be a formally étale algebra over a smooth $W(k)$-algebra, and let Φ_R be a σ-linear endomorphism of R (resp. of $R^\wedge$) which lifts the Fobenius of R/pR. We have in mind mainly the case when the only open subscheme of $\mathrm{Spec}(R)$ containing all the $\bar{k}$-valued points of $\mathrm{Spec}(R)$, is $\mathrm{Spec}(R)$ itself; but for the sake of generality we do not assume this. A finitely generated projective R-module (resp. $R^\wedge$-module) M, together with a decreasing filtration $(F^i)_{i\in\mathbb{Z}}$ (with F^i a direct summand of M) and a Φ_R-linear endomorphism $\varphi : M \to M$, is said to be a p-divisible object of $\mathcal{MF}(R)$ if for any $i \in \mathbb{Z}$, φ restricted to F^i is of the form $p^i\varphi_i$, with $\varphi_i : F^i \to M$, and the R-linear map $\oplus_{i\in\mathbb{Z}} F^i \otimes_{R\ \Phi_R} R \to M$, whose restriction to the summand $F^i \otimes_{R\ \Phi_R} R$ is $\varphi_i \otimes 1$ ($i \in \mathbb{Z}$), is an isomorphism. We denote such a p-divisible object by a triple

$$\mathcal{C} := (M, (F^i)_{i\in\mathbb{Z}}, \varphi).$$

So for any $n \in \mathbb{N}$, M/p^nM, together with $(F^i/p^nF^i)_{i\in\mathbb{Z}}$ and the induced $(\varphi_i)_{i\in\mathbb{Z}}$ is an object of $\mathcal{MF}(R)$. We denote it by $\mathcal{C}/p^n\mathcal{C}$ and call it the truncation mod p^n of $\mathcal{C}$. If M is a projective R-module, we identify this p-divisible object with the one obtained from it by passing from R to $R^\wedge$ (but we do not want to lose the information of being "algebraizable").

If $\mathcal{C}_1 = (M_1, (F_1^i)_{i\in\mathbb{Z}}, \varphi_1)$ is another such p-divisible object, then by a morphism from $\mathcal{C}$ to $\mathcal{C}_1$, we mean an R-linear map $M \to M_1$ (when M and M_1 are $R^\wedge$ modules, then we mean an $R^\wedge$-linear map $M \to M_1$) respecting the filtrations and the Frobenius. We get the category $p - \mathcal{MF}(R)$ of p-divisible objects of $\mathcal{MF}(R)$. If moreover we fix a family of tensors $(v_\alpha)_{\alpha\in\mathcal{J}}$ of the F^0-filtration of $\mathcal{T}(M[\frac{1}{p}])$ fixed by φ, then we speak about a p-divisible object with cycles of $\mathcal{MF}(R)$.

Similarly we speak about p-divisible objects (with cycles) of $\mathcal{MF}_{[a,b]}(R)$, $\mathcal{MF}_{\mathrm{big}}(R)$, $\mathcal{MF}^\nabla_{[a,b]}(R)$, or of $\mathcal{MF}^\nabla_{\mathrm{big}}(R)$, or of a p-divisible object (with cycles) of $\mathcal{MF}_{[a,b]}(X)$, etc., with X a smooth $W(k)$-scheme (in this last case the cycles are sections in locally free sheaves of finite rank over X or over the formal scheme defined by the p-adic completion of X). We also speak about the categories defined by such p-divisible objects: $p-\mathcal{MF}_{[a,b]}(R)$, etc. When we speak about a p-divisible object with cycles of $\mathcal{MF}^\nabla(R)$, we require that the cycles are horizontal, i.e. are annihilated by the connection involved. Warning: as we allowed R to be regular formally smooth (not only smooth) over $W(k)$, some precautions have to be taken. In the case of a p-divisible object of $\mathcal{MF}^\nabla(R)$, we work with the p-adic completion of the module of relative differentials of R over $W(k)$. For instance if $R = W(k)[[z_1, ..., z_m]]$, $m \in \mathbb{N}$, and if M is a free R-module of finite rank, then all the connections

$$\nabla : M \to M \otimes \Omega_{R/W(k)}$$

to be considered, factor through $M \otimes_R \bar{\Omega}_{R/W(k)}$, where $\bar{\Omega}_{R/W(k)}$ is the free R-submodule of $\Omega_{R/W(k)}$ generated by dz_1,...,dz_m.

We also speak about a principal quasi-polarization of a p-divisible object of $\mathcal{MF}_{[0,1]}(R)$ or of $\mathcal{MF}^\nabla_{[0,1]}(R)$. For instance, if $\mathfrak{C}$ is p-divisible object of $\mathcal{MF}_{[0,1]}(R)$, the dual $\mathfrak{C}^*$ of $\mathfrak{C}$ is a p-divisible object of $\mathcal{MF}_{[-1,0]}(R)$, and so $\mathfrak{C}^*(1)$ is a p-divisible object of $\mathcal{MF}_{[0,1]}(R)$; so by a principal quasi-polarization of $\mathfrak{C}$, we mean an isomorphism

$$\mathfrak{C} \tilde{\to} \mathfrak{C}^*(1)$$

producing a perfect alternating

$$\mathfrak{C} \otimes \mathfrak{C} \to R(1);$$

this form is a morphism in the category $p - \mathcal{MF}_{[0,2]}(R)$. Here

$$R(1) := R \otimes_{W(k)} W(k)(1).$$

Similarly we speak about principally quasi-polarized truncation mod p^n (this is an object) of a p-divisible object of $\mathcal{MF}_{[0,1]}(R)$ or of $\mathcal{MF}^\nabla_{[0,1]}(R)$; here $n \in \mathbb{N}$.

2.2.1.1. Remarks. 1) We might wonder if the terminology p-divisible objects is the right one. Our justification is: the philosophies of [Fa2, 7.1] and 3.4.18.5.1. Moreover the terminology formal is used in this paper in a different context (cf. 4.7).

2) We have a contravariant functor from the category of p-divisible groups over $R^\wedge$ (or R) and the category p-$\mathcal{MF}^\nabla_{[0,1]}(R)$ (this is a particular case of the Grothendieck-Messing theory; see [Me]). In particular to any p-divisible group over $R^\wedge$ or R it corresponds (or it is associated) a p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(R)$.

3) The above categories of objects or p-divisible objects depend on the choice of Φ_R. However, if $p > 2$ then the categories $\mathcal{MF}^{\nabla}_{[0,1]}(X)$ and $\mathcal{MF}^{\nabla}_{[0,1]}(X)$ do not depend on Φ_R, cf. [Fa2, 2.3].

4) Any circular diagonalizable σ-crystal has a unique slope.

2.2.2. Examples. The isocrystal of an abelian variety over k is 1/2-symmetric. If $k = \bar{k}$ then the filtered σ-crystal of the canonical lift of an ordinary abelian variety over k is diagonalizable and a p-divisible object of $\mathcal{MF}_{[0,1]}(W(k))$.

2.2.3. Definitions. 1) A Lie σ-crystal (resp. a Lie isocrystal over k) is a pair $(\mathfrak{g}, \varphi)$, with $\mathfrak{g}$ a Lie algebra over $W(k)$ (resp. over $B(k)$), which as a $W(k)$-module (resp. as a $B(k)$-vector space) is free of finite rank (resp. of finite dimension), and with φ a σ-linear Lie automorphism of $\mathfrak{g} \otimes B(k)$ such that $\varphi(p\mathfrak{g}) \subset \mathfrak{g}$ (resp. with φ a σ-linear Lie automorphism of $\mathfrak{g}$).

2) A Lie σ-crystal is said to be ordinary or of ordinary type, if the slopes of the isocrystal $(\mathfrak{g} \otimes B(k), \varphi)$ are (warning!) precisely -1, 0 and 1, with the multiplicity of the slope 1 equal to the multiplicity of the slope -1.

3) A filtered Lie σ-crystal is a quadruple $(\mathfrak{g}, \varphi, F^0(\mathfrak{g}), F^1(\mathfrak{g}))$ such that:

– the pair $(\mathfrak{g}, \varphi)$ is a Lie σ-crystal;

– $F^0(\mathfrak{g})$ and $F^1(\mathfrak{g})$ are Lie subalgebras of $\mathfrak{g}$ such that $F^1(\mathfrak{g})$ is an ideal of $F^0(\mathfrak{g})$;

– it defines a p-divisible object of the category $\mathcal{MF}_{[-1,1]}(W(k))$, i.e. $\varphi\big(\frac{1}{p}F^1(\mathfrak{g}) + F^0(\mathfrak{g}) + p\mathfrak{g}\big) = \mathfrak{g}$ and, as $W(k)$-submodules, $F^0(\mathfrak{g})$ and $F^1(\mathfrak{g})$ are direct summands of $\mathfrak{g}$ (we have $\mathfrak{g} = F^{-1}(\mathfrak{g})$ and $0 = F^2(\mathfrak{g})$).

2.2.4. Remarks. 1) The slopes of a filtered Lie σ-crystal are rational numbers of the interval [-1,1]. Most common, for Lie σ-crystals $(\mathfrak{g}, \varphi)$, $\mathfrak{g}$ is the Lie algebra of a reductive group and for filtered Lie σ-crystals $(\mathfrak{g}, \varphi, F^0(\mathfrak{g}), F^1(\mathfrak{g}))$, $\mathfrak{g}$ is the Lie algebra of a reductive group, $F^0(\mathfrak{g})$ is a parabolic Lie subalgebra of $\mathfrak{g}$ having $F^1(\mathfrak{g})$ as its nilpotent radical, and so $\big([\mathfrak{g}, \mathfrak{g}] \otimes B(k), \varphi\big)$ is a 0-symmetric isocrystal (i.e. the multiplicity of a slope α is the same as the multiplicity of the slope $-\alpha$). We sometimes need to use Lie σ-crystals $(\mathfrak{g}, \varphi)$, with $\mathfrak{g}$ as the Lie subalgebra of the Lie algebra of a reductive group.

2) We can introduce the notion of an a-Lie σ-crystal $(\mathfrak{g}, \varphi)$ (or of an $[a, b]$-filtered Lie σ-crystal $\big(\mathfrak{g}, \varphi, F^{a+1}(\mathfrak{g}), \dots, F^b(\mathfrak{g})\big)$) (with $a, b \in \mathbb{Z}$). For instance, for an a-Lie σ-crystal we require that $\mathfrak{g}$ is a Lie algebra over $W(k)$, that $\mathfrak{g}$ as a $W(k)$-module is free of finite rank, that φ is a σ-linear Lie automorphism of $\mathfrak{g} \otimes B(k)$, and that $\varphi(p^a\mathfrak{g}) \subset \mathfrak{g}$. But we do not need this for the present paper, except for a short intermezzo in 3.4.1.5-6.

3) Let $(\mathfrak{g}, \varphi)$ be a Lie isocrystal over k. Then there is a unique Lie subalgebra $\mathfrak{p}_{\geq 0}$ of $\mathfrak{g}$ left invariant by φ and such that all the slopes of the isocrystal $(\mathfrak{p}_{\geq 0}, \varphi)$ (resp. of $(\mathfrak{g}/\mathfrak{p}_{\geq 0}, \varphi)$) are non-negative (resp. are negative). If $\mathfrak{g}$ is a semisimple Lie algebra over $B(k)$, then $\mathfrak{p}_{\geq 0}$ is a parabolic Lie subalgebra of $\mathfrak{g}$. We call the isocrystal $(\mathfrak{p}_{\geq 0}, \varphi)$ (resp. $\mathfrak{p}_{\geq 0}$) the (Lie) subisocrystal of $(\mathfrak{g}, \varphi)$ (resp. the Lie subalgebra of $\mathfrak{g}$) corresponding to non-negative slopes of $(\mathfrak{g}, \varphi)$. Similarly we define things for non-positive (or positive, or negative) slopes. Similarly we define things for a Lie σ-crystal.

2.2.5. Definitions. Let X be a connected scheme.

1) A Shimura group pair over X is a pair $(G,[\mu])$, with G a reductive group over X, and with $[\mu]$ a $G(X')$-conjugacy class of cocharacters $\mu : \mathbb{G}_m \to G_{X'}$ defined over a connected étale cover X' of X, such that the induced action of $\mathbb{G}_m$ on $\mathrm{Lie}(G_{X'})$ is defined (warning!) precisely by the characters: the null, the identical and the inverse of the identical character of $\mathbb{G}_m$.

2) Two such Shimura group pairs over X $(G,[\mu_i])$, $i=\overline{1,2}$, (with $\mu_i : \mathbb{G}_m \to G_{X_i}$ defined over a connected étale cover X_i of X) are said to be identical if there is a connected étale cover X' of X, dominating X_1 and X_2, and over which μ_1 and μ_2 define the same $G(X')$-conjugacy class of cocharacters of $G_{X'}$.

3) For any such Shimura group pair $(G,[\mu])$ over X, we denote by $[\mathrm{Lie}(\mu)]$ the $G^{\mathrm{ad}}(X')$-conjugacy class of the (rank one) direct summand $\mu(\mathrm{Lie}(\mathbb{G}_m))$ of $\mathrm{Lie}(G_{X'})$. We call the pair $(\mathrm{Lie}(G),[\mathrm{Lie}(\mu)])$ the Shimura Lie pair (over X) attached to the Shimura group pair $(G,[\mu])$.

4) Two such Shimura Lie pairs over X $(\mathrm{Lie}(G_i),[\mathrm{Lie}(\mu_i)])$, $i=\overline{1,2}$, (with μ_i a cocharacter of G_{X_i}; X_i being an étale connected cover of X), are said to be isomorphic if:

a) there is an isomorphism of Lie algebras $\rho : \mathrm{Lie}(G_1)\tilde{\to}\mathrm{Lie}(G_2)$ producing an isomorphism $\mathrm{Lie}(G_1^{\mathrm{ad}})\tilde{\to}\mathrm{Lie}(G_2^{\mathrm{ad}})$ induced by an isomorphism $G_1^{\mathrm{ad}}\tilde{\to}G_2^{\mathrm{ad}}$;

b) over an étale connected cover X' of X, dominating X_1 and X_2, ρ takes $[\mathrm{Lie}(\mu_{1X'})]$ onto $[\mathrm{Lie}(\mu_{2X'})]$.

2.2.6. Remarks. 1) Most common, in 2.2.5 1), either G is split and X is local henselian, and then we can take $X'=X$, or $X=\mathrm{Spec}(\mathbb{Q}_p)$ (or $X=\mathrm{Spec}(\mathbb{Z}_p)$) and G is unramified over $\mathbb{Q}_p$, and then there is a smallest unramified cover X' of X, over which such a cocharacter $\mu\colon \mathbb{G}_m \to G_{X'}$ is defined.

2) We can work 2.2.5 in terms of the faithfully flat (or étale) topology of X, but for this paper, the present form of 2.2.5 is all that we need.

3) 2.2.5 1-2) allows us to introduce the notion of morphisms (maps) between two Shimura group pairs $(G_1,[\mu_1])$ and $(G_2,[\mu_2])$ over X: These are (group) homomorphisms $f : G_1 \to G_2$ such that for a simply connected pro-étale cover X' of X, $f_{X'}$ takes $[\mu_{1X'}]$ into $[\mu_{2X'}]$. We denote such a map by $f : (G_1,[\mu_1]) \to (G_2,[\mu_2])$. It is called an injective map (resp. an isomorphism), if as a group homomorphism is a closed embedding (resp. an isomorphism). We get the category $\mathrm{ShGp}(X)$ of Shimura group pairs over X.

4) A study of Shimura group pairs over $\mathbb{Z}_p$ together with their symplectic embeddings (i.e. injective maps into $\big(\mathrm{GSp}(W_{\mathbb{Z}_p},\psi),[\mu]\big)$; here $(W_{\mathbb{Z}_p},\psi)$ is a symplectic space over $\mathbb{Z}_p$, while $[\mu]$ is uniquely determined), is started in 3.10 and is fully developped in §7.

5) Any Shimura variety $\mathrm{Sh}(G,X)$ of dimension ≥ 1 gives birth to a Shimura group pair $(G,[\mu])$ over $\mathrm{Spec}(\mathbb{Q})$. Here $\mu : \mathbb{G}_m \to G_{E_1(G,X)}$ is a cocharacter, with $E_1(G,X)$ a finite field extension of $E(G,X)$, which over $\mathbb{C}$, under un embedding $E_1(G,X)\subset\mathbb{C}$ extending the inclusion $E(G,X)\subset\mathbb{C}$, is $G(\mathbb{C})$-conjugate to anyone of the cocharecters μ_x, $x\in X$, def. in [Va2, 2.2]. The association: $(G,[\mu])$ to $\mathrm{Sh}(G,X)$, is functorial.

2.2.7. Remark. If X is local henselian and if G is an X-simple adjoint split group, then the Shimura group pairs $(G,[\mu])$ are classified by [De2, 1.3.9]. We have $\left[\frac{\ell+1}{2}\right]$ isomorphism classes if G is of A_ℓ Lie type, 2 isomorphism classes if G is of D_ℓ Lie type ($\ell\geq 5$),

and one such class if G is of D_4, E_6, E_7, B_ℓ or C_ℓ Lie type. There are no Shimura group pairs $(G,[\mu])$ with G an adjoint group of E_8, F_4 or G_2 Lie type.

2.2.8. Definitions. **1)** A Shimura filtered σ-crystal over k is a quadruple (M,F^1,φ,G), with (M,F^1,φ) a filtered σ-crystal such that F^1 is a proper direct summand of M, and with G a quasi-split reductive subgroup of $GL(M)$ such that:

a) There is a family $(v_\alpha)_{\alpha\in\mathcal{J}}$ of tensors of the F^0-filtration (defined by the filtration F^1 of M) of $\mathcal{T}(M[\frac{1}{p}])$, fixed by G and φ, and such that $G_{B(k)}$ is the subgroup of $GL\big(M[\frac{1}{p}]\big)$ fixing v_α, $\forall\alpha\in\mathcal{J}$.

b) $\Big(\mathrm{Lie}(G),\varphi,F^0\big(\mathrm{Lie}(G)\big),F^1\big(\mathrm{Lie}(G)\big)\Big)$ is a filtered Lie σ-crystal; here

$$F^0\big(\mathrm{Lie}(G)\big):=\big\{x\in\mathrm{Lie}(G)\;\big|\;x(F^1)\subset F^1\big\}$$

and

$$F^1\big(\mathrm{Lie}(G)\big):=\big\{x\in\mathrm{Lie}(G)\;\big|\;x(F^1)=0\big\}.$$

c) There is an injective cocharacter $\mu:\mathbb{G}_m\hookrightarrow G$ producing a direct sum decomposition $M=F^1\oplus F^0$, with $\beta\in\mathbb{G}_m(W(k))$ acting through μ on F^i as the multiplication with β^{-i}, $i=\overline{0,1}$.

2) A Shimura σ-crystal over k is a triple (M,φ,G) which can be extended to a quadruple (M,F^1,φ,G) defining a Shimura filtered σ-crystal over k.

3) A generalized Shimura p-divisible object of $\mathcal{MF}_{[a,b]}(W(k))$ is a quadruple

$$(M,(F^i)_{a\leq i\leq b},\varphi,G),$$

where the triple $(M,(F^i)_{a\leq i\leq b},\varphi)$ is a p-divisible object of $\mathcal{MF}_{[a,b]}(W(k))$ (so $a,b\in\mathbb{Z}$, with $a\leq b$) and G is a quasi-split reductive subgroup of $GL(M)$, for which the conditions a) and b) are true, but for which condition c) is replaced by:

d) There is an injective cocharacter $\mu:\mathbb{G}_m\hookrightarrow G$ producing a direct sum decomposition $M=\oplus_{a\leq i\leq b}\tilde{F}^i$, with $\beta\in\mathbb{G}_m(W(k))$ acting through μ on $\tilde{F}^i$ as the multiplication with β^{-i}, $i=\overline{\overline{a,b}}$, with $F^i=\oplus_{i\leq j\leq b}\tilde{F}^j$, and with at least two of the $W(k)$-modules $\tilde{F}^i$, $i=\overline{a,b}$ being non-zero, and such that the induced cocharecter $\mu^{\mathrm{ad}}:\mathbb{G}_m\to G^{\mathrm{ad}}$ is either trivial, or gives birth to a Shimura group pair $(G^{\mathrm{ad}},[\mu^{\mathrm{ad}}])$.

4) A generalized Shimura p-divisible object over k is a triple (M,φ,G) which can be extended to a quadruple $(M,(F^i)_{a\leq i\leq b},\varphi,G)$ defining a generalized Shimura p-divisible object of $\mathcal{MF}_{[a,b]}(W(k))$.

5) A Shimura isocrystal over k is a triple $(M_{B(k)},\varphi,G_{B(k)})$, where $(M_{B(k)},\varphi)$ is an isocrystal over k, and where $G_{B(k)}$ is a reductive subgroup of $GL(M_{B(k)})$, such that there is a family of tensors $(v_\alpha)_{\alpha\in\mathcal{J}}$ of $\mathcal{T}(M_{B(k)})$ fixed by φ, and having the property that $G_{B(k)}$ is the subgroup of $GL(M_{B(k)})$ fixing its members.

6) A Shimura filtered isocrystal over k is a quadruple $(M_{B(k)},(F^i)_{i\in\mathbb{Z}},\varphi,G_{B(k)})$, where the triple $(M_{B(k)},\varphi,G_{B(k)})$ is a Shimura isocrystal over k, and where $(F^i)_{i\in\mathbb{Z}}$ is a decreasing

filtration of $M_{B(k)}$, such that, with the notations of 5), we can choose the family of tensors $(v_\alpha)_{\alpha\in\mathcal{J}}$ to be in the F^0-filtration (defined by $(F^i)_{i\in\mathbb{Z}}$) of $\mathcal{T}(M_{B(k)})$.

2.2.9. Remarks. 0) We often drop out mentioning "over k".

1) We refer to 2.2.8 1). The condition a) implies that $\varphi\big(\mathrm{Lie}(G)[\frac{1}{p}]\big) = \mathrm{Lie}(G)[\frac{1}{p}]$. It is almost implied by b), cf. [Fa2, rm. iii) of th. 10]: in the loc. cit. we do have to consider Tate-twists. The condition c) results from a) and b) (cf. [Wi]). So the main assumption is b).

1′) 1) above suggests the introduction of almost Shimura (filtered) σ-crystals: we assume just b) of 2.2.8 1).

2) If in 2.2.8 1) we have $F^1\big(\mathrm{Lie}(G)\big) \neq 0$, then the pair $(G,[\mu])$ defines a Shimura group pair over $W(k)$.

3) It is an easy exercise to see that the $G(W(k))$-conjugacy class of parabolic Lie subalgebras of $\mathrm{Lie}(G)$ defined by $F^0(\mathrm{Lie}(G))$ is uniquely determined by the Shimura σ-crystal (M,φ,G). We call it the filtration class of the Shimura σ-crystal (M,φ,G). About μ (or the $G(W(k))$-conjugacy class $[\mu]$ defined by μ) we say that it defines the conjugacy class. $[\mu]$ is uniquely determined by (M,φ,G), and so also the $G^{\mathrm{ad}}(W(k))$-conjugacy class $[\mu^{\mathrm{ad}}]$ defined by the cocharacter $\mu^{\mathrm{ad}}:\mathbb{G}_m\to G^{\mathrm{ad}}$ induced by μ, is also uniquely determined by (M,φ,G). So, if G is a torus, then μ itself is uniquely determined.

4) The generalized Shimura p-divisible objects over k do not show up in this first part, except breefly in 3.1.7, 3.4.1.5-6 and 4.5.6 9). They will be used in the second part (cf. 4.5.4). We will use freely the terminology to be introduced for Shimura (filtered) σ-crystals and which depends only on their attached (cf. 2.2.13 below) Shimura Lie σ-crystals, in the context of generalized Shimura p-divisible objects over k or of $\mathcal{MF}(W(k))$.

5) When we want to emphasize a family of tensors $(v_\alpha)_{\alpha\in\mathcal{J}}$ as in a) of 2.2.8 1), we denote a Shimura σ-crystal by a quadruple $(M,\varphi,G,(v_\alpha)_{\alpha\in\mathcal{J}})$. Similarly we denote a Shimura filtered σ-crystal by a quintuple $(M,F^1,\varphi,G,(v_\alpha)_{\alpha\in\mathcal{J}})$. Strictly speaking we should refer to $(M,\phi,G,(v_\alpha)_{\alpha\in\mathcal{J}})$ as a Shimura σ-crystal with an emphasized family of tensors, but we rarely do so; the same about the filtered case.

6) If $(M,\varphi,G,(v_\alpha)_{\alpha\in\mathcal{J}})$ and $(M_1,\varphi_1,G_1,(t_\alpha)_{\alpha\in\mathcal{J}_1})$ (resp. $(M,F^1,\varphi,G,(v_\alpha)_{\alpha\in\mathcal{J}})$ and $(M_1,F^1_1,\varphi_1,G_1,(t_\alpha)_{\alpha\in\mathcal{J}_1})$) are two Shimura σ-crystals (resp. two Shimura filtered σ-crystals) then an isomorphism between them is given by an isomorphism $t: M\tilde{\to}M_1$ taking the triple $(\varphi,G,(v_\alpha)_{\alpha\in\mathcal{J}})$ into $(\varphi_1,G_1,(t_\alpha)_{\alpha\in\mathcal{J}_1})$ (resp. taking the quadruple $(F^1,\varphi,G,(v_\alpha)_{\alpha\in\mathcal{J}})$ into $(F^1_1,\varphi_1,G_1,(t_\alpha)_{\alpha\in\mathcal{J}_1})$). If $j:\mathcal{J}\tilde{\to}\mathcal{J}_1$ is a bijection, and if the isomorphism t takes v_α into $t_{j(\alpha)}$, then we call t a j-isomorphism. In particular a $1_{\mathcal{J}}$-isomorphism between the two Shimura σ-crystals $(M,\varphi,G,(v_\alpha)_{\alpha\in\mathcal{J}})$ and $(M,\varphi_1,G,(v_\alpha)_{\alpha\in\mathcal{J}})$ is given by an element $g\in G(W(k))$ such that $\varphi_1 = g\varphi g^{-1}$. Of course an isomorphism of (M,φ,G) does not need to fix the family of tensors $(v_\alpha)_{\alpha\in\mathcal{J}}$; easy examples can be constructed for instance if $k=\mathbb{F}_p$, and so, by natural extension over any k.

7) The condition G quasi-split, used in 2.2.8 1-4), is not really needed (cf. 3.8). It is inserted just to be able to give a concrete example of a G-ordinary Shimura filtered σ-crystal (cf. 3.2.3 below). Also proposition 2.3.9 makes it very convenient for applications

to Shimura varieties. When we do not know that G is quasi-split, or when it is not, we speak about a Shimura (filtered) σ-crystal of being not necessarily quasi-split.

8) If in 2.2.8 1) we have $k = \bar{k}$, then writing $\varphi = a \circ \mu(\frac{1}{p})$, a becomes a σ-linear automorphism of M and of $\mathrm{Lie}(G)$, and $M^a := \{x \in M | a(x) = x\}$ is a $\mathbb{Z}_p$-free module of rank $\dim_{W(k)}(M)$. Then $\mathrm{Lie}(G)^a := \mathrm{Lie}(G) \cap \mathrm{End}(M)^a \subset \mathrm{End}(M^a)$ is the Lie algebra of a reductive subgroup $G_{\mathbb{Z}_p}$ of $GL(M^a)$. Its extension to $W(k)$ is the reductive subgroup G of $GL(M)$. Let $W(\mathbb{F}_{p^d})[\frac{1}{p}]$ be the field of definition of the conjugacy class of the cocharacter $\mu_{B(k)} : \mathbb{G}_m \to G_{B(k)}$. Here $d \in \mathbb{N}$. We deduce easily the existence of an element $g \in G(W(k))$ such that the Shimura filtered σ-crystal $\left(M, F^1, g\varphi, G, (v_\alpha)_{\alpha \in \mathcal{J}}\right)$ over k is isomorphic to the extension to k of a Shimura σ-crystal

$$\left(M^a \otimes W(\mathbb{F}_{p^d}), F^1_{\mu_0}, \varphi_0, G_{W(\mathbb{F}_{p^d})}, (v_\alpha)_{\alpha \in \mathcal{J}}\right)$$

over $\mathbb{F}_{p^d}$, where $F^1_{\mu_0} \subset M^a \otimes W(\mathbb{F}_{p^d})$ is the F^1-filtration associated to a cocharacter $\mu_0 : \mathbb{G}_m \to G_{W(\mathbb{F}_{p^d})}$ which over $W(k)$ is $G(W(k))$-conjugate to μ (cf. [Mi3, 4.6-7]). So, from many points of view, for the study of Shimura σ-crystals over a perfect field k_0 of characteristic p, we can assume that k_0 is a finite field (if $k_0 \neq \overline{k_0}$, then passing to a finite field extension k_1 of k_0, M^a, defined as above but working with $g(\varphi \otimes 1)$ instead of φ for a suitable $g \in G(W(k_1))$, is still a free $\mathbb{Z}_p$-module of rank $\dim_{W(k)}(M)$).

9) Any filtered σ-crystal (M, F^1, φ), with F^1 a proper direct summand of M, can be identified with the Shimura filtered σ-crystal $(M, F^1, \varphi, GL(M))$, and any σ-crystal (M, φ) which can be extended to a filtered σ-crystal (M, F^1, φ), with F^1 a proper direct summand of M, can be identified with the Shimura σ-crystal $(M, \varphi, GL(M))$.

10) Any Shimura (filtered) σ-crystal over k gives birth to a Shimura (filtered) isocrystal over k, by making p invertible. But not any Shimura (filtered) isocrystal over k is obtained in this manner: the Shimura isocrystals over k can have slopes which do not belong to the interval $[0, 1]$ and the reductive groups involved are not necessarily quasi-split.

11) The condition F^1 is a proper direct summand of M mentioned in 2.2.8 1) as well as the condition at least two of $\tilde{F}^i$ are different from zero, mentioned in 2.2.8 3), are inserted just to avoid the trivial cases. So the cocharacter μ of 2.2.8 c) and d) does not factor through the center of $GL(M)$. When we have $F^1 = 0$ or $F^1 = M$ in 2.2.8 1) (resp. $M = \tilde{F}^i$ in 2.2.8 3), we speak about a trivial Shimura (filtered) σ-crystals (resp. a trivial generalized p-divisible object of $\mathcal{MF}(W(k))$). Without a special reference, we assume that were are in a non-trivial situation.

12) For the sake of flexibility it is important in 2.2.8 1) not to assume that M has a symplectic perfect form on it, or that $\dim(F^1) = \dim(F^0)$ (as we have in the cases emerging from principally polarized abelian varieties over $W(k)$). So we completely ignore the phenomenon of Tate-twists. However when we have a perfect form $p_M : M \otimes M \to W(k)(1)$ fixed by G, we speak about a principally quasi-polarized Shimura (filtered) σ-crystal. As notation, we put p_M as part of some n-tuple at the very end; example: $(M, F^1, \varphi, G, (v_\alpha)_{\alpha \in \mathcal{J}}, p_M)$. Moreover we do not assume that $G_{B(k)}$ is the subgroup of $GL(M[\frac{1}{p}])$ fixing v_α, $\forall \alpha \in \mathcal{J}$: we just assume that it is the subgroup of $\mathrm{GSp}(M[\frac{1}{p}], p_M)$ fixing these tensors.

2.2.10. Shimura filtered F-crystals. Let (M, F^1, φ, G) be a Shimura filtered σ-crystal. Let H be a smooth subscheme of G containing the origin, and let R be the ring of the formal completion of H at the origin. There is an isomorphism $\tilde{f} : R \tilde{\to} W(k)[[x_1, \ldots, x_d]]$, with d the relative dimension of H over $W(k)$ in the origin. Let Φ_R be the Frobenius of R extending σ on $W(k)$, and taking x_i to x_i^p, $i = \overline{1,d}$ (we use the identification defined by $\tilde{f}$). Then the filtered F-crystal over R/pR defined by the quadruple

$$(M \otimes R, F^1 \otimes R, \Phi, \nabla),$$

with $\Phi := h(\varphi \otimes id)$ defined by the universal element $h \in H(R)$ of H, is called a Shimura filtered F-crystal (defined by (M, F^1, φ, G)) ([Fa2, th. 10] guarantees that there is a unique connection ∇ on $M \otimes R$ such that Φ is ∇-parallel in the sense of [Fa2, ch. 7]; it is integrable and nilpotent mod p). We denote such a Shimura filtered F-crystal by a sextuple $(M, F^1, \varphi, G, H, \tilde{f})$. When $H = G$, we denote it by a quintuple $(M, F^1, \varphi, G, \tilde{f})$. Usually H is a smooth subgroup of G. When we want to emphasize the family $(v_\alpha)_{\alpha \in \mathcal{J}}$, then we denote a Shimura filtered F-crystal also by a septuple $\big(M, F^1, \varphi, G, H, \tilde{f}, (v_\alpha)_{\alpha \in \mathcal{J}}\big)$, and then, by abuse of notation, we still write $(v_\alpha)_{\alpha \in \mathcal{J}}$ for their images in the tensor algebra of $M \otimes R$. We have:

$\nabla(v_\alpha) = 0$, $\Phi(v_\alpha) = v_\alpha$, *and* v_α *belongs to the* F^0*-filtration of the tensor algebra of* $(M \oplus M^*) \otimes R[\frac{1}{p}]$, $\forall \alpha \in \mathcal{J}$.

This implies: for any point of R/pR with values in a perfect field k_1 we get, by pulling back, a Shimura σ_1-crystal over k_1 (here σ_1 is the Frobenius automorphism of $W(k_1)$).

Similarly we speak about principally quasi-polarized Shimura filtered F-crystals. Moreover, the conventions of 2.2.9 12) apply as well.

2.2.11. Definitions. 1) A non-trivial Shimura filtered Lie σ-crystal is a filtered Lie σ-crystal $\big(\mathfrak{g}, \varphi, F^0(\mathfrak{g}), F^1(\mathfrak{g})\big)$, with $\mathfrak{g}$ the Lie algebra of a reductive group G over $W(k)$, for which there is an injective cocharacter $\mu : \mathbb{G}_m \hookrightarrow G$ such that:

– $(G, [\mu])$ is a Shimura group pair over $\mathrm{Spec}(W(k))$;

– it defines $F^0(\mathfrak{g})$ and $F^1(\mathfrak{g})$: i.e. $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^{-1}$, with $\mathfrak{g}^1 \neq 0$, with $\alpha \in \mathbb{G}_m(W(k))$ acting through μ on $\mathfrak{g}^i$ as the multiplication with α^{-i}, $i = \overline{-1,1}$, and with $F^1(\mathfrak{g}) = \mathfrak{g}^1$ and $F^0(\mathfrak{g}) = \mathfrak{g}^0 \oplus \mathfrak{g}^1$.

2) A non-trivial Shimura Lie σ-crystal is a Lie σ-crystal $(\mathfrak{g}, \varphi)$ which can be enlarged to a non-trivial Shimura filtered Lie σ-crystal $\big(\mathfrak{g}, \varphi, F^0(\mathfrak{g}), F^1(\mathfrak{g})\big)$. Writing $\varphi = a \circ \mu(\frac{1}{p})$, a is a σ-linear automorphism of $\mathfrak{g}$, whose class $[a] \in \mathrm{Aut}_{\sigma-\mathrm{lin}}(\mathfrak{g})/G^{\mathrm{ad}}(W(k))$ (modulo inner automorphism of $\mathfrak{g}$) does not depend on the choice of μ. We call $[a]$ the automorphism class of the non-trivial Shimura Lie σ-crystal $(\mathfrak{g}, \varphi)$.

3) A Lie σ-crystal $(\mathfrak{g}, \varphi)$, with $\mathfrak{g}$ the Lie algebra of a reductive group G over $W(k)$, and with $\varphi(\mathfrak{g}) = \mathfrak{g}$ is called a trivial Shimura Lie σ-crystal. When we do not bother about trivial or non-trivial, we speak about Shimura (filtered) Lie σ-crystals.

4) If in 1-3) above G is an adjoint group then we speak about a Shimura adjoint (filtered) Lie σ-crystal.

2.2.12. Definitions. A Shimura filtered Lie σ-crystal $\big(\mathfrak{g}, \varphi, F^0(\mathfrak{g}), F^1(\mathfrak{g})\big)$ is said to be:

a) of parabolic type, if the parabolic Lie subalgebra of $\mathfrak{g}$ corresponding to non-negative slopes of $(\mathfrak{g}, \varphi)$ is contained in $F^0(\mathfrak{g})$;

b) of Borel type, if there is a Borel Lie subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ contained in $F^0(\mathfrak{g})$ and such that $\varphi(\mathfrak{b}) \subset \mathfrak{b}$;

b') of strong Borel type, if it is of Borel type and if the parabolic Lie subalgebra of $\mathfrak{g}$ corresponding to non-negative slopes of $(\mathfrak{g}, \varphi)$ is a Borel Lie subalgebra (so there is only one parabolic Lie subalgebra of $\mathfrak{g}$ contained in $F^0(\mathfrak{g})$ and left invariant by φ);

c) cyclic diagonalizable, if there is a $W(k)$-basis $\{e_i | i \in A\}$ of $\mathfrak{g}$ (here A is the set $\{1, \ldots, \dim_{W(k)}(\mathfrak{g})\}$) such that for subsets $C \subset B \subset A$, $\{e_i | i \in C\}$ is a $W(k)$-basis of $F^1(\mathfrak{g})$ and $\{e_i | i \in B\}$ is a $W(k)$-basis of $F^0(\mathfrak{g})$, and there is a permutation π of A for which we have $\varphi(e_i) = p^{\varepsilon_i} e_{\pi(i)}$, with $\varepsilon_i \in \{-1, 0, 1\}$ equal to 1 if $i \in C$, equal to 0 if $i \in B \backslash C$ and equal to -1 if $i \in A \backslash B$; if $\pi = 1_A$, we drop the word cyclic.

2.2.13. Remark. Any Shimura σ-crystal (M, φ, G) has attached to it a Shimura Lie σ-crystal $(\mathrm{Lie}(G), \varphi)$, as well as a Shimura adjoint Lie σ-crystal $\big(\mathrm{Lie}(G^{\mathrm{ad}}), \varphi\big)$, where $\mathrm{Lie}(G^{\mathrm{ad}}) \subset \mathrm{Lie}(G^{\mathrm{der}})[\frac{1}{p}] \subset \mathrm{Lie}(G)[\frac{1}{p}]$ (this results trivially from rm. 8) of 2.2.9, by passage to $W(\bar{k})$). Similarly, for a Shimura filtered σ-crystal we speak about the Shimura (adjoint) filtered Lie σ-crystal attached to it.

2.2.14. Remark. Whenever we want to emphasize k in 2.2.11, we speak about a Shimura (adjoint) (filtered) Lie σ-crystal of being over k. If the reductive group G is split, then we speak about a split Shimura (adjoint) (filtered) (Lie) σ-crystal. If just G^{ad} is split then we speak about a split Shimura (adjoint) (filtered) Lie σ-crystal. We also speak about Shimura (adjoint) (filtered) (Lie) isocrystals (over k), obtained from Shimura (adjoint) (filtered) (Lie) σ-crystals, by making p invertible. For instance with the notations of 2.2.8 1), $(M[\frac{1}{p}], \varphi, G_{B(k)})$ is a Shimura isocrystal.

2.2.15. Remark. Though G. Shimura never dealt with the type of crystals we have introduced, we still feel that the terminology Shimura crystals used in the above part of 2.2 is appropriate, as Shimura was the first one (see the report [Sh]) to consider abelian varieties endowed with (algebraic) cycles, of whose adequate integral crystalline realization, are providing the first examples of the type of σ-crystals we have defined in 2.2.8. Similarly we think that the terminology of a Shimura group or Lie pair is justified. Of course we could argue if 2.2.8 5-6) are justified. We think it is a convenient terminology, very much related to the one introduced in 2.2.8 1-4). The study of such isocrystals, to our knowledge, was started in [Ko1].

2.2.16. Lemma. *We assume that $k = \bar{k}$. Let (M, F^1, φ) be a filtered σ-crystal over k, with F^1 a proper direct summand of M. It is cyclic diagonalizable iff there is a subtorus T of $GL(M)$ such that (M, F^1, φ, T) is a Shimura filtered σ-crystal.*

Proof: If (M, F^1, φ) is cyclic diagonalizable, then we can take T to be the maximal torus of $GL(M)$ which leaves invariant the $W(k)$-submodules of M generated by any element e_i, $i = \overline{1, dim(M)}$, of a basis of M which is as in 2.2.1 b).

If there is a subtorus T of $GL(M)$ such that (M, F^1, φ, T) is a Shimura filtered σ-crystal, then the proof of cyclic diagonalizability of (M, F^1, φ) is entirely as of 4.1.2.

2.2.16.1. Remark. In practice we take T to be the smallest subtorus of $GL(M)$ such that $\mathrm{Lie}(T)$ contains $\mu(\mathrm{Lie}(\mathbb{G}_m))$, where $\mu : \mathbb{G}_m \to GL(M)$ is the canonical split defined in [Wi].

2.2.17. Definitions. Let (M, φ, G) be a Shimura σ-crystal over k. By a lift of it of quasi CM type we mean a direct summand (filtration) F^1 of M such that the quadruple (M, F^1, φ, G) is a Shimura filtered σ-crystal with the property that the intersection of $F^0(\mathrm{Lie}(G_{B(k)}))$ with the maximal Lie subalgebra $\mathfrak{p}_0$ of $\mathrm{Lie}(G_{B(k)})$ left invariant by φ and having the property that the isocrystal $(\mathfrak{p}_0, \varphi)$ has all the slopes zero, contains a φ-invariant Lie subalgebra, which is the Lie algebra of a maximal torus of G. We also say that (M, F^1, φ, G) is a lift of quasi CM type of (M, φ, G).

By a lift of an abelian variety A_k over k of quasi CM type we mean an abelian variety A over $W(k)$ lifting A_k and such that the filtered σ-crystal attached to A is a lift of quasi CM type of the σ-crystal attached to A_k (here we use the identifications of 2.2.9 9)). Here the abbreviation CM refers to complex multiplication.

2.2.17.1. Remark. A Shimura σ-crystal can have more than one lift of quasi CM type. For instance this is the case for the σ-crystal attached to a product of supersingular elliptic curves over k.

2.2.18. Remark. There are plenty of examples of filtered σ-crystals (M, F^1, φ), with $\dim(M) = 2\dim(F^1)$, such that the σ-crystal (M, φ) does not have any lift of quasi CM type. The idea of constructing such examples is: if $k = \bar{k}$, then a σ-crystal (M, φ) which has a lift of quasi CM type is a direct sum of σ-crystals having only one slope (cf. 2.2.19). Moreover, the different invariants (like Hodge numbers, the p-adic valuation of determinants, etc.) of these factors are of particular type, and so very easy computable. So we just have to pass to an adequate isogeny. Here is an example with slopes having quite small denominators.

Example. Let M be of rank 6 over $W(k)$, and let φ such that it sends the elements of an ordered basis $(e_1, ..., e_6)$ of M into the ordered set $(e_2, pe_3, e_4, pe_5, e_1, pe_6)$. Let M' be the lattice of $M[\frac{1}{p}]$ generated by M and $\frac{e_2+e_4}{p}$. It is easy to see (arguments involving determinants) that (M', φ) is not cyclic diagonalizable. Moreover, if F^1 is the $W(k)$-submodule of M generated by e_2, e_6 and $e_1 + e_3$, then the triple (M, F^1, φ) is a filtered σ-crystal.

We would also like to recall that there are filtered σ-crystals (M, F^1, φ) with $\dim(M) = 2\dim(F^1)$ such that there is a lattice M' of $M[\frac{1}{p}]$ with the property that the pair (M', φ) is a σ-crystal which can not be extended to a filtered σ-crystal.

Example. Let M be of rank 4 over $W(k)$, and let φ such that it sends the elements e_0, e_1, f_0 and f_1 of a basis of M, into pe_0, e_1, pf_1 and respectively f_0. Let M' be the lattice of $M[\frac{1}{p}]$ generated by e_0, e_1, $\frac{e_0+f_0}{p}$ and f_1. Obviously (M', φ) can not be extended to a filtered σ-crystal.

2.2.19. Proposition. *We assume that $k = \bar{k}$. Let F^1 be a lift of quasi CM type of a Shimura σ-crystal (M, φ, G). Then the filtered σ-crystal (M, F^1, φ) is cyclic diagonalizable.*

Proof: Let $\mu : \mathbb{G}_m \to GL(M)$ be the canonical split of (M, F^1, φ) as defined in [Wi]. From its functoriality we deduce (cf. def. 1) of 2.2.8) that μ factors through G. From the def. 2.2.17 we deduce that there is a $\mathbb{Q}_p$-Lie subalgebra of $F^0(\mathrm{Lie}(G_{B(k)}))$ formed by elements fixed by φ, and which generates the Lie algebra of a maximal torus $T_{B(k)}$ of $G_{B(k)}$. Again from the functoriality of μ we deduce that μ fixes the elements of this $\mathbb{Q}_p$-Lie algebra, and so the generic fibre of μ factors through $T_{B(k)}$. Let $T^1_{B(k)}$ be the smallest subtorus of $T_{B(k)}$ such that the generic fibre of μ factors through it and its Lie algebra is left invariant by φ. It is easy to check that the closure of $T^1_{B(k)}$ in $GL(M)$ is a subtorus T^1 of G. So μ factors through T. We get a Shimura filtered σ-crystal (M, F^1, φ, T^1): this can be checked easily starting from a $\mathbb{Z}_p$-structure as in 2.2.9 8). Now everything results from 2.2.16. This ends the proof of the proposition.

2.2.20. Shimura p-divisible groups. Let X be a regular formally smooth scheme over $W(k)$. A Shimura p-divisible group over X is a pair $(D, (v_\alpha)_{\alpha\in\mathcal{J}})$ comprising from a p-divisible group D over X, and a family of sections $(v_\alpha)_{\alpha\in\mathcal{J}}$ of the essential tensor algebra of the filtered isocrystal (sheaf) on $X^\wedge$ defined by the first crystalline cohomology group sheaf of $D \times D^t$, satisfying the following axiom:

A. There is an open affine covering of X, such that for any member $U = \mathrm{Spec}(R)$ of it whose special fibre is non-empty, denoting $M_R := H^1_{\mathrm{crys}}(D_U/U^\wedge)$, the mentioned sections produce sections (we keep the same notation) of $\mathcal{T}(M_R[\frac{1}{p}])$ satisfying:

i) they are crystalline sections, i.e they are annihilated by the natural connection on M_R, are in the F^0-filtration of $\mathcal{T}(M_R[\frac{1}{p}])$, and are fixed by any Frobenius of M_R defined by a Frobenius lift of $R^\wedge$;

ii) the closure of the subgroup of $GL(M_R[\frac{1}{p}])$ fixing v_α, $\forall\alpha \in \mathcal{J}$, in $GL(M_R)$ is a reductive group $\tilde{G}_R$ over R;

iii) the F^1-filtration of M_R is a proper direct summand in each geometric point of the special fibre of U.

2.2.21. Remarks. 1) Any p-divisible group whose associated crystal is a Shimura filtered σ-crystal or a Shimura filtered F-crystal with an emphasized family of tensors, gets naturally the structure of a Shimura p-divisible group.

2) The importance of this notion springs from 1.16. For the present paper we need it mostly just to simplify the presentation.

3) As in 2.2.9 6) we have similarly the notion of isomorphism between two Shimura p-divisible groups.

4) Sometimes we need to speak about principally quasi-polarized Shimura p-divisible groups. The extra thing we need for this is: D has a principal quasi-polarization p_D, which as a perfect form on M_R is fixed by $\tilde{G}_R$, for any U as above. Notation: $(D, (v_\alpha)_{\alpha\in\mathcal{J}}, p_D)$. Moreover we allow 2.2.20 ii) to be modified as in 2.2.9 12).

5) If X_1 is a regular formally smooth scheme over $W(k)$, then the pull back of a Shimura p-divisible group over X through a morphism $X_1 \to X$, is also a Shimura p-divisible group (over X_1).

6) Of course, instead of crystalline sections we can use étale sections. But in the case when X is not (the p-adic completion of) a pro-étale scheme over a smooth scheme over $W(k)$, it is more convenient to work with crystalline sections: see the way [Fa1, 7.1] is stated. Moreover, in §5 we define Shimura p-divisible groups over larger classes of schemes (including ones of positive characteristic), and so it is much more convenient to work in the crystalline context.

7) It is convenient to allow the family $(v_\alpha)_{\alpha\in\mathcal{J}}$ to vary under the constraint that the reductive groups $\tilde{G}_R$ of 2.2.20 ii) remain the same. In other words by enlarging $\mathcal{J}$ or by passing to a subset of $\mathcal{J}$ subject to the mentioned constraint we consider that we have essentially the same Shimura p-divisible group. This leads to the notion of quasi-isomorphism: two Shimura p-divisible groups over X, $E^1 = (D^1, (v^1_\alpha)_{\alpha\in\mathcal{J}})$ and $E^2 = (D^2, (v^2_\alpha)_{\alpha\in\mathcal{J}})$, are said to be quasi-isomorphic if there is an isomorphism $t : D^1 \tilde{\to} D^2$ such that for any U as in 2.2.20, the reductive group $\tilde{G}^1_R$ we get via E^1 is mapped through t isomorphically into $\tilde{G}^2_R$ obtained via E^2. Similarly for the principally quasi-polarized case.

8) We would like to point out that the terminologies of Shimura p-divisible groups and crystals do not quite match: in 2.2.20 we do not require $\tilde{G}_R$ to become quasi-split over all points of R with values in fields of positive characteristic, while in 2.2.8 the accent does not fall on the cycles. The reason we adopted these terminologies is of purely practical matters: in the case of Shimura p-divisible groups we are essentially interested in understanding the cycles and the refined structure of the reductive groups involved, while we use mainly the σ-crystals as a tool for achieving this goal (see §5).

The condition iii) of 2.2.20 A is inserted to match 2.2.9 11). The conventions of 2.2.9 11) extend naturally to the context of Shimura p-divisible groups.

9) If $\mathcal{D} = (D, (v_\alpha)_{\alpha\in\mathcal{J}})$ is a Shimura p-divisible group over X, then for any $n \in \mathbb{N}$, $D[p^n]$ defines an object $\mathcal{D}_n$ of $\mathcal{MF}^\nabla_{[0,1]}(X)$, with the property that the Lie object of $\mathcal{MF}^\nabla_{[-1,1]}(X)$ defined by gluing the objects defined naturally by $\mathrm{End}(M_R/p^nM_R)$, has a natural Lie subobject $\mathfrak{g}(\mathcal{D}_n)$ defined naturally by gluing $\mathrm{Lie}(\tilde{G}_R) \otimes R/p^nR$. We refer to the pair $(\mathcal{D}_n, \mathfrak{g}(\mathcal{D}_n))$ as the kernel of the multiplication by p^n of $\mathcal{D}$. Similarly, we use the same language in the principally quasi-polarized context.

2.2.22. Complements on Shimura σ-crystals. 1) A Shimura σ-crystal (M, φ, G) over k is said to be:

– essentially cyclic diagonalizable, if there is a torus T of G such that (M, φ, T) is a Shimura σ-crystal;

– cyclic diagonalizable, if there is a torus T as above, and we can proceed as in 4.1.2 to show that (M, F^1, φ) is cyclic diagonalizable (here F^1 is the unique direct summand of M such that (M, F^1, φ, T) is a Shimura filtered σ-crystal);

– potentially cyclic diagonalizable if over $\bar{k}$ it is essentially cyclic diagonalizable.

Here is what we mean by "we can proceed as in 4.1.2": if $\mu : \mathbb{G}_m \to T$ is the cocharacter defining the filtration class of (M, F^1, φ, T), then the σ-linear automorphism $a : M \tilde{\to} M$ defined by $a\mu(\frac{1}{p}) = \varphi$, defines a $\mathbb{Z}_p$-structure on M. If $k = \bar{k}$ then the essential cyclic diagonalizability implies the cyclic diagonalizability; this is not necessarily true if $k \neq \bar{k}$.

2) Let (M, F^1, φ, G) be a Shimura filtered σ-crystal. By the isogeny class of it we mean the set of isomorphism classes of Shimura filtered σ-crystals of the form $(g(M), g(F^1), \varphi, G)$, with $g \in G(B(k))$ (the closure of $G_{B(k)}$ in $GL(g(M))$ is a reductive group isomorphic to G -cf. [Ti, end of 2.5]-; so it is appropriate to write the things the way we did). Similarly we define the isogeny class of a Shimura σ-crystal or of a Shimura (adjoint) (filtered) Lie σ-crystal (even more generally: of a generalized Shimura p-divisible object over k or of $\mathcal{MF}(W(k))$).

3) Any isocrystal over $\bar{k}$ can be put in a cyclic diagonalizable form, cf. Dieudonné's classification of isocrystals over $\bar{k}$. So the symplest type of σ-crystals over k are those which can be put similarly in a cyclic diagonalizable form. What we defined (see 2.2.1 b) and 2.2.12 c)) in a context related to p-divisible groups can be defined in the general context of p-divisible objects of $\mathcal{MF}(W(k))$. For simplicity, we keep focussing our attention on filtered σ-crystals.

Let $n \in \mathbb{N}$. Here we consider only n-tuples $(e_1, e_2, ..., e_n)$ with $e_i \in \{0, 1\}$, $\forall i \in \{1, ..., n\}$. We are interested in them only from the circular point of view: for us $(e_1, ..., e_n)$ is "equally good" as $(e_j, e_{j+1}, ..., e_n, e_1, ..., e_{j-1})$, where $j \in \{2, ..., n\}$. We say that an n-tuple $(e_1, ..., e_n)$ is in standard form if

$$\sum_{i=1}^{n} e_i 2^{n-i} \geq \sum_{i=1}^{n} d_i 2^{n-i},$$

for any other n-tuple of the form $(d_1, ..., d_n) = (e_j, ..., e_n, e_1, ..., e_{j-1})$, for some $j \in \{2, ..., n\}$.

For any n-tuple $(e_1, ..., e_n)$ which is in standard form we can construct a circular diagonalizable filtered σ-crystal (M, F^1, φ) as follows:

M is a free $W(k)$-module of rank n, $\varphi(a_i) = p^{e_i} a_{i+1}$, where $n+1 = 1$ and $\{a_i | i = \overline{1, n}\}$ is a basis of M, while F^1 is the submodule of M generated by all a_i's with $e_i = 1$.

About (M, F^1, φ) we say: it is in standard form associated to $(e_1, ..., e_n)$; about $(e_1, ..., e_n)$ we say: it is a type of (M, F^1, φ). From the very def. 2.2.1 b) we get that any filtered σ-crystal which is cyclic diagonalizable is a direct sum of circular diagonalizable ones; moreover, any filtered σ-crystal which is circular diagonalizable has a type. Warning: there are circular diagonalizable filtered σ-crystals which do not have a unique type. For instance the quadruples $(1, 1, 0, 0)$ and $(1, 0, 1, 0)$ are producing isomorphic filtered σ-crystals.

4) By the deformation dimension of (M, φ, G) we mean the number

$$dd((M, \varphi, G)) := \dim_{W(k)}(\mathrm{Lie}(G)/F^1(\mathrm{Lie}(G))).$$

Here $F^1(\mathrm{Lie}(G))$ is defined as in 2.2.8 1) starting from any (it does not matter which one) direct summand F^1 of M such that (M, F^1, φ, G) is a filtered σ-crystal.

2.3. The standard Hodge situation.

2.3.1. Let (W,ψ) be a symplectic space over $\mathbb{Q}$, and let

$$f : \mathrm{Sh}(G,X) \hookrightarrow \mathrm{Sh}\big(\mathrm{GSp}(W\psi), S\big)$$

be an injective map. Let $p \geq 3$ be a rational prime. We assume the existence of a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W such that ψ induces a perfect form $\psi : L_{(p)} \otimes L_{(p)} \to \mathbb{Z}_{(p)}$ and the closure $G_{\mathbb{Z}_{(p)}}$ of G in $\mathrm{GSp}(L_{(p)},\psi)$ is a reductive group over $\mathbb{Z}_{(p)}$. We call such a $\mathbb{Z}_{(p)}$-lattice good with respect to the map f (cf. [Va2, 5.8.3]). This assumption implies that G is unramified over $\mathbb{Q}_p$. Let v be a prime of $E(G,X)$ dividing p. It is unramified over p [Mi3, 4.7]. Let $K := \mathrm{GSp}(W,\psi)(L_{(p)} \otimes \mathbb{Z}_p)$ and let $H := K \cap G(\mathbb{Q}_p)$. K is a hyperspecial subgroup of $\mathrm{GSp}(W,\psi)(\mathbb{Q}_p)$ (as $\psi : L_{(p)} \otimes L_{(p)} \to \mathbb{Z}_{(p)}$ is a perfect form) and H is a hyperspecial subgroup of $G(\mathbb{Q}_p)$ (as $G_{\mathbb{Z}_{(p)}}$ is a reductive group and as $H = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$). Let

$$e := \frac{\dim_{\mathbb{Q}}(W)}{2}.$$

Let $(v_\alpha)_{\alpha\in\mathcal{J}}$ be a family of homogeneous tensors of the essential tensor algebra of $W \oplus W^*$, such that G is the subgroup of $\mathrm{GSp}(W,\psi)$ fixing the tensors of this family (cf. [De3, 3.1 c)]). Sometimes it is more convenient to work with an enlarged family $(v_\alpha)_{\alpha\in\mathcal{J}'}$ (with $\mathcal{J} \subset \mathcal{J}'$) of such tensors, such that G is the subgroup of $GL(W)$ fixing its members. Also it is convenient to consider that all the elements of $\mathrm{Lie}(Z(G))$ are part of this enlarged family. As $G(\mathbb{Q})$ contains the group of scalar automorphisms of W, we deduce that for any $\alpha \in \mathcal{J}'$, $v_\alpha \in W^{\otimes n_\alpha} \otimes W^{*\otimes n_\alpha}$, with $n_\alpha := \frac{\deg(v_\alpha)}{2}$. Let $G^0_{\mathbb{Z}_{(p)}}$ be the maximal (connected) reductive subgroup of $G_{\mathbb{Z}_{(p)}}$ fixing ψ. So $G^0_{\mathbb{Z}_{(p)}} = \mathrm{Sp}(L_{(p)},\psi) \cap G_{\mathbb{Z}_{(p)}}$ if the center of $G^0_{\mathbb{Q}}$ has -1_W as a $\mathbb{Q}$–valued point.

To any $x \in X$ it corresponds a homomorphism $\mu_x : \mathbb{G}_m \to G_{\mathbb{C}}$ (see [Va2, 2.2]). We often prefer to view G as a subgroup of $GL(W^*)$ or of $\mathrm{GSp}(W^*,\tilde{\psi})$, where $\tilde{\psi}$ is the perfect form on W^* induced by ψ, and then we denote by $\mu_x^* : \mathbb{G}_m \to G_{\mathbb{C}}$ the homomorphism (in the dual context) induced by μ_x. So μ_x^* is the homomorphism defined, as in [Va2, 2.2], but for the Hodge structure on W^* defined by $x \in X$. It achieves a decomposition $W^* \otimes \mathbb{C} = F_x^1 \oplus F_x^0$, with $\beta \in \mathbb{G}_m(\mathbb{C})$ acting through it by multiplication with β^{-i} on F_x^i, $i = \overline{0,1}$.

2.3.2. We get an injective map $f : (G,X,H,v) \hookrightarrow \big(\mathrm{GSp}(W,\psi), S, K, p\big)$ of Shimura quadruples. Let $\mathcal{M}$ be the extension to $O_{(v)}$ of the integral canonical model of $\big(\mathrm{GSp}(W,\psi), S, K, p\big)$. It is known (see [Va2, 3.2.9]) that $\mathcal{M}$ parametrizes isomorphism classes of principally polarized abelian schemes of dimension e over $O_{(v)}$-schemes, having level-N symplectic similitude structure for any $N \in \mathbb{N}$ relatively prime to p (the choice of a $\mathbb{Z}$-lattice L of W such that $L \otimes \mathbb{Z}_{(p)} = L_{(p)}$ and $\psi : L \otimes L \to \mathbb{Z}$ is a perfect form is implicit here, cf. [Va2, 4.1]).

2.3.3. Let $\mathcal{N}$ be the integral canonical model of (G,X,H,v) (for $p \geq 5$ cf. [Va2, 6.4.1] and the definitions of [Va2, 3.2.6]; for $p = 3$ cf. §6). Due to the extension property of $\mathcal{M}$, we have a natural morphism $\mathcal{N} \to \mathcal{M}$ (cf. [Va2, rm. 4) of 3.2.7]). From [Va2, 3.2.12] we deduce that $\mathcal{N}$ is the normalization of the closure of $\mathrm{Sh}(G,X)/H$ in $\mathcal{M}$. From the proof of [Va2, 3.4.1] we deduce that the morphism $\mathcal{N} \to \mathcal{M}$ is finite, and not just pro-finite.

The universal principally polarized abelian scheme over $\mathcal{M}$ gives birth, by pull back, to a principally polarized abelian scheme $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ over $\mathcal{N}$. $\mathcal{A}$ is naturally endowed with a family $(w_\alpha^{\mathcal{A}})_{\alpha\in\mathcal{J}'}$ of Hodge cycles. The points of $\mathcal{N}$ with values in fields $\mathcal{K}$ of characteristic zero, give birth to principally polarized abelian varieties (obtained from $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ by pull back), having a family $(w_\alpha)_{\alpha\in\mathcal{J}'}$ of Hodge cycles (obtained from $(w_\alpha^{\mathcal{K}})_{\alpha\in\mathcal{J}'}$ by pull back; they are defined over $\mathcal{K}$, due to the existence of the level structures), and satisfying some additional conditions and level structures (cf. [Va2, 4.1]).

We denote by H_0 an arbitrary compact open subgroup of $G(\mathbb{A}_f^p)$ such that:

– the subgroup $H_0 \times H$ of $G(\mathbb{A}_f)$ is smooth for (G, X) (so the morphism $\mathcal{N} \to \mathcal{N}/H_0$ is a pro-étale cover, cf. [Va2, 3.4.8 and 6.4.2.1]);

– there is a universal principally polarized abelian scheme $(\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ over $\mathcal{N}/H_0$, with $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ obtained from $(\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ through the pro-étale cover $\mathcal{N} \to \mathcal{N}_{H_0}$.

$\mathcal{N}/H_0$ is a quasi-projective smooth scheme over $O_{(v)}$, cf. [Va1, 6.4.1].

2.3.3.1. If (G, X) is of compact type, then $\mathcal{N}/H_0$ is a projective smooth scheme over $O_{(v)}$, cf. [Va2, 6.4.1.1 2)] (the proof of loc. cit. applies to the case $p = 3$ as well).

2.3.4. Definition. The $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W is said to be crystalline well positioned for the map f w.r.t. v, if, besides the properties mentioned in 2.3.1 (i.e. it is good with respect to f), it satisfies the following property:

$(*)$ *for any perfect field k of characteristic p and for any $z \in \mathcal{N}(W(k))$, the principally polarized abelian variety (A, p_A) over $W(k)$ having a family $(w_\alpha)_{\alpha\in\mathcal{J}}$ of Hodge cycles (obtained from $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ and its family $(w_\alpha^{\mathcal{A}})_{\alpha\in\mathcal{J}}$ of Hodge cycles through the point z), has the property that the closure in $GL(M)$ (with $M := H^1_{\mathrm{crys}}\big(A/W(k)\big) = H^1_{dR}(A/W(k))$) of the subgroup of $GL(M[\frac{1}{p}])$ fixing the perfect alternating form $p_A : M \otimes M \to W(k)(1)$ (induced by the polarization p_A) and the crystalline (de Rham) component t_α of ω_α (t_α is a homogeneous tensor of $\mathcal{T}(M[\frac{1}{p}])$), $\forall\alpha \in \mathcal{J}$, is a reductive group $\tilde{G}$ over $W(k)$.*

If $L_{(p)}$ is crystalline well positioned for the map f w.r.t. any prime v of $E(G, X)$ dividing p, then we say that $L_{(p)}$ is crystalline well positioned for the map f.

2.3.5. Definition. A triple $(f, L_{(p)}, v)$ as in 2.3.1, with $L_{(p)}$ a $\mathbb{Z}_{(p)}$-lattice of W which is crystalline well positioned for the map f w.r.t. v, is called a standard Hodge situation, abbreviated SHS. If moreover there is a $\mathbb{Z}_{(p)}$-subalgebra $\mathcal{B}$ of $\mathrm{End}(L_{(p)})$, which over $W(\mathbb{F})$ is a product of matrix algebras, which is self dual with respect to ψ, and is such that the subgroup of $\mathrm{GSp}(L_{(p)}, \psi)$ fixing its elements has $G_{\mathbb{Z}_{(p)}}$ as its connected component of the origin, then we speak about a standard PEL situation, and denote it by $(f, L_{(p)}, v, \mathcal{B})$ (cf. [Va2, 4.3.11]).

2.3.5.1. Exercise. Let (G_0, X_0) be an adjoint Shimura variety whose simple factors are of A_n type for some $n \in \mathbb{N}$. Let $p > 2$ be a prime such that G_0 is unramified over $\mathbb{Q}_p$. Show that there is an injective map $f\colon (G, X) \hookrightarrow (\mathrm{GSp}(W, \psi), S)$, with $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) = (G_0, X_0)$, and there is a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W good for f, such that for any prime v of $E(G, X)$ dividing p we get a standard PEL situation $(f, L_{(p)}, v, \mathcal{B})$, for a suitable $\mathbb{Z}_{(p)}$-subalgebra $\mathcal{B}$ of $\mathrm{End}(L_{(p)})$. Hint: use [Va2, 6.5 and Case 2 of 6.6.5].

2.3.5.2. Remark. Let $(f, L_{(p)}, v)$ be a SHS as above. Let $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) = \prod_{i\in\mathcal{F}}(G_i, X_i)$ be the direct factor decomposition in simple Shimura varieties. Here $\mathcal{F}$ is a finite set of indices. For any $i \in \mathcal{F}$ there is an injective map $f_i : (G^i, X^i) \hookrightarrow (G, X)$ such that:

– $G^{i\mathrm{der}}$ is the semisimple subgroup of G naturally isogeneous to G_i (so $(G^{i\mathrm{ad}}, X^{i\mathrm{ad}}) = (G_i^{\mathrm{ad}}, X_i^{\mathrm{ad}})$);

– $(f \circ f_i, L_{(p)}, v_i)$ is a SHS, for any prime v_i of $E(G^i, X^i)$ dividing v.

This is an immediate consequence of [Va2, rm. 11) of 3.2.7]. So from many points of view for things involving a SHS $(f, L_{(p)}, v)$, we can assume that G^{ad} is a simple $\mathbb{Q}$–group. However, for the sake of generality, we never assume this apriori.

2.3.6. Example. If the family of tensors $(v_\alpha)_{\alpha\in\mathcal{J}}$ is $\mathbb{Z}_{(p)}$-well positioned with respect to ψ for the group G, if it is enveloped by $L_{(p)}$ (see [Va2, 4.3.4] for the meaning of these), and if it is of degree at most $2(p-2)$, then the triple $(f, L_{(p)}, v)$ is a SHS (cf. [Va2, 5.1 and 5.6.5]). More generally, if there is a family of tensors of $\mathcal{T}(L_{(p)} \otimes W(\mathbb{F}))$ which is $W(\mathbb{F})$-very well positioned for $G_{W(\mathbb{F})}$, then the triple $(f, L_{(p)}, v)$ is a SHS, cf. [Va1, 5.6.9].

2.3.7. Convention. Whenever we consider a SHS, we use freely the notations of 2.3.1-3.

2.3.8. Remarks. 1) We do not know when the converse of 2.3.6 is true. That is, we do not know for which SHS $(f, L_{(p)}, v)$ there is a family $(v_\alpha)_{\alpha\in\mathcal{J}}$ of homogeneous tensors of the essential tensor algebra of $W \oplus W^*$, which is $\mathbb{Z}_{(p)}$-well positioned with respect to ψ for G, is enveloped by $L_{(p)}$, and is of degree at most $2(p-2)$. Obviously for $p = 3$ we need a standard PEL situation.

2) A major result of §6 states that any $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W good with respect to f is automatically crystalline well positioned for the map f.

3) Usually when we mention the polarizations we use the set of indices $\mathcal{J}$ (i.e. we would like to "carry" as few Hodge cycles as possible); when we do not, then we use $\mathcal{J}'$. If this is confusing to the reader, the reader can assume that $\mathcal{J} = \mathcal{J}'$.

2.3.8.1. Exercise. Let (G, X, H, v) be a Shimura quadruple of Hodge type. Assuming 2.3.8 2) show that there is a SHS $(f, L_{(p)}, v)$ for which we do have $H = G(L_{(p)} \otimes \mathbb{Z}_p)$. Hint: just restate the proof of [Va2, 6.7.2].

2.3.9. Proposition. *a) $\tilde{G}$ is an inner form of $G_{W(k)}$; moreover $\tilde{G}^{\mathrm{ab}}$ is a split torus.*

b) The reductive group $\tilde{G}$ mentioned in 2.3.4 splits over a finite Galois cover $\mathrm{Spec}(W(k_1))$ *of* $\mathrm{Spec}(W(k))$ *of order dividing a fixed power of the order q_G of the center of the simply connected group cover of $G_{\mathbb{C}}^{\mathrm{der}}$. Moreover, the Galois group* $\mathrm{Gal}(k_1/k)$ *is of index of nilpotence at most two. If for which simple factor (G_i, X_i) of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ of $D_n^{\mathbb{R}}$ type with $n \in 2\mathbb{N}$, the semisimple subgroups of $G_{\mathbb{C}}^{\mathrm{der}}$ naturally isogeneos to a simply factor of $G_{i\mathbb{C}}$, are simply connected, then this Galois group is abelian.*

c) If all the factors of G^{ad} are of C_n or A_n type for some $n \in \mathbb{N}$, and if for any simple factor $G'_{\mathbb{C}}$ of $G_{\mathbb{C}}^{\mathrm{ad}}$ of A_n type, the semisimple subgroup of $G_{\mathbb{C}}^{\mathrm{der}}$ naturally isogeneous to is a simply connected semisimple group, then $\tilde{G}$ is isomorphic to $G_{W(k)}$, regardless of the choice of the $W(k)$-valued point z.

d) If G^{der} is a simply connected semisimple group, then the Galois group of b) is of exponent 2.

Proof: First we remark that $k \supset \mathbb{F} = \overline{k(v)}$. This is so due to the existence of level-N structures (of A_k) (so the N-th roots of unity are contained in k), $\forall N \in \mathbb{N}$ with $(N,p)=1$. So $B(k) \supset B(\mathbb{F})$. The proposition is a consequence of this last inclusion and of Fontaine' s comparison theory. We present the details.

a) $G_{B(\mathbb{F})}$ is a split group: G is unramified over $\mathbb{Q}_p$. So $G_{W(k)}$ is a split group. Fontaine's comparison theory implies that $\tilde{G}_{B(k)}$ is an inner form of $G_{B(k)}$. But $\tilde{G}$ splits over a finite étale extension of $W(k)$. So over this extension it becomes isomorphic to $G_{W(k)}$, cf. the uniqueness of a split reductive group associated to a given root datum (see [SGA3, Vol. 3, p. 305]). We deduce that $\tilde{G}$ is an inner form of $G_{W(k)}$. This implies that $\tilde{G}^{\mathrm{ab}}$ is a split torus.

Now we prove simultaneously the other parts. Any element of $\mathcal{T}(L^*_{(p)} \otimes \mathbb{Q}_p)$ fixed by $G_{\mathbb{Q}_p}$ gives birth (cf. the identification of [Va2, 5.1.2]) through Fontaine's comparison theory to an element of $\mathcal{T}(M[\frac{1}{p}])$ fixed by $\tilde{G}$. As $B(k) \supset B(\mathbb{F})$, we can use $B(\mathbb{F})$-linear combinations of such elements. We denote by $\mathrm{Fix}(G_{B(\mathbb{F})})$ the set of tensors of $\mathcal{T}(L^*_{(p)} \otimes B(\mathbb{F}))$ fixed by $G_{B(\mathbb{F})}$. So any element of $\mathrm{Fix}(G_{B(\mathbb{F})})$ has a counterpart in $\mathcal{T}(M[\frac{1}{p}])$. As $G_{B(\mathbb{F})}$ is a split group, and as all the simple Lie factors of $\mathrm{Lie}(G^{\mathrm{ad}}_{B(\mathbb{F})})$ are classical (split) simple Lie algebras over $B(\mathbb{F})$, we can often "capture" the fact that $G_{B(\mathbb{F})}$ is split using just the set $\mathrm{Fix}(G_{B(\mathbb{F})})$.

We first assume that for any simple factor of $G^{\mathrm{ad}}_{B(\mathbb{F})}$ coming from a simple factor of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ of A_n ($n \in \mathbb{N}$) or $D^{\mathbb{R}}_n$ type ($n \in 2\mathbb{N}$), the semisimple subgroup of $G^{\mathrm{der}}_{B(\mathbb{F})}$ naturally isogeneous to it, is simply connected. This is automatically so in cases c) and d). Using this assumption, [De3, 3.1 (a)], the Weyl complete reducibility theorem, the structure of the irreducible representations of a direct sum of absolutely simple split semisimple Lie algebras over $B(k)$, and the fact that the irreducible subrepresentations of $L^*_p \otimes B(k)$ of a simple Lie factor of $\mathrm{Lie}(G^{\mathrm{ad}}_{B(\mathbb{F})})$ are given by minimal weights we deduce:

Fact. *For any simple factor $\mathfrak{g}_0$ of $\mathrm{Lie}(G^{\mathrm{der}}_{B(\mathbb{F})})$, we can single out the tensor product $\mathcal{TP}$ of the classical representation W_0 of $\mathfrak{g}_0$ (over $B(\mathbb{F})$) (its complex version is presented in [He, §8 of ch. 2]) with other representations of the same type of other simple factors of $\mathrm{Lie}(G^{\mathrm{ad}}_{B(k)})$, just by using $\mathrm{Fix}(G_{B(\mathbb{F})})$; moreover, the bilinear form (if it is not zero, then it is non-degenerate and unique up to a non-zero scalar) b_0 on W_0 annihilated by $\mathfrak{g}_0$ (if $\mathfrak{g}_0$ is of A_n Lie type, $n \in \mathbb{N} \setminus \{1\}$, then this bilinear form is zero), gives birth to a bilinear form on $\mathcal{TP}$ fixed by $G_{B(\mathbb{F})}$.*

As $\tilde{G}$ is a inner form of $G_{W(k)}$ we deduce that all the factors of $\tilde{G}^{\mathrm{ad}}$ are absolutely simple factors. From the above fact and Fontaine's comparison theory, we get the existence of an absolutely simple factor $\tilde{\mathfrak{g}}_0$ of $\mathrm{Lie}(\tilde{G}_{B(k)})$, which is an inner form of $\mathfrak{g}_0$, and which has a representation on $\tilde{W}_0 := W_0 \otimes B(k)$, annihilating a bilinear form $\tilde{b}_0$ on $\tilde{W}_0$. $\tilde{b}_0$ is a non-degenerate form iff the Lie type of $\mathfrak{g}_0$ is not A_n, $n \in \mathbb{N}$, $n \geq 2$.

We first treat the case when all the factors of G^{ad} are of A_n or C_n Lie type ($n \in \mathbb{N}$). By reasons of dimensions (resp. due to the existence of $\tilde{b}_0$) in case $\mathfrak{g}_0$ is of A_n Lie type

(resp. in case $\mathfrak{g}_0$ is of C_n Lie type), we get that $\tilde{\mathfrak{g}}_0$ is isomorphic to $\mathfrak{g}_0 \otimes B(k)$. We deduce that $\tilde{\mathfrak{g}}_0$ is a split Lie algebra over $B(k)$. We conclude that $\tilde{G}^{\mathrm{der}}$ is a split group. So (cf. a)) $\tilde{G}$ is a split group. As its generic fibre is an inner form of $G_{B(k)}$ we deduce that $G_{B(k)}$ is isomorphic to $\tilde{G}_{B(k)}$. We conclude as above that $\tilde{G}$ is isomorphic to $G_{W(k)}$. This ends the proof of c) and d).

Coming back to the general case, it is enough to show: if $\tilde{b}_0$ is a non-degenerate symmetric form (so $\mathfrak{g}_0$ is of B_n or D_n Lie type), then $\tilde{\mathfrak{g}}_0$ and $\mathfrak{g}_0 \otimes B(k)$ become isomorphic over a finite unramified abelian extension $B(k_1)$ of $B(k)$.

To see this, let $\tilde{G}_0$ be the semisimple group over $W(k)$ whose generic fibre is $\mathrm{Aut}(\tilde{W}_0, \tilde{b}_0)$. We just need to show that $\tilde{G}_{0k}$ splits over a finite abelian extension k_1 of k. We consider a $W(k)$-lattice $\tilde{L}_0$ of $\tilde{W}_0$ such that $\tilde{G}_0$ is a subgroup of $GL(\tilde{L}_0)$, cf. [Ja, 10.4]. The existence of $\tilde{b}_0$ implies the existence of a non-degenerate bilinear form on $\tilde{L}_0 \otimes k$ fixed by $\tilde{G}_{0k}$. Writing it as a sum $\sum_{i=1}^{m} a_i x_i^2$ with respect to some basis of $\tilde{L}_0 \otimes k$ (here $m = \dim(\tilde{W}_0)$, while $a_i \in k$ are non-zero elements), we get immediately that $\tilde{G}_{0k}$ splits over the field extension k_1 of k obtained by adjoining square roots of not more than $[\frac{m}{2}]$ elements of k. So $[k_1 : k]$ divides a fixed power of 2. This ends the proof of b) under the above assumption.

To prove b) in the general case we have to use the standard techniques of [Va2, 6.2.3.1, 6.2.3, 6.4.2 and 6.4.5.1] (for the case $p = 3$ cf. §6). In other words, cf. 2.3.6 and loc. cit., there is another SHS $(f^1, L^1_{(p)}, v^1)$ such that:

– the adjoint quadruple of the quadruples (G^1, X^1, H^1, v^1) and (G, X, H, v) are isomorphic;

– the above assumption is satisfied;

– there is an isogeny $G^{1\mathrm{der}} \to G^{\mathrm{der}}$.

As $\mathbb{F} \hookrightarrow k$, we get easily from [Va2, 6.2.3 and 3.2.7 10)] that, we can assume that each connected component of $\mathcal{N}^1_{W(k)}/H^1_0$ is an abelian cover of a connected component of $\mathcal{N}_{W(k)}/H_0$. Moreover the degree $d(H_0, \tilde{H}_0)$ of this cover divides a fixed power of q_G, cf. [Va2, 6.2.3.1]. So the standard trick using Segre's embeddings as in [Va2, Example 3 of 2.5] (it is fully recalled in 4.9), allows us to shift our attention to the new SHS. But for the new situation we can apply the above part. So we get easily b). This ends the proof of the proposition.

2.3.9.1. Remark. We do not know for which SHS $(f, L_{(p)}, v)$ the reductive group $\tilde{G}$ of 2.3.4 is automatically isomorphic to $G_{W(k)}$. The fixed power of q_G referred in 2.3.9 b) is effectively computable, as the degree $d(H_0, \tilde{H}_0)$ is effectively computable. For instance, if G^{der} is a simply connected semisimple group, then the Galois group of 2.3.9 b) we get is the product of $r(G^{\mathrm{ad}})$ copies of $\mathbb{Z}/2\mathbb{Z}$. Here $r(G^{\mathrm{ad}})$ is the sum of the indices of the Lie types of the simply factors of $G^{\mathrm{ad}}_{\mathbb{C}}$ of B_n or D_n Lie type (like $r(SO(2,m)^{\mathrm{ad}}) = [\frac{m+2}{2}]$, etc.).

2.3.9.2. Convention. From now on, for the sake of simplifying the notations, we assume that whenever we consider a point $\mathrm{Spec}(k) \to \mathcal{N}$ or a point $\mathrm{Spec}(W(k)) \to \mathcal{N}$, the reductive group we get as in 2.3.4 is (isomorphic to) $G_{W(k)}$. This convention is motivated

by 2.3.9 and 2.3.9.1. It does not apply to the similar points of a quotient of finite type of $\mathcal{N}$, like $\mathcal{N}/H_0$.

2.3.10. Remark. Let $(f, L_{(p)}, v)$ be a SHS. With the notations of 2.3.4, the triple $(M, \varphi, G_{W(k)})$ (with φ the Frobenius of $M = H^1_{\mathrm{crys}}(A/W(k))$; the family $(t_\alpha)_{\alpha\in\mathcal{J}'}$ is fixed by φ) is a Shimura σ-crystal (see 2.2.8), called the Shimura σ-crystal attached to the point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ defined by z. This means that there is an injective cocharacter $\tilde{\mu} : \mathbb{G}_m \hookrightarrow G_{W(k)}$ which gives birth to a direct sum decomposition $M = F^1 \oplus F^0$ (with $\dim(F^0) = \dim(F^1) = 1/2\dim(M)$, and with $\beta \in \mathbb{G}_m(W(k))$ acting through $\tilde{\mu}$ on F^i as the multiplication with β^{-i}, $i = \overline{0,1}$), such that $(M, F^1, \varphi, G_{W(k)})$ is a Shimura filtered σ-crystal (the proof of the existence of such a cocharacter $\tilde{\mu}$ is entirely the same as the construction of μ in [Va2, 5.3.1]). The cocharacter $\tilde{\mu}$ can be chosen so that F^1 is the Hodge filtration of M defined by A, and then the above Shimura filtered σ-crystal is said to be attached to the point $z \in \mathcal{N}(W(k))$. When we want to emphasize the family of tensors $(t_\alpha)_{\alpha\in\mathcal{J}'}$ then we also write down $(M, \varphi, G_{W(k)}, (t_\alpha)_{\alpha\in\mathcal{J}'})$ and $(M, F^1, \varphi, G_{W(k)}, (t_\alpha)_{\alpha\in\mathcal{J}'})$. We also speak about the Shimura (adjoint) Lie σ-crystal attached to y, and of the Shimura (adjoint) filtered Lie σ-crystal attached to z (cf. 2.2.13). We also speak about Shimura (adjoint) (Lie) isocrystal attached to y, and of Shimura (adjoint) filtered (Lie) isocrystal attached to z.

When we want to emphasize the polarization, we speak about the principally quasi-polarized Shimura σ-crystal attached to y, and denote it by $(M, \varphi, G_{W(k)}, p_A)$ or by $(M, \varphi, G_{W(k)}, (t_\alpha)_{\alpha\in\mathcal{J}}, p_A)$. Similarly for z.

Let now $z : \mathrm{Spec}(W(k)) \to \mathcal{N}/H_0$, and let A be the abelian variety over $W(k)$ obtained from $\mathcal{A}_{H_0}$ by pull back through z. Over a finite étale extension $W(k_1)$ of $W(k)$, $A_{W(k_1)}$ gets a family $(w_\alpha)_{\alpha\in\mathcal{J}'}$ of Hodge cycles. As the set of all Hodge cycles of $A_{W(k_1)}$ is invariant under the Galois group $\mathrm{Gal}(W(k_1)/W(k)) = \mathrm{Gal}(k_1/k)$, we still get that the (principally quasi-polarized) filtered σ-crystal (M, F^1, φ) attached to A has a natural structure of a (principally quasi-polarized) Shimura filtered σ-crystal $(M, F^1, \varphi, \tilde{G}_{W(k)})$ which is not necessarily quasi-split. As in the proof of 2.3.9, $\tilde{G}_{W(k)}$ is an inner form of $G_{W(k)}$.

This allows us to speak about the not necessarily quasi-split Shimura (adjoint) (filtered) (Lie) σ-crystal (isocrystal) attached to points of $\mathcal{N}/H_0$ (with values in perfect fields and in Witt rings).

2.3.10.1. Comment. The Shimura Lie σ-crystals attached to points $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ play a central role on the whole of this paper. We can not refrain from commenting why they have not been used before. We think that mainly due to two reasons. The first reason is: in the case of special fibres of integral canonical models of Siegel modular varieties, all the information obtained from a Shimura Lie σ-crystal attached to points, can be read out from the Shimura σ-crystal producing it. The second reason is: the way of thinking of points of $\mathcal{N}_{k(v)}$ was mainly rational, i.e. working with the isocrystals defined by Shimura σ-crystals attached to these points, even though for the standard PEL situation (see [Ko2, §5]) was quite obvious (cf. [Va2, 4.3.11]) that we get Shimura σ-crystals.

2.3.10.2. Example. We consider a special point $\mathrm{Spec}(B(k)) \to \mathcal{N}$. From the extension property of $\mathcal{N}$ we get it extends to a morphism $z_1 : \mathrm{Spec}(W(k)) \to \mathcal{N}$. Following

the proof of 2.2.19 we get immediately that the Shimura filtered σ-crystal attached to z_1 is essentially cyclic diagonalizable.

2.3.11. Let $(f, L_{(p)}, v)$ be a SHS. Then all the local deformation theory presented in [Va2, 5.4] can be applied for it (for $\mathcal{N}$). In particular, the fact that for $p \geq 5$ $\mathcal{N}$ is the normalization of the closure of $\mathrm{Sh}(G,X)/H$ in $\mathcal{M}$, results also from [Va2, 5.4]. For the case $p = 3$, cf. also §6.

We have variants of [Va2, 5.4]: we work over an arbitrary perfect field k (instead of $\mathbb{F}$), or we work with $G^0_{W(k)}$ (instead of $G^{\mathrm{der}}_{W(k)}$), or we work with $\mathcal{N}/H_0$ (instead of $\mathcal{N}$; so we work with a finite level symplectic similitude structure, instead of all level-N symplectic similitude structure, $N \in \mathbb{N}$, $(N,p) = 1$). This is so due to the fact that [Fa2, th. 10] is true for any perfect field.

Moreover, for any morphism $U \to \mathcal{N}$, with U a regular formally smooth scheme over a Witt ring of a perfect field, we get naturally a principally quasi-polarized Shimura p-divisible group over U. The change of the family $(v_\alpha)_{\alpha \in \mathcal{J}}$ corresponds to a passage to quasi-isomorphic principally quasi-polarized Shimura p-divisible groups, cf. 2.2.21 7).

2.3.12. We start with a point $z : \mathrm{Spec}(W(k)) \hookrightarrow \mathcal{N}/H_{0W(k)}$. Let $y : \mathrm{Spec}(k) \hookrightarrow \mathcal{N}/H_{0W(k)}$ be defined by z. We use the notations of 2.3.10. So F^1 is the Hodge filtration of M defined by A. For simplicity let us assume that the family of Hodge cycles $(w_\alpha)_{\alpha \in \mathcal{J}}$ is defined over $W(k)$, and not only over $W(\bar{k})$, and that the group $\tilde{G}_{W(k)}$ of 2.3.10 is in fact $G_{W(k)}$. We can always achieve this by replacing k by a finite field extension k_1 of it. Let O_y be the local ring of y. So O^h_y and $\widehat{O_y}$ are its henselization and respectively its completion. Let $(A^h_y, p_{A^h_y})$ be the principally polarized abelian scheme over $\mathrm{Spec}(O^h_y)$ obtained from $(\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ by pull back under the natural morphism $\mathrm{Spec}(O^h_y) \to \mathcal{N}/H_0$. Let $M^h_y := H^1_{dR}(A^h_y/O^h_y)$, and let F^{1h}_y be its Hodge filtration defined by A^h_y. Let $(\omega^h_\alpha)_{\alpha \in \mathcal{J}'}$ be the family of Hodge cycles with which $A^h_{yK^h_y}$ is naturally endowed (with K^h_y the field of fractions of O^h_y; $\mathcal{N}$ is formally smooth over $\mathbb{Z}_{(p)}$, so O^h_y is an integral domain). As we assumed that the family $(w_\alpha)_{\alpha \in \mathcal{J}'}$ is defined over $W(k)$, this implies that the family $(\omega^h_\alpha)_{\alpha \in \mathcal{J}'}$ is defined over O^h_y, and not only over its strict henselization. This is a consequence of the following obvious fact: a Hodge cycle of a deformation which is zero in a point is zero everywhere. Let t^h_α be the de Rham component of ω^h_α. It is a homogeneous tensor of $\mathcal{T}(M^h_y \otimes K^h_y)$.

2.3.13. Lemma. *We have an isomorphism $f_y : M^h_y \tilde{\to} M \otimes_{W(k)} O^h_y$ taking F^{1h}_y onto $F^1 \otimes O^h_y$, $p_{A^h_y}$ into p_A, and t^h_α into t_α, $\forall \alpha \in \mathcal{J}'$ (here we view $p_{A^h_y}$ and p_A as perfect forms on M^h_y and respectively on M).*

Proof: Such an isomorphism exists over $\widehat{O_y}$. This is the same as [Va2, 5.4]. The obstruction to the existence of f_y is measured by an element $\gamma \in H^1_{ff}(O^h_y, P_{O^h_y})$; here P is the parabolic subgroup of $G_{W(k)}$ leaving F^1 invariant, while the lower left index ff refers to the faithfully flat topology. So γ becomes zero over $\widehat{O_y}$. This implies $\gamma = 0$, as O^h_y is the henselization of the excellent ring O_y (so the Artin's approximation theorem applies). This proves the lemma.

2.3.13.1. Corollary. *There is an étale morphism $a : Y = \mathrm{Spec}(R) \to \mathcal{N}/H_{0W(k)}$ to which the point y lifts, and there is an isomorphism $M_R \tilde{\to} M \otimes_{W(k)} R$ taking F^1_R onto $F^1 \otimes_{W(k)} R$, p_{A_Y} into p_A, and t^R_α into t_α, $\forall \alpha \in \mathcal{J}'$ (here M_R, F^1_R, $(t^R_\alpha)_{\alpha\in\mathcal{J}'}$, p_{A_Y} have a meaning similar to the one in the case of O^h_y; for instance $M_R := H^1_{dR}(A_Y/R)$, where $A_Y = \mathcal{A}_{H_0} \times_{\mathcal{N}/H_0} Y$, the morphism $Y \to \mathcal{N}/H_0$ being naturally induced by a).*

2.3.14. Warning: Sp or GSp? Once for all we have to decide if we are going to make use of $\mathrm{GSp}(W,\psi)$ or of $\mathrm{Sp}(W,\psi)$ (and so of $G_{W(k)}$ or of $G^0_{W(k)}$). The use of Sp corresponds to a permanent mentioning of the polarizations (and so to a very precise presentation, but more complicated notations). The use of GSp corresponds to a loose treatment of the polarizations (and so to a less precise presentation, but simpler notations). We decided to go along with Sp.

The idea behind the above comment is: we can work out [Va2, 5.4.4-5] using not only Sp and G^{der} (or G^0), but also GSp and G. The use of GSp means that the cycle of the polarization is not exactly p_A, but p_A times some invertible element γ_p. This element γ_p can be made to be 1, by passing to an étale cover defined by the equation $y^2 = \gamma_p$ (as $p > 2$). So the difference is mostly of notations and of preciseness.

2.3.15. Proposition. *There is a smooth morphism $a : Y = \mathrm{Spec}(R) \to \mathcal{N}/H_{0W(k)}$ such that:*

a) the scheme $a^{-1}(y)$ contains a k-valued point y_1 lifting (to Y) y;

b) we have an isomorphism $M_R \tilde{\to} M \otimes_{W(k)} R$ taking F^1_R onto $F^1 \otimes_{W(k)} R$, p_{A_Y} into p_A, and t^R_α into t_α, $\forall \alpha \in \mathcal{J}'$ (here $M_R = H^1_{dR}(A_Y/R)$, with A_Y defined as in 2.3.13.1, etc.; the logical notations);

c) there is a Frobenius lift Φ_R of $R^\wedge$, compatible with the Frobenius automorphism σ of $W(k)$, and which is of the form $z_i \to z_i^p$, $i \in I$, where the elements of the subset $\{z_i | i \in I\} \cup \{p\}$ of R are forming a regular system of parameters of y_1 in Y;

d) the p-divisible object $\mathcal{C}$ of $\mathcal{MF}^\nabla_{[0,1]}(R)$ defined by the p-divisible group of A_Y, can be put in the form $\big(M \otimes R^\wedge, F^1 \otimes R^\wedge, g_Y(\varphi \otimes 1), \nabla_{R^\wedge}\big)$, with p_{A_Y} corresponding to p_A. Here $g_Y \in G^0_{W(k)}(R^\wedge)$ is defined by a morphism (still denoted by g_Y) $g_Y : Y^\wedge \to G^{0\wedge}_{W(k)}$, whose special fibre is étale, and for which the origin $\mathrm{Spec}(W(k)) \hookrightarrow G^{0\wedge}_{W(k)}$ can be lifted to a morphism $z_Y : \mathrm{Spec}(W(k)) \to Y^\wedge$ which mod p defines y_1 and we have $z = g_Y \circ z_Y$. The connection $\nabla_{R^\wedge}$ of $M \otimes R^\wedge$ annihilates t_α, $\forall \alpha \in \mathcal{J}'$.

Proof: Let $\hat{G}^0 = \mathrm{Spec}(\hat{R}^0)$ be the completion of $G^0_{W(k)}$ with respect to the origin. As in [Va2, 5.4.4] (see 2.3.11) we construct a (formally smooth) morphism $\hat{G}^0 \stackrel{\hat{a}}{\to} \mathcal{N}/H_{0W(k)}$ such that the Shimura principally quasi-polarized filtered F-crystal associated to the Shimura principally quasi-polarized p-divisible group over $\hat{G}^0$ obtained (see 2.3.11) by pull back through the composite of $\hat{a}$ with the natural morphism $\mathcal{N}/H_{0W(k)} \to \mathcal{N}/H_0$, is a principally quasi-polarized Shimura filtered F-crystal $(M, F^1, \varphi, G_{W(k)}, G^0_{W(k)}, \tilde{f}, p_A)$. Here $\tilde{f} : W(k)[[x_1, ..., x_{|I|}]] \tilde{\to} \hat{R}^0$.

Now 2.3.15 results from 2.3.13.1 and Artin's approximation theorem. We present the details. Let $n \geq 3$ be an integer, let I be the ideal of $\hat{R}^0$ defining the origin of $\hat{G}^0$ (so

$\hat{R}^0/I = W(k)$), and let (cf. [BLR, p. 91]) $Y = \mathrm{Spec}(R)$ be a smooth scheme over $W(k)$ such that:

i) there is a smooth morphism $a : Y \to \mathcal{N}/H_{0W(k)}$ such that the point z lifts to a point $z_1 : \mathrm{Spec}(W(k)) \to Y$;

ii) if I_1 is the ideal of R defining z_1, the morphism $\mathrm{Spec}(R/I_1^n) \to \mathcal{N}/H_{0W(k)}$ defined by a is obtained from the natural morphism $\mathrm{Spec}(\hat{R}^0/I^n) \to \mathcal{N}/H_{0W(k)}$ through an isomorphism $i_n : \hat{R}^0/I^n \tilde{\to} R/I_1^n$.

Replacing, if needed, Y with $Y' = \mathrm{Spec}(R')$ where $Y' \to Y$ is an étale morphism such that z_1 lifts to a $W(k)$-valued point of Y', we can achieve that the morphism a endows the abelian scheme A_Y with a family of Hodge cycles $(w_\alpha^R)_{\alpha\in\mathcal{J}'}$ and that b) and c) are satisfied (here z_i is such that $i_n(x_i)$ is congruent to z_i modulo I_1^n). The argument of this is entirely the same as in 2.3.13. This takes care of a), b) and c): we take y_1 to be defined by z_1.

Now the principally quasi-polarized p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(R)$ defined by the principally quasi-polarized p-divisible group of (A_Y, p_{A_Y}) (cf. [Fa1, 7.1] or 2.2.1.1 2)) can be put (cf. b)) in the form $\big(M\otimes R^\wedge, F^1\otimes R^\wedge, g_Y(\varphi\otimes 1), \nabla_{R^\wedge}, p_A\big)$, with $g_Y \in G^0_{W(k)}(R^\wedge)$ (as p_{A_Y} is fixed by g_Y, and as the Rham cycles t^R_α, $\alpha\in\mathcal{J}'$, are fixed by the Frobenius of this p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(R)$), and with $\nabla_{R^\wedge}$ a connection on $M\otimes R^\wedge$ annihilating t_α, $\forall\alpha\in\mathcal{J}'$. Moreover $g_Y(\varphi\otimes 1)$ is $\nabla_{R^\wedge}$-parallel. Due to property ii) above, as R is smooth over $W(k)$, we deduce easily that we can assume that g_Y is defined by a morphism $g_Y : Y^\wedge \to G^{0\wedge}_{W(k)}$, étale mod p, and having the property that the composite of the p-adic completion of z_1 with g_Y defines the origin of $G^{0\wedge}_{W(k)}$. This takes care of d), and ends the proof of the proposition.

From the proof of 2.3.15, by removing some extra regular parameters, we get:

2.3.15.1. Corollary. *There is an étale morphism $\tilde{a} : \tilde{Y} = \mathrm{Spec}(\tilde{R}) \to \mathcal{N}/H_{0W(k)}$ such that:*

a) the scheme $\tilde{a}^{-1}(y)$ contains a k-valued point $\tilde{y}_1$ lifting (to $\tilde{Y}$) y;

b) we have an isomorphism $M_{\tilde{R}}\tilde{\to}M\otimes_{W(k)}\tilde{R}$ taking $F^1_{\tilde{R}}$ onto $F^1\otimes_{W(k)}\tilde{R}$, $p_{A_{\tilde{Y}}}$ into p_A, and $t^{\tilde{R}}_\alpha$ into t_α, $\forall\alpha\in\mathcal{J}'$ (here $M_{\tilde{R}} = H^1_{dR}(A_{\tilde{Y}}/\tilde{R})$, with $A_{\tilde{Y}}$ defined as in 2.3.13.1, etc.; the logical notations);

c) there is a Frobenius lift $\Phi_{\tilde{R}}$ of $\tilde{R}^\wedge$, extending the Frobenius automorphism σ of $W(k)$, and which is of the form $z_i \to z_i^p$, $i\in\tilde{I}$, where the elements of the subset $\{z_i | i\in\tilde{I}\}\cup\{p\}$ of $\tilde{R}$ are forming a system of regular parameters of $\tilde{y}_1$ in $\tilde{Y}$ (so $|\tilde{I}| = d = \dim(Sh(G,X))$);

d) the p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(\tilde{R})$ defined by the p-divisible group of $A_{\tilde{Y}}$, can be put in the form $\big(M\otimes\tilde{R}^\wedge, F^1\otimes\tilde{R}^\wedge, g_{\tilde{Y}}(\varphi\otimes 1), \nabla_{\tilde{R}^\wedge}\big)$ in such a way that $p_{A_{\tilde{Y}}}$ corresponds to p_A. Here $g_{\tilde{Y}}\in G^0_{W(k)}(\tilde{R}^\wedge)$ is defined by a morphism (still denoted by $g_{\tilde{Y}}$) $g_{\tilde{Y}} : \tilde{Y}^\wedge \to G^{0\wedge}_{W(k)}$, for which the origin $\mathrm{Spec}(W(k)) \hookrightarrow G^{0\wedge}_{W(k)}$ can be lifted to a morphism $z_{\tilde{Y}} : \mathrm{Spec}(W(k)) \to \tilde{Y}^\wedge$ which mod p defines $\tilde{y}_1$ and we have $z = g_{\tilde{Y}}\circ z_{\tilde{Y}}$. Modulo the ideal $J := p((z_i)_{i\in\tilde{I}})^2$, $g_{\tilde{Y}}$

defines an isomorphism from $\mathrm{Spec}(\tilde{R}/J)$ *into the completion of a nilpotent subgroup* N *of* $G^0_{W(k)}$ *in the origin, taken modulo the ideal defined by* p *times the second power of the ideal sheaf of* N *defining its origin;* N *can be any such subgroup achieving an open embedding* $N \hookrightarrow G_{W(k)}/P_{W(k)}$*, with* $P_{W(k)}$ *the parabolic subgroup of* $G_{W(k)}$ *leaving invariant* F^1 *(so* $g_{\tilde{Y}}$ *mod* p *is a formal closed embedding in* $\tilde{y}_1$*);*

e) The connection $\nabla_{\tilde{Y}^\wedge}$ *of* $M \otimes \tilde{Y}^\wedge$ *annihilates* t_α*,* $\forall \alpha \in \mathcal{J}'$*;*

f) The special fibre of $\tilde{Y}$ *is connected.*

2.3.16. Corollary. *There is a dense open subscheme* U_k *of* G_k *such that: for any geometric point* $\mathrm{Spec}(k_1) \stackrel{y_1}{\to} U_k$ *(with* $k_1 = \bar{k}_1 \supset k$*), there is a point* $z_1 \in \mathcal{N}/H_0(W(k_1))$*, lifting a* k_1*-valued of the same connected component of* $\mathcal{N}_{\bar{k}}$ *to which the point* $y \circ i_k$ *belongs, with* $i_k : \mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(k)$ *the natural map, and having the property that the Shimura filtered* σ_1*-crystal (with* σ_1 *the Frobenius automorphism of* $W(k_1)$*) attached to the point* z_1 *is isomorphic to* $\big(M \otimes W(k_1), F^1 \otimes W(k_1), g_1(\varphi_0 \otimes \sigma_1), G_{W(k_1)}\big)$*, where* $g_1 \in G_{W(k)}\big(W(k_1)\big)$ *mod* p *is the point* y_1*.*

Proof: Working with G^0 instead of G, this is a direct consequence of 2.3.15 (we can take as U^0_k the image of the special fibre of $Y^\wedge$ through the étale morphism $g_Y : Y^\wedge \to G^{0\wedge}_{W(k)}$). Now, as U_k, we can take an open subscheme of G_k contained in the image of $U^0_k \times \mathbb{G}_m$ through the natural isogeny $G^0_k \times \mathbb{G}_m \to G_k$ (to be compared with 2.3.14: for any $\alpha \in \mathbb{G}_m(W(k))$, there is $\beta \in \mathbb{G}_m(W(k))$ such that $\alpha = \sigma(\beta)\beta^{-1}$). This proves the corollary.

Warning: working with $G_{W(k)}$ and not with $G^0_{W(k)}$ implies that we have to give up mentioning the principal quasi-polarization. So the isomorphism of 2.3.16 is of non principally quasi-polarized Shimura filtered σ-crystals. For applications to problems involving Newton polygons this is equally good.

2.3.17. Proposition. *With the notations of 2.3.12, the Shimura filtered* σ*-crystal* $(M, F^1_1, \varphi, G_{W(k)})$ *attached to another lift* $z_1 : \mathrm{Spec}(W(k)) \hookrightarrow \mathcal{N}/H_{0W(k)}$ *of* y*, satisfies* $F^1_1 = g(F^1)$*, with* $g \in G_{W(k)}(W(k))$ *leaving invariant* $F^1 \otimes k$*. Conversely, for any such* $g \in G_{W(k)}(W(k))$ *there is a lift* $z_g : \mathrm{Spec}(W(k)) \hookrightarrow \mathcal{N}_{W(k)}$ *of* y*, whose attached principally quasi-polarized Shimura filtered* σ*-crystal is* $(M, g(F^1), \varphi, G_{W(k)}, p_A)$*.*

Proof: The first statement is obvious, as the parabolic subgroups of $G_{W(k)}$ leaving invariant F^1 and respectively F^1_1 are $G_{W(k)}(W(k))$-conjugate (the argument of this is entirely similar to the ones used in [Va2, 5.3.1]), and as $F^1 \otimes k = F^1_1 \otimes k$.

The converse is a direct consequence of the smoothness of $G_{W(k)}$, of 2.3.11, and of the local deformation theory of abelian varieties (over k) (cf. [Va2, 5.5.1]).

2.3.17.1. Corollary. *The principally quasi-polarized Shimura filtered* σ*-crystal* $(M, g(F^1), \varphi, G_{W(k)}, p_A)$ *(here* $g \in G_{W(k)}(W(k))$ *mod* p *leaves invariant* $F^1 \otimes k$*) is induced from the principally quasi-polarized Shimura filtered F-crystal* $(M, F^1, \varphi, G_{W(k)}, G^0_{W(k)}, \tilde{f}, p_A)$ *(of the proof of 2.3.15) through a* $W(k)$*-epimorphism* $\hat{R}^0 \twoheadrightarrow W(k)$*.*

2.3.17.2. Exercise. Show that the above corollary is a property of (principally polarized) Shimura σ-crystals over k. Hint: if we feel more confortable with the context of principally polarized abelian scheme, follow the ideas of 4.12.12.

2.3.18. The $p = 2$ SHS. The case $p = 2$ presents some particular features, and so has to be singled aside. Working with $p = 2$, we can keep 2.3.1-2. But for 2.3.3 we have to proceed carefully: we define $\mathcal{N}$ as the normalization of $\mathrm{Sh}_H(G, X)$ in $\mathcal{M}$, and we assume that it is a smooth integral model. This implies that it is an integral canonical model of the Shimura quadruple (G, X, H, v). We keep 2.3.4-5 (in particular 2.3.5.1 remains true for $p = 2$), but we prefer to abbreviate the things differently: the resulting triple $(f, L_{(2)}, v)$ (resp. quadruple $(f, L_{(2)}, v, \mathcal{B})$) is referred to as a $p = 2$ SHS (resp. as a $p = 2$ standard PEL situation). Conventions 2.3.7 and 2.3.9.2 apply, but not 2.3.6 or 2.3.8. The above parts 2.3.9 c) and 2.3.10-17 remain valid. This is so due to the fact that [Fa2, th. 10] and the remarks following it are still true for $p = 2$; so [Va2, 5.4] and its variants (see 2.3.11) remain true for $p = 2$. The only difference: for $p = 2$ [Va2, 5.5.1] has to be modified slightly (see the case $p = 2$ in [Me]); however it applies to 2.3.17 in the same way. Moreover, 2.3.9 d) applies as well: the part of the proof of 2.3.9 referring to [Ja] has to be modified; we have to express $\tilde{b}_0$ as a linear combination of squares with coefficients in $\mathbb{G}_m(W(k))$ (this is possible as $\tilde{G}_0$ is a semisimple group over $W(k)$).

§3 *The basic results*

Let k be a perfect field of characteristic $p \geq 3$, and let σ be the Frobenius automorphism of the Witt ring $W(k)$. Let (M, φ, G) be a Shimura σ-crystal over k (cf. 2.2.8 and 2.2.14). From its definition, we deduce the existence of a cocharacter $\mu : \mathbb{G}_m \to G$ which produces a direct sum decomposition $M = F^1 \oplus F^0$ (with $\beta \in \mathbb{G}_m(W(k))$ acting through μ on F^i as the multiplication with β^{-i}, $i = \overline{0,1}$) such that (M, F^1, φ, G) is a Shimura filtered σ-crystal. The $G(W(k))$-conjugacy class $[\mu]$ of the cocharacter μ is uniquely determined by (M, φ, G) (cf. 2.2.9 3)). From 2.2.8 a) we deduce the existence of a family $(v_\alpha)_{\alpha \in \mathcal{J}}$ of tensors of $\mathcal{T}(M)[\frac{1}{p}]$, fixed by G and φ, and such that $G_{B(k)}$ is the subgroup of $GL\big(M[\frac{1}{p}]\big)$ fixing v_α, $\forall \alpha \in \mathcal{J}$. If desirable, we can assume that the tensors v_α are homogeneous. We also deduce that F^0 and F^1 are proper direct summands of M. Let $\mathfrak{g} := \mathrm{Lie}(G)$.

Fixing the group G, the family of tensors $(v_\alpha)_{\alpha \in \mathcal{J}}$, and the $G(W(k))$-conjugacy class $[\mu]$, any other Shimura σ-crystal (M, φ_1, G), with φ_1 fixing v_α, $\forall \alpha \in \mathcal{J}$, and with $[\mu]$ defining its filtration class, up to isomorphism (cf. 2.2.9 6)), can be put in the form $\varphi_1 = g\varphi$, with $g \in G(W(k))$. To see this, we can consider two Shimura filtered σ-crystals $(M, F^1, \varphi, G, (v_\alpha)_{\alpha \in \mathcal{J}})$ and $(M, F_1^1, \varphi_1, G, (v_\alpha)_{\alpha \in \mathcal{J}})$. Changing by an isomorphism, we can assume that $F^1 = F_1^1$. But then $\varphi_1 = g\varphi$ with $g \in GL(M)$. As φ_1 and φ are both fixing v_α, $\forall \alpha \in \mathcal{J}$, we get that actually $g \in G(W(k))$.

3.0. In the whole of §3, we consider only Shimura σ-crystals (M, φ_1, G) or Shimura filtered σ-crystals (M, F^1, φ_1, G), with φ_1 fixing $v_\alpha, \forall \alpha \in \mathcal{J}$, and with $[\mu]$ defining their filtration class.

3.1. The first group of basic results.

3.1.0. Theorem. a) *Among all σ-crystals $(M, g\varphi)$ with $g \in G(W(k))$ (i.e. among all Shimura σ-crystals $(M, \varphi_1, G, (v_\alpha)_{\alpha \in \mathcal{J}})$ whose filtration class is defined by $[\mu]$), the ones which have the smallest Newton polygon $\mathcal{P}$ (in the sense that all others have a Newton polygon strictly above it) are precisely the ones $(M, g_0\varphi)$ for which there is a Shimura filtered σ-crystal $(M, F_0^1, g_0\varphi, G)$ such that the parabolic Lie subalgebra $\mathfrak{p}_0$ of $\mathfrak{g}$, corresponding to non-negative slopes of $(\mathfrak{g}, g_0\varphi)$, is contained in the parabolic Lie subalgebra $F_0^0(\mathfrak{g})$ of $\mathfrak{g}$ leaving invariant the filtration F_0^1 of M.*

b) *For any Shimura σ-crystal $(M, g_0\varphi, G)$ having $\mathcal{P}$ as its Newton polygon, there is a unique filtration F_0^1 of M, such that $(M, F_0^1, g_0\varphi, G)$ is a Shimura filtered σ-crystal satisfying (with the notations of a)) $\mathfrak{p}_0 \subset F_0^0(\mathfrak{g})$.*

c) *Among all σ-crystals $(M, g\varphi)$ with $g \in G(W(k))$ (i.e. among all Shimura σ-crystals $(M, \varphi_1, G, (v_\alpha)_{\alpha \in \mathcal{J}})$ whose filtration class is defined by $[\mu]$), the ones which have the smallest Newton polygon $\mathrm{Lie}_G(\mathcal{P})$ (in the sense that all others have a Newton polygon strictly above it) of their attached Shimura Lie σ-crystals $(\mathfrak{g}, g\varphi)$, are precisely the ones having $\mathcal{P}$ as their Newton polygon.*

d) *There is a Shimura filtered F-crystal $\mathfrak{C}_F := (M, F^1, \varphi, G, \tilde{f})$ such that the Shimura σ_1-crystal we get (cf. 2.2.10) over a geometric point over the generic point of the scheme over which this $\mathfrak{C}_F$ is, has $\mathcal{P}$ as its Newton polygon.*

3.1.1. Definitions. Any Shimura σ-crystal $\big(M, \varphi_1, G, (v_\alpha)_{\alpha\in\mathcal{J}}\big)$ whose filtration class is defined by $[\mu]$ and has $\mathcal{P}$ as its Newton polygon, is called a G-ordinary σ-crystal (or a Shimura-ordinary σ-crystal, or, even better, a $(G,[\mu],[a])$-ordinary σ-crystal, as it depends on the filtration class defined by $[\mu]$ and on the automorphism class [a] (cf. def. 2) of 2.2.11)). The formal isogeny type of a G-ordinary σ-crystal $(M, g_0\varphi, G)$ (resp. such a G-ordinary σ-crystal), is called the G-ordinary type (resp. a G-ordinary σ-crystal) produced by (or attached to) the Shimura σ-crystal (M,φ,G) we started with. Any Shimura filtered σ-crystal (M, F_1^1, φ_1, G) with (M,φ_1,G) a $(G,[\mu])$-ordinary σ-crystal, and with F_1^1 such that the parabolic Lie subalgebra of $\mathfrak{g}$ leaving it invariant, contains the parabolic Lie subalgebra of $\mathfrak{g}$ corresponding to non-negative slopes of $(\mathfrak{g},\varphi_1)$, is called a G-canonical lift (or a Shimura-canonical lift) of (M,φ_1,G).

3.1.1.1. Exercise. If $G = GL(M)$, then the G-ordinary type we get is an ordinary type (i.e. its associated Newton polygon has only the slopes 0 and 1).

3.1.2. Remark. The terminology G-ordinary σ-crystals and G-canonical lifts, has been suggested to us by F. Oort and R. Pink. We have a slightly more preference for the terminology Shimura-ordinary σ-crystals and Shimura-canonical lifts.

3.1.2.1. Convention. Whenever we pass to a perfect field k_1 containing k, for not overloading the notations, we still use the terminology G-ordinary, instead of $G_{W(k_1)}$-ordinary. But we keep using "the phrase" σ_1-crystals, where σ_1 is the Frobenius automorphism of $W(k_1)$.

3.1.3. Reformulation of 3.1. 3.1.0 asserts five things:

a) Among all Newton polygons of σ-crystals $(M, g\varphi)$ with $g \in G(W(k))$, there is one $\mathcal{P}$ which is the smallest; let A_G be the set of σ-crystals $(M, g\varphi)$ having $\mathcal{P}$ as their Newton polygon.

b) Among all Newton polygons of Shimura Lie σ-crystals $(\mathfrak{g}, g\varphi)$ with $g \in G(W(k))$, there is one $\mathrm{Lie}_G(\mathcal{P})$ which is the smallest; let B_G be the set of σ-crystals $(M, g\varphi)$ having $\mathrm{Lie}_G(\mathcal{P})$ as the Newton polygon of their attached Shimura Lie σ-crystal $(\mathfrak{g}, g\varphi)$.

c) $A_G = B_G = C_G$, where C_G is the set of σ-crystals $(M, g\varphi)$, with $g \in G(W(k))$, having the property that there is a Shimura filtered σ-crystal $(M, F^1, g\varphi, G)$ such that the parabolic Lie subalgebra of $\mathfrak{g}$ corresponding to non-negative slopes of $(\mathfrak{g}, g\varphi)$ is contained in the parabolic Lie subalgebra of $\mathfrak{g}$ leaving invariant F^1 (i.e. there is a Shimura filtered σ-crystal $(M, F^1, g\varphi, G)$ such that its Shimura filtered Lie σ-crystal $\big(\mathfrak{g}, g\varphi, F^0(\mathfrak{g}), F^1(\mathfrak{g})\big)$ is of parabolic type; cf. def. 2.2.12 a)).

d) If $(M, g\varphi)$ is an element of C_G, then there is a unique Shimura filtered σ-crystal $(M, F^1, g\varphi, G)$ as in c) (i.e. whose attached Shimura filtered Lie σ-crystal $\big(\mathfrak{g}, g\varphi, F^0(\mathfrak{g}), F^1(\mathfrak{g})\big)$ is of parabolic type).

e) Any Shimura σ-crystal can be deformed (using Shimura F-crystals) so that it is the specialization of a G-ordinary σ_1-crystal over a perfect field k_1 containing k.

3.1.4. Proposition. *For any G-ordinary σ-crystal $(M, \tilde{\varphi}, G)$, there is a unique cocharacter $\tilde{\mu} : \mathbb{G}_m \to G$, with $[\tilde{\mu}] = [\mu]$, producing a direct sum decomposition $M = F^1 \oplus F^0$ (with $\beta \in \mathbb{G}_m(W(k))$ acting through $\tilde{\mu}$ on F^i as the multiplication with β^{-i},*

$i = \overline{0,1}$) such that the parabolic Lie subalgebra of $\mathfrak{g}$ corresponding to non-negative slopes (resp. non-positive slopes) of $(\mathfrak{g}, \tilde{\varphi})$ leaves invariant F^1 (resp. leaves invariant F^0).

For a first proof of this proposition see 3.11.1-2. A second proof of it can be obtained entirely similar to the proof of a) of 4.4.1 3) (where we work in the context of Shimura varieties of Hodge type) and so, for making §4 as little dependent on §3 as possible, it is reproduced only there.

3.1.5. Exercise. The cocharacter $\tilde{\mu}$ is the one defined in [Wi] for the filtered isocrystal $(M[\frac{1}{p}], F^1[\frac{1}{p}], \tilde{\varphi})$ (Hint: pass to $k = \bar{k}$ and make use of 3.11 and of the proof of part b) of 4.4.1 3)).

Warning: 3.1.4-5 are not referred before 4.5, so no vicious circle is created (the terminology of 3.1.6 is used in 3.11.2 just to introduce some extra terminology in 3.11.3-4).

3.1.6. Definition. The cocharacter $\tilde{\mu}$ of 3.1.4 is called the canonical split of $(M, \tilde{\varphi}, G)$.

3.1.7. Remark. It is an easy exercise to see that the whole of 3.1 (as well as the greatest part -i.e. the part which does not mention p-divisible groups or Shimura varieties- of this chapter §3) remains valid in the context of generalized Shimura p-divisible objects over k. We do not stop to reformulate the results in this extended context.

From 3.1.0 d) and 2.2.9 9) we get (cf. 3.1.1.1):

3.1.8. Corollary. *Any p-divisible group D over a perfect field $\tilde{k}$ of characteristic $p > 2$ is the specialization of an ordinary p-divisible group.*

3.1.8.1. Comment. The above corollary, for the case when D has a quasi-polarization over a finite field extension of $\tilde{k}$, was obtained in [NO]. The case when the a-number of D is at most one was handled in [Oo2, 6.7]. The general case, including the case $p = 2$, can be easily deduced from [Fa2, th. 10] through a simple algebraization process as in 3.4.18.4.1 (i.e., if $GL(M) = \mathrm{Spec}(R_{GL})$, then we work with the Frobenius of $M \otimes R_{GL}^{\wedge}$ defined by $\mathrm{gl}(\varphi \otimes 1)$, with gl as the universal element of $GL(M)(R_{GL}^{\wedge})$), based on an anlysis of the Hasse-Witt invariants and on example 3.2.3 below.

3.1.8.2. Exercise. Using 3.1.8.1, show that any polarized p-divisible group over $\tilde{k}$ is the specialization of an ordinary polarized p-divisible group.

This exercise reobtains the first main result of [NO].

3.2. The outline of the proof of 3.1.0.

3.2.1. Lemma. *If (N, φ_i), $i = \overline{1,2}$, are two σ-crystals such that $\forall m \in \mathbb{N}$, $\varphi_1^m = h_m \varphi_2^m$, with $h_m \in GL(N)(W(k))$, then they have the same Newton polygon. Moreover, for any $\alpha \in [0, \infty)$, the direct summand of N corresponding to slopes of (N, φ_i) bigger or equal to α, does not depend on $i \in \{1, 2\}$.*

Proof: The condition $\varphi_1^m = h_m \varphi_2^m$, with $h_m \in GL(N)(W(k))$, implies that the Hodge numbers of the σ^m-crystals (N, φ_1^m) and (N, φ_2^m) are the same. But the Newton polygon of a σ-crystal is determined by the Hodge numbers of its iterates [Ka2, 1.4.4]. This takes care of the first part.

To see the second part we can assume that $k = \bar{k}$. Let $n \in \mathbb{N}$ such that φ_i^n acts diagonally with respect to a basis β_i of $N \otimes B(k)$ formed by elements of $N \setminus pN$, $i \in \{1, 2\}$. From the hypothesis, applied to all multiplies m of n, we deduce easily by induction (there are only a finite number of slopes of (N, φ_1) and of (N, φ_2)) that for any $\alpha \in [0, \infty)$ the maximal $B(k)$-vector subspace $W_i(\alpha)$ of $N \otimes B(k)$ with the property that it is left invariant by φ_i, and the slopes of the isocrystal $(W(\alpha), \varphi_i)$ are all greater or equal to α, does not depend on i, i.e. we have $W_1(\alpha) = W_2(\alpha)$. This proves the lemma.

3.2.1.1. Corollary. *If (N, φ_i, G_N) are two Shimura σ-crystals such that $\forall m \in \mathbb{N}$, $\varphi_1^m = h_m \varphi_2^m$, with $h_m \in G_N(W(k))$, then the Lie subalgebra of $\mathrm{Lie}(G_N)$ corresponding to non-negative (resp. positive) slopes of $(\mathrm{Lie}(G_N), \varphi_i)$, does not depend on $i \in \{1, 2\}$.*

Proof: We just need to apply 3.2.1 to the σ-crystals $(\mathrm{Lie}(G_N), p\varphi_i)$, $i \in \{1, 2\}$.

3.2.2. We come back to the Shimura filtered σ-crystal (M, F^1, φ, G) considered in the beginning of §3. For the meaning of $F^0(\mathfrak{g})$ and $F^1(\mathfrak{g})$ see 2.2.8 1). The following assertions are (obviously) equivalent:

a) $F^1(\mathfrak{g}) = 0$;
b) $F^0(\mathfrak{g}) = \mathfrak{g}$;
c) the cocharacter $\mu : \mathbb{G}_m \to G$ factors through the centre of G;
d) the $G(W(k))$-conjugacy $[\mu]$ of μ, has a unique representative $\mu : \mathbb{G}_m \to G$.

If the above four assertions are true (for example in the case when G is a torus), then for any $g \in G(W(k))$, $\varphi \circ g = g_1 \circ \varphi$, with $g_1 \in G(W(k))$. By induction, $(g \circ \varphi)^m = g_m \circ \varphi^m$, with $g_m \in G(W(k))$, $\forall m \in \mathbb{N}$. So in this case, 3.1.0 is a direct consequence of 3.2.1.

3.2.2.1. Convention. In all that follows, till the end of §3, we assume that the assertions a) to d) of 3.2.2 are not true. So $(G, [\mu])$ is a Shimura group pair over $\mathrm{Spec}(W(k))$ (cf. def. 2.2.3).

3.2.3. A special example. Let P be the parabolic subgroup of G, having $F^0(\mathfrak{g})$ as its Lie algebra. Let $T \subset B \subset P$, with T a maximal torus containing the image of μ (so μ can be viewed as a cocharacter $\mu : \mathbb{G}_m \to T$), and with B a Borel subgroup of G (the existence of B is implied by our hypotheses in 2.2.8: G is quasi-split). Let $\mathfrak{t}$ (resp. $\mathfrak{b}$) be the Lie algebra of T ((resp. of B). We have inclusions $\mathfrak{t} \subset \mathfrak{b} \subset F^0(\mathfrak{g})$, and $F^1(\mathfrak{g}) \subset \mathfrak{b}$. We get $\varphi(\mathfrak{t}) \subset \varphi(\mathfrak{b}) \subset \mathfrak{g}$.

Let T_0 be a maximal torus of G having $\varphi(\mathfrak{t})$ as its Lie algebra. Such a torus exists due to the fact that a regular element of $\mathfrak{t} \otimes W(\bar{k})$ is mapped through $\varphi \otimes 1$ into a regular element of $\mathfrak{g} \otimes W(\bar{k})$, and a regular element of $\mathfrak{t} \otimes B(k)$ is mapped through φ into a regular element of $\mathfrak{g} \otimes B(k)$. Let B_0 be the Borel subgroup of G such that its Lie algebra contains $\varphi(\mathfrak{b})$ ($\varphi(\mathfrak{b}) \otimes B(k)$ is a Borel Lie subalgebra of $\mathfrak{g}_{B(k)}$). There is an element $g_0 \in G(W(k))$ such $g_0 B_0 g_0^{-1} = B$ (any two Borel subgroups of G are $G(W(k))$-conjugate). Let $T_1 := g_0 T_0 g_0^{-1} \subset B$. There is $g_1 \in B(W(\bar{k}))$ such that $g_1 T_{1W(\bar{k})} g_1^{-1} = T_{W(\bar{k})}$.

Let $\varphi_0 := g_0 \varphi$, and let τ_0 be the formal isogeny type of the σ-crystal (M, φ_0). We consider the following Shimura $\bar{\sigma}$-crystal $\left(M \otimes W(\bar{k}), \varphi_1, G_{W(\bar{k})}\right)$ defined by

$$\varphi_1 := g_1(\varphi_0 \otimes 1).$$

We have $\varphi_0(\mathfrak{b}) \subset \mathfrak{b}$ and $\varphi_0(\mathfrak{b} \otimes B(k)) = \mathfrak{b} \otimes B(k)$. We have $\varphi_1(\mathfrak{b} \otimes W(\bar{k})) \subset \mathfrak{b} \otimes W(\bar{k})$ and $\varphi_1(\mathfrak{t} \otimes W(\bar{k})) = \mathfrak{t} \otimes W(\bar{k})$.

3.2.4. The theorem 3.1.0 is a direct consequence of the claims 3.2.5-8 below and of the semicontinuity theorem [Ka2, 2.3.1]: 3.2.5-6 impliy 3.1.3 b); 3.2.6 implies 3.1.3 a) and e), and that $\mathcal{P}$ is the Newton polygon attached to the formal isogeny type τ_0; 3.2.5-6 imply $B_G \subset A_G$; 3.2.7 b) implies that $A_G \subset C_G \cap B_G$ and 3.2.7 c) implies $C_G \subset A_G$; so $A_G = B_G = C_G$; 3.2.8 implies 3.1.3 d).

3.2.5. Claim. *Among all Shimura Lie σ-crystals $(\mathfrak{g}, g\varphi)$ with $g \in G(W(k))$, there is no one whose Newton polygon is strictly below the Newton polygon of the Shimura Lie σ-crystal $(\mathfrak{g}, \varphi_0)$. Moreover, any Shimura σ-crystal $(M, g\varphi, G)$, with $g \in G(W(k))$, having the property that the Shimura Lie σ-crystals $(\mathfrak{g}, g\varphi)$ and $(\mathfrak{g}, \varphi_0)$ have the same Newton polygon, has the formal isogeny type τ_0.*

3.2.6. Lemma. *For any Shimura filtered σ-crystal $(M, F^1, g\varphi, G)$, with $g \in G(W(k))$, there is a Shimura filtered F-crystal (M, F^1, φ, G, f) such that the resulting F-crystal over the generic point of the special fibre (of the completion of G in the origin) has the formal isogeny type τ_0.*

3.2.7. Proposition. a) *The Shimura filtered σ-crystal (M, F^1, φ_0, G) has the property that the parabolic Lie subalgebra $\mathfrak{p}_0$ of it corresponding to non-negative slopes of $(\mathfrak{g}, \varphi_0)$ is contained in $F^0(\mathfrak{g})$.*

b) *The Shimura σ-crystals (M, φ_2, G) having τ_0 as the formal isogeny type are precisely the ones, which, up to isomorphism, can be extended to a Shimura filtered σ-crystal of the form $(M, F^1, \tilde{p}\varphi_0, G)$, with $\tilde{p}$ a $W(k)$-valued point of the parabolic subgroup P_0 of G satisfying Lie$(P_0) = \mathfrak{p}_0$. For any $\tilde{p} \in P_0(W(k))$, the Lie subalgebra of $\mathfrak{g}$ corresponding to non-negative slopes of $(\mathfrak{g}, \tilde{p}\varphi_0)$ is $\mathfrak{p}_0$ (and so it is included in $F^0(\mathfrak{g})$), and the Shimura Lie σ-crystals $(\mathfrak{g}, \varphi_0)$ and $(\mathfrak{g}, \tilde{p}\varphi_0)$ have the same Newton polygon.*

c) *Any Shimura filtered σ-crystal (M, F_2^1, φ_2, G) having the property that the parabolic Lie subalgebra of $\mathfrak{g}$ corresponding to non-negative slopes of $(\mathfrak{g}, \varphi_2)$ is contained in $F_2^0(\mathfrak{g})$, (as a σ-crystal) has the formal isogeny type τ_0.*

3.2.8. The uniqueness of the filtration mentioned in 3.1.0, results from the fact that any two such filtrations are conjugate by an element of $G(W(k))$, and from the fact that two conjugate parabolic subgroups of a reductive group, containing the same parabolic subgroup, are the same (it is enough to see this over a field; for this case cf. [Bo, 14.22]).

3.2.9. Warning. We present two proves of 3.2.5-7. The first one ends in 3.5.14; it avoids completely the use of 3.4. The second proof ends in 3.7 for 3.2.6-7 and in 3.5.13 for 2.5; it relies on 3.4.0-11.

3.3. About the proof of 3.2.5. We can assume that G is split and that $g_1 \in G(W(k))$ (we achieve this by passing from $W(k)$ to $W(k_1)$, with k_1 a finite extension of k; as G is split, we can choose T in 3.2.3 to be split, and then automatically we can take $g_1 \in G(W(k))$).

3.3.1. The claim 3.2.5 is a direct consequence of the following facts 3.3.2-5.

3.3.2. There is $b \in B(W(k))$ such that $\varphi_0 = b\varphi_1$ (proof: we know that there is $g \in G(W(k))$ such that $\varphi_0 = g\varphi_1$; as $\varphi_0(\mathfrak{b} \otimes B(k)) = \varphi_1(\mathfrak{b} \otimes B(k)) = \mathfrak{b} \otimes B(k)$ we get that g normalizes $\mathfrak{b} \otimes B(k)$ and so $g \in B(B(k)) \cap G(W(k)) = B(W(k)))$. As $\forall b_1 \in B(W(k))$ we have $\varphi_1 b_1 = b_1' \varphi_1$, with $b_1' \in B(W(k))$, by induction on $n \in \mathbb{N}$, we get that $\varphi_0^n = b(n)\varphi_1^n$, for some $b(n) \in B(W(k))$.

3.3.3. $B \subset P_1$, where P_1 is the parabolic subgroup of G, such that $\mathrm{Lie}(P_1) \otimes B(k)$ is the parabolic Lie subalgebra of $\mathfrak{g} \otimes B(k)$ corresponding to non-negative slopes of $(\mathfrak{g} \otimes B(k), \varphi_1)$ (proof: as $\mathfrak{b} \subset F^0(\mathfrak{g})$, we have $\varphi_1(\mathfrak{b}) \subset \mathfrak{b}$; we deduce that $(\mathfrak{b}, \varphi_1)$ is a σ-crystal, and so its slopes are non-negative). Moreover $P_1 = P_0$ (this is a consequence of 3.2.1.1 and 3.3.2).

3.3.4. Among all Shimura filtered σ-crystals $(M, F^1, g_2\varphi_0, G)$ with $g_2 \in G(W(k))$, the ones having a Newton polygon of their attached Shimura Lie σ-crystals $(\mathfrak{g}, g_2\varphi)$ strictly smaller or equal to the one of $(\mathfrak{g}, \varphi_0)$, are precisely the ones of the form $g_2\varphi_0 = g_3 p_0 \varphi_0 g_3^{-1}$, with $g_3 \in G(W(k))$ such that mod p it leaves invariant $F^1 \otimes k$, and with $p_0 \in P_0(W(k))$; this is true without the assumption G is split (for a proof of this see 3.5.13).

3.3.5. We remark that any Shimura Lie σ-crystal $(\mathfrak{g}, g_2\varphi_1)$, with $g_2 \in G(W(k))$ of the form mentioned in 3.3.4, has the same Newton polygon as $(\mathfrak{g}, \varphi_1)$, and that $(M, g_2\varphi_1)$ and (M, φ_1) have the same Newton polygon (proof: we can assume that $g_3 = 1$; then, as in 3.3.2, we have $(p_0\varphi_0)^n = p_{n-1}\varphi_1^n$, with $p_{n-1} \in P_0(W(k)) = P_1(W(k))$, $\forall n \in \mathbb{N}$, and so 3.2.1 and 3.2.1.1 apply).

3.4. The second group of basic results: global deformations and some principles for different categories $\mathcal{MF}(*)$. Let $G = \mathrm{Spec}(R)$ and let $g_2 \in G(W(k))$. Let $U_0 = \mathrm{Spec}(R_0)$ and $U_2 = \mathrm{Spec}(R_2)$ be affine open subschemes of G containing the origin $\mathrm{Spec}(W(k)) \overset{a_0}{\hookrightarrow} U_0$ and respectively containing the point g_2 (defined, for the sake of symmetry, by a morphism $\mathrm{Spec}(W(k)) \overset{a_2}{\hookrightarrow} U_2$), and having the following properties:

i) There are étale morphisms $U_0 \overset{b_0}{\to} Y := \mathrm{Spec}\big(W(k)[z_1, \ldots, z_d]\big)$ and $U_2 \overset{b_2}{\to} Y$, with $d = \dim(G)$, such that the $W(k)$-morphisms $b_0 \circ a_0$ and $b_2 \circ a_2$ are defined at the level of rings by: z_i goes to 0, $i = \overline{1,d}$.

ii) There are étale morphisms $W_0 \overset{c_0}{\to} Z = \mathrm{Spec}\big(W(k)[z_1, \ldots, z_e]\big)$ and $W_2 \overset{c_2}{\to} Z$, with $e = \dim\big(GL(M)\big) = (\dim M)^2$, and with $W_j = \mathrm{Spec}(S_j)$ an affine open subscheme of $GL(M)$ containing U_j, $j \in \{0, 2\}$, such that $b_0 = \tilde{d} \circ c_0 \circ i_0$ and $b_2 = \tilde{d} \circ c_2 \circ i_2$, with $i_0 : U_0 \hookrightarrow W_0$ and $i_2 : U_2 \hookrightarrow W_2$ as the natural inclusions, and with $\tilde{d} : Z \to Y$ as the $W(k)$-morphism which at the level of rings takes z_i into z_i, $i = \overline{1,d}$ (we have $e \geq d$).

3.4.0. From now on, till the end of 3.4.8, we assume that k is an infinite field. For the case of a finite field see 3.4.9 3).

We choose a morphism $h : \mathrm{Spec}(W(k)) \hookrightarrow U_0 \cap U_2$ (R/pR is an integral domain cf. [Ti2, 3.8.1]; [Bo, 18.3] guarantees the existence of a point $\mathrm{Spec}(k) \hookrightarrow (U_0 \cap U_2)_k$) such that the $W(k)$-epimorphisms $W(k)[z_1, \ldots, z_d] \twoheadrightarrow W(k)$ associated to $b_0 \circ i_{02} \circ h$ and $b_2 \circ i_{20} \circ h$ (with i_{02} and i_{20} the inclusions of $U_0 \cap U_2$ into U_0 and respectively into U_2) send z_i, $i = \overline{1,d}$, into elements of $\mathbb{G}_m(W(k))$. Composing the morphismss b_0 and b_2 with automorphisms

of Y of the form $z_i \to u_i z_i$, with $u_i \in \mathbb{G}_m(W(k))$, $i = \overline{1,d}$, we can assume that these $W(k)$-epimorphisms send z_i to 1, $i = \overline{1,d}$.

Let U_0^0, U_2^0, W_0^0 and W_2^0 be the right translations of U_0, U_2, W_0 and respectively W_2 by h^{-1}. So the morphism $a_0 : \mathrm{Spec}(W(k)) \hookrightarrow GL(M)$ defining the origin (the identity element of $GL(M)(W(k))$) factors through U_0^0, U_2^0, W_0^0 and W_2^0. Let Φ be the Frobenius lift of Y and Z, extending σ on $W(k)$ and taking $z_i - 1$ to $(z_i - 1)^p$, $i = \overline{1,e}$. Let Φ_{R_0}, Φ_{R_2}, Φ_{S_0} and Φ_{S_2} be the Frobenius lifts of $U_0^{0\wedge}$, $U_2^{0\wedge}$, $W_0^{0\wedge}$ and respectively of $W_2^{0\wedge}$ such that the two diagrams below:

$$\begin{array}{ccccc} U_j^{0\wedge} & \xrightarrow{r_h^\wedge}_{\sim} & U_j^\wedge & \xrightarrow{b_j^\wedge} & Y \\ \downarrow{\scriptstyle \Phi_{R_j}} & & & & \downarrow{\scriptstyle \Phi^\wedge} \\ U_j^{0\wedge} & \xrightarrow{r_h^\wedge}_{\sim} & U_j^\wedge & \xrightarrow{b_j^\wedge} & Y \end{array}$$

and

$$\begin{array}{ccccc} W_j^{0\wedge} & \xrightarrow{r_h^\wedge}_{\sim} & W_j^\wedge & \xrightarrow{c_j^\wedge} & Z^\wedge \\ \downarrow{\scriptstyle \Phi_{S_j}} & & & & \downarrow{\scriptstyle \Phi^\wedge} \\ W_j^{0\wedge} & \xrightarrow{r_h^\wedge}_{\sim} & W_j^\wedge & \xrightarrow{c_j^\wedge} & Z^\wedge, \end{array}$$

$j \in \{0,2\}$, are cartesian (here r_h is the right translation by h, and $b_j^\wedge$, $r_h^\wedge$, $c_j^\wedge$ and $\Phi^\wedge$ refer to morphisms defined by the p-adic completion of b_j, r_h, c_j and respectively of Φ). The isomorphism r_h allows us to identify $U_j^0 = \mathrm{Spec}(R_j)$ and $W_j^0 = \mathrm{Spec}(S_j)$, $j \in \{0,2\}$. Let $t_i(j) \in S_j$ such that under the identification $W_j^0 = \mathrm{Spec}(S_j)$, the morphism $c_j^\wedge \circ \tilde{r}_h^\wedge$, at the level of rings, takes $z_i - 1$ into $t_i(j)$, $j \in \{0,2\}$, $i \in \{1,...,e\}$. So $\Phi_{S_j}(t_i(j)) = t_i(j)^p$. We can regard $t_i(j)$, $i \in \{1,...,d\}$, as an element of R_j.

3.4.1. Let

$$M_{R_j} := \big(M \otimes_{W(k)} R_j^\wedge, F^1 \otimes_{W(k)} R_j^\wedge, \Phi_j^r\big)$$

and

$$M_{S_j} := \big(M \otimes_{W(k)} S_j^\wedge, F^1 \otimes_{W(k)} S_j^\wedge, \Phi_j^s\big),$$

with $\Phi_j^r := r_j(h\varphi_0 \otimes 1)$ and $\Phi_j^s := s_j(h\varphi_0 \otimes 1)$, where $r_j \in G(R_j^\wedge)$ and $s_j \in GL(S_j^\wedge)$ are the universal elements of $G^\wedge$ and respectively of $GL(M)^\wedge$, defined by the p-adic completion of the inclusions $U_j^0 \hookrightarrow G$ and $W_j^0 \hookrightarrow GL(M)$, $j \in \{0,2\}$. For any $n \in \mathbb{N}$, $M_{R_j}/p^n M_{R_j}$ is an object of $\mathcal{MF}_{[0,1]}(R_j)$, and $M_{S_j}/p^n M_{S_j}$ is an object of $\mathcal{MF}_{[0,1]}(S_j)$, $j \in \{0,2\}$. So M_{R_j} (resp. M_{S_j}) are p-divisible objects of $\mathcal{MF}_{[0,1]}(R_j)$ (resp. of $\mathcal{MF}_{[0,1]}(S_j)$).

3.4.1.1. For any formally étale morphism $\mathrm{Spec}(\tilde{Q}_j^\wedge) \xrightarrow{d_j} \mathrm{Spec}(R_j^\wedge)$, with $\tilde{Q}_j$ a $W(k)$-algebra, we denote by

$$M_{\tilde{Q}_j} := \big(M \otimes_{W(k)} \tilde{Q}_j^\wedge, F^1 \otimes_{W(k)} \tilde{Q}_j^\wedge, r_j \circ d_j(h\varphi_0 \otimes 1)\big)$$

the pull back of M_{R_j} under d_j, for $\Phi_{\tilde{Q}_j}$ the only Frobenius lift of $\tilde{Q}_j^\wedge$ satisfying $d_j \circ \Phi_{\tilde{Q}_j} = \Phi_{R_j} \circ d_j$. It is a p-divisible object of $\mathcal{MF}_{[0,1]}(\tilde{Q}_j)$. We call it the extension of M_{R_j} to

$\mathrm{Spec}(\tilde{Q}_j)$. So for any $n \in \mathbb{N}$, $M_{\tilde{Q}_j}/p^n M_{\tilde{Q}_j}$ is an object of $\mathcal{MF}_{[0,1]}(\tilde{Q}_j)$. In particular, if $\hat{R}_j^0$ is the completion of R_j in the origin, we denote by $M_{\hat{R}_j^0} := (M \otimes \hat{R}_j^0, F^1 \otimes \hat{R}_j^0, \hat{\Phi}_j^r)$ the pull back of M_{R_j} under the canonical monomorphism $i_{R_j} : R_j \hookrightarrow \hat{R}_j^0$, for $\Phi_{\hat{R}_j^0}$ the natural Frobenius lift of $\hat{R}_j^0$ induced from Φ_{R_j} by completion, $j \in \{0, 2\}$. Here $\hat{\Phi}_j^r = \hat{r}_j(h\varphi_0 \otimes 1)$, with $\hat{r}_j \in G(\hat{R}_j^0)$, obtained from r_j via i_{R_j}. $M_{\hat{R}_j^0}$ is a p-divisible object of $\mathcal{MF}_{[0,1]}(\hat{R}_j^0)$.

Similarly, if k_1 is a perfect field containing k, we denote by $M_{R_{j\,W(k_1)}}$ its pull back under the canonical morphism $\mathrm{Spec}(R_{j\,W(k_1)}) \to \mathrm{Spec}(R_j)$, the Frobenius of $R_{j\,W(k_1)}$ being the logical one. We call it the extension of M_{R_j} to $W(k_1)$.

In a similar manner we define the extension of any object (or p-divisible object) of $\mathcal{MF}_{[\tilde{a},\tilde{b}]}(R_3)$, with R_3 a smooth $W(k)$-algebra, under a formally étale morphism $\mathrm{Spec}(\tilde{Q}_3^\wedge) \to \mathrm{Spec}(R_3)$, or under the natural morphism $\mathrm{Spec}(R_{3W(k_1)}) \to \mathrm{Spec}(R_3)$, with k_1 as above.

3.4.1.1.1. Definitions and notations. 1) Let $d_j : \mathrm{Spec}(\tilde{Q}_j^\wedge) \to \mathrm{Spec}(R_j^\wedge)$ be a formally étale morphism, with $\tilde{Q}_j$ a $W(k)$-algebra. We recall that we have a direct sum decomposition $M = F^1 \oplus F^0$. Let

$$\Phi(\tilde{Q}_j)^0 := r_j \circ d_j(h\varphi_0 \otimes 1)$$

be the Frobenius of $M \otimes \tilde{Q}_j^\wedge$ and let

$$\Phi(\tilde{Q}_j)^1 : (F^1 + pM) \otimes_{W(k)} \tilde{Q}_j^\wedge \to M \otimes \tilde{Q}_j^\wedge$$

be defined by the formula $p\Phi(\tilde{Q}_j)^1(m) = \Phi(\tilde{Q}_j)^0(m)$, $m \in (F^1 + pM) \otimes \tilde{Q}_j^\wedge$. For $n \in \mathbb{N}$ we still denote by $\Phi(\tilde{Q}_j)^i$ its reduction mod p^n ($i \in \{0, 1\}$).

2) We say that $M_{\tilde{Q}_j}/p^n M_{\tilde{Q}_j}$ potentially can be viewed as an object of $\mathcal{MF}^\nabla_{[0,1]}(\tilde{Q}_j)$ if there is a connection

$$\nabla : M \otimes \tilde{Q}_j/p^n\tilde{Q}_j \to M \otimes \tilde{Q}_j/p^n\tilde{Q}_j \otimes \Omega_{(\tilde{Q}_j/p^n\tilde{Q}_j)/W(k)}$$

such that we have:

$$\nabla \circ \Phi(\tilde{Q}_j)^0(m) = p\Phi(\tilde{Q}_j)^0 \circ d\Phi_{\tilde{Q}_j *}/p \circ \nabla(m), \tag{1}$$

if $m \in F^0 \otimes \tilde{Q}_j/p^n\tilde{Q}_j$, and

$$\nabla \circ \Phi(\tilde{Q}_j)^1(m) = \Phi(\tilde{Q}_j)^0 \circ d\Phi_{\tilde{Q}_j *}/p \circ \nabla(m), \tag{2}$$

if $m \in F^1 \otimes \tilde{Q}_j/p^n\tilde{Q}_j$. Here $d\Phi_{\tilde{Q}_j *}/p$ denotes the differential of the Frobenius $\Phi_{\tilde{Q}_j}$ of $\tilde{Q}_j^\wedge$ divided by p, and then taking mod p^n. About ∇ we say that it makes (or allows) $M_{\tilde{Q}_j}/p^n M_{\tilde{Q}_j}$ potentially to be viewed as an object of $\mathcal{MF}^\nabla_{[0,1]}(\tilde{Q}_j)$. Warning: the system

of equations obtained by putting (1) and (2) together, does not depend on the choice of the direct supplement F^0 of M; moreover as we chose $\Phi(\tilde{Q}_j)^i$ to be Frobenius maps, and not $\tilde{Q}_j^{\wedge}$-linear maps, we got $m \in F^i \otimes \tilde{Q}_j/p^n\tilde{Q}_j$, and not in $F^i \otimes_{W(k)\,\sigma} \tilde{Q}_j/p^n\tilde{Q}_j$, $i \in \{0,1\}$.

We say that $M_{\tilde{Q}_j}/p^n M_{\tilde{Q}_j}$ potentially is an object of $\mathcal{MF}^{\nabla}_{[0,1]}(\tilde{Q}_j)$ if there is precisely one connection ∇ which makes it potentially to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(\tilde{Q}_j)$; about this connection ∇ we say that it makes (or allows) $M_{\tilde{Q}_j}/p^n M_{\tilde{Q}_j}$ potentially to be an object of $\mathcal{MF}^{\nabla}_{[0,1]}(\tilde{Q}_j)$.

We use the same language for the case of p-divisible objects, i.e. when $n = \infty$. In such a situation we also say that the Frobenius $\Phi(\tilde{Q}_j)^0$ of $M_{\tilde{Q}_j}$ is ∇-parallel.

If in any of the above cases the connection ∇ is integrable, then we drop the word potentially. We prove later on (cf. 3.4.18.4.1), that any connection ∇ which makes $M_{\tilde{Q}_j}/p^n M_{\tilde{Q}_j}$ potentially to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(\tilde{Q}_j)$, is automatically integrable, and so the word potentially is just for the time being.

3) Let $\mathcal{Z}_k$ be the closed subscheme of U_0^0, U_2^0, G, W_0^0, W_2^0 and of $GL(M)$, defining the origin of the special fibre of G_k. So $\mathcal{Z}_k = \mathrm{Spec}(k)$.

4) For an arbitrary smooth algebra R_3 over $W(k)$, and for any (p-divisible) object of $\mathcal{MF}_{[\tilde{a},\tilde{b}]}(R_3)$, with $\tilde{a}, \tilde{b} \in \mathbb{Z}$, $\tilde{a} < \tilde{b}$, we are going to use the same language as in 2) above, but always working with connections satisfying the Griffiths-transversality condition. The equations needed to be satisfied are similar to the ones of (1) and (2) of 2) above (see [Fa1, p. 33]).

3.4.1.1.2. The convenience assumption. Sometimes it is convenient to make the assumption that the closed subscheme of $\mathrm{Spec}(R_j^0/pR_j^0)$ defined by $t_i(j) = 0$, i running through the elements of an arbitrary subset of $\{1, ..., d\}$, is connected (in particular the origin $\mathrm{Spec}(k)$ of the special fibre of U_j^0 is the closed subscheme defined by $t_i(j) = 0$, $i \in \{1, ..., d\}$).

This can always be achieved by removing a proper closed subscheme of U_j^0. This should be done before the choice of h, so we do not lose track of a_0 and a_2. It can be done as follows: we choose an open affine subscheme $\mathcal{U} = \mathrm{Spec}(R_{\mathcal{U}})$ containing the origin of G, together with an étale morphism $b_{\mathcal{U}} : \mathcal{U} \to Y$, such that the closed subscheme $\mathrm{Spec}(R_{\mathcal{U}}/(p, (z_i)_{i \in I_d}))$ of $\mathcal{U}$, with I_d an arbitrary subset of $\{1, ..., d\}$, is connected, and $\mathrm{Spec}(R_{\mathcal{U}}/((z_i)_{i \in I}))$ is the origin $\mathrm{Spec}(W(k))$ of G. We can take now $U_0 = \mathcal{U}$, and $U_2 = g_2\mathcal{U}$; as b_0 and b_2 we take $b_{\mathcal{U}}$ and respectively its translation by g_2.

Warning: unless specifically stated we do not work under the convenience assumption.

3.4.1.2. Lemma (the unicity principle for Shimura σ-crystals). *For any formally étale morphism $l_j : \mathrm{Spec}(\tilde{Q}_j^{\wedge}) \to \mathrm{Spec}(R_j^{\wedge})$, with $\tilde{Q}_j$ a $W(k)$-algebra such that any connected component of $\mathrm{Spec}(\tilde{Q}_j/p\tilde{Q}_j)$ has a non-empty intersection with $\ell_j^{-1}(\mathcal{Z}_k)$, and for any $n \in \mathbb{N}$, $M_{\tilde{Q}_j}/p^n M_{\tilde{Q}_j}$ potentially can be viewed in at most one way as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(\tilde{Q}_j)$. Similarly $M_{\tilde{Q}_j}$ potentially can be viewed in at most one way as a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(\tilde{Q}_j)$.*

3.4.1.3. Theorem (the ∇, the moduli, and the surjectivity principle for Shimura σ-crystals). **1)** *For any $n \in \mathbb{N}$ there is a formally étale morphism $\mathrm{Spec}(Q_{j,n}^{\wedge}) \stackrel{\ell_{j,n}}{\to} \mathrm{Spec}(R_j^{\wedge})$, $j \in \{0,2\}$, with $Q_{j,n}$ a smooth $W(k)$-algebra, such that:*

i) $\ell_{j,n}^{-1}(\mathcal{Z}_k) = \mathrm{Spec}(k)$;

ii) $M_{Q_{j,n}}/p^n M_{Q_{j,n}}$ *is an object of* $\mathcal{MF}^{\nabla}_{[0,1]}(Q_{j,n})$, *i.e. there is a unique connection*

$$\nabla_j^n : M \otimes_{W(k)} Q_{j,n}/p^n Q_{j,n} \to M \otimes_{W(k)} Q_{j,n}/p^n Q_{j,n} \otimes \Omega_{(Q_{j,n}/p^n Q_{j,n})/W(k)}$$

which makes $M_{Q_{j,n}}/p^n M_{Q_{j,n}}$ potentially to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(Q_{j,n})$; the connection ∇_j^n is integrable (so the word potentially can be disregarded), nilpotent mod p, and respects the G-action;

iii) for any other formally étale morphism $\mathrm{Spec}(\tilde{Q}_j^{\wedge}) \stackrel{\tilde{\ell}_j}{\to} \mathrm{Spec}(R_j^{\wedge})$, with $\tilde{Q}_j$ a smooth $W(k)$-algebra, such that $M_{\tilde{Q}_j}/p^n M_{\tilde{Q}_j}$ potentially can be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(\tilde{Q}_j)$ and any connected component of $\mathrm{Spec}(\tilde{Q}_j/p\tilde{Q}_j)$ has a non-empty intersection with its closed subscheme $\tilde{\ell}_j^{-1}(\mathcal{Z}_k)$, there is a unique $R_j^{\wedge}$-morphism $\mathrm{Spec}(\tilde{Q}_j^{\wedge}) \stackrel{\tilde{\ell}_j^n}{\to} \mathrm{Spec}(Q_{j,n}^{\wedge})$ such that $\tilde{\ell}_j = \ell_{j,n} \circ \tilde{\ell}_j^n$ and the connection (it is unique cf. 3.4.1.2) on $M \otimes \tilde{Q}_j/p^n\tilde{Q}_j$ which makes $M_{\tilde{Q}_j}/p^n M_{\tilde{Q}_j}$ potentially to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(\tilde{Q}_j)$ is the pull back of ∇_j^n through $\tilde{\ell}_j^n$ mod p^n;

iv) the special fibre of $\mathrm{Spec}(Q_{j,n})$ is a geometrically connected scheme over $\mathrm{Spec}(k)$.

2) *The unique $R_j^{\wedge}$-morphism (cf. ii)) $\ell_j^n : \mathrm{Spec}(Q_{j,n+1}^{\wedge}) \to \mathrm{Spec}(Q_{j,n}^{\wedge})$ such that $\ell_{j,n+1} = \ell_{j,n} \circ \ell_j^n$ and the pull back of ∇_j^n under the morphism ℓ_j^n mod p^n is ∇_j^{n+1} mod p^n, mod p (i.e. the resulting morphism at the level of special fibres) is: étale, quasi-finite, generically an étale cover of degree $p^{d(n)}$, with $d(n) \leq d^2$. Similarly, the morphism $\ell_{j,1}$ mod p has all these properties.*

3) *The construction of $\mathrm{Spec}(Q_{j,n}^{\wedge})$ commutes with (perfect) base field extensions, i.e. if k_1 is a perfect field containing k, then the p-adic completion of the extension of $\ell_{j,n}$ to $W(k_1)$ is the morphism of 1) obtained for the same $n \in \mathbb{N}$ and for the situation where G is replaced by $G_{W(k_1)}$, R_j is replaced by $R_{j\,W(k_1)}$, and M_{R_j} is replaced by its extension to $W(k_1)$.*

4) *Let $\mathrm{Spec}(Q_j)$ be the p-adic completion of the $\mathbb{N}$-projective limit of $\mathrm{Spec}(Q_{j,n}^{\wedge})$ under the transition morphisms ℓ_j^n, $n \in \mathbb{N}$, and let $\ell_j : \mathrm{Spec}(Q_j) \to U_j^0 = \mathrm{Spec}(R_j)$ be the resulting morphism. There is a unique morphism $a^j : \mathrm{Spec}(W(k)) \to \mathrm{Spec}(Q_j)$ which composed with l_j defines the origin a_0 of U_j^0. M_{Q_j} is a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(Q_j)$. $\mathrm{Spec}(Q_j)$ has a universal property similar to the one described in iii) of 1). The k-scheme $\mathrm{Spec}(Q_j/pQ_j)$ is geometrically connected.*

5) *There is a reduced closed subset $\mathcal{B}_j(k)$ of G_k, not containing the origin, depending on $h\varphi_0$, on Φ_{R_j}, but not on how small or big U_j^0 is, and such that the morphism ℓ_j has non-empty fibres over (geometric) points of $\mathrm{Spec}(R_j/pR_j)$ which do not factor through $\mathcal{B}_j(k)$. For any perfect field k_1 containing k, we get similarly (cf. 3) above) a reduced*

closed subscheme $\mathcal{B}_j(k_1)$ *of* G_{k_1} *having similar properties as* $\mathcal{B}_j(k)$. *We can take* $\mathcal{B}_j(k_1) = \mathcal{B}_j(k)_{k_1}$.

The proves of 3.4.1.2-3 are presented in 3.4.8.

3.4.1.4. Remarks. 1) Based on 3.4.18.4.2, it can be checked out that in many cases we can take $Q_{j,n} = R_j$ (i.e. $d(n) = 1$), $\forall n \in \mathbb{N}$. All these cases are going to be listed in §7, after we present (in §7) the classification of Shimura group pairs over $\mathbb{Z}_p$, and introduce (see 3.10) some extra terminology. See also 3.11.4.

2) The morphism $\ell_{j,n}$ is an étale cover iff $Q^{\wedge}_{j,n} = R^{\wedge}_j$ (cf. i) of 3.4.1.3 1)). A simple example when $\ell_{j,1}$ is not an étale cover was constructed by A. Ogus.

Let M be of dimension 2 over $W(k)$, and let $G = GL(M)$. We assume that $(M, F^1, h\varphi_0)$ is the filtered σ-crystal of the canonical lift of an ordinary elliptic curve over k. We assume also the existence of a $W(k)$-basis $\{x, y\}$ of M such that $h\varphi_0$ takes x and y respectively into x and py. We choose a Frobenius of $G^{\wedge}$ compatible with the Frobenius σ of $W(k)$ and such that under the canonical identification $G = \mathrm{Spec}(W(k)[x_{11}, x_{12}, x_{21}, x_{22}][\frac{1}{x_{11}x_{22} - x_{21}x_{12}}])$ defined by the above ordered basis of M, x_{12} goes to x^p_{12}, x_{21} goes to x^p_{21}, while $x_{ii} - 1$ goes to $(x_{ii} - 1)^p$, $i = \overline{1,2}$. We can take $U_0 = U_2 = G$. To see that $l_{j,1}$ is not an étale cover, it is enough to see that the pull back of M_{R_j}/pM_{R_j} under the homomorphism $\mathbb{G}_a \to G$ defining the subgroup of G acting trivially on x and on the quotient $M/xW(k)$, can not get a connection over its pull back under an étale cover of $\mathbb{G}^{\wedge}_a$. This is easy: identifying $\mathbb{G}_a = \mathrm{Spec}(W(k)[t])$, with $t = x_{12}$, the connection we get on $M \otimes W(k)[[t]]$ annihilates x and takes y into $\alpha(t)xdt$, where $\alpha(t) = -\sum_{i=0}^{\infty} t^{p^i-1}$. So $\alpha(t) \bmod p$ satisfies the equation (in u)

$$t^{p-1}u^p = u + 1$$

(we recall that $p > 2$). This equation defines an étale scheme over $\mathbb{G}_{a\,k} = \mathrm{Spec}(k[t])$ which is not an étale cover: it is obtained from the étale cover $\mathrm{Spec}(k[t, v]/(v^p - v - t))$ of $\mathbb{G}_{a\,k}$ by removing its closed subscheme defined by $t = 0$, $v \neq 0$. This closed subscheme is a disjoint union of $p - 1$ copies of $\mathrm{Spec}(k)$.

In this example, coming back to the general notations, the morphism

$$\ell'_j : \ell_j^{-1}(\mathrm{Spec}(Q_j/pQ_j[\frac{1}{\prod_{i=1}^{d} t_i(j)}]) \to \mathrm{Spec}(R_j/pR_j[\frac{1}{\prod_{i=1}^{d} t_i(j)}]) := \mathcal{W}_j$$

is an $\mathbb{N}$-pro-étale morphism of infinite degree. Here $d = 4$, $t_i(j) = x_{ii} - 1$ and $t_{i+1}(j) = x_{i,3-i}$, with $i \in \{1, 2\}$. It defines an $\mathbb{N}$-pro-étale cover above the open subscheme of $\mathcal{W}_j$ defined by $t_1(j) \neq -1$, as it can be easily checked starting from 3.4.18.4. So if we take U^0_0 to be $\mathcal{W}_0$ and not the whole group G, then we can take $\mathcal{B}_0(k)$ to be defined by $x_{11} = 0$.

3) The estimate $d(n) \leq d^2$ is not the best possible. For instance if $G = GL(M)$ it can be checked that we can replace d^2 by $\frac{d^2}{4}$. For better estimates of the numbers $d(n)$, see 3.11.4. For the time being, let us just remark that the bound d^2 is independent on n. We do not know (if or) when $d(n) = d(1)$, $\forall n \in \mathbb{N}$.

4) We do not know if (or when) the map ℓ_j is surjective. Similarly about the maps $\ell_{j,n}$ or ℓ_j^n, $n \in \mathbb{N}$, $j \in \{0,2\}$. 3.4.18.5.2 motivates why it is natural to expect (that under some conditions) such maps to be surjective.

5) $M_{Q_{j,m}}/p^n M_{Q_{j,m}}$, $m,n \in \mathbb{N}$, $n \leq m$, are rigid objects in the sense they have few endomorphism. For instance, restricting to endomorphism (called G-endomorphism) fixing the family of tensors $(v_\alpha)_{\alpha\in\mathcal{J}}$ introduced at the beginning of §3, we often get only endomorphisms coming from elements of the Lie algebra of $Z(G)$ which are fixed by $h\varphi_0$. This remains true in many cases for the set of all endomorphisms, the group $Z(G)$ being replaced by the centralizer of G in $GL(M)$. This important phenomenon is elaborated in §7, being an essential tool in the strategy of proving the split criterion of 1.19. There are variants of it when we pass to étale schemes over $\mathrm{Spec}(Q_{j,m})$. In these variants extra endomorphisms show up: like the ones springing from the factors of $G^{\mathrm{ad}}_{W(k_1)}$ having a zero intersection with $F^1(\mathrm{Lie}(G)\otimes W(k_1))$ and left invariant by $h\varphi_0$; here k_1 is a finite field extension of k.

3.4.1.5. σ-$\mathcal{S}$-crystals. Let $\tilde{a},\tilde{b} \in \mathbb{Z}$, with $\tilde{a} < \tilde{b}$. Let $S(\tilde{a},\tilde{b}) := \{\tilde{a},...,\tilde{b}\}$. Let $(\tilde{M},(\tilde{F}^i)_{i\in S(\tilde{a},\tilde{b})},\tilde{\varphi})$ be a p-divisible object of $\mathcal{MF}_{[\tilde{a},\tilde{b}]}(W(k))$. Let $\tilde{G}$ be a smooth subgroup of $GL(\tilde{M})$, having connected fibres, and such that:

– $\mathrm{Lie}(\tilde{G})[\frac{1}{p}]$ is stable under the action of $\tilde{\varphi}$;

– $\mathrm{Lie}(\tilde{G})$ together with the induced filtration (from the natural induced filtration of $\mathrm{End}(\tilde{M})$) and Frobenius $\tilde{\varphi}$, is a p-divisible object of $\mathcal{MF}(W(k))$.

The quadruple $(\tilde{M},(\tilde{F}^i)_{i\in S(\tilde{a},\tilde{b})},\tilde{\varphi},\tilde{G})$ is called a filtered σ-$\mathcal{S}$-crystal (over k). As in 2.2.8 2) we get σ-$\mathcal{S}$-crystals. As examples we mention the pseudo Shimura σ-crystals and the generalized Shimura p-divisible objects over k, cf. def. 4) of 2.2.8. We would like to point out that we did not assume the existence of a cocharacter $\mu : \mathbb{G}_m \to \tilde{G}$, leaving invariant $\tilde{F}^i$, $i \in S(\tilde{a},\tilde{b})$, with $\beta \in \mathbb{G}_m(W(k))$ acting on $\tilde{F}^i/\tilde{F}^{i+1}$ as the multiplication with β^{-i} (to be compared with 2.2.8, where we assumed this). When such a cocharacter does exist, we speak about (filtered) σ-$\mathcal{S}$-crystals for which the W-condition is satisfied (the letter W stands to honor [Wi]).

3.4.1.6. Remark. It is expected that everything in 3.4.1.3 (when properly formulated; see below) remains true in the context of σ-$\mathcal{S}$-crystals. This is true if $\tilde{a}=0$ and $\tilde{b}=1$, as well as for generalized Shimura p-divisible objects of $\mathcal{MF}(W(k))$. The proof in the case when $\{\tilde{a},\tilde{b}\}=\{0,1\}$ is entirely the same. In the other case we have to interpret 3.4.1.3 as a property of Shimura filtered Lie σ-crystals, and to construct a Shimura filtered σ-crystal whose Shimura adjoint filtered Lie σ-crystal is isomorphic to the one of our Shimura p-divisible object of $\mathcal{MF}(W(k))$; for details see §7. In these two situations we can introduce as in 2.2.10 the notion of filtered F-$\mathcal{S}$-crystals and respectively of generalized Shimura filtered p-divisible objects of some category $\mathcal{MF}_{[\tilde{a},\tilde{b}]}(*)$; here $*$ stands for regular rings of formal power series with coefficients in $W(k)$.

What we mean by "properly formulated": following the discussion of 3.4.8.9, we might have to work with a different Frobenius of the module underlying the p-divisible objects involved (i.e. we do not use r_j of 3.4.1, but some other element of $\tilde{G}(*)$, carefully chosen).

3.4.2. Before proving 3.4.1, let us see some of its immediate implications. It is known [Fa1, 7.1] (here is the first place where we need $p \geq 3$) that the category $\mathcal{MF}^{\nabla}_{[0,1]}(Q_{j,n})$ is antiequivalent to the category of finite flat group schemes of p-power rank on $\mathrm{Spec}(Q^{\wedge}_{j,n})$, $j \in \{0,2\}$. So corresponding to $M_{Q_{j,n}}/p^n M_{Q_{j,n}}$ we get a finite flat group scheme $D_{j,n}$ on $\mathrm{Spec}(Q^{\wedge}_{j,n})$ of rank $p^{n\dim(M)}$. Using the uniqueness of ∇^n_j we get that the pull back of $D_{j,n}$ under the morphism ℓ^n_j is isomorphic to the kernel of the multiplication with p^n on $D_{j,n+1}$, $\forall n \in \mathbb{N}$, $j \in \{0,2\}$. So the objects (of $\mathcal{MF}^{\nabla}_{[0,1]}(Q_j)$) $M_{Q_j}/p^n M_{Q_j}$, $n \in \mathbb{N}$, put together give birth to a p-divisible group D_j over $\mathrm{Spec}(Q_j)$. Let ∇_j be the connection on $M \otimes Q_j$ which makes M_{Q_j} to be a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(Q_j)$. As the connections ∇^n_j, $n \in \mathbb{N}$, respect the G-action, we deduce that ∇_j also respects the G-action. We conclude: the pair

$$D(j) := (D_j, (v_\alpha)_{\alpha \in \mathcal{J}})$$

is a Shimura p-divisible group over Q_j.

3.4.2.1. For simplifying the notations we assume now, till the end of 3.4.5, that $k = \bar{k}$. We also assume that $h^{-1} \bmod p$ does not define a point of $\mathcal{B}_0(k)$, and that $g_2 h^{-1} \bmod p$ does not define a point of $\mathcal{B}_2(k)$. We have:

a) The pull back of $D(j)$ under the morphism a^j is the Shimura p-divisible group over $W(k)$ associated to the Shimura filtered σ-crystal $(M, F^1, h\varphi_0, G, (v_\alpha)_{\alpha \in \mathcal{J}})$, $j \in \{0,2\}$.

b) The pull back of $D(0)$ under a morphism $\mathrm{Spec}(W(k)) \to \mathrm{Spec}(Q_0)$ which through ℓ_0 produces the morphism $\mathrm{Spec}(W(k)) \hookrightarrow U^0_0$ defining h^{-1}, is the Shimura p-divisible group over $W(k)$ associated to the Shimura filtered σ-crystal $(M, F^1, \varphi_0, G, (v_\alpha)_{\alpha \in \mathcal{J}})$.

c) The pull back of $D(2)$ under a morphism $\mathrm{Spec}(W(k)) \to \mathrm{Spec}(Q_2)$ which through ℓ_2 produces the morphism $\mathrm{Spec}(W(k)) \hookrightarrow U^0_2$ defining $g_2 h^{-1}$, is the Shimura p-divisible group over $W(k)$ associated to the Shimura filtered σ-crystal $(M, F^1, g_2\varphi_0, G, (v_\alpha)_{\alpha \in \mathcal{J}})$.

d) We denote by $O_0 = W(k)[[\omega_1, \ldots, \omega_d]]$ the ring of the completion of G (or of U^0_0, or of U^0_2) in the origin. It is the same as the ring of the completion of Q_j in a^j, $j \in \{0,2\}$; so we have $O_0 = \hat{R}^0_0 = \hat{R}^0_2$. The pull backs of $D(0)$ and $D(2)$ to $\mathrm{Spec}(O_0)$ give us two Shimura p-divisible groups: D^0_0 and respectively D^0_2. The p-divisible objects with cycles of $\mathcal{MF}^{\nabla}_{[0,1]}(O_0)$ they define are both Shimura filtered F-crystals: $(M, F^1, h\varphi_0, G, \tilde{f}_0)$ and respectively $(M, F^1, h\varphi_0, G, \tilde{f}_2)$. Warning: the Frobenius of O_0 corresponding to D^0_0 is different from the Frobenius of O_0 corresponding to D^0_2. But these two Shimura filtered F-crystals are induced one from each other, cf. [Fa2, rm. iii) after th. 10] (the two Frobenius of O_0 are as needed for [Fa2, th. 10], cf. the way we defined Φ_{R_j}). This means that there are $W(k)$-homomorphisms $O_0 \overset{\omega_0}{\underset{\omega_2}{\rightrightarrows}} O_0$ such that the Shimura filtered F-crystal associated to the pull back of D^0_j under the morphism associated to ω_j is (isomorphic to) the Shimura filtered F-crystal associated to D^0_{2-j}, $j \in \{0,2\}$.

3.4.2.2. Remark. We call this last fact, that D^0_0 and D^0_2 are induced one from each other, the very weak gluing principle. It is trivial to check that we can actually choose ω_j, $j \in \{0,2\}$, to be isomorphisms of $W(k)$-algebras, and we call this the weak gluing principle.

3.4.3. We consider (cf. 3.4.1.3) the following p-divisible object of $\mathcal{MF}^{\nabla}_{[-1,1]}(Q_j)$:

$$\mathrm{End}(M_{Q_j}) := \big(\mathrm{End}(M)\otimes_{W(k)}Q_j, F^0(\mathrm{End}(M))\otimes_{W(k)}Q_j, F^1(\mathrm{End}(M))\otimes_{W(k)}Q_j, r_j\circ\ell_j(h\varphi_0\otimes 1), \nabla_j\big).$$

Similarly, we define $\mathfrak{g}_{Q_j}$ by working with $\mathfrak{g} = \mathrm{Lie}(G)$ instead of $\mathrm{End}(M)$. For any $n \in \mathbb{N}$, $\mathrm{End}(M_{Q_j}/p^nQ_j) := \mathrm{End}(M_{Q_j})/p^n\mathrm{End}(M_{Q_j})$ is an object of $\mathcal{MF}^{\nabla}_{[-1,1]}(Q_j)$, having (as ∇_j respects the G-action) $\mathfrak{g}_{Q_j}/p^n\mathfrak{g}_{Q_j}$ as a subobject in $\mathcal{MF}^{\nabla}_{[-1,1]}(Q_j)$, $j \in \{0,2\}$. We deduce: for any map $z : \mathrm{Spec}(W(k_1)) \to \mathrm{Spec}(Q_j)$, with k_1 a perfect field, the pull back of M_{Q_j} through z gets a natural structure of a Shimura filtered σ_1-crystal $\big(M\otimes_{W(k)} W(k_1), F^1\otimes_{W(k)} W(k_1), \varphi^1_1, G_{W(k_1)}, (v_\alpha)_{\alpha\in\mathcal{J}}\big)$, with $\varphi^1 = g^1\varphi_0$ for some element $g^1 \in G(W(k_1))$ (this is just a particular case of the principle of 2.2.20 5); here σ_1 is the Frobenius automorphism of $W(k_1)$. So we can speak about its Shimura filtered Lie σ_1-crystal defined by $\mathrm{Lie}(G_{W(k_1)}) = \mathfrak{g}\otimes W(k_1) \subset \mathrm{End}(M\otimes W(k_1))$.

3.4.4. Taking a point $z^j : \mathrm{Spec}(W(k_j) \to \mathrm{Spec}(Q_j)$, with k_j a perfect field, such that the induced point $\mathrm{Spec}(k_j) \to \mathrm{Spec}(Q_j/pQ_j)$ sits over the generic point of $\mathrm{Spec}(Q_j/pQ_j)$, we get a Shimura filtered σ_j-crystal

$$\mathcal{C}_j := \big(M\otimes_{W(k)} W(k_j), F^1\otimes_{W(k)} W(k_j), \varphi^j, G_{W(k_j)}\big),$$

with $\varphi^j = g^j\varphi_0$ for some element $g^j \in G(W(k_j))$; here σ_j is the Frobenius automorphism of $W(k_j)$. The Shimura filtered Lie σ_j-crystal

$$\mathrm{Lie}(\mathcal{C}_j) := \big(\mathfrak{g}_{W(k_j)}, \varphi^j, F^0(\mathfrak{g})_{W(k_j)}, F^1(\mathfrak{g})_{W(k_j)}\big)$$

has a Newton polygon which does not depend on $j \in \{0,2\}$ (cf. 3.4.2 d)) and is below the Newton polygon of the Shimura Lie σ-crystal $(\mathfrak{g}, \varphi_0)$ (as $\mathcal{C}_0$ specializes to (M, F^1, φ_0), cf. 3.4.2 b)). Moreover $\mathcal{C}_2$ specializes to $(M, F^1, g_2\varphi_0)$, cf. 3.4.2 c).

3.4.5. Remarks. 1) We could have worked to achieve the connection of the Shimura σ-crystal $(M, g_2\varphi_0, G)$ to the Shimura σ-crystal (M, φ_0, G) (expected to be G-ordinary) working with just one deformation, cf. 3.4.7 below. We preferred to work with two global deformations (over Q_0 and Q_2) as we hope that this method will be useful in other contexts (we have in mind applications to the construction of integral models of Shimura varieties of special type by making use of moduli schemes of p-divisible groups, as well as applications to p-adic uniformizations of Shimura varieties), cf. 3.4.19.

2) It can happen that h^{-1} mod p belongs to $\mathcal{B}_0(k)$, or that g_2h^{-1} mod p belongs to $\mathcal{B}_2(k)$ (see 3.4.1.4 2)). Then we can not go ahead with 3.4.2-4. So we have to proceed more carefully, cf. 3.4.7 below.

3) We denote the p-divisible group D_j as well as the Shimura p-divisible group $D(j)$ over $\mathrm{Spec}(Q_j)$ by

$$\mathcal{D}(U^0_j, \Phi_{R_j}, M, F^1, h\varphi_0, G),$$

in order to emphasize the data needed to construct them.

3.4.6. Lemma. *The set of elements $\tilde{h} \in G(W(k))$ with the property that $\tilde{h}\varphi_0 = gp_0\varphi_0 g^{-1}$, with $g \in G(W(k))$ leaving invariant $F^1 \otimes k$, and with $p_0 \in P_0(W(k))$ (P_0 was defined in 3.2.7), maps into a Zariski dense set $\mathcal{L}$ of k-valued points of G_k.*

Proof: Let H be the Zariski closure of $\mathcal{L}$ in G_k. For a Galois extension k_1 of k, we denote by $\mathcal{L}_1 \subset G(k_1)$ and $H_1 \hookrightarrow G_{k_1}$ the analogues of $\mathcal{L}$ and H obtained by working over $W(k_1)$ instead of $W(k)$. It is an easy exercise to see that $\mathcal{L}_1$ is stable under the natural action of the Galois group $\mathrm{Gal}(k_1/k)$ on $G(k_1)$ and that $\mathcal{L}$ is the subset of $\mathcal{L}_1$ fixed by the Galois action of $\mathrm{Gal}(k_1/k)$. So if $H_1 = G_{k_1}$, from 3.4.0 and [Bo, 18.3] we deduce that $H = G$. So we can assume that $k = \bar{k}$ (in fact it is enough to assume that G is split). This allows us to replace φ_0 by φ_1 (cf. 3.2.3 and 3.3.2-3).

We use the notations of 3.5 below. It is enough to work with a subset I_0 of I corresponding to a cycle γ_0 of γ (obviously H is $Z(G_k)$-invariant and $Z(G_k) \subset H$). We want to show that H contains the subgroup of G generated by G_i, $i \in I_0$.

We have to consider the following two cases: 1) $|I_0| = 1$, and 2) $|I_0| \geq 2$. Let N_i be the abelian subgroup of G_i having $F^1(\mathfrak{g}_i)$ as its Lie algebra, and let N_{-i} be the abelian subgroup of G_i whose Lie algebra is

$$\mathrm{Lie}(N_{-i}) = \bigoplus_{\substack{\beta \in \Phi_i^+ \\ \mathfrak{g}_\beta \subset \mathrm{Lie}(N_i)}} \mathfrak{g}_{-\beta}.$$

We first treat the case 1). So $I_0 = \{1\} = \{i\}$. Working with an arbitrary element $p_0 \in P_0(i)(W(k))$, and with

$$g \in \exp\big(p\mathrm{Lie}(N_{-i})\big) \times \prod_{\substack{\alpha \in \Phi_1^- \\ \mathfrak{g}_\alpha \not\subset \mathrm{Lie}(N_{-i}) \oplus \mathrm{Lie}(P_0(i))}} \mathbb{G}_{a,\alpha}(W(k)),$$

(here exp is the exponential map, $P_0(i) := G_i \cap P_0$, and $\mathbb{G}_{a,\alpha}$ is the $\mathbb{G}_a$-subgroup of G_i having $\mathfrak{g}_\alpha$ as its Lie algebra) we deduce that H contains an open subscheme of G_{ik}, and so it contains G_{ik}.

The case 2) is very much the same. We also use the notations of 3.7.2. We choose $g = \prod_{i \in I_0} g_i$, with $g_i \in G_i(W(k))$ of the form $g_i = v_i w_i$, where $v_i \in \exp\big(p\mathrm{Lie}(N_{-i})\big)$ and

$$w_i \in \prod_{\substack{\alpha \in \Phi_i^- \\ \tilde{g}_i(\alpha) \not\subset \mathrm{Lie}(N_{-i}) \oplus \mathrm{Lie}(P_0(i))}} \mathbb{G}_{a,\alpha(i)}(W(k)),$$

$\forall i \in I_0$. $\mathbb{G}_{a,\alpha(i)}$ is the subgroup of G having $\tilde{g}_i(\alpha)$ as its Lie algebra. We choose $p_0 = \prod_{i \in I_0} p_0^i$, with $p_0^i \in P_0(i)(W(k))$ (here $P_0(i)$ has the same significance as above). The equality $\tilde{h}\varphi_1 = gp_0\varphi_1 g^{-1}$ gives us an element $\tilde{h} = \prod_{i \in I_0} h_i$ of $G(W(k))$, with $h_i \in G_i(W(k))$ of the form

$$h_i = w_i p_0^i \varphi_1(w_{i-1}^{-1})\varphi_1(v_{i-1}^{-1}),$$

where for $\tilde{g} \in G(W(k))$ leaving invariant $F^1 \otimes k$, $\varphi_1(\tilde{g}) := \varphi_1 \tilde{g} \varphi_1^{-1} \in G(W(k))$. It is enough to show that the Zariski closure Z_H of the k-valued points of the form $\prod_{i \in I_0} \tilde{h}_i$ (where $\tilde{h}_i$ is the image of h_i in $G_i^{\mathrm{ad}}(k)$) in $\prod_{i \in I_0} G_i^{\mathrm{ad}}{}_k$, is $\prod_{i \in I_0} G_i^{\mathrm{ad}}{}_k$.

Let $\tilde{N}_i$ be the connected subgroup of G_i having $\varphi_1(\mathrm{Lie}(N_{-(i-1)}))$ as its Lie algebra. The role of $\varphi_1(v_{i-1}^{-1})$ is just that of an element of $\tilde{N}_i(W(k))$. Using [Bo, 14.14] we get that $h_i = w_i \varphi_1(w_{i-1}^{-1}) \tilde{p}_0^i \tilde{n}_i$, with $\tilde{n}_i \in \tilde{N}_i(W(k))$, and $\tilde{p}_0^i \in P_0(i)(W(k))$. For this we need p_0^i to be a $W(k)$)-valued point of an open subscheme of $P_0(i)$, containing the origin, and small enough. The key point is: the one dimensional $B(k)$-vector spaces $\tilde{g}_i(\alpha) \otimes B(k)$, $i \in I_0$, which are not included in $\mathrm{Lie}(P_1) \otimes B(k) = \mathrm{Lie}(P_0) \otimes B(k)$ (see 3.3.3 for this identification), are permuted by φ_1 in such a way that in any resulting cycle there is such a $B(k)$-vector space contained in $\mathrm{Lie}(N_{-i}) \otimes B(k)$, for some $i \in I_0$.

Writing $w_i = \tilde{w}_i u_i \varphi_1(w_{i-1})$, with $\tilde{w}_i \in \prod \mathbb{G}_{a,\alpha(i)}(W(k))$, where the product runs through all $\alpha \in \Phi_i$ such that $\tilde{g}_i(\alpha)$ is not included in $\mathrm{Lie}(P_1) \oplus \mathrm{Lie}(N_{-i}) \oplus \varphi_1(p\mathrm{Lie}(N_{-(i-1)})$, and with $u_i \in \prod \mathbb{G}_{a,\alpha(i)}(W(k))$, where the product runs through all $\alpha \in \Phi_i$ such that $\tilde{g}_i(\alpha)$ is not included in $\mathrm{Lie}(P_1) \oplus \varphi_1(p\mathrm{Lie}(N_{-(i-1)}))$ but $\tilde{g}_i(\alpha) \subset \mathrm{Lie}(N_i)$, we get that

$$h_i = \tilde{w}_i u_i \tilde{p}_i \tilde{n}_i .$$

The above key point implies: $\tilde{w}_i$, u_i, $\tilde{p}_i$ and $\tilde{n}_i$ mod p are allowed to run independently through the k-valued points of open subschemes of subgroups of G_{ik} whose Lie algebras put together are forming a direct sum decomposition of $\mathfrak{g}_i \otimes k$, $\forall i \in I_0$. This implies (again cf. [Bo, 14.14]) that Z_H is $\prod_{i \in I_0} G_i^{\mathrm{ad}}{}_k$, ending the proof of the lemma.

3.4.6.0. Exercise. If $\tilde{h} \in G(W(k))$ is obtained as in 3.4.6, and if $g_3 \in G(W(k))$ mod p is the identity element, then $(M, g_3 \tilde{h} \varphi_0)$ has τ_0 as its formal isogeny type, and $(\mathfrak{g}, g_3 \tilde{h} \varphi_0)$ has the same Newton polygon as $(\mathfrak{g}, g_3 \tilde{h} \varphi_0)$. (Hint: use the ideas of the proof of 3.4.6 and of b) of 4.4.1 2); by induction on $m \in \mathbb{N}$ show that we can assume that g_3 mod p^m is the identity element, and so, chosing m big enough [Ka2, 1.4.4] applies).

3.4.6.1. Remarks. 1) Another proof of 3.4.6 (and actually that $\mathcal{L} \supset U_1(k)$, with U_1 an open subscheme of G_k; the above proof of 3.4.6 also gives this result) can be obtained using 3.5.11 (the whole of 3.5-6 is independent of 3.4) and 3.4.10.

2) A fast proof of 3.1.3 a) and b) together with the inclusion $A_G \subset B_G$ can be obtained by putting together 3.4.6, 3.3.5, 3.5.11 and 3.4.10. In fact 3.5.11 can be substituted by 3.4.6.0.

3.4.7. Section 3.4 started with the desire of connecting the (arbitrary) Shimura filtered σ-crystal $(M, F^1, g_2 \varphi_0, G)$ with the Shimura filtered σ-crystal (M, F^1, φ_0, G) (or with another such Shimura filtered σ-crystal which is -supposed to be- a G-ordinary σ-crystal). This is achieved as follows.

Let $\mathcal{U} = \mathrm{Spec}(R_1)$ be an open affine subscheme of G, with $\mathcal{U}(W(k))$ containing the identity element of $G(W(k))$. We assume that there is an étale morphism $b_{\mathcal{U}} : \mathcal{U} \to Y$, with the origin of G factoring through the closed subscheme of $\mathcal{U}$ obtained as the inverse image through $b_{\mathcal{U}}$ of the closed subscheme of Y defined by $z_i = 0$, $i = \overline{1, d}$. If needed, we can shrink it a little bit, so that the condition ii) of the begining of 3.4 is satisfied. Let

Φ_{R_1} be the Frobenius of $\mathcal{U}^\wedge$, obtained through $b_{\mathcal{U}}$, from the Frobenius of Y which takes z_i to z_i^p, $i = \overline{1,d}$.

Let $D_1 := \mathcal{D}(\mathcal{U}, \Phi_{R_1}, M, F^1, g_2\varphi_0, G)$ be the p-divisible group over the p-adic completion $\mathrm{Spec}(Q_1)$ of an affine $\mathbb{N}$-pro-étale scheme over $\mathcal{U}$, obtained as in 3.4.1.3, cf. 1) and 3) of 3.4.5. Sometimes, not to complicate the notation, we refer to D_1 as a Shimura p-divisible group, the Shimura structure being the natural one. From 3.4.1.3 5) and 3.4.6 we deduce (as in 3.4.3-4 and 2.3.16) that:

3.4.7.1. Proposition. *There is a Shimura σ_1-crystal $\mathcal{C}_1$ over the algebraic closure of the field of fractions of Q_1/pQ_1, specializing to $(M, g_2\varphi_0, G)$ and to a Shimura $\bar{\sigma}$-crystal of the form $(M \otimes W(\bar{k}), g_3\varphi_0, G_{W(\bar{k})})$, with $g_3 \in G(W(\bar{k}))$ which mod p belongs to the set $\mathcal{L}$ of 3.4.6, but defined for $\bar{k}$ (here σ_1 is the Frobenius automorphism of $W(k_1)$).*

3.4.8. The proves of 3.4.1.2-3. We work with just one index: 0. We have:

1) We recall that $\Phi_{\hat{R}_0^0}$ is induced from Φ_{R_0}. There is a unique connection $\nabla_{\hat{R}_0^0}$ on $M \otimes_{W(k)} \hat{R}_0^0$ which makes $M_{\hat{R}_0^0}$ potentially to be viewed as a p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(\hat{R}_0^0)$; it is integrable and nilpotent mod p. For these facts cf. [Fa2, th. 10].

2) $\nabla_{\hat{R}_0^0}$ respects the G-action, i.e. is of the form $\delta + \beta$, with

$$\beta \in \hat{R}_0^0 \otimes_R (\mathrm{Lie}(G) \otimes_{W(k)} \Omega_G)$$

(cf. [Fa2, rm. ii) of th. 10]). Here δ is the connection on $M \otimes \hat{R}_0^0$ that takes any element of M into 0.

3) Let $S(M) := \{1, ..., dim(M)\}$, and let $S(G) := \{1, ..., d\}$. We choose a basis $\{e_i | i \in S(M)\}$ of M, and let $\{e_{is} | i, s \in S(M)\}$ be the canonical basis of $\mathrm{End}(M)$ with respect to the chosen basis of M. So $e_{is}(e_s) = e_i$. For the sake of generality, for the time being, we make the choice arbitrarily. Later on we specialize to a proper choice (see 3.4.8.5) suitable for computations. $\Omega_{R_0/W(k)}$ is a free R_0-module, having $\{dz_i | i \in S(G)\}$ as a basis.

Let $\tilde{l}_0 : \mathrm{Spec}(\tilde{Q}_0^\wedge) \to \mathrm{Spec}(R_0)$ be a formally étale morphism, with $\tilde{Q}_0$ a $W(k)$-algebra. An arbitrary connection ∇ on $M \otimes \tilde{Q}_0^\wedge$ can be written in the form

$$\nabla = \delta + \sum_{(i,s,l) \in S(M) \times S(M) \times S(G)} x_{isl} e_{is} \otimes dz_l, \tag{0}$$

with $x_{isl} \in \tilde{Q}_0^\wedge$, $\forall (i, s, l) \in S(M) \times S(M) \times S(G)$.

4) We first treat the case $n = 1$. The condition that ∇ makes $M_{\tilde{Q}_0}/pM_{\tilde{Q}_0}$ potentially to be viewed as an object of $\mathcal{MF}^\nabla_{[0,1]}(\tilde{Q}_0)$, is expressed by the following equations

$$\nabla \circ \Phi(\tilde{Q}_0)^0(m) = p\Phi(\tilde{Q}_0)^0 \circ d\Phi_{\tilde{Q}_0 *}/p \circ \nabla(m), \tag{1}$$

if $m \in F^0 \otimes \tilde{Q}_0/p\tilde{Q}_0$, and

$$\nabla \circ \Phi(\tilde{Q}_0)^1(m) = \Phi(\tilde{Q}_0)^0 \circ d\Phi_{\tilde{Q}_0 *}/p \circ \nabla(m), \tag{2}$$

if $m \in F^1 \otimes \tilde{Q}_0/p\tilde{Q}_0$.

5) The above equations are replaced, using the chosen form of ∇, by equations involving x_{isl}, $(i,s,l) \in S(M) \times S(M) \times S(G)$, and their Frobenius transforms taken mod p (i.e. their p-powers). These equations have coefficients in R_0/pR_0 and so they define an affine scheme $\mathcal{S}^1$ of finite type over $\mathcal{S}^0 := \mathrm{Spec}(R_0/pR_0)$. $\mathcal{S}^1$ is a moduli scheme of connections. Let

$$\ell(0) : \mathcal{S}^1 \to \mathcal{S}^0$$

be the morphism we get.

The nice thing is that these equations involving x_{isl} take the form

$$x_{isl} = L_{isl}(x^p_{111}, x^p_{112}, ..., x^p_{\dim(M)\,\dim(M)d}) + a_{isl}, \tag{3}$$

where the expression L_{isl} is homogeneous and linear in each variable and has coefficients in R_0/pR_0, and where $a_{isl} \in R_0/pR_0$. We have such an equation for any triple $(i,s,l) \in S(M) \times S(M) \times S(G)$. This nice shape of the equations involving x_{isl} is obtained as follows. Let $i \in S(M)$. There are $e_i(0) \in F^0 \otimes \tilde{Q}_0/p\tilde{Q}_0$ and $e_i(1) \in F^1 \otimes \tilde{Q}_0/p\tilde{Q}_0$ such that $e_i = \Phi(\tilde{Q}_0)^0(e_i(0)) + \Phi(\tilde{Q}_0)^1(e_i(1))$ ($M_{\tilde{Q}_0}$ is a p-divisible object of $\mathcal{MF}_{[0,1]}(\tilde{Q}_0)$). Summing up the equations obtained by plugging $e_i(0)$ and $e_i(1)$ in (1) and respectively in (2) of 4) above, we get the mentioned shape for the equations involving x_{isl}, $(s,l) \in S(M) \times S(G)$, in the left hand side of (3). By doing this for any $i \in S(M)$, we obtain "all" the equations in the variables x_{isl} (in the sense that any other equation is a linear combination of these ones).

6) The existence of another connection on $M \otimes \tilde{Q}_0/p\tilde{Q}_0$ making $M_{\tilde{Q}_0}/pM_{\tilde{Q}_0}$ potentially to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(\tilde{Q}_0)$ corresponds to a non-trivial solution (with values in $\tilde{Q}_0/p\tilde{Q}_0$) of the system of equations

$$x_{isl} = L_{isl}(x^p_{111}, x^p_{112}, ..., x^p_{\dim(M)\,\dim(M)d}), \tag{4}$$

$(i,s,l) \in S(M) \times S(M) \times S(G)$. Let $\mathcal{I}$ be the ideal of R_0/pR_0 generated by $t_i(0)$, $i \in \{1, ..., d\}$. The coefficients of the linear forms L_{isl} are elements of $\mathcal{I}^{p-1}$. This is so due to the fact that $\Phi_{R_0}(t_i(0)) = t_i(0)^p$.

To prove 3.4.1.2 for $n = 1$, we can assume that $\tilde{Q}_0$ is a complete ring with the maximal ideal generated by $\mathcal{I}$ and residue field $\bar{k}$. By induction on $m \in \mathbb{N}$, we get that any solution of the above system (4) of equations, belongs to $\mathcal{I}^{p-1+pm}\tilde{Q}_0/p\tilde{Q}_0$. So we get that any such solution must be the trivial one, given by $x_{isl} = 0$, $\forall (i,s,l) \in S(M) \times S(M) \times S(G)$. This proves 3.4.1.2, for $n = 1$.

7) From the shape of the equations of (3) we deduce that $\Omega_{\mathcal{S}^1/\mathcal{S}^0} = 0$ and that the criterion of formal smoothness is satisfied for $\ell(0)$. This implies that $\mathcal{S}^1$ is étale over $\mathcal{S}^0$. Let $\mathcal{S}^1_0$ be the maximal open closed subscheme of $\mathcal{S}^1$ with the property that any connected component of it has a non-empty intersection with the closed subscheme $\ell(0)^{-1}(\mathcal{Z}_k)$ of $\mathcal{S}^1$. From 6) above we get that $\ell(0)^{-1}(\mathcal{Z}_k)$ is either empty or $\mathrm{Spec}(k)$. So $\mathcal{S}^1_0$ is connected; below we show that it is non-empty. Lifting $\mathcal{S}^1_0$ to an affine smooth $W(k)$-scheme (this is

standard), we get a formally étale morphism $\ell_{0,1} : \mathrm{Spec}(Q^{\wedge}_{0,1}) \to \mathrm{Spec}(R^{\wedge}_0)$, with $Q_{0,1}$ a smooth $W(k)$-algebra such that $\mathrm{Spec}(Q_{0,1}/pQ_{0,1}) = \mathcal{S}^1_0$. From its construction and from 6) above we deduce that ii) and iii) of 1) of 3.4.1.3 are satisfied for $n = 1$ (the fact that the connection ∇^1_0 on $M \otimes Q_{0,1}/pQ_{0,1}$ we get is integrable, nilpotent mod p and respects the G-action results from the fact that there is a morphism $\mathrm{Spec}(\hat{R}^0_0/p\hat{R}^0_0) \to \mathcal{S}^1$ factoring through $\mathcal{S}^1_0$, cf. 1) above). Next we show that $\mathcal{S}^1_0$ is non-empty.

8) M_{R_0} is the pull back of M_{S_0} under the natural inclusion $i^0_0 : U^0_0 \hookrightarrow W^0_0$ respecting the Frobenius lifts. So to prove that $\mathcal{S}^1_0$ is non-empty, we can work with M_{S_0} instead of M_{R_0}, i.e. we can assume that $G = GL(M)$ and that $R_0 = S_0$. So $d = \dim(M)^2$. But in this case in 5) above we have $d^2 = \dim(M)^4$ equations in d^2 variables; so the fact that S^1_0 is non-empty is handled by the following simple lemma.

3.4.8.1. Lemma. *Let $m, l \in \mathbb{N}$, and let k_1 be a perfect field of characteristic $p > 0$. We consider an affine scheme $\mathcal{Y}_0$ over k_1 defined by a system of m equations in $m + s$ variables $x_1, ..., x_{m+s}$, $s \in \mathbb{N} \cup \{0\}$, of the form*

$$x_i = L_i(x_1, ..., x_{m+s})^{p^l} + c(i), \tag{5}$$

$i = \overline{1, m}$, where $\forall i \in \{1, ..., m\}$, L_i is a homogeneous polynomial of degree 1 with coefficients in k_1, and $c(i) \in k_1$. Then $\mathcal{Y}_0$ is a non-empty smooth scheme over k_1. If $s = 0$ then $\mathcal{Y}_0$ is étale over k_1 and the number of elements of the set $\mathcal{Y}_0(\overline{k_1})$ is finite and equal to p^{lm_1}, with $m_1 \in \mathbb{N} \cup \{0\}$, $m_1 \leq m$. Moreover m_1 depends only on the coefficients of the homogeneous linear forms L_i, $i \in \{1, ..., m\}$, and not on the free coefficients $c(i)$, $i \in \{1, ..., m\}$. We have: m_1 is less or equal to the rank r_L of the $m \times m$ matrix obtained using the coefficients of L_i, $i \in \{1, ..., m\}$.

Proof: It is trivial to check the criterion of formal smoothness for the morphism $\mathcal{Y}_0 \to \mathrm{Spec}(k_1)$. As $\mathcal{Y}_0$ is of finite type over k_1 we deduce that $\mathcal{Y}_0$ is smooth over k_1. For the other things we can assume that $s = 0$; this implies that $\mathcal{Y}_0$ is étale over k_1, as the sheaf of relative differentials $\Omega_{\mathcal{Y}_0/k_1}$ is zero. We can assume that k_1 is algebraically closed. The case $m = 1$ is obvious. We can introduce a new variable x_0, so that the equations become homogeneous:

$$x_i x_0^{p^l - 1} = L_i(x_1, ..., x_m)^{p^l} + c(i)x_0^{p^l}. \tag{6}$$

These equations are defining a closed subscheme $\mathcal{Y}$ of $\mathbb{P}^m_{k_1}$. By repeatedly using the projective dimension theorem (cf. [Ha, p. 48]), we deduce that it is non-empty. If it has no point in the hypersurface defined by $x_0 = 0$, then we are done: $\mathcal{Y}_0 = \mathcal{Y}$ and $|\mathcal{Y}_0(k_1)| = p^{ml}$, cf. the intersection theory. If $\mathcal{Y}$ has a point with $x_0 = 0$, then we get that the homogeneous polynomials L_i of degree 1, $i = \overline{1, m}$, are linearly dependent, i.e. there are constants $a(i) \in k_1$, not all zero, such that $\sum_{i=1}^m a(i)L_i = 0$. So $\sum_{i=1}^m a(i)^{p^l} L_i^{p^l} = 0$. We can assume that $a(m)$ is different from zero, and so we can assume that it is 1.

So the initial system of (non-homogeneous) equations is equivalent to the system of equations, where we preserve the first $m - 1$ equations, while we replace the last equation

(obtained for $i = m$) by the linear equation

$$x_m + \sum_{i=1}^{m-1} a(i)^{p^l} x_i = c(m) + \sum_{i=1}^{m-1} c(i)a(i)^{p^l}.$$

This last equation allows us to eliminate x_m. We come across a system of $m-1$ equations in $m-1$ variables of the same type as the original one. Moreover the coefficients of the new homogeneous linear forms in the variables x_1, x_2,..., x_{m-1} depend on L_i, $i \in \{1, ..., m\}$, but do not depend on $c(i)$, $i \in \{1, ..., m\}$. So we can proceed by induction. The last part about the estimate of m_1 is obvious: as above, we can eliminate $m - r_L$ variables at once. This proves the lemma.

3.4.8.1.0. m_1 can be smaller than r_L. In fact easy examples show that, if $r_L < m$, then m_1 can take any value of the set $\{0, ..., r_L\}$. Here, we just mention one example to illustrate the phenomenon. We take $m = 2$ and we consider the system defined by the following two equations $x_1 = x_1^{p^l} - x_2^{p^l}$ and $x_2 = x_1^{p^l} - x_2^{p^l} + 1$. Obviously $r_L = 1$. But $m_1 = 0$ as the only solution of this system is $(x_1, x_2) = (-1, 0)$.

If $r_L = m$ then the above system (5) for $s = 0$ is equivalent to a system of the form

$$y_i^{p^l} = \tilde{L}_i(y_1, ..., y_m), \tag{7}$$

$i = \overline{1, m}$, with $\tilde{L}_i$ as non-homogeneous linear forms; so in this case $\mathcal{Y} = \mathcal{Y}_0$ and $m_1 = m$.

3.4.8.1.1. Corollary. *Let $m \in \mathbb{N}$ and let (E, τ) be a pair comprising from a field E and an automorphism τ of it. Let $L_i(x_1, ..., x_m)$ be linear forms in m variables with coefficients in E, $i = \overline{1, m}$. We consider the difference variety defined by the system of equations*

$$y_i = L_i(\tau(y_1), ..., \tau(y_m)), \tag{8}$$

$i = \overline{1, m}$. The conclusion is: it is non-empty, i.e. it has a solution in the difference closure of (E, τ).

Proof: We can assume that (E, τ) is equal to its difference closure. The imerging philosophy of E. Hrushovski and A. Macintyre (see [Hr]) allows us to assume that $E = \mathbb{F}$ and that τ is a power of its Frobenius automorphism having $\mathbb{F}_p$ as its fixed field. But this case is handled by the above lemma.

3.4.8.1.2. Corollary. *Let $m, n \in \mathbb{N}$. Let $R(0)$ be a reduced $\mathbb{F}_p$-algebra. Let $L_i(x_1, x_2, ..., x_m)$ be homogeneous linear forms in m variables with coefficients in $R(0)$, $i = \overline{1, m}$.*

a) (the abstract surjectivity principle) *There is a Zariski dense open subscheme $U(0)$ of $\mathrm{Spec}(R(0))$, such that for any m-tuple $(c(1), ..., c(m))$ of elements of $R(1)$, with $R(1)$ a reduced $\mathbb{F}_p$-algebra containing $R(0)$, the system of equations*

$$x_i = L_i(x_1^{p^n}, x_2^{p^n}, ..., x_m^{p^n}) + c(i), \tag{9}$$

$i = \overline{1,m}$, defines an étale cover of $U(1) := \mathrm{Spec}(R(1)) \times_{\mathrm{Spec}(R(0))} U(0)$.

b) (the abstract constructibility property) *Let $R(\infty)$ be an inductive limit of $\mathbb{F}_p$-algebras $R(q)$, $q \in \mathbb{N}$, with $R(q)$ as the $R(q-1)$-algebra defined by a system of equations of the type (9) above, but with $c(i)$, $i = \overline{1,m}$, replaced by some $c(i)_q \in R(q-1)$. Then the image of any connected component of $\mathrm{Spec}(R(\infty))$ in $\mathrm{Spec}(R(0))$ is a constructible dense subset. Moreover the image of any open closed subscheme of $\mathrm{Spec}(R(\infty))$ in $\mathrm{Spec}(R(0))$ is an open subscheme.*

c) *If $R(0)$ is a strictly henselian local ring, than the additive map*

$$\mathcal{L}(n) : R(0)^m \to R(0)^m$$

defined by

$$\mathcal{L}(n)(x_1, ..., x_m) := \left(L_1(x_1^{p^n}, ..., x_m^{p^n}), ..., L_m(x_1^{p^n}, ..., x_m^{p^n})\right)$$

is surjective.

Proof: a) As the number of the coefficients of the forms $L_1, ..., L_m$ are finite, we can assume that $R(0)$ is a finitely generated reduced $\mathbb{F}_p$-algebra. In particular $R(0)$ is a noetherian ring. We can assume that $R(0)$ is an integral domain. We already know that the system (9) of equations defines an étale scheme $\mathrm{Spec}(R(2))$ over $\mathrm{Spec}(R(1))$. So we just have to show that $U(2) := \mathrm{Spec}(R(2)) \times_{\mathrm{Spec}(R)} U(0)$ is finite over $U(1)$, with $U(0)$ a Zariski dense open affine subscheme of $R(0)$ depending only on the forms L_i, $i = \overline{1,m}$. We can assume that the p^n-th roots of the coefficients of the linear forms L_i are in $R(0)$. So we can write $L_i(x_1^{p^n}, ..., x_m^{p^n}) = \tilde{L}_i(x_1, ..., x_m)^{p^n}$, with $\tilde{L}_i$ as homogeneous linear forms. Now we can follow the proof of 3.4.8.1. We proceed by induction on $m \in \mathbb{N}$. The case $m = 1$ is obvious. If the linear forms $\tilde{L}_i$ are linearly dependent over the field of fractions $K(0)$ of $R(0)$, by localizing $R(0)$ with respect to some non-zero element of it, we can eliminate one variable; so the induction applies. If the linear forms $\tilde{L}_i$ are linearly independent over $K(0)$, then they are linearly independent, over any point of an open (non-empty) affine subscheme $U(0)$ of $R(0)$. From 3.4.8.1.0 we get: $U(2)$ is an étale cover of $U(0)$. This proves the a) part.

b) This is a consequence of a) using standard noetherian induction on the topological space underlying the spectrum of the $\mathbb{F}_p$-subalgebra of $R(0)$ generated by the coefficients of $L_1, ...L_m$.

c) This is a direct consequence of 3.4.8.1. This ends the proof of the corollary.

3.4.8.1.3. Definitions. The largest open subscheme $U(0)$ of $\mathrm{Spec}(R(0))$ having the property of 3.4.8.1.2 a) is called the Artin-Schreier open subscheme defined by the linear homogeneous forms L_1,...,L_m. Using the restrictions of L_1,...,L_m to the reduced subscheme defined by $\mathrm{Spec}(R(0)) \setminus U(0)$, we get similarly an Artin-Schreier open subscheme $U(1)$ of $\mathrm{Spec}(R(0)) \setminus U(0)$. In this way, by induction, we get a finite stratification of $\mathrm{Spec}(R(0))$ in locally closed subschemes. We call it the Artin-Schreier stratification of $\mathrm{Spec}(R(0))$ defined by the linear forms L_1,...,L_m.

We similarly define the refined Artin-Schreier stratification of $\mathrm{Spec}(R(0))$ defined by the linear homogeneous forms L_1,...,L_m. The only difference: instead of $U(0)$, we take

its open closed subscheme $U(0)'$ such that the number of geometric points of (let's say) $\mathrm{Spec}(R(1))$ above geometric points of $U(0)'$ is maximal. So it is the stratification indexed by the number of geometric points of $\mathrm{Spec}(R(1))$ we get above geometric points of $\mathrm{Spec}(R(0))$.

3.4.8.1.4. Lemma (the purity property). *All Artin-Schreier stratifications and all refined Artin-Schreier stratifications satisfy the purity condition.*

Proof: We use the notations of 3.4.8.1.2. We can assume that $R(0)$ is a finitely generated integral $\mathbb{F}_p$-algebra. So $U(0) = U(0)'$. Let $R(0)^n$ (resp. $R(1)^n$) be the normalization of $R(0)$ in its field of fractions (resp. in the ring of fractions of $R(1)$). $\mathrm{Spec}(R(1))$ is an affine open subscheme of $\mathrm{Spec}(R(1)^n)$. So its complement is either empty or of pure codimension 1, as we have $R(1) = R(1)^n[\frac{1}{a(1)}]$, for some $a(1) \in R(1)^n$. Let $b(1) \in R(0)^n$ be the norm of $a(1)$. Let $P(x)$ be a monic polynomial with non-zero free coefficient $a(0)$ such that $P(b(1)) = 0$. Obviously $\mathrm{Spec}(R(0)) \setminus U(0)$ is the locus of the zeroes of $a(0)$. This implies that either $U(0)$ is $\mathrm{Spec}(R(0))$, or its complement in $\mathrm{Spec}(R(0))$ is of pure codimension one. This proves the lemma.

3.4.8.1.5. Remark. All the abobe part of 3.4.8.1 remains valid if L_i's are not necessarily (homogeneous) linear forms (in m variables), but sums of monomials which are some p-powers of the variables involved. The reason is: we can introduce new variables and equations of the form $y_i = x_i^p$, so that the new system of equations is of the same form as the one of 3.4.8.1.2 a), with $n = 1$ and with an increased number of variables.

We come back to the proof of 3.4.1.2-3.

3.4.8.2. Lemma (the explicit form of the surjectivity principle). *There is a reduced closed subscheme $\mathcal{B}_0(k)$ of G_k not containing the origin, depending only on the linear forms L_{isl}, but not on the coefficients a_{isl}, $(i,s,l) \in S(M) \times S(M) \times S(G)$, and such that the fibres of $\ell_{0,1}$ above points of $\mathrm{Spec}(R_0/pR_0)$ not belonging to $\mathcal{B}_0(k)$, are non-empty.*

Proof: All the schemes to be introduced below are reduced. First of all let us remark that there is a closed subscheme $\mathcal{R}(0)$ of $\mathrm{Spec}(R_0/pR_0)$, different from $\mathrm{Spec}(R_0/pR_0)$, and such that denoting by $\mathcal{T}(0)$ its open complement, the morphism $\ell_{0,1}^{-1}(\mathcal{T}(0)) \to \mathcal{T}(0)$ is an étale cover. By elementary matrix operations we see that $\mathcal{R}(0)$ depends only on the linear forms L_{isl}, and not on the coefficients a_{isl} (cf. 3.4.8.1.2 a)). Now we are going to construct $\mathcal{B}_0(k)$.

In what follows $\mathcal{J}(\tilde{i})$ are finite set of indices, $\tilde{i} = \overline{0, d-1}$. All the connected components of $\mathcal{R}(0)$ not passing through the origin of G_k should be subschemes $S_l(0)$, $l \in \mathcal{J}(0)$, of $\mathcal{B}_0(k)$. For any connected component of $\mathcal{R}(0)$ passing through the origin of G_k, we have to repeat the argument. Let $\mathcal{C}^1$ be the union of such connected components. There is (cf. 3.4.8.1.2 a)) a Zariski dense open subscheme $\mathcal{T}(1)$ of $\mathcal{C}^1$, again depending only on the restriction of the forms L_{isl} to $\mathcal{C}^1$, and not on the restriction of the coefficients a_{isl} to $\mathcal{C}^1$, $(i,s,l) \in S(M) \times S(M) \times S(G)$, such that the morphism $\ell_{0,1}^{-1}(\mathcal{T}(1)) \to \mathcal{T}(1)$ is an étale cover. Let $\mathcal{R}(1)$ be the subscheme of $\mathcal{C}^1$ defined by the complement of $\mathcal{T}(1)$ in $\mathcal{C}^1$. The connected components of $\mathcal{R}(1)$ not passing through the origin of G_k, should be subschemes $S_l(1)$, $l \in \mathcal{J}(1)$, of $\mathcal{B}_0(k)$. We keep going with those connected components of $\mathcal{R}(1)$ passing through the origin of G_k.

By induction on $\tilde{i} \in \{0, ..., d-1\}$ we build up closed schemes $S_l(\tilde{i})$, $l \in \mathcal{J}(\tilde{i})$, not containing the origin of G_k, and which are connected components of a closed subscheme $\mathcal{R}(\tilde{i})$ of $\mathrm{Spec}(R_0/pR_0)$. $\mathcal{B}_0(k)$ is defined as the closure in G_k of the (finite) union of these subschemes $S_l(\tilde{i})$, $l \in \mathcal{J}(\tilde{i})$, with $\tilde{i}$ running from 0 to $d-1$. It does not contain the origin of G_k.

From its construction we deduce that it does not depend on the coefficients a_{isl}, $(i,s,l) \in S(M) \times S(M) \times S(G)$, and that for any perfect field k_1 containing k we have $\mathcal{B}_0(k_1) = \mathcal{B}_0(k)_{k_1}$. This ends the proof of the lemma.

9) From the fact that the coefficients of L_{isl} are in $\mathcal{I}$ we deduce that the closed subscheme $\ell_{0,1}^{-1}(\mathcal{Z}_k)$ of $\mathcal{S}_0^1$ is $\mathrm{Spec}(k)$. This implies iv) of 3.4.1.3 1). The part of 3.4.1.3 2) referring to $\ell_{0,1}$ is a consequence of the proof of 3.4.8.1 (cf. also 13) below). This ends the proof of 3.4.1.3 1) and 2) involving $n = 1$.

10) By induction on $n \in \mathbb{N}$ we prove 3.4.1.3 1). Let $n \in \mathbb{N}$. We assume that we managed to build up $Q_{0,m}$, for any $m \in \mathbb{N}$ smaller than $n+1$. We now construct $Q_{0,n+1}$.

11) First we arbitrarily lift ∇_0^n to a connection respecting the G-action, still denoted by ∇_0^n, on $M \otimes Q_{0,n}/p^{n+1}Q_{0,n}$. Let $\tilde{\ell}_0 : \mathrm{Spec}(\tilde{Q}_0^{\wedge}) \to \mathrm{Spec}(Q_{0,n}^{\wedge})$ be a formally étale morphism, with $\tilde{Q}_0$ a $W(k)$-algebra. The general form of a connection on $M \otimes \tilde{Q}_0/p^{n+1}\tilde{Q}_0$ lifting the pull back of ∇_0^n, is of the form

$$\nabla = \nabla_0^n + \sum_{(i,s,l) \in S(M) \times S(M) \times S(G)} p^n x_{isl} e_{is} \otimes dz_l, \tag{10}$$

with all $x_{isl} \in \tilde{Q}_0/p\tilde{Q}_0$.

12) The condition that this connection ∇ on $M \otimes \tilde{Q}_0/p^{n+1}\tilde{Q}_0$ makes $M_{\tilde{Q}_0}/p^{n+1}M_{\tilde{Q}_0}$ potentially to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(\tilde{Q}_0)$ is expressed by a system of equations involving x_{isl}, which is of the same type as the system (3) of 5) above. We get the same linear forms (!) L_{isl}, $(i,s,l) \in S(M) \times S(M) \times S(G)$. The only difference is that the free coefficients a_{isl} of 5) above are replaced by new coefficients $a_{isl}(n)$. The fastest way to see this, is to use the same trick as in 8) above (for $G = GL(M)$ this is obvious).

As the coefficients a_{isl} have played no role in the steps 5-8) above, we can repeat the arguments of 5-8) above. So we get a moduli scheme $\mathcal{S}^{n+1}$ of connections, which is an étale scheme over $\mathcal{S}_0^n := \mathrm{Spec}(Q_{0,n}/pQ_{0,n})$. We define similarly $\mathcal{S}_0^{n+1}$. This takes care of 3.4.1.2 and of 1) of 3.4.1.3 for $n+1$ (again iv) is implied by i), while 1) above implies that ∇_0^{n+1} is integrable, nilpotent mod p, and respects the G-action).

13) 3.4.1.3 4) results from 3.4.1.3 1). 3.4.1.3 3) is obvious. The fact that the morphism ℓ_j^n mod p is:

– étale, results from the same argument used in 7) above (cf. also 3.4.8.1);

– of finite type results from the construction of $\mathrm{Spec}(Q_{0,n+1}/pQ_{0,n+1})$ (as a closed subscheme of a scheme of finite type over $\mathcal{S}_0^n$;

– quasi-finite, with geometric fibres of cardinality at most p^{d^2}, results from the fact that ∇_0^{n+1} respects the G-action, and so we actually have only d^2 variables, and from the intersection theory (cf. the proof of 3.4.8.1).

This takes care of 3.4.1.3 2).

14) 3.4.1.3 5) is a consequence of the proof of 3.4.8.2 and of 12) above (cf. 3.4.8.1.2 a)). More precisely, for any $n \in \mathbb{N}$ we construct an open subscheme $\mathcal{S}_1^n$ of $\mathcal{S}_0^n$, such that, under the morphism ℓ_j^n, $\mathcal{S}_1^{n+1}$ projects surjectively onto $\mathcal{S}_1^n$, as well as $\mathcal{S}_1^1$, under the morphism $\ell_{0,1}$, projects surjectively onto the open subscheme of $\mathrm{Spec}(R_0/pR_0)$ defined by the complement of $\mathcal{B}_0(k)$.

$\mathcal{S}_1^n$ is formed by removing from $\mathcal{S}_0^n$ the closed subscheme formed by the union of all connected components of $\ell_{0,n}^{-1}(\mathcal{C}_{\tilde{i}}^1)$ not containing the closed subscheme $\ell_{0,n}^{-1}(\mathcal{Z}_k) = \mathrm{Spec}(k)$, for $\mathcal{C}_{\tilde{i}}^1$ running through all connected components not containing $\mathcal{Z}_k$ of the scheme $\mathcal{R}(\tilde{i})$ defined in 3.4.8.2, with $\tilde{i}$ as its turn running through the set $\{0, ..., d-1\}$.

This ends the proof of 3.4.1.2-3.

3.4.8.3. Remark. A natural question arises: how to compute the connections ∇_j^n, and so how to construct the morphisms ℓ_j^n (write down the equations defining $Q_{j,n}$)? This is not difficult (cf. 5) of 3.4.8; see also 3.4.8.5 and 3.4.18.4). Here, as a digression, we show how using Shimura varieties of A_n type ($n \in \mathbb{N}$) we can construct a formally étale morphism $\tilde{Q}_{j,1}^{\wedge} \to \mathrm{Spec}(R_j^{\wedge})$ which factors through $\ell_{j,1}$. We can assume that k is the algebraic closure of a countably generated field over $\mathbb{F}_p$; so in particular we have $k = \bar{k}$. We are going to work again with only one index: 0.

1) M_{R_0} is the pull back of M_{S_0} under the inclusion $i_0^0 : U_0^0 \hookrightarrow W_0^0$. So we can work with M_{S_0} instead of M_{S_0}, i.e. we can assume from now on that $G = GL(M)$ and that $R_0 = S_0$.

2) Let us first remark that there is a SHS $(f, L_{(p)}, v)$ defined by an injective map $f : (G_0, X) \hookrightarrow \big(\mathrm{GSp}(W, \psi), S\big)$ (here we put exceptionally a lower left index 0, i.e. we write G_0 and $2e_0 = \dim_{\mathbb{Q}}(W)$, not to create confusion with the meaning of $G = GL(M)$ and $e = \dim(G) = \dim(GL(M))$) such that $G_{0W(k)} \supset G = GL(M)$ has the property that ${G_0^{\mathrm{der}}}_{W(k)} = SL(M)$ (the confusion between G and $G_{0W(k)}$ would have not been too serious) and the Shimura Lie pairs attached to the Shimura group pairs $(G, [\mu])$ and $(G_1, [\tilde{\mu}_x^*])$, are isomorphic (cf. def. 3-4) of 2.2.5). Here G_1 is the subgroup of $G_{0W(k)}$ generated by $G_{0W(k)}^{\mathrm{der}}$ and by the image of any cocharacter $\tilde{\mu}_x^* : \mathbb{G}_m \to G_{0W(k)}$, which, under an embedding of $W(k)$ in $\mathbb{C}$, extending the inclusion $E(G_0, X) \subset \mathbb{C}$, is $G_0(\mathbb{C})$-conjugate to the cocharacters μ_x^*, $x \in X$.

We need to have $G_{0\mathbb{R}}^{\mathrm{der}} = SU(p_0, q_0)_{\mathbb{R}}$, with $p_0 = \dim(F^1)$ and $q_0 = \dim(F^0) = \dim(M/F^1)$. We can take G_0^{der} to be an arbitrary $\mathbb{Q}$–form of $G_{0\mathbb{R}}^{\mathrm{der}}$ which splits over $\mathbb{Q}_p$ (to be compared with 4.6.4). [Va2, 6.4.2] guarantees the existence of a SHS as above if $p \geq 5$; the existence of such a SHS for $p = 3$ is not a problem: we can assume that we have a standard PEL situation $(f, L_{(p)}, v, \mathcal{B})$, cf. 2.3.5.1 (see also §6). For example, if $p_0 = 1$ and $q_0 = 2$, we can take a Picard surface (G_0, X) (cf. [Go]) having a reflex field $E(G_0, X)$ in

which p is totally split. The general case is just an obvious extension of the case $p_0 = 1$, $q_0 = 2$ (we have $p_0 q_0 \neq 0$ due to 3.2.2.1).

3) Secondly, we assume the existence of a $W(k)$-point z of $\mathcal{N}/H_0$ (cf. the standard notations 2.3.1-3 for the SHS introduced in 2) above), such that its Shimura filtered σ-crystal $\big(L^*_{(p)} \otimes W(k), F^1_z, \varphi_z, G_{0W(k)}\big)$ (cf. 2.3.10 and the assumption $k = \bar{k}$) has the property that the induced Shimura Lie filtered σ-crystal $\big(\mathfrak{g}_1, \varphi_z, F^0_z(\mathfrak{g}), F^1_z(\mathfrak{g})\big)$, with $\mathfrak{g}_1 := \mathrm{Lie}(G_1) \subset \mathrm{Lie}(G_{0W(k)})$, is isomorphic to the Shimura Lie filtered σ-crystal $(\mathrm{End}(M), h\varphi_0, F^0(\mathrm{End}(M)), F^1(\mathrm{End}(M)))$.

4) We have $G_1 = \mathbb{G}_m \times SL(M)$. So we have a natural isogeny $G_1 \to G = GL(M)$; it is not an étale morphism if p divides $\dim(M)$. Substituting M with $\tilde{M} = M \oplus W(k)$, and the direct sum decomposition $M = F^1 \oplus F^0$ with $\tilde{M} = F^1 \oplus \tilde{F}^0$, where $\tilde{F}^0 = F^0 \oplus W(k)$, we replace the Shimura σ-crystal $(M, \varphi, GL(M))$ with the Shimura σ-crystal $(\tilde{M}, \tilde{\varphi}, GL(\tilde{M}))$ (with $\tilde{\varphi}$ acting as φ on M, and as the Frobenius σ on the $W(k)$ summand of $\tilde{M}$). This goes as follows:

a) We choose arbitrarily a $W(k)$-basis $\{v_1, ..., v_{\dim(M)}\}$ of M; it gives birth to a $W(k)$-basis $\{v_1,...,v_{\dim(M)+1}\}$ of $\tilde{M}$, where $v_{\dim(M)+1}$ is 1 of the summand $W(k)$ of $\tilde{M}$.

b) We view $GL(M)$ as a subgroup of $GL(\tilde{M})$ (with $b_M \in GL(M)(W(k))$ acting as b_M on the summand M of $\tilde{M}$, and as identity on the other summand $W(k)$ of M).

c) We choose $\tilde{S}_0$ to be an open affine subscheme of $GL(\tilde{M})$ of the form $S_0 \times \mathbb{G}_a^{2\dim(M)} \times \mathbb{G}_m$ (the factors $\mathbb{G}_a$ correspond to elements of $GL(\tilde{M})(W(k))$, viewed as matrices, having 1 on the diagonal, and 0 anywhere else, except at the intersection of an i_1-th row with an i_2-th column, where i_1 and i_2 are distinct elements of the set $S_M = \{1, 2, ..., \dim(M) + 1\}$ such that the intersection $\{i_1, i_2\} \cap \{\dim(M) + 1\}$ is non-empty; the factor $\mathbb{G}_m$ is the subgroup of $GL(\tilde{M})$ leaving invariant the summand $W(k)$ of $\tilde{M}$ and fixing M).

d) We choose a Frobenius lift of the p-adic completion of $\tilde{S}_0$ of the form $\Phi_{S_0} \times \Phi_1$ (corresponding to the above product decomposition of $\tilde{S}_0$), with Φ_1 as the Frobenius lift of the p-adic completion of $\mathbb{G}_a^{2\dim(M)} \times \mathbb{G}_m = \mathrm{Spec}(W(k)[x_1, ..., x_{2\dim(M)+1}][\frac{1}{x_{2\dim(M)+1}}])$ which takes x_s to x_s^p, $\forall s \in \{1, ..., 2\dim(M)\}$, and $x_{2\dim(M)+1} - 1$ to $(x_{2\dim(M)+1} - 1)^p$, and which acts as σ on $W(k)$.

5) The same argument which allowed us to assume that G is $GL(M)$, allows us to replace $GL(M)$ with $GL(\tilde{M})$, if p divides $\dim(M)$. So we can assume that p does not divide $\dim(M)$. So $G_1 \to G$ is an étale cover and $\mathfrak{g}_1 = \mathfrak{g}$. This allows us to identify the completion of G_1 in the origin with the completion $\mathrm{Spec}(O_0)$ of G in the origin.

6) From the assumed existence of the point $z : \mathrm{Spec}(W(k)) \to \mathcal{N}/H_0$, we deduce that $\mathfrak{g}_{R_0} = \mathrm{End}(M)_{S_0}$ is (p-adic) algebraizable. To explain what we mean by this, we start with a formally étale morphism $\mathrm{Spec}(O_0) \to \mathcal{N}/H_0$ such that the pull back through it of the abelian scheme $\mathcal{A}_{H_0}$ over $\mathcal{N}/H_0$ gives birth to a Shimura filtered F-crystal $\mathfrak{C} = \big(L^*_{(p)} \otimes W(k), F^1_z \otimes W(k), \varphi_z, G_1, \bar{f}\big)$ over O_0 (with $\bar{f}$ induced by the morphism c_0 of ii) before 3.4.0); using the Artin's approximation theorem (cf. 2.3.15), we get:

7) a smooth morphism $\mathrm{Spec}(S_1) \stackrel{\ell_1}{\to} \mathcal{N}/H_0$ and a formally étale morphism $\mathrm{Spec}(S_1^\wedge) \stackrel{\ell}{\to} W_1^0 := \mathrm{Spec}(S_0) \times_G G_1$ such that the p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(S_1)$ defined by the pull back of $\mathcal{A}_{H_0}$ through ℓ_1 can be put in the form $\mathfrak{C}_{\ell_1} = \big(L^*_{(p)} \otimes S_1^\wedge, F^1_z \otimes S_1^\wedge, h_1(\varphi_z \otimes 1), \nabla_z\big)$, with $h_1 \in G_{0\,W(k)}(S_1^\wedge)$ which mod p is defined by ℓ (we view $\mathrm{Spec}(S_0) \times_G G_1$ as a subscheme of G_1, and so as a subscheme of $G_{0\,W(k)}$), and with Φ_{S_1} as the Frobenius of $S_1^\wedge$ for which the diagram

$$\begin{array}{ccc} \mathrm{Spec}(S_1^\wedge) & \stackrel{pr_1}{\longrightarrow} & \mathrm{Spec}(S_0^\wedge) \\ \big\downarrow \Phi_{S_1} & & \big\downarrow \Phi_{S_0} \\ \mathrm{Spec}(S_1^\wedge) & \stackrel{pr_1}{\longrightarrow} & \mathrm{Spec}(S_0^\wedge) \end{array}$$

is cartesian; here $pr_1 : \mathrm{Spec}(S_1^\wedge) \to \mathrm{Spec}(S_0^\wedge)$ is the p-adic completion of the composite of ℓ with the projection $W_1^0 \to \mathrm{Spec}(S_0)$;

8) a point $\mathrm{Spec}(W(k)) \stackrel{d_0}{\hookrightarrow} \mathrm{Spec}(S_1)$ such that $\ell \circ d_0^\wedge$ is the p-adic completion of the origin of $W_1^0 \subset G_1$;

9) modulo a power $I^n_{d_0}(n \in \mathbb{N}\backslash\{1\})$ of the ideal I_{d_0} defining d_0, $\mathfrak{C}_{\ell_1}$ is isomorphic to $\mathfrak{C}$ modulo the n-th power of the ideal of O_0 defining the origin of $\mathrm{Spec}(O_0)$.

10) We deduce that the p-divisible object of $\mathcal{MF}^\nabla_{[-1,1]}(S_1)$ defined by the Lie part of $\mathfrak{C}_{\ell_1}$ corresponding to $\mathfrak{g}$ is $\mathfrak{g}_{S_1} = \big(\mathfrak{g} \otimes S_1^\wedge, F^0(\mathfrak{g}) \otimes S_1^\wedge, F^1(\mathfrak{g}) \otimes S_1^\wedge, h_1(\varphi_z \otimes 1)\big)$.

11) So the object $\mathrm{End}(M_{S_0})/p\mathrm{End}(M_{S_0}) = \mathfrak{g}_{S_0}/p\mathfrak{g}_{S_0}$ of $\mathcal{MF}_{[-1,1]}(S_0)$ becomes by extension to $\mathrm{Spec}(S_1)$, through the morphism ℓ, an object of $\mathcal{MF}^\nabla_{[-1,1]}(S_1)$. This easily implies the existence of an étale morphism $\mathrm{Spec}(S_1/pS_1) \to \mathcal{S}_0^1$ which composed with $\ell_{0,1}$ is the morphism $\mathrm{Spec}(S_1/pS_1) \to \mathrm{Spec}(S_0/pS_0)$ naturally induced by $pr_1 \circ \ell$ (i.e. we can regain the connection of M_{S_1}/pM_{S_1} from the one of $\mathrm{End}(M_{S_1})/p\mathrm{End}(M_{S_1})$ and from ∇_z; in fact from the proof of [Va2, 6.4.2], as p does not divide $\dim(M)$, ($G_{0\,\mathbb{Z}_p}$ is a split group!) we get that there is an injective map $i_z : (M, F^1, h\varphi_0, G) \hookrightarrow (L^*_{(p)} \otimes W(k), F^1_z, \varphi_z, G_{0\,W(k)})$, with $i_z(M)$ as a direct summand of $L^*_{(p)} \otimes W(k)$ left invariant by $G_{0\,W(k)}$, and with $i_z(F^1) = i_z(M) \cap F^1_z$).

12) If there is no $W(k)$-point z of $\mathcal{N}/H_0$ for which 6) above takes place, we have to choose a $W(k)$-point z_0 of $\mathcal{N}/H_0$ such that its attached Shimura filtered σ-crystal $\big(L^*_p \otimes W(k), F^1_{z_0}, \varphi_{z_0}, G_{0\,W(k)}\big)$ has the property that the induced Shimura Lie σ-crystal $\big(\mathfrak{g}, \varphi_{z_0}, F^0_{z_0}(\mathfrak{g}), F^1_{z_0}(\mathfrak{g})\big)$ is isomorphic to a Shimura Lie filtered σ-crystal

$$\big(\mathrm{End}(M), g_3\varphi_0, F^0(\mathrm{End}(M)), F^1(\mathrm{End}(M))\big),$$

with $g_3 \in GL(M)(W(k))$ a $W(k)$-valued point of a "small" open subscheme U_3 of $U_0 \cap U_2$ containing h (cf. 2.3.16). Let U_3^0 be the right translate of U_3 by h^{-1}. Repeating the arguments of 3-11) we conclude that the spectrum of the the normalization of the S_0/pS_0-subalgebra of $\hat{R}_0^0$ (we recall that $R_0 = S_0$) generated by the coefficients of β of 3.4.8 1) taken mod p is generically étale over $\mathrm{Spec}(S_0/pS_0)$. As some trouble might result if $\mathcal{S}^1$ is not connected, this can be seen in two ways: either by remarking that the system (4) of

equations of 3.4.8 6) gives birth to an étale scheme over S_0/pS_0 (so somehow we go back to the general ideas), or by using the Kodaira-Spencer map (here is the place where we need U_3 to be "small": we can assume that this map defined using the coefficients of the mentioned β still has a image of the right rank in g_3 mod p) and an argument of gluing entirely similar to the one of 3.4.2.1 d), but performed mod p. The need of working mod p is implied by the fact that we do not know apriori that the spectrum of the S_0-algebra generated by the coefficients of β, is smooth over $W(k)$ in a $W(k)$-valued point lifting g_3; however the lifting property of 3.4.18.5.2 used in a relative context (see 3.4.18.7) implies that the gluing argument of 3.4.2.1 d) can be performed mod p as well. We do not detail this second approach here, as actually a point z as in 3) always exists: as $k = \bar{k}$, 3.4.15 below applies, as we can assume that (G_0, X) has the completion property (cf. the proof of 4.12.12).

3.4.8.4. Remarks. 1) The special form of the Frobenius of $M \otimes R_0$ played no role in the proof of 3.4.1.2-3. This allows us to use the method employed in 3.4.8 in (all) other situations (for instance in 3.4.18.4).

2) In the prove of the surjectivity principle, the special shape of U_0 played no role. So it can be used in all other situations (for instance in 3.4.11).

3) The (ring) Frobenius Φ_{R_0} did play a role in the above proof of 3.4.1.2-3. However, this role is not essential. For the case of a general type of (ring) Frobenius see 3.4.18.8.3.

4) Let $(f, L_{(p)}, v)$ be a SHS defined by an injective map $f : (G', X) \hookrightarrow (\mathrm{GSp}(W, \psi), S)$ (here exceptionally we put a right upper index $'$ for G and e, so we can make use of the other notations introduced in §3, without creating confusion). If we already know that the Shimura filtered σ-crystal $(M, F^1, h\varphi, G)$ is isomorphic to the Shimura filtered σ-crystal attached to a point $z : \mathrm{Spec}(W(k)) \to \mathcal{N}/H_0$, then 3.4.8.3 is significantly simplified: the replacement of G by $GL(M)$ is not needed. This implies that we do not need the property ii) of the beginning of 3.4, as well as in 3.4.0 we do not need the "complication" with W_0, W_2, W_0^0, W_2^0, Φ_{S_0}, Φ_{S_2}, etc.

5) We do not know if (or when) we have $\mathrm{Spec}(Q_{0,n}/pQ_{0,n}) = \mathcal{S}^n$. We do expect this equality (at least in the majority of cases).

3.4.8.5. The closed subscheme $\mathcal{B}_0(k)$. It is very much desirable to have a good description of the closed subscheme $\mathcal{B}_0(k)$ of G_k introduced in 3.4.8.2. In what follows we explain a simpler approach for determining its intersection with U_0^0.

From 3.4.8.2 we deduce that it is enough to determine $\mathcal{B}_0(\bar{k})$. So we can assume that $k = \bar{k}$. We choose the basis $\{e_i | i \in S(M)\}$ of M in such a way that:

There is a permutation $\tilde{\pi}$ of the set $S(M)$ with the property that $h\varphi_0(e_i)$ mod p is either zero or $e_{\tilde{\pi}(i)}$ mod p.

The existence of such a basis is guaranteed by [Mu, cor. p. 143], applied to the σ-linear map $M/pM \to M/pM$ defined by $h\varphi_0$ mod p, and by the standard Jordan form of a nilpotent σ-linear map (this is the same as in the case of a linear map).

1) Rewriting the equations of 5) above skilfully, for the new basis, we come across equations of the form

$$t_l(o)^{p-1} x_{isl}^p = \tilde{L}_{isl}(x_{11l}, x_{12l}, ..., x_{\dim(M)\,\dim(M)l}) + \tilde{b}_{isl}, \tag{11}$$

$\forall (i,s,l) \in S(M) \times S(M) \times S(G)$ such that $h\varphi_0(e_s)$ is zero mod p, and of the form

$$\tilde{L}_{isl}(x_{11l}, ..., x_{\dim(M)\,\dim(M)l}) = \tilde{b}_{isl}, \tag{12}$$

$\forall (i,s,l) \in S(M) \times S(M) \times S(G)$ such that $h\varphi_0(e_s)$ is non-zero mod p. Here $\tilde{L}_{isl}$ is a homogeneous linear form, while $\tilde{b}_{isl} \in R_0/pR_0$, $\forall (i,s,l) \in S(M) \times S(M) \times S(G)$.

These equations are obtained by plugging e_i in the equations (1) and (2) of 3.4.8 4), and identifying the coefficients of $r_0 \circ \tilde{l}_0(e_s)dz_l$ (we recall that $\Phi(\tilde{Q}_0)^0 = r_0 \circ \tilde{l}_0(h\varphi_0 \otimes 1)$).

2) Denoting $y_{isl} = t_l(o)x_{isl}$, $(i,s,l) \in S(M) \times S(M) \times S(G)$, this system over $\mathrm{Spec}(R_0/pR_0[\frac{1}{\prod_{i=1}^{d} t_i(0)}])$ is equivalent to a system of equations of the form

$$y_{isl}^p = \tilde{L}_{isl}(y_{11l}, ..., y_{\dim(M)\,\dim(M)l}) + t_l(0)\tilde{b}_{isl}, \tag{13}$$

if $s \in S(M)$ is such that $h\varphi_0(e_s)$ mod p is zero, and of the form

$$\tilde{L}_{isl}(y_{11l}, ..., y_{\dim(M)\,\dim(M)l}) = t_l(0)\tilde{b}_{isl}, \tag{14}$$

if $s \in S(G)$ is such that $h\varphi_0$ mod p is non-zero.

This system is much simpler than the one in 3.4.8 5). It can be used to get a very good description of the intersection of $\mathcal{R}(0)$ with $\mathrm{Spec}(R_0/pR_0[\frac{1}{\prod_{i=1}^{d} t_i(0)}])$.

3) Similarly, for any subset $S(t)$ of $S(G)$, restricting the system of equations of 1) above to $\mathrm{Spec}(R_0/(p,(t_i(0))_{i\in S(t)})R_0)$, we get a system of equations similar to the one in 2) above, involving y_{isl}, $(i,s,l) \in S(M) \times S(M) \times \big(S(G) \setminus S(t)\big)$. It is easier to determine the intersection of $\mathcal{B}_0(k)$ with $\mathrm{Spec}(R_0/(p,(t_i(0))_{i\in S(t)})[\frac{1}{\prod_{i\in S(G)\setminus S(t)} t_i(0)}])$ using it instead of another one, similar to the one in 3.4.8 5). This motivates why we get simplifications if we work under 3.4.1.1.2.

4) Let now $\tilde{G}$ be a smooth subgroup of G having connected fibres and such that its Lie algebra is stable under the action of the σ-linear map $h\varphi_0$. The whole of 3.4.1.3 can be redone in this context (cf. 3.4.1.6), with the group G replaced by $\tilde{G}$. We are going to use the same notations, not to complicate the story, with G being replaced everywhere by $\tilde{G}$.

We assume now that $\tilde{G}$ is an abelian group. We speak about the abelian situation.

The nice thing in this case is (due to the fact $\tilde{G}$ is abelian):

$\forall (i,s,l) \in S(M) \times S(M) \times S(\tilde{G})$, *the coefficients of* $\tilde{L}_{isl}$ *are elements of* $W(k)$.

This is so due to the fact that $\tilde{G}$ is abelian, and the connections ∇_j^n we get respect the $\tilde{G}$-action. So in the abelian situation $\mathcal{T}(0)$ contains $\mathrm{Spec}(R_0/pR_0[\frac{1}{\prod_{i\in S(\tilde{G})} t_i(0)}])$. The argument can be repeated for the closed subscheme $\mathrm{Spec}(R_0/(p,(t_i(0))_{i\in S(t)})$, with $S(t)$ an arbitrary subset of $S(G)$, to get the following result:

3.4.8.7. Proposition. *Under the convenience assumption, in the abelian situation, the morphisms* $\ell_{j,n}$ *mod* p, $n \in \mathbb{N}$, *are defining étale covers of the locally closed subscheme*

of $\mathrm{Spec}(R_j/pR_j)$ *defined by making some of the variables* $t_i(j)$ *to be zero, and inverting all others. In particular* $\mathcal{B}_j(k)$ *is empty.*

3.4.8.8. Remark. In the abelian situation $\mathcal{S}_0^n$ can be different from $\mathcal{S}^n$, i.e. $\mathcal{S}^n$ is not connected. This can be seen by examples similar to the one in 2) of 3.4.1.4, but working with σ-crystals of ordinary abelian varieties of dimension at least three.

3.4.8.9. The constancy property. We call the fact that when we pass in the above proof (of 3.4.1.2-3) from things mod p to things mod p^n, we get the same homogeneous forms (see (3), (4), (11), (12), (13) and (14) above, as well as 12) of 3.4.8), as the constancy property. The system of equations defined by (3) and (4) is said to be of the first type. The system of equations defined by (11) and (12) is said to be of additive second type. The system of equations defined by (13) and (14) is said to be of adjusted additive second type. For extra terminology see 3.4.18.4.6.

As mentioned in 3.4.8.4, this remains true in the context of a (truncation of a) p-divisible object of $\mathcal{MF}_{[0,1]}(R)$, with R a regular formally smooth $W(k)$-algebra. Of course, we can replace anywhere $\mathcal{MF}_{[0,1]}$ by $\mathcal{MF}_{[a,a+1]}$, $a \in \mathbb{Z}$. So, the natural question arises:

Q. *What about (truncations of) p-divisible objects of* $\mathcal{MF}_{[a,b]}(R)$*, with* $a, b \in \mathbb{Z}$*,* $b \geq a+2$*?*

Due to the fact that the connections have to satisfy the Griffiths transversality condition, the counting argument (of equations and variables) performed before 3.4.8.1, gives us in this situation, more equations then variables. We obtain systems of equations of the following form:

$$x_i + c_i = L_i(x_1^p, ..., x_{m+s}^p), \tag{15}$$

for $i \in \{1, ..., m\}$, and

$$c_i = L_i(x_1^p, ..., x_{m+s}^p), \tag{16}$$

for $i \in \{m+1, ..., m+s\}$. Here $c_i \in R/pR$, L_i are homogeneous linear forms with coefficients in R/pR, $i \in \{1, ..., m+s\}$, while $m, s \in \mathbb{N} \cup \{0\}$. If $s > 0$ we say that the above system of equations is of the third type.

In general $ms > 0$, and so it easy to see that there are systems of the above type which do not have any solution, or which do not define a smooth scheme over $\mathrm{Spec}(R/pR)$. However the constancy property remains. This is a very useful fact: it is easy to put conditions on the coefficients of L_i, $i \in \{1, ..., m+s\}$, so that the above system has a solution over any geometric point of R/pR, or that it defines a smooth scheme over $\mathrm{Spec}(R/pR)$, regardless of the values of the coefficients c_i, $i \in \{1, ..., m+s\}$. These conditions can be interpreted in terms of a chosen Frobenius of R, and of the Frobenius (of an $R^\wedge$-module) used in defining the (truncation) of the p-divisible object of $\mathcal{MF}_{[a,b]}(R)$ we started with. We get nice criteria of when 3.4.8 above can be worked out for such a (trucation of a) p-divisible object. We hope to come back to these ideas in a future paper.

3.4.8.9.1. It is worth commenting why initially we thought that we can work out the things using (mod p) just étale morphisms instead of $\mathbb{N}$-pro-étale morphisms in 3.4.1.3.

Refering to 3.4.1.3, once we obtained $Q_{j,1}$, we thought that the free coefficients $a_{isl}(n)$ of 3.4.8 12) we obtain for $n \in \mathbb{N}$, $n \geq 2$, do not depend on $n \geq 2$. But this is not the case.

However, in many cases, proceeding as follows, we can avoid the passage (mod p) to $\mathbb{N}$-pro-étale morphisms. Warning: what follows has to be interpreted up to a passage to an étale scheme over $\mathrm{Spec}(Q_{j,n})$, for some $n \in \mathbb{N}$; so $\mathrm{Spec}(Q_{j,m})$, with $m \geq n$, are replaced by the resulting pull back. We have to make a decision: either we work with "wild" connections (their behaviour modulo high powers of the prime p is uncontrolable) but with very nice Frobenius of the module underlying M_{R_j} (as in 3.4.1), and so later on we hope to get (sort of) a completion property (see 3.4.15) and a slice principle (see §7), or we work with very nice connections and with very "wild" Frobenius of the underlying modules. We explain this. The homogeneous linear forms L_{isl} of 3.4.8 (3) depend only on the expressions of $\Phi(R_j)^0$ and $\Phi(R_j)^1$ mod p and of the expression of Φ_{R_j} mod p^2. But the coefficients $a_{isl}(n)$ depend on the expressions of these three maps modulo higher powers of p. But, in many cases, we can modify $\Phi(Q_{j,1})^0$ and $\Phi(Q_{j,1})^1$ with something which is zero mod p (so we do not change the forms L_{isl}, and so we do not affect the étale property expressed in 3.4.1 through r_j mod p being a universal element), to make $a_{isl}(n) = 0$, $\forall (i,s,l) \in S(M) \times S(M) \times S(G)$, and $\forall n \in \mathbb{N}$, $n \geq m$, for some $m \in \mathbb{N}$: first we try to modify $\Phi(Q_{j,1})^0$ and $\Phi(Q_{j,1})^1$ modulo p^2 to get all $a_{isl}(1)$ to be zero, then what results we try to modify it mod p^3 by something which is zero p^2 to get all $a_{isl}(2)$ to be zero, etc.

Why should this be possible? Answer: there are two reasons. First, $\forall (l,n) \in S(G) \times \mathbb{N}$, the endomorphism of $M \otimes Q_{j,n}/pQ_{j,n}$ defined by the matrix whose coefficients are $a_{isl}(n)$, $i,s \in S(M)$, is an element of $\mathrm{Lie}(G) \otimes Q_{j,n}/pQ_{j,n}$. Second, any time we have to pass from things mod p^n to things mod p^{n+1}, we lift the connection (we got mod p^n) (as in 3.4.8 11)), not blindly, but in such a way that the resulting coefficients $a_{isl}(n)$ are linear combinations (with fixed coefficients of R_j/pR_j) of p-power elements of $Q_{j,n}/pQ_{j,n}$; after this careful lift we start to modify $\Phi(Q_{j,n})^i$, $i = \overline{0,1}$, by trying to achieve all $a_{ijl}(n)$ to be zero.

So the situation should get stabilized: we expect the existence of $m \in \mathbb{N}$ such that we can take $Q_{j,n} = Q_{j,m}$ $\forall n \in \mathbb{N}$, $n \geq m$. Sometimes it is preferable to have (if possible) $a_{isl}(n) = a_{isl}$, $\forall n \in \mathbb{N}$, $n \geq m$. If such stabilization is possible, then we do not have to pass (mod p) to $\mathbb{N}$-pro-étale covers. This is so due to the fact that $Q_{j,n}/pQ_{j,n}$, $n \geq m$, is a connected component of the étale scheme over $\mathrm{Spec}(Q_{j,m}/pQ_{j,m})$ defined by the system of equations (4) of 3.4.8. This gives a very nice expression (fully controlable) of the resulting connections ∇_j^n, but the control on $\Phi(Q_{j,n})^0$ and $\Phi(Q_{j,n})^1$ is much less. Still the situation looks better from many points of view, but we can not regain the (sort of) completion property or the slice principle. Once we gained the completion property and the slice principle, the second approach (trying to avoid the use of $\mathbb{N}$-pro-étale morphisms), looks, from many points of view, more convenient. In all the concrete cases we analysed (for instance, this is automatic if we have a quasi-polarization, as we can pass from p-divisible groups to abelian varieties) the situation gets stabilized. We will come back to this in §10; see also 4.12.12.

3.4.9. Remarks. 1) We could have worked 3.4.1 with U_0 and U_2 open subschemes of G^{der}. This is the most common case we need for applications. For example, a σ-crystal

$(M, tg_2\varphi_0)$, with $t \in Z(G)(W(k))$ and $g_2 \in G(W(k))$, is very much the same as the σ-crystal $(M, g_2\varphi_0)$; so we often (for instance when p does not divide the order of the center of G^{der}) can consider only elements $g_2 \in G^{\mathrm{der}}(W(k))$.

2) It can be checked easily, that the property ii) of the beginning of 3.4, is implied by the property i). This makes life easier (cf. 3.4.9.1 and 3.4.10 below), and in particular 3.4.11 below can be expressed in a simpler form, without mentioning U or b.

3) Looking back to 3.4.1-8 we see that the only places we used the fact that k is an infinite field are:

– 3.4.0, where we needed the existence of an element $h \in \big(U_0 \cap U_2\big)(k)$, which as a point of U_0 is different from the point obtained by taking the special fibre of a_0, and as a point of U_2 is different from the point obtained by taking the special fibre of g_2;

– 3.4.6, where we needed $\mathcal{L}$ to be infinite.

So if we are not bothered about gluing things ("over $W(k)$") (cf. 3.4.5), we can work with only one deformation, cf. 3.4.7. So, in this context of just one deformation, the whole of 3.4.1 makes sense and remains true even in the case when k is a finite field.

3.4.9.1. Potential-deformation sheets. Let k be a perfect field of characteristic $p > 2$. An affine locally closed subscheme U of a reductive group G over $W(k)$ (resp. an affine scheme U over $W(k)$), with $U(W(k))$ containing the identity element of $G(W(k))$ (resp. such that there is a point $y \in U(W(k))$), is said to be a potential-deformation sheet, if there is an étale morphism (resp. a formally étale morphism)

$$b_U : U \to \mathrm{Spec}(W(k)[x_1, ..., x_{d(U)}]),$$

for some $d(U) \in \mathbb{N}$, such that the identity element of $G(W(k))$ (resp. y) factors through the closed subscheme of U defined by $x_i = 0$, $i = \{1, ..., d(U)\}$. If moreover we have:

– the closed subscheme of U defined by the identity element of $G(W(k))$ (resp. defined by y) is the closed subscheme of U defined by $x_i = 0$, $i = \overline{1, d}$, and

– the closed subscheme of U_k defined by $x_i = 0$, with i running through an arbitrary subset of $\{1, ..., d(U)\}$, is connected,

then we speak about a convenient potential-deformation sheet.

Very often we denote a potential-deformation sheet by a pair (U, b_U) (resp. by a triple (U, b_U, y)). For any potential-deformation sheet we denote by Φ_U the Frobenius of $U^\wedge$ which takes x_i into x_i^p, $\forall i \in \{1, ..., d(U)\}$.

This definition of a (convenient) potential-deformation sheet is suggested by 3.4.9 2) and the convenience assumption 3.4.1.1.2.

3.4.10. Corollary. *For any $g_3 \in G(W(k))$ and for any open affine subscheme $U_3 = \mathrm{Spec}(R_3)$ of G which is a potential-deformation sheet, there is an $\mathbb{N}$-pro-étale morphism*

$$\ell_4 : U_4 = \mathrm{Spec}(R_4) \to U_3$$

such that:

– *there is a closed subscheme of G_k not containing the origin of G_k, and having the property that the fibres of ℓ_4 above points of the special fibre of U_3 not belonging to it, are non-empty;*

– *the special fibre of U_4 is geometrically connected over k;*

– *$\ell_4^{-1}(\mathcal{Z}_k) = \mathrm{Spec}(k)$, and so the origin of U_3 can be lifted uniquely to a point $a_4 \in U_4(W(k)) = U_4^\wedge(W(k))$;*

– *there is a p-divisible group $\mathcal{D}(U_3, \Phi_{U_3}, M, F^1, g_3\varphi_0, G)$ over $U_4^\wedge$, whose corresponding p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(U_4)$ (the Frobenius on $U_4^\wedge$ being the one induced from a fixed Frobenius Φ_{U_3} of $U_3^\wedge$ as in 3.4.9.1), is isomorphic to*

$$(M_{R_4}, \nabla_4) = \big(M \otimes R_4^\wedge, F^1 \otimes R_4^\wedge, h_4(g_3\varphi_0 \otimes 1), \nabla_4\big),$$

with h_4 as the universal element of G defined by $U_4^\wedge$, and with ∇_4 an integrable connection on $M \otimes R_4^\wedge$ respecting the G-action;

Moreover we have the following universal property: For any formally étale morphism $\ell_5 : U_5 = \mathrm{Spec}(R_5) \to U_3$, such that the closed subscheme $\ell_5^{-1}(\mathcal{Z}_k)$ has a non-empty intersection with any connected component of the special fibre of U_5, and for which we get a p-divisible group $\mathcal{D}_5$ over $U_5^\wedge$, defining a p-divisible object of $\mathcal{MF}_{[0,1]}(R_5)$ isomorphic to M_{R_5}, there is a unique $U_3^\wedge$-morphism $\ell : \mathrm{Spec}(U_5^\wedge) \to \mathrm{Spec}(U_4^\wedge)$ such that $\ell_4^\wedge = \ell_5^\wedge \circ \ell$ and $\mathcal{D}_5$ is isomorphic to $\ell^*(\mathcal{D}(U_3, \Phi_{U_3}, M, F^1, g_3\varphi_0, G)$.

Here M_{R_3} is defined as in 3.4.1.1, and allows us to get M_{R_4} and M_{R_5} as in 3.4.1.1.

3.4.11. Corollary. *Let $\tilde{H}$ be a smooth subgroup of G, having a connected special fibre, and let $U_{\tilde{H}} = \mathrm{Spec}(R_5)$ be an open affine subscheme of $\tilde{H}$ containing the origin a_0 of $G(W(k))$, and having the property that there is an open subscheme U of G containing $U_{\tilde{H}}$, and such that there are étale morphisms b and c making the diagram*

$$\begin{array}{ccc} U_{\tilde{H}} & \xrightarrow{i_{\tilde{H}}} & U \\ \big\downarrow b & & \big\downarrow c \\ \mathrm{Spec}\big(W(k)[z_1, \ldots, z_{e_{\tilde{H}}}]\big) & \xrightarrow{a} & \mathrm{Spec}\big(W(k)[z_1, \ldots, z_d]\big) \end{array}$$

to be cartesian (with $e_{\tilde{H}} := \dim(\tilde{H})$, with $i_{\tilde{H}}$ as the natural inclusion, with a as the natural closed embedding defined by: z_i goes to z_i, for $i \in \{1, ..., e_{\tilde{H}}\}$, and to 0 otherwise), and with the origin a_0 factoring through the closed subscheme of $U_{\tilde{H}}$ defined by $z_i = 0$, $i = \overline{1, e_{\tilde{H}}}$. Let the Frobenius Φ_{R_5} of $U_{\tilde{H}}^\wedge$ (compatible with σ) be defined by: z_i goes to z_i^p, $i = \overline{1, e_{\tilde{H}}}$. Then for any $\tilde{g} \in G(W(k))$, there is a formally étale morphism

$$\ell_6 : U_6 = \mathrm{Spec}(R_6) \to U_{\tilde{H}},$$

with the special fibre of U_6 a geometrically connected k-scheme, with $\ell_6^{-1}(\mathcal{Z}_k) = \mathrm{Spec}(k)$ (so the origin of $U_{\tilde{H}}$ lifts uniquely to a morphism $a_6 : \mathrm{Spec}(W(k)) \to U_6^\wedge$), and such that there is a p-divisible group over $U_6^\wedge$ whose corresponding p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(R_6)$ is isomorphic to

$$(M_{R_6^\wedge}, \nabla_6) = \big(M \otimes R_6^\wedge, F^1 \otimes R_6^\wedge, h_6(\tilde{g}\varphi_0 \otimes 1, \nabla_6)\big),$$

with h_6 as the universal element of $\tilde{H}$ defined by the p-adic completion of ℓ_6. ∇_6 respects the G-action (in fact, here we can replace G by the smallest smooth subgroup of G, containing $\tilde{H}$, and whose Lie algebra is left invariant by $p\tilde{g}\varphi_0$). We have a universal property similar to the one described in 3.4.10. Moreover, there is a closed subscheme of $\tilde{\mathcal{B}}_k$, not containing the origin of $\tilde{H}_k$, and such that the fibres of ℓ_6 above points of the special fibre of $U_{\tilde{H}}$ not belonging to it, are non-empty.

The results 3.4.10-11 are a direct consequence of 3.4.1.3 and its proof, of 3.4.9 2-3), and of 3.4.8.4 2).

3.4.11.1. Notation. We denote by $\mathcal{D}(U_{\tilde{H}}, \Phi_{R_5}, M, F^1, \tilde{g}\varphi_0, G)$ the p-divisible group over $U_6^\wedge$ mentioned in 3.4.11. For any $n \in \mathbb{N}$ we denote by $U_{6,n} = \mathrm{Spec}(Q_{6,n}^\wedge)$ the schemes constructed as in 3.4.1.3, starting from the data of the sextuple $(U_{\tilde{H}}, \Phi_{R_5}, M, F^1, \tilde{g}\varphi_0, G)$. We are going to use morphisms $\ell_{6,n}$, and ℓ_6^n, $n \in \mathbb{N}$, having the same significance as in 3.4.1.3. The limit of the $\mathbb{N}$-projective system ℓ_6^n, $n \in \mathbb{N}$, is U_6. Let Q_6 be an $W(k)$-algebra such that $U_6^\wedge = \mathrm{Spec}(Q_6)$.

3.4.12. Remark. In 3.4.11 it is enough to assume that $U_{\tilde{H}}$ is a locally closed subscheme of G containing the origin a_0 of $G(W(k))$, having a geometrically connected special fibre, and being smooth over $W(k)$, without assuming that it is the open subscheme of a smooth subgroup $\tilde{H}$ of G^{der}. In our work either $\tilde{H}$ is G^{der} (or G) or $\tilde{H}$ is an abelian subgroup of G isomorphic to $\mathbb{G}_a^{e_{\tilde{H}}}$, and in this situation we can take $U_{\tilde{H}} = \tilde{H}$. We can also work out the case when the special fibre of $U_{\tilde{H}}$ is just connected, but then U_6 is not going to have a geometrically connected special fibre: the number of (geometrically) connected components of $U_{6\bar{k}}$ is the same as the number of (geometrically) connected components of $U_{\tilde{H}} \times_k \mathrm{Spec}(\bar{k})$ (this can be deduced from 3.4.1.2).

3.4.13. Let $(f, L_{(p)}, v)$ be a SHS defined by an injective map $f : (\tilde{G}, X) \hookrightarrow \big(\mathrm{GSp}(W, \psi), S\big)$. We use the standard notation of 2.3.1-3, but with G replaced everywhere by $\tilde{G}$. Let k be a perfect field of characteristic p and let

$$z : \mathrm{Spec}(W(k)) \to \mathcal{N}_{W(k)}/H_0.$$

Let $(M, F^1, g\varphi_0, G', \tilde{\psi}_M)$ be its attached principally quasi-polarized not necessarily quasi-split Shimura filtered σ-crystal (cf. 2.3.10). Here G' is an inner form of $\tilde{G}_{W(k)}$. For simplicity of presentation we assume that $G' = \tilde{G}_{W(k)}$ (so we do not have to mention any more not necessarily quasi-split). Let G be the subgroup of G' such that under the previous identification it corresponds to $\tilde{G}^0_{W(k)}$ (see 2.3.1 for the meaning of $\tilde{G}^0_{\mathbb{Z}_{(p)}}$). Here $g \in G(W(k))$; in fact φ_0 is obtained from $g\varphi_0$, as in 3.2.3. We are going to use the previous notations of this 3.4. In particular we have a direct sum decomposition $M = F^1 \oplus F^0$, and a_0 denotes the origin of G. Let U_3 be an affine open subscheme of $G = \tilde{G}^0_{W(k)}$ (containing the origin) which is a potential-deformation sheet. Let $U_3 = \mathrm{Spec}(R_3)$ and let $(M_{R_4}, \nabla_4) = \big(M \otimes R_4^\wedge, F^1 \otimes R_4^\wedge, h_4(g\varphi_0 \otimes 1), \nabla_4\big)$ be the p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(R_4)$ mentioned in 3.4.10 for $g_3 = g \in G(W(k))$. We use the notations of 3.4.10; strictly speaking $(M, F^1, g\varphi_0, G, \tilde{\psi}_M)$ is not a principally quasi-polarized Shimura filtered σ-crystal, but for the sake of not complicating the notations, it is "close enough" (cf. 2.2.9 1) and 1′))

to a principally quasi-polarized Shimura filtered σ-crystal, so that 3.4.10 makes sense (cf. 3.4.1.6 and 3.4.11). Also, for a considerable simplification of the presentation, we assume that the subgroup H_0 of $G(\mathbb{A}_f^p)$ is small enough (with respect to z) so that the family of Hodge cycles $(\omega_\alpha)_{\alpha\in\mathcal{J}'}$ with which the abelian variety $A_z \times_{W(k)} \mathrm{Spec}(W(\bar{k}))$ over $W(\bar{k})$ is naturally endowed (through any lift $\mathrm{Spec}(W(\bar{k})) \to \mathcal{N}_{W(k)}$ of z), where $A_z := \mathrm{Spec}(W(k))_z \times_{\mathcal{N}_{W(k)}/H_0} \mathcal{A}_{H_0}$, is defined over $W(k)$, and so it gives birth to a family $(t_\alpha)_{\alpha\in\mathcal{J}'}$ of de Rham components of the F^0-filtration of $\mathcal{T}(M)$, and implicitely of the F^0-filtration of $\mathcal{T}(M\otimes R_4)$. This assumption is not really needed (cf. 3.4.14.2 below).

We have: $\nabla_4(t_\alpha)=0$, and $h_4(g\varphi_0\otimes 1)(t_\alpha)=t_\alpha$, $\forall\alpha\in\mathcal{J}'$. The fact that G fixes $\tilde{\psi}_m$, implies that ψ_M defines a principal quasi-polarization of M_{R_4}. Let

$$\tilde{M}_{R_4} := (M_{R_4}, \nabla_4, \tilde{\psi}_M).$$

It is a principally quasi-polarized p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R_4)$. Sometimes we prefer to regard it as being with cycles (they are t_α, $\alpha\in\mathcal{J}'$).

We denote by m_{nat} the natural morphism $(\mathcal{N}_{W(k)}/H_0)^\wedge \to \mathcal{N}_{W(k)}/H_0$.

SA1. Let $\widehat{G} = \mathrm{Spec}(\widehat{R_3}) = \mathrm{Spec}(\widehat{R_4})$ be the completion of G in the origin (or the completion of U_4 in a_4). From [Va2, 5.4.4] (see the proof of 2.3.15) we deduce the existence of a morphism $z_{\widehat{G}} : \widehat{G} \to \mathcal{N}_{W(k)}/H_0$ such that the completion of a_4 is mapped into z, and the two principally quasi-polarized Shimura p-divisible groups over $\widehat{G}$ obtained by pulling back the ones over $\mathcal{N}_{W(k)}/H_0$ and $\mathrm{Spec}(R_4^\wedge)$ are isomorphic, through an isomorphism lifting the identity in the completion of a_4. We assume that $z_{\widehat{G}} \bmod p$ factors through the henselization of the localization of $\mathrm{Spec}(R_4/pR_4)$ in the k-vakued point defined by a_4.

3.4.14. Theorem (the deformation principle: the expanded form). *There is an integral affine scheme $V_G = \mathrm{Spec}(S_G)$, which is the p-adic completion of an $\mathbb{N}$-pro-étale scheme over a smooth scheme over $W(k)$, and there are morphisms $a : \mathrm{Spec}(W(k)) \hookrightarrow V_G$, $b : V_G \to (\mathcal{N}_{W(k)}/H_0)^\wedge$, $c : V_G \to U_4^\wedge$ such that:*

a) $a_4 = c\circ a$ and the p-adic completion of z is $b\circ a$;

b) b is obtained by taking the p-adic completion of the composite of an $\mathbb{N}$-pro-étale morphism with a smooth morphism, and c is the p-adic completion of an $\mathbb{N}$-pro-étale morphism;

c) The special fibre of V_G is connected, and the image of $c \bmod p$ contains an open dense subset of the special fibre of $U_4^\wedge$;

d) If $N_{S_G} = (N, F^1_{S_G}, \Phi_N, \nabla_N, \tilde{\psi}_N)$ is the principally quasi-polarized p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(S_G)$ defined by the principally quasi-polarized p-divisible group of the principally polarized abelian scheme $V_{G\,b_0} \times_{\mathcal{N}_{W(k)}/H_0} (\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ over V_G, then there is an isomorphism of N_{S_G} into $\tilde{M}_{S_G}$ (the pull back of $\tilde{M}_{R_4}$ through the morphism c) taking the de Rham component $t_\alpha^{S_G}$ of the Hodge cycle $\omega_\alpha^{S_G}$ of $V_{G\,b_0} \times_{\mathcal{N}_{W(k)}/H_0} \mathcal{A}_{H_0}$ into t_α, $\forall\alpha\in\mathcal{J}'$ (here $b_0 := m_{\mathrm{nat}}\circ b$).

Let $\tilde{H}$ be the unipotent radical of the parabolic subgroup of $\tilde{G}^0_{W(k)}$ leaving F^0 invariant. Let $U_{\tilde{H}} = \mathrm{Spec}(R_5)$ be an open affine subscheme of $\tilde{H}$ such that the conditions of

3.4.11 are satisfied. We use the notations of 3.4.11 and 3.4.11.1, with $\tilde{g} = g$. Similarly we define a principally quasi-polarized p-divisible object (sometimes viewed with cycles) of $\mathcal{MF}^{\nabla}_{[0,1]}(R_6)$

$$\tilde{M}_{R_6} := (M_{R_6}, \nabla_6, \tilde{\psi}_M).$$

SA2. As above we assume that the similarly defined morphism from the completion of $U_{\tilde{H}}$ in the origin into $\mathcal{N}_{W(k)}/H_0$, modulo p, it factors through the henselization of the localization of $\mathrm{Spec}(Q_6/pQ_6)$ in the k-valued point defined by a_6.

We refer to this assumption (as well as of the one of SA1) as the starting assumption.

3.4.14.1. Theorem (the deformation principle). *There is an integral affine scheme $V_{\tilde{H}} = \mathrm{Spec}(S_{\tilde{H}})$, which is the p-adic completion of an $\mathbb{N}$-pro-étale scheme over a smooth scheme over $W(k)$, and there are morphisms $a : \mathrm{Spec}(W(k)) \hookrightarrow V_{\tilde{H}}$, $b : V_{\tilde{H}} \to (\mathcal{N}_{W(k)}/H_0)^{\wedge}$, $c : V_{\tilde{H}} \to U_6^{\wedge}$ such that:*

a) $a_6 = c \circ a$ and the p-adic completion of z is $b \circ a$;

b) b and c are p-adic completions of $\mathbb{N}$-pro-étale morphisms;

c) The special fibre of $V_{\tilde{H}}$ is connected, and the image of c mod p contains an open dense subset of the special fibre of $U_6^{\wedge}$;

d) If $N_{S_{\tilde{H}}} = (N, F^1_{S_{\tilde{H}}}, \Phi_N, \nabla_N, \tilde{\psi}_N)$ is the principally quasi-polarized p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(S_{\tilde{H}})$ defined by the principally quasi-polarized p-divisible group of the principally polarized abelian scheme $V_{\tilde{H}\, b_0} \times_{\mathcal{N}_{W(k)}/H_0} (\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ over V_G, then there is an isomorphism of $N_{S_{\tilde{H}}}$ into $\tilde{M}_{S_{\tilde{H}}}$ (the pull back of $\tilde{M}_{R_6^{\wedge}}$ through the morphism c) taking the de Rham component $t_\alpha^{S_{\tilde{H}}}$ of the Hodge cycle $\omega_\alpha^{S_{\tilde{H}}}$ of $V_{\tilde{H}\, b_0} \times_{\mathcal{N}_{W(k)}/H_0} \mathcal{A}_{H_0}$ into t_α, $\forall \alpha \in \mathcal{J}'$ (here $b_0 := m_{\mathrm{nat}} \circ b$).

Proof: It is easy to see that 3.4.14.1 can be deduced from 3.4.14 (also 3.4.14 can be deduced from 3.4.14.1). The proof of 3.4.14 is entirely similar to the proof of 3.4.14.1, just the presentation in the case of 3.4.14.1 is slightly more complicated. So we prove here only 3.4.14.1. We do not hesitate (though this makes the proof longer) to take the necessary time to explain in detail what the systems of equations of first type (can) bring new (besides 3.1.3.1) in the theory of deformations. As mentioned in 1.15, we do not hesitate to refer to parts of 3.5-12.

Let $Y_1 = \mathrm{Spec}(S^1_{\tilde{H}})$ be an affine smooth scheme over $W(k)$, with a connected special fibre, and such that there are morphisms $a_1 : \mathrm{Spec}(W(k)) \to Y_1$, $b_1 : Y_1 \to \mathcal{N}_{W(k)}/H_0$, $c_1 : Y_1^{\wedge} \to U_{\tilde{H}}$ satisfying the properties:

A) $a_6 = c_1 \circ a_1$ and $z = b_1 \circ a_1$;

B) c_1 mod p and b_1 are étale;

C) The principally polarized abelian scheme $(A_{Y_1}, p_{A_{Y_1}}) := Y_1 \times_{\mathcal{N}_{W(k)}/H_0} (\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ has the Hodge cycle $(\omega_\alpha^{S^1_G})_{\alpha \in \mathcal{J}'}$ (with which it is naturally endowed) defined over Y_1 (and not only over an étale scheme over Y_1), and there is an isomorphism $d_1 : N := H^1_{dR}(A_{Y_1}/S^1_{\tilde{H}}) \tilde{\to} M \otimes S^1_{\tilde{H}}$ taking the Hodge filtration of N defined by A_{Y_1} into $F^1 \otimes S^1_{\tilde{H}}$, $p_{A_{Y_1}}$ into $\tilde{\psi}_M$, and the de Rham component $t_\alpha^{S^1_{\tilde{H}}}$ of $\omega_\alpha^{S^1_{\tilde{H}}}$ into t_α, $\forall \alpha \in \mathcal{J}'$;

C') There is a Frobenius of $S^1_{\tilde{H}}$ compatible with σ and taking z_i into z_i^p, $i \in \{1, ..., e\}$, where the elements of the set $\{z_1, ..., z_e\}$ are forming a system of regular parameters of a_1 in Y_1, such that the resulting Frobenius of $M \otimes S^{1\wedge}_{\tilde{H}}$ (cf. the identification d_1 of C) above) is $g_{Y_1}(g\varphi_0 \otimes 1)$, where $g_{Y_1} \in \tilde{G}^0_{W(k)}(S^{1\wedge}_{\tilde{H}})$ is defined by c_1;

D) Denoting by I_z the ideal of $S^1_{\tilde{H}}$ defining the point a_1, the principally quasi-polarized p-divisible group we get over $\mathrm{Spec}(S^1_{\tilde{H}}/I_z^2)$ together with its de Rham cycles is isomorphic to the principally quasi-polarized p-divisible group over $\mathrm{Spec}(R_5/I_0^2) = \mathrm{Spec}(Q_6/I_0^2Q_6)$ together with its cycles, where I_0 is the ideal of $U_{\tilde{H}}$ defining a_6.

The existence of such a scheme Y_1 and of morphismms a_1, b_1 and c_1 with the above properties is just 2.3.15.1.

Over

$$Y_{16} := Y_1 \times_{W(k)} \mathrm{Spec}(Q_6)$$

we have two principally quasi-polarized p-divisible groups $(\mathcal{D}_1, \mathcal{P}_{\mathcal{D}_1})$ and $(\mathcal{D}_2, \mathcal{P}_{\mathcal{D}_2})$, obtained by pulling back under the natural projections from this product into its factors, the principally quasi-polarized p-divisible group of the principally polarized abelian scheme $(A_{Y_1}, p_{A_{Y_1}})$ over Y_1, and respectively the principally quasi-polarized p-divisible group

$$\big(\mathcal{D}(U_{\tilde{H}}, \Phi_{R_5}, M, F^1, g\varphi_0, G), \tilde{\psi}_M\big)$$

over $\mathrm{Spec}(Q_6)$. They can be viewed naturally as principally quasi-polarized Shimura p-divisible groups. For any $n \in \mathbb{N}$, let $Y_{16,n}$ be the moduli scheme over Y_{16} parametrizing isomorphisms between the two principally quasi-polarized finite flat group schemes over Y_{16} defined by the kernels of the multiplication with p^n on them. It is an affine scheme of finite type over Y_{16}. We get an $\mathbb{N}$-projective system defined by the morphisms

$$t^n : Y_{16,n+1} \to Y_{16,n}.$$

Let $Y_{16,\infty}$ be its projective limit. Property D) implies: there is a morphism

$$h_Z : Z := \mathrm{Spec}(W(k)[x_1, ..., x_e]/(x_1, ..., x_e)^2) \to Y_{16,\infty}$$

inducing isomorphisms $Z \tilde{\to} \mathrm{Spec}(S^1_{\tilde{H}}/I_z^2)$ and $Z \tilde{\to} \mathrm{Spec}(Q_6/I_0^2Q_6)$.

From D) and the proof of 2.3.15 we deduce the existence of a morphism

$$m_{\tilde{H}} : \tilde{H}^0 = \mathrm{Spec}(S^0_{\tilde{H}}) \to Y_{16,\infty},$$

where $\tilde{H}^0$ is the completion of $\tilde{H}$ (or of $U_{\tilde{H}}$) in the origin. The key point is:

E) The morphism $m_{\tilde{H}}$ mod p factors through the Y_{16}-scheme $\tilde{Y}_1 \times_{Y_{16}} Y_{16,\infty}$, where $\tilde{Y}_1$ is defined by the following fibre product:

$$\begin{array}{ccc} \tilde{Y}_1 & \xrightarrow{\tilde{c}_1} & \mathrm{Spec}(Q_6) \\ \big\downarrow{\scriptstyle \tilde{\ell}_6} & & \big\downarrow{\scriptstyle \ell_6} \\ Y_1 & \xrightarrow{c_1} & U_{\tilde{H}}. \end{array}$$

This is a consequence of the starting assumption. We stop to digress why the statement of the starting assumption is not automatically satisfied. Let us first remark that we have a canonical closed embedding $a_Z : Z \hookrightarrow \tilde{Y}_1$. The completion of $\tilde{Y}_1$ with respect to the ideal defining this embedding, can be identified with $\tilde{H}^0$ (cf. B)). Let $\tilde{c}_1^0$ and $\tilde{\ell}_6^0$ be the natural morphisms (obtained through this identification) from $\tilde{H}^0$ to $\mathrm{Spec}(Q_6)$ and respectively to Y_1. Let $m \in \mathbb{N}$, $m \geq 2$. Refering to [Fa2, th. 10], when we have to pass from things modulo $I_z^m S_{\tilde{H}}^0$ to things modulo $I_z^{m+1} S_{\tilde{H}}^0$, we do not know apriori that the morphism $\tilde{c}_1^0$ mod p is not changed. To argue this, we recall that P is the parabolic subgroup of G leaving invariant F^1. So $\mathrm{Lie}(P) = F^0(\mathrm{Lie}(G))$. The obstruction (as in loc. cit.) of liftings things modulo I_z^m to things modulo I_z^{m+1}, is given by some element

$$\gamma_m \in p\mathrm{Lie}(G) \otimes I_z^m / I_z^{m+1}$$

(cf. C') and D) for the case $m = 2$). We can modify (as in loc. cit.) $\tilde{c}_1^0$ by something which is zero mod p so that we have $\gamma_m \in pF^0(\mathrm{Lie}(G)) \otimes I_z^m / I_z^{m+1}$. But this element corresponds (as in loc. cit.) to a replacement of a Frobenius $\Phi_{\tilde{H}}^0$ of $M \otimes S_{\tilde{H}}^0$ (defined for instance via $\mathcal{D}_1$) by another one $(1 - p\gamma_m)\Phi_{\tilde{H}}^0(1 - p\gamma_m)^{-1}$ isomorphic to it. The new obstruction we get is given by some new element in

$$\gamma_{m+1} \in \mathrm{Lie}(G) \otimes_{W(k)} I_z^{m+1} / I_z^{m+2}.$$

But γ_{m+1} is not apriori divisible by p (however: if γ_m is divisible by p^2, then γ_{m+1} is divisible by p). So the induction does not applies. That is why we needed the starting assumption.

Let now $n \in \mathbb{N}$. From [BLR, Th. 12 of p. 83 and Th. 16 of p. 91], and from the fact that $Y_{16,n}$ is obtained from a scheme over $Y_1 \times_{W(k)} \mathrm{Spec}(Q_{16,n}^{\wedge})$ by pull back through the natural morphism $\mathrm{Spec}(Q_6) \to \mathrm{Spec}(Q_{6,n}^{\wedge})$, we deduce that the morphism $\tilde{H}^0 \to Y_{16,n}$ we get factors through an affine $Y_{16,n}$-scheme which (under the natural projection $\mathrm{Spec}(Y_{16,n}) \to \mathrm{Spec}(Q_6)$) is smooth over $\mathrm{Spec}(Q_6)$. From this and D), using a slice, we get an affine morphism

$$t(n) : \mathrm{Spec}(T_n) \to Y_{16,n}$$

such that, when composed with the projections of $Y_{16,n}$ on Y_1 and respectively on $\mathrm{Spec}(Q_6)$, it gives birth to a formally étale affine morphism $q(n)$ and respectively an affine étale morphism $s(n)$. Moreover we can assume that the induced morphism $Z \to Y_{16,n}$ factors through $\mathrm{Spec}(T_n)$, producing an affine morphism

$$h_n : Z \to \mathrm{Spec}(T_n),$$

and that $t(n)$ mod p factors through $\tilde{Y}_1 \times_{Y_{16}} Y_{16,n}$. Also we can assume that $\mathrm{Spec}(T_n / pT_n)$ is a connected scheme.

By induction on $n \in \mathbb{N}$ we check that we can choose the things in such a way that there is an étale affine morphism

$$t_n : \mathrm{Spec}(T_{n+1}) \to \mathrm{Spec}(T_n)$$

for which we have:

F) $s(n+1)\circ t_{n+1} = s(n)\circ t_n$, $t_{n+1}\circ h_{n+1} = h_n$ and $t^n \circ t(n+1)$ is congruent to $t(n)\circ t_n$ mod p^{n+1}.

The proof of F) in the ordinary case. To prove F) we just need to check that we can not have two formally étale morphisms from $\mathrm{Spec}(T_{n+m})$ into Y^1 which are different mod p^n, but identical mod p, and whose images mod p contain an open subset U^1 of the special fibre of Y^1, and such that the pull backs of $A_{Y_1}[p^{n+m}]$ (and its cycles, including its principal quasi-polarization) under these two morphisms are the same. We refer to this statement as the statement (n,m) ($m \in \mathbb{N}\cup\{0\}$).

We first prove F) in the case the the ordinary points are dense in $\mathcal{N}$ (see 4.2.1; or just 2.3.15 and 3.12), based on [Ka3, ch. 2] .

The idea is: it is enough to check a statement (n,m) at $\tilde{G}_{W(k)}$-ordinary points of U^1 (due to the fact that $\mathrm{Spec}(T_n/pT_n)$ is connected) using (good) $\tilde{G}_{W(k)}$-multiplicative coordinates. As we did not develop this theory yet (see 4.1-4 and 4.7) we have to work around. Such a theory of ordinary points and canonical coordinates is available (cf. [De2] and [No1]) for the particular case in which the Shimura-ordinary type associated to $\mathcal{N}$ (see 4.1) is an ordinary type. So we first prove statement (n,n) and F) in this particular case.

Obviously we can assume that $s(n+1)\circ t_{n+1} = s(n)\circ t_n$ and that $t_{n+1}\circ h_{n+1} = h_n$. We first need to check that $q(n)\circ t_{n+1}$ is congruent to $q(n+1)$ modulo $p^{m(n)}$ for a sufficiently large $m(n) \in \mathbb{N}$. From [Ka3, p. 151] we deduce that, at the ring levels, $q(n)\circ t_{n+1}$ is congruent to $q(n+1)$ modulo the $[\frac{n+1}{2}]$-power of any maximal ideal of T_n corresponding to an ordinary point $\mathrm{Spec}(T_n)(k)$ (with the notations of the loc. cit., if $x \in A(k)[p^n]$, then $p^{2n}\tilde{x} = 0$). But the ordinary points of $\mathrm{Spec}(T_n/pT_n)$ are Zariski dense (cf. 3.12 and the fact that ordinary points can be recognized working mod p). So we can take $m(n) = [\frac{n+1}{2}]$. This takes care of the statement (n,n). After a relabeling (like T_{2^n} becomes T_n), we get F). This ends the proof of F) in the ordinary case.

We come back to the general case, and we assume (for the time being) that property F) holds. Let $\mathrm{Spec}(T_\infty)$ be the limit of the $\mathbb{N}$-projective system defined by t_n, $n \in \mathbb{N}$. From F) above, we deduce the existence of a morphism

$$\mathrm{Spec}(T_\infty^\wedge) \to Y_{16,\infty}$$

such that h_Z factors through it. Moreover $\mathrm{Spec}(T_\infty^\wedge)$ together with the logical morphisms $\mathrm{Spec}(T_\infty^\wedge) \to U_6^\wedge$, $\mathrm{Spec}(T_\infty^\wedge) \to (\mathcal{N}_{W(k)}/H_0)^\wedge$ and the morphism $\mathrm{Spec}(W(k)) \to \mathrm{Spec}(T_\infty^\wedge)$ defined via h_Z, satisfies all the requirements of 3.4.14.1, except c). So to get 3.4.14.1 c), we have to work in a more refined way: we have to show that $\mathrm{Spec}(T_\infty/pT_\infty)$ can be obtained from $\mathrm{Spec}(T_1/pT_1)$ using systems of equations "closed in spirit" to those of the first type, in the same manner as in 3.4.8.1.2 b); on the way we prove F) in the general case.

The needed refinement. We work in the general case. We reached the situation: over $\mathrm{Spec}(T_1)$ there are two principally quasi-polarized Shimura p-divisible groups such that their kernels under the multiplication by p are isomorphic; as $\mathrm{Spec}(T_1)$ is a formally smooth regular $W(k)$-scheme it makes sense to speak about isomorphisms preserving the

cycles: by this we mean filtered isomorphisms of the filtered vector bundles underlying their filtered F-crystals, which mod p result in an isomorphism between truncations mod p of principally quasi-polarized p-divisible objects of $\mathcal{MF}^{\nabla}_{[0,1]}(T_1)$. Let $\mathcal{I}_1$ be this isomorphism. We first remark that:

G) There is a formally étale affine $\mathrm{Spec}(T_1)$-scheme $\tilde{Z}_\infty$ through which the morphism h_1 factors, such that over the special fibre $\tilde{Y}_\infty$ of $\tilde{Z}_\infty$, the two principally quasi-polarized p-divisible groups we get over $\mathrm{Spec}(T_1/pT_1)$ are isomorphic, under an isomorphism, which viewed as a non-filtered isomorphism between p-divisible objects of $\mathcal{MF}^{\nabla}_{[0,1]}(\tilde{Z}_\infty)$, lifts the extension of $\mathcal{I}_1$ to $\tilde{Z}_\infty$, and which in the Z-valued point defined by the factorization of h_1, is the one defined via h_Z.

This is a consequence of F). As we proved property F) only in the ordinary case, we need to point out that G) is also a consequece of SA2. To see this we just need to remark that the integral closed subscheme of the $\mathrm{Spec}(T_1/pT_1)$-scheme parametrizing isomorphisms between the kernels of the multiplication by p^n on the two p-divisible groups we got over $\mathrm{Spec}(T_1/pT_1)$, through which the morphism $\tilde{m}_H$ mod p factors naturally, is generically finite, and so étale in its k-valued point obtained from the one of $\tilde{H}^0$, via this factorization.

This results from 3.1.0 a) and d), and from 3.11.1 c), being a property of Shimura-ordinary σ-crystals. In other words, we just need to remark: if $k_1 \subset k_2$ are two algebraically closed fields containing k, then any isomorphism over $W(k_2)$ between the truncations mod p^n of two Shimura-canonical p-divisible groups over $W(k_1)$, is automatically defined over $W(k_1)$. Moreover the number of such isomorphisms is obviously finite. Obviously G) implies F), as we have a morphism $\tilde{Z}^{\wedge}_\infty \to Y_{16,infty}$ through which h_Z factors. This ends the proof of F) in the general case.

To explain in detail this property G), we need some notations. Let Φ_2 be the Frobenius of $M \otimes T_1^{\wedge}$ induced from the Frobenius of $M \otimes Q_6$, and let Φ_1 be the Frobenius of $M \otimes T_1^{\wedge}$ defined by the pull back of the natural p-divisible group on Y_1 (through the resulting morphism $\mathrm{Spec}(T_1) \to Y_1$) (cf. C)). We choose the Frobenius of the ring $T_1^{\wedge}$ to be the one induced from the one of Q_6. By property G), we mean: there is $\tilde{h}_\infty \in G(T_1^{h\wedge})$ (here $\mathrm{Spec}(T_1^h)$ is the henselization of $\mathrm{Spec}(T_1)$ with respect to the k-valued point defined naturally by h_1) such that mod p is the identity and we have

$$\tilde{h}_\infty \circ \Phi_1 = \Phi_2 \circ \tilde{h}_\infty. \tag{17}$$

Working as in 3.4.1.3, modulo powers of p, this equation (17) defines an $\mathbb{N}$-pro-étale scheme over $\mathrm{Spec}(T_1/pT_1)$ as follows. We start working mod p^2. So we are trying to find an element $\tilde{h}_1 \in G(T_1^{\wedge})$ which mod p is the identity, and for which we have:

$$\Phi_2 \circ \tilde{h}_1(m/p) = \tilde{h}_1 \circ \Phi_1(m/p) \ \mathrm{mod}\ p^2, \tag{18}$$

for $m \in F^1$, and

$$\Phi_2 \circ \tilde{h}_1(m) = \tilde{h}_1 \circ \Phi_1(m) \ \mathrm{mod}\ p^2, \tag{19}$$

if $m \in F^0$. The above congruences are between elements of $M \otimes T_1^{\wedge}$.

For solving this system, only the expression of $\tilde{h}_1 \bmod p^3$ is important. Moreover as we already know the existence of $\mathcal{I}_1$, we can assume that $\tilde{h}_1 \bmod p^2$ defines an element of $P(T_1/pT_1)$. So let $\gamma_{\tilde{h}_1} \in \mathrm{Lie}(P) \otimes T_1/pT_1$ such that $\tilde{h}_1$ is congruent to $1 + p\gamma_{\tilde{h}_1} \bmod p^2$. We choose arbitrarily a basis $\{\tilde{e}_1, ..., \tilde{e}_{\dim(P)}\}$ of $\mathrm{Lie}(P)$, and a basis $\{n_1, ..., n_{\dim(\tilde{H})}\}$ of $\mathrm{Lie}(\tilde{H})$. We can assume that $\{\tilde{e}_1, ...\tilde{e}_{\dim(\tilde{H})}\}$ is a basis of $F^1(\mathrm{Lie}(P))$. We write

$$\gamma_{\tilde{h}_1} = \sum_{i=1}^{\dim(P)} e_i x_i,$$

with x_i as variables. Moreover we can write $\tilde{h}_1 \bmod p^3$ as follows

$$1 + p\gamma_{\tilde{h}_1} + p^2 \sum_{j=1}^{\dim(\tilde{H})} n_j y_j,$$

with y_j as extra variables. The above equations (18) and (19) get translated in a system of equations involving x_i's and y_j's. It defines an affine $\mathrm{Spec}(T_1/pT_1)$-scheme $\mathrm{Spec}(T_1')$.

3.9.3 allows us to speak about $\tilde{G}_{W(k)}$-ordinary points of $\mathrm{Spec}(T_1/pT_1)$. Their definition does not depend on which of the two p-divisible groups over $\mathrm{Spec}(T_1)$ we use to define them. 3.12 implies that they are Zariski dense. We have

H) $\mathrm{Spec}(T_1')$ has finite fibres over the $\tilde{G}_{W(k)}$-ordinary points of $\mathrm{Spec}(T_1/pT_1)$ with values in $\bar{k}$.

We get this by looking at the system of equations in x_i's and y_j's as in 3.4.8.5; it is obtained by combining the equations of (20) and (21) below

$$x_i = L_i(x^p_{\dim(\tilde{H})+1}, ..., x^p_{\dim(P)}, y^p_1, ..., y^p_{\dim(\tilde{H})}), \tag{20}$$

$i \in \{1, ..., \dim(P)\}$, and

$$L_i(x^p_{\dim(\tilde{H})+1}, ..., x^p_{\dim(P)}, y^p_1, ..., y^p_{\dim(\tilde{H})}) = 0, \tag{21}$$

$i \in \{\dim(P) + 1, ..., \dim(G)\}$. Here L_i are non-homogeneous linear forms with coefficients in T_1/pT_1, $i \in \{1, ..., \dim(G)\}$. Over the perfection $\mathrm{Spec}(T_1/pT_1)^{\mathrm{perf}}$ (of $\mathrm{Spec}(R_1/pR_1)$), the last $\dim(P) - \dim(\tilde{H})$ equations of (20) together with the p-th root of the equations of (21), define a system of equations SE, which is very close to one of the first type. In fact, in any $\tilde{G}_{W(k)}$-ordinary point of $\mathrm{Spec}(T_1/pT_1)$ with values in $\bar{k}$, SE becomes a system of equations (with coefficients in k) which is equivalent to one of the first type. Using a Teichmüller lift $\mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(T_1^\wedge)$ of such a point $y_G : \mathrm{Spec}(k) \to \mathrm{Spec}(T_1/pT_1)$, everything results from 3.11.1 and 3.7.2. In other words, replacing the old variables by new once, "modelled" on the filtration $F^1(y_G)$ of M such that we get, by pulling back $\mathcal{D}_2$ via y_G, a Shimura canonical lift $(M, F^1(y_G), g(y_G)\varphi)$, we get a system of equations equivalent (i.e. it defines the same affine scheme over k) to the one obtained from SE via y_G, which is of the f the first type; this last property can be easily checked (the argument

is very similar to the one used in the proof of 3.4.18.4 B)), using the notations of 3.7.2 (3.11.1 allows to refer to 3.7.2). So 3.4.8.1 applies to end the proof of property H).

Moreover G) guarantees that $\mathrm{Spec}(T_1')$ is not an empty scheme, and that h_1 mod p factors through an integral closed subscheme $\mathrm{Spec}(T_1^{0\prime})$ of $\mathrm{Spec}(T_1')$ which is a connected scheme, étale over an open subscheme $\mathcal{U}_1$ of $\mathrm{Spec}(T_1/pT_1)$ (cf. also H)). But the same argument as the one used in 3.4.8.1.2 a) applies to give us that we can choose $\mathcal{U}_1$ to depend only on the coefficients of the homogeneous parts of the forms L_i, $i = \overline{1, \dim(G)}$. So we can repeat the arguments, for things modulo p^n ($n \in \mathbb{N}$, $n > 1$). We obtain new systems of equations, involving the same type of linear forms $L_i(n)$; moreover the homogeneous part of $L_i(n)$ is the same as of L_i, $\forall i \in \{1, ..., \dim(G)\}$ and $\forall n \in \mathbb{N} \setminus \{1\}$. From 3.4.8.1.2 a) we deduce that we can assume that $\tilde{Y}_\infty$ is an affine $\mathrm{Spec}(T_1/pT_1)$ such that its fibres over points of $\mathcal{U}_1$ are non-empty. In other words, coming back to the last paragraph before the needed refinement, we can assume that the morphism $\mathrm{Spec}(T_\infty/pT_\infty) \to \mathrm{Spec}(Q_6/pQ_6)$ has an imagine which contains an open subscheme of $\mathrm{Spec}(Q_6/pQ_6)$. This takes care of 3.4.14.1 c), ending the proof of 3.4.14.1.

3.4.14.2. Exercise. Show that 3.4.14 and 3.4.14.1 remain true, as mentioned in 3.4.13, without the assumptions made in 3.4.13. Hint: the only difference is that we have to work with different de Rham cycles, which become linear combination of de Rham components of Hodge cycles, after passage to an étale morphism.

3.4.14.3. Exercise. Restate 3.4.14.1 and 3.4.14.1 in terms of a k-valued point y of $\mathcal{N}_{W(k)}/H_0$, instead of a $W(k)$-valued point z of $\mathcal{N}_{W(k)}/H_0$.

3.4.14.4. Remark. We do not know when the starting assumptions SA1-2 are satisfied. When they are not satisfied, in 3.4.14 d) and 3.4.14.4.1 d) we have to work with truncations mod p^n of the p-divisible objects involved ($n \in \mathbb{N}$); of course in such a situation we do not need to pass to $\mathbb{N}$-pro-ètale morphisms.

It seems to us possible to prove directly, using just the techniques of 3.4 (at least in the case (G, X) is of compact type), the following expected result.

Expectation. *We assume $k = \bar{k}$. There is a $W(k)$-valued point of $\mathcal{N}_{W(k)}/H_0$ such that the kernel of the multiplication by p^n of the Shimura p-divisible group we get over $W(k)$ through it, is isomorphic to the same type of kernel of the Shimura p-divisible group we get through an arbitarry $W(k)$-valued point of* $\mathrm{Spec}(Q_6)$.

3.4.15. For the sake of presenting the group of ten fundamental principles in a reasonable compact form, we include the following expected result (it is proved in §11; see also 4.12) here.

Let $y : \mathrm{Spec}(k) \to \mathcal{N}_{W(k)}/H_0$ be defined by z. We continue to assume that $k = \bar{k}$.

Expectation (the completion property). *Giving $g_1 \in \tilde{G}^0_{W(k)}(W(k))$, there is a point $z_1 : \mathrm{Spec}(W(k)) \to \mathcal{N}_{W(k)}/H_0$ lifting a k-valued point of the same connected component $\mathcal{C}^0$ of $\mathcal{N}_k/H_0$ through which y factors, and such that its attached principally quasi-polarized Shimura filtered σ-crystal $\left(M_1, F_1^1, \varphi_1, \tilde{G}_{W(k)}, \tilde{\psi}_{M_1}\right)$ is isomorphic to $\left(M \otimes W(k), F^1 \otimes W(k), g_1\varphi_0, \tilde{G}_{W(k)}, \tilde{\psi}_M\right)$.*

The motivation for this expectation is based on 3.4.14.4 and on the following exercise.

A. Exercise. Let (M, φ, G) be an arbitrary Shimura σ-crystal. Show that there is $n \in \mathbb{N}$, such that for any $h_n \in G(W(k))$ congruent to 1 mod p^n, $(M, h_n\varphi, G)$ is isomorphic to (M, φ, G). The same remains true in a principally quasi-polarized context.

3.4.16. Remarks. 1) We say that $\mathcal{N}$ has the completion property, if the expectation of 3.4.15 holds.

2) We think it is interesting problem to determine in 3.4.15 A, the smallest value of n (or at least sharp estimates of it, perhaps in terms of the Hodge numbers of the iterates of (M, φ) and of $(\mathfrak{g}, \varphi)$).

3.4.17. Terminology. We propose that all connections used in 3.4.1-16 as well as in 2.2-3 to be called Faltings connections, in the honour of their source [Fa2, th. 10 and the remarks after].

3.4.18. Universal principles for connections in Fontaine's categories $\mathcal{MF}$. Let $m \in \mathbb{N}$. The notations to be used in 3.4.18-20 are entirely independent from the previous notations of §3. In this section $R = W(k)[[z_1, ..., z_m]]$ is the ring of formal power series in m variables with coefficients in $W(k)$. Let Φ_R be an arbitrary Frobenius of R, compatible with the Frobenius σ on $W(k)$.

As any object of a category $\mathcal{MF}_{[a,b]}(R)$ lifts to a p-divisible object (simple argument at the level of lifting matrices, based on the fact that R is a local ring), we prefer to look at such objects as finite consecutive extensions of truncations (mod p^n, for some $n \in \mathbb{N}$) of p-divisible objects. So we start with a p-divisible object

$$\mathfrak{C} := (M \otimes_{W(k)} R, F^1 \otimes_{W(k)} R, \Phi),$$

of $\mathcal{MF}_{[0,1]}(R)$. Here M is a free $W(k)$-module of finite rank, F^1 is a direct summand of M, while Φ is a Φ_R-linear endomorphism of $M \otimes R$ producing an isomorphism (still denoted by Φ) of R-modules

$$\Phi : (M + \frac{1}{p}F^1) \otimes_{R\ \Phi_R} R \tilde{\to} M \otimes R.$$

A natural question arises:

Q Is there a p-divisible group over R, whose corresponding p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$, when viewed as a p-divisible object of $\mathcal{MF}_{[0,1]}(R)$, is $\mathfrak{C}$?

If the answer is yes, we say that $\mathfrak{C}$ is induced from a p-divisible group. Similarly we speak about $\mathfrak{C}$ being induced from a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R_1)$, with R_1 a regular ring of formal power series with coefficients in $W(k)$.

3.4.18.0. Types of Frobenius of R. [Fa2, th. 10] can be restated: the answer to **Q** is yes provided the Frobenius Φ_R is of the form $\Phi_R(z_i) = z_i^p$. This assumption (on the special shape of the Frobenius of R) has been made in [Fa2, ch. 7] (quoting) "for technical reasons".

Let $S(m) := \{1, ..., m\}$ and let $S(M) := \{1, ..., \dim(M)\}$. An arbitrary Frobenius Φ_R of R compatible with σ of $W(k)$ is of the form

$$\Phi_R(z_i) = z_i^p + pP_i(z_1, ..., z_m) + pL_i(z_1, ..., z_m) + p\gamma_i,$$

where $P_i = \sum_{j=2}^{\infty} Q_{ij}$, with Q_{ij} a homogeneous polynomial of degree $j \geq 2$, where L_i is a linear homogeneous polynomial, and where $\gamma_i \in W(k)$, $\forall i \in S(m)$. It is an easy exercise to see that by replacing z_i by $z_i + p\delta_i$, with $\delta_i \in W(k)$ conveniently chosen, $i \in S(m)$, we can assume that $\gamma_i = 0$, $\forall i \in S(m)$. We assume this from now on.

We can write the Frobenius Φ of $M \otimes R$ in the form

$$\Phi = g_R(\varphi \otimes 1),$$

where $g_R \in GL(M)(R)$ modulo the ideal $(z_1, ..., z_m)$ of R is the identity element of $G(W(k))$, and where φ is the σ-linear endomorphism of M obtained from Φ by making $z_i = 0$, $i \in S(m)$. So what we are trying to do (for answering $\mathbf{Q}$), is nothing else but to deform over R the p-divisible group over $W(k)$ corresponding to the filtered σ-crystal (M, F^1, φ), in the way "prescribed" by $\mathfrak{C}$.

We write

$$L_i(z_1, ..., z_m) = \sum_{j=1}^{m} a_{ij} z_j,$$

with $a_{ij} \in W(k)$, $\forall i, j \in S(m)$. We obtain a matrix $A(\Phi_R) \in M_m(W(k))$, having a_{ij} as its entries. Let $\bar{A}(\Phi_R) \in M_m(k)$ be the matrix defined by $A(\Phi_R)$ mod p. We call $A(\Phi_R)$ (resp $\bar{A}(\Phi_R)$) the linear part (resp. the k-linear part) of Φ_R with respect to $z_1, ..., z_m$. Replacing z_i by $\sum_{j=1}^{m} b_{ij} z_j$, with $b_{ij} \in W(k)$ defining an invertible matrix $B \in GL_m(W(k))$, the matrix $A(\Phi_R)$ is replaced by $\sigma(B)A(\Phi_R)B^{-1}$. In what follows it is important only how $\bar{A}(\Phi_R)$ looks like.

3.4.18.0.1. Terminology. If $\Phi_R(z_i) = z_i^p$, $\forall i \in S(m)$, we speak about an additive type Frobenius of R. If $\Phi(z_i) = (z_i + 1)^p - 1$, then we speak about a multiplicative type Frobenius of R. If $A(\Phi_R)$ is an invertible matrix then we speak about an essentially multiplicative type Frobenius of R. If $\bar{A}(\Phi_R)$ is a nilpotent matrix, then we speak about an essentially additive type Frobenius of R.

3.4.18.1. Notations. Let $f(0) := \dim(M/F^1)$, and $f(1) := \dim(F^1)$. We denote by $s(i)$ the multiplicity of the slope i for (M, φ), $i = \{0, 1\}$. So $s(0)s(1)$ is the multiplicity of the slope -1 of $(\mathrm{End}(M), \varphi)$. Let r be the number of non-zero eigenvalues (counted with their multiplicities) in $\bar{k}$ of $\bar{A}(\Phi_R)$. It is easy to see that r is well defined regardless of all choices (cf. also 3.4.18.4).

3.4.18.2. The generic situation. This is obtained by requiring that:

a) (M, φ) to be an ordinary σ-crystal, and (M, F^1, φ) to be diagonalizable;

b) $\Phi_R(z_i) = (z_i + 1)^p - 1$, $\forall i \in S(m)$;

c) $k = \bar{k}$.

It is a simple exercise to see that for any $n \in \mathbb{N}$, there are precisely

$$p^{nmf(0)f(1)}$$

connections on $M \otimes R/p^n R$ which make $\mathfrak{C}/p^n\mathfrak{C}$ potentially to be viewed as an object of $\mathcal{MF}_{[0,1]}(R)$. Moreover all these connections are integrable, and (cf. also [Fa1, p. 34])

nilpotent mod p. The above number for connections mod p^n, for instance, can be obtained using 3.4.8 (3.4.8.9 and 3.4.8.1 show that it is enough to check it only for $n = 1$).

The reason we call this the generic situation is related to [De2, 1.4.7]. There are two philosophies at play:

– any p-divisible group over k is the specialization of an ordinary p-divisible group (cf. 3.1.8 and 3.1.8.1);

– whatever we can prove for the standard situation, and does not involve the precise power of p of the number of solutions we get for different system of equations (in the other cases we get smaller powers of p), can be proved in the general case (in particular we always get solutions).

As examples of this second philosophy see 3.4.18.4. It works also in the more general context where the category $\mathcal{MF}_{[0,1]}(R)$ is replaced by another one $\mathcal{MF}_{[a,b]}(R)$, with $a, b \in \mathbb{Z}$, $a < b$ (of course we keep b) and c), while a) is modified accordingly).

3.4.18.3. It is easy to see that the question **Q** does not always have a positive answer. Here is a simple example for which the answer to **Q** is no. Let $R = \mathbb{Z}_p[[t]]$ with $\Phi_R(t-1) = (t-1)^p$. We take $\mathfrak{C}$ to be defined as follows. Let M be a free $\mathbb{Z}_p$-module of dimension 2, and let $\{x, y\}$ be a basis of M, with $\{y\}$ a basis of F^1. Let $\Phi(x) = x$ and $\Phi(y) = py + px + pty$. It is easy to see that there is no connection ∇ on $M \otimes R$ for which Φ is ∇-parallel. However it can be checked that, moving from $\mathbb{Z}_p$ to $W(\mathbb{F})$ (i.e. we replace $\mathbb{F}_p$ by its algebraic closure), $(M \otimes W(\mathbb{F})[[t]], F^1 \otimes W(\mathbb{F})[[t]], \Phi \otimes 1)$ is induced from a p-divisible group (over $W(\mathbb{F})[[t]]$). So the obstruction is: the field $\mathbb{F}_p$ is not algebraically closed.

3.4.18.3.1. Lemma. *If there is an integrable connection ∇ on $M \otimes R$ which makes $\mathfrak{C}$ to be viewed as a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$, then $\mathfrak{C}$ is induced from a p-divisible group.*

Proof: The connection ∇ is also nilpotent mod p [Fa1, p. 34]. So it allows us to change the Frobenius of R to one of additive type, cf. [De2, (1.1.2.1)]. Now everything results from [Fa2, th. 10].

3.4.18.4. Theorem (the universal local ∇ principle for $\mathcal{MF}_{[0,1]}$). *We assume $k = \bar{k}$.*

A) There are connections ∇ on $M \otimes R$ which make $\mathfrak{C}$ potentially to be viewed as a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$.

B) Let $n \in \mathbb{N}$. The number of connections which make $\mathfrak{C}/p^n\mathfrak{C}$ potentialy to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$ is $p^{nrs(0)s(1)}$. This number is smaller or equal to $p^{nmf(0)f(1)}$.

Proof: The proof of part A) is contained in the proof of 3.4.1.3, cf. 3.4.8.4 and 3.4.8.9. From 3.4.8.1 and 3.4.8.9 we deduce that for the counting part of B) it is enough to consider the case $n = 1$. As $k = \bar{k}$, from 3.4.18.0 we deduce that we can assume that $\bar{A}(\Phi_R)$ is upper triangular. The existence of a connection on $M \otimes R/pR$ making $\mathfrak{C}/p\mathfrak{C}$ potentially to be viewed as an object of $\mathcal{MF}_{[0,1]}(R)$ is given (see 3.4.8 (3)) by a system of equations of the form

$$x_{ijl} = L_{ijl}(x^p_{111}, ..., x^p_{\dim(M)\dim(M)m}) + c_{ijl},$$

$(i,j,l) \in S(M) \times S(M) \times S(m)$, where L_{ijl} are linear homogeneous forms in $m\dim(M)^2$ variables, and where $c_{ijl} \in R/pR$, $\forall (i,j,l) \in S(M) \times S(M) \times S(m)$.

This system defines an affine étale scheme over $\mathrm{Spec}(R/pR)$, cf. 3.4.8.1. We consider the ideal $I := (z_1, ..., z_m)$ of R/pR. As R/pR is a strictly henselian ring, it is enough to look at this system only modulo I. Due to the upper triangular form of $\bar{A}(\Phi_R)$, this system mod I can be "separated":

In the expression of L_{ijl} only those variables $x_{ijl'}$ show up for which $l' \leq l$.

So this allows us to assume that $m = 1$: first we solve the system we get involving just x_{ij1}'s; we use any solution we get for it to get a new system involving just x_{ij2}'s (in other words x_{ij1}'s become constants for this new system involving the x_{ij2}'s); etc. (the induction applies). From now on, as $m = 1$, we drop the index l. So modulo I we get a system $\mathcal{S}$ of $\dim(M)^2$ equations in $\dim(M)^2$ variables with coefficients in k.

If $\bar{A}(\Phi_R) \in k$ is zero, then the system $\mathcal{S}$ is of the form $x_{ij} = d_{ij}$, $(i,j) \in S(M) \times S(M)$, with $d_{ij} \in k$. Obviously it has a unique solution. So we can assume that $\beta := \bar{A}(\Phi_R) \in k$ is non-zero (in fact we can assume that $\beta = 1$).

Modulo I, g_R contributes only to the coefficients d_{ij} (obtained by taking c_{ij} modulo I). From 3.4.8.1 we deduce that we can assume that g_R is the identity element of $GL(M)(R)$.

Now we are essentially done. We present first the particular case when (M, F^1, φ) is cyclic diagonalizable. We consider a basis $\{e_i | i \in S(M)\}$ of M. From def. 2.2.1 b) we deduce that we can assume that there is a subset $S(F^1)$ of $S(M)$ such that $\{e_i | i \in S(F^1)\}$ is a basis of F^1, and that there is a permutation π of $S(M)$ such that $\varphi(e_i) = pe_{\pi(i)}$, if $i \in S(F^1)$, and $\varphi(e_i) = e_{\pi(i)}$, if $i \notin S(F^1)$. So the system $\mathcal{S}$ is of the form

$$x_{\pi(i)\pi(j)} + d_{ij} = \beta\alpha_{ij}x_{ij}^p,$$

$i, j \in S(M)$, where α_{ij} is 1 or 0 depending on the fact that (i,j) is or is not in $(S(M) \setminus S(F^1)) \times S(F^1)$, and where $d_{ij} \in k$. So if $\pi(i) \neq i$ or if $\pi(j) \neq j$, or if $\pi(i) = i$ and $\pi(j) = j$, but $(i,j) \notin (S(M) \setminus S(F^1)) \times S(F^1)$, then in the sequence of elements α_{ij}, $\alpha_{\pi(i)\pi(j)}$,..., $\alpha_{\pi^d(i)\pi^d(j)}$, at least one is zero; so x_{ij} is uniquely determined. For any pair (i,j), with $\pi(i) = i$ and $\pi(j) = j$, and with $i \notin S(F^1)$ and $j \in S(F^1)$, we have precisely p (independent) choices for x_{ij}. The number of such pairs is precisely $s(0)s(1)$. This handles the particular case.

We come back to the general case. Let $B(\varphi)$ be a $\dim(M) \times \dim(M)$-matrix with coeficients in k formed naturally (i.e. it acts naturally on M/pM) by a solution of the system $\mathcal{S}$. From the shape of the equations of 3.4.1.1.1 2) we deduce that $B(\varphi)$ annihilates the image of $\varphi(M)$ in M/pM. This easily implies that $B(\varphi)$ annihilates the image in M/pM of the maximal σ-subcrystal (M, φ) corresponding to slopes smaller than 1 (the fastest way to see this: we can assume that $\Phi_R(z_1) = z_1^p + pz_1$; so working modulo (z_1) instead of modulo (p, z_1), we get similarly a matrix $\tilde{B}(\varphi)$ lifting $B(\varphi)$ to $W(k)$, such that, when viewed as an endomorphism of M, we have $p\varphi(\tilde{B}(\varphi)) = \tilde{B}(\varphi)$, cf. 3.4.1.1.1 2)). So we just need to look at the action of $B(\varphi)$ on the image in M/pM of elements $e \in M$ for which $\varphi(e) = pe$: (M, φ) is a direct sum of three σ-crystals having respectively only the slope 1, only the slope 0, only slopes of the interval $(0,1)$ (cf. the existence of the filtration

F^1 of M). From the same equations we deduce (the argument is as in the above particular case) that $B(\varphi)(e)$ belongs to the image in M/pM of the maximal σ-subcrystal of (M,φ) corresponding to the slope 0. So we have at most $p^{s(0)s(1)}$ possibilities for $B(\varphi)$. From this we deduce, cf. 3.4.18.2, that there are precisely $p^{s(0)s(1)}$ solutions of $\mathcal{S}$. This ends the proof of the B) part, and so of the theorem.

3.4.18.4.1. Theorem (the integrability principle for $\mathcal{MF}_{[0,1]}(R)$). *All the connection ∇ proved to exist in 3.4.18.4 are integrable and nilpotent mod p.*

Proof: Let $R^{\mathrm{al}} := W(k)[z_1,...,z_m]$. Here al stands for algebraization. The fact that a connection is integrable is expressed through some algebraic equations. In our case it is enough to check (regardless of which $n \in \mathbb{N}$ we work with) that these equations are satisfied modulo the ideal $J_r := (p, z_1, ..., z_m)^r$ of R, for any $r \in \mathbb{N}$. But these equations modulo J_r depend only on the expressions of Φ and Φ_R modulo J_{r+2}. This allows us to algebraize the things. From the Chinese remainder theorem we deduce the existence:

– of an element $g_{R^{\mathrm{al}}} \in GL(M)(R^{\mathrm{al}})$ such that modulo the ideal $J^{\mathrm{al}}_{r+2} := (p, z_1, ..., z_m)^{r+2}$ of R^{al} it is g_R modulo J_{r+2}, and modulo the ideal I_1^{r+2} of R^{al} (with $I_1 = (p, z_1 - 1, ..., z_m - 1)$) is an element $g_1 \in GL(M)(k) \subset GL(M)(R^{\mathrm{al}}/I_1^{r+2})$ for which the Frobenius $g_1(\varphi \otimes 1)$ of $M \otimes R^{\mathrm{al}}/I_1^{r+2}$ is isomorphic to one coming from a generic situation, as in 3.4.18.2 above;

– of a Frobenius $\Phi_{R^{\mathrm{al}}}$ of R^{al} compatible with σ, which modulo J^{al}_{r+2} is the same as the Frobenius Φ_R modulo J_{r+2}, and which modulo I_1^{r+2} is isomorphic to the one coming from a generic situation as in 3.4.18.2 above.

Let $n \in \mathbb{N}$, and let ∇_n be a connection on $M \otimes R/p^nR$ which makes $\mathfrak{C}/p^n\mathfrak{C}$ potentially to be viewed as an object of $\mathcal{MF}_{[0,1]}(R)$. As in 3.4.1.3 we construct for the object

$$\mathfrak{C}^{\mathrm{al}} := (M \otimes R^{\mathrm{al}}/p^nR^{\mathrm{al}}, F^1 \otimes R^{\mathrm{al}}/p^nR^{\mathrm{al}}, g_{R^{\mathrm{al}}}(\varphi \otimes 1))$$

of $\mathcal{MF}_{[0,1]}(R^{\mathrm{al}})$, where the Frobenius of R^{al} is $\Phi_{R^{\mathrm{al}}}$, a moduli scheme $M_n(\mathfrak{C}^{\mathrm{al}}) = \mathrm{Spec}(R_n)$ of connections which make the extension of $\mathfrak{C}^{\mathrm{al}}/p^n\mathfrak{C}^{\mathrm{al}}$ to $*$ potentially to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(*)$. Here $*$ stands for a p-adically complete formally étale R^{al}-algebra. Mod p it defines an étale affine scheme $\mathcal{R}(n) := \mathrm{Spec}(R_n/pR_n)$ over $\mathcal{R}(0) := \mathrm{Spec}(R^{\mathrm{al}}/pR^{\mathrm{al}})$. Moreover the fibres of $\mathcal{R}(n)$ over k-valued points of $\mathcal{R}(0)$ have a number of points less or equal to the number of points of the fibre $\mathcal{F}_1$ above the k-valued point of $\mathcal{R}(0)$ defined by $z_i = 1$, $i \in S(m)$, cf. 3.4.18.4 B). From this and the surjectivity principle 3.4.8.2 (cf. 3.4.8.4) we deduce that:

Any connected component of $\mathcal{R}(n)$ contains points of $\mathcal{F}_1$.

From this and from 3.4.18.2 we get:

$M_n(\mathfrak{C}^{\mathrm{al}})$ is a moduli scheme of integrable connections, nilopent mod p.

From this and the fact that ∇_n modulo the ideal $(p, z_1, ..., z_m)^r$ of R/p^nR, is equal to a connection obtained from the universal one on $M \otimes R_n/p^nR_n$, through a morphism $\mathrm{Spec}(R/p^nR) \to M_n(\mathfrak{C}^{\mathrm{al}})$, we get that ∇_n is integrable and nilpotent mod p (we recall that $r \in \mathbb{N}$ was arbitrary). This ends the proof of the theorem.

3.4.18.4.2. Corollary (the moduli principle). *For any separated regular scheme (resp. regular affine scheme) S which is formally smooth over $W(k)$ and has a Frobenius compatible with σ of $W(k)$, any object (resp. any p-divisible object) $\mathfrak{C}$ of $\mathcal{MF}_{[0,1]}(S)$ defines a moduli scheme $M(\mathfrak{C})$ which is the p-adic completion of an affine étale scheme over S (resp. of an $\mathbb{N}$-pro-étale scheme over S) of connections (over p-adically complete formally étale affine schemes over S) which makes the extension of $\mathfrak{C}$ to such a scheme S_1 to be an object (resp. a p-divisible object) of $\mathcal{MF}^{\nabla}(S_1)$. Moreover the fibres of $M(\mathfrak{C})$ over points with values in fields of the special fibre of S are not empty. In the case of an object of $\mathcal{MF}_{[0,1]}(S)$, these fibres are étale schemes over fields defined by algebras of dimension (as vector spaces over these fields) a power of p.*

Proof: This is a direct consequence of 3.4.18.4.1 and 3.4.8.1 (with $l = 1$). When we pass from p-divisible objects to objects, there is an extra feature showing up, which is worth being pointed out. If $S = \mathrm{Spec}(R_1)$ is an affine scheme, if $\mathcal{C}$ is an object of $\mathcal{MF}_{[0,1]}(S)$, then the underlying R_1-module M_1 of $\mathcal{C}$ might not be a projective R_1/p^nR_1-module, for some $n \in \mathbb{N} \cup \{0\}$. So sometimes, when we pass from things mod p^n to things mod p^{n+1}, we get system of equations in less variables, and so of completely different type of what we got working mod p^m, $m \in \mathbb{N} \cup \{0\}$, $m \leq n$. But the situation is the same: we still get system of equations of the first type. This is an easy consequence of the structure of the R_1-module M_1, cf. [Fa1, ii) of th. 2.1] (the proof in the loc. cit. works in the case of regular formally smooth schemes over $W(k)$).

The fact that for the case of an object of $\mathcal{MF}_{[0,1]}(S)$, we can work out the things just for S being a separated regular scheme (not necessarily affine) is a consequence of the fact that the things we obtain for an affine open cover of S can be glued together, cf. [Fa1, 2.3]. Here is the second place where we need $p > 2$. This ends the proof of the corollary.

As examples of such moduli schemes for connections, we have the schemes $\mathcal{S}^n$, $n \in \mathbb{N}$, of 3.4.8. As in the proof of 3.4.18.4.1 we get:

3.4.18.4.3. Corollary (the touching property). *We assume that the special fibre of S is connected and that $\mathfrak{C}$ is an object of $\mathcal{MF}_{[0,1]}(S)$. Let p^m be the maximal number of points $\mathrm{Spec}(k_1) \to M(\mathfrak{C})$ lifting a k_1-valued point of S, with k_1 an algebraically closed field containing k. Then any connecting component of $M(\mathfrak{C})$ intersects all the fibres of $M(\mathfrak{C})$ which are defined by étale algebras of relative dimension, as vector spaces over residue fields of the special fibre of S, equal to p^m.*

3.4.18.4.4. Remarks. 1) 3.4.18.4 remains true for the case when R/pR is a local strictly henselian ring and R is a regular formally smooth $W(k)$-algebra.

2) From now on we do not need (cf. 3.4.18.4.1) to use anymore the word potentially (see 3.4.1.1.1).

From 3.4.18.4.2, the construction of the schemes $M(\mathfrak{C})$ (see 3.4.8), and 3.4.8.1.2 b) we get immediately:

3.4.18.4.5. Corollary (the constructibility property). *Under the assumptions of 3.4.18.4.2, if the special fibre of S is a noetherian scheme, then any connected component of the special fibre of $M(\mathfrak{C})$, with $\mathfrak{C}$ a p-divisible object, maps into a constructible dense subset of a connected component of the special fibre of S.*

3.4.18.4.6. Extra terminology. What follows in this section is an algebraic digression supplementing the first paragraph of 3.4.8.9. In that paragraph we defined system of equations of the second type, while working with an additive type ring Frobenius. Similarly, if the ring Frobenius involved is of multiplicative (resp. essentially multiplicative or additive) type, we define systems of equations of multiplicative (resp. essentially multiplicative or additive) second type. In the proof of 3.4.18.4 we used such systems. The difference from what we got in 3.4.8.5 is: the coefficients of x_{isl}^p we get are not any more some $p-1$ power of a non-invertible element (see 3.4.8.5 (11)), but it is of more general nature; in particular it can be invertible.

A. Let now $\bar{R}$ be an arbitrary $\mathbb{F}_p$-algebra. A system of equations of the form

$$(*) \qquad x_i = L_i(x_1^p, x_2^p, ..., x_n^p) + c_i,$$

$i = \overline{1,n}$, where L_i is a linear homogeneous form with coefficients in $\bar{R}$, and $c_i \in \bar{R}$, $\forall i \in \{1, ..., n\}$, is said to be of first type with coefficients in $\bar{R}$ (cf. terminology of 3.8.9); we also call it a quasi Artin-Schreier system in n variables with coefficients in $\bar{R}$. It defines naturally an affine étale $\mathrm{Spec}(\bar{R})$-scheme (cf. 3.4.8.1). If this system defines a finite étale morphism (at the level of schemes), then we omit the word quasi. The $n \times n$ matrix obtained naturally (as in 3.4.18) from the coefficients of the linear forms L_i, $i = \overline{1,n}$, is called the matrix of the system $(*)$.

So (see 3.4.8) the special fibres of the schemes $M(\mathfrak{C})$ of 3.4.18.4.2 are obtained as (the $\mathbb{N}$-projective limit of) a sequence of morphisms defined by quasi Artin-Schreier systems; in the case of an object $\mathfrak{C}$ the sequence is finite, while in the case of a p-divisible $\mathfrak{C}$ it is infinite.

B. Artin-Schreier fundamental groups. Let $n \in \mathbb{N}$. We say that an étale cover $q_1 : \mathrm{Spec}(\bar{R}_1) \to \mathrm{Spec}(\bar{R})$ is crystalline elementary n-admissible if it is defined by an open closed subscheme of an étale cover of $\mathrm{Spec}(\bar{R})$ obtained using an Artin-Schreier system in at most n variables (cf. also the extra feature of the proof of 3.4.18.4.2). We say that q_1 is crystalline n-admissible if it is obtained through a finite sequence of consecutive crystalline elementary n-admissible étale covers.

We assume that $\mathrm{Spec}(\bar{R})$ is connected. The Galois group of the Galois pro-étale cover of $\mathrm{Spec}(\bar{R})$ generated by the connected components of crystalline n-admissible étale covers of $\mathrm{Spec}(\bar{R})$, is called the level n Artin-Schreier fundamental group of $\bar{R}$ or of $\mathrm{Spec}(\bar{R})$, and is denoted by $\Pi_n^{AS}(R)$. We have natural epimorphisms

$$\Pi_{n+1}^{AS}(\bar{R}) \twoheadrightarrow \Pi_n^{AS}(\bar{R}).$$

Variant: above we consinder only finite sequences of length at most m, where $m \in \mathbb{N}$ is fixed; we obtain pro-finite Galois groups $\Pi_{n,m}^{AS}(\bar{R})$, and natural epimorphisms

$$\Pi_{n_1,m_1}^{AS}(\bar{R}) \twoheadrightarrow \Pi_{n,m}^{AS}(\bar{R}),$$

with $n_1, m_1 \in \mathbb{N}$, $n_1 \geq n$, and $m_1 \geq m$. We denote by $\Pi_\infty^{AS}(\bar{R})$ the projective limit of $\Pi_n^{AS}(\bar{R})$. It is a quotient of the first fundamental group of $\mathrm{Spec}(\bar{R})$ (in all above, we can fix

a geometric point of $\mathrm{Spec}(\bar{R})$ in the usual way; not to complicate the notations we do not mention it explicitly). We call it the Artin-Schreier fundamental group of $\bar{R}$ or of $\mathrm{Spec}(\bar{R})$. We have:

C. Lemma (the fundamental lemma of the π_1-theory in positive characteristic). *If k_1 is a field (not necessarily perfect) containing $\mathbb{F}_p$, then $\Pi^{AS}_{\infty}(k_1)$ is exactly the Galois group of k_1.*

Proof: We can assume that k_1 is perfect (this is standard). Let k_2 be a finite Galois extension of k_1. Let $n_0 := [k_2 : k_1]$ and let $\{e_1, ..., e_{n_0}\}$ be an arbitrary basis of k_2 over k_1. Let σ_2 be the Frobenius automorphism of k_2. An arbitrary element $x \in k_2$ can be written uniquely as $x = \sum_{i=1}^{n_0} x_i e_i$, with $x_i \in k_1$, $i = \overline{1, n_0}$. We look at the equation $\sigma_2(x) = x$. This equation gets translated into an Artin-Schreier system of the form

$$x_i = L_i(x_1^p, ..., x_{n_0}^p),$$

$i = \overline{1, n_0}$, with L_i as linear homogeneous forms. The matrix formed by the coefficients of these linear forms is invertible. So 3.4.8.1 and 3.4.8.1.0 (with $l = 1$) apply. We get that k_2 splits over the Galois extension of k_1 generated by the connected k-schemes obtained using this Artin-Schreier system. This ends the proof of the lemma.

D. Remarks. 1) In the above lemma the condition k_1 is a field can be weaken: we just need that any projective k_1-module of finite rank is free. So to understand the first fundamental groups of arbitrary connected $\mathbb{F}_p$-schemes we need to understand how to "glue" Artin-Schreier systems.

2) From the proof of the above lemma we deduce that $\mathrm{Gal}(k_2/k_1)$ is naturally a quotient of $\Pi^{AS}_{n_0,1}(k_1)$. It is very much desirable to replace here the lower index $n_0, 1$ with another one $n, 1$, with n as small as possible. We do not know how to get better upper bounds for such an n.

3) Let $l \in \mathbb{N}$. Using p^l-powers instead of p-powers (as in 3.4.8.1) we define similarly groups $\Pi^{AS}_{n,m,l}(\bar{R})$. Obviously (see 3.4.8.1.5) we have a natural epimorphism:

$$\Pi^{AS}_{nl,m}(\bar{R}) \twoheadrightarrow \Pi^{AS}_{n,m,l}(\bar{R}).$$

4) There are $n(n+1)$ coefficients involved in the definition of a quasi Artin-Schreier system in n variables. Of course they are entirely independent. One can learn a lot by looking at Artin-Schreier systems in a smaller number of variables, like 2 or 3. For $n = 2$ we have 6 coefficients, while for $n = 3$ we have 12.

5) If a quasi Artin-Schreier system over $\bar{R}$ defines a Galois cover of an open dense subscheme of $\mathrm{Spec}(\bar{R})$, then the étale scheme over $\mathrm{Spec}(\bar{R})$ naturally asociated to it is an open subscheme of a Galois cover of $\mathrm{Spec}(\bar{R})$, cf. 3.4.8.1.

E. Definition. The Artin-Schreier dimension of $\bar{R}$, denoted $d^{AS}(\bar{R})$, is the smallest number $q \in \{0\} \cup \mathbb{N} \cup \{\infty\} \cup \{\infty + 1\}$ such that $\Pi^{AS}_q(\bar{R})$ is the whole fundamental group of $\mathrm{Spec}(\bar{R})$ (here $\Pi^{AS}_0(\bar{R})$ is the trivial group, while $\Pi^{AS}_{\infty+1}(\bar{R})$ is the fundamental group of $\mathrm{Spec}(\bar{R})$.

F. Exercise. Compute the level n Artin-Schreier fundamental groups and the Artin-Schreier dimensions of algebraic field extensions of $\mathbb{F}_p$.

We hope to come back in a future paper with the main (expected or known to be true) properties of these Artin-Schreier fundamental groups and dimensions.

3.4.18.5. Theorem (the inducing property). *The answer to the question* **Q** *is yes if one of the following conditions is satisfied:*

a) $k = \bar{k}$;

b) Φ_R *is essentially of additive type;*

c) the Newton polygon of (M, φ) *does not have either the slope 0 or the slope 1.*

This is a trivial consequence of 3.4.18.3-4.

3.4.18.5.1. Corollary. *We assume that the Frobenius* Φ_R *is essentially of additive type. Then we have an equivalence of categories between the category of p-divisible objects of* $\mathcal{MF}_{[0,1]}(R)$ *and the category of p-divisible objects of* $\mathcal{MF}^{\nabla}_{[0,1]}(R)$*, and we have an antiequivalence of categories between these two categories and the category of p-divisible groups over* R.

Proof: The antiequivalence part is implied by the equivalence part and by 3.4.18.5: we can change the Frobenius Φ_R (cf. [De2, (1.1.2.1)]) to one of additive type $\Phi_R(z_i) = z_i^p$; so the Grothendieck-Messing theory and 3.4.18.0 apply (we construct uniquely, starting from a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$, by induction on $n \in \mathbb{N} \cup \{0\}$, a p-divisible group over $R/(z_1, ...z_m)^n$).

For the equivalence part, from 3.4.18.4 (we have $r = 0$) we deduce that there is only one way to associate to a p-divisible object of $\mathcal{MF}_{[0,1]}(R)$ a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$. So we just need to check that this association is well behaved with respect to maps (morphisms) between p-divisible objects. Everything boils down to (keeping the previous notations): if $f \in F^0(\mathrm{End}(M)) \otimes R$ is fixed by Φ, then it is annihilated by the (unique) connection ∇ on $M \otimes R$ for which Φ is ∇-parallel. Let $\nabla(f) = f_1 \in \mathrm{End}(M) \otimes \bar{\Omega}_{R/W(k)}$. It satisfies the equation

$$f_1 = p\Phi \circ d\Phi_{R*}/p(f_1) = \Phi \circ d\Phi_{R*}(f_1).$$

We need to show that $f_1 = 0$. It is enough to show that $f_1 \bmod p$ is zero. But the above equation mod p defines a system of equations of the first type. As in the proves of 3.4.1.2 and of 3.4.18.4 (see also 3.4.18.8), due to the fact that Φ_R is essentially of additive type, we get that $f_1 \bmod p$ is zero. This ends the proof of the corollary.

3.4.18.5.2. Corollary (the liftability property). *We assume that* $k = \bar{k}$ *or that* Φ_R *is essentially of additive type. Let* $n \in \mathbb{N}$ *and let* $\mathfrak{C}$ *be an object (resp. a p-divisible object) of* $\mathcal{MF}_{[0,1]}(R)$*. Then any connection which makes* $\mathfrak{C}/p^n\mathfrak{C}$ *to be an object of* $\mathcal{MF}^{\nabla}_{[0,1]}(R)$*, can be lifted (not necessarily uniquely for the case* $k = \bar{k}$*) to a connection making* $\mathfrak{C}$ *to be an object (resp. a p-divisible object) of* $\mathcal{MF}^{\nabla}_{[0,1]}(R)$.

3.4.18.5.3. Corollary. *We assume that the Frobenius lift* Φ_R *is essentially of additive type. Then we have an equivalence of categories between* $\mathcal{MF}_{[0,1]}(R)$ *and* $\mathcal{MF}^{\nabla}_{[0,1]}(R)$*,*

and we have an antiequivalence of categories between these two categories and the category of finite flat group schemes of p-power rank over R.

For 3.4.18.5.2-3, cf. the proof of 3.4.18.4.2 and 3.4.18.5. We need to use the second paragraph of 3.4.18: any object of $\mathcal{MF}_{[0,1]}(R)$ lifts to a p-divisible object of $\mathcal{MF}_{[0,1]}(R)$.

3.4.18.5.4. Remark. It is easy to see that the results 3.4.18.5.1-3 extend to a more general context, inspired from 3.4.18.4.2. We leave this to the reader (to be compared to 3.4.18.8). 3.4.18.5.1 and 3.4.18.5.3 are the third place where we need $p > 2$.

3.4.18.5.5. A natural question arises. What about the categories $\mathcal{MF}_{[a,b]}(R)$, with $a, b \in \mathbb{Z}$, $b > a + 1$? The second philosophy of 3.4.18.2 says that it is enough to look at the case of a generic situation. The conclusion is: due to the reasons explained in 3.4.8.9, none of the results 3.4.18.4-5 above remain true, except one. We explain this exception.

Let $\mathfrak{C}$ be a p-divisible object of $\mathcal{MF}_{[a,b]}(R)$. As in 3.4.18.1 we define numbers $f(i) \in \mathbb{N} \cup \{0\}$, $\forall i \in \{a, ..., b\}$. Then the number of connections that make $\mathfrak{C}/p^n\mathfrak{C}$ potentially to be viewed as an object of $\mathcal{MF}^{\nabla}_{[a,b]}(R)$ is at most

$$p^{nr\sum_{i=a}^{b-1} f(i)f(i+1)},$$

cf. 3.4.8.9 and the estimate of m_1 in 3.4.8.1.

3.4.18.5.6. Problem. With the notations of 3.4.18.0-2, if $r \geq 1$, study the behaviour of connections (obtained as in 3.4.18.4) with respect to exact (short) sequences of objects.

3.4.18.6. Theorem. *Let now* $\mathrm{Spec}(R_1)$ *be a formally étale affine scheme over a smooth scheme over* $W(k)$*. We assume that its special fibre is connected, and that it is part of a triple* $(\mathrm{Spec}(R_1), \Phi_{R_1}, y)$ *defining a potential-deformation sheet. Let* $\mathfrak{C}_1$ *be a p-divisible object of* $\mathcal{MF}_{[0,1]}(R_1)$*.*

a) (the universal global ∇ principle for $\mathcal{MF}_{[0,1]}(*)$). *The whole of 3.4.1.3 remains true (except the fact that we get connections respecting the G-action which does not make any sense here) in this context. In particular* $\mathfrak{C}_1$ *becomes a p-divisible object of* $\mathcal{MF}^{\nabla}_{[0,1]}(Q_1)$*, with* Q_1 *a* $W(k)$*-algebra having a connected special fibre, and such that there is a morphism* $d_1 : \mathrm{Spec}(Q_1^{\wedge}) \to \mathrm{Spec}(R_1^{\wedge})$*, which is the p-adic completion of an* $\mathbb{N}$*-pro-étale morphism* ℓ_1*. Moreover the point y lifts to a point* $\mathrm{Spec}(W(k)) \to \mathrm{Spec}(Q_1^{\wedge})$*.*

b) (the surjectivity principle) *The fibres of* ℓ_1 *over an open subscheme of the special fibre of* $\mathrm{Spec}(R_1)$ *containing the k-valued point defined by y, are not empty.*

The same remains true for objects: we just need to replace the words $\mathbb{N}$*-pro-étale by the word étale (the b) part becomes trivial).*

The proof of this is entirely the same as the proof of 3.4.1.3 (cf. 3.4.18.4.1 for the integrability part), and so it is omitted.

3.4.18.7. The relative situation. 3.4.18.4-6 works in the relative context, i.e. when we start with a Shimura filtered σ-crystal, or even more generally with a filtered σ-$\mathcal{S}$-crystal of $\mathcal{MF}_{[0,1]}(W(k))$. So here we assume that there is a smooth subgroup $\tilde{G}$ of $GL(M)$ having connected fibres and the property that there is a family $(v_\alpha)_{\alpha \in \mathcal{J}}$ of the

F^0-filtration of $\mathcal{T}(M[\frac{1}{p}])$ fixed by φ and such that $\tilde{G}_{B(k)}$ is the subgroup of $GL(M[\frac{1}{p}])$ fixing its members. We have:

Fact. *If $g_R \in \tilde{G}(R)$, then there is an integrable connection ∇ on $M \otimes R$, which respects the $\tilde{G}$-action and makes $\mathfrak{C}$ to be viewed as a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$. Moreover, this p-divisible object is induced from a filtered F-$\mathcal{S}$-crystal of the form $(M, F^1, \varphi, \tilde{G}, \tilde{f})$ (defined as in 2.2.10).*

Proof: This is a consequence of 3.4.18.4.1 and [Fa2, rm, iii) after th. 10] (to be compared with 3.4.18.3), once we show that there is a connection on $M \otimes R$ which makes $\mathfrak{C}$ to be viewed as a p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$ and which respects the $\tilde{G}$-action. To see this, following the ideas of 3.4.8 4-7) and 11-12), we get that it is enough to show that there is a connection on $M \otimes R/pR$ which makes $\mathfrak{C}/p\mathfrak{C}$ to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$ (i.e. the liftability property works in this relative context also). We need to start with a conection $\nabla = \delta + \beta$, where δ is the connection on $M \otimes R/pR$ annihilating M, and where

$$\beta \in LIE := \mathrm{Lie}(\tilde{G}) \otimes \Omega_{(R/pR)/W(k)}.$$

The equations of 3.4.1.1.1 2), in this case, express the equality between two elements of LIE. So we get a system of $m \dim((\mathrm{Lie}(\tilde{G}))$ equations in the same number of variables; the resulting system of equations is of the first type, and so 3.4.8.1 applies. This ends the proof of the fact.

Warning: not all connections making $\mathfrak{C}/p\mathfrak{C}$ to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$ respect the $\tilde{G}$-action; simple examples are provided by the generic situation 3.4.18.2. However we have:

3.4.18.7.0. Formula. *The number of connections that respect the $\tilde{G}$-action and make $\mathfrak{C}/p^n\mathfrak{C}$ to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(R)$ is p to the power n times the number of slopes -1 of the Lie σ-crystal $(\mathrm{Lie}(\tilde{G}), \varphi)$.*

The proof of this formula, is entirely the same as the proof of 3.4.18.4 B).

3.4.18.7.1. It is worth detailing what 3.4.18.4.6 becomes in the relative context. There are two variants: one modelled on the faithful representation ρ of $\tilde{G}_k$ on M/pM, and another one modelled on $(M, F^1, \varphi, \tilde{G})$. We first detail the first one, then we hint the idea of the second one.

Let $\tilde{m} := \dim(M)$. Let $\bar{R}$ be a k-algebra. In the proof of 3.4.18.4.6 C we came across Artin-Schreier systems in n variables ($n \in \mathbb{N}$) with coefficients in $\bar{R}$, whose $n \times n$ matrix A_L is invertible. So A_L defines an element of $GL_n(\bar{R})$. If $\tilde{m} = n$, and if $A_L \in \tilde{G}_k(\bar{R})$ (under the representation ρ), then we say that such an Artin-Schreier system is modelled on ρ, and any connected component of the $\mathrm{Spec}(\bar{R})$-scheme defined by it is said to be a crystalline elementary ρ-admissible étale cover of $\mathrm{Spec}(\bar{R})$. As in 3.4.18.4.6 we define crystalline ρ-admissible étale covers of $\mathrm{Spec}(\bar{R})$, as well as Galois groups

$$\Pi^{\rho-AS}_{\tilde{m},n}$$

and

$$\Pi_{\tilde{m}}^{\rho-AS},$$

by using only Artin-Schreier systems which are modelled on ρ. Warning: all the Artin-Schreier systems modelled on ρ are in precisely $\tilde{m}$ variables. We refer to $\Pi_{\tilde{m}}^{\rho-AS}$ as the ρ Artin-Schreier fundamental group of $\mathrm{Spec}(\bar{R})$ (the index $\tilde{m}$ just having the role of keeping track of the dimension of the faithful representation ρ).

The most interesting cases are when ρ is an irreducible representation. For instance, we would like to understand the case when ρ is the spinorial (resp. semi-spinorial) representation of a split semisimple group over k of B_l (resp. of D_l) Lie type, with $l \in \mathbb{N}$, $l \geq 3$ (resp. $l \in \mathbb{N}$, $l \geq 4$).

The second variant immitates the first variant as well as the case described in the equations (3) and (4) of 3.4.18; in other words we require that some specific entries of A_L are zero, without requiring that A_L is invertible or that it defines an element of $\tilde{G}_k$. We intend to come back in a future paper to detail this second variant, as we think it is equally important as the first one.

3.4.18.8. Theorem (the universal unicity principle). *Let* $\mathrm{Spec}(R_1)$ *be a formally étale affine scheme over a smooth scheme over* $W(k)$ *such that* $\mathrm{Spec}(R_1/pR_1)$ *is a connected scheme. Let* Φ_{R_1} *be a Frobenius of* R_1 *compatible with the Frobenius* σ *of* $W(k)$*, and such that there is a maximal ideal* m_{R_1} *of* R_1*, with* R_1/m_{R_1} *a finite field extension of* k*, and with the property that the resulting Frobenius of the completion* $\widehat{R_1}$ *of* R_1 *with respect to it is essentially of additive type. Then for any object (or p-divisible object)* $\mathfrak{C}_1$ *of* $\mathcal{MF}_{[\tilde{a},\tilde{b}]}(R_1)$*, there is at most one connection which makes this object (resp. this p-divisible object) potentially to be viewed as an object (resp. as a p-divisible object) of* $\mathcal{MF}^{\nabla}_{[\tilde{a},\tilde{b}]}(R_1)$ *(here* $\tilde{a}, \tilde{b} \in \mathbb{Z}$*, with* $\tilde{a} < \tilde{b}$*).*

The proof of this is the same as the proof of 3.4.1.2: We work inductively moduko p^n, $n \in \mathbb{N}$. The difference from 3.4.8 6) is that we need to apply the same upper triangular trick used in the proof of 3.4.18.4 (for the resulting Frobenius of $\widehat{R_1}$) for "separating" the variables.

3.4.18.8.1. Theorem (the universal slope type unicity principle). *Let* $\mathrm{Spec}(R_1)$ *be a formally étale affine scheme over a smooth scheme over* $W(k)$ *such that* $\mathrm{Spec}(R_1/pR_1)$ *is a connected scheme. Let* Φ_1 *be a Frobenius of* R_1 *such that it leaves invariant the ideal of a* $W(k)$*-epimorphism* $R_1 \twoheadrightarrow W(k_1)$*, with* k_1 *a finite field extension of* k*. Let* $\mathfrak{C}_1$ *be a p-divisible object of* $\mathcal{MF}_{[0,1]}(R_1)$ *such that the p-divisible object of* $\mathcal{MF}_{[0,1]}(W(k_1))$ *obtained from it through this* $W(k)$*-epimorphism does not have slope zero or slope one. Then for any object (or p-divisible object)* $\mathfrak{C}'_1$ *of* $\mathcal{MF}_{[0,1]}(R_1)$ *which mod p is isomorphic to* $\mathfrak{C}_1/p\mathfrak{C}_1$*, there is at most one connection which makes it to be an object (resp. a p-divisible object) of* $\mathcal{MF}_{[0,1]}(R_1)$*.*

This is a consequence of the proof 3.4.18.4: everything (see the second paragraph of 3.4.18 and 3.4.18.5.2) gets reduced to the case of p-divisible objects.

3.4.18.8.2. Remark. We refer to 3.4.18.6. As usual we can speak about the p-rank of $\mathfrak{C}_1/p\mathfrak{C}_1$ in any maximal (or geometric) point of $\mathrm{Spec}(R_1/pR_1)$. The p-rank is zero in

such a point iff the F-crystal we get over a point of $\mathrm{Spec}(Q_1/pQ_1)$ lifting it does not have slope zero. So if $\mathfrak{C}_1$ has p-rank zero with respect to any maximal point of $\mathrm{Spec}(R_1/pR_1)$ then we do not need to pass to $\mathbb{N}$-pro-étale covers (or in the case of an object to étale covers). This is a consequence of 3.4.18.4 and of 3.4.18.4.2. Similarly, we can "detect" the number of slopes one by just looking at $\mathfrak{C}_1/p\mathfrak{C}_1$; their number is equal to the p-rank of $(\mathfrak{C}/p\mathfrak{C})^*(1)$.

3.4.18.8.3. Remark. 3.4.1.2-3 remain true for any Frobenius Φ_{R_j} of R_j such that $\Phi_{\hat{R}_j^0}$ is essentially of additive type. For an arbitrary Frobenius Φ_{R_j}, we lose the unicity part, so 3.4.1.2-3 have to be modified to a significant extend (for instance not all the connections we get are G-invariant, $\ell_{j,n}^{-1}(\mathcal{Z}_k)$ is not necessarily $\mathrm{Spec}(k)$, etc.). We do not stop to rewrite what remains true out of 3.4.1.2-3, in this general context.

3.4.18.9. Let R_1 be a regular formally smooth $W(k)$-algebra. We assume that $Y_1 := \mathrm{Spec}(R_1/pR_1)$ is connected and that the only open subscheme of Y_1 containing all the $\bar{k}$-valued points of Y_1, is Y_1 itself. We also assume the existence of a Frobenius Φ_{R_1} of $R_1^\wedge$ compatible with σ. Let $\mathcal{C}$ be an object of $\mathcal{MF}_{[0,1]}(R_1)$. We denote by $\nabla(\mathcal{C})$ the number of connections which make $\mathcal{C}$ to be viewed as an object of $\mathcal{MF}^\nabla_{[0,1]}(R_1)$.

Let $y \in Y_1(\bar{k})$. Either from the very definitions of 3.4.18.4.0-2, or from 3.4.18.4, we deduce that a similar number $r_y \in \mathbb{N}\cup\{0\}$ as in 3.4.18.1, can be defined starting just from Φ_{R_1}. As in 3.4.18.1 we define uniquely numbers $s_y(0), s_y(1) \in \mathbb{N}\cup\{0,1\}$. Let $n \in \mathbb{N}\cup\{0\}$ be the smallest number such that $p^n\mathcal{C} = 0$. Let

$$d_y(\mathcal{C}) := nr_y s_y(0)s_y(1).$$

From 3.4.18.4 we get:

$$\nabla(\mathcal{C}) \leq min\{d_y(\mathcal{C}) | y \in Y_1(\bar{k})\}.$$

From 3.4.18.4.2 and 3.4.8.1.2 a) we get directly:

Fact. *If the number $r_y s_y(0)s_y(1)$ does not depend on $y \in Y_1(\bar{k})$, and if the underlying module of $\mathcal{C}$ is a projective R_1/p^nR_1-module, then the special fibre of the moduli scheme $M(\mathcal{C})$ is an étale cover of Y_1.*

3.4.18.10. p-divisible objects of bounded ∇-deviations. We assume that R_1, Y_1 and Φ_{R_1} are as in 3.4.18.9. Let $\mathcal{C}$ be a p-divisible object of $\mathcal{MF}_{[0,1]}(R_1)$. We say that $\mathcal{C}$ is of bounded (resp. totally bounded) ∇-deviation if at least one connected component (resp. if all connected components) of the special fibre of $M(\mathcal{C})$ is (resp. are) of finite type over Y_1. The interesting cases are when Y_1 is of finite type over k, or when $R_1 = W(k)[[z_1, ..., z_m]]$, with $k \neq \bar{k}$. Plenty of examples of p-divisible objects of bounded ∇-deviation can be obtained following the strategy of 3.4.8.9.1. Similarly, using 3.4.18.8-9, we get plenty of examples of p-divisible objects of totally bounded ∇-deviation. Two problems are dear to us.

P1. In the interesting cases mentioned above, determine all p-divisible objects of bounded ∇-deviation.

P2. Determine conditions under which, in the case when R_1 is a smooth $W(k)$-algebra, a p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(R_1)$:

a) comes from a p-divisible group over R_1 (not only from one over $R_1^\wedge$);

b) viewed as an object of $\mathcal{MF}_{[0,1]}(R_1)$ is of totally bounded ∇-deviation.

3.4.19. The gluing principle. 3.4.14 and 3.4.14.1 are particular cases of a more general principle pertaining to Shimura p-divisible groups, which we call the gluing principle. However, we singled them out, as they are in the context of abelian varieties, are the most important cases needed in applications, and moreover we had to give a motivation for the expectation of 3.4.15.

A. Let $\mathrm{Spec}(R^{i\wedge})$ be the p-adic completion of a pro-étale affine scheme $\mathrm{Spec}(R^i)$ over a smooth scheme over $W(k)$, $i \in \overline{1,2}$. Let $z^i : \mathrm{Spec}(W(k)) \to \mathrm{Spec}(R^i)$, and let

$$\mathcal{D}^i = (D^i, (v^i_\alpha)_{\alpha\in\mathcal{J}})$$

be a Shimura p-divisible group over $\mathrm{Spec}(R^{i\wedge})$. We denote the p-adic completion of z^i still by z^i. We assume that:

i) the two Shimura p-divisible groups over $W(k)$ obtained from $\mathcal{D}^1$ and $\mathcal{D}^2$ through z^1 and respectively through z^2 are isomorphic;

ii) the relative dimension d_1 of $\mathrm{Spec}(R^1)$ in z^1 is bigger or equal to the relative dimension d_2 of $\mathrm{Spec}(R^2)$ in z^2; moreover $\mathcal{D}^i$ is a versal deformation in z^i;

iii) $\mathrm{Spec}(R^i/pR^i)$ is a connected scheme, $i = \overline{1,2}$.

Here the versality condition is defined as follows.

B. Definitions. Let $\mathrm{Spec}(R)$ be a formally étale scheme over a smooth scheme over $W(k)$. Let Φ_R be a Frobenius of $R^\wedge$ compatible with σ. A Shimura p-divisible group $\mathcal{D}_R$ over $\mathrm{Spec}(R)$ is said to be a versal deformation, if for each point $y_R : \mathrm{Spec}(k_1) \to \mathrm{Spec}(R/pR)$, with k_1 an algebraically closed field, the dimension of the image of the Kodaira-Spencer map attached to $\mathcal{D}_R$ in this point is equal to the deformation dimension of the Shimura σ_1 crystal we get naturally over k_1 through this point.

Similarly, an object or a p-divisible object of $\mathcal{MF}^\nabla_{[0,1]}(R)$ is said to be a quasi-versal deformation, if the Kodaira-Spencer map in each point y_R as above is injective, having an imagine whose dimension depends only on the connected component of $\mathrm{Spec}(R)$ through which y_R factors.

C. If $\mathcal{D}_i$ has a principal quasi-polarization $\mathcal{P}_{\mathcal{D}_i}$, then we say that we are in a principally quasi-polarized context; if we are in such a context then we assume that the isomorphism of i) respects the principal quasi-polarizations involved.

D. Let (M, F^1, φ, G) be the Shimura filtered σ-crystal associated to the two Shimura p-divisible groups obtained in i). Let $\mathrm{Spec}(\widehat{R^1})$ (resp. $\mathrm{Spec}(R^{1h})$) be the completion of $\mathrm{Spec}(R^1)$ (resp. the henselization of $\mathrm{Spec}(R^1)$) in z^1. We assume the existence of a formally smooth morphism $f_{12} : \mathrm{Spec}(\widehat{R^1}) \to \mathrm{Spec}(R^2)$ such that:

– $f_{12} \circ z^1 = z^2$;

– the image of R^2/pR^2 in $\widehat{R^1}/p\widehat{R^2}$ is contained in R^{1h}/pR^{1h};

– the pull back of $\mathcal{D}^2$ through f_{12} is isomorphic to the pull back of $\mathcal{D}^1$ through the natural morphism $\mathrm{Spec}(\widehat{R^1}) \to \mathrm{Spec}(R^1)$, through an isomorphism, lifting the one of i). Following the proves of 2.3.15 and 3.4.14.1 we can state (cf. also 3.4.2.2):

Theorem (the gluing principle: the global form). *There is an affine scheme* $\mathrm{Spec}(R^3)$ *over* $W(k)$*, with* $R^{3\wedge} = R^3$*, and there is a formally étale morphism* $q_1 : \mathrm{Spec}(R^3) \to \mathrm{Spec}(R^{1\wedge})$ *and a formally smooth morphism* $q_2 : \mathrm{Spec}(R^3) \to \mathrm{Spec}(R^{2\wedge})$*,* $i = \overline{1,2}$*, and there is a morphism* $a^3 : \mathrm{Spec}(W(k)) \to \mathrm{Spec}(R^3)$*, such that:*

a) $q_i \circ a^3 = z^i$*,* $i = \overline{1,2}$*;*

b) The pull back of $\mathcal{D}^1$ *through* q_1 *is isomorphic to the pull back of* $\mathcal{D}^2$ *through* q_2*, through an isomorphism lifting the one of i);*

c) $\mathrm{Spec}(R^3/pR^3)$ *is a connected scheme; it is a pro-étale scheme over an affine smooth* k*-scheme.*

If R^i *is a smooth* $W(k)$*-algebra (*$i \in \{1,2\}$*), then we can also assume that:*

d) the image of $\mathrm{Spec}(R^3/pR^3)$ *in* $\mathrm{Spec}(R^{3-i}/pR^{3-i})$ *contains an open dense subscheme of it, through which the point* $y^{3-i} : \mathrm{Spec}(k) \to \mathrm{Spec}(R^{3-i})$ *defined by* z^{3-i} *factors.*

The proof of this theorem is contained in the proof 3.4.14.1. Only two extra things we need to add. First, [BLR, Theorem 12, p. 83] applies as well, due to the following fact.

Fact. *If* X *is a projective limit of quasi-compact separated étale schemes* X_α*,* $\alpha \in \mathcal{J}(X)$*, over a smooth scheme over* $W(k)$*, then the categorie* $\mathcal{MF}^{\nabla}_{[0,1]}(X)$ *is the projective limit of the categories* $\mathcal{MF}^{\nabla}_{[0,1]}(X_\alpha)$*. If all these schemes are affine, and if we fix Frobenius lifts compatible with* σ *and with the projective system, then the same thing can be said for the categorie* $\mathcal{MF}(X)$*.*

The proof of this fact is standard (see [EGA]), based on [Fa1, 2.1 (ii)] and on the fact that any finitely generated projective module over a ring is finitely presented.

The second thing we need to add is related to the fact that d_i can be bigger than the deformation dimension d of the Shimura p-divisible groups of i) above. But the refinement part of the proof of 3.4.14.1 applies entirely in this situation. Using these two additions we get immediately this theorem.

E. Variants. There are many variants of the gluing principle:

iv) we work in a context involving Shimura F-crystals instead of Shimura p-divisible groups;

v) we weaken the part of ii) above involving the versality condition;

vi) we work in a principally quasi-polarized context;

vii) we work modulo a power of p.

We detail what we mean by this v). Let Q^i be the completion of R^i in z^i. Let $\tilde{D}^i$ be the pull back of $\mathcal{D}^i$ through the natural morphism $\mathrm{Spec}(Q^i) \to \mathrm{Spec}(R^i)$. Let Q^0 be the regular ring of the versal deformation space of the Shimura p-divisible $\mathcal{D}(W(k))$ showing up in i) above (see also 3.11 and [Va2, 5.4-5]). We get natural $W(k)$-homomorphisms $q_i : Q^0 \to Q^i$, $i = \overline{1,2}$. What we want to say in v): it is enough to assume that $\ker(q_1) = \ker(q_2)$, that over

dominant geometric points $\mathrm{Spec}(\tilde{k}) \to \mathrm{Spec}(Q^i/pQ^i)$ we get naturally Shimura-ordinary $\tilde{\sigma}$-crystals, and that the monomorphism $Q^0/\ker(q_i) \to Q^i$ is formally smooth. We express this fact (ment by v)) by: $\mathcal{D}^1$ and $\mathcal{D}^2$ unfold the same quasi-versal deformation of (the Shimura filtered σ-crystal underlying) $\mathcal{D}(W(k))$, which generically (in the special fibre) is Shimura-ordinary.

3.4.20. Final remarks for 3.4. 1) We might wonder why we preferred to work 3.4 with open subschemes of the group G, and not with open subschemes of the (abelian) subgroup of G acting trivially on F^0 and on M/F^1. The reason is: the full generality is useful in other situations, gives us more flexibility, and as we have seen a couple of times, the particular case (of an abelian group) can be obtained from the general case, by taking slices (cf. also §7).

2) We needed (3.1.8 and 3.5.14 are afterthoughts) the ∇ principle for Shimura σ-crystals for proving 3.1.0 and the conjecture of Milne (cf. 1.16) in their full generality (not only in the context of a SHS $(f, L_{(p)}, v)$, where it is not needed: 2.3.16 is enough, as the scheme $\mathcal{N}$ iteself provides the needed deformation). The other principles (of 3.4.18.4-8) are just a natural extension of it. The deformation principle is a logical continuation of the ∇-principle. Its importance stems from the fact that, when applies, we can take slices (cf. §7; see also the proof of 3.4.14.1) and its version 3.4.14.4 is very convenient for different global and local computations, cf. §10. Also the rigidity property (see 3.4.1.4 5) and §7) is a useful tool. We will use it in §8, in connection to 1.19. The gluing principle is our straightforward approach to prove the existence of integral canonical models of Shimura varieties of special type (cf. also [Va2, rm. 8) of 3.2.7]). No doubt we have been influenced (and inspired) by [Fa1, 7.1] and [Fa2, th. 10].

3) In parts of 3.4 we used [Fa2, th. 10], but not in 3.4.18.4. It is an easy exercise to check that using 3.4.18.4 and [Fa1, 7.1] we obtain another (completely different) proof of [Fa2, th. 10]. But the proof in the loc. cit. is of unsurpassable beauty. Also [Fa1, 7.1] can be deduced from [Fa2, th. 10] (Hint: follow the proof of 3.4.14.1; another approach is via 3.4.19 vii)).

4) There are variants of many of the above principles. For instance, by working with quasi p-divisible objects (these are extensions of objects by non-trivial p-divisible objects), or working with finitely generated modules, which are not necessarily projective (see 2.2.1 c)), or working with rings (algebras) which are not necessarily regular formally smooth (over $W(k)$). We do not stop to state the results in this generality, as anyone can state them easily, when needed.

5) It seems to us possible to use the ideas of 3.4 and to reobtain the result stated in [Va2, 3.2.2 1)], at least when we deal with a DVR faithfully flat over $\mathbb{Z}_{(p)}$, and of index of ramification 1. Even for $p = 2$ we hope to gain in this way a good understanding of which regular formally smooth scheme over such a DVR are healthy regular schemes.

6) It seems to us (see also 3.4.18.10) that there is a great similarity between 3.4.1.3, the theory of p-jets of A. Buium (see [Bu]), and the theory of almost étale extensions of G. Faltings (see [Fa1] and [Fa3]). We think that it is possible to obtain significant progress in the related areas by "fitting" these three theories together.

7) We view 3.4.8.1 (the part $s = 0$) and 3.4.8.1.2 c) as the algebraic analogue of the well known theorem of S. Lang on connected algebraic groups over finite fields (see [Bo, 16.3-4]).

8) There are many other ways (besides 3.4.19 B) to express the versality condition of 3.4.19 ii). One way is to use the proof of 3.12.1. Another way (more fashionable) is inspired from 2.3.17: any Shimura p-divisible group over $W(k)$, whose Shimura σ-crystal over k is isomorphic to the one obtained from $\mathcal{D}^i$ through the k-valued point of $\mathrm{Spec}(R^{i\wedge})$ defined by z^i, is induced from $\mathcal{D}^i$ through a $W(k)$-valued point of $\mathrm{Spec}(R^{i\wedge})$. The fact that all these definitions are equivalent is proved in the same manner as for the case of a SHS (see 2.3.17 and [Va2, 5.4-5]).

9) In §6 we use the gluing principle (see 3.4.2.2 and 3.4.19 D) to construct the theory of Shimura envelopes. This theory generalizes the theory of (quotients of) integral canonical models of Shimura varieties.

10) In positive characteristic, regardless of the fact that we are interested in some fundamental group (see 3.4.18.4.6 C), or on some connection (see the whole of 3.4.8), or on some stratification (see 3.4.8.1.4 and 4.5.17), or on some deformation (see 3.4.14), or on some homomorphism between two p-divisible groups (see the proves of 3.4.14.1 and 3.11.1), the quasi Artin-Schreier systems are at the root of the subject.

3.5. Split Shimura Lie σ-crystals. We come back to the situation mentioned at the beginning of 3.3: G is split and $g_1 \in G(W(k))$. We have:

$$\mathfrak{g}[\frac{1}{p}] = \mathfrak{z}[\frac{1}{p}] \bigoplus \oplus_{i\in I} \mathfrak{g}_{\mathfrak{i}}[\frac{1}{p}]$$

as Lie algebras, where $\mathfrak{z}$ is the Lie algebra of the connected component $Z(G)^0$ of the origin of the center $Z(G)$ of G, and where $\mathfrak{g}_{\mathfrak{i}}$ is the Lie algebra of a semisimple subgroup G_i of G, such that

$$G^{\mathrm{ad}} = \prod_{i\in I} G_i^{\mathrm{ad}}$$

(cf. [Ti1, p. 46]), with G_i^{ad} absolutely simple split adjoint groups, $\forall i \in I$. We have an injective linear map

$$\mathfrak{z} \oplus \bigoplus_{i\in I} \mathfrak{g}_{\mathfrak{i}} \hookrightarrow \mathfrak{g}$$

which becomes an isomorphism by inverting p, but which is not always surjective (as we have assumed $p \geq 3$, the failure of surjectivity can happen only when $p = 3$ and there are factors G_i of E_6 Lie type or when p is arbitrary and there are factors G_i of A_ℓ Lie type, with $p|\ell + 1$). We have an isogeny

$$Z(G)^0 \prod \prod_{i\in I} G_i \to G.$$

3.5.0. The σ-linear endomorphism φ_1 of M leaves invariant $\mathfrak{z}$ and permutes the factors $\mathfrak{g}_i \otimes B(k)$, $i \in I$, producing a permutation γ of I. We write

$$\gamma = \prod_{s\in\mathcal{J}_0} \gamma_s$$

as a product of disjoint cyclic permutations γ_s. The σ-crystal $(\mathfrak{z}, \varphi_1)$ has all the slopes zero. We start working with just one cycle γ_0 of γ (so $0 \in \mathcal{J}_0$) permuting cyclically a subset I_0 of I. We can assume that $I_0 = \{1, 2, \dots, n\}$ and that γ_0 is the permutation: $i \to i+1$ (with $n+1 = 1$). Let

$$\mathfrak{g}_0 := \bigoplus_{i \in I_0} \mathfrak{g}_i.$$

Let us fix an element $g_2 \in G(W(k))$ and let $\varphi_2 := g_2 \circ \varphi_1$.

Let $\bar{h} \in \mathfrak{t}$ be the element defined by: $\bar{h}(x) = 0$ if $x \in F^0$, and $\bar{h}(x) = x$ if $x \in F^1$. We recall that the cocharacter $\mu : \mathbb{G}_m \to G$ produces a direct sum decomposition $M = F^1 \oplus F^0$, with $\beta \in \mathbb{G}_m(W(k))$ acting through μ on F^i as the multiplication with β^{-i}, $i = \overline{0,1}$; so

$$\bar{h} \in \mu(\mathrm{Lie}(\mathbb{G}_m)) \subset \mathrm{Lie}(T) = \mathfrak{t}.$$

Let $\bar{h}_i$ be the component of $\bar{h}$ in $\mathfrak{g}_i\big[\frac{1}{p}\big]$ (corresponding to the direct sum decomposition $\mathfrak{g}[\frac{1}{p}] = \mathfrak{z}[\frac{1}{p}] \bigoplus \oplus_{i \in I} \mathfrak{g}_i[\frac{1}{p}]$). If $\bar{h}_i = 0$ then $\mathfrak{g}_i \subset F^0(\mathfrak{g})$. If $\bar{h}_i \neq 0$ then $\bar{h}_i$ is a semisimple element of $\mathfrak{g}_i\big[\frac{1}{p}\big]$. When viewed as an endomorphism of $\mathfrak{g}_i[\frac{1}{p}]$ (via $z_i \in \mathfrak{g}_i$ is mapped into $[\bar{h}_i, z_i]$), its eigenvalues are precisely -1, 0 and 1.

We have $F^1(\mathfrak{g}) = \big\{x \in \mathfrak{g} \;\big|\; [\bar{h}, x] = x\big\}$, and so

$$F^1(\mathfrak{g}_i) := F^1(\mathfrak{g}) \cap \mathfrak{g}_i = \big\{x \in \mathfrak{g}_i \;\big|\; [\bar{h}_i, x] = x\big\}.$$

Similarly, $F^0(\mathfrak{g}) = \big\{x \in \mathfrak{g} \;\big|\; [\bar{h}, x] = 0 \text{ or } x\big\}$, and so

$$F^0(\mathfrak{g}_i) := \mathfrak{g}_i \cap F^0(g) = \big\{x \in \mathfrak{g}_i \;\big|\; [\bar{h}_i, x] = 0 \text{ or } x\big\}.$$

Let $F^j(\mathfrak{g}_0) := \bigoplus_{i \in I_0} F^j(\mathfrak{g}_i)$, $j = \overline{0,1}$. We get:

3.5.1. Lemma. *The quadruple* $\big(\mathfrak{g}_0, \varphi_j, F^0(\mathfrak{g}_0), F^1(\mathfrak{g}_0)\big)$ *is a Shimura filtered Lie* σ*-crystal,* $j = \overline{1,2}$.

Proof: $\varphi_j\big(\frac{1}{p}F^1(\mathfrak{g}_i) + F^0(\mathfrak{g}_i) + p\mathfrak{g}_i\big)$ is a direct summand of $\mathfrak{g}$ of rank $\dim_{W(k)}(\mathfrak{g}_i) = \dim_{W(k)}(\mathfrak{g}_{i+1})$, and is included in $\mathfrak{g}_{i+1}\big[\frac{1}{p}\big]$; so it is $\mathfrak{g}_{i+1}$ ($\forall i \in I_0$, $\forall j \in \{1,2\}$). So $\big(\mathfrak{g}_0, \varphi_j, F^0(\mathfrak{g}_0), F^1(\mathfrak{g}_0)\big)$ is a p-divisible object of $\mathcal{MF}_{[-1,1]}(W(k))$. It is a Shimura filtered Lie σ-crystal, as the cocharacter of G^{ad}, obtained by composing μ with the natural homomorphism $G \to G^{\mathrm{ad}}$, can be lifted to a cocharacter of the isogeny cover $\times_{i \in I} G_i$ of G^{ad}. This ends the proof.

3.5.1.1. Remark. We can work equally well 3.5.1 (as well as all that follows) with $\mathrm{Lie}\big(\sum_{i \in I_0} G_i\big)$ (we have $\mathfrak{g}_0 \subset \mathrm{Lie}(\sum_{i \in I_0} G_i) \subset \mathfrak{g}_0[\frac{1}{p}]$), or with $\mathrm{Lie}\big((\sum_{i \in I_0} G_i)^{\mathrm{ad}}\big) = \oplus_{i \in I_0} \mathrm{Lie}(G_i^{\mathrm{ad}})$ instead of $\mathfrak{g}_0$ (cf. 2.2.13) (here $\sum_{i \in I_0} G_i$ is the subgroup of G generated by G_i, $i \in I_0$).

3.5.1.2. We assume that $F^1(\mathfrak{g}_0) \neq 0$, otherwise there is nothing to be done (cf. 3.2.2).

3.5.1.3. Let $\tilde{G}_0$ be the subgroup of G generated by the torus T and the semisimple subgroups G_i, $i \in I_0$. So $\tilde{G}_0^{\mathrm{ad}} = \times_{i\in I_0} G_i^{\mathrm{ad}}$. We have: $\mathfrak{g}_0$ is a Lie subalgebra of $\mathrm{Lie}(\tilde{G}_0)$. Moreover a $G(W(k))$-conjugate μ_1 of μ factors through $\tilde{G}_0$, giving birth to a Shimura group pair $(\tilde{G}_0, [\mu_1])$ over $W(k)$, and to a Shimura σ-crystal $(M, \varphi_1, \tilde{G}_0)$ (cf. 2.2.9 1)).

3.5.2. Let $\mathfrak{H}_2$ (resp. $\mathfrak{H}_1$) be the set of slopes of the isocrystal $\big(\mathfrak{g}_0[\frac{1}{p}], \varphi_2\big)$ (resp. of the isocrystal $\big(\mathfrak{g}_0[\frac{1}{p}], \varphi_1\big)$). The sets $\mathfrak{H}_2$ and $\mathfrak{H}_1$ are subsets of the real interval $[-1, 1]$. We have direct sum decompositions

$$\mathfrak{g}_0[\frac{1}{p}] = \bigoplus_{\alpha\in\mathcal{H}_2} g_2(\alpha)$$

and

$$\mathfrak{g}_0[\frac{1}{p}] = \bigoplus_{\alpha\in\mathcal{H}_1} g_1(\alpha),$$

with $g_j(\alpha)$ as the maximal $B(k)$-vector subspace of $\mathfrak{g}_0\big[\frac{1}{p}\big]$ left invariant by φ_j, and such that the isocrystal $\big(g_j(\alpha), \varphi_j\big)$ has pure slope α, $j = \overline{1,2}$. As φ_1^n and φ_2^n leave invariant any one of $\mathfrak{g}_i$, $i \in I_0$, we deduce that:

$$g_j(\alpha) = \bigoplus_{i\in I_0} g_j(\alpha) \cap \mathfrak{g}_i\big[\frac{1}{p}\big], \tag{3.5.2.1}$$

$\forall j \in \{1, 2\}$, and $\forall \alpha \in \mathcal{H}_j$. As φ_j maps $g_j(\alpha) \cap \mathfrak{g}_i[\frac{1}{p}]$ bijectively onto $g_j(\alpha) \cap \mathfrak{g}_{i+1}[\frac{1}{p}]$, $i \in I_0$, we have:

(3.5.2.2) *For $j \in \{1, 2\}$, the dimension of $g_j(\alpha) \cap \mathfrak{g}_i[\frac{1}{p}]$ does not depend on $i \in I_0$.*

3.5.3. Let $\mathfrak{t}_i$ ($\mathfrak{b}_i$) be the component of $\mathfrak{t}$ (resp. of $\mathfrak{b}$) in $\mathfrak{g}_i$, i.e. $\mathfrak{t}_i = \mathfrak{g}_i \cap \mathfrak{t}\big[\frac{1}{p}\big]$ (resp. $\mathfrak{b}_i = \mathfrak{g}_i \cap \mathfrak{b}\big[\frac{1}{p}\big]$). Let Δ_i be the base of roots of $\mathfrak{g}_i$ with respect to the maximal torus T_i of G_i having $\mathfrak{t}_i$ as its Lie algebra, corresponding to the Borel Lie subalgebra $\mathfrak{b}_i$ of $\mathfrak{g}_i$. Let Φ_i be the set of roots of $\mathfrak{g}_i$ with respect to the torus T_i. Then Φ_i^+ (the set of positive roots) and Φ_i^- (the set of negative roots) are defined with respect to Δ_i. Here $i \in I$.

3.5.3.1. φ_1^n leaves invariant $\mathfrak{g}_1[\frac{1}{p}]$, $\mathfrak{b}_1$ and $\mathfrak{t}_1$, and so it induces a permutation π_1 of Δ_1. Let e be the order of π_1. It is very convenient to work simultaneously with the Shimura Lie σ-crystal

$$(\mathfrak{g}_0^e, \varphi_j^{(e)}),$$

where $\mathfrak{g}_0^e := \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_0$ (e-factors), and where $\varphi_j^{(e)}$ is the Frobenius automorphism of $\mathfrak{g}_0^e$ acting like φ_j on any direct summand $\mathfrak{g}_i$, $i \in I_0$, of a $\mathfrak{g}_0$ copy of $\mathfrak{g}_0^e$, but sending the $\mathfrak{g}_n$ direct summand of the s-th $\mathfrak{g}_0$ copy to the $\mathfrak{g}_1$ direct summand of the $(s+1)$-th $\mathfrak{g}_0$ copy, $\forall s \in \mathbb{Z}/e\mathbb{Z}$. The set of slopes of $(\mathfrak{g}_0^e, \varphi_j^{(e)})$ is the set of slopes of $(\mathfrak{g}_0, \varphi_j)$, all multiplicities being e-times more, $j = \overline{1,2}$.

The permutation of Δ_1 defined by $(\varphi_1^{(e)})^{ne}$ is the trivial one.

e can be 1, 2 or 3. It can be 3 only in the case when $\mathfrak{g}_1$ is of D_4 Lie type, and it can be to 2 only when $\mathfrak{g}_1$ is of A_ℓ, D_ℓ or E_6 Lie type (and then π_1 is the usual involution of Δ_1).

Let $\tilde{I}_0 := \{1, ..., en\}$. We like to think of $\mathfrak{g}_0^e$ as a direct sum $\oplus_{i\in\tilde{I}_0}\mathfrak{g}_i$. Here for $s \in \{1, ..., e-1\}$ and $i \in I_0$, $\mathfrak{g}_{i+sn} := \mathfrak{g}_i$; similarly $\mathfrak{t}_{i+sn} := \mathfrak{t}_i$, $\Delta_{i+sn} := \Delta_i$, $\mathfrak{b}_{i+sn} := \mathfrak{b}_i$, and $\bar{h}_{i+sn} := (\varphi_1^e)^{sn}(\bar{h}_i) \in \mathfrak{g}_{i+sn}[\frac{1}{p}]$.

3.5.3.2. Let ℓ be the rank of $\mathfrak{g}_1$. For $i \in \tilde{I}_0$ let $\alpha_1(i), \ldots, \alpha_\ell(i)$ be the simple roots of Δ_i, numbered in such a way that φ_1 sends the (one dimensional) $B(k)$-vector subspace of $\mathfrak{g}_i[\frac{1}{p}]$ corresponding to $\alpha_s(i)$, $s \in \{1, ..., \ell\}$, to the (one dimensional) $B(k)$-vector subspace of $\mathfrak{g}_{i+1}[\frac{1}{p}]$ corresponding to $\alpha_s(i+1)$ (with $en+1 = 1$), and such that the numbering for $\mathfrak{g}_1$ is the one from [Bou, planche I-VI]. The maximal root of Δ_1 (dropping the lateral right index (1) from now on) depends on the Lie type of $\mathfrak{g}_1$. It is (cf. loc. cit.):

a) $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell$ if $\mathfrak{g}_1$ is of A_ℓ Lie type ($\ell \geq 2$);

b) $\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_\ell$ if $\mathfrak{g}_1$ is of B_ℓ Lie type ($\ell \geq 1$);

c) $2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$ if $\mathfrak{g}_1$ is of C_ℓ Lie type ($\ell \geq 2$);

d) $\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$ if $\mathfrak{g}_1$ is of D_ℓ Lie type ($\ell \geq 4$);

e6) $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ if $\mathfrak{g}_1$ is of E_6 Lie type;

e7) $2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$ if $\mathfrak{g}_1$ is of E_7 Lie type.

$\mathfrak{g}_1$ is one of these Lie types, cf. 2.2.7 and 3.5.1.2-3.

We usually prefer to consider $B_\ell (\ell \geq 1)$ and $A_\ell (\ell \geq 2)$, i.e. we prefer to consider B_1 instead of A_1.

3.5.3.3. Let $\mathfrak{g}_1 = \mathfrak{t}_1 \oplus \bigoplus_{\beta\in\Phi_1} \mathfrak{g}_\beta$ be the Weyl direct sum decomposition associated to the maximal split torus T_1 of G_1. So T_1 acts on $\mathfrak{g}_\beta$ through the character defining the root β.

In cases b), c) and e7), $\bar{h}_1 \in \mathfrak{t}_1$ is either zero or is defined by:

— in case b), $\bar{h}_1$ acts as zero on $\mathfrak{g}_{\alpha_s}$, $s \in \{2, ..., \ell\}$, and as identity on $\mathfrak{g}_{\alpha_1}$;

— in case c), $\bar{h}_1$ acts as zero on $\mathfrak{g}_{\alpha_s}$, $s \in \{1, ..., \ell-1\}$, and as identity on $\mathfrak{g}_{\alpha_\ell}$;

— in case e7), $\bar{h}_1$ acts as zero on $\mathfrak{g}_{\alpha_s}$, $s \in \{1, ..., 7\}$, and as identity on $\mathfrak{g}_{\alpha_7}$.

In case a), $\bar{h}_1 \in \mathfrak{t}_1$ is either zero or there is $s_1 \in \{1, 2, \ldots, \ell\}$ such that it acts as zero on $\mathfrak{g}_{\alpha_s}$, $s \in \{1, ..., \ell\} \setminus \{s_1\}$, and as identity on $\mathfrak{g}_{\alpha_{s_1}}$ (we have ℓ non-zero possibilities for $\bar{h}_1$).

In case d), $\bar{h}_1 \in \mathfrak{t}_1$ is either zero or there is $s_1 \in \{1, \ell-1, \ell\}$ such that it acts as zero on $\mathfrak{g}_{\alpha_s}$, $s \in \{1, ..., \ell\} \setminus \{s_1\}$, and as identity on $\mathfrak{g}_{\alpha_{s_1}}$ (we have 3 non-zero possibilities for $\bar{h}_1$).

In case e6), $\bar{h}_1$ is either zero or there is $s_1 \in \{1, 6\}$ such that it acts as zero on $\mathfrak{g}_{\alpha_s}$, $s \in \{1, ..., 6\} \setminus \{s_1\}$, and as identity on $\mathfrak{g}_{\alpha_{s_1}}$ (we have 2 non-zero possibilities for $\bar{h}_1$).

3.5.3.4. For $i \in \tilde{I}_0$, we define $\varepsilon_i = 0$ if $\bar{h}_i = 0$, and $\varepsilon_i = s$ if $\bar{h}_i$ acts as identity on the rank one direct summand of $\mathfrak{g}_i$ corresponding to the root $\alpha_s(i)$; here $s \in \{1, ..., \ell\}$. So, fixing i, ε_i can take more than one non-zero value only if we are in the cases a), d) or e6).

The assumption $F^1(\mathfrak{g}_0) \neq 0$ implies that the set

$$I_1 := \{i \in I_0 \mid \varepsilon_i \neq 0\}$$

is not empty. Let

$$\tilde{I}_1 := \{i \in \tilde{I}_0 \mid \varepsilon_i \neq 0\}.$$

3.5.4. We have:

Fact. *The sets $\mathcal{H}_1$ and $\mathcal{H}_2$ are subsets of $\left[-\frac{|I_1|}{n}, \frac{|I_1|}{n}\right]$. The smallest element of $\mathcal{H}_1$ is*

$$\delta := -\frac{1}{n}\,|I_1|\,.$$

For instance $\mathfrak{g}_{-\beta}$, with $\beta \in \Phi_1^+$ the maximal root with respect to the base Δ_1, is included in $g_1(\delta)$.

3.5.5. Let $\Psi_j^0 : \mathfrak{g}_0 \to \mathfrak{g}_0$, $j = \overline{1,2}$, be the σ-linear maps defined by: if $x \in \mathfrak{g}_i$, then

i) $\Psi_j^0(x) = \varphi_j(x)$ if $\varepsilon_i = 0$;

ii) $\Psi_j^0(x) = p\varphi_j(x)$ if $\varepsilon_i \neq 0$.

Taking everything mod p, we get maps $\bar{\psi}_j^0 : \mathfrak{g}_0/p\mathfrak{g}_0 \to \mathfrak{g}_0/p\mathfrak{g}_0$, $j = \overline{1,2}$. We have $\Psi_2^0 = g_2\Psi_1^0$ and $\bar{\psi}_2^0 = g_2\bar{\psi}_0^0$.

We can assume that $\varepsilon_1 \neq 0$. Let

$$N_2 := g_2(\delta) \cap \mathfrak{g}_1.$$

For any $i \in \tilde{I}_1$, let $\mathfrak{p}_{\varepsilon_i}$ be the parabolic Lie subalgebra of $\mathfrak{g}_1$ containing $\mathfrak{b}_1$, and such that its nilpotent radical $\mathfrak{s}_{\varepsilon_i}$ is the $W(k)$-submodule of $\mathfrak{g}_1$ generated by $\mathfrak{g}_\beta$, with $\beta \in \Phi_1^+$ having $+\alpha_{\varepsilon_i}$ in their expression. Let $\mathfrak{q}_{\varepsilon_i}$ be the parabolic Lie subalgebra of $\mathfrak{g}_1$, opposite to $\mathfrak{p}_{\varepsilon_i}$ (with respect to $\mathfrak{t}_i$ and Δ_i), and let $\mathfrak{r}_{\varepsilon_i}$ be its nilpotent radical. The multiplicity m_δ of the slope δ for $(\mathfrak{g}_0, \varphi_1)$ is n times the rank of the intersection $\bigcap_{i\in I_1} \mathfrak{r}_{\varepsilon_i}$ (as $(\mathfrak{g}_0, \varphi_1)$ gives birth to a 0-symmetric isocrystal). So we have:

$$m_\delta = \dim_{B(k)} g_1(\alpha).$$

The multiplicity $\dim_{B(k)} g_2(\alpha)$ of the slope δ for $(\mathfrak{g}_0, \varphi_2)$ is n times the rank of N_2 (cf. 3.5.2.2). We denote by $\mathfrak{p}_{\varepsilon_i}(i)$, $\mathfrak{q}_{\varepsilon_i}(i)$, $\mathfrak{r}_{\varepsilon_i}(i)$, respectively $\mathfrak{s}_{\varepsilon_i}(i)$ the intersection of $\mathfrak{g}_i$ with $\varphi_1^{i-1}\big(\mathfrak{p}_{\varepsilon_i}[\frac{1}{p}]\big)$, $\varphi_1^{i-1}\big(\mathfrak{q}_{\varepsilon_i}[\frac{1}{p}]\big)$, $\varphi_1^{i-1}\big(\mathfrak{r}_{\varepsilon_i}[\frac{1}{p}]\big)$, and respectively with $\varphi_1^{i-1}\big(\mathfrak{s}_{\varepsilon_i}[\frac{1}{p}]\big)$, $i = \overline{1, en}$.

3.5.6. Theorem. *If $\delta \in \mathcal{H}_2$, then the multiplicity of the slope δ of the Shimura Lie σ-crystal $(\mathfrak{g}_0, \varphi_2)$ is less or equal to m_δ. If it is m_δ, then $\forall i \in I_1$, we have*

$$\Big(\Big(\bigoplus_{\alpha\in\mathcal{H}_2\cap[0,1]} g_2(\alpha)\Big) \cap \mathfrak{g}_i\Big) \otimes k \subset \mathfrak{p}_{\varepsilon_i}(i) \otimes k. \tag{3.5.7}$$

Moreover, the fact that this multiplicity is m_δ or not depends only on the expression of g_2 mod p.

Before proving this, let us see first some of its immediate implications. Not to complicate the notations by introducing extra indices, in 3.5.8-12 we allow g_2 to vary as needed.

3.5.8. Proposition. *The Newton polygon of the Shimura Lie σ-crystal $(\mathfrak{g}_0, \varphi_1)$ is not strictly above the Newton polygon of any other Shimura Lie σ-crystal of the form $(\mathfrak{g}_0, g_2\varphi_1)$, with $g_2 \in \tilde{G}_0(W(k))$.*

Proof: We assume that 3.5.8 is false; let $g_2 \in \tilde{G}_0(W(k))$ for which 3.5.8 is not true. As $\delta \in \mathcal{H}_1$, from 3.5.4 and 3.5.6 we deduce that the multiplicity of the slope $\delta \in \mathcal{H}_2$ for $(\mathfrak{g}_0, \varphi_2)$ is m_δ.

Let P_2 be the parabolic subgroup of $\tilde{G}_0$ whose Lie algebra is the parabolic Lie subalgebra of $\mathrm{Lie}(\tilde{G}_0)$ corresponding to non-negative slopes of $(\mathrm{Lie}(\tilde{G}_0), \varphi_2)$. $p\varphi_2$ leaves invariant $\mathrm{Lie}(P_2)$. From 3.5.7 we deduce that P_{2k} is included in P_k^0 (with P^0 as the parabolic subgroup of $\tilde{G}_0$ having $F^0(\mathfrak{g}) \cap \mathrm{Lie}(\tilde{G}_0)$ as its Lie algebra). We deduce the existence of an element $g_3 \in \mathrm{Ker}\big(\tilde{G}_0(W(k)) \to \tilde{G}_0(k)\big)$ such that $P_3 = g_3 P_2 g_3^{-1} \subset P^0$. We can assume that $\mathfrak{b} \cap \mathrm{Lie}(\tilde{G}_0)$ is included in $\mathrm{Lie}(P_3)$.

Let $\varphi_3 := g_3\varphi_2 g_3^{-1}$. We can write $\varphi_3 = g_4\varphi_1$, with $g_4 \in \tilde{G}_0(W(k))$. Obviously $(\mathfrak{g}_0, \varphi_3)$ and $(\mathfrak{g}_0, \varphi_2)$ have the same Newton polygon. Let B_0 be the Borel subgroup of $\tilde{G}_0$ whose Lie algebra contains $\oplus_{i\in I_0}\mathfrak{b}_i$. We have $B_0 \subset P^0$. Repeating the argument of 3.2.3, we deduce the existence of an element $g_5 \in P_3(W(k))$ such that $\varphi_4 = g_5\varphi_3$ takes $\mathfrak{t}$ onto $\mathfrak{t}$ and $\mathrm{Lie}(B_0)$ into $\mathrm{Lie}(B_0)$. We get $\varphi_4^s = p_{3,s}\varphi_3^s$, $\forall s \in \mathbb{N}$, with $p_{3,s} \in P_3(W(k))$, and that $\varphi_4^s = b_s\varphi_1^s$, $\forall s \in \mathbb{N}$, with $b_s \in B_0(W(k))$. This implies that $\mathrm{Lie}(P_3) = \mathrm{Lie}(\tilde{G}_0 \cap P_1)$, cf. 3.2.1.1; so $P_3 = \tilde{G}_0 \cap P_1$.

From 3.2.1.1 we get also that $(\mathfrak{g}_0, \varphi_2)$ and $(\mathfrak{g}_0, \varphi_1)$ have the same Newton polygon. Contradiction. This ends the proof of the proposition.

3.5.9. Corollary. *If $g_2 \in \tilde{G}_0(W(k))$ is such that the Shimura Lie σ-crystals $(\mathfrak{g}_0, g_2\varphi_1)$ and $(\mathfrak{g}_0, \varphi_1)$ have the same Newton polygon, then $(M, g_2\varphi_1)$ and (M, φ_1) have the same Newton polygon, and $g_2\varphi_1$ is of the form $g_3 p_1 \varphi_1 g_3^{-1}$, with $p_1 \in P_{10}(W(k))$ where P_{10} is the parabolic subgroup of $\tilde{G}_0$ having $\mathrm{Lie}(\tilde{G}_0) \cap \mathrm{Lie}(P_1)[\frac{1}{p}]$ as its algebra, and with $g_3 \in \tilde{G}_0(W(k))$ such that it leaves invariant $F^1 \otimes k$.*

3.5.10. Corollary. *If $h_0 \in \tilde{G}_0(W(k))$, then the multiplicity of the slope δ for $(\mathfrak{g}_0, h_0\varphi_1)$ is less or equal to m_δ. If we have equality, then $(\mathfrak{g}_0, h_0\varphi_1)$ and $(\mathfrak{g}_0, \varphi_1)$ have the same Newton polygon.*

3.5.9-10 are consequences of the proof of 3.5.8. 3.5.11-12 result from 3.5.6-10, by working with all cyclic permutations γ_s, $s \in \mathcal{J}_0$, at once.

3.5.11. Corollary. *If $g_2 \in G(W(k))$ is such that the Shimura Lie σ-crystals $(\mathfrak{g}, g_2\varphi_1)$ and $(\mathfrak{g}, \varphi_1)$ have the same Newton polygon, then the σ-crystals $(M, g_2\varphi_1)$ and (M, φ_1) have the same Newton polygon, and $g_2\varphi_1 = g_3 p_1 \varphi_1 g_3^{-1}$, with $g_3 \in G(W(k))$ leaving invariant $F^1 \otimes k$, and with $p_1 \in P_1(W(k))$. Moreover the fact that $(\mathfrak{g}, g_2\varphi)$ has the same Newton polygon as $(\mathfrak{g}, \varphi_1)$ depends only on the expression of g_2 mod p.*

3.5.12. Corollary. *The Newton polygon of the Shimura Lie σ-crystal $(\mathfrak{g}, \varphi_1)$ is not strictly above the Newton polygon of any other Shimura Lie σ-crystal of the form $(\mathfrak{g}, \mathfrak{g}_2\varphi_1)$, with $g_2 \in G(W(k))$.*

3.5.13. Remark. 3.5.11-12 remain true if we replace φ_1 by φ_0, and do not assume that k is algebraically closed (assumption made at the beginning of 3.5). This is an easy consequence of 3.3.2 and of the fact that two parabolic subgroups of G which over $W(\bar{k})$ are $G(W(\bar{k}))$-conjugate are $G(W(k))$-conjugate (this results easily from [Bo, 20.9] and [Ti, 3.3.2]). So in 3.5.11 (as every parabolic subgroup is its own normalizer), if $g_2 \in G(W(k))$, then we can choose $g_3 \in G(W(k))$ (not just $g_3 \in G(W(\bar{k}))$); so we get $g_2\varphi_0 = g_3 p_0 \varphi_0 g_3^{-1}$, with g_3 mod p leaving invariant $F^1 \otimes k$, and so $p_0 \in G(W(k)) \cap P_0(W(\bar{k})) = P_0(W(k))$. This proves 3.3.4 and ends the proof of 3.2.5.

3.5.14. Exercise. Using just the ideas and results of 3.1.8.1 and 3.5.6-12 prove the whole of 3.1.0 (i.e. end the proof of 3.2.6-7).

This exercise ends the first proof of 3.2.6-7, and so (cf. 3.2.4) of 3.1.0.

3.6. The proof of 3.5.6. With the notations of 3.5.4, for proving the part of 3.5.6 not involving the expression of g_2 mod p, it is enough to show that

$$\dim_{W(k)}(N_2) \leq \dim_{W(k)}(\bigcap_{i \in \tilde{I}_1} \mathfrak{r}_{\varepsilon_i}) = \frac{m_\delta}{n}, \tag{0}$$

and, if equality holds, then (3.5.7) is true, with $i = 1$.

For $i \in \tilde{I}_0$ let $\bar{\mathfrak{g}}_i := \mathfrak{g}_i \otimes k$. For every $i \in \tilde{I}_1$, let Π_{ε_i} be the projector of $\bar{\mathfrak{g}}_1$ of whose kernel is $\bar{\mathfrak{p}}_{\varepsilon_i} := \mathfrak{p}_{\varepsilon_i} \otimes k$ and of whose image is $\bar{\mathfrak{r}}_{\varepsilon_i} := \mathfrak{r}_{\varepsilon_i} \otimes k$. Let $\bar{\mathfrak{s}}_{\varepsilon_i} := \mathfrak{s}_{\varepsilon_i} \otimes k$, and let $\bar{\mathfrak{q}}_{\varepsilon_i} := \mathfrak{q}_{\varepsilon_i} \otimes k$.

Following the image of $\bar{\mathfrak{g}}_1$ under the s-th iterate of $\bar{\psi}_2^0$, and identifying $\bar{\mathfrak{g}}_i$ and $\bar{\mathfrak{g}}_i \otimes_k k$ (the map $k \to k$ being the Frobenius automorphism σ) with $\bar{\mathfrak{g}}_1$, we come across k-vector spaces of the form

$$^{h_s}\Pi_{\varepsilon_{i_s}}\big(^{h_{s-1}}\Pi_{\varepsilon_{i_{s-1}}}\big(\cdots\big(\Pi_{\varepsilon_{i_2}}(^{h_1}\bar{\mathfrak{r}}_{\varepsilon_1})\cdots\big)\big)\big), \tag{1}$$

where $h_1, \ldots, h_s \in G_1(k)$. Here the left upper index ^{h_l}, $l = \overline{1,s}$, means conjugation with h_l. Taking $s \in \mathbb{N}$ big enough and a multiple of $\left|\tilde{I}_1\right|$, the k-vector space of (1) becomes nothing else but $N_2 \otimes k$. We allow such an s to vary. We first prove (0), i.e. that the dimension of the k-vector spaces of (1) computing $\dim_k(N_2 \otimes k)$, is less or equal to $\dim_{W(k)}(\bigcap_{i \in \tilde{I}_1} \mathfrak{r}_{\varepsilon_i})$.

3.6.1. Let $A := \{\varepsilon_i | i \in \tilde{I}_1\}$. So $\cap_{i \in \tilde{I}_1} \mathfrak{r}_{\varepsilon_i} = \cap_{a \in A} \mathfrak{r}_a$. The set A can have more than two elements only when $\mathfrak{g}_1$ is in case a), d) or e6). In case d) A has at most 3 elements. In case e6) A has at most 2 elements. In case a) A can have more than 3 elements, but in this case

$$\bigcap_{a \in A} \mathfrak{r}_a = \mathfrak{r}_{a_0} \cap \mathfrak{r}_{a_1},$$

with a_0 as the smallest element of A, and with a_1 as the greatest element of A. This is so because any positive root containing α_{a_0} and α_{a_1} in their expression, automatically contains α_{a_2} in their expression, for any $a_2 \in \mathbb{N}$ such that $1 \leq a_0 < a_2 < a_1 \leq \ell$ (see [Bou, planche I]).

3.6.2. Let $\bar{G}_1 := G_{1k}$ and let $\bar{T}_1 := T_{1k}$. We call an element $\omega \in \bar{G}_1(k)$ normalizing $\bar{T}_1$, a Weyl element. For any $\varepsilon_i \in A$, let $\bar{Q}_{\varepsilon_i}$ (resp. $\bar{P}_{\varepsilon_i}$) be the parabolic subgroup of $\bar{G}_1$ having $\bar{\mathfrak{q}}_{\varepsilon_i}$ (resp. $\bar{\mathfrak{p}}_{\varepsilon_i}$) as its Lie algebra.

3.6.3. Let us study closer an endomorphism of $\bar{\mathfrak{g}}_1$ of the form $\Pi_a\, {}^g\Pi_b$, with $a, b \in A$. The Bruhat decomposition allows us to write $g = q_a w q_b$, with $q_b \in \bar{Q}_b(k)$, with $w \in \bar{G}_1(k)$ a Weyl element, and with $q_a \in \bar{Q}_a(k)$. We get:

$$\mathrm{Im}(\Pi_a\, {}^g\Pi_b) = \mathrm{Im}({}^{\tilde{g}}\Pi_a\, {}^w\Pi_b), \quad \text{with} \quad \tilde{g} \in \bar{G}_1(k). \tag{2}$$

We also have the following inversion property:

$$\Pi_a\, {}^w\Pi_b = {}^w\Pi_b\, {}^w\Pi_a\, {}^{w^{-1}}, \tag{3}$$

for any Weyl element $w \in \bar{G}_1(k)$.

For any $h \in G_1(k)$ we have the following elimination property

$$\mathrm{Im}(\Pi_a\, {}^g\Pi_a) \subset \mathrm{Im}(\Pi_a); \tag{4}$$

So for proving (0) we can assume that any two consecutive ε_{i_j} and $\varepsilon_{i_{j+1}}$ in (1) above are distinct.

We are now ready to prove the first statement of 3.5.6.

1) If $|A| = 1$, we have nothing to prove.

2) If $|A| = 2$, then in any sequence ${}^{h_s}\Pi_{\varepsilon_{i_s}} \ldots {}^{h_1}\Pi_{\varepsilon_1}$ "computing" $N_2 \otimes k$, we get a subsequence $\Pi_a\, {}^g\Pi_b$, with $A = \{a, b\}$ and $g \in \bar{G}_1(k)$. We deduce that $\dim_{W(k)}(N_2) \leq \dim_{W(k)} \mathfrak{r}_a \cap \mathfrak{r}_b$, as $\dim_k(\Pi_a({}^g\bar{\mathfrak{r}}_b)) \leq \dim_k(\bar{\mathfrak{r}}_a \cap \bar{\mathfrak{r}}_b)$. This can be proved in many ways. Using property (2), this can be reduced to the case when g is a Weyl element, and then by a simple computation

$$\dim(\Pi_a({}^g\bar{\mathfrak{r}}_b)) = \dim(\bar{\mathfrak{r}}_b) - \dim({}^g\bar{\mathfrak{r}}_b \cap \bar{\mathfrak{p}}_a) = \dim(\bar{\mathfrak{r}}_b) - \dim(\bar{\mathfrak{p}}_a) + \dim({}^g\bar{\mathfrak{p}}_b \cap \bar{\mathfrak{p}}_a)$$

(the second equality due to the fact that g is a Weyl element). And so the dimension of $\Pi_a({}^g\bar{\mathfrak{r}}_b)$ is maximal when the dimension of the intersection ${}^g\bar{\mathfrak{p}}_b \cap \bar{\mathfrak{p}}_a$ of parabolic Lie subalgebras of $\bar{\mathfrak{g}}_1$ is maximal, i.e. when the intersection ${}^g\bar{\mathfrak{p}}_b \cap \bar{\mathfrak{p}}_a$ contains a Borel Lie subalgebra of $\bar{\mathfrak{g}}_1$. This can be checked very easily, case by case, starting from the fact that we can assume that g is a Weyl element, and from [Bou, planche I, IV, and V]; the most involved case (i.e. the case $A = \{1, \ell - 1\}$, with $\mathfrak{g}_1$ of D_ℓ lie type) is treated in 4) below (the other cases can be treated similarly). From this and [Bo, 14.22] we get $\dim({}^g\bar{\mathfrak{p}}_b \cap \bar{\mathfrak{p}}_a) \leq \dim(\bar{\mathfrak{p}}_b \cap \bar{\mathfrak{p}}_a)$.

3) If $|A| \geq 3$, then using repeatedly properties (2) and (3), we can bring together a short sequence of the form $\Pi_a\, {}^g\Pi_b$, if $\mathfrak{g}_1$ is in case a), with a the smallest element of A and with b the greatest element of A, or of the form $\Pi_a\, {}^g\Pi_b\, {}^h\Pi_c$ if $\mathfrak{g}_1$ is in case d), with $A = \{a, b, c\}$. This proves (0) if $\mathfrak{g}_1$ is in case a) (cf. 2) above).

4) Let us assume that we are in case d) with $|A| = 3$.

Let $\mathfrak{g}_1$ be of type D_ℓ ($\ell \geq 4$). Then $A = \{1, \ell-1, \ell\}$. Using property (2), we can assume that $\Pi_a\, {}^g\Pi_b\, {}^h\Pi_c$ is such that h is a Weyl element. Due to the circular form of the expressions of (1) above, we can assume that $c = 1$ and $b = \ell - 1$. But in this case the dimension of the image of $\Pi_a\, {}^g\Pi_b\, {}^h\Pi_c$ is less or equal to $\ell - 2$.

This can be checked using [Bou, planche IV]. Using the notations of the loc. cit. for this paragraph (so ε_i are not any more some integers in the set $\{0, 1, ..., \ell\}$, but some vectors in $\mathbb{R}^\ell$), the roots of Φ_1 corresponding to $\bar{\mathfrak{r}}_1$ are of the form $-\varepsilon_1 \pm \varepsilon_s$, with $2 \leq s \leq \ell$. The roots corresponding to $\mathfrak{r}_{\ell-1}$ are of the form $-\varepsilon_i + \varepsilon_\ell$, with $i < \ell$, or of the form $-(\varepsilon_i + \varepsilon_s)$, with $1 \leq i < s < \ell$. So if the element h of the Weyl group of loc. cit. sends ε_1 into an element of the set $\{\varepsilon_1, \varepsilon_2, \ldots \varepsilon_{\ell-1}, -\varepsilon_\ell\}$, then $\bar{\mathfrak{r}}_{\ell-1} \cap {}^h\bar{\mathfrak{r}}_1$ has dimension exactly $\ell - 1$. If h sends ε_1 into an element of the set $\{-\varepsilon_1, -\varepsilon_2, \ldots, -\varepsilon_{\ell-1}, \varepsilon_\ell\}$ then we have $\bar{\mathfrak{r}}_{\ell-1} \cap {}^h\bar{\mathfrak{r}}_1 = 0$. If $\bar{\mathfrak{r}}_{\ell-1} \cap {}^h\bar{\mathfrak{r}}_1 = 0$, we are done. If $\dim(\bar{\mathfrak{r}}_{\ell-1} \cap {}^h\bar{\mathfrak{r}}_1) = \ell - 1$, then the intersection $\bar{\mathfrak{q}}_{\ell-1} \cap {}^h\bar{\mathfrak{q}}_1$ contains a Borel subalgebra of $\bar{\mathfrak{g}}_1$, and we can apply the same argument as in case 2) above to conclude that the k-vector space $\bar{\mathfrak{r}}_\ell \cap {}^g(\bar{\mathfrak{r}}_{\ell-1} \cap {}^h\bar{\mathfrak{r}}_1)$ has dimension $\leq \ell - 2$ (the arguments of (2) above show that we can assume that g is a Weyl element).

This ends the proof of (0), and so of the first statement of 3.5.6.

3.6.4. We assume now that $\dim(N_2) = \frac{m_\delta}{n}$. To prove (3.5.7) we first remark:

Fact. *The Lie subalgebra* $\big(g_2(-\delta) \cap \mathfrak{g}_1\big) \otimes k$ *of* $\bar{\mathfrak{g}}_1$ *is of the form* ${}^g\big(\cap_{a\in A}\bar{\mathfrak{s}}_a\big)$, *with* $g \in \bar{P}_{\varepsilon_1}$, *and so its normalizer is* ${}^g\big(\bigcap_{a\in A}\bar{P}_a\big) \subset \bar{P}_{\varepsilon_1}$.

Proof: This fact is obvious if $|A| = 1$. If $|A| \geq 2$ we prove it working with notations involving negative roots. If $|A| = 2$, with $A = \{a, b\}$, we just have to show: if the image of the endomorphism $\Pi_a\, {}^w\Pi_b$, with $w \in G_1(k)$ a Weyl element, has dimension $\dim(\bar{\mathfrak{r}}_a \cap \bar{\mathfrak{r}}_b)$, then $\bar{\mathfrak{r}}_b \cap {}^{w^{-1}}\bar{\mathfrak{r}}_a$ is $G_1(k)$-conjugate to $\bar{\mathfrak{r}}_a \cap \bar{\mathfrak{r}}_b$, and the normalizer of $\bar{\mathfrak{r}}_a \cap \bar{\mathfrak{r}}_b$ is $\bar{Q}_a \cap \bar{Q}_b$.

The first part is obvious as $\bar{Q}_a$ and ${}^w\bar{Q}_b$ must contain a Borel subgroup of $\bar{G}_1$ (cf. 3.6.3 2)). Let $\bar{\mathfrak{g}}_\alpha := \mathfrak{g}_\alpha \otimes k$. For the second part it is enough to show that the Lie subalgebra of $\mathfrak{g}_1$ normalizing $\mathfrak{r}_a \cap \mathfrak{r}_b$ is $\mathrm{Lie}(\bar{Q}_a \cap \bar{Q}_b)$. We use the following facts:

a) $[\bar{\mathfrak{g}}_\alpha, \bar{\mathfrak{g}}_{-\alpha}] \neq 0$ *(this is the third place where we need* $p > 2$*);*

b) *If* $\alpha, \beta, \alpha + \beta \in \Phi_1$, *then* α *and* β *generate a root system of rank 2 of type* A_2 *(the inequality* $|A| \geq 2$ *implies that all roots of* Φ_1 *have the same length, cf. [Bou, planche I, IV, and V]).*

So everything results from [BT, 4.3] (this is also part of the third place where we need $p > 2$). Of course we can also use directly the formula

$$[\bar{\mathfrak{g}}_\alpha, \bar{\mathfrak{g}}_\beta] = \bar{\mathfrak{g}}_{\alpha+\beta},$$

where we have $\alpha, \beta, \alpha + \beta \in \Phi_1$. This formula is a consequence of b).

If $|A| \geq 3$ and $\mathfrak{g}_1$ is of A_ℓ Lie type, then we have to work with $\Pi_a\, {}^w\Pi_b$, with a and b the smallest and respectively the biggest element of A, and with $w \in \bar{G}_1(k)$ a Weyl element. So this case is reduced to the case when $|A| = 2$.

If $|A| = 3$ and $\mathfrak{g}_1$ is of D_ℓ Lie type, then the same argument at the level of Lie algebras applies: we have to work with $\Pi_a\, {}^w\Pi_b\, {}^{w_0}\Pi_c$, where $A = \{a, b, c\}$, and $w, w_0 \in G_1(k)$ are two Weyl elements (cf. 3.6.3 4)). This ends the proof of the fact.

3.6.4.1. We are now ready to prove (3.5.7). We recall that $(\mathfrak{g}_0, \varphi_j)$, $j = \overline{1,2}$, are giving birth to 0-symmetric isocrystals and so we can work equally well with the slope $-\delta$.

3.6.4 implies that the parabolic subgroup of $\bar{G}_1$ having $\left(\bigoplus_{\alpha \in \mathcal{H}_1 \cap [0,1]} g_0(\alpha) \cap \mathfrak{g}_1\right) \otimes k$ as its Lie algebra, is included in $\bar{P}_{\varepsilon_1}$. This ends the proof of (3.5.7). The last part of 3.5.6 (involving g_2 mod p) is a consequence of the fact that the σ-linear map $\bar{\psi}_2^0$ used in computing the multiplicity of m_δ of $(\mathfrak{g}_0, \varphi_2)$ depends only on g_2 mod p. This ends the proof of the Theorem 3.5.6.

3.6.5. Remark. It is easy to see that 3.5.1-8, 3.5.10 as well as the whole of 3.6 remain true if, instead of $g_2 \in G(W(k))$ (resp. $h_0 \in \tilde{G}_0(W(k))$ for 3.5.10) we work with any inner automorphism of $G_{W(k)}$ (resp. of $\tilde{G}_0$). Warning: this is not so if we use outer automorphism, as such an automorphism changes the automorphism class of (M, φ, G), producing a different G-ordinary type τ.

In particular we get: if $g \in G^{\mathrm{ad}}(W(k))$ is such that $(\mathfrak{g}, \varphi)$ and $(\mathfrak{g}, g\varphi_0)$ have the same Newton polygon, then at the level of Lie automorphisms of $\mathfrak{g}$ we have $g\varphi_0 = g_3 p_4 \varphi_0 g_3^{-1}$, where g_3 is a $W(k)$-valued point of G^{ad} which mod p defines a k-valued point of the parabolic subgroup of G^{ad} having $F^0(\mathfrak{g})[\frac{1}{p}] \cap \mathrm{Lie}(G^{\mathrm{ad}})$ as its Lie algebra, and where p_4 is a $W(k)$-valued point of the parabolic subgroup of G^{ad} having $\mathrm{Lie}(P_1)[\frac{1}{p}] \cap \mathrm{Lie}(G^{\mathrm{ad}})$ as its Lie algebra. The same remains true if instead of $\mathfrak{g}$ we work with $\mathrm{Lie}(G^{\mathrm{ad}})$; moreover we have a variant of this when we work with $\mathfrak{g}_0$ instead of $\mathfrak{g}$.

3.6.6. We expect that the following statement is true:

Expectation. *Let H be a semisimple split group H over a field k, and let $P_H(1)$,..., $P_H(n)$ be n parabolic subgroups of it ($n \in \mathbb{N}$). Then the dimension of the intersection of the Lie algebras of n parabolic subgroups $\tilde{P}_1$,..., $\tilde{P}_n$ of H which are respectively $H(k)$-conjugate to $P_H(1)$,..., $P_H(n)$, is maximal precisely in the case when there is a Borel subgroup of H contained in $\cap_{i=1}^n \tilde{P}_i$.*

The case $n = 2$ is easy (to be compared with 3.6.3 2)). For $n > 2$ we can prove it in many situations, but we do not have a proof which works in the general case.

3.7. The proves of 3.2.6-7.

3.7.1. Let $g_2 \in G(W(k))$. We use the Shimura σ_1-crystals $\mathcal{C}_1$ considered in 3.4.7.1. As $\mathcal{C}_1$ specializes to $(M \otimes W(\bar{k}), g_3(\varphi_0 \otimes 1), G_{W(\bar{k})})$, with $g_3 \in G_{W(\bar{k})}(W(\bar{k}))$ such that mod p belongs to the subset $\mathcal{L}$ of $G(\bar{k})$ defined in 3.4.6, but for $\bar{k}$, we deduce from 3.5.11-13 that the Shimura Lie σ-crystal attached to $\mathcal{C}_1$ has the same Newton polygon as $(\mathfrak{g}, \varphi_1)$, and so as $(\mathfrak{g}, \varphi_0)$ (cf. 3.3.2 and 3.3.5). From the fact that $\mathcal{C}_1$ specializes to $(M, g_2\varphi_0)$, cf. 3.4.7.1,

we deduce that the Newton polygon of $(\mathfrak{g}, g_2\varphi_0)$ is above the Newton polygon of $(\mathfrak{g}, \varphi_0)$. The same remains true working just with $\mathfrak{g}_0$.

From 3.3.4-5, and from the fact that Shimura Lie σ-crystal attached to $\mathcal{C}_1$ and $(\mathfrak{g}, \varphi_1)$ have the same Newton polygon, we get (cf. 3.5.11) that the formal isogeny type of $\mathcal{C}_1$ is τ_0. As $\mathcal{C}_1$ specializes to $(M, g_2\varphi_0)$, and as $g_2 \in G(W(k))$ was arbitrary, we get 3.2.6.

3.7.2. We assume here that $k = \bar{k}$. Let

$$\mathfrak{t} \oplus \bigoplus_{\substack{i\in I\\ \alpha\in\Phi_i}} \tilde{g}_i(\alpha) \subset \mathrm{Lie}(G)$$

be a Weyl direct sum decomposition of $\mathrm{Lie}(Z(G)^0 \times \times_{i\in I} G_i)$ associated to the maximal torus T of G. As $\varphi_1(\mathfrak{t}) = \mathfrak{t}$ we deduce that φ_1 permutes the $B(k)$-vector spaces $\tilde{g}_i(\alpha)\otimes B(k)$. We get a permutation γ_Φ of the disjoint union $\cup_{i\in I}\Phi_i$. So φ_1 takes any $p\tilde{g}_i(\alpha)$ into $\tilde{g}_{\gamma(i_0)}\big(\gamma_\Phi(\alpha)\big)$. More precisely we have

$$\varphi_1\big(\tilde{g}_i(\alpha)\big) = p^s \tilde{g}_{\gamma(i)}\big(\gamma_\Phi(\alpha)\big),$$

where $s \in \{-1, 0, 1\}$ is such that the cocharacter $\mu : \mathbb{G}_m \to T$ has the property that $\mu(\beta)$ acts on $\tilde{g}_i(\alpha)$ as the multiplication with β^{-s}, $\forall \beta \in \mathbb{G}_m(W(k))$. From this we deduce that $\mathrm{Lie}(P_1) \subset F^0(\mathfrak{g})$.

For any $p_1 \in P_1(W(k))$, the parabolic subalgebra $\mathfrak{p}^1_{\geq 0}$ of $\mathfrak{g}$ corresponding to non-negative slopes of $(\mathfrak{g}, p_1\varphi_1)$ is $\mathrm{Lie}(P_1)$. To see this we first remark that $(p_1\varphi_1)^s = p_s\varphi_1^s$, with $p_s \in P_1(W(k))$, $\forall s \in \mathbb{N}$. From this and 3.2.1 we get that $\mathrm{Lie}(P_1) \subset \mathfrak{p}^1_{\geq 0}$. The same argument applies to get the reverse inclusion. So $\mathrm{Lie}(P_1) = \mathfrak{p}^1_{\geq 0}$ (of course we can also quote directly 3.2.1.1).

3.7.3. 3.2.7 a) results directly from 3.7.2 and 3.3.2.

3.7.4. Let (M, F^1_2, φ_2, G) be a Shimura filtered σ-crystal with the property that the parabolic subalgebra ${}_2\mathfrak{p}$ of $\mathfrak{g}$ corresponding to non-negative slopes of $(\mathfrak{g}, \varphi_2)$, is contained in $F^0_2(\mathfrak{g})$. Let ${}_2B$ be a Borel subgroup of G containing the image of a cocharacter $\mu_2 : \mathbb{G}_m \to G$ of the form $g\mu g^{-1}$ (with $g \in G(W(k))$, and such that $g(F^1) = F^1_2$, and $F^1_2(\mathfrak{g}) \subset \mathrm{Lie}({}_2B) \subset {}_2\mathfrak{p} \subset F^0_2(\mathfrak{g})$. Let $g_3 \in {}_2P(W(k))$, with ${}_2P$ as the parabolic subgroup of G having ${}_2\mathfrak{p}$ as its Lie algebra, such that $g_3\varphi_2\big(\mathrm{Lie}({}_2B)\big) \subset \mathrm{Lie}({}_2B)$ (as in 3.2.3). But now $(M, F^1_2, g_3\varphi_2, G)$ has the same properties as (M, F^1, φ_0, G) which allowed us to prove 3.2.5-6. We get that $(M, g_3\varphi_2)$ has the formal isogeny type τ_0. As $(g_3\varphi_2)^s = g_{3,s}\varphi_2^s$, with $g_{3,s} \in {}_2P(W(k))$, $\forall s \in \mathbb{N}$, we deduce from 3.2.1, that (M, φ_2) has the formal isogeny type τ_0. This proves 3.2.7 c).

3.7.5. Let (M, F^1_2, φ_2, G) be a Shimura filtered σ-crystal having τ_0 as the formal isogeny. From 3.2.6, we deduce that the Lie σ-crystal $\big(\mathrm{End}(M), \varphi_2\big)$ has the smallest Newton polygon $\mathrm{End}(\mathcal{P})$ among all Lie σ-crystals $\big(\mathrm{End}(M), g\varphi_2\big)$, with $g \in G(W(k))$. But for any $g \in G(W(k))$, the Newton polygon of $\big(\mathrm{End}(M), g\varphi_2\big)$ is obtained from the Newton polygon of $(\mathfrak{g}, g\varphi_2)$ and from the Newton polygon of $\big(\mathrm{End}(M)/\mathfrak{g}, g\varphi_2\big)$. Again from 3.2.6 we get that the smallest Newton polygon $\mathrm{End}(\mathcal{P})$ is obtained from the smallest

Newton polygon $\mathrm{Lie}_G(\mathcal{P})$ of all Lie σ-crystals $(\mathfrak{g}, g\varphi_2)$, $g \in G(W(k))$, and from the smallest Newton polygon $\mathrm{Lie}_G^{\perp}(\mathcal{P})$ of all quotients $\big(\mathrm{End}(M)/\mathfrak{g}, g\varphi_2\big)$, $g \in G(W(k))$ (these quotients can be extended to p-divisible objects $\big(\mathrm{End}(M)/\mathfrak{g}, g\varphi_2, F^0(\mathrm{End}(M)/\mathfrak{g}), F^1(\mathrm{End}(M)/\mathfrak{g})\big)$ of $\mathcal{MF}_{[-1,1]}(W(k))$, but they are not Lie σ-crystals). We deduce that $(\mathfrak{g}, \varphi_2)$ has the same Newton polygon as $(\mathfrak{g}, \varphi_0)$. Changing everything by an isomorphism (given by an element of $G(W(k))$) we can take $F_2^1 = F^1$. But ${}_2P$ and P_0 are conjugate by an element $g_3 \in G(W(k))$ leaving invariant $F^1 \otimes k$ (this is so over $W(\bar{k})$, cf. 3.3.4, and so over $W(k)$ as it can be easily checked starting from [Bo, 20.9]; see 3.5.13). So, up to isomorphism, we can assume that ${}_2P = P_0$. But then obviously $\varphi_2 = p_0\varphi_0$, with $p_0 \in P_0(W(k))$. This, together with 3.3.2 and 3.7.2, ends the proof of 3.2.7 b), and so it ends the proof of 3.2.7.

3.7.6. End of the second proof of 3.1.0. From 3.2.4 we deduce 3.1.0.

3.8. The case when the reductive group G is not quasi-split. The basic results of 3.1 remain true, if we do not assume that G is quasi-split (assumption made in the definition 2.2.8). To see this, we first remark that if G is not quasi-split, then the field k must be infinite (any reductive group over a finite field is quasi-split, cf. [Bo, 16.6]). Let k_1 be a finite Galois extension of k such that G_{k_1} is quasi-split. Let σ_1 be the Frobenius automorphism of $W(k_1)$. We know that 3.1.0 is true for $G_{W(k_1)}$ (i.e. when we work with the Shimura σ_1-crystals $(M \otimes W(k_1), g(\varphi \otimes 1), G_{W(k_1)})$, with $g \in G(W(k_1))$). So to get 3.1.0 for G we just have to remark two things:

3.8.1. There is $g \in G(W(k))$ such that $(M \otimes W(k_1), g(\varphi \otimes 1), G_{W(k_1)})$ is a $G_{W(k_1)}$-ordinary σ_1-crystal.

3.8.2. If $(M, g_0\varphi)$, with $g_0 \in G(W(k))$, is a G-ordinary σ-crystal, due to the uniqueness assertion of b) of 3.1.0, the filtration F_0^1 of $M \otimes W(k_1)$ describing the $G_{W(k_1)}$-canonical lift of the $G_{W(k_1)}$-ordinary σ_1-crystal $\big(M \otimes W(k_1), g_0(\varphi \otimes 1), G_{W(k_1)}\big)$, is definable over $W(k)$, once it is defined over $W(k_1)$ (i.e. $F_0^1 = \tilde{F}_0^1 \otimes_{W(k)} W(k_1)$, with $\tilde{F}_0^1 \subset M$).

3.8.2 is trivial. 3.8.1 results from the fact that $G(k)$ is Zariski dense in G_{k_1} (cf. [Bo, 18.3]), and from the fact that there is (cf. 3.4.6.1 1)) an open subscheme U of G_{k_1} such that for any $g \in G(W(k_1))$ lifting a k_1-valued point of U, $\big(M \otimes W(k_1), g(\varphi_0 \otimes 1)\big)$ is a $G_{W(k_1)}$-ordinary σ_1-crystal.

3.9. A supplement to 3.1.

3.9.1. The Lie stable p-rank. Let (M, φ_2, G) be a Shimura σ-crystal. We use the previous notations of §3. For not overloading the notation, we assume $k = \bar{k}$. So G is split.

Let $\Psi : \bigoplus_{i\in I} \mathfrak{g}_i \to \bigoplus_{i\in I} \mathfrak{g}_i$ be the σ-linear map defined by the rule:

$$\Psi(x) := \begin{cases} p\varphi_2(x) & \text{if } x \in \mathfrak{g}_i \text{ and } F^1(\mathfrak{g}_i) \neq 0 \\ \varphi_2(x) & \text{if } x \in \mathfrak{g}_i \text{ and } F^1(\mathfrak{g}_i) = 0\,. \end{cases}$$

Let $\bar{\Psi} : \bigoplus_{i\in I} \mathfrak{g}_i \otimes k \to \bigoplus_{i\in I} \mathfrak{g}_i \otimes k$ be obtained by taking $\bar{\Psi}$ mod p. Let

$$N_{\varphi_2} := \bigcap_{m\in\mathbb{N}} \mathrm{Im}(\bar{\Psi}^m).$$

We define

$$R_{\varphi_2} := \dim_k(N_{\varphi_2}).$$

We call N_{φ_2} the Lie stable k-vector space of (M, φ_2, G) and we call R_{φ_2} the Lie stable p-rank of (M, φ_2, G).

Similarly we define the Lie stable k-vector space and the Lie stable p-rank of any Shimura Lie σ-crystal over k.

3.9.1.1. Remark. The Lie stable p-ranks and the Lie stable k_1-vector spaces are well defined for any Shimura σ_1-crystal (M_1, φ^1, G_1) over a perfect field k_1. This is so due to the fact that the map Ψ_1 defined as above but starting from the extension to $\overline{k_1}$ of (M_1, φ^1, G_1) is in fact defined over $W(k_1)$, cf. the existence of a cocharacter $\mu_1 : \mathbb{G}_m \to G_1$ defining the filtration class of (M_1, φ^1, G_1) and of the structure (see [Ti1]) of $W(k_1)$-simple adjoint groups. In other words: $\mathrm{Lie}(G_1^{\mathrm{ad}})$ is a direct sum of $W(k_1)$-simple factors, each of them having the property that all its simple factors over $W(\bar{k}_1)$ are producing (as in 3.5.3.4) either only positive or only zero ε_i's numbers.

3.9.2. Theorem. *A Shimura σ-crystal (M, φ_2, G) is a G-ordinary σ-crystal iff its Lie stable p-rank is maximal (i.e. $R_{\varphi_2} \geq R_{\varphi_3}$, for any $\varphi_3 = g_3\varphi_2$, with $g_3 \in G(W(k))$), and so equal to R_{φ_0}.*

Proof: We remark that $(\mathfrak{g}, \varphi_2)$ has the smallest Newton polygon among all Shimura Lie σ-crystals $(\mathfrak{g}, g\varphi_2)$, iff for any cycle γ_0 of the permutation γ of I, $(\mathfrak{g}_0, \varphi_2)$ has the smallest Newton polygon among all Shimura Lie σ-crystals $(\mathfrak{g}_0, g\varphi_2)$ (here $g \in G(W(k))$, γ_0 is a cyclic permutation of a subset I_0 of I, and $\mathfrak{g}_0 = \oplus_{i \in I_0} \mathfrak{g}_i$). So everything results from 3.5.10, once we interpret R_{φ_2} as the multiplicity of the slope δ for $(\mathfrak{g}, \varphi_2)$.

3.9.3. Corollary. *The fact that a Shimura σ-crystal $(M, g\varphi, G)$ is G-ordinary depends only on the value of g mod p.*

Proof: This results from 3.9.2 once we remark that the Lie stable p-rank of $(M, g\varphi)$ depends only on the value of g module p.

3.9.3.1. We refer to 3.4.1.3. From 3.4.18.4 B), 3.1.0 and 3.9.3 we get directly:

Corollary. *If $y \in \mathrm{Spec}(R_j)(\bar{k})$ gives birth to a G-ordinary σ-crystal (3.9.3 guarantees that this makes sense), and if $r_y = \dim(G)$, then y lifts to a point $y_\infty \in \mathrm{Spec}(Q_j)(\bar{k})$.*

3.9.4. The refined Lie stable p-rank. We use the notations from the beginning of 3.5. We choose a bijection $f_{\mathcal{J}_0}$ between $\mathcal{J}_0$ and the set $\{1, 2, \ldots, |\mathcal{J}_0|\}$. Then for any $s \in \mathcal{J}_0$, we can define the Lie stable p-rank $R_{\varphi_2}(s)$ of (M, φ_2, G) with respect to s. For instance, if we work with the cycle γ_0, $R_{\varphi_2}(f_{\mathcal{J}_0}(0))$ is nothing else but the Lie stable p-rank of $(\mathfrak{g}_0, \varphi_2)$. We denote by $R^r_{\varphi_2}$ the sequence $(a_1, \ldots, a_{|\mathcal{J}_0|})$ of $|\mathcal{J}_0|$ non-negative integers defined by the formula:

$$a_l := \frac{R_{\varphi_2}\left(f^{-1}_{\mathcal{J}_0}(l)\right)}{\mathrm{length}\left(\gamma_{f^{-1}_{\mathcal{J}_0}(l)}\right)},$$

for any $l \in \{1, 2, \ldots, |\mathcal{J}_0|\}$ (a_l are non-negative integers cf. (3.5.2); for instance if $l = f_{\mathcal{J}_0}(0)$, then $R_{\varphi_2}(0)$ is the multiplicity of the slope δ of $(\mathfrak{g}_0, \varphi_2)$, and so is a multiple of $|I_0|$).

On the set of sequences of $|\mathcal{J}_0|$ non-negative integers we have a partial order defined by $(b_1, b_2, \ldots, b_{|\mathcal{J}_0|}) \leq (c_1, c_2, \ldots, c_{|\mathcal{J}_0|})$ if $b_i \leq c_i$, $\forall i \in \{1, 2, \ldots, |\mathcal{J}_0|\}$. From the proof of 3.9.2 we get:

3.9.5. Theorem. *A Shimura σ-crystal (M, φ_2, G) is a G-ordinary σ-crystal iff its refined Lie stable p-rank is maximal, equal to $R^r_{\varphi_0}$.*

3.9.6. Remark. It is an easy exercise to see that the refined Lie stable p-rank of a Shimura σ-crystal or of a Shimura Lie σ-crystal over k is the same, if instead of working with Shimura Lie σ-crystals of the above form $(\mathfrak{g}_0, \varphi_2)$, we work with their adjoint Lie σ-crystal $(\mathfrak{g}_0^{\mathrm{ad}}, \varphi_2)$, where $\mathfrak{g}_0^{\mathrm{ad}} := \mathrm{Lie}(\tilde{G}_0^{\mathrm{ad}}) \subset \mathrm{Lie}(\tilde{G}_0)[\frac{1}{p}]$ (cf. 2.2.13).

3.9.7. The zero, the positive, and the non-negative rank and type. Let k again be an arbitrary perfect field. By the non-negative rank (resp. type) of a Shimura σ-crystal (M, φ, G) over k (or of its attached Shimura Lie σ-crystal) we mean the rank (resp. the $G(W(k))$-conjugacy class) of the parabolic Lie subalgebra $\mathfrak{p}_{\geq 0}$ of $\mathrm{Lie}(G)$ corresponding to non-negative slopes of $(\mathrm{Lie}(G), \varphi)$. Similarly we define the positive and the zero ranks and types, by using positive slopes and respectively the slope zero (of course the wording "parabolic" has to be replaced accordingly). This definition extends to Shimura Lie σ-crystals, or to a filtered context.

3.10. Terminology and formulae.

3.10.0. Cyclic Shimura Lie σ-crystals. The notations that follows are similar to the previous ones, but, without specifically stating, they are independent. A Shimura adjoint Lie σ-crystal $(\mathfrak{g}, \varphi)$ is said to be cyclic, if $\mathfrak{g}$ is the Lie algebra of an adjoint group G over $W(k)$ such that $G_{W(\bar{k})} = \prod_{i \in \tilde{I}} G'_i$, with all ${G'_i}^{\mathrm{ad}}$ as simple adjoint groups over $W(\bar{k})$, and with φ permuting cyclically the direct summands $\mathrm{Lie}\,(G'_i)[\frac{1}{p}]$ of $\mathfrak{g} \otimes W(\bar{k})[\frac{1}{p}]$.

$(\mathfrak{g}, \varphi)$ is said to be trivial if $\varphi(\mathfrak{g}) = \mathfrak{g}$; otherwise it is said to be non-trivial. $(\mathfrak{g}, \varphi)$ is said to be of A_ℓ, B_ℓ, C_ℓ, D_ℓ, E_6 or E_7 Lie type if the factors G'_i are of this Lie type. We do not bother here about the other Lie types, as they force $(\mathfrak{g}, \varphi)$ to be trivial, cf. 2.2.7.

Let $g \in G(W(\bar{k}))$ such that $g\varphi$ leaves invariant the Lie algebras $\mathfrak{b}$ and $\mathfrak{t}$ of a Borel subgroup $B_{W(\bar{k})}$ of $G_{W(\bar{k})}$, and respectively of a maximal torus $T_{W(\bar{k})}$ of $B_{W(\bar{k})}$ (cf. 3.2.3). The permutation of the basis of roots of $\mathrm{Lie}(G'_i)$ with respect to the maximal torus T_i of G'_i contained in $T_{W(\bar{k})} \cap G'_i$, and corresponding to the Borel subgroup $B'_i := B_{W(\bar{k})} \cap G'_i$ of G'_i, induced by $(g\varphi)^{|\tilde{I}|}$, can be non-trivial. This permutation can be:

– an involution (this can happen only when $\mathfrak{g}$ is of A_ℓ, D_ℓ or E_6 Lie type);

– a cycle of order 3 (this can happen only when $\mathfrak{g}$ is of D_4 Lie type);

– trivial.

We say respectively that $(\mathfrak{g} \otimes W(\bar{k}), g\varphi)$ (or $(\mathfrak{g}, \varphi)$) is with involution, or with a 3-cycle, or without a cycle.

The above definition will be used as well in a filtered context, or in a non-necessarily adjoint context (i.e. when we work with a semisimple group over $W(k)$ which over $W(\bar{k})$ is a product of semisimple groups having simple adjoint groups).

3.10.0.1. Remark. If we require just that the simple factors of $\mathrm{Lie}(G)$ to be permuted cyclically by φ, then we say that $(\mathfrak{g}, \varphi)$ is weakly cyclic. The notions of weakly cyclic and cyclic coincide over finite fields, but not in general. In what follows, without specifically stating, we will deal only with the cyclic notion.

3.10.1. Definitions. Any split Shimura adjoint (filtered) Lie σ-crystal $\mathfrak{C}^{\mathrm{ad}}$ is isomorphic to a direct sum of cyclic Shimura adjoint (filtered) Lie σ-crystals, which are called the cyclic factors of $\mathfrak{C}^{\mathrm{ad}}$. The direct sum of the non-trivial cyclic factors is called the non-trivial part of $\mathfrak{C}^{\mathrm{ad}}$, while the direct sum of the trivial ones is called the trivial part of $\mathfrak{C}^{\mathrm{ad}}$.

Similarly, for any Shimura σ-crystal (M, φ, G) or Shimura filtered σ-crystal (M, F^1, φ, G) (resp. for any Shimura Lie σ-crystal $(\mathrm{Lie}(G), \varphi)$, or any Shimura filtered Lie σ-crystal $(\mathrm{Lie}(G), \varphi, F^0(\mathrm{Lie}(G)), F^1(\mathrm{Lie}(G))))$, with G^{ad} a split group, we define its cyclic Lie factors (resp. its cyclic factors), as well as its cyclic adjoint factors. For instance, the Shimura filtered Lie σ-crystal $(\mathfrak{g}_0, \varphi_2, F^0(\mathfrak{g}_0), F^1(\mathfrak{g}_0))$ of 3.5.1 is cyclic, and is called a cyclic Lie factor of (M, F^1, φ_2) (or a cyclic factor of $(\mathfrak{g}, \varphi_2, F^0(\mathfrak{g}), F^1(\mathfrak{g}))$); its adjoint (defined by the Lie algebra $\oplus_{i\in I_0}\mathrm{Lie}(G_i^{\mathrm{ad}})$) is called a cyclic adjoint factor of (M, F^1, φ_2) or of $(\mathfrak{g}, \varphi_2, F^0(\mathfrak{g}), F^1(\mathfrak{g}))$. Similarly we speak about the non-trivial part (or the trivial part) of a split Shimura (filtered) Lie σ-crystal.

Similarly, for any Shimura (adjoint) (filtered) Lie σ-crystal (resp. any Shimura (filtered) σ-crystal), we speak about its weakly cyclic factors (resp. about its weakly cyclic Lie factors and its weakly cyclic adjoint factors), and about these factors being trivial or non-trivial. The difference is: the adjoint groups we get are not always product of absolutely simple adjoint groups (cf. 3.10.0.1).

3.10.2. From now on we work with a cyclic Shimura adjoint filtered Lie σ-crystal $\left(\mathfrak{g}, \varphi, F^0(\mathfrak{g}), F^1(\mathfrak{g})\right)$ which is non-trivial, and has the property that there are Lie algebras $\mathfrak{b}$ and $\mathfrak{t}$ of a Borel subgroup B of G and respectively of a maximal torus T of B, such that $F^1(\mathfrak{g}) \subset \mathfrak{b} \subset F^0(\mathfrak{g})$, $\varphi(\mathfrak{t}) = \mathfrak{t}$, and $\varphi(\mathfrak{b}) \subset \mathfrak{b}$ (cf. 3.2.3). Here G is as in 3.10.0.

All the terminology to be introduced below for the case G split adjoint will be also used in the context of 3.10.2 (by passage to $W(\bar{k})$), for the cyclic factors of split Shimura filtered Lie σ-crystals defined by Shimura-canonical lifts (i.e. for factors like $\left(\mathfrak{g}_0, \varphi_1, F^0(\mathfrak{g}_0), F^1(\mathfrak{g}_0)\right)$ of 3.2.3), as well as for the adjoints of such factors.

3.10.3. From now on we also assume that G is a split group. For not overloading the notations, we assume that $G = \times_{i\in I_0} G_i$, $\mathfrak{g} = \mathfrak{g}_0, F^0(\mathfrak{g}) = F^0(\mathfrak{g}_0), F^1(\mathfrak{g}) = F^1(\mathfrak{g}_0)$, and $\varphi = \varphi_1$ (of 3.2.3). So we use fully the notations and conventions introduced in 3.5.0-3 and 3.5.5. So $\tilde{I} = I_0$ and $G'_i = G_i$, $\forall i \in I_0$. Warning: here G_i are adjoint groups; so only the notations are as in the mentioned places.

3.10.4. The ε-type. The sequence $(\varepsilon_1, \ldots, \varepsilon_m)$ is called the ε-type of $(\mathfrak{g}_0, \varphi_1)$. Here $m = \left|\tilde{I}_0\right| = e\,|I_0| \in \{|I|_0\,, 2\,|I_0|\,, 3\,|I_0|\}$. If $(\mathfrak{g}_0, \varphi_1)$ is with involution or with a 3-cycle, then we replace $(\mathfrak{g}_0, \varphi_1)$ by $(\oplus_{j\in\{1,\ldots,e\}}\mathfrak{g}_0, \varphi_1^{(e)})$ (cf. 3.5.3.2) (e being 2 or 3, depending on π_1 being an involution or a 3-cycle). So $\varepsilon_i \in \{1, ..., \ell\}$.

3.10.4.1. Warning. We replace $(\mathfrak{g}_0, \varphi_1)$ by $\left(\oplus_{j\in\{1,\ldots,e\}}\mathfrak{g}_0, \varphi_1^{(e)}\right)$ only for defining the ε-type of $(\mathfrak{g}_0, \varphi_1)$. For the formulae that follows this is not allowed.

3.10.4.2. If $(\mathfrak{g}_0, \varphi_1)$ is without a cycle, then $(\varepsilon_1, \ldots, \varepsilon_m)$ is uniquely defined up to a cyclic permutation, and up to a renumbering of the roots of Δ_1 (cf. 3.5.3.2). We have $m = n = |I_0|$.

If $(\mathfrak{g}_0, \varphi_1)$ is with involution then $(\varepsilon_1, \ldots, \varepsilon_m)$ is uniquely determined up to a cyclic permutation, and, in the case when $\mathfrak{g}_0$ is of D_4 Lie type, up to a renumbering of the roots of Δ_1. We have $m = 2n$.

If $(\mathfrak{g}_0, \varphi_1)$ is with a 3-cycle then $(\varepsilon_1, \ldots, \varepsilon_m)$ is uniquely determined up to cyclic permutation, and up to a renumbering of the roots of Δ_1. We have $m = 3n$.

3.10.4.3. Convention. We say that an ε-type $(\varepsilon_1, ..., \varepsilon_m)$ is in standard form if $\sum_{i=1}^{m} \varepsilon_i(\ell+1)^{m-i}$ is the biggest number among all the numbers obtained in this manner using an ε-type obtained from $(\varepsilon_1, ..., \varepsilon_m)$ through the rules of this 3.10.4.2 (i.e. up to renumbering and cyclic permutation). Here the role of $\ell + 1$ is irrelevant: we just need an integer bigger then all ε_i's. We always prefer to use ε-types which are in standard form. For instance we prefer to work with the ε-type $(1, 1, 0, 0)$ rather than the ε-type $(0, 0, 1, 1)$.

3.10.4.4. Spreadings. We define $A := \{\varepsilon_i | i \in I_1\}$ and $B := \{\varepsilon_i | i \in \tilde{I}_1\}$. We have $|A| \leq |B| \leq 2\,|A|$. The numbers $|A|$ and $|B|$ are called the apparent spreading and respectively the spreading of the cyclic Shimura Lie σ-crystal $(\mathfrak{g}_0, \varphi_1)$.

3.10.5. Definitions. If $\varepsilon_i \neq 0$, $\forall i \in \tilde{I}_0$, then $(\mathfrak{g}_0, \varphi_1)$ is said to be of totally non-compact type. If there is $i \in I_0$ such that $\varepsilon_i = 0$, then we say that $(\mathfrak{g}_0, \varphi_1)$ has compact factors.

If all ε_i, $i \in \{1, ..., m\}$, are equal and different from zero, we say that $(\mathfrak{g}_0, \varphi_1)$ is totally non-compact of constant type, or totally non-compact of type:

i) 1 if $\mathfrak{g}_0$ is of B_ℓ, C_ℓ, D_4, E_6 or E_7 Lie type, or of D_ℓ Lie type with $\varepsilon_1 = 1$.

ii) 2 if $\mathfrak{g}_0$ is of D_ℓ Lie type, with $\ell \geq 5$, and $\varepsilon_1 \in \{\ell - 1, \ell\}$.

iii) $\min\{\varepsilon_i, 1 + \ell - \varepsilon_i\}$ if $\mathfrak{g}_0$ is of A_ℓ Lie type.

In case i) for $\mathfrak{g}_0$ of type D_ℓ, with $\ell \geq 4$, we also say that $(\mathfrak{g}_0, \varphi_1)$ is totally non-compact of $D_\ell^{\mathbb{R}}$ type. In case ii) we also say that $(\mathfrak{g}_0, \varphi_1)$ is of simple $D_\ell^{\mathbb{H}}$ type. Similarly if all non-zero ε_i are equal ($i \in \{1, ..., m\}$) but there is an $\varepsilon_{i_0} = 0$ ($i_0 \in I_0$), we say that $(\mathfrak{g}_0, \varphi_1)$ is of constant type, or of type 1, or 2, or $\min\{\varepsilon_i, 1+\ell-\varepsilon_i\}$ (or of $D_\ell^{\mathbb{R}}$ type or of simple $D_\ell^{\mathbb{H}}$ type) depending on which of the situations i)-iii) above we are, but dropping the expression of totally non-compact type.

If $|B| \geq 2$ we say that $(\mathfrak{g}_0, \varphi_1)$ is of non-constant type. This can happen only when $\mathfrak{g}_0$ is of A_ℓ, D_ℓ or E_6 Lie type.

If $\mathfrak{g}_0$ is of D_ℓ Lie type, with $B = \{\ell - 1, \ell\}$, or if $\mathfrak{g}_0$ is of D_4 Lie type, with $|B| = 2$, we say that $(\mathfrak{g}_0, \varphi_1)$ is of (non-simple) $D_\ell^{\mathbb{H}}$ type. In the cases when $\mathfrak{g}_0$ is of D_4 Lie type with $|B| = 3$, or of D_ℓ Lie type, with $\ell \geq 5$ and $|B| \geq 2$, but $B \neq \{\ell - 1, \ell\}$, we say that $(\mathfrak{g}_0, \varphi_1)$ is of mixed D_ℓ type. If $\mathfrak{g}_0$ is of D_ℓ Lie type with $|B| = 3$, that is $B = \{1, \ell - 1, \ell\}$, we say that $(\mathfrak{g}_0, \varphi_1)$ is of strong mixed D_ℓ type.

If $\mathfrak{g}_0$ is of A_ℓ Lie type with $|B| \geq 2$, we say that $(\mathfrak{g}_0, \varphi_1)$ is of $\{a, b\}$ A_ℓ type, where $1 \leq a < b \leq \ell$ and $a \leq \ell + 1 - b$, and, if potentially changing $\ell - i$ with $i + 1$, $i = \overline{0, \ell - 1}$, a is the smallest element of B and b is the biggest element of B.

$q_1 = |I_1|$ is called the non-compact length of $(\mathfrak{g}_0, \varphi_1)$, and $q_0 = n = |I_0|$ is called the cyclic length of $\mathfrak{g}_0$. As we have assumed that $(\mathfrak{g}_0, \varphi_1)$ is non-trivial, we have $q_1 \geq 1$. By the rank of $(\mathfrak{g}_0, \varphi_1)$ we mean the rank $\ell \in \mathbb{N}$ of any simple Lie factor of $\mathfrak{g}_0$.

3.10.6. Formulae. The formulae below are based on the list of hermitian symmetric domains and of their real dimension (see [He, p. 518]), and on the standard tables of roots of the simple split semisimple Lie algebras over a field of characteristic zero (see [Bou, planche I-VI]). There is nothing deep about the computations involved, so we just list the results.

i) Let $\mathfrak{g}_0$ be of B_ℓ, C_ℓ or E_7 Lie type.

The slopes of $(\mathfrak{g}_0, \varphi_1)$ are precisely $-\frac{q_1}{q_0}$, 0 and $\frac{q_1}{q_0}$. The multiplicity of $\frac{q_1}{q_0}$ is $q_0 \dim(X)$, where X is a hermitian symmetric domain associated to an $\mathbb{R}$-simple adjoint group of the same Lie type as $\mathfrak{g}_0$.

If $\mathfrak{g}_0$ is of C_ℓ Lie type, then $\dim(X) = \frac{\ell(\ell+1)}{2}$.

If $\mathfrak{g}_0$ is of B_ℓ Lie type, then $\dim(X) = 2\ell - 1$.

If $\mathfrak{g}_0$ is of E_7 Lie type, then $\dim(X) = 27$.

ii) Let $\mathfrak{g}_0$ be of E_6 Lie type.

Case 1. $(\mathfrak{g}_0, \varphi_1)$ is of type 1; then everything is as in i), with $\dim(X) = 16$.

Case 2. $(\mathfrak{g}_0, \varphi_1)$ is of non-constant type, but without a cycle, and with $|I_2| \neq |I_1 \backslash I_2|$, where $I_2 := \{i \in I_1 | \varepsilon_i = 1\}$. We can assume that $|I_2| > |I_1 \backslash I_2|$. Let $q_2 := |I_2|$. The slopes of $(\mathfrak{g}_0, \varphi_1)$ are precisely $-\frac{q_1}{q_0}$, $-\frac{q_2}{q_0}$, $\frac{-q_1+q_2}{q_0}$, 0, $\frac{q_1-q_2}{q_0}$, $\frac{q_2}{q_0}$ and $\frac{q_1}{q_0}$.

The multiplicities of the slopes $\frac{q_1}{q_0}$, $\frac{q_2}{q_0}$ and $\frac{q_1-q_2}{q_0}$ are all equal to $8q_0$.

Case 3. $(\mathfrak{g}_0, \varphi_1)$ is of non-constant type with involution, or is without a cycle, but with $|I_2| = |I_1 \backslash I_2|$ (with I_2 as above). The slopes of $(\mathfrak{g}_0, \varphi_1)$ are precisely $\frac{-q_1}{q_0}$, $\frac{-q_1}{2q_0}$, 0, $\frac{q_1}{2q_0}$ and $\frac{q_1}{q_0}$.

The multiplicity of $\frac{q_1}{2q_0}$ is $16q_0$. The multiplicity of $\frac{q_1}{q_0}$ is $8q_0$.

iii) Let $\mathfrak{g}_0$ be of D_ℓ Lie type.

Case 1. $(\mathfrak{g}_0, \varphi_1)$ is of type 1 (i.e. of $D_\ell^{\mathbb{R}}$ type); then the situation is as in i), with $\dim(X) = 2(\ell - 1)$.

Case 2. $(\mathfrak{g}_0, \varphi_1)$ is of type 2 (i.e. it is of simple $D_l^{\mathbb{H}}$ type); then the situation is as in i), with $\dim(X) = \frac{\ell(\ell-1)}{2}$.

Case 3. $(\mathfrak{g}_0, \varphi_1)$ is of $D_\ell^{\mathbb{H}}$ type. We can assume that $B = \{\ell - 1, \ell\}$ even for $\ell = 4$. Let $I_2 := \{i \in I_1 | \varepsilon_i = \ell - 1\}$. We can assume that $|I_2| \geq |I_1 \backslash I_2|$. Let $q_2 := |I_2|$.

Subcase 1. $(\mathfrak{g}_0, \varphi_1)$ is without a cycle and $q_2 \neq q_1 - q_2$. The slopes are like in case 2 of ii).

The multiplicity of the slope $\frac{q_1}{q_0}$ is $\frac{(\ell-1)(\ell-2)}{2}q_0$.

The multiplicity of the slope $\frac{q_2}{q_0}$ is $(\ell-1)q_0$ and is equal to the multiplicity of $\frac{-q_2+q_1}{q_0}$.

Subcase 2. $(\mathfrak{g}_0, \varphi_1)$ is without a cycle and $|I_2| = |I_1 \backslash I_2|$, or is with involution. The slopes are like in case 3 of ii).

The multiplicity of $\frac{q_1}{q_0}$ is $\frac{(\ell-1)(\ell-2)}{2}q_0$.

The multiplicity of $\frac{q_1}{2q_0}$ is $2(\ell-1)q_0$.

Case 4. $(\mathfrak{g}_0, \varphi_1)$ is of mixed type.

Subcase 1. $(\mathfrak{g}_0, \varphi_1)$ is of strong mixed type. Let $I_2 := \{i \in I_1 | \varepsilon_i = 1\}$, $I_3 := \{i \in I_1 | \varepsilon_i = \ell - 1\}$, and $I_4 := \{i \in I_1 | \varepsilon_i = \ell\}$. Let $q_j := |I_j|$ for $j = \overline{2,4}$. We have $q_1 = q_2 + q_3 + q_4$. We can assume $q_3 \geq q_4$.

Subcase 1.1. $(\mathfrak{g}_0, \varphi_1)$ is without a cycle. The set of slopes is formed from 0 and $\pm a$, where a is of the form $\frac{\sum_{j \in \mathcal{J}_0} q_j}{q_0}$, with $\mathcal{J}_0$ a non-empty subset of $\{q_2, q_3, q_4\}$.

The multiplicity of $a = \frac{\sum_{j \in \mathcal{J}_0} q_j}{q_0}$ is equal to $\big(\sum_{\mathcal{J} \in \mathcal{J}_1} r_{\mathcal{J}}\big)q_0$, where $\mathcal{J}_1$ is the set of non-empty subsets of $\{2,3,4\}$, with the property that $\sum_{j \in \mathcal{J}_1} q_j = aq_0$. Here $r_{\{2,3,4\}} = \ell - 2$, $r_{\{2,3\}} = r_{\{2,4\}} = 1$, $r_{\{3,4\}} = \frac{(\ell-3)(\ell-2)}{2}$, and $r_{\{2\}} = r_{\{3\}} = r_{\{4\}} = \ell - 2$.

Subcase 1.2. $(\mathfrak{g}_0, \varphi_1)$ is with a 3-cycle. This implies $\ell = 4$. The slopes are $-\frac{q_1}{q_0}, \frac{-2q_1}{3q_0}, \frac{-q_1}{3q_0}, 0, \frac{q_1}{3q_0}, \frac{2q_1}{3q_0}$ and $\frac{q_1}{q_0}$.

The multiplicity of the slope $\frac{q_1}{q_0}$ is $2q_0$.

The multiplicity of the slope $\frac{2q_1}{3q_0}$ is $3q_0$.

The multiplicity of the slope $\frac{q_1}{3q_0}$ is $6q_0$.

Subcase 1.3. $(\mathfrak{g}_0, \varphi_1)$ is with involution. If $\ell = 4$, then we assume that the involution of the basis of roots of $\mathfrak{g}_1$ takes $\alpha_{\ell-1}$ into α_ℓ (to fit the assumption $q_3 \geq q_4$). The slopes are $0, \pm a$, with a of the form $\big(\sum_{j \in \mathcal{J}_0} q_j\big)\frac{1}{2q_0}$, for $\mathcal{J}_0$ one of the sets $\{2,3,4,5,6,7\}, \{2,3,4,5\}, \{3,4,6,7\}, \{2,5\}$ or $\{3,4\}$ (here $q_{3+i} = q_i$ for $i = \overline{2,4}$).

The multiplicity of

$$a = \frac{1}{2q_0}\Big(\sum_{j \in \mathcal{J}_0} q_j\Big)$$

is

$$\Big(\sum_{\mathcal{J}_1 \in \mathcal{J}_2} r_{\mathcal{J}_1}\Big)q_0,$$

where $\mathcal{J}_2$ is the set of those sets $\mathcal{J}_0$ as above for which $\sum_{j \in \mathcal{J}} q_j = 2q_0 a = \sum_{j \in \mathcal{J}_0} q_j$. Here $r_{\mathcal{J}_1}$ is $\ell - 2$ if $\mathcal{J}_1 = \{2,3,4,5,6,7\}$ or $\mathcal{J}_1 = \{2,5\}$, is 2 if $\mathcal{J}_1 = \{2,3,4,5\}$, is $\frac{(\ell-3)(\ell-2)}{2}$ if $\mathcal{J}_1 = \{3,4,6,7\}$, and is $2(\ell-2)$ if $\mathcal{J}_1 = \{3,4\}$.

Subcase 2. $(\mathfrak{g}_0, \varphi_1)$ is not of strong mixed type. This implies that $(\mathfrak{g}_0, \varphi_1)$ is without an involution and that $\ell \geq 5$. This subcase is entirely analogous to the subcase 1.1, for the value $q_4 = 0$.

iv) Let $\mathfrak{g}_0$ be of A_ℓ type ($\ell \geq 2$).

Case 1. $|B| = 1$**.** This happens when $|A| = 1$ and $(\mathfrak{g}_0, \varphi_1)$ is without a cycle, or when $(\mathfrak{g}, \varphi_1)$ is with involution and $A = \{\frac{\ell+1}{2}\}$ (in this second situation ℓ must be odd).

The slopes are $-\frac{q_1}{q_0}$, 0 and $\frac{q_1}{q_0}$.

The multiplicity of the slope $\frac{q_1}{q_0}$ is $i_0(\ell + 1 - i_0)q_0$, where $B = \{i_0\}$.

Case 2. $|B| = 2$**.** We distinguish three subcases.

Subcase 1. $A = \{i_0\}$ with $i_0 \neq \frac{\ell+1}{2}$, and $(\mathfrak{g}_0, \varphi_1)$ is with involution. We can assume $i_0 < \frac{\ell+1}{2}$. The slopes are $-\frac{q_1}{q_0}$, $\frac{-q_1}{2q_0}$, 0, $\frac{q_1}{2q_0}$ and $\frac{q_1}{q_0}$.

The multiplicity of $\frac{q_1}{q_0}$ is $i_0^2 q_0$.

The multiplicity of $\frac{q_1}{2q_0}$ is $2i_0(\ell + 1 - 2i_0)q_0$.

Subcase 2. $A = \{i_0, i_1\}$ with $1 \leq i_0 \leq \frac{\ell+1}{2}$, $i_0 < i_1$, and $(\mathfrak{g}_0, \varphi_1)$ is without a cycle. Let $I_2 := \{j \in I_1 | \varepsilon_j = i_0\}$ and let $q_2 := |I_2|$.

Subcase 2.1. $2q_2 \neq q_1$. The slopes are $\pm\frac{q_1}{q_0}$, $\pm\frac{q_2}{q_0}$, $\pm\frac{q_1-q_2}{q_0}$ and 0.

The multiplicity of $\frac{q_1}{q_0}$ is $i_0(\ell + 1 - i_1)q_0$.

The multiplicity of $\frac{q_2}{q_0}$ is $i_0(1 + i_1 - i_0)q_0$.

The multiplicity of $\frac{q_1-q_2}{q_0}$ is $(\ell + 1 - i_1)(-i_0 + i_1)q_0$.

Subcase 2.2. $2q_2 = q_1$. The slopes are $\pm\frac{q_1}{q_0}, \pm\frac{q_1}{2q_0}$, and 0.

The multiplicity of $\frac{q_1}{q_0}$ is $i_0(\ell + 1 - i_1)q_0$.

The multiplicity of $\frac{q_1}{2q_0}$ is $\big[i_0(i_1 - i_0 + 1) + (i_1 - i_0)(\ell + 1 - i_1)\big]q_0$.

Subcase 3. $A = \{i_0, \ell+1-i_0\}$ and $(\mathfrak{g}_0, \varphi_1)$ is with involution. This is entirely similar to the subcase 1. The slopes are $-\frac{q_1}{q_0}, -\frac{q_1}{2q_0}, 0, \frac{q_1}{2q_2}$ and $\frac{q_1}{q_0}$.

The multiplicity of $\frac{q_1}{q_0}$ is $i_0^2 q_0$.

The multiplicity of $\frac{q_1}{2q_0}$ is $2i_0(\ell + 1 - 2i_0)q_0$.

Case 3. $|B| \geq 3$**.** So $|A| \geq 2$. We do not treat this case. We just mention that the multiplicity of the biggest positive slope $\frac{q_1}{q_0}$ is $i_0(\ell + 1 - i_1)$, where i_0 (resp. i_1) is the smallest (resp. the biggest) element of B.

3.10.7. Remark. If the ε-type of $(\mathfrak{g}_0, \varphi_1)$ is $(\varepsilon_1, \ldots, \varepsilon_m)$, then the parabolic Lie subalgebra $\mathfrak{p}_0$ of $\mathfrak{g}_0$ corresponding to non-negative slopes of $(\mathfrak{g}_0, \varphi_1)$ is a direct sum of its intersection with the factors $\mathfrak{g}_i$ of $\mathfrak{g}_0$ ($i \in I_0$), while its intersection with $\mathfrak{g}_i$ is the $W(k)$-module generated by $\mathfrak{t}_i$, and by the (one dimensional) $W(k)$-submodules corresponding to all positive roots (of Φ_i^+), and to those negative roots (of Φ_i^{-1}), not having $-\alpha_{\varepsilon_j}(i)$ in their expression, for any $j \in \{1, \ldots, m\}$ such that $\varepsilon_j \neq 0$.

Using this we get directly the following facts:

F1. $\mathfrak{p}_0 \cap \mathfrak{g}_i$ *is a maximal parabolic Lie subalgebra of* $\mathfrak{g}_i$ *if and only if* $(\mathfrak{g}_0, \varphi_1)$ *is of constant type.*

F2. $\mathfrak{p}_0 = F^0(\mathfrak{g}_0)$ *if and only if* $(\mathfrak{g}_0, \varphi_1)$ *is totally non-compact of constant type.*

F3. $\mathfrak{p}_0$ *is a Borel Lie subalgebra of* $\mathfrak{g}$ *if and only if* $\mathfrak{g}_0$ *is of* A_ℓ *Lie type with* $B = \{1, 2, \ldots, \ell\}$. *Here* $\ell \geq 1$.

When the situation of $F3$ takes place, we say that $\big(\mathfrak{g}_0, \varphi_1, F^0(\mathfrak{g}_0), F^1(\mathfrak{g}_0)\big)$ is of strong Borel type (cf. c) of 2.2.12), or that it has maximal spreading.

3.11. The case $k = \bar{k}$. We use the notations of 3.1-2. We assume that $(M, \varphi, G, (v_\alpha)_{\alpha\in\mathcal{J}})$ is a G-ordinary σ-crystal. What follows relies (cf. the policy of 1.15) on completely independent parts of the proof of 4.4.1.

3.11.1. Theorem. *Let* $\mathfrak{C} = (M, F^1, \varphi, G, (v_\alpha)_{\alpha\in\mathcal{J}})$ *be the* G*-canonical lift of* $(M, \varphi, G, (v_\alpha)_{\alpha\in\mathcal{J}})$. *Then:*

a) $\mathfrak{C}$ *is cyclic diagonalizable;*

b) *its Shimura filtered Lie* σ*-crystal is cyclic diagonalizable, of Borel and parabolic type;*

c) $\mathfrak{C}$ *is isomorphic to* $\big(M, F^1, \varphi_1, G, (v_\alpha)_{\alpha\in\mathcal{J}}\big)$ *mentioned in 3.2.3.*

Proof: There are many ways in proving c). One way is entirely the same as the proof of b) of 4.4.1 2) (see the extra details needed for the proof of b) of 4.4.1 3)), made in the context of a SHS. As the referred proof is self contained (based on 3.5.11), this way is not repeted here. Another way is to use 3.5.11 and 3.1.0 c). So we can assume that $\varphi = g\varphi_1$, with $g \in \mathrm{Ker}(P_1(W(k)) \to P_1(k))$ (see 3.3.3 for the meaning of P_1). We need to find $h \in \mathrm{Ker}(P_1(W(k)) \to P_1(k))$ such that $hg\varphi_1 = \varphi_1 h$. As φ_1 leaves invariant $\mathrm{Lie}(P_1)$ and as P_1 leaves invariant F^1, we can by induction, working modulo powers of p. We come across quasi Artin-Schreier systems of equations. So 3.4.8.1.2 c) applies, as $k = \bar{k}$, to give us the existence of such an element h.

a) results from c) and 2.2.19 (cf. the def. of φ_1 in 3.2.3). b) results from c) (cf. 3.7.2). This proves the theorem.

3.11.2. Remark. From 3.11.1 c) we get immediately 3.1.4 (the unicity part allows to assume that in 3.1.4 we have $k = \bar{k}$). So let $\mu_y : \mathbb{G}_m \to G$ be the canonical split of (M, φ, G) (cf. 3.1.6). We write $\varphi = a\mu_y(\frac{1}{p})$. Then a is a σ-linear automorphism of M and $M^a := \{x \in M | a(x) = x\}$ defines a $\mathbb{Z}_p$-structure on M. Let $\bar{h}_1 \in \mathrm{End}(M)$ be defined by: $\bar{h}_1(x) = x$ if $x \in F^1$, and $\bar{h}_1(x) = 0$ if $\mathbb{G}_m$ through μ_y acts trivially on x. We have $\bar{h}_1 \in \mathrm{End}\big(M^a \otimes W(\mathbb{F})\big)$ (cf. 3.11.1 c)). Let $d \in \mathbb{N}$ be such that $\varphi^d(\bar{h}_1) = \bar{h}_1$, but $\forall i \in \mathbb{N}$, $i < d$, we have $\varphi^i(\bar{h}_1) \neq \bar{h}_1$. So μ_y can be viewed as a cocharacter $\mu_y : \mathbb{G}_m \to GL\big(M^a \otimes W(\mathbb{F}_{p^d})\big)$. Let $\varphi^i(\bar{h}_1) = \bar{h}_{i+1}$, $\forall i \in \mathbb{N}$. So $\bar{h}_{d+1} = \bar{h}_1$. The $\bar{h}_i$ are commuting among themselves. Let $G_{\mathbb{Z}_p}$ be the subgroup of $GL(M^a)$ which over $W(k)$ becomes G. So (M, φ, G) is isomorphic to the extension to $W(k)$ of the following Shimura σ-crystal $\big(M^a \otimes_{\mathbb{Z}_p} W(\mathbb{F}_{p^d}), \sigma\mu_y(\frac{1}{p}), G_{W(\mathbb{F}_{p^d})}\big)$ (σ acts trivially on M^a and as Frobenius on $W(\mathbb{F}_{p^d})$).

Let ${}_A\bar{h}_1$ (resp. ${}_T\bar{h}_1$) be the image of $\bar{h}_1$ in $\mathrm{Lie}(G^{\mathrm{ad}})$ (resp. in $\mathrm{Lie}(G^{\mathrm{ab}})$), and let $d_1 \in \mathbb{N}$ (resp. $d_2 \in \mathbb{N}$) be the smallest positive integer such that ${}_A\bar{h}_1$ (resp. ${}_T\bar{h}_1$) is the image of $\bar{h}_{d_1+1}$ in $\mathrm{Lie}(G^{\mathrm{ad}})$ (resp. the image of $\bar{h}_{d_2+1}$ in $\mathrm{Lie}(G^{\mathrm{ab}})$). Then d_1 and d_2 divide d, and in fact d is the least common multiple of d_1 and d_2.

All the above facts are a consequence of 3.11.1 c) (cf. 3.7.2).

3.11.3. Degrees of definition. The number d of 3.11.2 is called the degree of definition of (M, φ, G), d_1 is called the degree of definition of $\big(\mathrm{Lie}(G^{\mathrm{ad}}), \varphi\big)$, or the A-degree of definition of (M, φ, G), and d_2 is called the degree of definition of $\big(\mathrm{Lie}(G^{\mathrm{ab}}), \varphi\big)$, or the T-degree of definition of (M, φ, G). Here A- and T- stand for adjoint and respectively toric.

3.11.4. The numbers $d(n)$ of 3.4.1.3. We come back to the notations of 3.1-4. 3.11.1 allows us to define the number $r((M, \varphi, G)) \in \mathbb{N} \cup \{0\}$, defined as the number of slopes -1 of any G-canonical lift produced by (M, φ, G). Directly from 3.4.18.4 B) we get: $\forall n \in \mathbb{N}$ we have the estimate

$$d(n) \leq \dim(G) r((M, \varphi, G)).$$

The numbers $r((M, \varphi, G))$ can be easily computed using 3.10.6. As a gross estimate we get $d(n) \leq \frac{\dim(G)^2}{2}$. In particular, if $r((M, \varphi, G)) = 0$ (there are plenty of such examples, even in the context of a SHS; see 4.6.8 1)), then $d(n) = 1$, $\forall n \in \mathbb{N}$. This result is a consequence of 3.4.18.8.1. So we can take $Q_{j,n} = R_j$, $\forall n \in \mathbb{N}$.

3.12. Shimura F-crystals and generic points. Let $\big(M, F^1, \varphi, G, (v_\alpha)_{\alpha \in \mathcal{J}}\big)$ be a Shimura filtered σ-crystal. Let $\mathfrak{C}_H = (M, F^1, \varphi, G, H, \tilde{f})$ be a Shimura filtered F-crystal with H a locally closed subscheme of G such that the morphism $H \to G/P$ is smooth in a neighborhood of the origin of H (here P is the parabolic subgroup of G leaving invariant F^1). With the notations of 2.2.10, let $\eta : \mathrm{Spec}(k_\eta) \to \mathrm{Spec}(R)$ be a geometric point sitting over the generic point of the special fibre of $\mathrm{Spec}(R)$. Let

$$\mathfrak{C}_\eta := \big(M \otimes W(k_\eta), \varphi_\eta, G_{W(k_\eta)}\big)$$

be the Shimura σ_η-crystal obtained from $(M, F^1, \varphi, G, H, \tilde{f})$ by pull back through η (here σ_η is the Frobenius automorphism of $W(k_\eta)$).

3.12.1. Theorem. *$\mathfrak{C}_\eta$ is a $G_{W(k_\eta)}$-ordinary σ_η-crystal.*

Proof: There is a Shimura filtered F-crystal $\mathfrak{C}_G := (M, F^1, \varphi, G, \tilde{f}_1)$ with the property that the Shimura σ-crystals obtained from it by pull back through geometric points over the generic point of the special fibre of the scheme over which $\mathfrak{C}_G$ is defined are G-ordinary σ-crystals (cf. 3.2.6 and 3.1.0 d)).

Let $\ell := dd((M, \varphi, G))$ (see 2.2.22 4)). We consider a filtered F-crystal $\mathfrak{C}_1$ with cycles over the special fibre of $\mathrm{Spec}(R_1)$, where $R_1 := W(k)[[x_1, ..., x_\ell]]$, which is a versal deformation of $(M, F^1, \varphi, G, (v_\alpha)_{\alpha \in \mathcal{J}})$. It can be constructed as follows. Let N be a smooth abelian subgroup of G such that the natural morphism $N \to G/P$ is an open embedding.

Then $\mathfrak{C}_1$ (cf. the remarks following [Fa2, th. 10]) is isomorphic to any Shimura filtered F-crystal $\mathfrak{C}_2 = (M, F^1, \varphi, G, N, \tilde{f}_2)$ (i.e. it is induced from $\mathfrak{C}_2$ through an isomorphism $R_2 \tilde{\to} R_1$; here $\mathrm{Spec}(R_2)$ is the completion of N at the origin).

Now the theorem results from the fact that $\mathfrak{C}_H$ and $\mathfrak{C}_G$ are both induced from $\mathfrak{C}_1$ through formally smooth $W(k)$-homomorphisms $R_1 \hookrightarrow R$ and respectively $R_1 \hookrightarrow R_G$ (as the natural morphism $H \to G/P$ is smooth around the origin and as the natural morphism $G \to G/P$ is smooth) (to be compared with [Va2, 5.4.4-8]). This ends the proof of the theorem.

3.13. Deviations of Shimura (filtered) (Lie) σ-crystals. In this section we express our hopes for building up a reasonable theory of good "G-pseudo-canonical lifts" of Shimura σ-crystals (M, φ, G) which are not potentially cyclic diagonalizable. We restrict to present some definitions, to restate some of the previous results and problems in terms of them, and to list some of the most interesting problems arising.

Let $\mathfrak{C}_0 = (M, \varphi, G)$ be a Shimura σ-crystal over k. Let $\mathcal{F}$ be the set of direct summands F^1 of M such that the quadruple (M, F^1, φ, G) is a Shimura filtered σ-crystal. From 2.2.9 3) we deduce that $\mathcal{F}$ is the $G(W(k))$-conjugacy class of any $F^1 \in \mathcal{F}$. To fix the notations we chose (arbitrarily) an element $F^1 \in \mathcal{F}$. We get a Shimura filtered σ-crystal $\mathfrak{C} := (M, F^1, \varphi, G)$. As usual, let $F^0(\mathrm{Lie}(G))$ be the parabolic Lie subalgebra of $\mathrm{Lie}(G)$ leaving invariant F^1. We assume that G is not a torus. Let $\mathfrak{p}$ be the parabolic Lie subalgebra of $\mathfrak{sl}(M))$ defined by the property that $\mathfrak{p}[\frac{1}{p}]$ is left invariant by φ and the isocrystal $(\mathfrak{p}[\frac{1}{p}], \varphi)$ is the maximal isocrystal of $(\mathfrak{sl}(M \otimes B(k)), \varphi)$ having only non-negative slopes. Let

$$\mathfrak{p}_G := \mathrm{Lie}(G^{\mathrm{der}}) \cap \mathfrak{p}.$$

Let π be a projector of $\mathfrak{gl}(M[\frac{1}{p}])$ on $\mathrm{Lie}(G_{B(k)})$ fixed by φ and G. For instance we can take π such that its kernel is the perpendicular $B(k)$-vector space on $\mathrm{Lie}(G_{B(k)})$ with respect to the trace form on $\mathfrak{gl}(M[\frac{1}{p}])$ (see [Va2, 4.2]). Let $\mathfrak{p}_G^0 := \mathfrak{p} \cap \ker(\pi)$. The $B(k)$-vector spaces $\mathfrak{p}_G^0[\frac{1}{p}]$ and $\mathfrak{p}_G[\frac{1}{p}]$ are left invariant by φ.

3.13.1. Definitions and notations. 1) By the deviation of $\mathfrak{C}$ we mean the rational number

$$d(\mathfrak{C}) = \frac{\dim\big(\mathfrak{p}_G \cap F^0(\mathrm{Lie}(G))\big)}{\dim(\mathfrak{p}_G)}.$$

2) By the complementary deviation of $\mathfrak{C}$ we mean the rational number

$$d^{\perp}(\mathfrak{C}) = \frac{\dim\big(\mathfrak{p}_G^0 \cap F^0(\mathfrak{sl}(M))\big)}{\dim(\mathfrak{p}_G^0)}.$$

3) By the global deviation of $\mathfrak{C}$ we mean the rational number

$$gd(\mathfrak{C}) = \frac{\dim\big(\mathfrak{p} \cap F^0(\mathfrak{sl}(M))\big)}{\dim(\mathfrak{p})}.$$

4) By the deviation of $\mathfrak{C}_0$ we mean the rational number $d(\mathfrak{C}_0)$ defined as the maximum of all numbers $d(\mathfrak{C})$, with the filtration F^1 running through the set $\mathcal{F}$.

5) By the complementary deviation of $\mathfrak{C}_0$ we mean the rational number $d^{\perp}(\mathfrak{C}_0)$ defined as the maximum of all rational numbers $d^{\perp}(\mathfrak{C})$, with the filtration F^1 of M running as in 4) above.

6) By the global deviation of $\mathfrak{C}_0$ we mean the rational number $gd(\mathfrak{C}_0)$ defined as the maximum of all numbers $gd(\mathfrak{C})$, with the filtration F^1 running as in 4) above.

7) By a G-pseudo-canonical lift of $\mathfrak{C}_0$ we mean a Shimura filtered σ-crystal $\tilde{\mathfrak{C}} = (M, \tilde{F}^1, \varphi, G)$ such that $d(\mathfrak{C}_0) = d(\tilde{\mathfrak{C}})$.

8) By the CM-deviation of $\mathfrak{C}_0$, we mean the number $CM - d(\mathfrak{C}_0) \in \mathbb{N} \cup \infty$, defined as follows: if there is a finite flat DVR extension V of $W(\bar{k})$ such that there is a maximal $\mathbb{Q}_p$-torus of $G_{B(\bar{k})}$ formed by elements of $\mathrm{Lie}(G_{B(\bar{k})})$ fixed by φ, and included in a parabolic subgroup of G_V, lifting the parabolic subgroup of $G_{\bar{k}}$ leaving invariant $F^1 \otimes \bar{k}$, then $CM - d(\mathfrak{C}_0)$ is the smallest possible value of $[V : W(\bar{k})]$ for such a V; if such a V does not exist, then $CM - d(\mathfrak{C}_0) = \infty$.

3.13.2. Remarks. 1) $\mathfrak{C}_0$ is a G-ordinary σ-crystal iff $d(\mathfrak{C}_0) = 1$. $\mathfrak{C}$ is a G-canonical lift iff $d(\mathfrak{C}) = 1$. This motivates our terminology: deviations of Shimura (filtered) σ-crystals. So we view the numbers introduced in 1-6) of 3.13.1 as a measure of how far a Shimura σ-crystal (M, φ, G) (resp. a Shimura filtered σ-crystal (M, F^1, φ, G)) is from being a G-ordinary σ-crystal (resp. a G-canonical lift).

2) A G-pseudo-canonical lift is not unique. This can be seen looking at σ-crystals associated to supersingular abelian varieties over k.

3) The CM-deviations are measuring how far a Shimura σ-crystal is from being potentially cyclic diagonalizable: the CM-deviation of a Shimura σ-crystal over $\bar{k}$ is 1 iff it has a lift of quasi CM type (cf. 2.2.19).

3.13.3. Variants. 1) 3.13.1 has been worked out using non-negative slopes. The same thing can be worked out using positive slopes, or negative slopes, or non-positive slopes. In such a case, as well as in others (to be defined below), we use a left upper index of the form $+$, $-$, $0+$, or $0-$ for the deviations introduced, to emphasize that they have been defined using positive, negative, non-negative, or respectively non-positive slopes. For instance we get rational numbers ${}^{+}d(\mathfrak{C})$, $p^n - {}^{0-}d(\mathfrak{C})$, etc.

2) In some cases it is useful to define deviations of higher order. By this we mean that in 1-6) of 3.13.1 we replace $F^0(\mathrm{Lie}(G^{\mathrm{der}}))$ or $F^0(\mathfrak{sl}(M))$, with their iterates under the action of φ (for instance, we replace $F^0(\mathfrak{sl}(M))$ by $\mathfrak{sl}(M) \cap \varphi^i(F^0(\mathfrak{sl}(M[\frac{1}{p}])))$, $i \in \mathbb{N} \cup \{0\}$). So we obtain sequences of deviation numbers $d_i(\mathfrak{C})$, $d_i^{\perp}(\mathfrak{C}_0)$, $gd_i(\mathfrak{C})$, $i \in \mathbb{N} \cup \{0\}$, which for $i = 0$ become the numbers introduced in 1-6) of 3.13.1.

$2'$**)** The same as above but we replace $\varphi^i(F^0(\mathfrak{sl}(M[\frac{1}{p}])))$ by $\cap_{n=0}^{i} \varphi^n(F^0(\mathfrak{sl}(M[\frac{1}{p}])))$. We obtain $\cap$ (intersection) type deviations of higher order. The notations to be used: $d_{\cap i}(\mathfrak{C})$, $d_{\cap i}^{\perp}(\mathfrak{C}_0)$, etc.

3) All the definitions of 3.13.1 make sense for Shimura (filtered) isocrystals over k. Of course, for a Shimura isocrystal (M^1, φ^1, G^1) we need to choose a set $\mathcal{F}^1$ of "admissible"

filtrations of M^1. Usually it is a $G^1(B(k))$-conjugacy class of a fixed filtration. Warning: an element of $\mathcal{F}^1$ is not always just a $B(k)$-subspace of M^1. So we speak about deviations of Shimura isocrystals.

Sometimes, for the sake of refinement, we work also with p^n-deviations of Shimura (filtered) σ-crystals, $n \in \mathbb{N}$. Not to make the story too long, here we define just the p^n-deviation of $\mathfrak{C}$, $n \in \mathbb{N}$. It is the rational number defined by:

$$p^n - d(\mathfrak{C}) = \frac{\text{length}(\mathfrak{p}_G \otimes W_n(k) \cap F^0(\text{Lie}(G) \otimes W_n(k)))}{\text{length}(\mathfrak{p}_G \otimes W_n(k))}.$$

Here the length$(*)$ is the usual length of finitely generated modules over the artinian local ring $W_n(k)$.

4) All the deviations introduced above which are not of global or complementary type, make sense for Shimura (filtered) Lie σ-crystals and (the "conventions" of 3.13.3 3) apply) for Shimura Lie isocrystals.

5) We can define isogeny deviations. They will be denoted by putting $iso-$ in front of the notation of the involved deviation. For instance, by the isogeny deviation of $\mathfrak{C}_0$, denoted $iso - d(\mathfrak{C}_0)$, we mean the biggest number $d(\mathfrak{C}_0^i)$, with $\mathfrak{C}_0^i$ running through the representatives of the isogeny class of $\mathfrak{C}_0$.

6) We can define strong CM-deviations, denoted by replacing CM by SCM, by replacing (in 3.13.1 8)) the expression "there is a maximal" by: "for any maximal".

3.13.3.1. Remarks 1). $\mathfrak{C}_0$ is a G-ordinary σ-crystal iff $p - d(\mathfrak{C}_0) = 1$. $\mathfrak{C}$ is a G-canonical lift iff $p^n - d(\mathfrak{C}) = 1$ for all $n \in \mathbb{N}$. These two facts are a consequence of the proof of 3.5.8.

2) We can introduce also the numbers

$$p^n - d_f(\mathfrak{C}) = \frac{\text{length}(\mathfrak{p}_G \otimes W_n(k) \cap F^0(\text{Lie}(G) \otimes W_n(k)))}{\text{length}(F^0(\text{Lie}(G)) \otimes W_n(k))}.$$

But these numbers can be bigger or smaller than the ones attached to a G-canonical lift. Using this we can easily construct examples for which these numbers for a G-canonical lift are not 1, and do not characterize the G-canonical lifts.

3) In 1995 we conjectured: all the $iso - SCM$-deviations are finite numbers. This conjecture is very much related to the Langlands-Rapoport conjecture referred to in 1.24, and so it will play a central role in §14.

4) We do believe that the deviations introduced in this 3.13 allow a uniform treatment of all Shimura (adjoint) (filtered) (Lie) σ-crystals, regardless of the fact they are or are not cyclic diagonalizable.

3.13.4. Problems. Many problems (questions) are arising. We mention just few of them.

1) Can a G-pseudo-canonical lift be defined in terms of $d^{\perp}(*)$, or in terms of $gd(*)$?

2) Fixing $\mathfrak{C}_0$, can we determine all the possible values of the numbers $d(\mathfrak{C})$, $d^{\perp}(\mathfrak{C})$, $gd(\mathfrak{C})$, where the filtration F^1 of M runs as in 3.13.1 4)?

3) Determine all Shimura σ-crystals $\mathfrak{C}_0$ for which we can define uniquely a G-pseudo-canonical lift of them, by using the numbers introduced in 2) and 3) of 3.13.3.

4) How are p^n-deviations ($n \in \mathbb{N}$) related to deviations?

5) How deviations (p^n-deviations, CM-deviations) vary when we replace φ by $g\varphi$ with $g \in G(W(k))$? What are their possible values?

6) What is the connection between CM-deviations and deviations?

7) Let A be an abelian variety over a number field E. For any unramified prime v of E of good reduction for A, we can define a CM-deviation and a SCM-deviation (of the σ-crystal of the reduction of A with respect to v; here σ is the Frobenius of $W(k(v))$). How are they varying in terms of v? Is it true that all these CM-deviations are 1, except a finite number?

3.13.5. Wintenberger groups and deviations. We consider the canonical split (cocharacter) $\mu(F^1) : \mathbb{G}_m \to GL(M)$ defined (cf. [Wi]) by the filtered σ-crystal (M, F^1, φ). Due to its functoriality (see loc. cit.) $\mu(F^1)$ factors through G. Let $\mathcal{W}_1$ be the subgroup of G generated by $\mu(F^1)$, with F^1 running as in 3.13.1 4). Let $\mathcal{W}_2$ be the closure in G of the smallest subgroup of $G_{B(k)}$ with the property that its Lie algebra is stable under φ and contains $\mathrm{Lie}(\mathcal{W}_1)$. Let $\mathcal{W}_{i+2}$ be the closure in $GL(M)$ of the centralizer of the generic fibre of $\mathcal{W}_i$ in $GL(M[\frac{1}{p}])$.

We call $\mathcal{W}_i$ as the i-th Wintenberger group of $\mathfrak{C}_0$.

Exercise. If (M, φ) is an ordinary σ-crystal, show that $\mathcal{W}_1$ is the extension of $\mathbb{G}_m$ by a smooth commutative unipotent group of dimension equal to $dd(\mathfrak{C}_0)$.

3.13.5.1. Versal endomorphisms of $\mathfrak{C}_0$. We consider an endomorphism $h : M \to M$ fixed by φ. We say that h is a versal endomorphism of $\mathfrak{C}_0$ if one of the following equivalent conditions is satisfied:

– h leaves invariant any element of $\mathcal{F}$;

– h is fixed by $\mathcal{W}_1$;

– h is fixed by $\mathcal{W}_2$;

– h defines naturally an endomorphism of the versal deformation of the Shimura p-divisible group over $W(k)$ defined by $\mathfrak{C}$ over a formal power series ring in $dd(\mathfrak{C}_0)$ variables with coefficients in $W(k)$.

Similarly we speak about versal automorphisms of $\mathfrak{C}_0$; in this case the above equivalent conditions are also equivalent to:

– $h \in \mathcal{W}_3(W(k))$;

– $h \in \mathcal{W}_4(W(k))$.

3.13.5.2. Comments. We do not know when $\mathcal{W}_i$ is a smooth group over $W(k)$. We do not know when the special fibre of $\mathcal{W}_i$ is a connected group. We do not know when the generic fibre of $\mathcal{W}_3$ or $\mathcal{W}_4$ is a connected group. We do not know how to compute

the relative dimension w_i of $\mathcal{W}_i$. We do not know how to determine the structure of the generic fibre of $\mathcal{W}_i$. It is easy to check that w_i is an isogeny invariant.

3.13.5.3. $\mathcal{W}$-deviations. Let $g \in G(W(k))$ such that $(M, g\varphi, G)$ is a G-ordinary σ-crystal. We similarly define $w_i(G-ord)$ as the relative dimension of the i-th Wintenberger group of $(M, g\varphi, G)$. We call the rational number

$$\frac{w_i}{w_i(G-ord)}$$

as the i-th $\mathcal{W}$-deviation or the i-th Wintenberger deviation of $\mathfrak{C}_0$. Some of the previous questions and problems can be easily adapted to the context of $\mathcal{W}$-deviations.

3.13.5.4. Relative rigidity. Let G_1 be a reductive subgroup of $GL(M)$ containing G and such that (M, φ, G_1) is a Shimura σ-crystal. We say that $\mathfrak{C}_0$ is rigid with respect to G_1 if for any versal automorphism of $\mathfrak{C}_0$ defined by an $h \in G_1(W(k))$ we have: h centralizes $\mathrm{Lie}(G^{\mathrm{der}})$. When $G_1 = GL(M)$ we drop the words "with respect to G_1".

Exercise. We assume that $\dim(M) = 8$, that G contains the group of scalar automorphisms of M, and that G is a split GL_2 group whose representation on M is a direct sum of 4 copies of the standard two-dimensional irreducible representation of G. If $dd(\mathfrak{C}_0) = 1$ (i.e. if it is non-zero), then $\mathfrak{C}_0$ is rigid.

3.14. The case $p = 2$. Looking back to 3.1-13 we see that we used the fact that $p > 2$ in precisely five places: in 3.4.2, in 3.4.18.4.2, in 3.4.18.5.1 and 3.4.18.5.3, and twice in 3.6.4.1. In 3.4.18.4.2, for the case $p = 2$, we have to restrict to the case: S is affine regular scheme. In 3.4.18.5.1 and 3.4.18.5.3 remain true for $p = 2$, due to the fact that [Fa2, th. 10] is true as well for $p = 2$. It is easy to see that in 3.6.4.1 it was not essential that $p > 2$: we can replace the argument at the level of Lie algebras in charactersistic p by an argument at the level of reductive groups in characteristic zero. But in 3.4.2 it was crucial that $p > 2$. So everything in 3.1-15, which is not based on 3.4.2, remains true for the case $p = 2$. In other words, everything in 3.1-15 except 3.4.2, 3.4.14-15, remains true for the case $p = 2$.

However, it is possible to modify the statement and the proves of 3.4.14.1-4 accordingly (i.e. without relying on 3.4.2). For this we just have to replace the assumptions SA1-2, by the statement of property G) of the proof of 3.4.14.1. So 3.4.14 and 3.4.14.1, under this restatement, are true as well for a $p = 2$ SHS.

3.15. Some conclusions. 3.4.1.3 allows us (cf. 3.4.9 3)) to strengthen the result of 3.1.8. Here $p \geq 2$ is an arbitrary prime. We have:

Fact 1. *Any p-divisible group over a perfect field $\tilde{k}$ of characteristic p admits a global deformation over the p-adic completion of an $\mathbb{N}$-pro-étale scheme* $\mathrm{Spec}(R_1)$ *over a smooth affine $W(\tilde{k})$-scheme* $\mathrm{Spec}(R)$ *(and so also over an $\mathbb{N}$-pro-étale scheme $\bar{R}_1$ over a smooth $\tilde{k}$-scheme $\bar{R}$), of whose Kodaira-Spencer map is an isomorphism. We can assume that* $\mathrm{Spec}(R_1/pR_1)$ *is connected, and that the morphism* $\mathrm{Spec}(R_1/pR_1) \to \mathrm{Spec}(R/pR)$ *is surjective. Moreover the p-divisible group over the generic point of* $\mathrm{Spec}(R_1/pR_1)$ *(or of $\bar{R}$) is ordinary.*

Similarly, in the relative context, we have:

Fact $1'$**.** *Let $\mathcal{D}(W(\tilde{k}))$ be a Shimura p-divisible group over a perfect field $\tilde{k}$ of characteristic p. Then there is a smooth $W(\tilde{k})$-algebra R, an $\mathbb{N}$-pro-étale affine R-scheme $\mathrm{Spec}(R_1)$, and a Shimura p-divisible group $\mathcal{D}$ over $\mathrm{Spec}(R_1^{\wedge})$ such that:*

a) in a $W(\tilde{k})$-valued point z of $\mathrm{Spec}(R_1^{\wedge})$, $\mathcal{D}$ becomes $\mathcal{D}(W(\tilde{k}))$;

b) $\mathcal{D}$ is a versal deformation in each geometric point of $\mathrm{Spec}(R_1/pR_1)$;

c) $\mathrm{Spec}(R_1/pR_1)$ is connected, and the morphism $\mathrm{Spec}(R_1/pR_1) \to \mathrm{Spec}(R/pR)$ is surjective.

Moreover, the resulting Shimura σ-crystal $\mathcal{C}$ over the algebraic closure of the field of fractions of R_1/pR_1 is Shimura-ordinary.

The parts involving (Shimura) ordinary properties of the above facts are a consequence of 3.12.1. For why the above facts are true also for $p = 2$, cf. the last part of 3.14. The expectation of 3.4.8.9.1 gets translated: in the above facts, it should be possible to avoid the complication with $\mathbb{N}$-pro-étale morphisms (and so to avoid introducing R_1). For the case $k = \bar{k}$ this is achieved in 4.12.12.

§4 *Applications of the basic results to integral canonical models of Shimura varieties of preabelian type, to p-divisible groups, and to abelian varieties*

§3 contains much more than what is needed for applications to integral canonical models of Shimura varieties of preabelian type. In 4.1-8 and 4.11 which form a sequence, continuing 2.3, we try to make as little use of §3 as possible. In particular, we avoid the use of 3.4, except of its independent lemma 3.4.6 (the use of 3.4.6 is in fact entirely avoidable cf. 3.5.14).

We start with a SHS $(f, L_{(p)}, v)$.

4.1. The G-ordinary type. Let $L_p^* := L_{(p)}^* \otimes \mathbb{Z}_p$. We view $G_{\mathbb{Z}_p}$ also as a subgroup of $GL(L_p^*)$. Let T be a maximal torus of $G_{\mathbb{Z}_p}$ such that:

1) there is a cocharacter $\mu : \mathbb{G}_m \hookrightarrow T_{W(k(v))}$ which over an embedding of $W(k(v))$ in $\mathbb{C}$, extending the inclusion $E(G,X) \subset \mathbb{C}$, is $G(\mathbb{C})$-conjugate to the cocharacters $\mu_x^* : \mathbb{G}_m \hookrightarrow G_{\mathbb{C}}$, $x \in X$;

2) there is a Borel subgroup B of $G_{\mathbb{Z}_p}$, whose Lie algebra leaves invariant the F^1 filtration of $L_p^* \otimes W(k(v))$ defined by μ (i.e. we have $\mathrm{Lie}(B) \subset F^0\big(\mathrm{Lie}(G_{W(k(v))})\big)$).

For instance, we can take T to contain a maximal split $\mathbb{Z}_p$-torus of $G_{\mathbb{Z}_p}$, and over $W(k(v))$ to contain a maximal split $W(k(v))$-torus of $G_{W(k(v))}$, and then we can take B to be a Borel subgroup of $G_{\mathbb{Z}_p}$, containing T (cf. the definition of the reflex field: $W(k(v))\big[\frac{1}{p}\big]$ is the field of definition of the conjugacy class of cocharacters of $G_{W(\overline{k(v)})}$ which under an embedding of $W(\overline{k(v)})[\frac{1}{p}]$ into $\mathbb{C}$, extending the inclusion $E(G,X) \subset \mathbb{C}$, are $G(\mathbb{C})$-conjugate to the cocharacters μ_x, $x \in X$; see [Mi3, 4.6-7]).

B not necessarily contains T, but it must contain the smallest subtorus T_μ of T with the property that the cocharacter μ of $T_{W(k(v))}$ factors through $T_{\mu\, W(k(v))}$.

We recall that μ gives birth to a direct sum decomposition $L_{(p)}^* \otimes W(k(v)) = F^1 \oplus F^0$, with $\beta \in \mathbb{G}_m\big(W(k(v))\big)$ acting through μ on F^i as the multiplication with β^{-i}, $i = \overline{0,1}$.

4.1.1. The G-ordinary type τ associated to (or defined by) the SHS $(f, L_{(p)}, v)$ is the formal isogeny type associated to the σ-crystal

$$(L_{(p)}^* \otimes W(k(v)), \varphi)$$

defined by $\varphi := \sigma \circ \mu\big(\frac{1}{p}\big)$. Here σ acts as identity on L_p^* and as Frobenius on $W(k(v))$, and $\frac{1}{p} \in \mathbb{G}_m\big(W(k(v)\big[\frac{1}{p}\big])\big)$.

4.1.1.0. Convention. We usually write $\sigma\mu(\frac{1}{p})$ instead of $\sigma \circ \mu(\frac{1}{p})$.

4.1.1.1. Let $M := L_{(p)}^* \otimes W(k(v))$, and let $\mathfrak{t} := \mathrm{Lie}\big(T_{W(k(v))}\big)$. Let $\bar{h}_1 : M \to M$ be the endomorphism defined by $\bar{h}_1(x) = 0$ if $x \in F^0$, and by $\bar{h}_1(x) = x$ if $x \in F^1$. $\bar{h}_1$ is an

element of $\mathfrak{t}$. Let $d \in \mathbb{N}$ such that $|k(v)| = p^d$; so $k(v) = \mathbb{F}_{p^d}$. Then $\varphi^d(\bar{h}_1) = \bar{h}_1$, and $\forall i \in \{1, 2, \ldots, d-1\}$, $\varphi^i(\bar{h}_1) := \bar{h}_{i+1} \neq \bar{h}_1$. Let

$$h^0 = \frac{1}{d} \sum_{i=1}^{d} \bar{h}_i.$$

We have $\varphi(h^0) = h^0$. Then the slopes of the σ-crystal (M, φ) are the eigenvalues of h^0 (acting on M), and the multiplicities are the same. This results from the fact that if $x \in M_i := \{x \in M | h^0(x) = \frac{i}{d}x\}$, with $i \in \{0, 1, \ldots, d\}$, and $x \notin pM$, then $\varphi^d(x) = p^i y$, with $y \in M_i$, but $y \notin pM$. We have $M = \bigoplus_{0 \leq i \leq d} M_i$ (as $\bar{h}_i \in \mathfrak{t}$, $\forall i \in \{1, ..., d\}$).

4.1.1.2. Let $\mathrm{Lie}_G(\tau)$ be the formal isogeny type associated to the Lie σ-crystal

$$\big(\mathrm{Lie}\big(G_{W(k(v))}\big), \varphi\big).$$

As above, the slopes of it are the eigenvalues of h^0 (acting on $\mathrm{Lie}\big(G_{W(k(v))}\big)$), and the multiplicities are the same. We call it the Lie G-ordinary type associated to (or defined by) the SHS $(f, L_{(p)}, v)$.

4.1.1.3. The G-ordinary type τ depends on f and v, but not on $L_{(p)}$, or on the choice of T or B. The Lie G-ordinary type $\mathrm{Lie}_G(\tau)$ depends only on the Shimura pair (G, X) and on v.

4.1.1.4. For any $i \in \{1, \ldots, d\}$ and $j \in \{0, 1\}$ we denote by ${}^iF^j$ the free $W(k(v))$-submodule of M on which $\bar{h}_i$ acts trivially if $j = 0$, and identically if $j = 1$. For any function $\bar{f} : \{1, \ldots, d\} \to \{0, 1\}$ we denote by

$$F_{\bar{f}} := \bigcap_{i \in \{1, \ldots, d\}} {}^iF^{\bar{f}(i)}.$$

Let $\mathcal{L}$ be the set of functions $\bar{f}$ as above for which $F_{\bar{f}} \neq 0$. We get a direct sum decomposition

$$M = \oplus_{\bar{f} \in \mathcal{L}} F_{\bar{f}}$$

due to the fact that all endomorphisms $\bar{h}_i$ of M are commuting among themselves. Let

$$\bar{\varphi} : \mathcal{L} \to \mathcal{L}$$

be the bijection defined by the rule: $\bar{\varphi}(\bar{f})(i) = \bar{f}(i-1)$; here $\bar{f}(0) := \bar{f}(d)$. We get: $\varphi(F_{\bar{f}}) \subset F_{\bar{\varphi}(\bar{f})}$, and $\varphi(F_{\bar{f}}) = F_{\bar{\varphi}(\bar{f})}$ if $\bar{f}(1) = 0$, and $\varphi(F_{\bar{f}}) = pF_{\bar{\varphi}(\bar{f})}$ if $\bar{f}(1) = 1$.

Let $\bar{\varphi} = \prod_{i \in I_{\bar{\varphi}}} \bar{\varphi}_i$ be written as a product of disjoint cyclic permutations. We allow trivial cyclic permutations. So we have a disjoint union $I_{\bar{\varphi}} = I^1_{\bar{\varphi}} \cup I^0_{\bar{\varphi}}$; $i \in I_{\bar{\varphi}}$ belongs to $I^1_{\bar{\varphi}}$ iff $\bar{\varphi}_i$ is a non-trivial permutation (and then any $\bar{f} \in \mathcal{L}$ such that $\bar{\varphi}_i(\bar{f}) \neq \bar{f}$ is said to be associated to $\bar{\varphi}_i$). So $I^0_{\bar{\varphi}} = \{\bar{f} \in \mathcal{L} | \bar{\varphi}(\bar{f}) = \bar{f}\}$; any $\bar{f} \in I^0_{\bar{\varphi}}$ is said to be associated to $\bar{\varphi}_{\bar{f}}$. $I^0_{\bar{\varphi}}$ has either zero or two elements. As $\varphi^d = 1_{\mathcal{L}}$, we get that the order d_i of the cyclic

permutation $\bar{\varphi}_i$ divides d. For any $i \in I_{\bar{\varphi}}$, we choose arbitrarily an element $\bar{f}_i$ of $\mathcal{L}$ which is associated to $\bar{\varphi}_i$. Then $\varphi^{d_i}(F_{\bar{f}_i}) = p^{\frac{d_i}{d}\ell(\bar{f}_i)}F_{\bar{f}_i}$, with $\ell(\bar{f}_i) := \sum_{s=1}^{d}\bar{f}_i(s)$, and φ^{d_i} acts on $F_{\bar{f}_i}$ as $p^{\frac{d_i}{d}\ell(\bar{f}_i)}\sigma^{d_i}$. Let

$$ {}_pF_{\bar{f}_i} := \big\{x \in F_{\bar{f}_i} | \sigma^{d_i}(x) = x\big\}.$$

It is a $W(\mathbb{F}_{p^{d_i}})$-free submodule of $F_{\bar{f}_i}$ such that $F_{\bar{f}_i} = {}_pF_{\bar{f}_i} \otimes_{W(\mathbb{F}_{p^{d_i}})} W(k(v))$.

4.1.2. Proposition. *The filtered σ-crystal (M, F^1, φ) is cyclic diagonalizable.*

Proof: We choose arbitrarily a $W(\mathbb{F}_{p^{d_i}})$-basis $\{\ell_s | s \in \mathcal{J}_i\}$ of ${}_pF_{\bar{f}_i}, \forall i \in I_{\bar{\varphi}}$. We view it as a $W(k(v))$-basis of $F_{\bar{f}_i}$. For any cyclic permutation $\bar{\varphi}_i$ of length ≥ 2 (i.e. when we deal with an $i \in I^1_{\bar{\varphi}}$), and for any element $\bar{f} \in \mathcal{L}$ associated to $\bar{\varphi}_i$, but different from $\bar{f}_i$, let j be the smallest natural number such that $\bar{f} = \bar{\varphi}^j(\bar{f}_i)$. We get a $W(k(v))$-basis of $F_{\bar{f}}$ by taking $\frac{1}{p^{e_j}}$ of the images of $\varphi^j(\ell_s)$, $s \in \mathcal{J}_i$, with $e_j = \sum_{n=1}^{j}\bar{f}_i(n)$. With respect to the $W(k(v))$-basis of M obtained by putting together the chosen bases of $F_{\bar{f}}$, $\bar{f} \in \mathcal{L}$, (M, F^1, φ) gets the desired cyclic diagonalizable form. This ends the proof of the proposition.

4.1.3. Example. $\mathcal{L}$ has two elements iff τ is an ordinary type, i.e. iff $d = 1$.

4.1.4. Proposition. *The Shimura filtered σ-crystal*

$$\big(\mathrm{Lie}(G_{\mathbb{Z}_p}) \otimes W(k(v)), \varphi, F^0\big(\mathrm{Lie}(G_{W(k(v))})\big), F^1\big(\mathrm{Lie}(G_{W(k(v))})\big)\big)$$

and its adjoint are cyclic diagonalizable and of Borel type.

Proof: The fact that the mentioned two Shimura filtered Lie σ-crystals are of Borel type results from the fact that $\varphi(\mathrm{Lie}(B_{W(k(v))})) \subset \mathrm{Lie}(B_{W(k(v))})$ (we recall that B is a Borel subgroup of $G_{\mathbb{Z}_p}$). The proof of their cyclic diagonalizability is entirely analogous to the proof of 4.1.2, and so it is omitted.

4.1.5. Let $N \subset G_{\mathbb{Z}_p}$ be the normalizer of T. Let

$$W_G := (N/T)\big(W(\overline{k(v)})\big)$$

be the Weyl group (of $G_{W\left(\overline{k(v)}\right)}$) with respect to $T_{W(\overline{k(v)})}$. It depends only on G. For any $\omega \in W_G$, let $g_\omega \in N(W(\overline{k(v)}))$ be such that its image in W_G is ω. Let

$$\mathfrak{C}_\omega := (M \otimes W(\overline{k(v)}), F^1 \otimes W(\overline{k(v)}), g_\omega\sigma\mu(\frac{1}{p}), G_{W(\overline{k(v)})})$$

be the Shimura filtered σ-crystal defined by g_ω. We denote by $\mathcal{P}_\omega$ the Newton polygon of $\mathfrak{C}_\omega$, and by $\mathrm{Lie}_G(\mathcal{P}_\omega)$ the Newton polygon of the Shimura filtered Lie σ-crystal attached to $\mathfrak{C}_\omega$. We have:

4.1.6. Proposition. *$\mathfrak{C}_\omega$ is cyclic diagonalizable, its attached Shimura filtered (adjoint) Lie σ-crystal is cyclic diagonalizable, and $\mathcal{P}_\omega$ and $\mathrm{Lie}_G(\mathcal{P}_\omega)$ depend only on ω and not on the choice of g_ω.*

Proof: The proof of the cyclic diagonalizability of $\mathfrak{C}_\omega$ or of its attached Shimura filtered (adjoint) Lie σ-crystal is entirely analogous to the proof of 4.1.2, and so it is omitted. If $t \in T\big(W(\overline{k(v)})\big)$, then $\big(tg_\omega\sigma\mu(\frac{1}{p})\big)^n = t_n\big(g_\omega\sigma\mu(\frac{1}{p})\big)^n$, $\forall n \in \mathbb{N}$, with $t_n \in T\big(W(\overline{k(v)})\big)$, so the last part of 4.1.6 results from 3.2.1.

4.1.7. We view the alternating form $\tilde{\psi}$ of 2.3.1 also as a perfect alternating form $\tilde{\psi} : M \otimes M \to W(k(v))(1)$.

4.2. G-ordinary points. In all that follows, k is a perfect field containing $k(v)$.

4.2.1. Theorem. *There is a $G(\mathbb{A}_f^p)$-invariant dense open subscheme $\mathcal{U}$ of $\mathcal{N}_{k(v)}$ such that:*

a) a point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ has τ as the formal isogeny type of the Shimura σ-crystal attached to it (cf. 2.3.10) if and only if y factors through $\mathcal{U}$;

b) a point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ has $\mathrm{Lie}_G(\tau)$ as the formal isogeny type of the Shimura Lie σ-crystal attached to it (cf. 2.3.10), if and only y factors through $\mathcal{U}$.

Proof: Let $\mathbb{F} := \overline{k(v)}$ and let $V_0 := W(\mathbb{F})$. We still denote by σ the Frobenius automorphism of V_0. Let $\mathcal{N}_{V_0} = \mathcal{N} \times_{O(v)} V_0$ and let $\mathcal{N}_\mathbb{F}$ be its special fibre. Let $y_0 : \mathrm{Spec}(\mathbb{F}) \hookrightarrow \mathcal{N}_\mathbb{F}$ be a closed point and let $z_0 : \mathrm{Spec}(V_0) \hookrightarrow \mathcal{N}_{V_0}$ be a lift of it. Let $\mathcal{N}_\mathbb{F}^0$ be the connected component of $\mathcal{N}_\mathbb{F}$ containing y_0. Let (A_0, p_{A_0}) be the principally polarized avelian variety over V_0 obtained from $(\mathcal{A}, \mathcal{P}_\mathcal{A})$ by pull via z_0. Let $\big(M_0, F_0^1, \varphi_0, G_{V_0}, (t_\alpha)_{\alpha\in\mathcal{J}'}, p_{A_0}\big)$ be the principally quasi-polarized Shimura filtered σ-crystal attached to the point z_0 (cf. 2.3.10). So $M_0 = H^1_{dR}(A_0/V_0)$, F_0^1 is its Hodge filtration defined by A_0, and $(t_\alpha)_{\alpha\in\mathcal{J}'}$ is the family of de Rham components of the family $(\omega_\alpha)_{\alpha\in\mathcal{J}'}$ of Hodge cycles with which A_0 is naturally endowed (cf. 2.3.3). From 2.3.13.1 we deduce that for any point $z_1 : \mathrm{Spec}(V_0) \hookrightarrow \mathcal{N}_{V_0}$ lifting a point $y_1 : \mathrm{Spec}(\mathbb{F}) \hookrightarrow \mathcal{N}_\mathbb{F}^0$, the Shimura filtered σ-crystal attached to it is of the form $\big(M_0, F_0^1, \varphi_1, G_{V_0}, (t_\alpha)_{\alpha\in\mathcal{J}'}\big)$. We have $\varphi_1 = g_1\varphi_0$, with $g_1 \in G^0_{V_0}(V_0)$ (cf. the beginning of §3). From 2.3.16 we deduce that:

(4.2.1.1) *For any $\mathbb{F}$-valued point $\bar{g}_1$ of an open subscheme of $G_\mathbb{F}$ there is $g_1 \in G_{V_0}(V_0)$ lifting it, and there is an V_0-valued point z_1 of $\mathcal{N}_{V_0}$ lifting an $\mathbb{F}$-valued point y_1 of $\mathcal{N}_\mathbb{F}^0$, such that we have $\varphi_1 = g_1\varphi_0$.*

Let $\mathcal{U}_0$ (resp. $\mathcal{U}_1$) be the open subscheme of $\mathcal{N}_\mathbb{F}^0$ of whose geometric points have the smallest Newton polygon of their attached Shimura σ-crystals (resp. of their attached Shimura Lie σ-crystals). Such open subschemes due exist, cf. the semicontinuity theorem [Ka2, 2.3.1].

As all σ-crystals (M_0, φ_1) give birth to a 1/2-symmetric isocrystals we deduce:

(4.2.2) *The fact that the σ-crystal (M_0, φ_1) attached to the point $y_1 \in \mathcal{N}_\mathbb{F}^0(\mathbb{F})$ has the smallest Newton polygon among all σ-crystals attached to $\mathbb{F}$-valued points of $\mathcal{N}_\mathbb{F}^0$, is equivalent to the fact that the Lie σ-crystal $\big(\mathrm{End}(M_0), \varphi_1\big)$ has the smallest Newton polygon among all Lie σ-crystals $\big(\mathrm{End}(M_0), \tilde{\varphi}\big)$ defined naturally by the σ-crystals attached to $\mathbb{F}$-valued points of $\mathcal{N}_\mathbb{F}^0$.*

As the Newton polygon of $\big(\mathrm{End}(M_0), \varphi_1\big)$ is obtained from the Newton polygon of $\big(\mathrm{Lie}(G_{V_0}), \varphi_1\big)$ and from the Newton polygon of $\big(\mathrm{End}(M_0)/\mathrm{Lie}(G_{V_0}), \varphi_1\big)$, we deduce that

$\mathcal{U}_0 \subset \mathcal{U}_1$. $\big(\mathrm{End}(M_0)/\mathrm{Lie}(G_{V_0}), \varphi_1\big)$ is not a Lie σ-crystal, but

$$\big(\mathrm{End}(M_0)/\mathrm{Lie}(G_{V_0}), \varphi_1, F^0(\mathrm{End}(M_0))/F^0(\mathfrak{g}), F^1(\mathrm{End}(M_0))/F^1(\mathfrak{g})\big)$$

is a p-divisible object of $\mathcal{MF}_{[-1,1]}(V_0)$.

4.2.3. Let $K_0 := V_0[\frac{1}{p}]$. It is known that there is an isomorphism

$$H^1_{\acute{e}t}(A_{0\bar{K}_0}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K_0 \tilde{\to} M_0 \otimes_{V_0} K_0$$

taking the p-component of the étale component u_α of ω_α into t_α, $\forall \alpha \in \mathcal{J}'$, and taking $\tilde{\psi}$ into p_{A_0} (cf. [Va2, 5.1.4 and 5.2.17.2]). But $H^1_{\acute{e}t}(A_{0\bar{K}_0}, \mathbb{Q}_p)$ and the tensors $(u_\alpha)_{\alpha\in\mathcal{J}'}$ of $\mathcal{T}\big(H^1_{\acute{e}t}(A_{0\bar{K}_0}, \mathbb{Q}_p)\big)$ can be identified with $L_p \otimes \mathbb{Q}_p$ and the tensors $(v_\alpha)_{\alpha\in\mathcal{J}'}$ of $\mathcal{T}(L_p \otimes \mathbb{Q}_p)$ (cf. [Va2, 5.1.2]). So we get an isomorphism

$$\rho : M \otimes_{W(k(v))} K_0 \tilde{\to} M_0 \otimes_{V_0} K_0$$

taking v_α into t_α, $\forall \alpha \in \mathcal{J}'$, and $\tilde{\psi}$ into p_{A_0}.

The isomorphism ρ allows us to identify the Frobenius automormism φ of $M[\frac{1}{p}]$ defined in 4.1 with a σ-linear automorphism of $M_0 \otimes_{V_0} K_0$. So we have $\varphi = \tilde{g}\varphi_0$ with $\tilde{g} \in G^0(K_0)$. There is a cocharacter $\mu_0 : \mathbb{G}_m \to G_{V_0}$ which produces a direct sum decomposition $M_0 = F^1_0 \oplus F^0_0$, with $\beta \in \mathbb{G}_m(V_0)$ acting through μ on F^i_0 as the multiplication with β^{-i} (cf. 2.3.10). Under an embedding of V_0 in $\mathbb{C}$, extending the embedding $E(G, X) \hookrightarrow \mathbb{C}$, μ_0 becomes $G(\mathbb{C})$-conjugate to the cocharacters μ^*_x, $x \in X$. We deduce that μ_0 and μ are conjugate over K_0, as G_{K_0} is a split group. So we can assume that

$$\rho(F^1 \otimes_{W(k(v)} K_0) = F^1_0 \otimes K_0.$$

If G is a torus, 4.2 is trivial: we have $\mathcal{U} = \mathcal{N}_{k(v)}$ (cf. [Va2, 3.2.8 and 3.3.1-2]). So we can assume that G is not a torus.

4.2.4. We got two particular hyperspecial subgroups of $G(K_0) : G_{V_0}(V_0)$ (defined by the V_0-lattice M_0 of $M_0 \otimes K_0$) and $G_{\mathbb{Z}_{(p)}}\big(M \otimes_{W(k(v))} V_0\big)$ (defined by the V_0-lattice $M \otimes_{W(k(v))} V_0$ of $M \otimes K_0$). But any two hyperspecial subgroups of $G(K_0)$ are conjugate by an element $g_0 \in G^{\mathrm{ad}}(K_0)$ (cf. [Ti, p. 47]). Changing the Shimura adjoint filtered Lie σ-crystal attached to z_0 by the inner automorphism defined by g_0, we deduce that it can be put in the form $\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0}), \varphi'_0, F^0_0\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big), F^1_0\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big)\big)$. As μ and μ_0 are conjugate over K_0, changing everything by an isomorphism (defined by an element of $G^{\mathrm{ad}}_{V_0}(V_0)$), we can assume that $F^0_0\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big) = F^0\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big)$ and $F^1_0\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big) = F^1\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big)$. As $\varphi = \tilde{g}\varphi_0$ with $\tilde{g} \in G^0(K_0)$, we deduce that the Shimura adjoint filtered Lie σ-crystal attached to z_0 is isomorphic to $\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0}), g\varphi, F^0\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big),\ F^1\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big)\big)$, with $g \in G^{\mathrm{ad}}_{V_0}(K_0)$. As g leaves invariant $\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})$, and as $G^{\mathrm{ad}}_{V_0}(V_0)$ is a maximal compact subgroup of $G^{\mathrm{ad}}_{V_0}(K_0)$ (being a hyperspecial subgroup of it), we deduce that $g \in G^{\mathrm{ad}}_{V_0}(V_0)$.

This last fact allows us, via 3.6.5, to appeal to the results of 3.1-3, 3.4.6 and 3.5.

4.2.5. 4.2.1.1 implies: there is an open subscheme U of $G^{\mathrm{ad}}_{\mathbb{F}}$ with the property that for any element $\bar{g}_0 \in U(\mathbb{F})$, there is a V_0-valued point of $\mathcal{N}_{V_0}$ lifting an $\mathbb{F}$-valued point of $\mathcal{N}^0_{\mathbb{F}}$ such that its attached Shimura adjoint filtered Lie σ-crystal is isomorphic to

$$\Big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0}), g_0 g\varphi, F^0\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big), F^1\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0})\big)\Big),$$

with g_0 mod p equal to $\bar{g}_0$.

4.2.6. Now 4.2.1 for $\mathcal{N}^0_{\mathbb{F}}$ is a direct consequence of 4.2.5, of the proof of 3.1.0, of 3.4.6, of 3.5.11, of 3.6.5, and of 4.2.7 below. We present the details. From §3 we need:

– 3.3.4 which implies that $\big(M, \varphi, G_{W(k(v))}\big)$ is a $G_{W(k(v))}$-ordinary σ-crystal.

– 3.4.6 and 3.5.11 which together with 4.2.5 imply (via 3.6.5) that there are $\mathbb{F}$-valued points of $\mathcal{N}^0_{\mathbb{F}}$ whose attached Shimura Lie σ-crystals have $\mathrm{Lie}_G(\tau)$ as their formal isogeny type.

– 3.2.5 and 3.6.5 which together with 4.2.5 imply that the $\mathbb{F}$-valued points of $\mathcal{U}_1$ have $\mathrm{Lie}_G(\tau)$ as the formal isogeny type of their attached Shimura Lie σ-crystals; we recall that $\mathrm{Lie}\big(Z(G_{V_0})\big)$ contributes only with slopes zero to the formal isogeny type of $\big(\mathrm{Lie}(G_{V_0}), g\varphi_0\big)$, $\forall g \in G_{V_0}(V_0)$, cf. 3.5.0.

– 3.2.5 which implies that $\mathcal{U}_0 = \mathcal{U}_1$.

In fact the use of 3.4.6 can be avoided, cf. 3.5.14. So we need to have $\mathcal{U}_{\mathbb{F}} \cap \mathcal{N}^0_{\mathbb{F}} = \mathcal{U}_1 = \mathcal{U}_0$.

4.2.7. The fact that the $\mathbb{F}$-valued points of $\mathcal{U}_1$ give birth to Shimura σ-crystals having precisely τ as their formal isogeny type results form the fact that under the isomorphism ρ of 4.2.3, the slopes of (M, φ) together with their multiplicities, and the slopes of the Shimura σ-crystal attached to a point of $\mathcal{U}_1(\mathbb{F})$ together with their multiplicities, are both computed using the eigenvalues (and their multiplicities) of two semisimple elements s_1 and respectively s_2 of $G_{K_0}(K_0)$ which are conjugate by an element of $G^{\mathrm{ad}}_{V_0}(K_0)$, and so are conjugate by an element of $G_{V_0}(\overline{K_0})$ as the homomorphism $G_{V_0}(\overline{K_0}) \to G^{\mathrm{ad}}_{V_0}(\overline{K_0})$ is surjective.

To see this, for being able to use the previous notations, we assume that y_0 factors through $\mathcal{U}_0$. A sufficiently high power n of φ (resp. of φ_0) acts diagonally with respect to a $W(\mathbb{F})$-basis of $M \otimes W(\mathbb{F})$ (resp. of M_0), producing (via 3.11.2) the semisimple element s_1 (resp. s_2) of $G_{K_0}(K_0)$ (we use the identification achieved by ρ). As the cocharacter of $G^{\mathrm{ab}}_{K_0}$ obtained by composing μ with the quotient homomorphism $m_q : G_{K_0} \to G^{\mathrm{ab}}_{K_0}$, is the same as the one obtained by composing μ_0 with m_q (as μ and μ_0 are $G_{K_0}(K_0)$-conjugate), we deduce that the images of s_1 and s_2 in $G^{\mathrm{ab}}_{K_0}(K_0)$ are the same. Moreover the images of s_1 and s_2 in $G^{\mathrm{ad}}_{K_0}(K_0)$ are $G^{\mathrm{ad}}_{K_0}(K_0)$-conjugate: this is an immediate consequence of 4.2.4-5, via 3.5.11 and 3.6.5. Let q be the order of the isogeny $G^{\mathrm{der}} \to G^{\mathrm{ad}}$. We deduce that s_1^q and s_2^q are $G^{\mathrm{ad}}_{K_0}(K_0)$-conjugate.

4.2.8. The formal isogeny types τ and $\mathrm{Lie}_G(\tau)$ depend only on f and v, and so what we got for $\mathcal{N}^0_{\mathbb{F}}$ remains true for all other connected components of $\mathcal{N}_{\mathbb{F}}$. This implies that the open subscheme $\mathcal{U}$ of $\mathcal{N}_{k(v)}$ defined by 4.2.1 a) exists, is also defined by 4.2.1 b), and

moreover is Zariski dense in $\mathcal{N}_{k(v)}$. As the Newton polygon is an isogeny invariant we get that $\mathcal{U}$ is $G(\mathbb{A}_f^p)$-invariant. This ends the proof of 4.2.1.

Another way of passing from one connected component of $\mathcal{N}_{\mathbb{F}}$ to all others is to use the following fact:

4.2.8.1.* Fact. *The connected components of $\mathcal{N}_{\mathbb{F}}$ are permuted transitively by $G(\mathbb{A}_f^p)$ (cf. [Va2, 3.3.2] and the existence of smooth toroidal compactifications of $\mathcal{N}$, see [Va3]; if (G,X) is of compact type, then this fact is a consequence of 2.3.3.1).*

4.2.9. Definition. The points of $\mathcal{U}_{k(v)}$ (resp. of $\mathcal{U}_{k(v)}/\tilde{H}_0$) with values in fields are called G-ordinary points, or Shimura-ordinary points of $\mathcal{N}_{k(v)}$ (resp. of $\mathcal{N}_{k(v)}/\tilde{H}_0$). Here $\tilde{H}_0$ is an arbitrary compact subgroup of $G(\mathbb{A}_f^p)$.

4.2.10. Corollary. *The automorphism class (see 2.2.11 for a definition of this) of the Shimura adjoint σ-crystals attached to points of $\mathcal{N}_{k(v)}$ with values in a perfect field k is given by the Frobenius σ-automorphism of* $\mathrm{Lie}(G^{\mathrm{ad}}_{W(k)}) = \mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p}) \otimes W(k)$ *acting trivially on* $\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p})$.

Proof: This is nothing else but 4.2.5. The fact that we can assume that k is just perfect is implied by the convention 2.3.9.2, and by the fact that $G_{W(k)}$ is split (cf. 2.3.9); so the isomorphism ρ of 4.2.3 exists as well in this case.

4.3. The (refined) Lie stable p-rank.

4.3.1. Let $G^{\mathrm{ad}}_{\mathbb{Z}_p} = \times_{i\in\mathcal{H}} G_i$ be the product decomposition of $G^{\mathrm{ad}}_{\mathbb{Z}_p}$ in $\mathbb{Z}_p$-simple factors. So any $G_{i\,\mathbb{Q}_p}$ is a $\mathbb{Q}_p$-simple adjoint group. Without specifically stating, we assume that $k = \bar{k}$. For any point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ the Shimura adjoint Lie σ-crystal $\big(\mathrm{Lie}(G^{\mathrm{ad}}_{W(k)}), \varphi_y\big)$ attached to y is a product of $|\mathcal{H}|$-factors $\big(\mathrm{Lie}(G_{i\,W(k)}), \varphi_y\big)$, $i \in \mathcal{H}$, each one being a cyclic Shimura adjoint Lie σ-crystal. In other words φ_y permutes (cyclically) transitively the Lie algebras of the $W(k)$-simple factors of $G_{i\,W(k)}$ (this is a consequence of 4.2.10 and of the fact that for any $i \in \mathcal{H}$, $G_{i\mathbb{Q}_p}$ is the Weil restriction of an absolutely simple adjoint group over a finite field unramified extension of $\mathbb{Q}_p$). We call these factors the cyclic adjoint factors attached to the point y. We also speak about the cyclic Lie factors atached to y (cf. def. 3.10.1). We naturally extend the terminology to points $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$, or to the context of the quotient $\mathcal{N}/H_0$, or to the case when k is not algebraically closed, but just perfect (in this last case the adjoint groups involved are still products of absolutely simple adjoint groups; this is a consequence of 4.2.5 and of 3.10.0.1).

4.3.1.1. We write $G_{i\,W(k)} = \times_{j\in\mathcal{H}_i} G_j$, with G_j an absolutely simple adjoint $W(k)$-group. So $G^{\mathrm{ad}}_{W(k)} = \times_{\substack{i\in\mathcal{H}\\ j\in\mathcal{H}_i}} G_j = \times_{\ell\in I} G_\ell$, where I is the (disjoint) union of $\mathcal{H}_i$, $i \in \mathcal{H}$. So φ_y permutes the factors $\mathrm{Lie}(G_\ell)$, $\ell \in I$, achieving a permutation γ of I. γ is a product of $|\mathcal{H}|$-disjoint cyclic permutations γ_i, with γ_i the cyclic permutation of $\mathcal{H}_i$ defined by φ_y.

4.3.1.2. Notations. We denote by R_y (resp. R_y^r) the Lie stable p-rank (resp. the refined Lie stable p-rank) of the Shimura adjoint σ-crystal attached to y (cf. def. 3.9.1 and 3.9.4).

We denote by $R(\tau)$ (resp. $R^r(\tau)$) the Lie stable p-rank (resp. the refined Lie stable p-rank) of the Shimura adjoint filtered Lie σ-crystal attached to the Shimura σ-crystal $\big(M, F^1, \varphi, G_{W(k(v))}\big)$ defined in 4.1.

4.3.2. R_y is a non-negative integer. Fixing a bijection $\mathcal{H} \overset{f_{\mathcal{H}}}{\to} \{1, 2, \ldots, |\mathcal{H}|\}$, R_y^r is a sequence of non-negative integers, indexed by the elements of the set $\{1, ..., |\mathcal{H}|\}$. These elements index the $\mathbb{Q}_p$-simple factors of $G^{\mathrm{ad}}_{\mathbb{Q}_p}$. The n-th number of the sequence is the Lie stable p-rank of the cyclic Shimura adjoint Lie σ-crystal $\Big(\mathrm{Lie}\big(G_{f_{\mathcal{H}}^{-1}(n)}W(k)\big), \varphi_y\Big)$ (here $n \in \{1, \ldots, |\mathcal{H}|\}$). If $k \neq \bar{k}$, we still denote by R_y (resp. R_y^r) the Lie stable p-rank (resp. the refined Lie stable p-rank) of the point $\bar{y}$ defined by the composite morphism $\mathrm{Spec}(\bar{k}) \overset{i_k}{\to} \mathrm{Spec}(k) \overset{y}{\to} \mathcal{N}_{k(v)}$, with i_k defined by the natural inclusion $k \hookrightarrow \bar{k}$ (see also 3.9.1.1).

4.3.3. Remark. The set of Lie stable p-ranks of points of $\mathcal{N}_{k(v)}$ is not a sequence of consecutive non-negative integers. For example in the case of special fibres of integral canonical models of a Siegel modular variety $\mathrm{Sh}\big(\mathrm{GSp}(W, \psi), S\big)$, the set of values of Lie stable p-ranks is $\{1, 3, 6, \ldots, \frac{e(e+1)}{2}\}$ (we recall that $\dim_{\mathbb{Q}}(W) = 2e$).

4.3.4. Theorem. *Let $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ be an arbitrary point. The following statements are equivalent:*

a) *y factors through $\mathcal{U}$;*

b) *$R_y = R(\tau)$;*

c) *$R_y^r = R^r(\tau)$.*

Proof: This is a direct consequence of 3.9.2, 3.9.4, 4.2.1 and 4.2.4, applied to any connected component of $\mathcal{N}_{\mathbb{F}}$ (cf. also 3.9.6).

4.3.5. Corollary. *$R(\tau)$ (resp. $R^r(\tau)$) is the biggest number (resp. the biggest sequence) among all numbers R_y (resp. among all sequences of numbers $R^r_{\tilde{y}}$) associated to points $\tilde{y} \in \mathcal{N}_{k(v)}(\tilde{k})$) (with $\tilde{k}$ a perfect field).*

Proof: This is a consequence of 4.2.1, 4.2.4 and 3.5.6.

4.3.6. Corollary. *The complement $\mathcal{N}_{k(v)} \backslash \mathcal{U}$ of $\mathcal{U}$ in $\mathcal{N}_{k(v)}$, is of pure codimension 1 in $\mathcal{N}_{k(v)}$.*

Proof: The proof of this is entirely analogous to the one in the case of the open subscheme of ordinary points of the special fibre of an integral canonical model of a Siegel modular variety. If $\dim(X) = 0$ then the corollary is obvious: we have $\mathcal{U} = \mathcal{N}_{k(v)}$.

We assume now that $\dim(X) \geq 1$. 4.3.4-5 assures (to be compared with 4.3.7) that $\mathcal{N}_{k(v)} \backslash \mathcal{U}$ is of pure codimension 1, if it is nonvoid.

To see that $\mathcal{N}_{k(v)} \backslash \mathcal{U}$ is nonvoid, we first treat the case when $\mathcal{N}$ has the completion property, just to emphasize how 3.4.15 can be applied in practice. So, assuming that $\mathcal{N}$ has the completion property, the fact that $\mathcal{N}_{k(v)} \backslash \mathcal{U}$ is nonvoid can be checked, by

just remarking: If $\big(M_y, F^1_z, \varphi_y, G_{W(k)}\big)$ is the Shimura filtered σ-crystal attached to a G-canonical lift z (defined in 4.4.2 below; its definition relies just on 2.3.17 and 3.1.0) of a G-ordinary point $y \in \mathcal{U}(k)$ then $F^1_z\big(\mathrm{Lie}(G_{W(k)}\big) \neq 0$ (corresponding to the fact $\dim(X) > 0$), and there is $g \in G^0_{W(k)}(W(k))$ such that $\big(M_y, g\varphi_y, G_{W(k)}\big)$ is not a $G_{W(k)}$-ordinary σ-crystal.

To see this let G_{ℓ_0} be a $W(k)$-simple factor of $G^{\mathrm{ad}}_{W(k)}$, with $\ell_0 \in I$, such that $F^1_z\big(\mathrm{Lie}(G_{l_0})\big) \neq 0$. Let T and B be as in 4.1, and let $\mathrm{Lie}(B) = \mathfrak{b}$. Let $\omega \in G^0_W(k)(W(k))$ be a Weyl element of $G_{W(k)}$ with respect to $T_{W(k)}$ (i.e. $\omega T_{W(k)}\omega^{-1} = T_{W(k)}$) such that its action on the G_{ℓ_0} factor of $G^{\mathrm{ad}}_{W(k)}$ takes the nilpotent radical of the opposite Borel Lie subalgebra of $\mathfrak{b}_{\ell_0} := \mathfrak{b} \otimes W(k)[\frac{1}{p}] \cap \mathrm{Lie}(G_{\ell_0})$ with respect to T_{ℓ_0} (defined as the image of $T_{W(k)}$ in G_{l_0}) into the nilpotent radical of $\mathfrak{b}_{\ell_0}$. In other words ω takes the negative roots of the Weyl decomposition of $\mathrm{Lie}(G_{\ell_0})$ with respect to T_{ℓ_0} and $\mathfrak{b}_{\ell_0}$, into positive roots.

We can assume that $\mathfrak{b}$ leaves invariant F^1_z, and that φ_y leaves invariant $\mathrm{Lie}(T_{W(k)})$ and $\mathfrak{b} \otimes W(k)$ (cf. 3.4.15 and 3.2.3, or cf. 3.11.1 c)). Obviously the Lie stable p-rank of the cyclic Shimura adjoint Lie σ-crystal $\big(\mathrm{Lie}(G_{iW(k)}), \omega\varphi_1\big)$ (with G_i a $\mathbb{Z}_p$-simple factor of $G^{\mathrm{ad}}_{\mathbb{Z}_p}$ such that $G_{\ell_0} \subset G_{iW(k)}$) is zero (cf. also 4.3.7 below), and so (cf. 4.3.4) it is not a $G_{W(k)}$-ordinary σ-crystal.

To treat the general case, we use the above notations. We use special quadruples (cf. [Va2, 3.2.10]). Let $\tilde{T}$ be a maximal $\mathbb{Q}$–torus of G such that:

i) over $\mathbb{Z}_p$ is a maximal torus of $G_{\mathbb{Z}_p}$ with the property that its image in G_i has rank 0 over $W(k(v_i))$ (here $k(v_i)$ is a finite field such that $B(k(v_i))$ is the field of definition of the conjugacy class of the cocharacter of $G_{iB(k(v))}$ obtained by taking the generic fibre of the composite of μ (of 4.1) with the natural quotient map $G_{W(k(v))} \to G_{iW(k(v))}$);

ii) over $\mathbb{R}$ is a compact torus.

The existence of such a torus is implied by [Har, 5.5.3], by the structure of simple adjoint groups over fields (see [Ti1]), and by the simple fact that that any absolutely simple adjoint group $\tilde{S}$ over a finite field $\tilde{k}$ has a maximal torus whose rank over $\tilde{k}_1$ is zero; here $\tilde{k}_1$ is $\tilde{k}$ or the field extension of $\tilde{k}$ of degree two, depending on the fact that $\tilde{S}$ is split or not. As in [Va2, 3.2.11], we deduce the existence of a special quadruple $(\tilde{T}, \{\tilde{h}\}, \tilde{H}_{\tilde{T}}, \tilde{v})) \hookrightarrow (G, X, H, v)$. Obviously the $\tilde{T}$-ordinary type we get using special quadruple is not the G-ordinary type of 4.4.1 (to be compared with 4.11). This implies that $\mathcal{N}_{k(v)} \backslash \mathcal{U}$ is non-empty. This ends the proof of the corollary.

4.3.7. Examples. We concentrate just on one $\mathbb{Z}_p$-simple factor G_i of $G^{\mathrm{ad}}_{\mathbb{Z}_p}$. We can assume that $\mathcal{H}_i = \{1, 2, \ldots, d_i\}$, with $d_i \in \mathbb{N}$. So $|\mathcal{H}_i| = d_i$. Let $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$ be a lift of y and let

$$ {}_i\mathcal{C}_y = \Big(\mathrm{Lie}\big(G_{iW(k)}\big), \varphi_y, F^0\big(\mathrm{Lie}(G_{iW(k)})\big), F^1\big(\mathrm{Lie}(G_{iW(k)})\big)\Big) $$

be the i-th cyclic factor of the Shimura adjoint filtered Lie σ-crystal attached to z. We can assume that $F^1\big(\mathrm{Lie}(G_1)\big) \neq 0$ (if $F^1\big(\mathrm{Lie}(G_{iW(k)}\big) = 0$, there is nothing to be done). Let $\bar{\psi} : \mathrm{Lie}(G_{ik}) \to \mathrm{Lie}(G_{ik})$ (defined as in 3.9.1) be the map used in computing the Lie stable p-rank of ${}_i\mathcal{C}_y$.

Case 1. ${}_i\mathcal{C}_y$ is without involution. The σ^{d_i}-linear map

$$ {}_1\overline{\psi_i}^{d_i} : \big(\mathrm{Lie}(G_1)/F^0(\mathrm{Lie}(G_1))\big) \otimes k \to \big(\mathrm{Lie}(G_1)/F^0(\mathrm{Lie}(G_1))\big) \otimes k $$

obtained from $\psi_i^{d_i}$ by passage to quotients, still computes $\frac{1}{d_i}$ times the Lie stable p-rank of ${}_i\mathcal{C}_y$.

If moreover ${}_i\mathcal{C}_y$ is of B_ℓ, C_ℓ or $D_\ell^{\mathbb{R}}$ type, then this Lie stable p-rank is the $f_{\mathcal{H}}(i)$-element of the sequence describing the refined Lie stable p-rank of a G-ordinary point of $\mathcal{N}_{k(v)}$ iff the σ^{d_i}-linear map ${}_1\overline{\psi_i}^{d_i}$ has a non-zero determinant (cf. with the formulae of 3.10.6 i)). So if all the simple factors of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ are of B_ℓ, C_ℓ or $D_\ell^{\mathbb{R}}$ type, then the fact that for a point $y \in \mathcal{N}_{k(v)}(k)$ we have $R_y = R(\tau)$, can be expressed in terms of some determinants being non-zero. This is very close to the intuition provided by the Hasse-Witt matrices.

Case 2. ${}_i\mathcal{C}_y$ is with involution. So ${}_i\mathcal{C}_y$ is of A_ℓ or of $D_\ell^{\mathbb{H}}$ type (cf. the classification of adjoint Shimura varieties of abelian type [De1]). The σ^{2d_i}-linear map

$$ {}_1\overline{\psi_i}^{2d_i} : \big(\mathrm{Lie}(G_1)\big)/F^0\big(\mathrm{Lie}(G_1)\big) \otimes k \to \big(\mathrm{Lie}(G_1)\big)/F^0\big(\mathrm{Lie}(G_1)\big) \otimes k $$

obtained from $\psi_i^{2d_i}$ by passage to quotients, still computes $\frac{1}{d_i}$ times the Lie stable p-rank of ${}_i\mathcal{C}_y$.

So in the cases when ${}_i\mathcal{C}_y$ is of A_ℓ or of $D_\ell^{\mathbb{H}}$ type, with or without involution, the fact that the Lie stable p-rank of this σ^{m_i}-linear map ${}_1\overline{\psi_i}^{m_i}$, with $m_i \in \{d_i, 2d_i\}$, is the $f_{\mathcal{H}}(i)$-element of the sequence describing the refined Lie stable p-rank of a G-ordinary point of $\mathcal{N}_{k(v)}$, can be expressed (cf. 4.3.5) in terms of some matrices of rank at most 1 being non-zero (such matrices are obtained by taking an adequate exterior power of ${}_1\overline{\psi_i}^{m_i}$).

As a conclusion we can always express the fact that for a point $y \in \mathcal{N}_{k(v)}(k)$ we have $R_y = R(\tau)$, in terms of some matrices of rank at most 1 being non-zero.

4.3.8. Remarks. 1) For any point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$, we denote by $p - \mathrm{Lie}_G(y)$, the Lie p-rank, defined as the multiplicity of the slope -1 of the Shimura Lie σ-crystal attached to y. If $p - \mathrm{Lie}_G(y) \neq 0$, then the Lie stable p-rank of y is different from zero, and the p-rank of the abelian variety A_y over k, obtained from $\mathcal{A}$ by pull back through y, is different from zero. But the converse of the first implication is not true, i.e. there are examples ($G_{\mathbb{R}}^{\mathrm{ad}}$ must have compact factors) when the Lie stable p-rank of y (or the p-rank of A_y) is different from zero, but the Lie p-rank of y is zero.

2) In the case of special fibres of integral canonical models of Siegel modular varieties, to know the Lie p-rank r^y of a k-valued point y (of such a fibre) is equivalent to knowing the p-rank r_y of A_y: we have

$$ r^y = \frac{r_y(r_y+1)}{2}. $$

But this is not true in the general context of Shimura varieties of Hodge type.

4.3.9. Problem. For any abelian variety over a perfect field $\tilde{k}$, interpret the a-number of it in terms of the Lie $\tilde{\sigma}$-crystal attached to it (defined by End( ) of its $\tilde{\sigma}$-crystal), and

then use this to define G-a-numbers for abelian varieties over $\tilde{k}$, obtained from $\mathcal{A}$ by pull back through points $\mathrm{Spec}(\tilde{k}) \to \mathcal{N}_{k(v)}/H_0$ (here $\tilde{\sigma}$ is the Frobenius of $W(\tilde{k})$).

4.3.10. Definition. The σ^{m_i}-linear maps ${}_1\bar{\psi}_i^{m_i}$, $m_i \in \{d_i, 2d_i\}$, defined in 4.3.7, as well as their exterior powers are called Shimura–Hasse–Witt maps, and the matrices defined by them are called Shimura–Hasse–Witt matrices. Similarly, for any cyclic Shimura Lie σ-crystal (so we are not necessarily over an algebraically closed field; see 3.9.1.1) we define such maps and matrices. The only difference: m_i can be also $3d_i$ (cf. 3.10.0).

4.4. G-canonical lifts.

4.4.1. Theorem. **1)** *A point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ factors through $\mathcal{U}$ iff y is a parabolic point, i.e. iff there is a lift $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$ of y with the property that the Shimura filtered Lie σ-crystal attached to z is of parabolic type.*

2) *We assume now that y factors through $\mathcal{U}$. We have:*

a) There is a unique lift $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$ of y with the property that the Shimura filtered Lie σ-crystal attached to z is of parabolic type;

b) If $k = \bar{k}$, then the Shimura adjoint filtered Lie σ-crystal attached to this unique lift z is isomorphic to $\Big(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p}) \otimes W(k), \sigma \circ \mu(\frac{1}{p}), F^0\big(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p}) \otimes W(k)\big), F^1\big(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p}) \otimes W(k)\big)\Big)$, and so it is cyclic diagonalizable and of Borel type (μ being as in 4.1);

c) If $k = \bar{k}$ then the Shimura filtered Lie σ-crystal attached to z is isomorphic to

$$\Big(\mathrm{Lie}(G_{\mathbb{Z}_p}) \otimes W(k), \sigma \circ \mu(\frac{1}{p}), F^0\big(\mathrm{Lie}(G_{\mathbb{Z}_p}) \otimes W(k)\big), F^1\big(\mathrm{Lie}(G_{\mathbb{Z}_p}) \otimes W(k)\big)\Big),$$

and so it is cyclic diagonalizable and of Borel type.

3) *Let $\big(M_y, F^1_z, \varphi_y, G_{W(k)}, (t_\alpha)_{\alpha \in \mathcal{J}'}, \psi_y\big)$ be the principally quasi-polarized Shimura filtered σ-crystal attached to z. We have:*

a) There is a unique cocharacter $\mu_y : \mathbb{G}_m \hookrightarrow G_{W(k)}$, with $[\mu_y] = [\mu]$, which produces a direct sum decomposition $M_y = F^1_z \oplus F^0_z$ such that the parabolic Lie subalgebra of $\mathrm{Lie}(G_{W(k)})$ corresponding to non-negative (resp. non-positive) slopes of $\big(\mathrm{Lie}(G_{W(k)}), \varphi_y\big)$ leaves invariant F^1_z (resp. F^0_z);

b) If $k = \bar{k}$, then the filtered σ-crystal (M_y, F^1_z, φ_y) is cyclic diagonalizable, and $\big(M_y, F^1_z, \varphi_y, G_{W(k)}, (t_\alpha)_{\alpha \in \mathcal{J}'}, \psi_y\big)$ is isomorphic to any other principally quasi-polarized Shimura filtered σ-crystal $\big(M_{y_1}, F^1_{z_1}, \varphi_{y_1}, G_{W(k)}, (t^1_\alpha)_{\alpha \in \mathcal{J}'}, \psi_{y_1}\big)$ defined similarly but for another G-ordinary point $y_1 : \mathrm{Spec}(k) \to \mathcal{U}$ factoring through the same connected component of $\mathcal{U}_k$ through which y factors;

c) If $k = \bar{k}$, then there is $n \in \mathbb{N}$ such that the principally quasi-polarized Shimura filtered σ^n-isocrystal $\big(M_y[\frac{1}{p}], F^1_z[\frac{1}{p}], \varphi^n_y, G_{W(k)}[\frac{1}{p}], (t_\alpha)_{\alpha \in \mathcal{J}'}, \psi_y\big)$ is isomorphic to the principally quasi-polarized Shimura filtered σ^n-isocrystal $\big(M \otimes W(k)[\frac{1}{p}], F^1 \otimes W(k)[\frac{1}{p}], (\varphi \otimes 1)^n, G_{W(k)}[\frac{1}{p}], (v_\alpha)_{\alpha \in \mathcal{J}'}, \tilde{\psi}\big)$;

d) $\big(M_y, F^1_z, \varphi_y, G_{W(k)}, (t_\alpha)_{\alpha \in \mathcal{J}'}, \psi_y\big)$ is isomorphic to $\big(M \otimes W(k), F^1 \otimes W(k), \varphi \otimes 1, G_{W(k)}, (v_\alpha)_{\alpha \in \mathcal{J}'}, \tilde{\psi}\big)$. So in c) above we can take $n = 1$.*

Proof: The part 1) is a direct consequence of 3.1.0 and 2.3.17. What is actually needed from §3 is 3.5.11 and 3.2.7 c) (the proof of 3.2.7 c), cf. 3.7.4, does not use 3.4). Part a) of 2) follows from 3.2.8 and 2.3.17. For the part b) of 2) we can assume already (cf. 4.2.4, 3.5.11 and 3.6.5) that the Shimura adjoint filtered Lie σ-crystal attached to z is

$$\Big(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p})\otimes W(k), g\sigma\mu(\frac{1}{p}), F^0\big(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p})\otimes W(k)\big), F^1\big(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p})\otimes W(k)\big)\Big),$$

with $g \in \tilde{P}_0(W(k))$, where $\tilde{P}_0$ is the parabolic subgroup of $G^{\mathrm{ad}}_{W(k)}$ having as its Lie algebra the Lie subalgebra $\tilde{\mathfrak{p}}_0$ of $\mathrm{Lie}\big(G^{\mathrm{ad}}_{W(k)}\big)$ corresponding to non-negative slopes of $\Big(\mathrm{Lie}\big(G^{\mathrm{ad}}_{W(k)}\big), \sigma\mu(\frac{1}{p})\Big)$. Let $\varphi_y := g\sigma\mu(\frac{1}{p})$. Let $\mathfrak{n}_0$ be the nilpotent radical of $\tilde{\mathfrak{p}}_0$. Then $(\mathfrak{n}_0, \varphi_y)$ and $\big(\mathfrak{n}_0, \sigma\mu(\frac{1}{p})\big)$ are Lie σ-crystals as well as p-divisible objects of $\mathcal{MF}_{[1,1]}(W(k))$. Let $\mathfrak{p}^0_0$ be the Lie subalgebra of $\tilde{\mathfrak{p}}_0$ corresponding to the slope zero for $\big(\tilde{\mathfrak{p}}_0, \sigma\mu(\frac{1}{p})\big)$, and let $\mathfrak{p}^1_0$ be the Lie subalgebra of $\tilde{\mathfrak{p}}_0$ corresponding to slope zero for $(\tilde{\mathfrak{p}}_0, \varphi_y)$. Now $\mathfrak{p}^0_0$ and $\mathfrak{p}^1_0$ are both Lie algebras of two Levi subgroups $\tilde{P}^0_0$ and respectively $\tilde{P}^1_0$ of $\tilde{P}_0$. For $(\tilde{\mathfrak{p}}_0, \sigma\mu(\frac{1}{p}))$ this is obvious, cf. its definition; for $(\tilde{\mathfrak{p}}_0, \varphi_y)$ this is a consequence of the fact that for any $n \in \mathbb{N}$, $\varphi^n_y = a_n\varphi^n$, with $a_n \in \tilde{P}_0(W(k))$. So both these two Lie algebras project bijectively onto $\tilde{\mathfrak{p}}_0/\mathfrak{n}_0$.

But any two Levi subgroups are conjugate by an element $g_0 \in \tilde{P}_0(W(k))$. Changing φ_y with $g_0^{-1}\varphi_y g_0$ (i.e. changing it by an isomorphism, as $\tilde{\mathfrak{p}}_0 \subset F^0(\mathrm{Lie}(G^{\mathrm{ad}}_{W(k)}))$), we can assume that $\mathfrak{p}^0_0 = \mathfrak{p}^1_0$. But this implies that $g \in \tilde{P}^0_0$. This means that g commutes with the images of the elements (introduced in 4.1) $\bar{h}_i$ in $\mathrm{Lie}\big(G^{\mathrm{ad}}_{W(k)}\big)$, $i \in \{1, \ldots, d\}$. Let $\sigma_1 := g\sigma$, and let $\tilde{G}^{\mathrm{ad}}_{\mathbb{Z}_p}$ be the adjoint group over $\mathbb{Z}_p$ having

$$\mathrm{Lie}(G^{\mathrm{ad}}_{W(k)})^{\sigma_1} := \big\{x \in \mathrm{Lie}(G^{\mathrm{ad}}_{W(k)}) | \sigma_1(x) = x\big\}$$

as its Lie algebra (the fact that $k = \bar{k}$, implies that $\mathrm{Lie}(G^{\mathrm{ad}}_{W(k)})^{\sigma_1}$ has the right dimension, and that a $\mathbb{Z}_p$-basis of it is a $W(k)$-basis of $\mathrm{Lie}(G^{\mathrm{ad}}_{W(k)})$).

Claim. *$\tilde{G}^{\mathrm{ad}}_{\mathbb{Z}_p}$ is isomorphic to $G^{\mathrm{ad}}_{\mathbb{Z}_p}$.*

Proof: Let $\tilde{\mu}$ be the composite of μ with the quotient homomorphism $G_{W(k(v))} \twoheadrightarrow G^{\mathrm{ad}}_{W(k(v))}$. As $\tilde{\mu}\big(\mathrm{Lie}(\mathbb{G}_m)\big) \subset \tilde{G}^{\mathrm{ad}}_{V_0}$ (we recall that $V_0 = W(\mathbb{F})$), we can assume that $k = \mathbb{F}$. As the map $\tilde{P}^0_0(\mathbb{F}) \to \tilde{P}^0_0(\mathbb{F})$ defined by $\tilde{g} \to \tilde{g}\sigma(\tilde{g})^{-1}$ is surjective (cf. [Bo, 16.4]), we can assume that g mod p is the identity element of $\tilde{P}^0_0(\mathbb{F})$ (as $\tilde{P}^0_0$ commutes with $\tilde{\mu}$, when we replace $g\sigma\tilde{\mu}(\frac{1}{p})$ with $g_1 g\sigma\tilde{\mu}(\frac{1}{p})g_1^{-1}$, with $g_1 \in \tilde{P}^0_0(V_0)$, we get $g_1 g\sigma(g_1)^{-1}\sigma\tilde{\mu}(\frac{1}{p})$; moreover $\tilde{P}^0_0$ is defined over $\mathbb{Z}_p$). But $g \equiv 1$ mod p implies $\tilde{G}^{\mathrm{ad}}_{\mathbb{F}_p} = G^{\mathrm{ad}}_{\mathbb{F}_p}$. As two adjoint semisimple groups over $\mathbb{Z}_p$ having isomorphic special fibres, are isomorphic, we get that $\tilde{G}^{\mathrm{ad}}_{\mathbb{Z}_p}$ is isomorphic to $G^{\mathrm{ad}}_{\mathbb{Z}_p}$. This proves the claim.

But now $\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0}), \sigma_1\tilde{\mu}(\frac{1}{p})\big)$ with respect to the new $\mathbb{Z}_p$-structure defined by $\mathrm{Lie}(\tilde{G}^{\mathrm{ad}}_{\mathbb{Z}_p})$, is exactly $\big(\mathrm{Lie}(G^{\mathrm{ad}}_{V_0}), \sigma\tilde{\mu}(\frac{1}{p})\big)$ with the old $\mathbb{Z}_p$-structure defined by $\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p})$. This ends the

proof of part b) of 2), as obviously $\big(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p})\otimes V_0, \sigma\tilde{\mu}(\frac{1}{p})\big)$ is of Borel type and cyclic diagonalizable (cf. 4.1.4). c) of 2) is a consequence of b) of 2).

For the proof of part a) of 3) we first remark that μ_y, if exists, is unique (F^0_z is uniquely determined, cf. the argument of 3.2.8 for the uniqueness of F^1_z), and due to the uniqueness assertion, we can assume $k=\bar{k}$. But in this case, it results directly from the part b) of 2) (i.e. μ_y is determined once we know $\tilde{\mu}_y:\mathbb{G}_m\to G^{\mathrm{ad}}_{W(k)}$, obtained by composing μ_y with the natural quotient homomorphism $G_{W(k)}\twoheadrightarrow G^{\mathrm{ad}}_{W(k)}$).

The proof of part b) of 3) not involving the cyclic diagonalizable part is entirely analogous to the proof of part b) of 2), starting from 3.5.11 and 2.3.13.1: a reductive group over $\mathbb{Z}_p$ is uniquely determined by its special fibre, while a torsor of a reductive group over $\mathbb{Z}_p$ which is trivial over $\mathbb{F}_p$, is trivial. It is an easy exercise to verify that the Shimura filtered σ-crystal $(M_y, F^1_z, \varphi_y, G_{W(k)})$ is cyclic diagonalizable (hint: $\mu_y:\mathbb{G}_m\to G_{W(k)}$ defines an element $\bar{h}^z_1\in\mathrm{End}(M_y)$ by $\bar{h}^z_1(x)=x$ if $x\in F^1_z$, and $\bar{h}^z_1(x)=0$ if $x\in F^0_z$; denoting $\bar{h}^z_i=\varphi_y^{i-1}(\bar{h}^z_1)$, $i\in\mathbb{N}$, we again get that $\bar{h}^z_{d+1}=\bar{h}^z_1$, i.e. $\varphi^d_y(\bar{h}^z_1)=\bar{h}^z_1$, and the whole argument of the proof of 4.1.2 can be repeated).

To see c) we just have to remark that there is an isomorphism of quadrples

$$\big(M\otimes B(k), F^1\otimes B(k), (v_\alpha)_{\alpha\in\mathcal{J}'}, \tilde{\psi}\big)\tilde{\to}\big(M_y[\frac{1}{p}], F^1_z[\frac{1}{p}], (t_\alpha)_{\alpha\in\mathcal{J}'}, \psi_y\big),$$

taking $\bar{h}_i$ into $\bar{h}^z_i$, $i=\overline{1,d}$ (for the first part cf. 4.2.3; for the second part cf. part b) of 2): the fact that μ and μ_y are $G_{W(k)}(W(k))$-conjugate takes care of $G^{\mathrm{ab}}_{W(k)}$).

In other words, writing $\varphi_y=a_y\mu_y(\frac{1}{p})$, a_y becomes a σ-linear automorphism of M_y; M_y and $G_{W(k)}$ get a natural $\mathbb{Z}_p$-structure $M_y(\mathbb{Z}_p)$ and respectively $\tilde{G}_{\mathbb{Z}_p}$, by considering the elements fixed by a_y (for $G_{W(k)}$, we look at the elements of $\mathrm{Lie}(G_{W(k)})$ fixed by a_y: they form $\mathrm{Lie}(\tilde{G}_{\mathbb{Z}_p})$). It is easy to see that in fact $\tilde{G}_{\mathbb{Z}_p}$ is isomorphic to $G_{\mathbb{Z}_p}$ (cf. b) of 2) for the adjoint parts; the fact that $G^{\mathrm{ab}}_{\mathbb{Z}_p}=\tilde{G}^{\mathrm{ab}}_{\mathbb{Z}_p}$ is standard). This is not important here; what we need: there is $n\in\mathbb{N}$ such that over $W(\mathbb{F}_{p^n})[\frac{1}{p}]$ the representation of $G_{\mathbb{Q}_p}$ on L^*_p becomes isomorphic to the representation of the generic fibre $\tilde{G}_{\mathbb{Q}_p}$ of $\tilde{G}_{\mathbb{Z}_p}$ on $M_y(\mathbb{Z}_p)\otimes\mathbb{Q}_p$, and under this isomorphism $\rho(y,n)$, $\bar{h}_i$ goes into $\bar{h}^z_i$, etc., as mentioned above. For instance we need n such that both $G_{\mathbb{Z}_p}$ and $\tilde{G}_{\mathbb{Z}_p}$ split over $W(\mathbb{F}_{p^n})$. The fact that $\rho(y,n)$ takes φ^n into φ^n_y is obvious.

The proof of part d) of 3) will be presented in §5 as it is related to the Milne's conjecture referred to in 1.16. Here we just remark that we do not need to assume that $k=\bar{k}$ (cf. 2.3.9.2).

4.4.2. Definition. The unique lift $z:\mathrm{Spec}(W(k))\to\mathcal{N}$ (resp. $z:\mathrm{Spec}(W(k))\to\mathcal{N}/H_0$) of a G-ordinary point $y:\mathrm{Spec}(k)\to\mathcal{U}$ (resp. of a G-ordinary point $y:\mathrm{Spec}(W(k))\to\mathcal{N}/H_0$) with the property that its attached Shimura filtered Lie σ-crystal is of parabolic type is called the G-canonical lift of y, or the Shimura canonical lift of y.

4.4.2.1. Remark. The translation of a G-canonical lift by an element of $G(\mathbb{A}^p_f)$ is again a G-canonical lift. This allows to replace H_0 in the above definition with any compact subgroup $\tilde{H}_0$ of $G(\mathbb{A}^p_f)$.

4.4.3. Definition. Let $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}/H_0$ (resp. $z : \mathrm{Spec}(W(k)) \to \mathcal{N}/H_0$) be an arbitrary point, and let A_y (resp. A_z) be the abelian variety over k (resp. over $W(k)$) obtained from $\mathcal{A}_{H_0}$ by pullback through y (resp. through z). An endomorphism f_y (resp. f_z) of A_y (resp. of A_z) is said to be a G-endomorphism if, as an element of $\mathrm{End}\big(H^1_{\mathrm{crys}}(A_y/W(k))\big)$, annihilates the de Rham components of the Hodge cycles $(\omega_\alpha)_{\alpha\in\mathcal{J}'}$ of A_z (i.e. it belongs to the Lie algebra $\mathrm{Lie}(G_{W(k)})$ defining the Shimura Lie σ-crystal attached to y). We denote the set of G-endomorphisms of A_y (A_z) by $\mathrm{End}_G(A_y)$ (resp. by $\mathrm{End}_G(A_z)$). $\mathrm{End}_G(A_y)$ and $\mathrm{End}_G(A_z)$ have a natural structure of Lie algebras over $\mathbb{Z}$.

An automorphism a_y (resp. a_z) of A_y (resp. A_z) is said to be a G-automorphism if, as an automorphism of $H^1_{\mathrm{crys}}(A_y/W(k))$, is an element of $G_{W(k)}(W(k))$. Notation $\mathrm{Aut}_G(A_y)$ (resp. $\mathrm{Aut}_G(A_z)$). Obviously $\mathrm{Aut}_G(A_y)$ and $\mathrm{Aut}_G(A_z)$ have a group structure. Warning: $\mathrm{Aut}_G(*)$, as a set, is not a subset of $\mathrm{End}_G(*)$; here $* \in \{A_y, A_z\}$.

Similarly we speak about G-$\mathbb{Q}$–endomorphisms or about G-$\mathbb{Q}$–automorphisms.

4.4.4. Corollary. *Let $y : \mathrm{Spec}(k) \to \mathcal{U}/H_0$ be a G-ordinary point and let $z : \mathrm{Spec}(W(k)) \to \mathcal{N}/H_0$ be its G-canonical lift. Then*

$$\mathrm{End}_G(A_y) = \mathrm{End}_G(A_z)$$

and

$$\mathrm{Aut}_G(A_y) = \mathrm{Aut}_G(A_z)$$

(i.e. any G-endomorphism and any G-automorphism of A_y lifts to an endomorphism of A_z).

Proof: This is a direct consequence of a) of 4.4.1 2).

4.4.4.1. Remark. 4.4.3-4 can be easily extended to the case of Shimura-ordinary σ-crystals and of their Shimura-canonical lifts. The essence behind them is 3.1.0 a) and b).

4.4.5. Corollary. *If $(f, L_{(p)}, v, \mathcal{B})$ is a standard PEL situation (cf. [Ko2, §5]), then any G-canonical lift $z : \mathrm{Spec}\big(W(\mathbb{F})\big) \to \mathcal{N}/H_0$ of a G-ordinary point $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{U}$ gives birth to a special point $z_0 : \mathrm{Spec}\big(W(\mathbb{F})[\frac{1}{p}]\big) \to \mathrm{Sh}(G,X)/H_0$ (i.e. A_z has complex multiplication).*

Proof: With the notations of 2.3.5 (see also [Ko2, §5]), $\mathrm{End}_G(A_y)$ is the intersection of the centralizer of $\mathcal{B}$ in $\mathrm{End}(A_y)$ with the normalizer of the $B(\mathbb{F})$-vector space of the principal polarization p_{A_y} of A_y, obtained from $\mathcal{P}_{\mathcal{A}}$ by pull back through y, and viewed as a cycle in the crystalline context. So the $\mathbb{Q}$–group defined by the invertible elements of $\mathrm{End}_G(A_y) \otimes \mathbb{Q}$ contains a maximal torus of rank equal to the rank of G. Now everything results from the identity $\mathrm{End}_G(A_y) = \mathrm{End}_G(A_z)$ (cf. 4.4.4).

4.4.6. Remark. In §12 we will see that 4.4.5 remains true without assuming that $(f, L_{(p)}, v)$ comes from a standard PEL situation $(f, L_{(p)}, v, \mathcal{B})$. This implies that $\mathcal{U}$ is a locally closed subscheme of $\mathcal{M}_{k(v)}$ (cf. §13).

4.4.7. Remark. Let $\bar{k} = k$. With the notations of c) of 4.4.1 3), $M_y(\mathbb{Z}_p) := \{x \in M_y | a_y(x) = x\}$ is a $\mathbb{Z}_p$-structure on M with respect to which μ_y can be viewed as an

injective cocharacter $\mu_y : \mathbb{G}_m \hookrightarrow GL\big(M^a \otimes W(k(v))\big)$, and $t_\alpha \in \mathcal{T}(M^a)$, $\forall \alpha \in \mathcal{J}$. So $\big(M_y, F^1_z, \varphi_y, (t_\alpha)_{\alpha\in\mathcal{J}'}\big)$ is isomorphic to $(M_y(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k), {}'F^1_z, \sigma_y\mu_y(\frac{1}{p}), (t_\alpha)_{\alpha\in\mathcal{J}'})$, with σ_y as the σ-linear automorphism of $M^a \otimes W(k)$ fixing $M_y(\mathbb{Z}_p)$, and with ${}'F^1_z := F^1_z \cap M_y(\mathbb{Z}_p) \otimes W(k(v))$.

Moreover, the subgroup of $GL(M^a \otimes \mathbb{Q}_p)$ fixing t_α, $\forall \alpha \in \mathcal{J}$, is $G_{\mathbb{Q}_p}$, and the degree of definition of $\big(M_y, F^1_z, \varphi_y, G_{W(k)}\big)$ is d (we recall that $k(v) = \mathbb{F}_{p^d}$). This is an easy consequence of b) of 4.4.1 2) and of the fact that the cocharacters of $G^{\mathrm{ab}}_{W(k)}$ defined naturally by μ and μ_y are the same (the resulting $\mathbb{Z}_p$-structures of $G^{\mathrm{ab}}_{W(k)}$ are as well the same).

4.4.8. Exercises. 1) We assume 4.4.6. If $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{U}$, prove that there is a unique lift (the G-canonical lift of y, cf. 4.4.6) $z : \mathrm{Spec}\big(W(\mathbb{F})\big) \to \mathcal{N}$ such that A_z has complex multiplication. Hint: if $k(v) = \mathbb{F}_p$ this is well known, cf. property P1 of 4.6, and cf. the theory of CM-lifts of ordinary points; if $k(v) \neq \mathbb{F}_p$, first copy the proof of 2.2.19, and then use 3.7.2.

2) Show that the property express in 1) is a property of Shimura-ordinary σ-crystals (of course we need to work in the context of lifts of quasi CM type).

2′) We assume that $k = \bar{k}$. Let $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$ be the G-canonical lift of a point $y : \mathrm{Spec}(k) \to \mathcal{U}$. Let $\big(\mathrm{Lie}\big(G_{W(k)}\big), \varphi_y\big)$ be the Shimura Lie σ-crystal attached to y. If the parabolic Lie subalgebra of $\mathrm{Lie}(G_{W(k)})$ corresponding to non-negative slopes of $(\mathrm{Lie}(G_{W(k)}), \varphi_y)$ is a Borel Lie subalgebra, then z is the unique lift of y to $W(k)$, whose attached Shimura filtered σ-crystal is a lift of quasi CM type (cf. def. 2.3.17) of the Shimura σ-crystal attached to y. (Hint: use property P7 of 4.6).

4.4.9. Remark. Coming back to 4.4.4, we would like to mention that not always we have $\mathrm{End}(A_y) = \mathrm{End}(A_z)$. This is always the case when $k(v) = \mathbb{F}_p$, cf. properties $P1$ and $P2$ of 4.6. But when $k(v) \neq \mathbb{F}_p$, we can have $\mathrm{End}(A_y) \underset{\neq}{\supseteq} \mathrm{End}(A_z)$ for some G-ordinary points y. For instance this is so when G is a torus and the G-ordinary type is a supersingular type (concret example: when $\mathrm{Sh}\big(\mathrm{GSp}(W,\psi), S\big)$ is the elliptic modular curve, G is a torus and the prime v is such that the G-ordinary type is (1,1)). There are many examples with G a torus such that $\mathrm{End}(A_y) \neq \mathrm{End}(A_z)$, and starting from this we can build up examples with G a reductive group which is not a torus and when not always $\mathrm{End}(A_y)$ is equal to $\mathrm{End}(A_z)$.

However, in §14, we will see that

$$\mathrm{End}(A_y) \otimes \mathbb{Q} = \mathrm{End}_G(A_y) \otimes \mathbb{Q} \oplus \mathrm{End}(A_y)^{\perp_G} \otimes \mathbb{Q}$$

and similarly

$$\mathrm{End}(A_z) \otimes \mathbb{Q} = \mathrm{End}_G(A_z) \otimes \mathbb{Q} \oplus \mathrm{End}(A_z)^{\perp_G} \otimes \mathbb{Q},$$

where $\mathrm{End}(A_y)^{\perp_G}$ (resp. $\mathrm{End}(A_z)^{\perp_G}$) is the set of endomorphisms of A_y (resp. of A_z) with the property that inside $\mathrm{End}\Big(H^1_{\mathrm{crys}}\big(A_y/W(k)\big)\big[\frac{1}{p}\big]\Big)$ they are perpendicular to $\mathrm{Lie}(G_{W(k)})$ with respect to the trace form on $\mathrm{End}\big(H^1_{\mathrm{crys}}(A_y/W(k))\big)$.

4.4.10. Problem. Referring again to 4.4.4, study the structure of the Lie algebras $\mathrm{End}_G(A_y)\otimes\mathbb{Q}$ and $\mathrm{End}_G(A_y)$ for various G-ordinary points $y:\mathrm{Spec}(k)\to\mathcal{U}$ or $y:\mathrm{Spec}(k)\to\mathcal{U}/H_0$. The most interesting case is when $k=\mathbb{F}$ or when k is a finite field.

4.4.11. Remark. Not only $d(\mathfrak{C})$, but also the numbers $d^{\perp}(\mathfrak{C})$ and $gd(\mathfrak{C})$ (defined in 3.13.1), with $\mathfrak{C}$ the Shimura filtered σ-crystal attached to a G-canonical lift $z:\mathrm{Spec}(W(k))\to\mathcal{N}$, do not depend on the choice of the G-canonical lift z (cf. c) of 4.4.1 3)).

4.4.12. Remark. We assume d) of 4.4.1 3) and that $k=\bar{k}$. Let $V(k)$ be a complete DVR which is a finite faithfully flat extension of $W(k)$. Let $m_{V(k)}:\mathrm{Spec}(V(k))\to\mathcal{N}$ be a morphism. Let $(A_{V(k)},p_{A_{V(k)}})=m^*_{V(k)}(\mathcal{A},\mathcal{P}_{\mathcal{A}})$, and let $(t_\alpha^{V(k)})_{\alpha\in\mathcal{J}'}$ be the family of Hodge cycles with which $A_{V(k)}$ is naturally endowed. From 2.3.13.1 and the density part of 4.2.1, we deduce the existence of a $V(k)$-isomorphism

$$L^*_{(p)}\otimes_{\mathbb{Z}_{(p)}}V(k)=M\otimes_{W(k(v)}V(k)\tilde{\to}H^1_{dR}(A_{V(k)}/V(k))$$

taking $\tilde{\psi}$ into $p_{A_{V(k)}}$ and v_α into $t_\alpha^{V(k)}$, $\forall\alpha\in\mathcal{J}'$.

4.5. Different stratifications of $\mathcal{N}_{k(v)}$.

4.5.1. Definition. The stratification of $\mathcal{N}_{k(v)}$ in $G(\mathbb{A}_f^p)$-invariant locally closed (reduced) subschemes indexed and defined by the Newton polygons of the Shimura Lie σ-crystals attached to points $\mathrm{Spec}(k)\to\mathcal{N}_{k(v)}$ is called the canonical Lie stratification.

4.5.2. Definition. The stratification of $\mathcal{N}_{k(v)}$ in $G(\mathbb{A}_f^p)$-invariant locally closed (reduced) subschemes, indexed and defined by sequences of length $|\mathcal{H}|$ formed by the Newton polygons of the cyclic factors (their number is $|\mathcal{H}|$) of the Shimura adjoint σ-crystals attached to points $\mathrm{Spec}(k)\to\mathcal{N}_{k(v)}$ is called the refined canonical Lie stratification.

4.5.3. Remark. $\mathcal{U}$ is the Zariski dense open stratum of both Lie stratifications of $\mathcal{N}_{k(v)}$ defined above (cf. 4.2).

4.5.4. Remark. We assume $k=\bar{k}$. Any representation $\rho:G_{\mathbb{Z}_p}\to GL(N)$, with N a free $\mathbb{Z}_p$-module of finite rank, allows us to attach to any point $z:\mathrm{Spec}(W(k))\to\mathcal{N}$, a p-divisible object ${}_\rho\mathfrak{C}_z$ of $\mathcal{MF}_{[a_\rho,b_\rho]}(W(k))$. Here $a_\rho,b_\rho\in\mathbb{Z}$ are defined by the fact that

$$\mu_\rho:=\rho\circ\mu:\mathbb{G}_m\to GL\big(N\otimes W(k(v))\big)$$

produces a direct sum decomposition $N\otimes W(k(v))=\oplus_{i\in\mathcal{I}}F^i_\rho$ (with all $F^i_\rho\neq 0$, with $a_\rho,b_\rho\in\mathcal{I}$ such that $\mathcal{I}\subset\{a_\rho,\dots,b_\rho\}$, and where $\beta\in\mathbb{G}_m(W(k))$ acts through μ_ρ on F^i_ρ as the multiplication with β^{-i}. This is done as follows.

We make use of an isomorphism of the quadruple $\big(M\otimes W(k),\mu,(v_\alpha)_{\alpha\in\mathcal{J}'},\tilde{\psi}\big)$ with the quadruple $\big(M_y,\mu_y,(t_\alpha)_{\alpha\in\mathcal{J}'},p_{A_y}\big)$ (cf. d) of 4.4.1 3) and 4.4.12; if we do not want to use this d) of 4.4.1 3), then we can use 4.4.7: at the end we get the same thing). Here (A_z,p_{A_z}) is the principally polarized abelian variety over k obtained from $(\mathcal{A},\mathcal{P}_{\mathcal{A}})$ through z, A_y is the special fibre of A_z, while $M_y=H^1_{\mathrm{crys}}\big(A_y,W(k)\big)$; $(t_\alpha)_{\alpha\in\mathcal{J}'}$ are de Rham components of

the Hodge cycles with which A_z is naturally endowed. $\mu_y : \mathbb{G}_m \to GL(M_y)$ is a cocharacter fixing t_α, $\forall \alpha \in \mathcal{J}'$; it produces a direct sum decomposition $M_y = F_z^1 \oplus F_z^0$, with F_z^1 as the Hodge filtration of M_y defined by A_z; $\beta \in \mathbb{G}_m(W(k))$ acts through μ_y on F_z^i as the multiplication with β^{-i}.

We can write the Frobenius φ_y of M_y, under this isomorphism, as $g\sigma\mu(\frac{1}{p})$ with $g \in G_{\mathbb{Z}_p}(W(k)) \subset GL\big(M \otimes W(k)\big)(W(k))$. Now

$$\big(N \otimes W(k), (F_\rho^i \otimes W(k))_{i\in\mathcal{I}}, \varphi_{y,\rho}\big),$$

with

$$\varphi_{y,\rho} := \rho(g)\sigma\mu_\rho(\frac{1}{p}),$$

is the p-divisible object of $\mathcal{MF}_{[a_\rho,b_\rho]}(W(k))$ attached to ρ and z. In §5 we will see that there is a p-divisible object ${}_\rho\mathfrak{C}\mathcal{N}$ of $\mathcal{MF}^\nabla_{[a_\rho,b_\rho]}(\mathcal{N})$ attached to ρ in such a manner that in z it becomes ${}_\rho\mathfrak{C}_z$. ${}_\rho\mathfrak{C}_z$ is called the ρ p-divisible object of $\mathcal{MF}_{[a_\rho,b_\rho]}(W(k))$ attached to z. The p-divisible object ${}_\rho\mathfrak{C}\mathcal{N}$ defines a Newton polygon stratification of $\mathcal{N}_{k(v)}$ in locally closed subschemes (cf. §5). We call it the ρ-stratification of $\mathcal{N}_{k(v)}$.

4.5.5. Definition. The intersection of all ρ-stratifications of $\mathcal{N}_{k(v)}$ is called the absolute stratification of $\mathcal{N}_{k(v)}$.

4.5.6. Exercises. 0) The ρ-stratification of $\mathcal{N}_{k(v)}$ is trivial (i.e. formed just by one stratum) iff μ_ρ factors through the center of $GL(N \otimes W(k(v)))$.

1) Any ρ-stratification is $G(\mathbb{A}_f^p)$-invariant.

2) The canonical Lie stratification is the ρ^{ad}-stratification, where $\rho^{\mathrm{ad}} : G_{\mathbb{Z}_p} \to GL(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p}))$ is the adjoint representation.

3) If $\rho_i^{\mathrm{ad}} : G_{\mathbb{Z}_p} \to GL(\mathrm{Lie}(G_i))$, $i \in \mathcal{H}$, is the (irreducible) subrepresentation of ρ^{ad} induced on the $\mathbb{Z}_p$-direct summand $\mathrm{Lie}(G_i)$ of $\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{Z}_p})$ (we recall that $G^{\mathrm{ad}}_{\mathbb{Z}_p} = \times_{i\in\mathcal{H}} G_i$), then the refined canonical Lie stratification is the intersection of all the ρ_i^{ad}-stratifications, $i \in \mathcal{H}$.

4) The absolute stratification is well-defined, i.e. its strata are locally closed subschemes, and their number is finite.

5) In the case of special fibres of integral canonical models of Siegel modular varieties, any non-trivial ρ-stratification coincides with the usual Newton polygon stratification.

6) If $\rho_0 : G_{\mathbb{Z}_p} \hookrightarrow GL(L_p^*)$ is the faithful representation introduced in 4.1, then the ρ_0-stratifications is the stratification of $\mathcal{N}_{k(v)}$ defined by the Newton polygons of the Shimura σ-crystals attached to points of $\mathcal{N}_{k(v)}$ with values in perfect fields.

7) Show by examples that 5) above is not true for an arbitrary SHS $(f, L_{(p)}, v)$ (Hint: either make use of products of Shimura varieties of Hodge type or look at the case when $|\mathcal{H}| \geq 2$).

8) $\mathcal{U}$ is an open subscheme of the (generic) Zariski dense open stratum of any ρ-stratification, and so it is the (generic) open dense stratum of the absolute stratification

(Hint: use 3.1.0 in the context of generalized Shimura p-divisible objects (cf. 3.1.7); another way -a simpler one- is to make use of b) of 4.4.1 2)).

All the above exercises are easy.

4.5.6.1. Proposition. *The absolute stratification coincides with the refined Lie canonical stratification. Moreover the statement of 5) above remains true if* $|\mathcal{H}| = 1$.

Proof: We assume $k = \bar{k}$. Let $y_i : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$, $i = \overline{1,2}$, be two points of the same stratum of the refined canonical Lie stratification of $\mathcal{N}_{k(v)}$. We need to show that they belong to the same stratum of any ρ-stratification of $\mathcal{N}_{k(v)}$. Let $(M_i, \varphi_i, G_{W(k)})$ be the Shimura σ-crystal attached to y_i. Let $m \in \mathbb{N}$ such that φ_i^m acts diagonally on the elements of a $W(k)$-basis of M_i. We get a semisimple element $b_i \in G(B(k))$, whose eigenvalues, as an endomorphism of M_i are p to powers m times the slopes of (M, φ_i), multiplicities corresponding. We choose m to be a multiple of 6 times the product of the length of the cyclic adjoint factors of the Shimura Lie σ-crystal attached to y_1 (or y_2; 4.4.1 2) implies that this number does not depend on the k-valued point of $\mathcal{N}_{k(v)}$). The key fact is:

Replacing m by a suitable multiple of itself, b_1 and b_2 become conjugate by an element of $G(B(k))$.

To prove this, let $\tilde{G}$ be the product of the simple connected semisimple group cover of G^{der} and of the toric part of $Z(G)$. Replacing m by a multiple of itself, b_i is replaced by a power of itself; so we can assume that b_i lifts to an element $\tilde{b}_i \in \tilde{G}(B(k))$. We consider a representation $\tilde{G}_{B(k)} \to GL(V_{B(k)})$ such that restricted to $\tilde{G}^{\mathrm{der}}_{B(k)}$ is a direct sum, indexed by the simple factors of $G^{\mathrm{ad}}_{B(k)}$, of the standard embeddings (see [He, §8]) of the classical groups defined by the simple connected group covers of these adjoint factors (these group covers are quotients of $\tilde{G}^{\mathrm{der}}_{B(k)}$). For instance, if we have a factor $SL(\tilde{V}_{B(k)})$ of $\tilde{G}^{\mathrm{der}}_{B(k)}$, with $\tilde{V}_{B(k)}$ a $B(k)$-vector space of finite dimension, then we get the natural representation of it on $\tilde{V}_{B(k)}$. Similarly, if we have factors of B_ℓ, C_ℓ or D_ℓ Lie type, we get respectively the classical representation of the split group $SO(2\ell+1)$, $\mathrm{Sp}(\ell)$ or $SO(2\ell)$ (over $B(k)$), described at the level of Lie algebras, in the loc. cit.

$\tilde{b}_i$ acts diagonally on $V_{B(k)}$; replacing m again by a multiple of itself, we can choose $\tilde{b}_i$ so that its eigenvalues, as an endomorphism of $V_{B(k)}$, are integral powers of p. It also acts diagonally on the Lie algebra of the image of $\mathrm{Lie}(\tilde{G}^{\mathrm{der}}_{B(k)})$ in $\mathrm{End}(\tilde{V}_{B(k)})$. With its eigenvalues, as an endomorphism of this image, we build up a Newton polygon NP_i. The fact that y_1 and y_2 are points of the same stratum of the refined canonical Lie stratification of $\mathcal{N}_{k(v)}$, gets restated in: $NP_1 = NP_2$. This (together with the principle expressed in 3.5.1 and 3.5.2.2) implies that the images of $\tilde{b}_1$ and $\tilde{b}_2$ in $GL(\tilde{V}_{B(k)})$ are conjugate by an element of the group of $B(k)$-valued points of the image of $\tilde{G}^{\mathrm{der}}_{B(k)}$ in $GL(\tilde{V}_{B(k)})$. This is a simple exercise (for $\tilde{G}^{\mathrm{der}}$ having only simple factors of A_ℓ Lie type this is well known; to be compared with 4.2.2). 3.5.1 and 3.5.2.2 imply that the images of b_i in the simple factors of $G^{\mathrm{ad}}_{B(k)}$ of whose Lie algebras are permuted by φ_i, up to canonical identification of these groups, are the same. So the above exercise can be restated:

Let C be a classical semisimple group over $B(k)$ (so C is $SO(n)$, $SL(n)$ or $\mathrm{Sp}(n)$, for some adequate $n \in \mathbb{N}$). Let b^1 and b^2 be semisimple elements of $C(B(k))$ such that

their eigenvalues (under the classical representation of C) are integral powers of p. If the Newton polygons obtained with them acting on $\mathrm{Lie}(C)$ *are the same; then b^1 and b^2 are $C(B(k))$-conjugate.*

The simple proof of this fact is left to the reader.

We can assume that the images of $\tilde{b}_1$ and $\tilde{b}_2$ in $\tilde{G}^{\mathrm{ab}}$ are the same. From the above fact we deduce that $\tilde{b}_1^{m_{\mathrm{iso}}^2}$ and $\tilde{b}_2^{m_{\mathrm{iso}}^2}$ are $\tilde{G}(B(k))$-conjugate; here m_{iso} is the degree of natural isogeny from $\tilde{G}$ into the product of $\tilde{G}^{\mathrm{ab}}$ with the image of $\tilde{G}^{\mathrm{der}}$ in $GL(V_{B(k)})$. This obviously implies that y_1 and y_2 are points of the same stratum of any ρ-stratification of $\mathcal{N}_{k(v)}$ (cf. the constructions in 4.5.4).

The second part (of when $|\mathcal{H}| = 1$) is trivial: for any representation $\rho : G_{\mathbb{Z}_p} \to GL(N)$ over $\mathbb{Z}_p$, either ρ factors through $G^{\mathrm{ab}}_{\mathbb{Z}_p}$ (and so the ρ-stratification is trivial), or the restriction of ρ to $G^{\mathrm{der}}_{\mathbb{Z}_p}$ is an isogeny (and so the above part of the proof applies). This ends the proof of the proposition.

4.5.7. Let $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}_{k(v)}$ be an arbitrary point and let $\big(M_0, \varphi, G_{W(\mathbb{F})}\big)$ be its attached Shimura σ-crystal. Let $NP(\mathcal{N}_{k(v)})$ be the set of Newton polygons defined by the σ-crystals $(M_0, g\varphi)$ with $g \in G^0_{W(\mathbb{F})}\big(W(\mathbb{F})\big)$, and let $LNP(\mathcal{N}_{k(v)})$ be the set of Newton polygon defined by the Shimura adjoint Lie σ-crystals $\big(\mathrm{Lie}(G^{\mathrm{ad}}_{W(\mathbb{F})}), g\varphi\big)$ with $g \in G^0_{W(\mathbb{F})}\big(W(\mathbb{F})\big)$. Similarly we can define $RLNP(\mathcal{N}_{k(v)})$ as the set of sequences of length $|\mathcal{H}|$ of Newton polygons attached to the cyclic factors (their number is $|\mathcal{H}|$) of the above Shimura adjoint Lie σ-crystals.

Let v^{ad} be the prime of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ divided by v. Then $N(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}}) := \big|LNP(\mathcal{N}_{k(v)})\big|$ is the number of the strata of the canonical Lie stratification of $\mathcal{N}_{k(v)}$, and $N_1(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}}) := \big|RLNP(\mathcal{N}_{k(v)})\big|$ is the number of the strata of the refined canonical Lie stratification of $\mathcal{N}_{k(v)}$, provided $\mathcal{N}$ has the completion property. This is a consequence of b) of 4.4.1 2) and of 3.4.15. We also write $N(\mathcal{N}_{k(v)}) = N(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$, and $RLNP(\mathcal{N}_{k(v)}) = RLNP(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$.

4.5.8. Remarks. *1)* The natural numbers $N(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$ and $N_1(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$ depend only on the isomorphism class of the Shimura group pair $(G^{\mathrm{ad}}_{\mathbb{Z}_p}, \mu^{\mathrm{ad}})$, where $\mu^{\mathrm{ad}} : \mathbb{G}_m \to G^{\mathrm{ad}}_{W\left(k(v^{\mathrm{ad}})\right)}$ has the similar meaning as μ of 4.1, but for Shimura variety $\mathrm{Sh}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$. The same about the set $RLNP(\mathcal{N}_{k(v)})$. This justifies our notations.

2) in §10 we will use different ρ-stratifications of $\mathcal{N}_{k(v)}$ to compute the numbers $N(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$ and $N_1(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$.

4.5.9. For more about the values of $N(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$ and $N_1(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$, and the dimensions and local geometry of strata of different stratifications of $\mathcal{N}_{k(v)}$ see §10.

4.5.10. Remark. The canonical stratification of the special fibre $\mathcal{M}_{k(v)}$ of $\mathcal{M}$ (cf. [EO]) gives birth (by pull back) to an f-canonical stratification of $\mathcal{N}_{k(v)}$, in $G(\mathbb{A}_f^p)$-invariant locally closed (reduced) quasi-affine subschemes (as $\mathcal{N} \to \mathcal{M}$ is a finite morphism, $\mathcal{N}_{k(v)} \to \mathcal{M}_{k(v)}$ is a finite morphism). We do not deal here with the interplay between

the f-canonical stratification of $\mathcal{N}_{k(v)}$ and the canonical Lie stratification of $\mathcal{N}_{k(v)}$. We just mention that $\mathcal{U}$ is the generic (open dense) stratum of the f-canonical stratification of $\mathcal{N}_{k(v)}$ (cf. b) or d) of 4.4.1 3) and the definition of the canonical stratification of $\mathcal{M}_{k(v)}$). It is expected that the interpretation of the f-canonical stratification of $\mathcal{N}_{k(v)}$ in terms of Shimura Lie σ-crystals attached to points of $\mathcal{N}_{k(v)}$ with values in perfect fields leads to a refined f-canonical stratification.

4.5.11. Remark. We do not stop here to prove that if $y_1, y_2 \in \mathcal{N}_{k(v)}(k)$ are two distinct points giving birth to the same point $y \in \mathcal{M}_{k(v)}(k)$, then y_1 and y_2 are always in the same stratum, for any ρ-stratification of $\mathcal{N}_{k(v)}$ (defined by a $\mathbb{Z}_p$-representation ρ of $G_{\mathbb{Z}_p}$), as a proof of the Langlands–Rapoport conjecture for $\mathcal{N}_{k(v)}$ in the context provided by $(f, L_{(p)}, v)$ (see [Mi2] and [Va2, 1.7]) implies that such a situation $y_1 \to y \leftarrow y_2$ does not occur (cf. §14; to be compared with [Va2, 5.6.4] and 4.4.6).

4.5.12. Remark. Here k is again an arbitrary perfect field. The isomorphism of b) leads us:

a) to define a map $f_G : W_G \to NP(\mathcal{N}_{k(v)})$ from the Weyl group W_G of $G_{W(\mathbb{F})}$ (introduced in 4.1.5) into $NP(\mathcal{N}_{k(v)})$, associating to $\omega \in W_G$, $\mathcal{P}_\omega$ (of 4.1.5);

b) to define for any $\omega \in W_G$ the set of $G(\omega)$-ordinary points $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$, defined by the property that the Shimura adjoint Lie σ-crystal attached to y, over $\bar{k}$ it is isomorphic to the extension to $\bar{k}$ of the Shimura adjoint Lie σ-crystal of some $\mathfrak{C}_\omega$ of 4.1.5.

If $\mathcal{N}$ has the completion property, than the set of $G(\omega)$-ordinary points is non-empty.

4.5.12.1. We assume that the set of $G(\omega)$-ordinary points is non-empty. The $G(\omega)$-ordinary points are points of a stratum $\mathfrak{s}_\omega$ of the refined canonical Lie stratification of $\mathcal{N}_{k(v)}$. We do not know (if or) when all the points of $\mathfrak{s}_\omega$ are $G(\omega)$-ordinary points. We do not know when the $G(\omega)$-ordinary points are points of an open dense subscheme of $\mathfrak{s}_\omega$, or (at the opposite pole) when any connected component of $\mathfrak{s}_\omega$ has $G(\omega)$-ordinary points.

4.5.12.2. Definition. The points of $\mathcal{N}_{k(v)}$ with values in perfect fields which are $G(\omega)$-ordinary, for some $\omega \in W_G$, are called the toric points of $\mathcal{N}_{k(v)}$.

4.5.12.2.1. Exercise. Show that the toric points can be easily defined in terms of Galois representations: we can assume that $k = \bar{k}$; a point $\mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ is a toric point, iff it admits a $W(k)$-lift, whose attached Galois representation $\rho : \mathrm{Gal}(\overline{B(k)}/B(k)) \to G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ (we view $G_{\mathbb{Z}_p}$ as a subgroup of $GL(L_p^*)$, and identify, cf. [Va2, 5.1.2], $L_{(p)}^* \otimes \mathbb{Z}_p$ with a first group of the étale cohomology with $\mathbb{Z}_p$ coefficients) factors through the group of $\mathbb{Z}_p$-valued points of a maximal torus of $G_{\mathbb{Z}_p}$.

4.5.12.3. Remark. We will see in §10 that the toric points are the "pillars" of the (refined) canonical Lie stratification of $\mathcal{N}_{k(v)}$.

4.5.13. Problem. The map f_G is usually not injective. Compute the number of elements of the fibres of f_G and of its image $f_G(W)$.

4.5.14. Problem. Compute the number of the strata and the dimensions of the connected components of the strata of the f-canonical stratification of $\mathcal{N}_{k(v)}$.

4.5.15. Other Lie type stratifications. Using the (refined) Lie stable p-ranks attached to points of $\mathcal{N}_{k(v)}$ with values in fields, we obtain a $G(\mathbb{A}_f^p)$-invariant (refined) Lie-stable stratification of $\mathcal{N}_{k(v)}$ in locally closed subschemes. We can use also the Lie p-ranks to define another Lie type stratification of $\mathcal{N}_{k(v)}$, but the refined Lie-stable stratification is a more refined one (in general it is a much more refined one). Variants: using the maps ${}_1\bar{\psi}_i^{m_i}$ of 4.3.7, or their exterior or symmetric powers, or the σ-linear maps of 3.9.1, we can define a whole bunch of Lie type $G(\mathbb{A}_f^p)$-invariant stratifications of $\mathcal{N}_{k(v)}$. For instance: using the images of the maps of 4.3.7, and not their stable images, we get the Lie non-stable stratication. There are many questions arising, and which involve these Lie type stratifications (for instance how are they related?, etc.). To us, the most interesting ones are:

Q_1) Are the connected components of a stratum of the Lie-stable stratification of $\mathcal{N}_{k(v)}$ (or of $\mathcal{N}_{\mathbb{F}}$) permuted transitively by $G(\mathbb{A}_f^p)$?

Q_2) Using all the stratifications hinted at in this 4.5.15, we can define their intersection. We call it the ultra stratification of $\mathcal{N}_{k(v)}$. It is easy to see that in general it is a refinement of the absolute stratification. How more refined it is? How is it related to the f-canonical stratification?

Q_3) These stratifications are very much related to the incipient theory of deviations of 3.13: the σ-linear maps of 3.9.1, and other ones constructed naturally from them - like the ones of 4.3.7- give birth to a big bunch of stable or non-stable -or mixed-stable-deviations of Shimura (Lie) (filtered) σ-crystals. Which other type of "points" (besides the G-ordinary ones), and of their lifts, can be singled out using the σ-linear maps of 3.9.1 (and eventually some deviations similar to the ones introduced in 3.13)? Can we in this way single out the toric points?

4.5.16. Non Lie type stratifications. Of course, forgetting the extra (Shimura) structure of the σ-crystals attached to points of $\mathcal{N}_{k(v)}$ with values in perfect fields, we can define other (not necessarily of Lie type!) stratifications of $\mathcal{N}_{k(v)}$, which are closer to the spirit coming from the case of Siegel modular varieties (like the one using the p-rank, see [NO]). The slight unpleasant feature of such stratifications of $\mathcal{N}_{k(v)}$: very often they are not canonical (i.e. they depend on the SHS producing them; on the contrary the Lie type stratifications are canonical, see 4.9), or they are not refined enough, as it can be seen easily through examples (like when $|\mathcal{H}| \geq 2$).

4.5.17. Artin-Schreier stratifications. Let X_p be an arbirary $\mathbb{F}_p$-scheme. A stratification of it in locally closed reduced subschemes is called an Artin-Schreier stratification (to be abbreviated as: an AS stratification), if there is $m \in \mathbb{N}$, such that locally in the Zariski topology of X_p, it can be defined as in 3.4.8.1.3, using quasi Artin-Schreier systems in m variables (see 3.4.18.4.6 A for the definition of these systems). Similarly we define a refined Artin-Schreier stratification (to be abbreviated as RAS stratification) of X_p.

Example 1. We assume that X_p is a smooth connected k-scheme, and that there is a a principally quasi-polarized p-divisible group (D_p, p_{D_p}) over X_p. We consider the stratification of X_p defined by the p-ranks of D_p over geometric points of X_p. From 3.4.18.4 B) we deduce immediately that it is a RAS stratification (we can take m to be $\dim(X_p)\mathrm{rk}(D_p)^2$). In the case X_p is a connected component of $\mathcal{M}_{k(v)}$ and we are dealing

with the standard principally quasi-polarized p-divisible group over it, from [NO, 4.1] we deduce that this RAS stratification is in fact an AS stratification.

Example 2. The Lie p-rank stratification of $\mathcal{N}_{k(v)}$ is a RAS stratification (cf. 3.4.18.7.0). Similarly, using the proof of 3.4.18.4.6 C we get that the refined Lie-stable stratification of $\mathcal{N}_{k(v)}$ is a RAS stratification.

The above two examples are particular samples: 3.4.20 4) shows that in fact always a p-rank and a Lie p-rank stratification (in a context involving Shimura p-divisible groups), are RAS stratifications. Warning: not always a RAS stratification is an AS stratification.

Example 3. Let $R := k[x_1, x_2]$. We consider the quasi Artin-Schreier system in two variables with coefficients in R defined by the equations $z_1 = x_1x_2z_1^p + x_1z_2^p$ and $z_2 = x_1x_2z_1^p + 2x_1z_2^p$. Obviously the RAS stratification of $\mathrm{Spec}(R)$ defined by it can not be an AS stratification.

Expectation. *We expect that (at least in many cases) a Newton polygon stratification associated to p-divisible groups is a RAS stratification.*

The motivation of this expectation is basen on examples, on 3.4.20 4) and on the fact that taking different exterior powers of objects of $\mathcal{MF}_{[0,1]}(*)$, we expect formulas similar to the ones of 3.4.18.4 B) and 3.4.18.7, and which involves slopes which are not just integral values. These different exterior powers should be able to "capture" the shape of the Frobenius of the vector bundles involved modulo higher powers of p (we recall that in getting the formula of 3.4.18.4 B) only the value of Φ mod p was important). In particular using 3.4.8.1.4 one should be able to regain the purity of Newton polygon stratifications of de Jong and Oort.

Remark. One of the advantages of AS stratifications in comparison to the Newton polygon ones is: using different determinant or norm maps, we can put different natural scheme structures on the strata, which are not apriori reduced (there is always logic in the choice of m and of the quasi Artin-Schreier systems involved).

We hope to come back to the ideas of this paragraph in a future paper.

4.6. Examples and main properties. Here we list the main properties of the theory of G-ordinary types, of G-ordinary points, and of their G-canonical lifts, and present some examples to illustrate them and the complexities which are arising.

P1 *The G-ordinary type τ is the (usual) ordinary type iff $k(v) = \mathbb{F}_p$.*

Proof: From the description of the G-ordinary type given in 4.1, we deduce (with the notations of 4.1.1.1) that h^0 has only the eigenvalues 0 and 1 iff $\bar{h}_i = \bar{h}_1$, $\forall i \in \{1, \dots, d\}$, and this is equivalent to $\sigma(\bar{h}_1) = \bar{h}_1$, i.e. it is equivalent to μ being defined over $\mathbb{Z}_p$. But $W(k(v))[\frac{1}{p}]$ is the field of definition of the $G_{\mathbb{Z}_p}(\mathbb{Z}_p^{sh})$-conjugacy class of cocharacters of $G_{\mathbb{Z}_p^{sh}}$ (which over an embedding of the field of fractions of $\mathbb{Z}_p^{sh}$ in $\mathbb{C}$, extending the inclusion $E(G, X) \subset \mathbb{C}$, are $G(\mathbb{C})$-conjugate to $\mu_x^*, x \in X$). Now everything results from [Mi3, 4.7].

4.6.1. Example 1. In the case of a standard PEL situation $(f, L_{(p)}, v, \mathcal{B})$, if $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is of simple C_ℓ or $D_\ell^{\mathbb{H}}$ type, we have $E(G, X) = \mathbb{Q}$. So we get only usual ordinary types. Similarly, for the classical Spin modular varieties of odd dimension and rank 2 considered in [Va2, 5.7.5], we get only usual ordinary types.

P2 *If $k(v)=\mathbb{F}_p$, then the abelian variety A_z over $W(k)$ obtained from $\mathcal{A}$ by pull back through a G-canonical lift z of a G-ordinary point $y:\mathrm{Spec}(k)\to\mathcal{U}$ (i.e. of an ordinary point of $\mathcal{N}_{k(v)}$, cf. P1) is the (usual) canonical lift of the (ordinary) abelian variety over k, defined by the special fibre of A_z.*

Proof: This is a direct consequence of c) of 4.4.1 3) and of the description of the σ-crystal (M,φ) of 4.1.1 giving birth to the G-ordinary type.

P3 *If the G-ordinary type τ is $e(1,1)$, i.e. if τ is the formal isogeny type associated to supersingular abelian varieties of dimension e, then G is a torus.*

Proof: If τ is $e(1,1)$, then the Shimura Lie σ-crystal attached to a G-ordinary point y of $\mathcal{N}_{k(v)}$ has only slopes zero. Now everything results from the fact that the Shimura filtered Lie σ-crystal of a G-canonical lift z of y is of parabolic type, and from 3.2.2.

P4 *The slopes of the Shimura σ-crystals attached to G-ordinary points of $\mathcal{N}_{k(v)}$ are of the form $\frac{i}{d}$, with $i\in\{0,1,\ldots,d\}$.*

Proof: This is a consequence of the form of h_0 in 4.1. We recall that the eigenvalues of h_0 are the slopes of the G-ordinary σ-crystal $(M,\varphi,G_{W(k(v))})$ defined in 4.1.1.

4.6.2. Example 2. If $k(v)=\mathbb{F}_{p^2}$ and if G is not a torus, then τ is of the form $r(1,0)+s(1,1)+r(0,1)$, with $r,s\in\mathbb{N}$ (i.e. the slopes of a Shimura σ-crystal attached to a G-ordinary point of $\mathcal{N}_{k(v)}$ are precisely 0, 1/2 and 1).

Proof: This is a direct consequence of the properties P1, P3 and P4.

4.6.3. Example 3. We consider now the case of Picard surfaces, with f the map defined in [Go]. If $k(v)=\mathbb{F}_{p^2}$, then τ is $2(1,0)+(1,1)+2(0,1)$.

4.6.4. Example 4. This is the generalization of 4.6.3. Let E be an imaginary quadratic field, let V be a $(m+n)$-dimensional E-vector space (with $m,n\in\mathbb{N}$), and let $\mathcal{J}_V:V\times V\to E$ be a non-degenerate hermitian form on V which has signature (m,n) over $\mathbb{R}$. We can assume $m\geq n$. Let $\mathrm{Sh}(G,X)$ be the Shimura variety defined as in [Go], for this situation. Similarly to [Go] it can be embedded in a Siegel modular variety $f:\mathrm{Sh}(G,X)\hookrightarrow\mathrm{Sh}\big(\mathrm{GSp}(W,\psi),S\big)$ with $\dim_{\mathbb{Q}}(W)=2(m+n)$. We have $E(G,X)=E$ if $m>n$, and $E(G,X)=\mathbb{Q}$, if $m=n$. Moreover $G^{\mathrm{der}}_{\mathbb{R}}=SU(m,n)_{\mathbb{R}}$. If v is a prime of E relatively prime with 2 and such that $k(v)=\mathbb{F}_{p^2}$, and if $(f,L_{(p)},v)$ is a SHS, then the G-ordinary type we get is $2n(1,0)+(m-n)(1,1)+2n(0,1)$.

P5 *The Lie G-ordinary type $\mathrm{Lie}_G(\tau)$ is an ordinary type (i.e. it has precisely only the slopes $-1,0$ and 1) iff G is not a torus and $k(v^{\mathrm{ad}})=\mathbb{F}_p$, where v^{ad} is the prime of $E(G^{\mathrm{ad}},X^{\mathrm{ad}})$ divided by v.*

Proof: This is a direct consequence of b) of 4.4.1 2), and of 3.2.2.

Let $\mathfrak{g}:=\mathrm{Lie}(G_{W(k)})$ and let $\big(\mathfrak{g},\varphi_0,F^0(\mathfrak{g}),F^1(\mathfrak{g})\big)$ be the Shimura filtered Lie σ-crystal attached to the G-canonical lift of a G-ordinary point $y:\mathrm{Spec}(k)\to\mathcal{N}_{k(v)}$.

P6 *The parabolic Lie subalgebra $\mathfrak{p}_0$ of $\mathfrak{g}$ corresponding to non-negative slopes of $(\mathfrak{g},\varphi_0)$ is $F^0(\mathfrak{g})$ iff $k(v^{\mathrm{ad}})=\mathbb{F}_p$.*

Proof: This is a direct consequence of c) of 4.4.1 3) and of the description of $\big(M, F^1, \sigma\mu(\frac{1}{p})\big)$ in 4.1 (cf. also with the proof of P1).

P7 *$\mathfrak{p}_0$ can be the Lie subalgebra of a Borel subgroup of $G_{W(k)}$.*

Proof: This is equivalent to: the rank of G is equal to the number of slopes zero of $(\mathfrak{g}, \varphi_0)$. From the formulae of 3.10.6 we deduce that, if this is so, then all simple adjoint factors of $G^{\mathrm{ad}}_{\mathbb{C}}$ are of A_ℓ Lie type, $\ell \in \mathbb{N}$. What we actually need is: any cyclic adjoint factor of $(\mathfrak{g}, \varphi_0, F^0(\mathfrak{g}), F^1(\mathfrak{g}))$ has maximal spreading (cf. def. 3.10.7).

Concrete example: Let F be a totally real number field, and let $\tilde{G}$ be an absolutely simple adjoint group over F of A_ℓ Lie type, such that $\tilde{G}_{\mathbb{R}}$ is the adjoint group of $\prod_{i=1}^{[\frac{\ell+1}{2}]} SU(i, \ell+1-i)_{\mathbb{R}}$. We assume the existence of a prime w of F unramified over a rational prime $p > 2$, and such that: $\tilde{G}$ is unramified over $\mathbb{Q}_p$, w is the only prime of F over p, and $\tilde{G}$ over the completion F_w of F with respect to w has non-trivial involution. Then for any SHS $(f, L_{(p)}, v)$, with $G^{\mathrm{ad}} = \mathrm{Res}_{F/\mathbb{Q}}\tilde{G}$, and with v dividing w (we have $F \subset E(G, X)$) and such that $k(v) = k(w)$, P7 above holds. Such a SHS does exist cf. [Va2, 6.5.1.1].

4.6.5. Example 5. P7 holds for the case considered in 4.6.3.

4.6.6. Example 6. If in 4.6.4 we have $m + n \geq 4$, then $\mathfrak{p}_0$ is not the Lie subalgebra of a Borel subgroup. If $m = n$, then $\mathrm{Lie}_G(\tau)$ is always an ordinary type.

P8 *For any rational number $r = a/d$, with $a \in \{1, \ldots, d\}$, there are situations when the slopes of the Shimura Lie σ-crystals attached to G-ordinary points of $\mathcal{N}_{k(v)}$ are precisely $-r, 0$ and r.*

Proof: We take (G, X) of B_ℓ (or C_ℓ or $D^{\mathbb{R}}_\ell$) type and such that $G^{\mathrm{ad}} = \mathrm{Res}_{F/\mathbb{Q}}G^1$, with F a totally real number field satisfying $[F : \mathbb{Q}] = d$, and with G^1 an absolutely simple F-group. We assume that for precisely a-embeddings of F into $\mathbb{R}$, the Lie group defined by the $\mathbb{R}$-values points of G^1 is non-compact. We also assume the existence of a prime v of $E(G, X) = F$ (cf. [De2, 2.3.12]) such that $k(v) = \mathbb{F}_{p^d}$, with $p > 2$ a prime for which G is unramified over $\mathbb{Q}_p$. Now for a SHS $(f, L_{(p)}, v)$ (cf. [Va2, 6.4.2]) defined by an injective map $f : (G, X) \hookrightarrow (\mathrm{GSp}(W, \psi), S)$, the slopes of the Shimura Lie σ-crystals attached to G-ordinary points of $\mathcal{N}_{k(v)}$ are precisely $-\frac{a}{d}$, 0 and $\frac{a}{d}$ (cf. 3.10.6 i)).

Obviously for any given $r \in \mathbb{Q} \cap (0, 1]$ we can choose a number field F and a reductive group G^1 over F such that the above assumptions are satisfied. Of course there are examples for P8 with (G, X) of A_ℓ or $D^{\mathbb{H}}_\ell$ type (starting from 3.10.6). We leave this to the reader.

P9 *If the slopes of the Shimura σ-crystal attached to G-ordinary points of $\mathcal{N}_{k(v)}$ are rational numbers of the interval $(0, 1)$, then no cyclic factor of the Shimura adjoint Lie σ-crystal attached to a G-ordinary point of $\mathcal{N}_{k(v)}$ is totally non-compact (see def. 3.10.5). The converse of this not true.*

Proof: The first part is a direct consequence of the formulae of 3.10.6. The second part can be seen through trivial examples.

As a corollary we get:

P10 *If there is a totally non-compact cyclic factor of the Shimura adjoint Lie σ-crystal attached to a G-ordinary point of $\mathcal{N}_{k(v)}$, then the abelian varieties obtained by pulling back $\mathcal{A}$ through G-ordinary points with values in algebraically closed fields, have non-trivial p-torsion.*

4.6.7. Example 7: The case of curves. We assume now that $\dim(X) = 1$, i.e. that $G^{\mathrm{ad}} = \mathrm{Res}_{F/\mathbb{Q}} G^1$, with F a totally real number field, and with G^1 a form of PSL_2 such that for exactly one embedding of F into $\mathbb{R}$, $G^1_{\mathbb{R}}$ is a non-compact (so split) group (cf. [De1, 2.3.4]). We make no restrictions on $(G^{\mathrm{ab}}, X^{\mathrm{ab}})$. Let $(f, L_{(p)}, v)$ be a SHS defined by an injective map $f : (G, X) \hookrightarrow (\mathrm{GSp}(W, \psi), S)$, and let v^{ad} be the prime of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}}) = F$ (cf. [De2, 2.3.12]) divided by v.

Let $G^{\mathrm{ad}}_{\mathbb{Q}_p} = \prod_{i \in I_F} \mathrm{Res}_{F^i/\mathbb{Q}_p} G^1_i$, where $F \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{i \in I_F} F_i$, with F_i an unramified finite field extension of $\mathbb{Q}_p$, and where G^1_i is an absolutely simple F_i-group. Let $i_0 \in I_F$ corresponding to the non-compact factor of $G^{\mathrm{ad}}_{\mathbb{R}}$, and let $d_0 = [F_{i_0} : \mathbb{Q}_p] \geq 1$. Then $k(v^{\mathrm{ad}}) = \mathbb{F}_{p^{d_0}}$, and the slopes of the adjoint Lie σ-crystal attached to a G-ordinary point of $\mathcal{N}_{k(v)}$ are precisely $-\frac{1}{d_0}$, 0 and $\frac{1}{d_0}$; the multiplicity of $-1/d_0$ (or of $1/d_0$) is d_0, while the multiplicity of 0 is $3\big(\dim_{\mathbb{Q}}(F) - d_0\big) + d_0 = 3\dim_{\mathbb{Q}}(F) - 2d_0$ (cf. 3.10.6 i)). So if $d_0 = 1$ P10 applies.

4.6.8. Remarks. 1) There are plenty of examples when the Newton polygon of the G-ordinary type of a SHS $(f, L_{(p)}, v)$ does not have the slopes 0 or 1. Such examples can be easily constructed starting from [Va2, 6.5.1.1] and its proof.

2) In [Va5] we also show that the results of 4.1-6 remain true (under proper formulation) for the integral canonical models to be constructed in §6 for quotients of Shimura varieties of Hodge type with respect to non-hyperspecial subgroups.

3) We reobtained the "ordinary theory" for special fibres of integral canonical models of Siegel modular varieties, with respect to primes $p \geq 2$ (for $p = 2$ cf. 4.14.3). In particular we obtained a completely new proof of the Zariski density (see [NO], [FC] and [EO]) of ordinary points in these special fibres.

4) The whole of 4.1 as well as properties P1 to P10 remain true under proper formulation in the context of Shimura σ-crystals (cf. 3.11.2). The only difference is: we have to change $k(v) = \mathbb{F}_{p^d}$ (with $d \in \mathbb{N}$) with the degree of definition ($= d$), and $k(v^{\mathrm{ad}}) = \mathbb{F}_{p^{d_0}}$, with the L-degree of definition ($= d_0$). Besides we have to change the condition G is a torus, of P3 (resp. G is not a torus, of P5), to any one of the equivalent conditions a) to d) of 3.2.2 (resp. with the condition that 3.3.2 a) to d) does not hold).

5) In particular 4.1 and P1 to P10 can be applied to the context of abelian varieties with Hodge cycles without an apriori defined principal polarization (for instance, there is no problem if we have a polarization which is not principal), as long as we are in a context of Shimura σ-crystals.

6) It can happen that we have two standard Hodge situations $(f, L_{(p)}, v_i)$, $i = \overline{1, 2}$, with v_1 and v_2 dividing the same rational prime $p \geq 3$, such that their attached G-ordinary types are different. For instance, this is the case if $k(v_1) = \mathbb{F}_p$, while $k(v_2) = \mathbb{F}_{p^d}$, with $d \geq 2$.

4.6.9. Application. Let E be a number field and let A be an abelian variety over E. Let $\mathrm{Spec}(O_E^r) \to \mathrm{Spec}(O_E)$ be the maximal open subscheme of the affine scheme defined by the ring of integers O_E of E, over which A extends to an abelian scheme A^r. Let $E(A)$ be the reflex field of the Shimura variety associated to the Mumford–Tate group of A (cf. [Va2, 2.12.3]).

4.6.9.1. Theorem *(The non-ordinary reduction criterion). If $E(A) \neq \mathbb{Q}$, then A^r has a non-ordinary reduction with respect to an infinite set of primes of O_E^r.*

Proof: We consider an embedding $E \hookrightarrow \mathbb{C}$, and let $\tilde{L}_{\mathbb{Z}} := H_1(A_{\mathbb{C}}, \mathbb{Z})$ be the first group of the Betti homology of $A_{\mathbb{C}}$. The Mumford–Tate group of A (of $A_{\mathbb{C}}$) is a reductive subgroup G_A of $GL(\tilde{L}_{\mathbb{Z}} \otimes \mathbb{Q})$. Now there is $N_0(A) \in \mathbb{N}$ such that for any prime $p \geq N_0(A)$, the closure of G_A in $GL(\tilde{L}_{\mathbb{Z}} \otimes \mathbb{Z}_{(p)})$ is a reductive group over $\mathbb{Z}_{(p)}$. If also $p \geq \dim(A)$ and $p \geq 5$, then there is a $\mathbb{Z}_{(p)}$-well positioned family of tensors (of $\mathcal{T}(\tilde{L}_{\mathbb{Z}} \otimes \mathbb{Z}_{(p)})$) for the group G_A (cf. [Va2, 5.8.6]) enveloped by $\tilde{L}_{\mathbb{Z}} \otimes \mathbb{Z}_{(p)}$. Let now v be a prime of O_E^r dividing a rational prime $p \geq \max\big(N_0(A), 5, \dim(A)\big)$. Let V be the completion of the maximal unramified cover of the completion of $O_{(v)}$. So V is a finite extension of $V_0 = W(\mathbb{F})$. Let A_V be the abelian variety over V, obtained from A^r through the logical morphism $\mathrm{Spec}(V) \to \mathrm{Spec}(O_E^r)$. All the Hodge cycles of A_V are defined over $V_1[\frac{1}{p}]$, with V_1 a finite DVR cover of V (cf. [De3]; in fact we can take $V_1 = V$, but this is irrelevant here). We choose a family $(t_\alpha)_{\alpha \in \mathcal{J}(V_1)}$ of Hodge cycles of A_{V_1} such that G_A is the subgroup of $GL(\tilde{L}_{\mathbb{Z}} \otimes \mathbb{Q})$ fixing the Betti realizations of the members of the family $(t_\alpha)_{\alpha \in \mathcal{J}(V_1)}$.

Let $\bar{e} := [V_1 : V_0]$. In what follows, to simplify the presentation, we use similar notations as before, though the situation is entirely unrelated to a fix SHS. Repeating the arguments of [Va2, 5.2-3], we get an abelian scheme $A_{\tilde{R}_{\bar{e}}}$ over the ring $\tilde{R}_{\bar{e}}$ defined in [Va2, 5.2.1] endowed with a family $({t_\alpha}^{\tilde{R}_{\bar{e}}})_{\alpha \in \mathcal{J}(V_1)}$ of Hodge cycles such that:

i) under the canonical V_0-epimorphism $\tilde{R}_{\bar{e}} \twoheadrightarrow V_1$ defined by a uniformizer of V_1 (see [Va2, 5.2]), it becomes A_{V_1}, together with $(t_\alpha)_{\alpha \in \mathcal{J}(V_1)}$;

ii) under the canonical V_0-epimorphism $\tilde{R}_{\bar{e}} \twoheadrightarrow V_0$, it becomes an abelian variety A_{V_0} for which $M := H^1_{\mathrm{crys}}(A_{V_0} \times \mathbb{F}, V_0)$, together with its Hodge filtration F^1 associated to A_{V_0}, and with its Frobenius φ and the family of de Rham components $(u^0_\alpha)_{\alpha \in \mathcal{J}(V_1)}$ of the Hodge cycles (obtained naturally from $(t_\alpha)_{\alpha \in \mathcal{J}(V_1)}$) $(t^0_\alpha)_{\alpha \in \mathcal{J}(V_1)}$ of A_{V_0}, is a Shimura filtered σ-crystal $\big(M, F^1, \varphi, G_{V_0}, (u^0_\alpha)_{\alpha \in \mathcal{J}(V_1)}\big)$ (here G_{V_0} is the closure in $GL(M)$ of the subgroup of $GL(M[\frac{1}{p}])$ defined by fixing u^0_α, $\forall \alpha \in \mathcal{J}(V_1)$).

Let $\mathcal{P}$ be the Newton polygon of the G_{V_0}-ordinary type associated to the G_{V_0}-ordinary σ-crystals produced by (M, φ, G_{V_0}). From 3.1.0 we get:

(4.6.9.2) *The Newton polygon of the abelian variety over $k(v)$ obtained from A by reduction with respect to v, is above $\mathcal{P}$.*

Let E_1 be the composite field of E and $E(A)$, and let ω be a prime of E_1 dividing v. Let ω_A be the prime of $E(A)$ divided by ω. We have:

(4.6.9.3) *If $k(\omega_A) \neq \mathbb{F}_p$, then $\mathcal{P}$ is not the Newton plygon of an ordinary type.*

This results from property $P1$ of 4.6: the principal polarization existing in the context of a SHS played no role in obtaining the theory of 4.1-6 (cf. rm. 5) of 4.6.8); of course [Mu, cor. 1, p. 234] allows us to assume that A is principally polarized.

Now 4.6.9.1 is a direct consequence of 4.6.9.2-3, as there are infinitely many primes of E_1 dividing primes of $E(A)$ not having a residue field with a prime number of elements.

4.6.9.4. Remark. We can make 4.6.9.1 more precise. Let v be a prime of O_E^r such that:

a) it divides a prime $p \geq 3$ such that G_A is unramified over $\mathbb{Q}_p$;

b) there is a prime ω of E_1 dividing v and dividing a prime ω_A of $E(A)$ for which $k(\omega_A)$ has more than p elements.

Then A has a non-ordinary reduction with respect to v (and actually there is a Newton polygon $\mathcal{P}$ of an abelian variety over $\mathbb{F}$, of dimension equal to $\dim(A)$, which is the Newton polygon of a G_A-ordinary type which is not an ordinary type, and such that the Newton polygon of the reduction of A with respect to v is above $\mathcal{P}$).

This will become transparent in the light of §5-8, cf. 2.3.8 2).

4.6.9.5. Question. Let $O_{E_1}^r$ be the normalization of O_E^r in E_1, and let $A_{E_1}^r$ be the abelian scheme over $O_{E_1}^r$, obtained from A^r by pull back. We do not assume that $E(A)$ is not $\mathbb{Q}$. Is it true that for an infinite set of primes $\mathcal{L}$ of $O_{E_1}^r$, $A_{E_1}^r$ has a G_A-ordinary reduction which is not ordinary?

4.6.9.6. Remark. Property P1 supports the well-known conjecture (cf. [Oo1, p. 11]) concerning the existence of ordinary reductions of an abelian variety over a number field E with respect to primes of E.

4.6.9.7. Remark. We regard the above non-ordinary reduction criterion as the natural generalization of the (well-known) fact that any abelian variety over a number field E, having complex multiplication over $\bar{E}$, has a non-ordinary reduction with respect to an infinite set of primes of E.

4.6.9.8. Exercise. Let $(f, L_{(p)}, v)$ be a SHS. Let τ be the G-ordinary type attached to it. Show that there is an infinite number of standard Hodge situations $(f, L_{(q)}, v_q)$ (with v_q a prime of $E(G, X)$ dividing an odd rational prime q) which have τ as their attached G-ordinary type. (Hint: If the reflex field $E(G, X)$ is a Galois extension of $\mathbb{Q}$, this is obvious; now consider the Galois extension $E_1(G, X)$ of $\mathbb{Q}$ generated by $E(G, X)$, and make use of 4.1 and of Tchebotarev density theorem).

This simple fact makes 4.6.9.5 an interesting question, and suggests that the conjectures referrred to in 4.6.9.6 should admit variants for G-ordinary types, which are not ordinary.

4.6.10. The case when the degree of definition is 1. For the sake of reference we point out the following two particular cases of P1 and P2.

A. The split case. *If $(f, L_{(p)}, v)$ is a SHS, and if $G_{\mathbb{Q}_p}$ is a split reductive group, then the G-ordinary type we obtain is the ordinary type, and the abelian varieties over Witt*

rings of perfect fields, obtained from $\mathcal{A}$ by pull back through G-canonical lifts of $\mathcal{N}$, are (usual) (ordinary) canonical lifts.

B. *If $(f, L_{(p)}, v)$ is a SHS, and if a $G_{W(\mathbb{F})}$-canonical lift of a $G_{W(\mathbb{F})}$-ordinary point of $\mathcal{N}_{k(v)}$ gives birth to a Shimura σ-crystal whose degree of definition is 1, then all abelian varieties over $W(\mathbb{F})$ obtained from $\mathcal{A}$ by pull back through any $G_{W(\mathbb{F})}$-canonical lift of $\mathcal{N}$, are (usual) canonical lifts, and so have complex multiplication.*

4.6.11. An application to [Va2, 6.8.6]. We consider the situation and notations desribed in [Va2, 6.8.6] in the case we have a non-trivial involution. So $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is a totally imaginary quadratic extension of $\mathbb{Q}$. We assume that we are dealing with a prime $p > 4$ which splits completely in $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$.

Exercise. Show that the criterion b) of [Va2, 6.8.2] is satisfied. Hint: use property P2 and [Mi1, 4.12].

4.6.12. Exercise. Prove d) of 4.4.1 3) for the case $k(v) = \mathbb{F}_p$. Hint: use P1, P2 and the Fontaine's comparison theory.

4.7. Crystalline coordinates.

4.7.1. Let $y : \mathrm{Spec}(k) \to \mathcal{U}/H_0$ be a G-ordinary point. For simplifying the notations we assume that attached to y we do have a principally quasi-polarized Shimura σ-crystal, and that it is of the form $\big(M_y, \varphi_y, G_{W(k)}, (t_\alpha)_{\alpha\in\mathcal{J}'}, \psi_y\big)$ (cf. 2.3.10). Let $\mu_y : \mathbb{G}_m \to G_{W(k)}$ be its canonical split. It produces a direct sum decomposition $M_y = F^1_y \oplus F^0_y$, with $\beta \in \mathbb{G}_m(W(k))$ acting through μ_y on F^i_y as the multiplication with β^{-i}. Let N be the subgroup of $G^{\mathrm{der}}_{W(k)}$ which acts trivially on F^0_y and on M_y/F^0_y. Let R be the ring of the formal completion of N at the origin. We choose an isomorphism $\tilde{f} : R \tilde{\to} W(k)[[x_1, \ldots, x_{e_0}]]$, with $e_0 = \dim(X)$ (see [Va2, 5.4.7]). We are mainly interested in the case when $k = \bar{k}$; but for a great part of what follows this is not needed. However, to simplify the notations we assume that k is big enough so that the group N is isomorphic to $\mathbb{G}_a^{e_0}$. We can choose the coordinates $x_1, \ldots, x_{e_0}$ (i.e. we can choose $\tilde{f}$) such that the i-th $\mathbb{G}_a$ copy of N is $\mathrm{Spec}\big(W(k)[x_i]\big)$. So N as a scheme is $\mathrm{Spec}\big(W(k)[x_1, \ldots, x_{e_0}]\big)$. We take the Frobenius lift of N to be compatible with σ, and such that at the rings level it takes $x_i \to x_i^p$, $i = \overline{1, e_0}$. It induces a Frobenius Φ_R on R.

As the case of a torus (of a zero dimensional Shimura variety) is trivial, we assume from now on that G is not a torus. Let N_0 be the smallest connected subgroup of $G_{W(k)}$ such that:

i) it contains N;

ii) its Lie algebra is $p\varphi_y$-invariant.

For all that follows, we have to rely heavily on the terminology and formulae of 3.10.6. It is easy to see (to be compared also with 3.7.2) that:

0) N_0 is a product indexed by the number of simple factors of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$.

1) N_0 is a nilpotent smooth subgroup of $G_{W(k)}$.

2) We get a Lie σ-crystal $(\mathrm{Lie}(N_0), \varphi_y, F^0(\mathrm{Lie}(N_0)), F^1(\mathrm{Lie}(N_0)))$, where $F^i(\mathrm{Lie}(N_0)) := F^i(\mathrm{End}(M)) \cap \mathrm{Lie}(N_0)$, $i = \overline{0, 1}$.

3) $N = N_0$ iff the Lie G-ordinary type $\mathrm{Lie}_G(\tau)$ we get is ordinary (i.e. it has precisely the slopes -1, 0 and 1), cf. properties P5 and P6 of 4.6.

4) N_0 is an abelian group iff the spreading of any cyclic adjoint factor attached to y is 0 or 1, cf. def. in 3.10.4.4 and 4.3.1. This automatically happens if all the simple factors of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ are of B_l, C_l or $D_l^{\mathbb{R}}$ type, but it is not necessarily true if there are simple factors of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ of A_ℓ or $D_\ell^{\mathbb{H}}$ type ($\ell \in \mathbb{N}$).

5) More generally, N_0 is a nilpotent group of order of nilpotency not bigger than some $m \in \mathbb{N}$, if the spreading of any cyclic adjoint factor attached to y is not bigger than m.

So if $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ has only simple factors of $D_\ell^{\mathbb{H}}$ type ($\ell \in \mathbb{N}$), then N_0 has the order of nilpotency at most 2; if $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is a simple adjoint Shimura variety of A_ℓ type, then the order of nilpotency of N_0 is smaller or equal to ℓ, and equality can take place (cf. property P7 of 4.6).

We prefer to regard y as a closed embedding $y : \mathrm{Spec}(k) \hookrightarrow \mathcal{N}_{W(k)}/H_0$. Let $\hat{O}_y$ be the completion of the local ring of y in $\mathcal{N}_{W(k)}/H_0$. Let $(A_{\hat{y}}, p_{A_{\hat{y}}})$ be the principally polarized abelian scheme over $\mathrm{Spec}(\hat{O}_y)$ obtained from $(\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ through the canonical morphism $\mathrm{Spec}(\hat{O}_y) \to \mathcal{N}/H_0$. We consider the following principally quasi-polarized Shimura filtered F-crystal

$$ {}_N M_{\hat{y}} := (M_y, F_y^1, \varphi_y, G_{W(k)}, N, \tilde{f}, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_y) $$

over R/pR. So there is a unique connection ${}_N\nabla_{\hat{y}}$ on $M_y \otimes R$ which is integrable, nilpotent mod p, and such that the Frobenius $\Phi_y := n(\varphi_y \otimes 1)$ of $M_y \otimes R$ is ${}_N\nabla_{\hat{y}}$-parallel, cf. 2.2.10 (here $n \in N(R)$ is the universal element defined by the natural morphism $\mathrm{Spec}(R) \to N$).

Similarly we define ${}_{G^{\mathrm{der}}_{W(k)}} M_{\hat{y}}$, starting from a Frobenius Φ_{R_1} of the ring R_1 of the completion of $G^{\mathrm{der}}_{W(k)}$ at the origin, leaving invariant the kernel of the canonical $W(k)$-epimorphism $R_1 \twoheadrightarrow R$, and inducing Φ_R on R).

Let $\tilde{e}_0$ be the dimension of N_0. As a scheme $N_0 = \mathrm{Spec}(W(k)[x_1, ..., x_{\tilde{e}_0}])$, cf. the above assumption on k. Such an identification can be chosen so that each closed subscheme $\mathrm{Spec}(W(k)[x_i])$, $i \in \{1, ...\tilde{e}_0\}$, is a $\mathbb{G}_a$ subgroup of N_0, to be referred as the i-th $\mathbb{G}_a$ copy of N_0. Let $\mathrm{Spec}(R_0)$ be the completion of the localization of N_0 in the origin. So $R_0 = W(k)[[x_1, ..., x_{\tilde{e}_0}]]$, with $\mathrm{Spec}(W(k)[[x_i]])$ being the completion of the i-th $\mathbb{G}_a$ copy of N_0, $i = \overline{1, \tilde{e}_0}$. We choose such an identification, so that the natural epimorphism $R_0 \twoheadrightarrow R$ is defined by x_i goes to x_i, if $i \in \{1, ..., e_0\}$, and into zero, if $i \in \{1 + e_0, ...\tilde{e}_0\}$. We assume that the Frobenius Φ_{R_1} of R_1 is such that it leaves invariant the kernel of the natural epimorphism $R_1 \twoheadrightarrow R_0$, inducing a Frobenius of R_0 which takes x_i into x_i^p, $i = \overline{1, \tilde{e}_0}$.

4.7.2. Lemma. *The principally quasi-polarized filtered F-crystal over $\mathrm{Spec}(\hat{O}_y/p\hat{O}_y)$ endowed with a family of crystalline tensors, defined by $(A_{\hat{y}}, p_{A_{\hat{y}}})$ and de Rham components of the Hodge cycles with which $A_{\hat{y}}$ is naturally endowed, is isomorphic to ${}_N M_{\hat{y}}$.*

Proof: In [Va2, 5.4.4-5] we saw that there is a morphism $\mathrm{Spec}(R_1) \xrightarrow{a_1} \mathcal{N}_{W(k)}/H_0$ lifting y, and such that the principally quasi-polarized filtered F-crystal over R_1/pR_1 endowed with tensors, and induced by the pull back through a_1 of $(A_{\hat{y}}, p_{A_{\hat{y}}})$ and de Rham

components of the mentioned Hodge cycles of $A_{\hat{y}}$, is isomorphic to ${}_{G^{\mathrm{der}}_{W(k)}}M_{\hat{y}}$. But the pull back of ${}_{G^{\mathrm{der}}_{W(k)}}M_{\hat{y}}$ under the canonical morphism $\mathrm{Spec}(R) \overset{b_1}{\hookrightarrow} \mathrm{Spec}(R_1)$ is ${}_{N}M_{\hat{y}}$ (due to the compatibility of the two Frobenius lifts of R_1 and R with the $W(k)$-epimorphism $R_1 \twoheadrightarrow R$). But the morphism $c_1 := b_1 \circ a_1 : \mathrm{Spec}(R) \to \mathrm{Spec}(\hat{O}_y)$ is an isomorphism, cf. [Va2, 5.4.7-8]. This proves the lemma.

We denote by $\bar{\Omega}_{R/W(k)}$ the free R-module generated by dx_i, $i = \overline{1, e_0}$. $\bar{\Omega}_{R/W(k)}$ is naturally a direct summand of the R-module $\Omega_{R/W(k)}$.

4.7.3. Lemma. *The connection ${}_{N}\nabla_{\hat{y}}$ is of the form $d + \beta_N$, where $\beta_N \in \mathrm{Lie}(N_0) \otimes \Omega_{N/W(k)} \otimes R = \mathrm{Lie}(N_0) \otimes \bar{\Omega}_{R/W(k)}$.*

Proof: We define similarly ${}_{N_0}M_{\hat{y}}$. ${}_{N}M_{\hat{y}}$ is obtained from ${}_{N_0}M_{\hat{y}}$ by pull back through the $W(k)$-epimorphism $R_0 \twoheadrightarrow R$. So the lemma is a consequence of [Fa2, rm. ii) of th. 10] applied to the subgroup N_0 of $GL(M_y)$.

4.7.3.1. We write

$$\beta_N = \sum_{j=1}^{\tilde{e}_0} n_j \otimes \varepsilon_j ,$$

with $\{n_j\}$ a $W(k)$-basis of the Lie algebra of the j-th copy $\mathbb{G}_a$ of N_0, and with $\varepsilon_j \in \bar{\Omega}_{R/V_0}$ (cf. 4.7.3). The elements n_j, $j \in \{1, ..., \tilde{e}_0\}$, are endomorphisms of $M_y \otimes R$.

4.7.4. Let v^{ad} be the prime of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ divided by v. To build up G-multiplicative coordinates, and then to extend this to the general case of 4.7.33.1, we distinguish five different cases (levels of complexity). They are:

A. $N = N_0$.

B. N_0 is an abelian group.

C. N_0 is a nilpotent group of order of nilpotency two.

D. N_0 is a nilpotent group of order of nilpotency greater than two.

E. The general case refered to in 4.7.33.1 below.

The condition $N = N_0$ is equivalent to $k(v^{\mathrm{ad}}) = \mathbb{F}_p$, as it can be easily checked starting from 3.10.6 and property P2 of 4.6. First we describe the case **A** (so $\tilde{e}_0 = e_0$). Warning: 7.5-29 below refer to this case only. The general theory of cristalline coordinates is elaborated in 4.7.30. In 4.7.31-37 we deal with different aspects of it.

From now on we assume $k = \bar{k}$.

4.7.5. Theorem. *We assume that $N = N_0$.*

i) *There is a basis $\{a_i | 1 \le i \le e\}$ of F^0_y and a basis $\{b_i | 1 \le i \le e\}$ of F^1_y and a permutation π of $\{a_1, \ldots, a_e, b_1, \ldots, b_e\}$ satisfying the conditions:*

(4.7.6) ${}_{N}\nabla_{\hat{y}} a_i = 0$, $\forall i \in \{1, \ldots, e\}$;

(4.7.7) ${}_{N}\nabla_{\hat{y}} b_i = \sum_{j=1}^{e_0} n_j(b_i) \otimes \varepsilon_j$;

(4.7.8) $\Phi_y(a_i) = \pi(a_i)$ *if* $\pi(a_i) \in \{a_1, \ldots, a_e\}$;

(4.7.9) $\Phi_y(a_i) = n\pi(a_i)$ *if* $\pi(a_i) \in \{b_1, \ldots, b_e\}$;

(4.7.10) $\Phi_y(b_i) = p\pi(b_i)$ *if* $\pi(b_i) \in \{a_1, \ldots, a_e\}$;

(4.7.11) $\Phi_y(b_i) = pn\pi(b_i)$ *if* $\pi(b_i) \in \{b_1, \ldots, b_e\}$.

Furthermore ε_i*'s are closed 1-forms verifying the equations:*

(4.7.12) *If* $i \in \{1, \ldots, e\}$ *is such that* $\pi(b_i) \in \{a_1, \ldots, a_e\}$, *then*

$$0 = \sum_{j=1}^{e_0} \varphi(n_j)\big(\pi(b_i)\big) \otimes \Phi_R^* \varepsilon_j \, ;$$

(4.7.13) *If* $i \in \{1, \ldots, e\}$ *is such that* $\pi(b_i) \in \{b_1, \ldots, b_e\}$, *then*

$$pdn\big(\pi(b_i)\big) + \sum_{j=1}^{e_0} p\, n_j\big(\pi(b_i)\big) \otimes \varepsilon_j = \sum_{j=1}^{e_0} \varphi(p\, n_j)\big(\pi(b_i)\big) \otimes \Phi_R^*(\varepsilon_j);$$

(4.7.14) *If* $i \in \{1, \ldots, e\}$ *is such that* $\pi(a_i) \in \{b_1, \ldots, b_e\}$ *then*

$$dn\,\big(\pi(a_i)\big) + \sum_{j=1}^{e_0} n\big(\, n_j\big(\pi(a_i)\big)\big) \otimes \varepsilon_j = 0;$$

ii) *There are formal power series* $\mu_j \in W(k)[\frac{1}{p}][[x_1, \ldots, x_{e_0}]]$ *such that:*

(4.7.15) $\varepsilon_j = d\mu_j \ \forall j \in \{1, \ldots, e\}$;

(4.7.16) $\forall i \in \{1, \ldots, e\}$ *such that* $\pi(b_i) \in \{a_1, \ldots, a_e\}$ *we have*

$$\big(\sum_{j=1}^{e_0} \varphi(n_j)\big(\pi(b_i)\big)\Phi_R^* \mu_j\big)(0) = 0;$$

(4.7.17) $\forall i \in \{1, \ldots, e\}$ *such that* $\pi(b_i) \in \{b_1, \ldots, b_e\}$ *we have*

$$\big((n - 1_{M_y \otimes R})(\pi(b_i)) + p \sum_{j=1}^{e_0} n_j\big(\pi(b_i)\big)\mu_j\big)(0) = \sum_{j=1}^{e_0} \big(\varphi(p\, n_j)\big(\pi(b_i)\big)\Phi_R^* \mu_j\big)(0);$$

(4.7.18) $\forall i \in \{i, \ldots, e\}$ *such that* $\pi(a_i) \in \{b_1, \ldots, b_e\}$ *we have*

$$\big((\log\, n)\big(\pi(a_i)\big) + \sum_{j=1}^{e_0} n_j\big(\pi(a_i)\big)\mu_j\big)(0) = 0.$$

We have $\mu_j(0) \in pW(k)$ *and the series* $q_j = \exp(\mu_j)$, $j \in \{1, \ldots \tilde{e_0}\}$, *belongs to* R *and satisfies* $q_j(0) = 1$ *modulo* $pW(k)$.

4.7.19. Corollary. *The elements $q_j - 1 \in R$, $j = \overline{1, e_0}$, form together with p a regular system of parameters of R, i.e. the $W(k)$-homomorphism $R \to R$ sending x_j to $q_j - 1$, $j = \overline{1, e_0}$, is an isomorphism. Furthermore if we take the Frobenius of R to be defined by $q_j \to q_j^p$, $j = \overline{1, e_0}$, then the Frobenius of ${}_N M_{\hat{y}}$ takes b_j into $p\pi(b_j)$, and takes a_j into $\pi(a_j)$, $\forall j \in \{1, \ldots, e\}$.*

4.7.20. Definition. The elements $q_j - 1 \in R$ form, due to 4.7.19, the G-canonical coordinates of R, or the Shimura-canonical coordinates of R.

4.7.21. Corollary. *The formal moduli scheme $\mathcal{M}_G(y)$ of G-deformations (i.e. deformations factoring through $\mathcal{N}/H_0$) of the principally polarized abelian variety (A_y, p_{A_y}), over artinian local $W(k)$-algebras having k as their residue field, is canonically isomorphic to the formal torus $\mathbb{G}_m^{e_0\wedge}/W(k)$; the principally polarized abelian variety over $W(k)$, obtained from $(\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}})$ by pull back through the G-canonical lift of y, corresponds to the unit element of $\mathbb{G}_m^{e_0\wedge}/W(k)(W(k))$.*

4.7.22. Corollary. *The permutation π is trivial if and only if $k(v) = \mathbb{F}_p$, i.e. A_y is an ordinary abelian variety. If this is the case then the formal torus of G-deformations of (A_y, p_{A_y}) is a formal subtorus of the formal torus of deformations of A_y, or of GSp-deformations of (A_y, p_{A_y}) (i.e. of deformations of the principal polarized abelian variety (A_y, p_{A_y})).*

4.7.23. About the proves of 4.7.5-22. The proves of 4.7.5 and of 4.7.19 are entirely analogous to the proves of [De2, 1.4.2 and 1.4.7]. As this is the only part of our work which is similar to the theory of ordinary points of special fibres of integral canonical models of Siegel modular varieties, here we just indicate which are the differences occurring. We do not bother to mention what happens when we choose different Frobenius lifts on R (see [De2]).

The first part b) of 4.4.1 3) guarantees that the filtered σ-crystal (M_y, F_y^1, φ_y) is cyclic diagonalizable. From the proof of it, we deduce the existence of a basis $\{a_i | 1 \leq i \leq e\}$ of F_y^0 and a basis $\{b_i | 1 \leq i \leq e\}$ of F_y^1, and of a permutation π of $B := \{a_1, \ldots, a_e, b_1, \ldots, b_e\}$ such that $\varphi_y(b_i) = p\pi(b_i)$ and $\varphi_y(a_i) = \pi(a_i)$, $i = \overline{1, e}$.

The difference from [De2, 1.4] comes from the fact that that π is not always the trivial permutation 1_B of B.

Now (4.7.6-7) just say what is the action of ${}_N\nabla_{\hat{y}}$ on the basis B of M_y, using the expression of ${}_N\nabla_{\hat{y}}$ given in 4.7.3.1. (4.7.8-11) just makes explicit the action of Φ_y on elements of B, starting from the mentioned expression of φ_y. The proof of the fact that ε_j are closed is as in [De2, 1.4.2], starting from the fact that the connection ${}_N\nabla_{\hat{y}}$ is integrable (this is also implied by the assumption $N = N_0$). (4.7.12-14) just writes down what means that Φ_y is ${}_N\nabla_{\hat{y}}$-parallel (with respect to Φ_R) (cf. [De2, 1.1.3.3 and 1.4.2.5]).

To see ii) of 4.5, we follow closely the proof of [De2, 1.4.2.5 ii)]. First the Poincaré lemma assures the existence of formal power series $\mu_j \in W(k)[\frac{1}{p}][[x_1, \ldots, x_{e_0}]]$ such that $d\mu_j = \varepsilon_j$, unique up to a constant. Now (4.7.12-14) can be rewritten as:

(4.7.24) $d\Big(\sum_{j=1}^{e_0} \varphi(n_j)\big(\pi(b_i)\big)\Phi_R^*\mu_j\Big) = 0;$

(4.7.25) $d\Big(pn\big(\pi(b_i)\big) + p\sum_{j=1}^{e_0} n_j\big(\pi(b_i)\big)\mu_j - \varphi(p\,n_j)\big(\pi(b_i)\big)\Phi_R^*\mu_j\Big) = 0;$

(4.7.26) $d\Big(\log\, n\pi(a_i) + \sum_{j=1}^{e_0} n_j\big(\pi(a_i)\big)\mu_j\Big) = 0.$

We remark that $\varphi(p\,n_j) \in M_y$ is non-zero mod p, and that N is an abelian group, and so we have no problems with $\log n$. The equations (4.7.24-26) are of similar type: they are obtained by taking the d-operator of linear combinations of functions; (4.7.25) is an equation similar to the ones obtained by taking the d-operator of [De2, 1.4.2.7]. The values of μ_j at zero are determined by the equations (4.7.16-18). To see why this is so, let $t_j = n_j\mu_j(0)$, $j \in \{1, ..., e_0\}$, and let $t = \sum_{j=1}^{e_0} t_j$, We have $t \in \mathrm{Lie}(N)[\frac{1}{p}]$. To determine $\mu_j(0)$, $j \in \{1, \ldots, e_0\}$, is equivalent to determine t. The equations (4.7.16-18) can be rewritten respectively as:

(4.7.27) $\varphi(t)\big(\pi(b_i)\big) = 0;$

(4.7.28) $\varphi(pt)\big(\pi(b_i)\big) = pt\big(\pi(b_i)\big);$

(4.7.29) $t\big(\pi(a_i)\big) = 0.$

Let $B_0 := \{a_1, \ldots, a_e\} \cup \big\{\pi(a_i), \ldots, \pi(a_e)\big\}$. Now t acts trivially on the subspace of M_y generated by B_0, i.e. on $F_y^0 \cup \varphi_y(F_y^0)$. This implies that if there is $t \in \mathrm{Lie}(N)[\frac{1}{p}]$ such that (4.7.27-29) holds, then t is unique, and $t \in p\mathrm{Lie}(N)$ (as $\frac{t}{p}$ takes M_y into M_y). It is easy to see that there is a unique $t \in \mathrm{End}(M_y)$ such that $t|F_y^0 = 0$ and 4.7.27-29 hold.

So we need to show that such a t annihilates t_α, $\forall \alpha \in \mathcal{J}'$. This is easy: repeating the arguments for $(M_y \otimes M_y^*)^{n_\alpha}$ (with $\alpha \in \mathcal{J}'$), instead of M_y, we get equations similar to the ones of 4.7.24-26. As the connection ${}_N\nabla_{\hat{y}}$ of $(M_y \otimes M_y^*)^{n_\alpha} \otimes R$ is uniquely determined (cf. 3.4.18.8), the new equations we get are linear combinations of those of 4.7.24-25 and respectively of the ones of 4.7.26. So 4.7.8 can be interpreted as $t(u_\alpha) = 0$ if $\varphi(u_\alpha) = u_\alpha$ and $u_\alpha \in F^0\big((M_{\mathbb{Z}_p} \otimes M_{\mathbb{Z}_p}^*)^{\otimes n_\alpha} \otimes W(k)\big)$, with $M_{\mathbb{Z}_p}$ the $\mathbb{Z}_p$-subspace of M_y generated by the base B). So $t(t_\alpha) = 0$, $\forall \alpha \in \mathcal{J}'$. In fact, due to the special form of Φ_y and of Φ_R, we can easily check that $t = 0$ (cf. also 4.7.30 1)).

The last fact of ii), to prove that $q_j \in R$ and that $q_j(0) = 1 \bmod pW(k)$, is similar to [De2, 1.4.5], starting from (4.7.23-25) and using the element

$$t_R := \sum n_j u_j \in \mathrm{Lie}(N) \otimes W(k)[\frac{1}{p}][[x_1, \ldots, x_{e_0}]].$$

The proof of 4.7.19 is entirely analogous to the proof of [De2, 1.4.7]: we are in the context of principally polarized abelian schemes and so the Kodaira–Spencer map

$$gr_N\nabla_{\hat{y}} : \bar{T}_{R/W(k)} \to \mathrm{Hom}_R(F_y^1 \otimes R, F_y^0 \otimes R)$$

(with $\bar{T}_{R/W(k)} = \bar{\Omega}^*_{R/W(k)}$) is injective, having $\mathrm{Lie}(N) \otimes R$ as its image (we have a canonical injective $W(k)$-linear map $\mathrm{Lie}(N) \hookrightarrow \mathrm{Hom}_{W(k)}(F_y^1, F_y^0)$).

Now 4.7.21 is obvious. 4.7.22 is a particular case of the principle expressed in 4.7.36; it reobtains in the case of a SHS $(f, L_{(p)}, v)$, with $k(v) = \mathbb{F}_p$, the main result of [No1].

4.7.30. The general theory of crystalline coordinates for p-divisible groups over k. We itemize the ideas.

1) Based on [De2, 1.4] and on the results pertaining to a SHS and presented till 4.7.4, the above part 4.7.5-29 is essentially trivial. What we did: we interpreted [De2, 1.4] as a property of deformations of Shimura adjoint Lie σ-crystals. In other words the p-divisible group associated to (M_y, φ_y) is a direct sum of two p-divisible group D_y^1 and D_y^2 such that:

– the slopes of D_y^1 are precisely 0 and 1, while D_y^2 does not have any one of these slopes;

– the G-deformations of (A_y, p_{A_y}) referred above correspond to deformations of D_y^1, while "keeping" D_y^2 fixed.

In other words, if $M_y = M_y^1 \oplus M_y^2$ is the direct sum decomposition corresponding to D_y^1 and D_y^2, then $\mathrm{Lie}(N) \subset \mathrm{End}(M_y^1)$. The main goal of including 4.7.5-29 here is: to get familiar with the new type of equations (differential or not) showing up when we pass from (Shimura) ordinary σ-crystals to cyclic diagonalizable Shimura σ-crystals. The equations 4.7.9-10, 4.7.12, 4.7.14, 4.7.16, 4.7.18, 4.7.24 and 4.7.26 are typical to the cyclic diagonalizable situation: they do not show up in [De2, 1.4].

2) Let D be a p-divisible group over k. Let m be the product of the dimensions of D and of its dual D^t. It is known that D has a versal deformation $\mathcal{D}$ over $R_m := W(k)[[z_1, ..., z_m]]$. For any $W(k)$-epimorphism $q : W(k)[[z_1, ..., z_m]] \twoheadrightarrow W(k)[[x_1, ..., x_n]]$, with $n \in \mathbb{N}$, $n \leq m$, we get a p-divisible group $\mathcal{D}_q$ over $R_n := W(k)[[x_1, ..., x_n]]$. The first goal of the general problem of crystalline coordinates (for p-divisible groups) is to find a good description (if possible canonical) of the p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R_n)$ defined by $\mathcal{D}_q$, for epimorphisms q of interest; we have freedom (as part of the good description) for the choice of a Frobenius Φ_{R_n} of R_n. If Φ_{R_n} is compatible with σ and satisfies $\Phi_{R_n}(x_i) = x_i^p$, $i = \overline{1, n}$, we speak about an additive type situation. If Φ_{R_n} is compatible with σ and satisfies $\Phi_{R_n}(x_i) = (x_i - 1)^p$, we speak about a multiplicative type situation. These are the most common (and handable) type of situations. The second goal of the problem of crystalline coordinates is to compare, in the case we do not have a canonical description, the different descriptions obtained in the process of achieving the first goal. In what follows we essentially concentrate in achieving the first goal.

Very often the σ-crystal $(\tilde{M}, \tilde{\varphi})$ defined by D gets naturally the structure of a Shimura σ-crystal $(\tilde{M}, \tilde{\varphi}, \tilde{G})$, and then we take q to be the maximal quotient of R_m preserving the cycles (cf. 2.2.8 1)) involved (for concrete examples see [Va2, 5.4-5]). So we get a p-divisible object with cycles $\mathcal{C}_q$ of $\mathcal{MF}^{\nabla}_{[0,1]}(R_n)$, defined by q. It is associated to a Shimura p-divisible group over R_n. The theorem 3.4.18.5 (as well as its Shimura context counterpart 3.4.18.7), in essence, achieves the first goal of the theory of crystalline coordinates. What we have to do is:

a) First we choose, according to the needs and desires, a Frobenius of R_n leaving invariant its ideal $I_n := (x_1, ..., x_n)$, and a Frobenius Φ of $\tilde{M} \otimes R_n$ of the form $g_{R_n}(\tilde{\varphi} \otimes 1)$, with $g_{R_n} \in \tilde{G}(R_n)$ which modulo I_n is zero. We choose also a filtration $\tilde{F}^1$ of $\tilde{M}$ such that

$(\tilde{M}, \tilde{F}^1, \tilde{\varphi}, \tilde{G})$ is a Shimura filtered σ-crystal. We get, as in 3.4.18, a p-divisible object with cycles $\mathcal{C}$ of $\mathcal{MF}_{[0,1]}(R_n)$.

b) We determine the connections which make $\mathcal{C}/p\mathcal{C}$ to be viewed as an object of $\mathcal{MF}^{\nabla}_{[0,1]}(R_n)$. We look at the resulting Kodaira-Spencer maps (one for each such connection), and we pick up one such connection of whose Kodaira-Spencer map is injective (an isomorphism on its image) modulo the maximal ideal of R_n; the choices in a) above should be made in a such a way that such a connection does exist. So $\mathcal{C}$ and $\mathcal{C}_q$, as p-divisible objects with cycles of $\mathcal{MF}^{\nabla}_{0,1]}(R_n)$, are isomorphic (cf. [Fa2, th. 10 and the remarks after]).

Using the steps a-b) above, we regain (cf. 3.4.18.2) the results of [De2, 2.1], without any computational effort. Moreover, the case of $K3$ surfaces (see [De2, 2.1 B]) for $p = 2$ (this is not covered in loc. cit.), is settled by [Fa2, rm. iii) after th. 10]. A. Ogus pointed out to us that his unpublished work on ordinary elliptic curves can also be used to settle this case.

3) Let us look closer at [De2] to understand the real principles underlying the main results of the loc. cit. We see that there are three ideas. We formulate them in the language of this paper.

a) We have a Shimura σ-crystal which admits a canonical split: with the notations of 2) above, this is a cocharacter $\mu : \mathbb{G}_m \to \tilde{G}$ with special properties; in particular it produces a direct sum decomposition $\tilde{M} = \tilde{F}^0 \oplus \tilde{F}^0$, with $\beta \in \mathbb{G}_m(W(k))$ acting through μ on $\tilde{F}^i$ as the multiplication with β^{-i}, $i = \overline{0,1}$ (to be compared with 3.1.4).

b) The resulting Shimura filtered σ-crystal $(\tilde{M}, \tilde{F}^1, \tilde{\varphi}, \tilde{G})$ is (cyclic) diagonalizable.

c) We can build up good additive coordinates. By this we mean [De2, 1.4.2] (see also 5) below).

3′) Using the cocharacter μ above, we get a natural $\mathbb{Z}_p$-structure of $(\tilde{M}, \tilde{F}^1, \tilde{\varphi}, \tilde{G})$, as in 4.4.7. So $\tilde{M} = M_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \otimes W(k)$, with $M_{\mathbb{Z}_p}$ a free $\mathbb{Z}_p$-module; under this identification, $\tilde{G}$ is obtained by extension of scalars of a reductive subgroup $\tilde{G}_{\mathbb{Z}_p}$ of $GL(M_{\mathbb{Z}_p})$. Moreover μ is defined over a finite unramified DVR extension $W(k_\mu)$ of $\mathbb{Z}_p$. Let T be the smallest subtorus of $\tilde{G}_{\mathbb{Z}_p}$ such that μ factors through $T_{W(k_\mu)}$. Let C be the centralizer of T in $\tilde{G}_{\mathbb{Z}_p}$. It is a reductive group over $\mathbb{Z}_p$.

The principle behind the choice of bases in [De2, 1.4.2] and in 4.7.5 is: we pick up a maximal torus T_C of C^{der}; only the $C(\mathbb{Z}_p)$-conjugacy class of T_C matters. Of course, if C^{der} is split, then the most natural choice for T_C is: a split maximal torus of C^{der}. When C^{der} is not split, still we have quite natural choices for T_C. For instance we can take T_C to contain a maximal split torus of C^{der} and to split over the smallest unramified extension of $\mathbb{Z}_p$ over which C^{der} splits (cf. [Ti2, 1.10]). For the most general applications it is important to take T_C arbitrarily. Let T^1_C be the maximal torus of $\tilde{G}_{\mathbb{Z}_p}$ generated by T_C and the toric part of $Z(C)$.

4) It is easy to see that choosing a multiplicative type Frobenius of R_n, we can not, for an arbitrary cyclic diagonalizable Shimura σ-crystal, choose g_{R_n} to be the identity element of $\tilde{G}(R_n)$. Even for the case of the p-divisible group of a supersingular elliptic curve over k (so $n = 1$) we can not choose $g_{R_1} = 1$. To get out from this deadlock, we have to either

use the additive situation (see 5) below), or work with the multiplicative situation, with a carefully chosen g_{R_n} (see 6) below).

By $\tilde{G}$-canonical (multiplicative) coordinates for $(\tilde{M}, \tilde{\varphi}, \tilde{G})$ we mean that, referring to a-b) of 2), we can choose:

– a lift $\tilde{F}^1$ of it of quasi CM type;

– $n = dd((\tilde{M}, \tilde{\varphi}, \tilde{G}))$ (see 2.2.22 4));

– a Frobenius of R_n of multiplicative type;

– $g_{R_n} = 1$;

– a connection which respects the $\tilde{G}$-action and of whose Kodaira-Spencer map modulo the maximal ideal of R_n is injective.

From the proof of 3.4.18.4 B) we get (cf. also 2) above) directly:

Fact. *A cyclic diagonalizable Shimura σ-crystal $(\tilde{M}, \tilde{\varphi}, \tilde{G})$ has $\tilde{G}$-canonical (multiplicative) coordinates iff its Shimura adjoint Lie σ-crystal is ordinary, with the multiplicity of the slope -1 being maximal, i.e. equal to $dd((\tilde{M}, \tilde{\varphi}, \tilde{G}))$.*

The above iff part can be reformulated: iff it is a $\tilde{G}$-ordinary σ-crystal and its A-degree of definition (cf. 3.11.3) is 1. In particular we reobtain, in a much faster manner, the results of 4.7.19-20.

5) Let $\tilde{N}$ be the (abelian) subgroup of $\tilde{G}$ acting trivially on $\tilde{F}^0$ and on $\tilde{M}/\tilde{F}^0$. We identify R_n with the completion of $\tilde{N}$ in the origin (obviously $\dim(\tilde{N}) = dd((\tilde{M}, \tilde{\varphi}, \tilde{G}))$). $\tilde{N}$ has a natural $W(k_\mu)$-structure; let $\tilde{N}_{W(k_\mu)}$ be the subgroup of $\tilde{G}_{W(k_\mu)}$ which over $W(k)$ is $\tilde{N}$. If the Frobenius Φ_{R_n} is of additive type, then we speak about additive coordinates. Due to condition on the Kodaira-Spencer map, we do not have too many choices: the p-divisible object with cycles $\mathcal{C}_q$ of $\mathcal{MF}^{\nabla}_{[0,1]}(R_n)$ is a Shimura filtered F-crystal of the form $(\tilde{M}, \tilde{F}^1, \tilde{\varphi}, \tilde{G}, \tilde{N}, \tilde{f})$; so the only choice we have is in choosing (the isomorphism) $\tilde{f}$ (of R_n). The choice of $\tilde{f}$ corresponds to a choice of an additive type Frobenius of R_n. [Fa2, rm. iii) after th. 10] tells us that it does not matter which choice we make. We propose one such choice, which to us looks convenient (at least from the point of view of 8) below).

T^1_C might not be split. It splits over an unramified finite DVR extension $W(k_1)$ of $\mathbb{Z}_p$. We choose the smallest such extension. Let $\mathbb{G}_a(i)$, $i = \overline{1, \dim(\tilde{N})}$, be the different subgroups of $\tilde{N}_{W(k_1)}$, whose Lie algebras are left invariant by T^1_C. We feel inclined of choosing Φ_{R_n} such that it leaves invariant the subring $R(i) = W(k_1)[[z_i]]$ of R_n defined by the projection (it is a group homomorphism) of $\tilde{N}_{W(k_1)}$ on $\mathbb{G}_a(i)$ having $\prod_{j \in \{1,\dots,\dim(\tilde{N})\} \setminus \{i\}} \mathbb{G}_a(j)$ as its kernel; of course $\Phi_{R_n}(z_i) = z_i^p$, $i = \overline{1, \dim(\tilde{N})}$. We can use T^1_C to determine uniquely z_n, through different considerations, but this (presently) looks irrelevant to us.

6) We move now to the case when Φ_{R_n} is of multiplicative type. A natural question arises: What happens if the conditions of the fact of 4) above are not satisfied? Is there a best choice of g_{R_n} in such a case? We propose here one good choice of g_{R_n} working out in all cases (i.e. in the case of the above mentioned fact, g_{R_n} becomes the identity). Let

$g^1_{R_n} \in \tilde{N}(R_n)$ be the universal element; so

$$g^1_{R_n} := 1_{\tilde{M}\otimes R_n} + \sum_{i=1}^{\dim(\tilde{N})} a_i z_i,$$

where $a_i \in \mathrm{Lie}(\mathbb{G}_a(i))$ generates this Lie algebra. Let

$$h_{R_n} := 1_{\tilde{M}\otimes R_n} + \sum_{i=1}^{\dim(\tilde{N})} \tilde{\varphi}(pa_i)z_i \in \tilde{G}(R_n).$$

We take

$$g_{R_n} := g^1_{R_n} h^{-1}_{R_n}.$$

It is not difficult to check that for such a g_{R_n} we can choose a connection on $\tilde{M}\otimes R_n$ whose Kodaira-Spencer map is an injection, having $\mathrm{Lie}(\tilde{N})$ as its image (cf. the proof of 3.4.18.4 B)). We usually choose a_i such that $\tilde{\varphi}^{j_i}(a_i)$ is a p-power multiple of a_i, j_i being the smallest positive integer such that $\tilde{\varphi}^{j_i}(\mathrm{Lie}(\mathbb{G}_a(i)) \subset \mathrm{Lie}(\mathbb{G}_a(i))[\frac{1}{p}]$. So in the case of the fact of 4) above, we can choose a_i such that $\tilde{\varphi}(pa_i) = a_i$; for such a choice we get $g_{R_n} = 1$ (as $\tilde{N}$ is an abelian group). In the general case it is still arguable which is the best choice for a_i and for z_i. Of course a_i and z_i are interrelated. We leave this problem of the best choice to get matured on its due time.

7) The ideas of 3.4 (for instance see 3.4.1.3 and 3.4.8.1.2) suggest that rings of the form $W(k)[[z_1, ..., z_m]]$ are not always the right rings of building up crystalline coordinates. Often they should be replaced by the p-adic completion of ind-étale affine algebras over smooth $W(k)$-algebras. For instance we can work with the p-adic completion of the henselization of the localization of $W(k)[z_1, ..., z_m]$ with respect to its maximal ideal $(p, z_1, ..., z_m)$. But this rings are still "loosing" a lot of information. Very often the rings we get have plenty of k-valued points. For instance, working with Shimura σ-crystals which are ordinary, using the fact of 3.4.18.9, we get plenty of rings whose spectra are the p-adic completion of $\mathbb{N}$-pro-étale covers of the p-adic completion of affine open subschemes of $\mathrm{Spec}(W(k)[z_1, ..., z_m])$ (like the one where we make all z_i to be invertible), such that we have versal deformations of Shimura p-divisible groups over them, in a "presribed way" (see 3.4.1.3 and 3.4.11). This opens a completely new horizon: variation of crystalline coordinates in families.

When we are in the context of a principally polarized abelian variety, we do not know for which "presribed ways" we still get principally polarized abelian schemes over the spectra of these rings; we have variants when we have polarizations which are not necessarily principal. Of course we do need a polarization in order to (hope to) get abelian schemes over such spectra.

We do expect that the study of the variation of coordinates leads to nice modular properties. We hope to come back to this idea in a future paper.

8) In 3-6) we had some choices: of μ (and so of $\tilde{F}^1$ and of $\tilde{N}$), of T_C, of z_i, and of a_i, $i = \overline{1, \dim(\tilde{N})}$. We do not stop to see if we can make all these choices to result in

something canonical. However, the $\tilde{G}$-canonical multiplicative coordinates of the fact of 4) above, are canonical (this can be checked in the same manner as in the case of an ordinary p-divisible group: there is a unique multiplicative type Frobenius of R_n for which g_{R_n} is the identity element); this motivates the terminology: G-canonical.

So 7) above gives us natural good structures of a formal torus of the formal moduli scheme $\mathsf{M}(\tilde{M},\tilde{\varphi},\tilde{G})$ of deformations of $(\tilde{M},\tilde{\varphi},\tilde{G})$ (see the proof of 3.11.1): we do not stop to see if there is a unique group structure on $\mathsf{M}(\tilde{M},\tilde{\varphi},\tilde{G})$ (the origin being fixed; it is defined by the F^1-filtration $\tilde{F}^1$ of $\tilde{M}$).

We assume now that there is a reductive subgroup $\tilde{G}_0$ of $\tilde{G}$ such that $(\tilde{M},\tilde{\varphi},\tilde{G}_0)$ is also a Shimura σ-crystal. It is not difficult to see that we can make all the choices, so that, the formal moduli scheme of deformations of $(\tilde{M},\tilde{\varphi},\tilde{G}_0)$ is a subtorus of the formal torus (scheme) of deformations of $(\tilde{M},\tilde{\varphi},\tilde{G}_0)$.

9) We come back to the general case of 3). Let r be the Lie p-rank of $(\tilde{M},\tilde{\varphi},\tilde{G})$. We have $r \leq dd((\tilde{M},\tilde{\varphi},\tilde{G}))$. If equality holds than we are in the situation of the above fact. We assume now that $1 \leq r < dd((\tilde{M},\tilde{\varphi},\tilde{G}))$. In this case there is a quotient $R_n \twoheadrightarrow R_r := W(k)[[x_1,...,x_r]]$ such that, in the sense of 4) above we can take (the same argument applies) g_{R_r} to be 1. So, the choices of 8) above can be made, so that we get a canonical subtorus of $\mathsf{M}(\tilde{M},\tilde{\varphi},\tilde{G})$ of dimension r.

10) Of course all the above 9) points can be entirely adapted to the case when k is just perfect.

4.7.31. Definition. We come back to the situation described in 4.7.5. The lifts of y (to DVR which are finite extensions of $W(k)$) corresponding to torsion points of the formal torus describing the G-deformations of A_y, are called G-quasi-canonical lifts of y.

4.7.32. Exercise. We assume 4.4.6. If $k=\mathbb{F}$, prove that the G-quasi-canonical lifts of a G-ordinary point $y:\mathrm{Spec}(k)\to\mathcal{U}$ are giving birth to special points of $\mathcal{N}$ (i.e. the abelian varieties obtained from $\mathcal{A}$ by pull back through these G-quasi-canonical lifts have complex multiplication). Show that they are the only lifts of y to DVR of mixed characteristic, which are special points of $\mathcal{N}$ (Hint: if $k(v)=\mathbb{F}_p$, this is well known; if $k(v)\neq\mathbb{F}_p$ then use 4.4.6 and just immitate the argument of the case $k(v)=\mathbb{F}_p$).

4.7.33. Remark. We do not assume anymore that $N=N_0$. From 4.7.30 3-6) we deduce that for any toric point $y_1:\mathrm{Spec}(k)\to\mathcal{N}_{k(v)}$ there are good G-multiplicative coordinates of the formal moduli scheme of G-deformations of $(A_{y_1},p_{A_{y_1}})=\mathrm{Spec}(k)_{y_1}\times_{\mathcal{N}}(\mathcal{A},\mathcal{P}_{\mathcal{A}})$. The same thing remains valid if we do not work in the Shimura (i.e. having a principal polarization, etc.) context. In particular:

a) For any point $y_1:\mathrm{Spec}(k)\to\mathcal{N}_{k(v)}$ whose attached Shimura adjoint Lie σ-crystal is isomorphic to the extension to k of the Shimura adjoint Lie σ-crystal of $\mathfrak{C}_\omega$ of 4.1.5 (i.e. for any $G(\omega)$-ordinary point y_1), we get good $G(\omega)$-multiplicative coordinates describing the formal moduli scheme of G-deformations of $(A_{y_1},p_{A_{y_1}})$.

b) For any point $y:\mathrm{Spec}(k)\to\mathcal{U}$, we get good GSp-multiplicative coordinates (resp. good multiplicative coordinates) describing the formal moduli scheme of deformations of the principally polarized abelian variety (A_y,p_{A_y}) (resp. of deformations of A_y).

c) For any supersingular elliptic curve A_{y_1} over k (so we reobtain an unpublished result of A. Ogus), we get a good multiplicative coordinate describing its formal moduli scheme of deformations.

d) For any abelian variety (over an algebraically closed field of characteristic $p > 2$), which is a product of abelian varieties A_{y_1} as in a) to c) above, we get good multiplicative coordinates describing its formal moduli scheme of deformations.

More generally we have (for instance using multiplicative coordinates):

4.7.33.1. Proposition. *If A is an abelian variety over k of dimension d, and if its attached σ-crystal has a lift (over $W(k)$) of quasi CM type, then the formal moduli scheme of deformations of A is naturally isomorphic to the formal torus $\mathbb{G}_m^{d^2\wedge}/W(k)$, with the abelian variety defined by the lift of pseodo CM type as its origin.*

Proof: This is a direct consequence of 2.2.19 and of 4.7.33.

4.7.33.2. Remarks. 4.7.33.1 is the most general context in which presently we can prove the existence of useful (crystalline) (additive or multiplicative) coordinates. From 2.2.18 we deduce easily that it does not handle the case of all abelian varieties over k. In the cases left, the hopes of building up any theory of multiplicative coordinates which has any significance at all (i.e. it is useful and motivated), are based on 3.13. So the problem 13 from [Oo1, p. 13] is still open for the cases left. We leave it to the reader to formulate 4.7.33.1 in the relative context (of abelian varieties over k whose attached σ-crystals have the extra structure of a cyclic diagonalizable Shimura σ-crystals) (cf. 4.7.30 5)).

4.7.33.3. In what follows we refer to 4.7.33 a). For any good $G(\omega)$-multiplicative coordinates, we can define $G(\omega)$-quasi-canonical lifts of y_1. It looks to us interesting to study them. In particular, if y_1 is a G-ordinary point, and if $N \neq N_0$, we do not know if these G-quasi-canonical lifts are uniquely determined (we had some choices in 4.7.30), and if the facts of 4.7.32 remain true.

4.7.34. Exercise. Coming back to 4.7.3, prove that $\forall n \in \mathbb{N}$, ${}_N\nabla_{\hat{y}} \bmod p^n$ is algebraic, i.e. the functions u_j, $j = \overline{1, e_0}$, are algebraic over $N_{W_n(k)}$ (Hint: use 4.4.7 to reduce to the case $k = \overline{k(v)}$; then use the fact that everything, like the bases $(a_i)_{1\leq i\leq e}$ and $(b_i)_{1\leq i\leq e}$, μ_y, etc., are definable over a finite extension of $k(v)$; another way: cf. 3.4.1.3).

4.7.35. Exercise. Show that all of [No2, section 2] remains valid in the context of G-ordinary points (Hint: if $N = N_0$ use 4.7.36 below; if $N \neq N_0$ use 4.7.30 6) and work formally).

4.7.36. Remark. Every formal subscheme of $\mathcal{M}_G(y)$ of 4.7.21 which can be "defined" starting from a smooth subgroup N_1 of N whose Lie-algebra is left invariant by φ_y is a formal subtorus. This can be checked using 4.7.5-29 or 4.7.30 8). The same thing remains true in the general context of 4.7.30 2). Here, for the sake of future reference, we state this principle explicitely, in the context of Shimura σ-crystals. With the notations of 4.7.30 2) to 4), we have:

Fact. *If there is a reductive subgroup $\tilde{G}_1$ of $\tilde{G}$ such that $(\tilde{M}, \tilde{\varphi}, \tilde{G}_1)$ is also a Shimura σ-crystal, and if the filtration defining its $\tilde{G}_1$-canonical lift is the same as the one defining*

the $\tilde{G}$-canonical lift of $(\tilde{M},\tilde{\varphi},\tilde{G})$, then the formal torus of dimension $n_1 := dd((\tilde{M},\tilde{\varphi},\tilde{G}_1))$ of deformations of $(\tilde{M},\tilde{\varphi},\tilde{G}_1)$ is canonically a formal subtorus of the formal torus of deformations of $(\tilde{M},\tilde{\varphi},\tilde{G})$.

Proof: From [Fa2, th.10 and the remarks after] and 4.7.30 2) we deduce that it is enough to show that there is a filtered F-crystal over the special fibre $\mathrm{Spec}(k[[x_1,....,x_n]])$ of $\mathrm{Spec}(R_n)$, with n as in 4.7.30 4), such that:

– it is described by a p-dibisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R_n)$ of the form $(\tilde{M}\otimes R_n, \tilde{F}_1\otimes R_n, \tilde{\varphi}\otimes 1, \tilde{\nabla})$, with $\tilde{\nabla}$ as in 4.7.30 4);

– the Frobenius of R_n is compatible with σ and takes x_i-1 to $(x_i-1)^p$;

– $\tilde{\nabla}$ modulo the ideal of R_n generated by $x_{n_1+1},...,x_n$ is $\tilde{G}_1$-invariant.

The construction of such a filtered F-crystal is trivial (cf. 4.7.30 1) to 4) and 3.4.18.2).

4.7.37. Problem. Obtain the general form of the main result of [No1] and of [No2] in the context of G-ordinary points (Hint: if $N=N_0$ use 4.7.5-29 and 4.7.32, and just immitate the arguments of [No1-2]; if $N\neq N_0$ then use 4.7.30).

4.7.38. Problem. State a theorem similar to 4.7.5 in the general case, i.e. when $k(v^{\mathrm{ad}})$ is not $\mathbb{F}_p$. Hint: use 4.7.4.

4.8. The Galois property of G-ordinary points.

4.8.1. We continue to assume that $k=\bar{k}$. We start with a morphism $y:\mathrm{Spec}(k)\to \mathcal{U}/H_0$. Let $\mathrm{Spec}(V)\stackrel{z_V}{\to}\mathcal{N}/H_0$ be a lift of y, with V a DVR which is a finite flat extension of $W(k)$. Let $K:=V[\frac{1}{p}]$ and let (A_K,p_{A_K}) be the principally polarized abelian variety over K obtained from $(\mathcal{A}_{H_0},\mathcal{P}_{\mathcal{A}_{H_0}})$ by pull back through z_V. Let

$$\rho:\mathrm{Gal}(\bar{K}/K)\to GL\big(H^1_{\acute{e}t}(A_{\bar{K}},\mathbb{Q}_p)\big)(\mathbb{Q}_p)$$

be the p-adic Galois representation associated to A_K. Let $G_{\mathbb{Q}_p}\hookrightarrow GSp\big(H^1_{\acute{e}t}(A_{\bar{K}},\mathbb{Q}_p),p_{A_K}\big)$ be the subgroup fixing the p-components of the étale components of the family of Hodge cycles $(t_\alpha)_{\alpha\in\mathcal{J}'}$ with which $A_{\bar{K}}$ is naturally endowed.

4.8.2. Theorem. *There is a finite extension K_1 of K such that the restriction $\rho_1:=\rho|\mathrm{Gal}(\bar{K}/K_1)$ factors through the group of $\mathbb{Q}_p$-valued points of a connected solvable subgroup of $G_{\mathbb{Q}_p}$.*

Proof: We can assume that K_1 is big enough so that the family $(t_\alpha)_{\alpha\in\mathcal{J}'}$ is defined over K_1. So ρ_1 factors throgh $G_{\mathbb{Q}_p}$. As $k=\bar{k}$ we have:

i) the residue field of the ring of integers of K_1 is k;

ii) there is a morphism $z:\mathrm{Spec}(W(k))\to\mathcal{N}/H_0$ lifting y, and such that the Shimura filtered σ-crystal attached to it is a $G_{W(k)}$-canonical lift;

iii) $G_{W(k)}$ splits (so 4.7.1-3 makes sense).

In fact the condition ii) is satisfied for any k (cf. 4.4.3).

Let $(M_y, F^1_y, \varphi_y, G_{W(k)})$ be the Shimura filtered σ-crystal attached to z. Let $\mu_y : \mathbb{G}_m \to GL(M_y)$ be its canonical split, cf. 3.1.4. Let $\bar{h}_i$, $i = \overline{1,d}$, be the elements of $\mathrm{End}(M_y)$ defined as in 4.1 but working with μ_y.

Let N_1 be the smallest subgroup of $G_{W(k)}$ such that $\mathrm{Lie}(N_1)$ contains $\mathrm{Lie}(N)$ (N was def. in 4.7.1), and the Lie algebra generated by $\bar{h}_i$, $i = \overline{1,d}$. N_1 is a connected solvable subgroup of $G_{W(k)}$. Let B_{N_1} be a Borel subgroup of $G_{W(k)}$ containing N_1 and such that its Lie algebra is left invariant by $p\varphi_y$. The existence of such Borel subgroups is a consequence of b) 4.4.1 3) (to be compared with 3.11.1 c) and 3.7.2). Let $\tilde{N}_1$ be the connected component of the origin of the intersection of all such Borel subgroups B_{N_1} of $G_{W(k)}$ containing N_1. Now $\mathrm{Lie}(N_1)$, $\mathrm{Lie}(\tilde{N}_1)$ and $\mathrm{Lie}(B_{N_1})$ are left invariant by $p\varphi_y$.

With the notations of 4.7.1 we get that $\mathrm{Lie}(N_1)\otimes R$ and $\mathrm{Lie}(B_{N_1})\otimes R$ are left invariant by $n(\varphi_y \otimes 1)$. So for the morphism $\mathrm{Spec}(V_1) \hookrightarrow \mathrm{Spec}(R)$ (with V_1 the ring of integers of K_1), corresponding to A_{V_1} (cf. 4.7.2), we get that the Galois representation ρ_1 leaves invariant $\mathrm{Lie}(N_2)$, $\mathrm{Lie}(\tilde{N}_2)$ and $\mathrm{Lie}(B_{N_2})$, with $N_2 \subset \tilde{N}_2 \subset B_{N_2} \subset G_{\mathbb{Q}_p}$ such that B_{N_2} is a Borel subgroup of $G_{\mathbb{Q}_p}$, and with N_2 and $\tilde{N}_2$ as solvable groups (N_2, $\tilde{N}_2$ and B_{N_2} are respectively the $\mathbb{Q}_p$-étale correspondents of the crystalline ones N_1, $\tilde{N}_1$ and B_{N_1}) . But as B_{N_2} is its own normalizer, we do get that ρ_1 factors through B_{N_2}. Repeating the argument for any Borel subgroup of $G_{W(k)}$ as above, we do get that, by extending again K_1 by a finite cover (as this intersection of all such Borel subgroups B_{N_1} might not be connected), ρ_1 factors through the group of $\mathbb{Q}_p$-valued points of $\tilde{N}_2$. This proves the theorem.

4.8.3. Remarks. a) 4.8.2 solves a question of C.-L. Chai. We would like to thank R. Pink who pointed out to us the construction of N_1 (of the proof), provided 4.7.2 and c) of 4.4.1 2) were known. We do not know when $\mathrm{Lie}(\tilde{N}_1) = \mathrm{Lie}(N_1) + \mathrm{Lie}(Z(G_{W(k)}))$. We just remark that the unipotent radical of $\tilde{N}_1$ is the unipotent radical of N_1.

b) Using [De3, 3.1 and 3.2 a)] it can be checked out easily that we do not need to use $\tilde{N}_1$, and so that ρ_1 factors through $N_2(\mathbb{Q}_p)$.

c) There are plenty of examples in which $\tilde{N}_1$ is a Borel subgroup of $G_{W(k)}$ (for instance if the parabolic Lie subalgebra $\mathfrak{p}_0$ of 4.6.4 is of Borel type, see P7 of 4.6; if moreover $G^{\mathrm{ad}}_{\mathbb{Z}_p}$ is a simple group then $N_1 \cap G^{\mathrm{der}}_{W(k)}$ is a Borel subgroup $G^{\mathrm{der}}_{W(k)}$) and plenty of examples when it is not (for instance if $G^{\mathrm{ad}}_{\mathbb{Z}_p}$ is a simple split group of C_ℓ Lie type, with $\ell \geq 2$).

d) A similar result as 4.8.1 can be proved for an arbitrary toric point $\mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$. The only difference is: we need to state explicitly that $\mathrm{Lie}(N_1)$ has to be also $p\varphi_y$-invariant. Moreover, if we still want $\tilde{N}_1$, then we have to define it using parabolic subgroups containing N_1. Warning: N_1 is not necessarily a solvable group. There are plenty of cases when its image in $G^{\mathrm{ad}}_{W(k)}$ is a parabolic subgroup of $G^{\mathrm{ad}}_{W(k)}$ which is not a Borel subgroup.

e) 4.8.2 and d) above are properties of Shimura-ordinary σ-crystals (instead of 4.7.2 we have to use 3.11.1). Moreover the condition $k = \bar{k}$ can be weaken: we just need the existence of a Borel subgroup B_{N_1} of $G_{W(k)}$ of the type mentioned above; but in this case we can not replace $\tilde{N}_2$ by N_2.

We assume now that k is a finite field, that we are in the context of 4.8.1, and that 4.4.6 is satisfied. Then, up to a passage to a finite field extension, the existence of a such a

Borel subgroup is automatique. This can be easily checked starting from 4.4.6, 3.3.4 and the proof of the Claim of the proof of 4.4.1. In this way we obtain plenty of quasi-ordinary crystalline representations, in the sense of [Pi, 2.10].

f) We stated this section 4.8 in terms of $\mathbb{Q}_p$-valued points; but obviously everything can be restated in terms of $\mathbb{Z}_p$-valued points.

4.9. The passage from Shimura varieties of Hodge type to Shimura varieties of preabelian type.

4.9.1. We start with an arbitrary adjoint Shimura variety $\mathrm{Sh}(G_0, X_0)$ of abelian type. Let (G_0, X_0, H_0, v_0) be a Shimura quadruple with v_0 dividing a rational prime $p \geq 3$. We fix a connected component X_0^0 of X_0 (it does not matter which one, cf. [Va2, 3.3.3]). Let (G_i, X_i, H_i, v_i), $i = \overline{1,3}$, be Shimura quadruples having (G_0, X_0, H_0, v_0) as their adjoint quadruples, and such that $X_0^0 \subset X_i$ (cf. [Va2, 2.4]). We assume the existence of two standard Hodge situations $(f_i, L_{i(p)}, v_i)$ defined by $f_i : (G_i, X_i) \hookrightarrow \big(\mathrm{GSp}(W_i, \psi_i), S_i\big)$, $i = \overline{1,2}$, such that $H_i = G_i(L_{i(p)} \otimes \mathbb{Z}_p)$ (for $p \geq 5$ cf. [Va2, 6.4.2]; for $p = 3$ cf. §6]). For the two standard Hodge situations considered we use the standard notations of 2.3.1-3, except that we put a right lower index i everywhere, $i = \overline{1,2}$.

We denote by $\mathcal{N}_0$ and $\mathcal{N}_3$ the integral canonical models of (G_0, X_0, H_0, v_0) and respectively of (G_3, X_3, H_3, v_3). We get natural morphisms $\mathcal{N}_i \xrightarrow{j_i} \mathcal{N}_0$ (cf. [Va2, rm. 4) of 3.2.7]), $i = \overline{1,3}$.

4.9.1.1. Let $V_0 := W(\mathbb{F})$ and let $\mathfrak{C}_0$ be the open closed subscheme of $\mathcal{N}_{0V_0}$ defined as the intersection of the images (through j_3) of $\mathcal{N}_{3V_0}$ into $\mathcal{N}_{0V_0}$ associated to all finite maps $i_3 : (G_3, X_3, H_3, v_3) \to (G_0, X_0, H_0, v_0)$ such that $X_0^0 \subset X_3$. The fact that $\mathfrak{C}_0$ is non-empty is implied by [Va2, rm. 3) of 3.2.7] and by the fact that $\mathfrak{C}_0 \times_{V_0} {}_j\mathbb{C}$ is non-empty (with $j : V_0 \hookrightarrow \mathbb{C}$ an embedding such that restricted to $O_{(v)}$ is the inclusion $O_{(v)} \subset E(G, X) \subset \mathbb{C}\big)$. The fact that $\mathfrak{C}_0$ is open closed is implied for $p \geq 5$ by [Va2, 6.4.5.1] (for $p = 3$ cf. §6). Similarly we define $\mathfrak{C}_i \hookrightarrow \mathcal{N}_{iV_0}$, $i = \overline{1,3}$ (by working with the connected component X_0^0 of X_i).

4.9.2. Let (G_4, X_4, H_4, v_4) be a Shimura quadruple which together with maps $h_i : (G_4, X_4, H_4, v_4) \to (G_i, X_i, H_i, v_i)$, $i = \overline{1,2}$, is a quasi fibre product of the finite maps q_1 and q_2, i.e. we have a commutative diagram

$$\begin{array}{ccc} (G_4, X_4, H_4, v_4) & \xrightarrow{h_1} & (G_1, X_1, H_1, v_1) \\ \big\downarrow{\scriptstyle h_2} & & \big\downarrow{\scriptstyle q_1} \\ (G_2, X_2, H_2, v_2) & \xrightarrow{q_2} & (G_0, X_0, H_0, v) \times \big(\mathbb{G}_m, \{x_0\}, \mathbb{G}_m(\mathbb{Z}_p), p\big) \end{array}$$

in the category $qf - Sh$ defined in [Va2, rm. 3) of 3.2.7], satisfying a quasi universal property as in [Va2, 2.4.0]. Here q_i is defined by the canonical map $(G_i, X_i, H_i, v_i) \to (G_0, X_0, H_0, v_0)$ and by the canonical map $h_i : (G_i, X_i) \to (GSp^{\mathrm{ab}}, S^{\mathrm{ab}})$ defined by f_i, $i = \overline{1,2}$, where $\{x_0\} := S^{\mathrm{ab}}$ and $GSp^{\mathrm{ab}} = \mathbb{G}_m$. We assume that $X_0^0 \subset X_4$, cf. loc. cit. and [Va2, 2.4.0]. We call (G_4, X_4) as a Hodge twist of (G_1, X_1) and (G_2, X_2) and denote it by $(G_1, X_1)\tilde{\times}(G_2, X_2)$.

We consider a Hodge quasi product $(G_1 \times_{\mathcal{H}} G_2, X_1 \times_{\mathcal{H}} X_2)$ of (G_1, X_1) and (G_2, X_2), and a Hodge quasi product $(\mathrm{GSp}(W_1, \psi_1) \times_{\mathcal{H}} \mathrm{GSp}(W_2, \psi_2), S_1 \times_{\mathcal{H}} S_2)$ of $(\mathrm{GSp}(W_i, \psi_1), S_1)$ and $(\mathrm{GSp}(W_2, \psi_2), S_2)$, cf. [Va2, Example 3 of 2.5]. We use a similar notation for quadruples. We can assume that $X_4 \subset X_1 \times_{\mathcal{H}} X_2 \subset S_1 \times_{\mathcal{H}} S_2$. So we get a SHS $(f_4, L_{1(p)} \oplus L_{2(p)}, v_4)$ defined by

$$f_4 : (G_4, X_4) \overset{\tilde{i}_4}{\hookrightarrow} \big(\mathrm{GSp}(W_1, \psi_1) \times_H \mathrm{GSp}(W_2, \psi_2), S_1 \times_H S_2\big) \overset{i}{\hookrightarrow} (\mathrm{GSp}(W_1 \oplus W_2, \psi_1 \oplus \psi_2), S),$$

where $\tilde{i}_4 := (f_1 \times_{\mathcal{H}} f_2) \circ i_4$, with $i_4 : (G_4, X_4) \hookrightarrow (G_1 \times_H G_2, X_1 \times_H X_2)$ and

$$f_1 \times_{\mathcal{H}} f_2 : (G_1 \times_{\mathcal{H}} G_2, X_1 \times_{\mathcal{H}} X_2) \hookrightarrow (\mathrm{GSp}(W_1, \psi_1) \times_{\mathcal{H}} \mathrm{GSp}(W_2, \psi_2), S_1 \times_{\mathcal{H}} S_2)$$

as the natural inclusions, and where i is the Segre embedding, as defined in [Va2, Example 3 of 2.5].

Let $\mathcal{N}_4$ be the integral canonical model of (G_4, X_4, H_4, v_4), and let $\mathfrak{C}_4 \subset \mathcal{N}_{4V_0}$ be defined similarly as $\mathfrak{C}_0$.

The fact that the triple $(f_4, L_{1(p)} \oplus L_{2(p)}, v_4)$ is indeed a SHS, is related to the prove of Milne's conjecture (see 1.16, d) of 4.4.1 3) and 4.4.12). Let $\mathcal{M}_4$ be the integral canonical model associated to the SHS $(1_{(\mathrm{GSp}(W_1 \oplus W_2), \psi_1 \oplus \psi_2), S)}, L_{1(p)} \oplus L_{2(p)}, p)$. From [Va2, rm. 4) of 3.2.7] we get a natural morphism $\tilde{m} : \mathcal{N}_4 \to \mathcal{M}_4$. We need to show that for any morphism $z : \mathrm{Spec}(W(k)) \to \mathcal{N}_4$, working with $\tilde{m}$, the condition $*$ of 2.3.4 is satisfied. From 2.3.12-13 and [BLR, th. 1 of p. 109] we deduce that it is enough to check this for a point z lifting a G_4-ordinary point of $\mathcal{N}_4$ (the set of such points is the same, regardless of which morphism $\mathcal{N}_4 \to \mathcal{N}_1$ or $\mathcal{N}_4 \to \mathcal{N}_2$ we use to define them -by being mapped into G_i-ordinary points of $\mathcal{N}_i$, $i \in \{1, 2\}$-: the prove of 9.4 below, at the level of isocrystals, applies). If we desire to simplify the things, we can define such G_4-ordinary points, as points mapping into G_i-ordinary points of $\mathcal{N}_i$, $\forall i \in \{1, 2\}$ (sort of intersection: [Va2, 6.4.5.1] and 4.2.1 guarantee that this intersection is an open dense subscheme of $\mathcal{N}_{k(v_4)}$). Moreover we can assume that z maps into a G_i-canonical lift of $\mathcal{N}_i$, $\forall i \in \{1, 2\}$ (cf. 4.9.4, performed with filtered isocrystals over k; b) of 4.4.1 2) implies that a Shimura-canonical lift can be read out from the attached Shimura adjoint filtered Lie isocrystal). This implies that the Galois representation defined by z (working with $\tilde{m}$) factors through the $\mathbb{Z}_p$-valued points of a torus of $G_{\mathbb{Z}_p}$ (see also 4.5.12.2.1). It is in this context where we can prove directly Milne's conjecture referred in 1.16 (for details see §5). So condition $*$ of 2.3.4 is satisfied for z.

As we did not include a proof of d) of 4.4.1 3), here we present an alternative (independent) proof of why $*$ of 2.3.4 is satisfied for z. We consider the Lie filtered isocrystal $LI \otimes B(k)$ associated to z. It is a Lie filtered subisocrystal of the direct sum of $(LI_1 \oplus LI_2) \otimes B(k)$, where LI_i is the Lie filtered σ-crystal associated to the morphism $\mathrm{Spec}(W(k)) \to \mathcal{N}_i$ defined by z ($i \in \{1, 2\}$). We just need to show that $LI := LI \otimes B(k) \cap (LI_1 \oplus LI_2)$ is the Lie algebra of a reductive subgroup of $G_1 \times G_2$. [Va2, 4.3.13] takes care of the toric part, while [Va2, 3.1.6] allows us to concentrate just on the "der" parts, and so on the adjoint parts. But for the adjoint groups everything results

from 4.2.3-5, once we remark that $G^{\rm ad}_{i\mathbb{Z}_{(p)}}$ does not depend on $i \in \{1,2,4\}$. In other words, the Galois representation

$$\mathrm{Gal}(\overline{B(k)}/B(k)) \to G^{\rm ad}_{4\mathbb{Z}_p}(\mathbb{Z}_p) \hookrightarrow \mathrm{Aut}(\mathrm{Lie}(G^{\rm ad}_{4\mathbb{Z}_p}))(\mathbb{Z}_p)$$

defined naturally by the $W(k)$-valued point of $\mathcal{M}_1 \times \mathcal{M}_2$ obtained naturally from z (we recall that, cf. 2.3.7, $\mathcal{M}_i$ is the integral canonical model of the SHS $(1_{\mathrm{GSp}(W_i,\psi_i),S_i}, L_{i(p)}, p)$), defines, via Fontaine's comparison theory, an adjoint Lie isocrystal

$$(\mathrm{LIE}, \tilde{\varphi}) := (\mathrm{Lie}(G^{\rm ad}_{4B(k)}), \varphi_{\rm ad})$$

over k. Moreover, using the Fontaine's comparison theory for the $W(k)$-valued point of $\mathcal{N}_i$ defined by z, we deduce the existence of a $W(k)$-lattice LIE^i_W of LIE, such that $(\mathrm{LIE}^i_W, \tilde{\varphi})$ is a Shimura adjoint Lie σ-crystal, $i = \overline{1,2}$. But $\mathrm{LIE}^1_W = \mathrm{LIE}^2_W$.

This last fact, for $p \geq 5$ is a consequence of [Fa2, §6], while for $p = 3$ is a consequence of loc. cit. and of the parabolic type property of 4.4.1 1) and of a) of 4.4.1 2). So, if $B^+(V)$ and β have the same significance as in [Fa2, §6], for $V := W(k)$, then we have (cf. loc. cit.) regardless of p the following inclusions of $B^+(V)$-modules:

$$\mathrm{LIE}^i_W \otimes_{W(k)} \beta^2 B^+(V) \subset \mathrm{Lie}(G^{\rm ad}_{4\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} B^+(V) \subset \mathrm{LIE}^i_W \otimes_{W(k)} \beta^{-2} B^+(V).$$

The part of 4.4.1 mentioned above implies directly that we can replace β^2 and β^{-2} by β and respectively by β^{-1}; so the things are fine for $p = 3$ as well.

4.9.2.1. For the SHS $(f_4, L_{1(p)} \oplus L_{2(p)}, v_4)$ we are going to use the standard notations of 2.3.1-3, except that we put a right lower index 4 everywhere. We can assume that $H_4 = G_4(\mathbb{Q}_p) \cap \mathrm{GSp}(W_1 \oplus W_2, \psi_1 \oplus \psi_2)\big((L_{1(p)} \oplus L_{2(p)}) \otimes \mathbb{Z}_p\big)$.

We (naturally) get a commutative diagram of pro-étale covers:

$$\begin{array}{ccc} \mathfrak{C}_4 & \xrightarrow{m_2} & \mathfrak{C}_1 \\ \big\downarrow{\scriptstyle m_1} & & \big\downarrow{\scriptstyle n_1} \\ \mathfrak{C}_2 & \xrightarrow{n_2} & \mathfrak{C}_0 \end{array}$$

(cf. the definition of $\mathfrak{C}_i$, $i \in \{1,2,4\}$, and [Va2, rm. 4) of 3.2.7 and 6.4.5.1]).

4.9.3. Let $z_0 \in \mathfrak{C}_0(V_0)$ be a lift of $y_0 \in \mathfrak{C}_0(\mathbb{F})$, and let $z_i \in \mathfrak{C}_i(V_0)$ such that $n_i \circ z_i = z_0$, $i = \overline{1,2}$. Let $z_4 \in \mathcal{C}_4(V_0)$ such that $m_1 \circ z_4 = z_2$, and let $m_2 \circ z_4 = z_3$. We denote by $y_i \in \mathfrak{C}_{s_i}(\mathbb{F})$ the point defined by $z_i \in \mathfrak{C}_{s_i}(V_0)$, with $i = \overline{0,4}$, with $s_i = i$ if $i \neq 3$, and with $s_3 = 1$. Let $\mathfrak{C}^{\rm ad}_{y_i}$ be the Shimura adjoint Lie σ-crystal attached to y_i, $i = \overline{1,4}$, obtained using the SHS $(f_{s_i}, L_{s_i(p)}, v_{s_i})$, with $L_{4(p)} := L_{1(p)} \oplus L_{2(p)}$. From the construction of f_4, we deduce the existence of a map $\mathfrak{C}^{\rm ad}_{y_4} \to \mathfrak{C}^{\rm ad}_{y_3} \oplus \mathfrak{C}^{\rm ad}_{y_2}$ between Shimura adjoint Lie σ-crystals, which produces an isomorphism of $\mathfrak{C}^{\rm ad}_{y_4}$ with $\mathfrak{C}^{\rm ad}_{y_2}$ and $\mathfrak{C}^{\rm ad}_{y_3}$.

4.9.4. Lemma. *$\mathfrak{C}^{\rm ad}_{y_i}$, $i = \overline{1,4}$, are all isomorphic.*

Proof: We have just seen that $\mathfrak{C}^{\mathrm{ad}}_{y_4}$ is isomorphic to $\mathfrak{C}^{\mathrm{ad}}_{y_2}$ and $\mathfrak{C}^{\mathrm{ad}}_{y_3}$. The fact that $\mathfrak{C}^{\mathrm{ad}}_{y_1}$ and $\mathfrak{C}^{\mathrm{ad}}_{y_3}$ are isomorphic results from the fact that z_1 and z_3 are two lifts to $\mathfrak{C}_1$ of z_0, and so $z_3 = a(z_1)h$, with $h \in G_1(\mathbb{A}_f^p)$, and with $a : (G_1, X_1, H_1, v_1) \xrightarrow{\sim} (G_1, X_1, H_1, v_1)$ an automorphism defined by an element of $G_0(\mathbb{Z}_{(p)}) = G_0(\mathbb{Q}) \cap H_0$ (cf. [Va2, 3.3.1]). But the Shimura adjoint Lie σ-crystal attached to y_1 is isomorphic to the one attached to $a(y_1)$ (the argument of this is entirely analogous to the one which produced an isomorphism of $\mathfrak{C}^{\mathrm{ad}}_{y_4}$ with $\mathfrak{C}^{\mathrm{ad}}_{y_2}$ and $\mathfrak{C}^{\mathrm{ad}}_{y_3}$), and so it is isomorphic to the one attached to $y_3 = a(y_1)h$.

4.9.5. We call $\mathfrak{C}^{\mathrm{ad}}_{y_1}$ the Shimura adjoint Lie σ-crystal attached to y_0. The above lemma guarantees that it is well defined.

4.9.5.1. For any algebraically closed field k of characteristic p, and for any $y_{01} \in \mathfrak{C}_0(k)$, we define similarly the Shimura adjoint Lie σ-crystal attached to y_{01}.

4.9.6. If now $z_0^0 \in \mathcal{C}_0(V_0)$ is the translate of z_0 by an element of $G_0(\mathbb{A}_f^p)$, and we work with $z_1^0, z_2^0, z_3^0, z_4^0$, and $y_1^0, y_2^0, y_3^0, y_4^0$ having a similar meaning, we get similarly that $y_i^0 = a_i^0(y_i)h_i$, with $h_i \in G_i(\mathbb{A}_f^p)$ and with $a_i^0 : (G_i, X_i, H_i, v_i) \xrightarrow{\sim} (G_i, X_i, H_i, v_i)$ an automorphism defined by an element of $G_0(\mathbb{Z}_{(p)})$, $\forall i \in \{1, 2, 4\}$. As in 4.9.4 we get:

4.9.7. Fact. *The Shimura adjoint Lie σ-crystals attached to y_1^0, y_2^0, y_3^0 and y_4^0 are all isomorphic to $\mathfrak{C}^{\mathrm{ad}}_{y_4}$.*

4.9.8. Theorem. *For any integral canonical model $\mathcal{N}$ of a Shimura quadruple (G, X, H, v) of preabelian type, with v diving a rational prime $p \geq 3$, there is a unique $G(\mathbb{A}_f^p)$-invariant stratification (called the refined canonical Lie stratification) of $\mathcal{N}_{k(v)}$ in locally closed (reduced) subschemes such that:*

a) the strata are indexed by the set $RLNP(G^{\mathrm{ad}}, X^{\mathrm{ad}}, v^{\mathrm{ad}})$ (defined in 4.5.7);

b) it is functorial with respect to finite morphism between quadruples;

c) there is a unique open stratum; it is Zariski dense in $\mathcal{N}_{k(v)}$;

d) any point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$, with k a perfect field, of the open dense stratum has a uniquely defined G-canonical lift $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$;

e) the G-canonical lifts are functorial with respect to finite maps between quadruples;

f) if $\mathcal{N}$ and (G, X, H, v) are related to a SHS $(f, L_{(p)}, v)$ as in 4.1, then we reobtain the refined canonical Lie stratification defined in 4.5 and the G-canonical lifts defined in 4.4;

g) it is invariant under isomorphisms, i.e. it is invariant under the natural action of the group $\mathrm{Aut}((G, X, H))$ (defined in [Va2, 9) of 3.2.7]) on $\mathcal{N}_{k(v)}$.

Proof: For the case $p = 3$ we refer to §6. We assume now that $p \geq 5$. Let (G_0, X_0, H_0, v_0) be the adjoint quadruple of (G, X, H, v). We use the notations of 4.9.1-7. A point $y_0 \in \mathfrak{C}_0(\mathbb{F})$ belongs to the stratum of $s \in RLNP(G_0, X_0, v_0)$ iff the sequence of Newton polygons defined by the cyclic factors of the Shimura adjoint Lie σ-crystal attached to y_0, is the same as that of s. The Hecke operators $G_0(\mathbb{A}_f^p)$ allows us to define the stratum of belongness for any point $y \in \mathcal{N}_0(\mathbb{F})$ (cf. [Va2, 3.3.2]). 4.9.4 and 4.9.7 guarantees that this is well defined. The fact that the things do not depend on the connected component X_0^0 of X_0 we chose is implied by [Va2, 3.3.3].

To see that the partition of $\mathcal{N}_0(\mathbb{F})$ in disjoints subsets defines locally closed subschemes we just have to remark that the pull back (of these disjoint sets) to $\mathcal{N}_1(\mathbb{F})$ through the morphism j_1 (of 4.9.1) are forming the partition of $\mathcal{N}_1(\mathbb{F})$ in subsets corresponding to the refined canonical Lie stratification of $\mathcal{N}_{1k(v_1)}$ defined by the SHS $(f_1, L_{1(p)}, v_1)$. So [Va2, 6.4.5.1] applies. This takes care of a) and f).

The stratification of $\mathcal{N}_{k(v)}$ is obtained from the one of $\mathcal{N}_{0k(v_0)}$ by the pull back operation under the natural morphism $\mathcal{N} \to \mathcal{N}_0$ (defined by the extension property of $\mathcal{N}_0$).

Now 4.9.4 and 4.9.7 guarantees that everything is well defined. This takes care of b), while c) results from 4.2.1, via [Va2, 6.4.5.1]. The existence and uniqueness of G-canonical lifts is a direct consequence of the following remark (it is just a particular case of the filtered version of 4.9.4):

(4.9.8.1) If in 4.9.4 z_2 is a G-canonical lift, then z_1 and z_3 are also G-canonical lifts.

So it makes sense to say that z_0 is a G_0-canonical lift of y_0 (4.9.7 guarantees that z_0 is well defined and unique). For a point $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$ of the open stratum, with k a perfect field, we first get a G-canonical lift $\bar{z} : \mathrm{Spec}(W(k_1)) \to \mathcal{N}$ (with k_1 a finite Galois extension of k) of the point $\bar{y} \in \mathcal{N}(k_1)$ defined by y, which due to uniqueness property is defined over $W(k)$, i.e. we get a G-canonical lift $z : \mathrm{Spec}(W(k)) \to \mathcal{N}_{k(v)}$ of y. This takes care of d) and e).

The argument why the refined canonical Lie stratification of $\mathcal{N}_{k(v)}$ is invariant under the action of $\mathrm{Aut}((G, X, H))$ has been presented in the proof of 4.9.4 (cf. the definition of $\mathrm{Aut}((G, X, H))$). This takes care of g), ending the proof of the theorem.

4.9.8.2. Corollary. *If* $\dim(X) > 0$*, then there is a* $G(\mathbb{A}_f^p)$*-invariant closed subscheme of* $\mathcal{N}_{k(v)}$ *different from* $\mathcal{N}_{k(v)}$*.*

Proof: [Va2, 6.4.2 and 6.4.5.1] reduces easily (cf. the proof of 4.9.8) the situation to the case when we have a SHS. But this case is handled by 4.3.6. This ends the proof of the corollary.

This corollary fulfils the promise of [Va2, 3.2.12].

4.9.9. Remark. A result similar to 4.9.8 can be obtained for the canonical Lie stratification, or for the Lie-stable stratification (defined for a SHS in 4.5).

4.9.10. Definitions. The points with values in perfect fields of the open dense stratum of $\mathcal{N}_{k(v)}/\tilde{H}_0$ (with $\tilde{H}_0$ a compact subgroup of $G(\mathbb{A}_f^p)$) are called G-ordinary points or Shimura-ordinary points.

4.9.11. Corollary. *We have a natural functor from the full subcategory of qf-Sh of whose objects are Shimura quadruples* (G, X, H, v) *of preabelian type, with* $(v, 2) = 1$*, to the category of stratified schemes, taking such a quadruple into the scheme defined by the special fibre of its integral canonical model, and endowed with the refined canonical Lie stratification (the action of the functor on morphisms being the logical one, cf. [Va2, rm. 4) of 3.2.7]).*

4.9.12. Corollary. *Under the Hecke operators of* $G(\mathbb{A}_f^p)$*, a* G*-canonical lift is taken into a* G*-canonical lift.*

4.9.13. Remark. Let $\tilde{H}_0$ be an arbitrary compact subgroup of $G(\mathbb{A}_f^p)$. As in 4.4.2.1 we define uniquely the G-canonical lift $\mathrm{Spec}(W(k)) \to \mathcal{N}/\tilde{H}_0$ of a G-ordinary point $\mathrm{Spec}(k) \to \mathcal{N}_{k(v)}/\tilde{H}_0$. In §12 we will see that a G-canonical lift $\mathrm{Spec}\big(W(\mathbb{F})\big) \to \mathcal{N}/\tilde{H}_0$ gives birth to a special point $\mathrm{Spec}\big(W(\mathbb{F})[\frac{1}{p}]\big) \to \mathrm{Sh}(G,X)/\tilde{H}_0$ (cf. 4.4.6).

4.9.14. Question. With the notations of 4.5, is it true that the f-canonical stratification of $\mathcal{N}_{k(v)}$ defined in 4.5.10 is indeed canonical, i.e. it does not depend on the SHS $(f, L_{(p)}, v)$ producing it?

The expectation (cf. 4.5.10) of the existence of a refined f-canonical stratification of $\mathcal{N}_{k(v)}$, suggests a negative answer to this question. We expect that the supposed existing refined f-canonical stratification of $\mathcal{N}_{k(v)}$ (cf. 4.5.10), does not depend on the SHS producing it (and so that we can drop mentioning "f-").

4.9.15. Problem. Prove a result similar to 4.9.8, but for the (refined) canonical stratifications.

4.9.16. Exercise. Show that the theory of G-multiplicative coordinates presented in 4.7.33 extends to the case of Shimura-ordinary points of an integral canonical model of a Shimura quadruple (G, X, H, v) of preabelian type. Moreover the G-multiplicative coordinates we get are uniquely determined, provided the prime v^{ad} of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ divided by v has $\mathbb{F}_p$ as its residue field. Hint: for the uniqueness part, use the idea of the proof of 4.9.4, and 4.7.30 4).

4.9.17. Definition. Let $\mathcal{N}$ be the integral canonical model of a Shimura quadruple (G, X, H, v) of preabelian type, with $(v, 2) = 1$. Let p be the characteristic of $k(v)$. Let $y \in \mathcal{N}(k)$, with k an algebraically closed field of characteristic p. Let $\mathcal{N}_0$ be the integral canonical model of the quadruple $(G_0, X_0, H_0, v_0) := (G^{\mathrm{ad}}, X^{\mathrm{ad}}, H^{\mathrm{ad}}, v^{\mathrm{ad}})$, and let $y_0 \in \mathcal{N}_0(k)$ be the composite of y with the natural morphism $\mathcal{N} \to \mathcal{N}_0$. Let $h \in G^{\mathrm{ad}}(\mathbb{A}_f^p)$ such that $y_{01} = y_0 h \in \mathfrak{C}_0(k)$ (cf. [Va2, 3.3]). The Shimura adjoint Lie σ-crystal $\mathfrak{C}^{\mathrm{ad}}_{y_{01}}$ (cf. 4.9.5.1) is called the Shimura adjoint Lie σ-crystal attached to y.

The reason why is this well defined is nothing else but 4.9.4 and 4.9.7 (performed for $W(k)$, instead of V_0). In the case of a SHS $(f, L_{(p)}, v)$ we recover the definition of a Shimura adjoint Lie σ-crystal attached to a point $y \in \mathcal{N}(k)$.

4.9.17.1. Remark. In the situation of 4.9.17, we similarly define the Shimura adjoint filtered Lie σ-crystal attached to a point $\mathrm{Spec}(W(k)) \to \mathcal{N}$. So the canonical lifts $\mathrm{Spec}(W(k)) \to \mathcal{N}$ can be recognized (see 4.4.1 2)) as follows: their Shimura adjoint filtered Lie σ-crystals are of parabolic (and Borel) type.

4.9.17.2. Definition. A point $y : \mathrm{Spec}(k_1) \to \mathcal{N}_{k(v)}$, with k_1 a perfect field, is called a toric point if the Shimura adjoint Lie $\bar{\sigma}_1$-crystal attached to the $\overline{k_1}$-valued point of $\mathcal{N}_{k(v)}$ naturally defined by y is isomorphic to the extension to $\overline{k_1}$ of the Shimura adjoint Lie $\bar{\sigma}_1$-crystal defined by a Shimura σ-crystal $\mathfrak{C}_\omega$ showing up in 4.1.5, by working with an adequately chosen SHS. Here σ_1 is the Frobenius automorphism of k_1 and of $W(k_1)$.

4.9.17.3. Remark. 4.9.4 shows that the notion of toric points is intrinsic (canonical). In particular, if (G, X, H, v) is obtained from a SHS $(f, L_{(p)}, v)$ then we "regain" the toric

points defined in 4.5.12.2. Moreover, 4.12.3 and 4.12.6 show that there are plenty of toric points (cf. as well to [Va2, 6.4.2] and 4.12.5). As in 4.5.12.2.1, the toric points can be defined in terms of Galois representations.

4.9.17.4. Remark. Based on 4.9.8, it makes sense to speak, as in 3.4.15-16, about $\mathcal{N}_0$ having the completion property (we just rephrase 3.4.15 in terms of Shimura adjoint Lie σ-crystals).

4.9.18. Remark. We think that the terminology (refined) canonical Lie stratification is justified. The word Lie should make distinction from other ρ-stratifications introduced in 4.5.4 (for a SHS), while the word canonical should emphasize that in the case of a Shimura quadruple (G, X, H, v) emerging from a SHS $(f, L_{(p)}, v)$, the (refined) (canonical) Lie stratification we get for $\mathcal{N}_{k(v)}$ does not depend on the choice of the SHS $(f, L_{(p)}, v)$. We might argue that the terminology the absolute stratifucation might have been better, cf. 4.5.6.1. But the proof of 4.5.6.1 works only in the context of deformations of Shimura σ-crystals whose attached cyclic adjoint factors are of A_ℓ, B_ℓ, C_ℓ or D_ℓ Lie type. So the cases of E_6 and E_7 Lie types are left unsolved (see also 4.14.2).

4.9.19. Exercise. Let (G_i, X_i, H_i, v_i), $i = \overline{1,2}$, be two Shimura quadruples of preabelian type, with v_1 and v_2 dividing the same prime $p \geq 3$. Let w be a prime of $E(G_1 \times G_2, X_1 \times X_2)$ dividing v_1 and v_2. So we get a Shimura quadruple $(G_1 \times G_2, X_1 \times X_2, H_1 \times H_2, w)$. Let $\mathcal{M}_i$ be the extension to $\mathrm{Spec}(O_{(w)})$ of the integral canonical model $\mathcal{N}_i$ of (G_i, X_i, H_i, v_i), $i = \overline{1,2}$. It is known that $\mathcal{M}_1 \times_{O_{(w)}} \mathcal{M}_2$ is the integral canonical model of $(G_1 \times G_2, X_1 \times X_2, H_1 \times H_2, w)$ (cf. [Va2, 3.2.16]). Show that the refined canonical Lie stratification of $\mathcal{M}_{k(w)}$ is the product stratification of the refined canonical Lie stratifications of $\mathcal{M}_{ik(w)}$.

4.9.20. Exercise. With the notations of 4.9.17, let $H_0 \subset G(\mathbb{A}_f^p)$ be a subgroup such that $H_0 \times H$ is a subgroup of $G(\mathbb{A}_f)$ smooth for (G, X). Show that for any formally smooth morphism $f_s : \mathrm{Spec}(R) \to \mathcal{N}/H_0$, we can define uniquely the Shimura adjoint filtered Lie F-crystal (over $\mathrm{Spec}(R/pR)$) attached to f_s.

A proof of this exercise will be given in §5. In what follows we need only the case when the first fundamental group of $\mathrm{Spec}(R)$ is trivial: this case can be treated entirely as in 4.9.2-4 (where we have $R = \mathbb{F}$).

4.9.21. The relative Oort-Moonen problem. Let (G, X, H, v) be a Shimura quadruple of preabelian type, with v dividing a prime $p > 2$. Let $\mathcal{N}$ be its integral canonical model (cf. 6.4.1 for $p > 3$, and cf. §6 for $p = 3$). Let F be a number field containing $E(G, X)$. Let $H_0 \subset G(\mathbb{A}_f^p)$ be a compact open subgroup such that the subgroup $H_0 \times H$ of $G(\mathbb{A}_f)$ is smooth for (G, X). Let Z be an irreducible subvariety of $\mathrm{Sh}_{H_0 \times H}(G, X)_F$. Let $Z_{(v)}$ be the closure of Z in $(\mathcal{N}/H_0)_{F_{(v)}}$ (here $F_{(v)}$ is the normalization of $O_{(v)}$ in F). We assume that there is a prime w of F dividing v and a G-ordinary point $y : \mathrm{Spec}(\overline{k(v)}) \to \mathcal{N}_{k(w)}$ factoring through $Z_{(v)}$, such that $Z_{(v)}$ has formally quasi-linear components at y. Then Z is of preabelian type, i.e. any irreducible component of $Z \times_F \mathbb{C}$ is a subvariety of preabelain type (cf. def. [Mo, 3.7]).

Due to the existence of G-multiplicative coordinates (cf. 4.9.16) the notion of having formally quasi-linear components of y can be defined as in [Mo]. It seems to us that we

can use 4.7.30 and 4.7.36, to prove that this notion is independent of all choices referred to in 4.7.30 8); if not, we have to define the notion of formally quasi-linearity by working with respect to all -allowed, i.e. logical- choices mentioned in 4.7.30 8)).

For not being too long, we simplify the things by just giving an example (without being particular about the precise choices defining G-multiplicative coordinates) in the case when w is unramified over v. If the closed subscheme (defined naturally by $Z_{(v)}$) of the spectrum of the completion of the local ring of the point $y_1 : \mathrm{Spec}(\overline{k(v)}) \to \mathcal{N}_{W(\overline{k(v)})}$ defined by y, is a formal subtorus of the formal torus (describing G-multiplicative coordinates) defined (cf. 4.9.16) by the G-ordinary point y_1, then (by the very definition) $Z_{(v)}$ has formally quasi-linear components at y.

4.9.21.1. Exercise. Using [Mo, 5.2] show that the relative Oort-Moonen problem is true if $k(v^{\mathrm{ad}}) = \mathbb{F}_p$ (here v^{ad} is the prime of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ divided by v). Hint: make use of 4.7.30, and of [Va2, 6.5.1.1].

4.9.22. Definition. A simple adjoint Shimura variety $\mathrm{Sh}(G, X)$ is said to be of $\mathbb{Z}$-End type if there is a principally polarized abelian variety over $\mathbb{C}$ with $\mathrm{End}(A) = \mathbb{Z}$, whose attached (see [Va2, 2.12 3)]) Shimura variety has $\mathrm{Sh}(G, X)$ as its adjoint variety.

We refer to the situation of 4.9.2. Let E be a number field. Let H_{00} be the subgroup of $G_0(\mathbb{A}_f^p)$ generated by the images of H_{01} and H_{02} (we recall that H_{0i} is a subgroup of $G_i(\mathbb{A}_f^p)$ subject to some conditions; $i \in \{1, 2\}$). We consider two morphisms $a(i) : \mathrm{Spec}(E) \to \mathrm{Sh}_{H_{0i} \times H_i}(G_i, X_i)$, $i \in \{1, 2\}$, giving birth to the same morphism $\mathrm{Spec}(E) \to \mathrm{Sh}_{H_{00} \times H_0}(G_0, X_0)$. Let A_i be the abelian variety over E naturally associated to $a(i)$ and the embedding f_i of 4.9.1. From 4.9.8 we deduce immediately:

4.9.23. Corollary. *The ordinary conjecture (see [Oo, p. 11]) is true for A_1 iff it is true for A_2.*

From this and [Pi, 7.2] we deduce:

4.9.23.1. Corollary. *Let A be an abelian variety over a number field E. We assume that the Shimura variety attached to it is such that all the factors of its adjoint variety are of $\mathbb{Z}$-End type and are not of C_n type with $n \geq 3$. Then the ordinary conjecture is true for A.*

4.9.23.2. Exercise. Show that there are situations as in 4.9.23.1 with A an absolutely simple variety such that $\mathrm{End}(A_{\bar{E}}) \neq \mathbb{Z}$. Hint: use Mumford curves and 2.3.5.1.

4.10. Hecke orbits of G-ordinary points. Let (G, H, H, v) be a Shimura quadruple of preabelian type, with $(v, 2) = 1$, and let $\mathcal{N}$ be its integral canonical model over $O_{(v)}$. Let $\mathcal{U} \hookrightarrow \mathcal{N}_{k(v)}$ be the open (Zariski dense) subscheme of G-ordinary points (cf. 4.9.8). From the proof of 2.3.9 we deduce that the field of fractions of any connected component of $\mathcal{N}_{k(v)}$ contains $\mathbb{F}$.

4.10.1. Conjecture. *$\mathcal{U}$ is the smallest non-empty open subscheme of $\mathcal{N}_{k(v)}$ which is $G(\mathbb{A}_f^p)$-invariant.*

4.10.1.1. Standard arguments show that another way to formulate 4.10.1 is: the Hecke orbit of a G-ordinary point y (of $\mathcal{U}$) with values in $k = \bar{k}$, i.e. the set of points obtained from y through $G(\mathbb{A}_f^p)$-translations, is dense in $\mathcal{N}_k$, i.e. it does not factor through a closed subscheme of $\mathcal{N}_k$ which is not $\mathcal{N}_k$ itself.

4.10.2. The expectations of the validity of the statement of 4.10.2 are based on:

a) 4.2.8.1;

b) exercise 8) of 4.5.6;

c) the part d) (or c)) of 4.4.1 3), and b) and c) of 4.4.1 2);

d) the validity of 4.10.1 in the case when $\mathrm{Sh}(G,X)$ is a Siegel modular variety [Ch];

e) 4.10.3 below;

f) 4.10.5 below.

4.10.3. Proposition. *The above conjecture is true if* $\dim(X) \leq 1$.

Proof: If $\dim(X) = 0$, i.e. if G is a torus, this is obvious (cf. [Va2, 3.2.8 and 3.3.1]). If $\dim(X) = 1$, then everything results from 4.2.8.1, from 4.9.12, and from the fact the $G(\mathbb{A}_f^p)$-orbit of any point of $\mathrm{Sh}(G,X)(\mathbb{C})$ is Zariski dense in $\mathrm{Sh}(G,X)_{\mathbb{C}}$. As we did not include a proof of 4.2.8.1, we would like to remark that for Shimura curves 4.2.8.1 is a consequence of [Va2, 6.4.11] (the case of non-compact Shimura curves, i.e. of Shimura curves having the same adjoint curve as the elliptic modular curve, is well known; see [FC, ch. IV, 5.10 and 6.12]).

4.10.4. Variants. There are variants of 4.10.1. For instance by working not with the full group $G(\mathbb{A}_f^p)$ of Hecke operators, but just with $G(\mathbb{Q}_q)$, where q is a rational prime, different from p, and such that G is unramified over $\mathbb{Q}_q$ (cf. [Ch]), or more generally with an adequate (i.e. of interest) subgroup of $G(\mathbb{A}_f^p)$. However in such a variant we have to be careful: $G(\mathbb{Q}_q)$ might not permute transitively the connected components of $\mathrm{Sh}_H(G,X)_{\mathbb{C}}$. Another possibility is of replacing G-ordinary points and $\mathcal{U}$ respectively by $G(\omega)$-ordinary points and the $\mathfrak{s}_\omega$ stratum (cf. 4.5.12). Again some precautions have to be taken: we might not get dense Hecke orbits. So in some of these variants 4.10.1 needs to be reformulated.

4.10.5. Remark. The gluing principle of 3.4.19 D can be interpreted as a local Hecke correspondance. Using this we prove (see [Va7]) the following IOTA (if one than all) criterion:

4.10.5.1. Criterion. *Let (G,X,H,v) be a quadruple of preabelian type, with $k(v^{\mathrm{ad}}) = \mathbb{F}_p$. We assume that either $p \geq 3$ or $p = 2$ and (G,X,H,v) is associated to a $p = 2$ SHS. Let $\mathcal{U}$ be the G-ordinary locus of the integral canonical model $\mathcal{N}$ of (G,X,H,v). If the $G(\mathbb{A}_f^p)$-orbit of a G-ordinary point of $\mathcal{U}$ is dense in $\mathcal{N}_{k(v)}$ (i.e. it is dense in $\mathcal{U}$), then the $G(\mathbb{A}_f^p)$-orbit of any G-ordinary point of $\mathcal{U}$ is dense in $\mathcal{N}_{k(v)}$.*

This criterion recovers all the previous known cases of density of Hecke orbits in positive characteristic and provides plenty of new examples of such densities (like the case of classical Spin modular varieties, see [Va2, 5.7.5] and [Va4]).

4.11. The functorial behaviour of G-ordinary points.

4.11.1. We consider now a SHS $(f, L_{(p)}, v)$ defined by an injective map $f : (G, X) \hookrightarrow \big(\mathrm{GSp}(W, \psi), S\big)$ which factors as the composite $f_1 \circ f_2$ of two injective maps, with $f_2 : (G, X) \hookrightarrow (G_1, X_1)$, and with $f_1 : (G_1, X_1) \hookrightarrow \big(\mathrm{GSp}(W, \psi), S\big)$. Let v_1 be the prime of $E(G_1, X_1)$ divided by v. We assume that $(f_1, L_{(p)}, v_1)$ is also a SHS. We still use for it the standard notations of 2.3.1-3, but everything having a right lower index 1, except $L_{(p)}$. We have a morphism $i_2 : \mathcal{N} \to \mathcal{N}_1$ (cf. [Va2, rm. 4) of 3.2.7]). Two natural questions arises:

Q1. When i_2 takes G-ordinary points of $\mathcal{N}_{k(v)}$ into G_1-ordinary points of $\mathcal{N}_{1k(v_1)}$?

Q2. If i_2 takes G-ordinary points of $\mathcal{N}_{k(v)}$ into G_1-ordinary points of $\mathcal{N}_{1k(v_1)}$, is it true that a G-canonical lift of a G-ordinary point y of $\mathcal{N}_{k(v)}$ is mapped by i_2 into a G-canonical lift of $i_2 \circ y$?

$Q1$ can be reformulated (cf. 4.2.1): when the dense open stratum of $\mathcal{N}_{k(v)}$ is the pull back through i_2 of the dense open stratum of $\mathcal{N}_{1k(v_1)}$?

The case $k(v_1) = \mathbb{F}_p$ is completely handled by properties P1 and P2 of 4.6. The anwser to $Q1$ is yes when $k(v) = \mathbb{F}_p$, and then $Q2$ also has a positive answer. What about $k(v_1) = \mathbb{F}_{p^{q_1}}$ with $q_1 \in \mathbb{N}$, $q_1 \geq 2$?

4.11.2. Theorem. *The morphism i_2 maps G-ordinary points of $\mathcal{N}_{k(v)}$ into G_1-ordinary points of $\mathcal{N}_{1k(v_1)}$, and G-canonical lifts of $\mathcal{N}$ into G_1-canonical lifts of $\mathcal{N}_1$ if and only if the following two conditions are satisfied:*

a) $k(v) = k(v_1)$;

b) there is a torus T of $G_{\mathbb{Z}_p}$, a cocharacter $\mu : \mathbb{G}_m \hookrightarrow T_{W(k(v))}$, and a Borel subgroup B_1 of $G_{1\mathbb{Z}_p}$ such that:

i) Over an embedding of $W(k(v))$ in $\mathbb{C}$, extending the inclusion $E(G, X) \subset \mathbb{C}$, μ becomes $G(\mathbb{C})$-conjugate to the cocharacters μ_x^*, $x \in X$;

ii) Under the direct sum decomposition $L_p^* \otimes W(k(v)) = F^1 \oplus F^0$ produced by μ ($\beta \in \mathbb{G}_m\big(W(k(v))\big)$ acts through μ on F^i as the multiplication with β^{-i}), F^1 is left invariant by B_1 (i.e $\mathrm{Lie}(B_1) \subset F^0(\mathrm{Lie}(G_{W(k(v))})$;

iii) $B_1 \cap G_{\mathbb{Z}_p}$ is a Borel subgroup B of $G_{\mathbb{Z}_p}$.

Proof: The fact that the conditions a) and b) imply that the i_2 takes G-ordinary points of $\mathcal{N}_{k(v)}$ into G_1-ordinary points of $\mathcal{N}_{1k(v_1)}$ results from 4.1-2: the G-ordinary type of the SHS $(f, L_{(p)}, v)$ and the G_1-ordinary type of the SHS $(f_1, L_{(p)}, v_1)$ can both be computed (cf. 4.1) starting from the data of T, B and μ (resp. of T, B_1 and μ), and so they are equal. The fact that a) and b) imply also that i_2 takes G-canonical lifts into G_1-canonical lifts is a direct consequence of c) of 4.4.1 3) and of 4.4.1 1): there is $n \in \mathbb{N}$ such that the filtered σ^n-isocrystals attached to G-canonical lifts $z : \mathrm{Spec}(W(k)) \to \mathcal{N}$, with $k = \bar{k}$, depend only on the data of b).

Let us now assume that i_2 takes G-ordinary points into G_1-ordinary points and G-canonical lifts into G_1-canonical lifts. Let $z : \mathrm{Spec}\big(W(\mathbb{F})\big) \to \mathcal{N}$ be a G-canonical lift of a G-ordinary point $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}_{k(v)}$, and let $\mathfrak{C} := (M_y, F_z^1, \varphi_y, G_{W(\mathbb{F})})$ be the Shimura filtered σ-crystal attached to z. We get another Shimura filtered σ-crystal $\mathfrak{C}_1 = (M_y, F_z^1, \varphi_y, G_{1W(\mathbb{F})})$; it is the one attached to $y_2 \circ z$.

Now the canonical split $\mu_y : \mathbb{G}_m \hookrightarrow G_{W(k)}$ of $\mathfrak{C}$ when composed with $G_{W(k)} \hookrightarrow G_{1W(k)}$, produces the canonical split of $\mathfrak{C}_1$ (cf. 3.1.5). This implies that the degree of definition of $\mathfrak{C}$ is equal to the degree of definition of $\mathfrak{C}_1$ and so $k(v) = k(v_1)$ (cf. 4.4.7). This proves a). Writing $\varphi_y = a\mu_y(\frac{1}{p})$, a becomes a σ-linear automorphism of M_y. Then $M_y^a := \{x \in M_y | a(x) = x\}$ is a $\mathbb{Z}_p$-structure on M_y. We get groups $\tilde{G}_{\mathbb{Z}_p} \overset{\tilde{f}_2}{\hookrightarrow} \tilde{G}_{1\,\mathbb{Z}_p} \hookrightarrow GL(M_y^a)$ and a cocharacter $\mu_y : \mathbb{G}_m \hookrightarrow \tilde{G}_{W(k(v))}$. Let $\tilde{B}$ be a Borel subgroup of $\tilde{G}_{\mathbb{Z}_p}$ leaving invariant F_z^1 and such that μ_y factors through $\tilde{B}_{W(k(v))}$, and let $\tilde{B}_1$ be a Borel subgroup of $\tilde{G}_{1\,\mathbb{Z}_p}$ leaving invariant F_z^1 and containing $\tilde{B}$ (the existence of $\tilde{B}$ and of $\tilde{B}_1$ is a direct consequence of b) of 4.4.1 2)). Let $\tilde{T}$ be a maximal torus of $\tilde{G}_{\mathbb{Z}_p}$ contained in $\tilde{B}$ and such that μ_y factors through $\tilde{T}_{W(k(v))}$.

Now b) results from the fact that we have isomorphisms $\tilde{G}_{\mathbb{Z}_p} \tilde{\to} G_{\mathbb{Z}_p}$ and $\tilde{G}_{1\,\mathbb{Z}_p} \tilde{\to} G_{1\mathbb{Z}_p}$, through which $\tilde{f}_2$ becomes the natural inclusion $\bar{f}_2 : G_{\mathbb{Z}_p} \hookrightarrow G_{1\mathbb{Z}_p}$ as subgroups of $GL(L_p^*)$. This is implied by d) of 4.4.1 3). As we postponed the proof of d) of 4.4.1 3), we present another argument of why b) is satisfied. To show the existence of an isomorphism $\tilde{G}_{\mathbb{Z}_p} \tilde{\to} G_{\mathbb{Z}_p}$, it is enough to show the existence of an isomorphism $\tilde{G}^{\mathrm{ad}}_{\mathbb{Z}_p} \tilde{\to} \tilde{G}^{\mathrm{ad}}_{\mathbb{Z}_p}$, as obviously $Z(\tilde{G})_{\mathbb{Z}_p} = Z(G)_{\mathbb{Z}_p}$) (cf. the definition of $\tilde{G}_{\mathbb{Z}_p}$) and as $\tilde{G}_{W(\mathbb{F})} = G_{W(\mathbb{F})}$. But this is implied by b) of 4.4.1 2). Similarly we get an isomorphism $\tilde{G}_{1\,\mathbb{Z}_p} \tilde{\to} G_{1\,\mathbb{Z}_p}$.

The fact that in c) of 4.4.1 3) we can take $n = 1$ or not for the SHS $(f, L_{(p)}, v)$ and for the point y, is expressed in terms of a torsor of $\tilde{G}_{\mathbb{Q}_p}$ being zero or not. But the resulting torsor of $\tilde{G}^{\mathrm{ad}}_{\mathbb{Q}_p} = G^{\mathrm{ad}}_{\mathbb{Q}_p}$ is trivial, cf. b) of 4.4.1 2). So it corresponds to a torsor of $Z(\tilde{G}_{\mathbb{Q}_p}) = Z(G)_{\mathbb{Q}_p}$. As this group is included in $\tilde{T}$, $\tilde{B}$ and $\tilde{B}_1$, we get that the condition b) is satisfied. This ends the proof of the theorem.

4.11.3. Example. We consider now the case when f_2 satisfies the conditions:

a) The toric part of $Z(G)$ is the toric part of $Z(G_1)$;

b) $G^{\mathrm{ad}} = \mathrm{Res}_{F/\mathbb{Q}} G^s$, with F a totally real number field, and with G^s an F-simple adjoint group;

c) f_2 induces an injective map $f_2^{\mathrm{ad}} : (G^{\mathrm{ad}}, X^{\mathrm{ad}}) \hookrightarrow (G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}})$ which is the injective map $f_{2\,F_1} : \mathrm{Sh}(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \hookrightarrow \mathrm{Sh}^{F_1}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ defined in [Va2, 6.6.1] (with F_1 a totally real number field containing F) (i.e. $G_1^{\mathrm{ad}} = \mathrm{Res}_{F_1/\mathbb{Q}} G^s_{F_1}$, and $f_2^{\mathrm{ad}} : G^{\mathrm{ad}} \hookrightarrow G_1^{\mathrm{ad}}$ is the canonical inclusion).

Then $E(G_1, X_1) = E(G, X)$, and a) and b) of 4.11.2 hold. We get:

i_2 takes G-ordinary points of $\mathcal{N}_{k(v)}$ into G_1-ordinary points of $\mathcal{N}_{1k(v_1)}$ and G-canonical lifts of $\mathcal{N}$ into G_1-canonical lifts of $\mathcal{N}_1$.

This result is going to be used in §12.

There are two other natural questions inspired by 4.11.2:

Q_3. If i_2 takes G-ordinary points into G_1-ordinary points, does it automatically take G-canonical lifts into G_1-canonical lifts?

Q_4. Does i_2 take G-ordinary points into G_1-ordinary points if we know only that $k(v) = k(v_1)$?

4.11.4. Exercise. Assuming 4.4.6 prove that the answer to Q_3 is yes (Hint: use 4.4.8).

4.11.5. Remark. The answer to Q_4 is no if $k(v) = k(v_1) \neq \mathbb{F}_p$. There are many examples with G a torus which provide a negative answer to Q_4. If $k(v) = k(v_1) = \mathbb{F}_p$ then the condition b) of 4.11.2 is automatically satisfied, and this motivates the form of properties P1 and P2 of 4.6.

4.11.6. Example. Here we concentrate just on Shimura σ-crystals. We assume that $G_{1\,\mathbb{Z}_p}^{\mathrm{ab}}$ is a split torus, and that $G_{1\,\mathbb{Z}_p}^{\mathrm{ad}}$ is an absolutely simple group of A_ℓ Lie type with non-trivial involution (so $\ell \geq 2$), and which splits over $W(\mathbb{F}_{p^2})$. Let T be a maximal torus of $G_{1\,\mathbb{Z}_p}$ which splits over $W(\mathbb{F}_{p^2})$, cf. [Ti, 1.10]. Let B a Borel subgroup of $G_{1\,\mathbb{Z}_p}$ containing T. Let $\mu : \mathbb{G}_m \to T_{W(\mathbb{F}_{p^2})}$ such that the conditions of 4.1 are satisfied. Let $\omega \in G_{\mathbb{Z}_p}\big(W(\mathbb{F}_{p^2})\big)$ normalizing $T_{W(\mathbb{F}_{p^2})}$ and taking $B_{W(\mathbb{F}_{p^2})}$ into its opposite (with respect to $T_{W(\mathbb{F}_{p^2})}$) Borel subgroup of $G_{1\mathbb{Z}_p}$. Let $\sigma_1 = \omega\sigma$ and let $T_1 \hookrightarrow \tilde{G}_{1\,\mathbb{Z}_p}$ be an inclusion of reductive groups which over $W(\mathbb{F}_{p^2})$ becomes the inclusion $T_{W(\mathbb{F}_{p^2})} \hookrightarrow G_{1\,W(\mathbb{F}_{p^2})}$ and which is defined at the level of Lie algebras by $\mathrm{Lie}(T_1) := \mathrm{Lie}\big(T_{W(\mathbb{F}_{p^2})}\big)^{\sigma_1} \hookrightarrow \mathrm{Lie}\big(G_{1\,W(\mathbb{F})}\big)^{\sigma_1} = \mathrm{Lie}(\tilde{G}_{1\,\mathbb{Z}_p})$ (the upper index σ_1 means we take the elements fixed by σ_1). We get:

– the Shimura σ-crystals $\big(L_p^* \otimes W(\mathbb{F}), \sigma_1\mu(\frac{1}{p}), T_{1\,W(\mathbb{F})}\big)$ and $\big(L_p^* \otimes W(\mathbb{F}), \sigma\mu(\frac{1}{p}), G_{1\,W(\mathbb{F})}\big)$ are both Shimura-ordinary, with the degree of definition equal to 2.

Obviously the $T_{1\,W(\mathbb{F})}$-ordinary type we get is different from the $G_{1\,W(\mathbb{F})}$-ordinary type we get.

4.11.7. Exercise. Under the assumptions a) and b) of 4.11.2, if the A-degree of definition of a G-canonical lift $\mathrm{Spec}(W(\mathbb{F})) \to \mathcal{N}$ is 1, prove that: if $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}$, with $k = \bar{k}$, is a G-ordinary point, then the formal torus (of the formal moduli scheme) of G-deformations of A_y is a subtorus of the formal torus of G_1-deformation of A_y (Hint: use 4.7.36).

This is a generalization in the case of an arbitrary SHS with the A-degree of definition 1 of the main result of [No1]. There are variants (cf. 4.7.30 8)) of this exercise for an arbitrary A-degree of definition.

4.12. The integral Manin problem. Let $e \in \mathbb{N}$, and let p be a rational prime. Let k be an algebraically closed field of characteristic p. Let σ be the Frobenius automorphism of $W(k)$ and of $B(k)$.

The (original) Manin problem asked for the proof of the predicted classification of isocrystals over k attached to (principally polarized) abelian varieties over k, of dimension e. This problem was positively solved by the Honda–Serre–Tate theory (cf. [Ta, p. 98]). For a second, more recent, proof, see [Oo2].

4.12.0. The goal of this section is to formulate (and solve in some cases; see §11 for the general case) the integral Manin problem for (special fibres of) integral canonical

models of Shimura varieties of preabelian type with respect to primes dividing a rational prime greater than 2. For the solution of this integral problem in the cases in which we know the completion property, we rely heavily on 3.4.15 and 4.4.1, and so we first treat the case $p \geq 3$. As a roundabout (cf. 4.12.12.1 and 3.14) we gain the case $p = 2$ as well. From now on, except for a special reference, $p \geq 3$.

4.12.1. The integral Manin problem for Siegel modular varieties. Determine all isomorphism classes of principally quasi-polarized σ-crystals over k attached to principally polarized abelian varieties over k of dimension e.

4.12.2. The integral Manin problem for a SHS $(f, L_{(p)}, v)$. Determine all isomorphism classes of Shimura σ-crystals and of Shimura Lie σ-crystals over k attached to k-valued points of $\mathcal{N}_{k(v)}$.

4.12.3. The integral Manin problem for a Shimura quadruple (G, X, H, v) of preabelian type, with v dividing a rational prime $p \geq 3$. Determine all isomorphism classes of Shimura adjoint Lie σ-crystals attached (cf. 4.9.13) to k-valued points of the special fibre $\mathcal{N}_{k(v)}$ of the integral canonical model $\mathcal{N}$ of the quadruple (G, X, H, v).

The answer to all these three versions of the integral Manin problem is provided by 4.4.1, provided 3.4.15 holds.

4.12.4. Solution of 4.12.1. Let $(\mathrm{GSp}(W, \psi), S)$ define a Siegel modular variety, with $\dim_{\mathbb{Q}}(W) = 2e$. Let $L_{(p)}$ be a $\mathbb{Z}_{(p)}$-lattice of W such that $\psi : L_{(p)} \otimes L_{(p)} \to \mathbb{Z}_{(p)}$ is a perfect form. We get a SHS $(1_{(\mathrm{GSp}(W,\psi),S)}, L_{(p)}, p)$. Let $I_{2e} := \{1, 2, ..., 2e\}$. Let $\{w_i | i \in I_{2e}\}$ be a $\mathbb{Z}_{(p)}$-basis of $L^*_{(p)}$ such that for $i, j \in I_{2e}$ satisfying $j \geq i$, $\tilde{\psi}(w_i, w_j)$ is equal to 1 if $j - i = e$, and is equal to 0 otherwise. Let $M_0 := L^*_{(p)} \otimes W(k)$, and let $F^1 \subset M_0$, be the $W(k)$-submodule of M_0 generated by w_i, $i \in \{1, ..., e\}$. Let φ be the σ-linear endomorphism of M_0 such that $\varphi(w_i) = pw_i$, if $i \in \{1, ..., e\}$, and $\varphi(w_i) = w_i$, if $i \in \{e + 1, ..., 2e\}$. We have:

Answer: *The set $\mathcal{S}_e(k)$ of isomorphism classes of σ-crystals attached to principally polarized abelian varieties over k of dimension e is precisely the set of isomorphism classes of principally quasi-polarized σ-crystals defined by elements of the set*

$$\{(M_0, g\varphi, \tilde{\psi}) | g \in Sp(M_0, \tilde{\psi})(W(k))\}.$$

This is an easy consequence of [Ta, p. 98] and of the following isogeny property:

The isogeny property. *If $(M'_0, g\varphi, \tilde{\psi})$, with M'_0 a $W(k)$-lattice of $M_0[\frac{1}{p}]$, is the principally quasi-polarized σ-crystal attached to a principally polarized abelian variety $(A', p_{A'})$ over k, then there is a principally polarized abelian variety (A, p_A) over k, $\mathbb{Z}[\frac{1}{p}]$-isogeneous to $(A', p_{A'})$ and of whose principally quasi-polarized σ-crystal, under this $\mathbb{Z}[\frac{1}{p}]$-isogeny, is identified with $(M_0, g\varphi, \tilde{\psi})$.*

4.12.5. Solution of 4.12.2 in the case $\mathcal{N}$ has the completion property. Let $M_0 := M \otimes_{W(k(v))} W(k)$, $F^1_0 := F^1 \otimes_{W(k(v))} W(k)$, and let $\varphi_0 := \varphi \otimes 1$ be the Frobenius endomorphism of M_0, where M, F^1 and φ have the same significance as in 4.1. We have:

Answer*: *The set* $\mathcal{S}_{(f,L_{(p)},v)}(k)$ *of isomorphism classes of Shimura* σ*-crystals attached to* k*-valued points of* $\mathcal{N}$ *is precisely the set of isomorphism classes of principally quasi-polarized Shimura* σ*-crystals defined by elements of the set*

$$\{(M_0, g\varphi_0, G_{W(k)}, (v_\alpha)_{\alpha\in\mathcal{J}}, \tilde{\psi})|g \in G^0_{W(k)}(W(k))\}.$$

Answer: *The set* $\mathrm{Lie}(\mathcal{S}_{(f,L_{(p)},v)})(k)$ *of isomorphism classes of Shimura Lie* σ*-crystals attached to* k*-valued points of* $\mathcal{N}$ *is precisely the set of isomorphism classes of Shimura Lie* σ*-crystals defined by elements of the set* $\{(\mathrm{Lie}(G_{W(k)}), g\varphi_0)|g \in G^0_{W(k)}(W(k))\}$.

The proof of these two statements is a direct consequence of 3.4.15, and of d) of 4.4.1 3) (resp. of c) of 4.4.1 2)) for the first statement (resp. for the second one), cf. also 4.4.12. As we postponed the proof of d) of 4.4.1 3), we invite the reader to formulate a version of the first statement, inspired from 3.4.15, for the k-valued points of any connected component $\mathcal{C}$ of $\mathcal{N}_k$ (in fact 4.2.8.1 takes care of all connected components of $\mathcal{N}_k$ at once). We can use 4.4.7 and 2.3.13.1: we get the above starred answer (for $\mathcal{C}$) without knowing that (M, F^1, φ) is the one introduced in 4.1.

4.12.6. Solution of 4.12.3 in the case the integral canonical model $\mathcal{N}^{\mathrm{ad}}$ **of the adjoint quadruple of** (G, X, H, v) **has the completion property.** We refer to 4.9.17.4 for the terminology. Let $G^{\mathrm{ad}}_{W(k)}$ be the reductive group whose special fibre is $G^{\mathrm{ad}}_{B(k)}$. Let v^{ad} be the prime of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ divided by v. Let $\mu^{\mathrm{ad}} : \mathbb{G}_m \to G^{\mathrm{ad}}_{W(k(v^{\mathrm{ad}}))}$ be a cocharacter which over an embedding of $W(k(v^{\mathrm{ad}}))$ in $\mathbb{C}$, extending the embedding $E(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \hookrightarrow \mathbb{C}$, is $G^{\mathrm{ad}}(\mathbb{C})$-conjugate to the cocharacters $\mu_x : \mathbb{G}_m \hookrightarrow G^{\mathrm{ad}}_{\mathbb{C}}$, with $x \in X^{\mathrm{ad}}$ defined in [Va2, 2.2]. We have:

Answer: *The set* $\mathcal{S}_{(G,X,H,v)}(k)$ *of isomorphism classes of Shimura adjoint Lie* σ*-crystals attached to* k*-valued points of* $\mathcal{N}$ *is precisely the set of isomorphism classes of Shimura adjoint Lie* σ*-crystals defined by elements of the set*

$$\{(\mathrm{Lie}(G^{\mathrm{ad}}_{W(k)}), g\sigma\mu^{\mathrm{ad}}(\frac{1}{p}))|g \in G^{\mathrm{ad}}_{W(k)}(W(k))\}.$$

To see this, we can assume that G^{ad} is a $\mathbb{Q}$–simple adjoint group. Let $o(G^{\mathrm{ad}})$ be the order of the center of the simply connected group cover of $G^{\mathrm{ad}}_{\mathbb{C}}$. The proof of the above statement in the case when p does not divide $o(G^{\mathrm{ad}})$ is a direct consequence of 4.9.17 and of b) of 4.4.1 2).

4.12.6.1. We would like to point out that $\mathcal{N}^{\mathrm{ad}}$ has the completion property, provided 3.4.15 is true for a certain standard Hodge situation, involving a Shimura quadruple having the same adjoint as (G, X, H, v). To see this, we can assume that G^{ad} is a $\mathbb{Q}$–simple adjoint group. Let $o(G^{\mathrm{ad}})$ be the order of the center of the simply connected group cover of $G^{\mathrm{ad}}_{\mathbb{C}}$. The proof of the above statement in the case when p does not divide $o(G^{\mathrm{ad}})$ is a direct consequence of 4.9.17 and of b) of 4.4.1 2) (if p does not divide $o(G^{\mathrm{ad}})$ then the natural homomorphism $G^{\mathrm{sc}}_{W(k)}(W(k)) \to G^{\mathrm{ad}}_{W(k)}(W(k))$ is surjective).

As we assumed $p \geq 3$, p divides $o(G^{\mathrm{ad}})$ only in the case when $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is of A_l Lie type, with p dividing $l+1$. But in such a case there is an injective map $f : (G_1, X_1) \hookrightarrow (\mathrm{GSp}(W, \psi), S)$, with $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) = (G^{\mathrm{ad}}, X^{\mathrm{ad}})$, and with G_1^{der} simply connected, such that we have a standard PEL situation $(f, L_{(p)}, v_1, \mathcal{B})$, with v_1 dividing v^{ad}, and with the homomorphism $G_{1W(k)}(W(k)) \to G^{\mathrm{ad}}_{W(k)}(W(k))$ surjective, cf. 2.3.5.1. The surjectivity parts results from the fact that for any simple factor of $G^{\mathrm{ad}}_{W(k)}$ there is a general linear group over $W(k)$ which is included in $G_{1W(k)}$ and which has the given factor as its adjoint group, cf. 2.3.5.1 and [Ko1, (A) of p. 395]. Again, directly from def. 4.9.17 we get the validity of the above statement in this case, provided $\mathcal{N}_1$ has the completion property.

4.12.7. The Manin problem for a SHS $(f, L_{(p)}, v)$. Classify all Shimura isocrystals (resp. all Shimura adjoint Lie isocrystals) over k defined by the Shimura σ-crystals (resp. by the Shimura adjoint Lie σ-crystals) attached to k-valued points of $\mathcal{N}$.

4.12.8. The Manin problem for a Shimura quadruple (G, X, H, v) of preabelian type, with v dividing a rational prime $p \geq 3$. Classify all adjoint Lie isocrystals over k defined by the Shimura adjoint Lie σ-crystals attached to k-valued points of the integral canonical model of (G, X, H, v).

4.12.9. Remarks. 1) To our knowledge the result (answer) of 4.12.4 has not been stated as such for Siegel modular varieties of dimension greater than 1. It says: any principally quasi-polarized σ-crystal (i.e. any principally quasi-polarized p-divisible group) over k, comes from a principally polarized abelian variety over k, and any principally quasi-polarized filtered σ-crystal (i.e. any principally quasi-polarized p-divisible group over $W(k)$), comes from a principally polarized abelian scheme over $W(k)$.

2) Of course, 4.12.5-6 provide a solution of the problems 4.12.7-8 (provided we assume the completion property). But it is very much desirable, to have a more concrete (and finite) answer to 4.12.7-8, similar to the answer of the (original) Manin problems, in terms of formal isogeny types. We will deal with this in §10.

4.12.10. Example. Let $n \geq m$ be natural numbers, and let $l := m + n$. Let $(f, L_{(p)}, v)$ be a SHS such that $G_{\mathbb{R}}^{\mathrm{der}} = SU(n, m)_{\mathbb{R}}$, $G_{\mathbb{Z}_p}^{\mathrm{der}} = SL(N)$, with N a free $\mathbb{Z}_p$-module of rank l, and $\mathcal{N}$ has the completion property. Then the set of isomorphism classes of Shimura adjoint Lie isocrystals over k defined by the Shimura adjoint Lie σ-crystals attached to k-valued points of $\mathcal{N}$, is the set obtained by taking $\mathfrak{sl}(*)$ of isomorphism classes of isocrystals $(\mathcal{I}, \varphi_{\mathcal{I}})$ over k having only slopes which are rational numbers of the interval $[0, 1]$, whose total sum is m, with $\mathcal{I}$ a free $B(k)$-vector space of dimension l.

For more examples see §10 and [Va4].

4.12.11. Problems. 1) Let (G, X, H_p, v) be a Shimura quadruple of preabelian type, with v dividing a rational prime $p \geq 3$. Let $k_1(v)$ be a finite field extension of $k(v)$, and let H_0 be a (compact open) subgroup of $G(\mathbb{A}_f^p)$, such that the subgroup $H_0 \times H_p$ of $G(\mathbb{A}_f)$ is smooth for the pair (G, X). Let σ_1 be the Frobenius automorphism of $W(k_1(v))$. Determine all Shimura adjoint Lie σ_1-crystals over $k_1(v)$ attached to $k_1(v)$-valued points of $\mathcal{N}/H_0$, where $\mathcal{N}$ is the integral canonical model of the quadruple (G, X, H_p, v). In

particular, determine conditions on H_0 and $k_1(v)$, which assures that $\mathcal{N}/H_0(k_1(v))$ is not empty.

2) In the above situation, for a fixed subgroup H_0 of $G(\mathbb{A}_f^p)$, determine the smallest algebraic (resp. finite) field extension $k_1(v)$ of $k(v)$ with the property that any isomorphism class of Shimura adjoint Lie σ-crystals (resp. isocrystals) over $\overline{k(v)}$ attached to $\overline{k(v)}$-valued points of $\mathcal{N}$, is defined (by passage to $\overline{k(v)}$) by the Shimura adjoint Lie σ-crystal (resp. isocrystal) attached to a $k_1(v)$-valued point of $\mathcal{N}/H_0$.

3) The same problems 1)-2) above, but for a SHS $(f, L_{(p)}, v)$.

4.12.11.1. Remark. For the problem 1) of 4.12.11, if there is a SHS $(f, L_{(p)}, v)$, with $H_p = G(L_{(p)} \otimes \mathbb{Z}_p)$, then for $k_1(v)$ big enough, the answer to the problem is of the same form as the one given in 4.12.6, but with k replaced by $k_1(v)$, and with g running through a subset $\mathcal{G}\big(k_1(v), \mathcal{N}/H_0\big)$ of $G_{W(k_1(v))}(W(k_1(v)))$. In §14 we will deal with the problem of determining this subset and how "big" $k_1(v)$ has to be.

4.12.12. Theorem. *Any p-divisible group D over k has a versal deformation to a p-divisible group over a smooth affine $W(k)$-scheme.*

Proof: Let (M, φ) be the σ-crystal defined by D, and let (M, F^1, φ) be the filtered σ-crystal defined by an arbitrary lift of D to a p-divisible group over $W(k)$. If F^1 is not a proper summand of M we have nothing to prove. So we can assume that we have a Shimura filtered σ-crystal $(M, F^1, \varphi, GL(M))$. Let $a := \dim_{W(k)}(F^1)$ and $b := \dim_{W(k)}(M/F^1)$. We have $ab \neq 0$. We can assume that $\varphi = g\varphi_0$, with $g \in SL(M)(W(k))$, and with φ_0 a Frobenius endomorphism of M such that $(M, F^1, \varphi_0, GL(M))$ is a $GL(M)$-canonical lift. So F^1 is left invariant by φ_0, and the slopes of the isocrystal defined by (M, φ_0) are 0 and 1 with multiplicities b and respecively a.

We consider a SHS $(f, L_{(p)}, v)$ such that:

a) $G_{\mathbb{R}}^{\mathrm{der}}$ is $SU(a, b)_{\mathbb{R}}$;

b) G splits over $\mathbb{Q}_p$; so $k(v) = \mathbb{F}_p$ and $G_{\mathbb{Z}_p}^{\mathrm{der}} = SL(\mathbb{Z}_p^{a+b})$;

c) G^{ab} is a one or two dimensional torus depending on the fact that a is equal to be b or not;

d) $E(G, X)$ is $\mathbb{Q}$ or a totally imaginary quadratic extension of $\mathbb{Q}$ depending on the fact that a is equal to be b or not;

e) the representation $G_{\mathbb{Z}_p} \to GL(L_{(p)}^* \otimes \mathbb{Z}_p)$ is a direct sum of two irreducible representations ρ_1 and ρ_2 which restricted to $G_{\mathbb{Z}_p}^{\mathrm{der}}$ are respectively the representations associated to the two minimal fundamental weights ω_1 and ω_{a+b-1} (we use the standard notations for weights); in other words we have a direct sum decomposition as $G_{\mathbb{Z}_p}$-modules $L_{(p)}^* \otimes \mathbb{Z}_p = \mathbb{Z}_p^{a+b} \oplus \mathbb{Z}_p^{a+b}$ such that $A \in SL(\mathbb{Z}_p^{a+b})(\mathbb{Z}_p)$ acts as A on the first summand and as the inverse of the transpose of A on the second direct summand.

It is well that the conditions a) to e) imply (for an elementary solution see [Ko, p. 392]) that (G, X) is of compact type.

The existence of such a SHS is an easy thing, being a simple property of unitary Shimura varieties (for the case $a = 1$ and $b = 2$ -or vice-verse- see [Go]). In fact we get

a standard PEL situation $(f, L_{(p)}, v, \mathcal{B})$, with $\mathcal{B}$ as the subalgebra of $\mathrm{End}(L_{(p)})$ formed by endomorphisms fixed by G. If $a \neq b$ then $\mathcal{B}$ is the localization of the ring of integers of $E(G, X)$ with respect to the ideal generated by p. So for $a \neq b$ we need a totally imaginary quadratic extension of $\mathbb{Q}$ which splits over p, together with a symplectic representation $\mathrm{Res}_{\mathcal{B}/\mathbb{Z}_{(p)}}\mathbb{G}_m \hookrightarrow \mathrm{GSp}(L_{(p)}, \psi)$ modelled on $SU(a, b)$. The case $a = b$ is even simpler, and so left as an exercise.

The p-divisible group over $\mathcal{N}_{W(k)}/H_0$ defined by $\mathcal{A}_{H_0}$ is a direct sum of two p-divisible groups $\mathcal{D}_1$ and $\mathcal{D}_2$, cf. e). In fact $\mathcal{D}_1 = \mathcal{D}_2^t$. Moreover $\mathcal{D}_1$ and $\mathcal{D}_2$ are versal deformations in each k-valued point of $\mathcal{N}_{W(k)}/H_0$. We can assume that $\mathcal{D}_1$ has dimension a in each k-valued point of $\mathcal{N}_{W(k)}/H_0$.

We check now that $\mathcal{N}$ has the completion property. We follow the idea of 4.12.4. The similar isogeny property can be easily checked in this context: with the notations of 4.12.4, if any invertible element of $\mathcal{B}$, when viewed naturally (via de Rham components) as an automorphism of M_0', is also an automorphism of M_0, then all these elements (assumed to define automorphisms of A') define automorphism of A', and so [Ko, ch. 5] applies. So we just need to check that for any cyclic diagonalizable Shimura σ-crystal $\mathcal{C}_N = (N, F_N^1, \varphi_N, GL(N))$, with N a free $W(k)$-module of rank $a + b$, and with F_N^1 a direct summand of it of rank b, there is a k-valued y_N point of $\mathcal{N}_{W(k)}/H_0$ such that the Shimura σ-crystal we get over k by pulling back $\mathcal{D}_1$ (see 2.2.9 9)), is isomorphic to $\mathcal{C}_N$.

To see this let T_N be a maximal torus of $GL(N)$ such that $(N, F_N^1, \varphi_N, T_N)$ is a Shimura σ-crystal (cf. 2.2.22). We consider the Galois representation associated to it; it can ve viewed as a homomorphism

$$\rho_N : \mathrm{Gal}(B(k)) \to \tilde{T}_N(\mathbb{Z}_p),$$

where $\tilde{T}_N$ is a maximal subtorus of the image of $G_{\mathbb{Z}_p}$ in the GL-group $GL(1)$ of the first $\mathbb{Z}_p^{a+b}$-summand of $L_{(p)}^* \otimes \mathbb{Z}_p$. Its extension to $W(\mathbb{F})$ is naturally equipped with a cocharacter μ_N. Let (cf. [Har]) T_0 be a maximal $\mathbb{Z}_{(p)}$-subtorus of $G_{\mathbb{Z}_{(p)}}$ such that:

– over $\mathbb{R}$ it is compact;

– its extension to $\mathbb{Z}_p$ has an image in $GL(1)$ which is $GL(1)(\mathbb{Z}_p)$-conjugate to $\tilde{T}_N$.

We get naturally (from μ_N) a cocharacter of it $\mu(T_0)$ over the maximal Galois cover of $\mathrm{Spec}(\mathbb{Z}_{(p)})$ having a field of fractions contained in $\mathbb{R}$. We get immediately a special quadruple $(T_{0\mathbb{Q}}, \{h_0\}, T_0(\mathbb{Z}_p), v_p)$, with $h_0 : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \to T_{0\mathbb{R}}$ defined by $\mu(T_0)$, and with v_p that prime of $E(T_0, \{h_0\})$ dividing v which has the property that the cocharacter of $T_{0\,W(k(v_p))}$ we naturally get (see 4.1) is modelled on μ_N. Now one can easily check that any k-valued point of $\mathcal{N}_{W(k)}/H_0$ factoring through the integral canonical model of this special quadruple, can be chosen as the desired y_N point.

Now the theorem is a consequence of 4.12.5: there is $y \in \mathcal{N}_{W(k)}/H_0$ such that $\mathcal{D}_1$ over y is isomorphic to D. As $k(v) = \mathbb{F}_p$, to get 4.12.5 for our SHS $(f, L_{(p)}, v)$ we can use 4.6.12 instead of d) of 4.4.1 3) . This ends the proof of the theorem.

4.12.12.0. Remark. In the above proof of 4.12.12 we can substitute arguments on unitary Shimura varieties by 2.3.5.1: what is important that over $\mathbb{Z}_p$ we get a right

direct sum decomposition into irreducible representations so that "we can single" out a p-divisible group $\mathcal{D}_1$ as above which is "related" to D. In fact we do not need a direct sum decomposition: we just need an adequate irreducible subrepresentation. So e) above can be replaced by more general versions, inspired from [Va2, 6.5-6].

We use this idea to show that 4.12.12 remains true for any Shimura p-divisible group (M, φ, G): for the case when G together with its filtration class is related to a Shimura variety of preabelian type see §7; for the case when the situation is related to Shimura varieties of special type see [Va6]. Moreover it seems to us that the condition $k = \bar{k}$ is not (really) needed: in other words, (in many situations) it is enough to work with a perfect field (as the combination of 3.15 and 4.12.12 suggests).

4.12.12.1. Remark. The natural question arises: what about the case $p = 2$? Defining the completion property for $p = 2$ as in 3.4.16.1 and 4.9.17.4, we have:

Answer: *4.12.5-6 remain true for the case of a $p = 2$ SHS $(f, L_{(2)}, v)$ whose integral canonical model $\mathcal{N}$ has the completion property.*

Moreover, for 4.12.6 it is enough to assume that $\mathcal{N}$ has the completion property. With the notations of 4.12.6, we just need to check that in 4.12.6 we can still use any element of $G^{\mathrm{ad}}_{W(k)}(W(k))$ for our $p = 2$ SHS $(f, L_{(2)}, v)$. This goes as follows. For any adjoint simple factor $\tilde{G}$ of $G^{\mathrm{ad}}_{\mathbb{Z}_p}$, there is a semisimple subgroup $\tilde{G}^1$ of $G_{\mathbb{Z}_p}$ such that we have a central isogeny $\tilde{G}^1 \to \tilde{G}$. If $\tilde{T}$ is a maximal torus of $\tilde{G}$, then $\tilde{G}(W(k)) = \tilde{G}^1(W(k))\tilde{T}(W(k))$; so the natural variant of 3.11.1 (see also 3.6.5) involving adjoint Lie σ-crystals applies to give us: with the notations of 4.12.6, any Shimura adjoint Lie σ-crystal $(\mathrm{Lie}(G^{\mathrm{ad}}_{W(k)}, g\sigma\mu^{\mathrm{ad}}(\frac{1}{p}))$, with $g \in G^{\mathrm{ad}}_{W(k)}(W(k))$ is isomorphic to one defined similarly, but for some $g^1 \in G^{\mathrm{der}}_{W(k)}(W(k))$.

From 4.12.12 and 2.3.18 we get directly:

4.12.12.2. Corollary. *4.12.12 remains true for $p = 2$: any 2-divisible group over an algebraically closed field $\tilde{k}$ of characteristic 2 has a versal deformation (to a 2-divisible group) over a smooth affine $W(\tilde{k})$-scheme.*

4.12.12.3. Tate deformation algebras and Shimura p-divisible groups. Let $\tilde{k}$ be a perfect field of characteristic $p \geq 2$. Let $\tilde{\sigma}$ be the Frobenius automorphism of $W(\tilde{k})$ and $\tilde{k}$. Let $(\tilde{M}, \tilde{\varphi}, \tilde{G})$ be the not necessary quasi-split Shimura $\tilde{\sigma}$-crystal associated to a Shimura p-divisible group $\tilde{D}$ over $\tilde{k}$. Let $\tilde{d} := dd(\tilde{M}, \tilde{\varphi}, \tilde{G}))$. Let $\tilde{R}$ be the p-adic completion of the henselization $\tilde{R}(h)$ of $W(\tilde{k})[x_1, ..., x_{\tilde{d}}]$. There is (cf. Fact $1'$ of 3.15) a versal Shimura p-divisible group $D_{\mathrm{def}}(\tilde{D})$ over $\mathrm{Spec}(\tilde{R})$, which in the $\tilde{k}$-valued point $\tilde{y}$ of $\mathrm{Spec}(\tilde{R})$ it becomes $\tilde{D}$. Warning: $D_{\mathrm{def}}(\tilde{D})$ is not uniquely determined; here unicity is up to isomorphism of $D_{\mathrm{def}}(\tilde{D})$ and of $\tilde{R}$.

We refer to $D_{\mathrm{def}}(\tilde{D})$ as a Tate deformation Shimura p-divisible group of $\tilde{D}$. Writing $D_{\mathrm{def}}[p^n] = \mathrm{Spec}(R_n(\tilde{D}))$, we refer to $R_n(\tilde{D})$ as a level-n Tate deformation algebra of $\tilde{D}$ or of $\tilde{D}[p^n]$. We think it is an interesting problem to study (classify) these finite flat $\tilde{R}$-algebras. We have a similar problem (much harder) to understand (classify) the similarly defined $\tilde{R}(h)$-algebras, in the case $D_{\mathrm{def}}(\tilde{D})$ is definable over $\tilde{R}(h)$

4.12.12.4. Exercise. Use the proof of 4.12.12 to get a new proof of the original Manin problem, and so of 4.12.4. Hint: using products of principally quasi-polarized

abelian varieties, reduce to the case when we have only the slopes α and $1-\alpha$, with $\alpha \in [0, \frac{1}{2}]$; writing $\alpha = \frac{a}{a+b}$, with $(a,b)=1$, $a \in \mathbb{N} \cup \{0\}$, $b \in \mathbb{N}$, $d \geq b \geq a$, make use of the SHS mentioned in the proof of 4.12.12.

4.12.13. Remark. 3.15 (or 3.4.7.1 and 3.11.1) can be interpreted as the solution of an integral Manin problem: Any Shimura filtered σ-crystal $(M, F^1, \varphi, \tilde{G})$ over k can be connected (through the p-adic completion of an $\mathbb{N}$-pro-étale scheme over a smooth scheme over $W(k)$, having a connected special fibre) with a G-canonical lift $(M, F^1, \varphi_1, \tilde{G})$.

4.12.14. The general local integral Manin problem for Siegel modular varieties. We consider a SHS of the form $(1_{(\mathrm{GSp}(W,\psi),S)}, L_{(p)}, p)$. So, with the standard notations, $\mathcal{M} = \mathcal{N}$. Let $m \in \mathbb{N} \cup \{0\}$. Let $R := W(k)[[x_1, ..., x_m]]$ and let Φ_R be a Frobenius of it compatible with σ. The problem is to determine all principally quasi-polarized p-divisible objects of $\mathcal{MF}_{[0,1]}(R)$ defined by principally quasi-polarized p-divisible groups over R, obtained from $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ through morphisms $\mathrm{Spec}(R) \to \mathcal{N}$.

4.12.15. The general local integral Manin problem for a SHS $(f, L_{(p)}, v)$. Let $(f, L_{(p)}, v)$ be a SHS. The problem is the same: which principally quasi-polarized p-divisible objects with cycles of $\mathcal{MF}_{[0,1]}(R)$ are obtained (from $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ and the Hodge cycles $(w_\alpha^{\mathcal{A}})_{\alpha \in \mathcal{J}'}$) through morphisms $\mathrm{Spec}(R) \to \mathcal{N}$?

4.12.16. The general local integral Manin problem for a Shimura quadruple (G, X, H, v) of preabelian type, with $(v, 2) = 1$. The problem is to determine which p-divisible objects of $\mathcal{MF}_{[-1,1]}(R)$ are isomorphic to Shimura adjoint filtered Lie F-crystals over R/pR attached to (cf. 4.9.20) morphisms $\mathrm{Spec}(R) \to \mathcal{N}$.

4.12.17. The solution of 4.12.14. The answer is: any principally quasi-polarized p-divisible object of $\mathcal{MF}_{[0,1]}(R)$. This is an easy consequence of 4.12.4, [Va2, 5.4] and 3.4.18.7.

4.12.18. The solution of 4.12.15 in the case $\mathcal{N}$ has the completion property. We use the notations of 4.12.5. The answer is: any such p-divisible object of the form $(M_0 \otimes R, F_0^1 \otimes R, g_R(\varphi_0 \otimes 1), (v_\alpha)_{\alpha \in \mathcal{J}}, \tilde{\psi})$, with g_R an arbitrary element of $G^0_{\mathbb{Z}_{(p)}}(R)$. This is an easy consequence of 4.12.5, [Va2, 5.4] and 3.4.18.7.

4.12.19. The solution of 4.12.16 in the context of 4.12.6. We use the notations of 4.12.6. The answer is: any such p-divisible object of the form

$$(\mathrm{Lie}(G_R^{\mathrm{ad}}), g_R^{\mathrm{ad}}(\sigma \mu^{\mathrm{ad}}(\frac{1}{p}) \otimes 1), F^0(\mathrm{Lie}(G_R^{\mathrm{ad}})), F^1(\mathrm{Lie}(G_R^{\mathrm{ad}}))),$$

with $g_R^{\mathrm{ad}} \in G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}(R)$. Here the filtrations F^0 and F^1 are obtained as usual from the cocharacter μ^{ad} of 4.12.6. This is an easy consequence of 4.12.6, of 3.4.18.7, of def. 4.9.17, of [Va2, 5.4] and of 2.3.17.

4.12.16-18 admit a non-filtered version. Not to be too long this is not going to be presented here.

4.12.20. Exercise. State and prove the analogues of 4.12.14-15 and 4.12.17-18 for the case when $p = 2$. Hint: use 2.3.18 and 4.12.12.1.

4.12.21. The general (integral) Manin problems. Given an arbitrary $\mathbb{Z}_{(p)}$-scheme Y we would like to understand which principally quasi-polarized p-divisible groups (resp. which isogeny classes of principally quasi-polarized p-divisible groups) over Y are coming from principally polarized abelian schemes (resp. from isogeny classes of principally polarized abelian schemes) over Y of relative dimension e. This forms the general integral Manin problem (resp. the general Manin problem) for the Siegel modular variety of 4.12.4. Similarly we can define the general (integral) Manin problem for an arbitrary SHS $(f, L_{(p)}, v)$: we need to replace p-divisible groups by Shimura p-divisible groups (see 2.2.20; the definition there can be adapted to any $O_{(v)}$-scheme). We can also define the general (integral) Manin problem in the context of 4.12.3: again 4.9.20 can be extended to any $O_{(v)}$-scheme (see §5 for details).

The general (integral) Manin problems look very hard in such a generality. In the cases when we are dealing with (the p-adic completion of) a (regular formally) smooth separated scheme Y (over DVR's of index of ramification 1 over $\mathbb{Z}_{(p)}$ or $O_{(v)}$) we hope that some partial solutions can be obtained based on 3.4.

4.13. The local form of the invariance principle. Let $p \geq 3$ be a rational prime, and let k be a perfect field of characteristic p. Let (G_i, X_i, H_i, v_i), $i = \overline{1,2}$, be two Shimura quadruples of preabelian type, with v_i dividing p. Let $\mathcal{N}_i$ be the integral canonical model of (G_i, X_i, H_i, v_i), $i = \overline{1,2}$. Let $H_{0i} \subset G_i(\mathbb{A}_f^p)$ be a subgroup such that $H_{0i} \times H_i$ is a subgroup of $G_i(\mathbb{A}_f)$ smooth for (G_i, X_i), $i = \overline{1,2}$. Let $y_i : \mathrm{Spec}(k) \to \mathcal{N}_{ik}/H_{0i}$, $i = \overline{1,2}$, such that the non-trivial part of the Shimura adjoint Lie σ-crystal attached to y_1 is isomorphic to the non-trivial part of the Shimura adjoint Lie σ-crystal attached to y_2. We fix such an isomorphism $\mathcal{I}$. Let O_{y_i} be the local ring of y_i (in $\mathcal{N}_{iW(k)}/H_{0i}$), and let $O_{y_i}^{h\wedge}$ be the p-adic completion of its henselization, $i = \overline{1,2}$. For simplicity we assume that $k = \bar{k}$. From 4.9.20 we deduce the existence of a Shimura adjoint filtered Lie crystal $\mathfrak{C}_{y_i}^{\mathrm{ad}}$ over $O(p)_{y_i}^h := O_{y_i}^{h\wedge}/pO_{y_i}^{h\wedge}$. Let $\mathfrak{C}_{y_i}^{\mathrm{nt}}$ be its non-trivial part (it is defined as in 3.10.1). The refined canonical Lie stratification of $\mathcal{N}_{ik(v_i)}$ makes $\mathrm{Spec}(O(p)_{y_i}^h)$ to be a stratified scheme.

Let $n \in \mathbb{N}$.

4.13.1. Theorem. *There is an isomorphism*

$$\rho : \mathrm{Spec}(O_{y_1}^{h\wedge}) \tilde{\to} \mathrm{Spec}(O_{y_2}^{h\wedge})$$

inducing an isomorphism

$$\rho(p) : \mathrm{Spec}(O(p)_{y_1}^h) \tilde{\to} \mathrm{Spec}(O(p)_{y_2}^h)$$

of stratified schemes, and such that $\rho^*(\mathfrak{C}_{y_2}^{\mathrm{nt}}/p^n\mathfrak{C}_{y_2}^{\mathrm{nt}})$ is isomorphic to $\mathfrak{C}_{y_1}^{\mathrm{nt}}/p^n\mathfrak{C}_{y_1}^{\mathrm{nt}}$, through an isomorphism lifting $\mathcal{I}$ mod p^n.

4.13.2. Theorem. *Let $(f^i, L_{(p)}^i, v^i)$, $i = \overline{1,2}$, be two standard Hodge situations. Using the standard notations of 2.3.1-3 for the SHS $(f^i, L_{(p)}^i, v^i)$, but putting everywhere an upper right index i, let $y_i : \mathrm{Spec}(k) \to \mathcal{N}_k^i/H_0^i$ be a morphism, $i = \overline{1,2}$. Here k is an arbitrary perfect field containing $k(v^1)$ and $k(v^2)$. We assume that the principally*

quasi-polarized Shimura σ-crystal attached to y_1 is isomorphic to the principally quasi-polarized Shimura σ-crystal attached to y_2. We fix such an isomorphism $\mathcal{I}$. Then there is an isomorphism

$$\rho : \mathrm{Spec}(O_{y_1}^{h\wedge}) \tilde{\to} \mathrm{Spec}(O_{y_2}^{h\wedge})$$

inducing an isomorphism

$$\rho(p) : \mathrm{Spec}(O(p)_{y_1}^{h}) \tilde{\to} \mathrm{Spec}(O(p)_{y_2}^{h})$$

of stratified schemes, and such that the pull back under ρ, of the kernel of the multiplication by p^n of the principally quasi-polarized Shimura p-divisible group $\mathcal{D}_{y_2}$ over $\mathrm{Spec}(O_{y_2}^h)$ obtained from the principally quasi-polarized Shimura p-divisible group over $\mathcal{N}^2/H_0^2$ of the principally polarized abelian scheme $(\mathcal{A}^2_{H_0^2}, \mathcal{P}_{\mathcal{A}^2_{H_0^2}})$ by pull back under the canonical morphism $\mathrm{Spec}(O_{y_2}^h) \to \mathcal{N}^2/H_0^2$, is isomorphic to the kernel of the multiplication by p^n of the principally quasi-polarized Shimura p-divisible group $\mathcal{D}_{y_1}$ over $\mathrm{Spec}(O_{y_1}^h)$, defined similarly as $\mathcal{D}_{y_2}$ (here $O_{y_i}^h$, $O(p)_{y_i}^h$ have the same meaning as in 4.13.1), through an isomorphism lifting $\mathcal{I}$ mod p^n.

4.13.1 is a consequence of 3.4.14.4; for the stratification part, we need to take n big enough and to apply [Ka2, 1.4.4]. 4.13.2 is a consequence of 3.4.14.4 (or of [Va2, 5.4], via Artin's approximation theorem); for both theorems cf. definitions 4.9.17 and 4.9.20 (in the case of a SHS, 4.9.20 is trivial, just introducing some terminology) (in the case of 4.13.2 cf. 2.2.21 9) as well).

4.13.3. Remarks. 1) The form of 4.13.2, where instead of taking henselization we take the completion, was obtained previously: the result of Serre-Tate that (in many situations) deforming (locally) an abelian variety is the same as deforming its p-divisible group implies this form for a standard PEL situation; for the general case cf. [Va2, 5.4].

2) The condition $k = \bar{k}$ in 4.13.1 can be weaken: see §5.

4.14. Final remarks. Here we include some remarks which are sort of overview of great parts of [Va2] and of this paper.

4.14.1. The generic situation. Many of the results presented here or in [Va2] are trivial or very easy, from the generic point of view, i.e. working with a prime p big enough. But the main inconvenience of this simple approach is that we do not have at all effective bounds (of how big p has to be), and so, from the point of view of concreteness, is void. We list now some of the results which generically "are easy", and motivate why this is so.

a) [Va2, 3.4.1 and 3.4.7] is all we need to get generically the existence of integral canonical models of Shimura varieties of Hodge type.

b) [Va2, 6.2.2 a), 6.2.3 and 6.8] is all we need to extend this to Shimura varieties of preabelian type.

c) [Va2, 6.4.4] generically is obvious.

d) The density part of 4.2.1 is obvious generically (due to the existence of smooth toroidal compactifications of canonical models of Shimura varieties and of c) above).

e) Using [Va2, 3.2.11] and d) above we get immediately 4.2.1.

f) By the same reason as in e) above, P1 of 4.6 is obvious generically.

The list can keep going on.

4.14.2. The special type case. There are plenty of Shimura σ-crystals of (so called) special type. So, even before the prove of the existence of integral canonical models of Shimura varieties of special type and of the validity of the expectation of [Va2, a) of 3.2.7 8)], we can speak about stratifications (as in 4.5) of schemes of moduli of Shimura p-divisible groups obtained from Shimura σ-crystals of special type, cf. 3.4.1.3 (and §7). For the study of these stratifications see [Va6].

4.14.3. The case of a $p = 2$ SHS. From 2.3.18 and 3.14 we get immediately that all the results of 4.1-8 and 4.11, except (perhaps) d) of 4.4.1 3), remain true for a $p = 2$ SHS $(f, L_{(2)}, v)$ (see also [Va5]). Warning: the unpublished work of A. Ogus on ordinary elliptic curves implies that in 4.7.30 8), with $p = 2$, the Frobenius of R_n which makes g_{R_n} to be the identity element, is not unique. The only extra thing we would like to point out is: the G-canonical lifts are defined at the level of filtrations (cf. 4.4.1); so, as $p = 2$, for their uniqueness we have to make use of the existence of G-crystalline (additive or multiplicative) coordinates (this is the same as getting the uniqueness of the canonical lift of an ordinary abelian variety over a perfect field of characteristic 2). Also part of 4.9-10 can be adapted to this situation: as we do not know if $\mathcal{N}$ is uniquely determined, we have to work out 4.9.8 and 4.10 using just $\mathcal{N}$ (and so we lose the functorial aspect of 4.9.8). Related to 4.12 see 4.12.12.1 and 4.12.20 1). Moreover the whole of 4.13 remains true (cf. 3.14).

Appendix

It is desirable to make the connection between the present work and [RR]. There are three results here related to [RR]. They were obtained independently, and their main part precedes [RR], cf. [Va1] and 1.25. These results are:

A) The part of 3.1.0 expressed in rm. 2) of 3.4.6.1. This is related to [RR, ch. 4]. The loc. cit. obtains rm. 2) of 3.4.6.1 in a more general context. However our ideas can be adapted to this more general context (see below).

B) The proof of b) of 4.4.1 2). Its idea is the same as in the general result [RR, 1.16].

C) 4.5.4 and 4.5.6.1. The idea of 4.5.4 is the same underlying [RR, def. 3.3]. The proposition 4.5.6.1 is related to the main result 3.6 of [RR]. It can be used to restate this main result in a simpler way, provided we restrict the class of groups. We use the notations of [RR, ch. 3]. We have:

Criterion. *If all the factors of G^{ad} are of A_n, B_n, C_n or D_n Lie type ($n \in \mathbb{N}$), then for two points s_1, s_2 of S, we have $\bar{b}(s_1) < \bar{b}(s_2)$, iff the Newton polygon of the Lie isocrystal over $\bar{s}_1$ (defined by* Lie(G)*, or in the sense of [RR, 3.3], corresponding to the adjoint representation of G on its Lie algebra) is above the Lie isocrystal over $\bar{s}_2$.*

The proof of this is entirely the same as of 4.5.6.1. We do not know if (or when) this criterion is true without the assumption on the Lie type of the simple factors of G^{ad}. In practice it is easier to work with the above criterion than with the partial ordering on $B(G)$ defined in [RR, 2.3]. However [RR, 2.4] is a very powerful theoretical tool (for instance, the proof of [RR, 4.2] is very elegant).

As a conclusion: the present work is integral, and so more refined (handling the deformation aspects and subtler things mod p, like 3.9) and more restricted (we work in a context of good reduction and of p-divisible groups), while [RR] works rationally, and so less refined, but more general. The greatest part of the ideas and results presented in 3.4 can be used to more general contexts. There are four steps in achieving this. Let us start with an arbitrary filtered σ-$\mathcal{S}$-crystal $\mathcal{C} = (\tilde{M}, (\tilde{F}^i_{i \in S(\tilde{a},\tilde{b})}), \tilde{\varphi}, \tilde{G})$. We do not assume that it satisfies the W-condition, or that $\tilde{G}$ is reductive group (see 3.4.1.5).

1) If $\mathcal{C}$ satisfies the W-condition and $\tilde{G}$ is a quasi-split reductive group, than 3.2.3 and 3.4.6 remain valid, with no essential change. So the obstructions of proving 3.1, 3.4.1.2-3, and 3.5-6 in this context come just from 3.8.9 (cf. the discussion in 3.4.1.6) and 3.6.6. The obstruction coming from 3.6.6 is not a very serious one (as we expect 3.6.6 to be a known result), as it can be substituted (cf. 3.4.6.0) by the expectation of 3.4.1.6. So the serious obstruction is in achieving global deformations of $\mathcal{C}$. We hope to come back to this in a future paper.

2) If $\mathcal{C}$ satisfies the W-condition and $\tilde{G}$ is a reductive group which is not quasi-split, then (provided we achieved the global deformations in 1) above) 3.8 applies.

3) If $\mathcal{C}$ satisfies the W-condition, but $\tilde{G}$ is not a reductive group, we have to proceed as follows. For the problems of Newton polygons (cf. [Ka2, 1.4.4]) it matters only how $\tilde{G}^{\wedge}$ looks like. This reduces easily the problem to the case when $\tilde{G}$ is the extention of a reductive group by a nilpotent group. So to work out 3.5-6 and 3.4.6 in this situation we have to combine the case of a reductive with the case of a nilpotent group. How this is accomplished is explained in [Va5] (cf. 4.6.8 2)).

4) The case when $\mathcal{C}$ does not satisfy the W-condition (we recall that the main result of [Wi] is obtained under some conditions). This is an obstacle in getting 3.4.6 and 3.5-6 in this context. Still 3.1.0 should be provable in this context, using global deformations. We hope to come back to this step in a future paper.

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