Points of Integral Canonical Models of Shimura Varieties of Preabelian Type, p-divisible Groups, and Applications

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part 2A out of three, first sketched version, Dec. 1997
first version, July 28 1999, Univ. of Utah, submitted for publication in Duke Math. J. on 9/21/99

ABSTRACT. We prove a (motivic) conjecture of Milne relating the étale cohomology with $\mathbb{Z}_p$ coefficients to the de Rham cohomology with integral coefficients, in the larger context of Shimura p-divisible groups, and list some of its immediate applications complementing a result of Faltings.

Key words: cohomologies with integral coefficients, Shimura varieties of Hodge type, abelian varieties and p-divisible groups.
Math. Subject Classification 2000: Primary 14F30, 14G35, 11S25, 11G10 and 11G18.

Contents

§1. Introduction

1.0. The purpose of this second part (which is a continuation of [Va2]) is double. First we aim to continue the programme of proving that many well known results for integral canonical models of Siegel modular varieties remain true (under proper formulation) for the integral canonical models of Shimura varieties of preabelian type. The part of this programme presented here consists of:

a) the proof of the existence of automorphic vector bundles on the integral canonical models of Shimura varieties of preabelian type;
b) the proof of the existence of integral canonical models of Shimura varieties of preabelian type with respect to primes dividing $p = 3$;

c) the generalization of the supersingular stratum of the special fibre of an integral canonical model of a Siegel modular variety, to the case of the special fibre of an integral canonical model of a Shimura variety of preabelian type (we obtain the null stratum or the G-supersingular stratum);
d) the (partial) results pertaining to these null strata similar to the results true for the supersingular strata.

Second we aim at some applications of the technniques and results presented in [Va1-2]. These applications lead to:

1) the introduction of new mathematical concepts (like Shimura p-divisible groups over non-formally smooth schemes over Witt rings of perfect fields, like twisted Shimura varieties);
2) the obtaining of some new properties of the integral canonical models of Shimura varieties of Hodge type (which are obvious in the case of Siegel modular varieties) as: the Milne's conjecture stated in [Va1, 5.6.6], the integrality theorem of Shimura varieties of Hodge type (cf. [Va2, 1.16]), and the fact that for any injective map $f: (G, X) \hookrightarrow (GSp(W, \psi), S)$ of Shimura pairs (cf. [Va1, 2.4]) (with (W, ψ) a symplectic space over $\mathbb{Q}$, and with $(GSp(W, \psi), S)$ a Shimura pair defining a Siegel modular variety, cf. [Va1, example 2 of 2.5]), any $\mathbb{Z}_{(p)}$-lattive $L_{(p)}$ of W (with $p \geq 3$ a rational prime, and with $\mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ with respect to p) good with respect to f (i.e. we get a perfect form $\psi: L_{(p)} \otimes L_{(p)} \to \mathbb{Z}_{(p)}$, and the closure of G in $GSp(L_{(p)}, \psi)$ is a reductive group over $\mathbb{Z}_{(p)}$) is automatically crystalline well positioned for the map f (see [Va2, 2.3.4] for the meaning of this);
3) the obtaining of some results for the integral canonical models of Shimura varieties of Hodge type which were not previously known even in the case of Siegel modular varieties (like: the folklor conjecture expressed in [Ch, end of $(Q1)_D$], and the existence of some integral canonical models of quotients of Shimura varieties by non-hyperspecial subgroups);
4) a stronger approach to the Mumford-Tate conjecture than the one in [Va3].

1.1. Conventions and notations. In the whole of the paper we use the notations and conventions of [Va2, 2.1 and 2.3.7]. Whenever we consider a standard Hodge situation (standard abbreviation: SHS) $(f, L_{(p)}, v)$ (cf. [Va2, 2.3]), we use freely the notations introduced in [Va2, 2.3.1-3]. Always by k we denote a perfect field of positive characteristic p. If not specifically stated otherwise, we assume that $p > 2$. $W(k)$ is the Witt ring of k, and σ is the Frobenius automorphism of $W(k)$. For F a field we denote by $\bar{F}$ its algebraic closure. The dual of a free module M of finite rank over a ring is denoted by M^*.

1.2. Contrary to the statements in [Va2], it has been felt desirable to split the second part in six smaller parts to be referred as: part 2A, part 2B,..., part 2F. Parts 2A and 2B will present [Va2, §5] (cf. [Va2, 1.16]), part 2C will present [Va2, §6] (cf. [Va2, 1.17]), etc. The first one (part 2A) is presented here. So here we prove the results announced in the first three paragraphs of [Va2, 1.16].

1.3. In §2 we elaborate on the notion (and properties) of Shimura p-divisible groups introduced in [Va2, 2.2.20]. In loc. cit. we worked over regular formally smooth $W(k)$-schemes. Here we also work over other basis like (the spectrum of) V, Re, or $\widetilde{Re}$, with V

a discrete valuation ring which is a finite flat extension of $W(k)$, with $e = [V : W(k)]$, and with Re and $\widetilde{Re}$ as the piano rings of k (so they are obtained as in [Va1, 5.2.1], but working with the field k instead of $\mathbb{F}$), see [Va3, 2.1]. Roughly speaking a Shimura p-divisible group over such a bases is nothing else but a p-divisible group endowed with a family of de Rham (crystalline) cycles, satisfying some axioms inspired from the properties emerging in the similar context of a SHS.

1.4. In §3-4 we prove two main results pertaining to Shimura p-divisible groups: Milne's conjecture (3.1-2) and the integrality and non-vanishing principle (4.1). In the context of Shimura varieties of Hodge type these two results are stated in weaker forms in [Va1, 5.6.6] (see also [Va2, d) of 4.4.1 3)]) and respectively (partially stated) in [Va2, 1.16]; for their statements in full form see V3 of 3.2 and 4.1.2 (see also 1.4.1 below).

There are many variants of the original Milne's conjecture of [Mi1, 0.1] (see [Va3, 2.2] and 3.2). [Va1, 5.6.6] is a slight restatement of the original Milne's conjecture.

The proof of Milne's conjecture for Shimura p-divisible groups over discrete valuation rings (see 3.1) is a consequence of the fact that a Shimura filtered σ-crystal $(M, F^1, \phi, G_{W(k)})$ (with $k = \bar{k}$ and with M a free $W(k)$-module of finite rank) defined by the $G_{W(k)}$-canonical lift of a $G_{W(k)}$-ordinary σ-crystal (see [Va2] for definitions) is cyclic diagonalizable (cf. [Va2, 3.11.1]).

In other words, using the theory of $G_{W(k)}$-canonical lifts of [Va2, 3.1] and the theory of global deformations of [Va2, 3.4] we reduce the study of the conjecture to the case when $G_{W(k)}$ is a torus T (see [Va2, Fact' of 3.15] for a "sort of" combined version of these two theories). But in such a simple case the integral version of Fontaine's theory (see [Fa2]) can be analysed straightforward (cf. the proof of [Fa2, th. 5]), so that the standard ideas of [Va3, 4.3.4.8 b)] and of [Va2, 4.1.2] apply directly.

1.4.1. The integrality and non-vanishing principle for Shimura p-divisible groups is a trivial corollary of Milne's conjecture. In the context of a SHS, it becomes the integrality and non-vanishing principle (supplemented by the non zero specialization property) for Shimura varieties of Hodge type. So we have (see 4.1.2 and 4.1.2.1):

Theorem. *Let $(f, L_{(p)}, v)$ be a SHS. Let v_α be a tensor of the tensor algebra of $W \oplus W^*$ fixed by G.*

a) If for a morphism $z\colon Spec(D) \to \mathcal{N}$ (with D a local integral faithfully flat $\mathbb{Z}_{(p)}$-algebra), the Hodge cycle w_α with which $A_D := \mathcal{A} \times_{\mathcal{N}} D$ is naturally endowed (cf. [Va2, 2.3.3]) has a de Rham component t_α^D which is integral (i.e. is an element of the tensor algebra of $H^1_{dR}(A_D/D) \oplus H^1_{dR}(A_D/D)^*$) (resp. is integral and non zero modulo the maximal ideal of D), then (with logical notations) for any other morphism $z_R\colon Spec(R) \to \mathcal{N}$ (with R a local integral faithfully flat $\mathbb{Z}_{(p)}$-algebra), t_α^R is integral (resp. is integral and non zero modulo the maximal ideal of R).

b) For such a morphism z, t_α^D is integral (resp. is integral and non zero modulo the maximal ideal of D), iff v_α is integral (resp. is integral and non zero mod p).

1.4.2. The integrality and non-vanishing principle for Shimura p-divisible groups complements [Fa2, cor. 9] (see 4.2 for a detail discussion).

1.5. We would like to thank UC at Berkeley and University of Utah for providing us with very good conditions for the writing of the paper. We are over and over oblidged to [Mi1, 0.1]: in our work on integral aspects of Shimura varieties it has been by far the most inspiring moment (thing). This research was partially supported by the NSF grant DMF 97-05376.

§2. Preliminaries

2.1. Shimura p-divisible groups. Let k be a perfect field of arbitrary characteristic $p>0$. The notion of Shimura p-divisible groups over regular formally smooth $W(k)$-schemes was introduced in [Va2, 2.2.20]. In [Va2, 4.12.21] we hinted to a more general definition, where we have as a base any $\mathbb{Z}_{(p)}$-scheme. Here we start making this extension of [Va2, 2.2.20] very precise in the cases needed in §3-4. So we restrict mostly to the affine case, as the phenomenon of Milne's conjecture (of 1.4) is a local one.

2.1.1. Definitions. a) Let $(M,\phi,G,(t_\alpha)_{\alpha\in\mathcal{J}})$ be a Shimura σ-crystal over k (cf. [Va2, 2.2.8]). Let D be a p-divisible group over k whose attached σ-crystal is (M,ϕ). The pair $(D,(t_\alpha)_{\alpha\in\mathcal{J}}))$ will be referred as a Shimura p-divisible group over k.

b) Let $Y=\mathrm{Spec}(R)$ be a complete, local, faithfully flat $W(k)$-scheme whose residue field is k. A Shimura p-divisible group over Y is nothing else but the pull back $(\mathcal{D}_Y,(t_\alpha^Y)_{\alpha\in\mathcal{J}})$ of a Shimura p-divisible group $(\mathcal{D},(t_\alpha)_{\alpha\in\mathcal{J}}))$ over $\mathrm{Spec}(W(k)[[x_1,...,x_d]])$, with $d\in\mathbb{N}\cup\{0\}$, under a morphism

$$Y\to\mathrm{Spec}(W(k)[[x_1,...,x_d]]).$$

So t_α^Y is a homogeneous tensor of the F^0-filtration of $\mathcal{T}(H^1_{dR}(\mathcal{D}_Y)\left[\frac{1}{p}\right])$, where $H^1_{dR}(\mathcal{D}_Y)$ is the free filtered R-module of rank equal to the height of $\mathcal{D}$, obtained by pulling back the free filtered $W(k)[[x_1,...,x_d]]$-module underlying (via the first crystalline cohomology group functor) the filtered F-crystal over $k[[x_1,...,x_d]]$ defined by $\mathcal{D}$ (cf. [Me]).

In practice $Y=\mathrm{Spec}(V)$, with V a finite, totally ramified, faithfully flat, DVR extension of $W(k)$, or $Y=\mathrm{Spec}(V/m_V^n)$, with m_V as the maximal ideal of V and $n\in\mathbb{N}$.

c) We assume now that $p>2$. Let $e\in\mathbb{N}$. Let $\widetilde{Re}$ (resp. Re) be the piano ring of k of resonance e (resp. of convergent resonance e); we identify them with $W(k)$-subalgebras of $B(k)[[T]]$ containing $W(k)[[T]]$, cf. [Va3, 2.1] (here T is an independent variable). They are endowed naturally with a Frobenius endomorphism, cf. loc. cit. Let Y be $\mathrm{Spec}(Re)$ or $\mathrm{Spec}(\widetilde{Re})$. Let D be a p-divisible group over Y. Let (M,F^1,Φ_M,∇) be the filtered Frobenius crystal over Y obtained by taking the dual of the Lie algebra of the universal vector extension of D (cf. [Me]; see also [Va1, 5.2]). Let $(t_\alpha)_{\alpha\in\mathcal{J}}$ be a family of homogeneous tensors of the F^0-filtration of the essential tensor algebra of $(M\oplus M^*)\left[\frac{1}{p}\right]$ such that:

– its members are fixed by Φ_M and are annihilated by ∇;

– the subgroup of $GL(M)$ obtained by taking the Zariski closure in $GL(M)$ of the subgroup of $GL(M\left[\frac{1}{p}\right])$ fixing its members, is a reductive group G_Y over Y.

The pair $(D,(t_\alpha)_{\alpha\in\mathcal{J}}))$ will be referred as a Shimura p-divisible group over Y.

d) We define as in [Va2, 2.2.21 4)] principally quasi-polarized Shimura p-divisible groups over the above type of affine schemes.

2.1.2. Examples. Let $(f,L_{(p)},v)$ be a SHS. Let Y be an affine scheme of the type mentioned in a) to c) of 2.1.1. Then for any morphism $Y\to\mathcal{N}$, the pull back (D,p_D) of the principally quasi-polarized p-divisible group (over $\mathcal{N}$) of $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$, together with the pull back $(t_\alpha)_{\alpha\in\mathcal{J}'}$ of the family $(t_\alpha^{\mathcal{A}})_{\alpha\in\mathcal{J}'}$ of de Rham components of the Hodge cycles (see [Va2, 2.3.3]) with which $\mathcal{A}$ is naturally endowed, defines a principally quasi-polarized Shimura p-divisible group $(D, (t_\alpha)_{\alpha\in\mathcal{J}'}, p_D)$ over Y.

If all the Hodge cycles of $\mathcal{A}$ are obtained by pulling back Hodge cycles of $\mathcal{A}_{H_0}$, then the say remains true in the finite level context, i.e. for morphisms $Y \to \mathcal{N}/H_0$.

Also it is worth pointing out that large classes of Shimura p-divisible groups, which are not principally quasi-polarized, show up naturally in the context of a SHS, cf. [Va2, 4.12]. We do not detail this part in full generality; we just mention that this phenomenon shows up when the representation of $G_{\mathbb{Z}_p}$ on $L_{(p)} \otimes \mathbb{Z}_p$ is not irreducible.

2.1.3. Remarks. 1) Denoting (for the sake of uniformizing the notations) $[V : W(k)]$ by e, an arbitary choice of a uniformizer of V defines naturally (see [Va3, 2.1]) $W(k)$-epimorphisms $Re \twoheadrightarrow V$ and $\widetilde{Re} \twoheadrightarrow V$. As Re is p-adically complete and as Re/pRe is a standard inductive limit (indexed by $\mathbb{N}$) of local artinian rings having k as their residue fields, we deduce easily that any $W(k)$-homomorphism $W(k)[[x_1, ..., x_d]] \to V$ lifts to a $W(k)$-homomorphism $W(k)[[x_1, ..., x_d]] \to Re$. So any (principally quasi-polarized) Shimura p-divisible group over V lifts to one over Re, and so to one over $\widetilde{Re}$, as we have (see loc. cit.) a natural $W(k)$-monomorphism $Re \hookrightarrow \widetilde{Re}$ compatible with the mentioned $W(k)$-epimorphisms.

2) It is easy to see that the definitions of 2.1.1 are compatible with the operation of pulling back. For instance we have: any (principally quasi-polarized) Shimura p-divisible group D over $\mathrm{Spec}(Re)$ or $\mathrm{Spec}(\widetilde{Re})$ pulls back to one:

a) over $W(k)$ (under the $W(k)$-epimorphism $\widetilde{Re} \twoheadrightarrow W(k)$ defined by T goes to zero), cf. [Va2, 2.2.9 1)];

b) over V, cf. the proof of the Proposition of [Va3, 2.4] (and [Va3, 2.4.1]).

3) It is not always easy to recognize when a p-divisible group "endowed with cycles" over a faithfully flat affine $W(k)$-scheme Y of the type showing up in 2.1.1, does or does not define a Shimura p-divisible group. Usually, it is desirable to "read this out" from the étale context with $\mathbb{Z}_p$ coefficients. The best general criterion we have in this direction, in the case $Y = \mathrm{Spec}(V)$, is the one of [Va3, 2.4]. However, in part 2C we will improve it in the context of a SHS, cf. the statement of [Va2, 2.3.8 2)] (to be compared with 4.3).

2.1.3.1. Exercise. We refer to 2.1.1 c). Show that there is a cocharacter $\mu : \mathbb{G}_m \to G_Y$, achieving a direct sum decomposition $M = F^1 \oplus F^0$, with $\beta \in \mathbb{G}_m(Y)$ acting as the multiplication with β^{-i} on F^i, $i = \overline{0,1}$. Hint: using [Va2, 2.2.9 1)] show that the intersection of $\mathrm{Lie}(G_Y)$ with $F^0(\mathrm{End}(M))$ is the Lie algebra of a parabolic subgroup of G_Y; then use the properties of Re referred to in 2.1.3 1) (resp. copy the arguments of [Va1, 5.3.2]) if $Y = \mathrm{Spec}(Re)$ (resp. if $Y = \mathrm{Spec}(\widetilde{Re})$).

2.1.4. The étale counterpart. We assume $p > 2$. Let Y be either:

a) $\mathrm{Spec}(V)$, with V as in 2.1.1 b), or

b) $\mathrm{Spec}(Re)$, or

b') $\mathrm{Spec}(\widetilde{Re})$, or

c) the p-adic completion of a smooth scheme over $W(k)$ having a connected special fibre.

Let $\mathcal{D} := (D, (t_\alpha)_{\alpha\in\mathcal{J}})$ be a Shimura p-divisible group over Y.

In case a) let $K := V\big[\frac{1}{p}\big]$ and let $H^1 := H^1_{\text{ét}}(D, \mathbb{Z}_p)$ be the dual of the Tate module of D_K. It is a free $\mathbb{Z}_p$-module of rank equal to the height of D and we get naturally a Galois representation

$$\rho : \mathrm{Gal}(\bar{K}/K) \to GL(H^1)(\mathbb{Z}_p).$$

Moreover the Fontaine's comparison theory (see [Fa1-2]) shows that we have a family $(v_\alpha)_{\alpha\in\mathcal{J}}$ of homogeneous tensors of $\mathcal{T}(H^1\big[\frac{1}{p}\big])$ fixed by the image of ρ. The subgroup $G_{\mathbb{Q}_p}$ of $GL(H^1\big[\frac{1}{p}\big])$ fixing its members is reductive. Let $G_{\mathbb{Z}_p}$ be its closure in $GL(H^1)$.

We refer now to case b). [Fa2, ch. 3 and 4] shows that we similarly have a free $\mathbb{Z}_p$-module H^1 of rank equal to the height of D, endowed with a Galois action by $\mathrm{Gal}(\bar{K}/K)$. H^1 is obtained by taking the dual of the image of the p-divisible object (M, F^1, Φ_M, ∇) of $\mathcal{MF}^{\nabla}_{[0,1]}(Re)$ defined by D (see 2.1.1 c)), under the functor $\mathbb{D}$ of [Fa2, ch. 4], cf. [Fa2, th. 5]. Warning: such a quadruple referred here as a p-divisible object, is referred in [Fa2, ch. 3] as an object of $\mathcal{MF}_{[0,1]}(V)$; however it does not depend on the choice of a $W(k)$-epimorphism $f_V\colon \widetilde{Re} \twoheadrightarrow V$ (via the choice of a uniformizer of V), and moreover this terminology matches the one of [Va2]. Of course, in getting H^1 and its Galois action we do need to fix f_V. Similarly to case a) we get a family $(v_\alpha)_{\alpha\in\mathcal{J}}$ of homogeneous tensors of $\mathcal{T}(H^1\big[\frac{1}{p}\big])$.

In [Fa2, th. 7] it is shown that the $\mathrm{Gal}(\bar{K}/K)$-module H^1 is the same as the one defined in case a), after we pull back D via f_V (cf. 2.1.3 2)). This allows us to define the $\mathrm{Gal}(\bar{K}/K)$-module H^1 and the family $(v_\alpha)_{\alpha\in\mathcal{J}}$ even in case b'). So in cases b) and b') we can define similarly (via the choice of f_V) a group scheme $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$.

In 4.4 1) we will see that the triple

$$\mathcal{D}_{\text{ét}} := (H^1, G_{\mathbb{Z}_p}, (v_\alpha)_{\alpha\in\mathcal{J}})$$

does not depend on the choice of f_V; once more: the $\mathrm{Gal}(\bar{K}/K)$-structure of H^1 depends on the choice of f_V. We refer to it as the étale counterpart of $\mathcal{D}$. We use the same notation and terminology in case a).

In case c) the étale counterpart of $\mathcal{D}$ can be defined entirely similarly, starting from [Fa1, 7.1]. The only difference: we get a Galois action by the Galois group of the field of fractions of Y.

2.1.4.1. The principally quasi-polarized context. We assume now that $\mathcal{D}$ has a principal quasi-polarization p_D. Then $\mathcal{D}_{\text{ét}}$ is also principally quasi-polarized; in other words, in all the above cases, there is a perfect alternating form (still denoted by p_D) invariant by $\mathrm{Gal}(\bar{K}/K)$

$$p_D\colon H^1 \otimes H^1 \to \mathbb{Z}_p(1).$$

We refer to the quadruple $(H^1, G_{\mathbb{Z}_p}, (v_\alpha)_{\alpha\in\mathcal{J}}, p_D)$ as the étale counterpart of $(D, (t_\alpha)_{\alpha\in\mathcal{J}}, p_D)$.

§3. The basic result: Milne's conjecture

Let k be a perfect field of characteristic $p > 2$. Let $e \in \mathbb{N}$. Let V be a totally ramified DVR extension of $W(k)$ of degree e. Let $\mathcal{D} := (D, (t_\alpha)_{\alpha\in\mathcal{J}})$ be a Shimura p-divisible group over V. Let $\mathcal{D}_{\text{ét}} = (H^1, G_{\mathbb{Z}_p}, (v_\alpha)_{\alpha\in\mathcal{J}})$ be its étale counterpart.

References to the Milne's conjecture we have in mind, in the context of abelian varieties, can be found (cronologically) in [Mi3, p. 511-2], [Mi1, 0.1], [Va1, 5.6.5 h) and 5.6.6], [Va2, d) of 4.4.1 3)] and in [Va3, 2.2]. Here we situate ourselves in a larger context than the one of abelian varieties: we work in the context of Shimura p-divisible groups, and then as corollaries we draw conclusions for the context of abelian varieties. So the goal of this chapter is to prove the following theorem and some of its numerous variants.

3.1. Theorem (the de Rham form of Milne's conjecture for Shimura p-divisible groups over DVR's). *There is an étale cover Spec(V_1) of Spec(V) and an isomorphism*

$$\rho_0 : H^1 \otimes_{\mathbb{Z}_p} V_1 \tilde{\to} H^1_{dR}(D) \otimes_V V_1$$

taking v_α into t_α, $\forall \alpha \in \mathcal{J}$.

Proof: A) We can assume that $k = \bar{k}$. So the only choice left for V_1 is V itself.

From 2.1.1 b) we deduce that $\mathcal{D}$ is obtained by pull back from a Shimura p-divisible group $\mathcal{D}_{\text{loc}}$ over $R = W(k)[[x_1, ..., x_d]]$, with $d \in \mathbb{N} \cup \{0\}$. Let

$$\mathcal{C}_0 := (M_0, F^1, \phi_0, G, (t^0_\alpha)_{\alpha\in\mathcal{J}})$$

be the Shimura filtered σ-crystal over k we get by restricting $\mathcal{D}_{\text{loc}}$ to the closed subscheme of Spec(R) defined by $x_i = 0$, $i = \overline{1, d}$. We recall that G is a reductive subgroup of $GL(M_0)$. From [Fa2, th. 10 and the remarks after] we deduce that we can assume that $\mathcal{D}_{\text{loc}}$ is a versal deformation in the sense of [Va2, 3.4.19 B)] and that d is the deformation dimension $dd(\mathcal{C}_0)$ of $\mathcal{C}_0$ (see [Va2, 2.2.22 4)] for its definition).

Multiplying for each $\alpha \in \mathcal{J}$, t_α and v_α by the same power of p, we can assume that $t^0_\alpha \in \mathcal{T}(M_0)$ and that $v_\alpha \in \mathcal{T}(H^1)$.

B) From [Va2, Fact' of 3.4.15] we deduce that $\mathcal{D}_{\text{loc}}$ is pro-algebraizable. In other words there is a smooth $W(k)$-algebra R^{al}, étale over $W(k)[x_1, ..., x_d]$, and there is an $\mathbb{N}$-pro-étale morphism Spec$(R^{\text{pro-al}}) \to$ Spec(R^{al}), such that:

1) $\mathcal{D}_{\text{loc}}$ is obtained from a Shimura p-divisible group $\mathcal{D}_{\text{pro-al}}$ over the p-adic completion Spec(R^1) of Spec$(R^{\text{pro-al}})$ by pull back under a formally étale morphism Spec$(R) \to$ Spec(R^1);

2) the p-divisible object of $\mathcal{MF}^{\nabla}_{[0,1]}(R^{\text{pro-al}})$ with cycles defined by $\mathcal{D}_{\text{pro-al}}$ is isomorphic to

$$\mathcal{C}_1 := (M_0 \otimes R^1, F^1 \otimes R^1, g(\phi_0 \otimes 1), \nabla, (t^0_\alpha)_{\alpha\in\mathcal{J}}),$$

with $g \in G(R^1)$;

3) mod p it is surjective.

C) Using [Va2, Fact of 3.4.19] and [Fa1, 2.6 and 7.1] we can replace $\mathcal{D}$ by any other Shimura p-divisible group over $W(k)$, obtained from $\mathcal{D}_{\text{pro-al}}$, through a $W(k)$-morphism

$\mathrm{Spec}(W(k)) \to \mathrm{Spec}(R^1)$. In other words, [Fa1, 2.6] allows us "to move" in a family in the étale context with $\mathbb{Z}_p$ coefficients. At the first glance it looks like we need to assume that $\deg(t_\alpha) \le p-2$, $\forall \alpha \in \mathcal{J}$, in order to apply [Fa1, 2.6]; but this assumption is not needed, as $t_\alpha^0 \in \mathcal{T}(M_0)$, and as $\mathcal{C}_1$ is a p-divisible object with cycles of $\mathcal{MF}_{[0,1]}^{\nabla}(R^{\text{pro-al}})$ (we recall that $p \ge 3$). So [Va2, Fact' of 3.15 and 2.3.17.1-2] allow us to assume that:

a) $e = 1$;

b) $\mathcal{C}_0$ is a G-canonical lift in the sense of [Va2, 3.1.1].

Based on a) and b), from [Va2, 3.11.1 c) and 3.2.3] we deduce that we can assume that G is a torus T_0. It is worth remarking that we can achieve this replacement, by just enlarging the family $(t_\alpha^0)_{\alpha \in \mathcal{J}}$ with tensors of degree 2 (i.e. by using just endomorphisms). Let $\mu: \mathbb{G}_m \to T_0$ be the canonical split of $\mathcal{C}_0$. We can assume (cf. [Va2, 2.2.9 1)]) that T_0 is the smallest subtorus of $GL(M_0)$ whose Lie algebra is left invariant by ϕ_0, and through which μ factors. Let a be the T-degree of definition of $\mathcal{C}_0$ (see [Va2, 3.11.3]). We have $a \in \mathbb{N}$.

As in the proof of [Va2, 4.1.2] we deduce the existence of a $W(k)$-bases

$$\mathcal{B} := \{a_1, ..., a_n, b_1, ..., b_m\}$$

of M_0 such that:

i) $\{b_1, ..., b_m\}$ is a $W(k)$-bases of F^1;

ii) $\phi_0(a_i)$ and $\frac{\phi(b_j)}{p}$ belong to $\mathcal{B}$.

In other words it is known that (M_0, F^1, T_0) is a cyclic diagonalizable Shimura filtered σ-crystal (cf. [Va2, 2.2.16 and 2.2.22]). Here $n, m \in \mathbb{N}$ (cf. [Va2, 2.2.9 11)], while $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$. Let π_0 be the permutation of $\mathcal{B}$ we get naturally via ϕ_0 (see ii) above and [Va2, 2.2.1 b)]).

D) Let $B^+(W(k))$ be the Fontaine's ring of $W(k)$ (see [Fa2]). We consider the integral version (see [Fa2]) of Fontaine's comparison theory for $\mathcal{C}_0$. We have a $B^+(W(k))$-monomorphism of filtered modules

$$\rho_1: M_0 \otimes_{W(k)} B^+(W(k)) \hookrightarrow H^1 \otimes_{\mathbb{Z}_p} B^+(W(k)),$$

taking t_α to v_α, $\forall \alpha \in \mathcal{J}$. ρ_1 respects the Frobenius and the Galois actions of $\mathrm{Gal}(\overline{B(k)}/B(k))$ (see [Fa2, th. 7]).

Claim. *There is an element $t \in T_0(B^+(W(k))[\frac{1}{p}])$ taking $\rho_1^{-1}(H^1 \otimes B^+(W(k)))$ onto $M_0 \otimes B^+(W(k))$.*

E) Before proving the claim, let us point out that it implies (in fact it is equivalent to) the existence of an isomorphism ρ_0 (as in the statement of the theorem) with the desired properties: the existence of ρ_0 (with $V_1 = W(k)$), due to the Claim, is expressed in terms of a torsor of T_0 being trivial; but as $k = \bar{k}$, and as T_0 is a split torus, we have $H^1_{ff}(T_0, W(k)) = 0$ (cf. also [Mi2, 3.9 of p. 114]), and so this torsor is indeed trivial.

F) Let C be the centralizer of T_0 in $GL(M_0)$, and let T_1 be the center of C. T_1 is a torus containing T_0. Before handling the general case we first treat some particular cases, just to present different techniques.

Case 0: $a = 1$**.** This case is entirely the same as [Va3, 2.2.3].

Case 1: $T_1 = T_0$**.** This situation is not very common. For instance, if $a = 1$ we have: T_0 is a one dimensional split torus, while T_1 is two dimensional. Of course, if we do not choose T_0 to be the "smallest", the situation becomes more common (like in the case of PEL type embeddings; see [Va1, 4.3.11]).

To handle this case, we can assume that $(t_\alpha^0)_{\alpha\in\mathcal{J}}$ is the family of all endomorphisms of M_0 fixed by ϕ_0. Let B be the $\mathbb{Z}_p$-subalgebra of $\mathrm{End}(M_0)$ generated by these endomorphisms. It is isomorphic to the $\mathbb{Z}_p$-subalgebra of $\mathrm{End}(H^1)$ generated by v_α, $\alpha \in \mathcal{J}$. So the above Claim in this case is a consequence of the fact that the two representations of $B\otimes_{\mathbb{Z}_p} B^+(W(k))$ on $H^1 \otimes B^+(W(k))$ and on $M_0 \otimes B^+(W(k))$ are isomorphic, as they are isomorphic by making p invertible. This case is an abstract, simplified form of [Mi1, (0.2) c)].

Case 2: $T_1 \neq T_0$**, but** v_α **has its partial degrees equal and smaller than** p**,** $\forall\alpha \in \mathcal{J}$**.** If $v_\alpha \in H^{1^{\otimes m_1^\alpha}} \otimes H^{1*^{\otimes m_2^\alpha}}$, then we refer to m_1^α and m_2^α as its partial degrees. So in this case we assume that

$$m_1^\alpha = m_2^\alpha \leq p-1, \forall\alpha \in \mathcal{J}.$$

We can assume that $(v_\alpha)_{\alpha\in\mathcal{J}}$ is the maximal family of étale cycles of $\mathcal{T}(H^1)$ of partial degrees equal and smaller than p, cf. [Fa2, cor. 9]. In other words from loc. cit. we deduce directly:

Fact. *For any* $\alpha \in \mathcal{J}$, t_α^0 *is not zero mod* p *iff* v_α *is not zero mod* p.

So this case is nothing else but a simpler version of the proof of [Va3, 4.3.4.8 b)].

Case 3: $T_1 \neq T_0$ **but** $\deg(t_\alpha^0) < p-1$**,** $\forall\alpha \in \mathcal{J}$**.** This case is handled as the previous one. Following the proof of [Fa2, th. 4] (or just quoting directly [Fa1, 2.6]), we deduce that the above Fact is still valid in this context; so the proof of [Va3, 4.3.4.8 b)] applies again.

Case 4: the general case. From Case 1 we deduce the existence of an element $t \in T_1(B^+(W(k))[\frac{1}{p}])$ taking $\rho_1^{-1}(H^1 \otimes B^+(W(k)))$ onto $M_0 \otimes B^+(W(k))$. It is uniquely determined up to multiplication by an element of $T_1(B^+(W(k)))$. We need to show that in fact we can assume that $t \in T_0(B^+(W(k))[\frac{1}{p}])$.

We first choose an arbitrary such $t \in T_1(B^+(W(K))[\frac{1}{p}])$. Obviously we have:

$$t(a_i) = u_i a_i$$

and

$$t(b_j) = v_j b_j,$$

with $u_i, v_j \in B^+(W(k))$ such that they are invertible in $B^+(W(k))[\frac{1}{p}]$, and with i, j running as in C).

We recall that we have (cf. [Fa2, ch. 4]) a Galois-invariant surjection

$$s_{\mathcal{V}}\colon B^+(W(k)) \twoheadrightarrow \mathcal{V},$$

where $\mathcal{V} := \bar{V}^\wedge$ denotes the p-adic completion of the normalization of $W(k)$ in $\overline{B(k)}$. Let $\mathcal{K} := \mathcal{V}[\frac{1}{p}]$.

Let $\tilde{u}_i$ be the image of u_i in $\mathcal{V}$. Similarly we define $\tilde{v}_j$. Using them we define naturally an element $\tilde{t} \in T_1(\mathcal{K})$. As $\mathcal{V}$ is a faithfully flat $W(k)$-algebra, it is enough to show that we can assume that $\tilde{t} \in T_0(\mathcal{K})$. Let y_i (resp. z_j) be the image of $\tilde{u}_i$ (resp. of $\tilde{v}_j$) in $\mathcal{V}/p\mathcal{V}$.

The explicite values of y_i's and of z_j's (modulo invertible elements of $\mathcal{V}/p\mathcal{V}$) can be easily computed using the ideas of [Fa2, proof of th. 5]. In other words using the dual form of the $B^+(W(K))$-monomorphism ρ_1 of D), we get a $B^+(W(k))$-monomorphism

$$\rho_1^*: H^{1*} \otimes B^+(W(k)) \hookrightarrow M_0^* \otimes_{W(k)} B^+(W(K)).$$

Using loc. cit. for $e = 1$ (inclusive the notations, except that we write $R1$ instead of R_1), it can be interpreted as the natural inclusion of loc. cit.

$$i_M: \mathbb{D}(M) \otimes_{\mathbb{Z}_p} B^+(W(k)) \hookrightarrow M_0^* \otimes_{W(k)} B^+(W(k));$$

here $M := (M_0 \otimes_{W(k)} R1, F^1 \otimes_{W(k)} R1, \phi_0 \otimes 1)$.

In all that follows we assume that different roots of p showing up are powers of the same (high order) root of p. Let $(c_1, ..., c_q)$ be a cycle under the permutation π_0 of $\mathcal{B}$. Let $(w_1, ..., w_q)$ be the ordered entities $\tilde{u}_i$ or $\tilde{v}_j$ we get using this cycle. To compute w_r's $(r = \overline{1, q})$ is almost the same as solving out the system (it is convenient to view it as being with coefficients in $\mathcal{V}$) $\mathcal{S}$ of q equations in q variables defined as follows

$$X_r^p = (-p)^{\frac{\varepsilon_r}{p}} X_{r+1},$$

$r = \overline{1, q}$ (with $X_{q+1} = X_1$), and with ε_r equal to 0 or 1 depending on the fact that $c_r \in \{a_1, ..., a_n\}$ or not. Here almost the same means: if $(x_1, ..., x_q)$ is a non zero solution of $\mathcal{S}$, then we can take:

$$w_r = p^{\frac{\varepsilon_r}{p}} x_r. \tag{1}$$

This is so as we have:

i) Via i_M, $M_0^* \otimes_{W(k)} B^+(W(k))$ is included in $\frac{1}{\beta}\mathcal{D}(M)$, cf. [Fa2, ii) of th. 5] (here $\beta \in F^1(B^+(W(k))$ is as in [Fa2, ch. 4]);

ii) The image of β in $\mathcal{V}/p\mathcal{V}$ is $p^{\frac{1}{p-1}}$ times a unit of $\mathcal{V}/p\mathcal{V}$, cf. [Fa2, ch.4] (the important thing is: as $p > 2$ we have $\frac{1}{p} < \frac{1}{p-1} < \frac{2}{p} < 1$);

iii) The map $\mathbb{D}(M) \otimes_{\mathbb{Z}_p} \mathcal{V} \to M_0^* \otimes_{W(k)} \mathcal{V}$, defined naturally by i_M and $s_{\mathcal{V}}$, is injective (cf. [Fa2, ii) of th. 5*]);

iv) Via i_M, with respect to the standard bases $\mathcal{B}^* := \{a_1^*, ..., a_n^*, b_1^*, ..., b_m^*\}$ of the $\mathcal{V}$-module $M_0^* \otimes_{W(k)} \mathcal{V}$, $\mathbb{D}(M) \otimes_{\mathbb{Z}_p} \mathcal{V}$ can be identified as the $\mathcal{V}$-module generated as follows. We can concentrate just on the ordered subbases $\{c_1^*, ..., c_q^*\}$ of $\mathcal{B}^*$. Then the intersection $\mathbb{D}(c-cycle)$ of $\mathbb{D}(M) \otimes_{\mathbb{Z}_p} \mathcal{V}$ with the $\mathcal{V}$-module generated by c_1^*,..., c_{q-1}^* and c_q^*, is generated by elements of the form

$$\sum_{r=1}^{q} \alpha_r c_r^*,$$

where $(\alpha_1, ..., \alpha_q)$ is a q-tuple of elements of $B^+(W(k))$ whose image in $\mathcal{V}/p\mathcal{V}$ is a solution of the above system $\mathcal{S}$ considered mod p (cf. [Fa2, proof of th. 5]).

v) Any other non zero solution of $\mathcal{S}$ is of the form $(\gamma x_1, \gamma^p x_2, ..., \gamma^{p^{q-1}} x_q)$, for some p^q-1-th root of unity γ; so using linear combinations of such solutions with coefficients in $W(k)$ (as all such γ's belong to $W(k)$!), we get immediately (via i) and ii) above)) that $\mathbb{D}(c-cycle)$ is generated by $w_r c_r^*$, $r \in \{1, ..., q\}$.

With this formula (1) we are essentially done. For each such cycle $(c_1, ..., c_q)$ of the permutation π_0 of $\mathcal{B}$ we choose $\tilde{u}_i$'s and $\tilde{v}_j$'s as prescribed by (1). To check that the resulting $\tilde{t} \in T_1(\mathcal{K})$ we get belongs to $T_0(\mathcal{K})$ is easy. We need to remark five simple things.

First we remark that all the above numbers w_i, up to units of $\mathcal{V}$, are of the form: p to a power $P_i(p)$ where P_i is a rational function of one variable. Second we remark that P_i depends only on π_0 and not on p. Third we remark that the fact that $\tilde{t}$ belongs to $T_0(\mathcal{K})$ can be expressed as some monomial expressions $E_1, ..., E_s$ (with $s \in \mathbb{N}$) in $\tilde{u}_i$'s, $\tilde{v}_j$'s and their inverses are equal to 1; the conditions $E_1 = ... = E_s = 1$ are equivalent to:

– p is the root of $2s$ rational functions $F_1, ..., F_{2s}$ in one variable.

The first s such functions are "computing" (keeping track of) the exponents of p in respectively E_1,..., E_s, while the last s functions are "computing" (keeping track of) the exponents of -1 in respectively E_1,..., E_s (of course we can always multiply $\tilde{u}_i$'s and $\tilde{v}_j$'s by units of $\mathcal{V}$, so we do not have to bother about the exponents of -1 in the expressions involved). Fourth we remark that the degrees of v_α, $\alpha \in \mathcal{J}$, can be assumed to be bounded, and that such an upper bound $UP(\pi_0) \in \mathbb{N}$ can be obtained just in terms of π_0 and not of p. Fifth we remark that if $p > 1 + UP(\pi_0)$ then we know that $\tilde{t} \in T_0(\mathcal{K})$, cf. Case 3.

From these five remarks we get directly: $F_1 = ... = F_{2s} = 0$. So $\tilde{t} \in T_0(\mathcal{K})$. This ends the proof of the Claim in the general case, and so of the theorem.

3.1.1. Corollary. *$G_{\mathbb{Z}_p}$ is a reductive group over $\mathbb{Z}_p$.*

3.1.2. Remark. It is worth remarking that in the context of principally polarized abelian varieties and of Hodge cycles, we do not need to refer to [Fa1, 2.6 and 7.1] or to the global deformation part of [Va2] in the proof of 3.1. In other words, if D is the p-divisible group of an abelian variety A over V, if there is a principal polarization p_A of A which makes $\mathcal{D}$ to be principally quasi-polarized, and if t_α is the de Rham component of a Hodge cycle u_α of A, $\forall \alpha \in \mathcal{J}$, then v_α is the p-component of the étale component of u_α (this is entirely the same as [Va1, 5.2.10]). So [Va1, 5.4] applies to give us the necessary local deformation of (A, p_A) and of its Hodge cycles. Using moreover [Va2, 3.12.1] (or [Va2, 4.2 and 4.4]) we achieve the desired "move" to a G-canonical lift (the rest, i.e. Case 4 of 3.1, is entirely the same).

3.1.3. Exercises. 1) Prove 3.1.1 directly without using Case 4 of the proof of 3.1.

2) Show that we have the following inequalities

$$\dim(T_0) \leq a,$$

and

$$\min(2, a) \leq \dim(T_1) \leq 2^a.$$

3) Determine a necessary and sufficient condition for having T_0 containing the one dimensional torus of scalar multiplications on M_0.

3.1.3.1. Problem. Determine the best lower value of the numbers $UP(\pi_0)$ introduced at the end of Case 4, in different concrete situations, as well as universal upper bounds for these numbers, in terms of different invariants of $\mathcal{C}_0$. Some of the invariants of $\mathcal{C}_0$ we have in mind are: its slopes, the rank of M_0 or of F^1, its T-degree of definition a, the denominators of its slopes, the lengths of the cycles of π_0, etc.

3.1.4. Example. We consider the case when $a = 2$, in the context of C) of 3.1. There are two posiibilities:

a) all the slopes of (M_0, ϕ_0) are $\frac{1}{2}$, or

b) the slopes of (M_0, ϕ_0) are elements of the set $\{0, \frac{1}{2}, 1\}$, and either the multiplicity of the slope 0 or of the slope 1 is positive.

In case a) M_0 is of even rank and $\dim(T_0) = \dim(T_1) = 2$. This is so as (cf. the proof of [Va2, 4.1.2]) we can asume that $\phi_0(a_i) = b_i$ and $\phi_0(b_i) = pa_i$ (so $n = m$), $i = \overline{1, n}$. So the Case 1 of 3.1 applies directly to give us the existence of ρ_0.

In case b) M_0 can be of odd rank as well. Again $\dim(T_0) = 2$, but $\dim(T_1)$ is 3 or 2 depending on the fact that (M_0, ϕ_0) has or has not the slope 1 with positive multiplicity. Let $q, r \in \mathbb{N} \cup \{0\}$ be such that $\phi_0(a_i) = a_i$ for $i = \overline{1, q}$ and $\phi_0(b_j) = pb_j$ for $j = \overline{1, r}$. We have $q + r > 0$. We can also assume that $\phi_0(a_{q+s}) = b_{r+s}$ and that $\phi_0(b_{r+s}) = pa_{q+s}$; here $s \in \{1, ..., n-q\}$ and we have $n - q = m - r > 0$.

We use the notations of Case 4 of 3.1. From the computations of Case 4 of 3.1 we deduce that we can choose:

i) $\tilde{u}_i = 1$ for $i = \overline{1, q}$;

ii) $\tilde{v}_j = (-1)^{\frac{1}{p(p-1)}} p^{\frac{1}{p-1}}$ for $j = \overline{1, r}$;

iii) $\tilde{u}_i = (-p)^{\frac{1}{p(p^2-1)}}$ for $i = \overline{q+1, n}$;

iv) $\tilde{v}_j = (-p)^{\frac{1}{p^2-1}} p^{\frac{1}{p}}$ for $j = \overline{r+1, m}$.

Obviously the product of values of i) and ii) is equal to the product of the values of iii) and iv). So indeed the element $\tilde{t} \in T_1(\mathcal{K})$ we get belongs to $T_0(\mathcal{K})$.

In case a) we have $UP(\pi_0) = 2$. In case b) we have $UP(\pi_0) = 3$ or $UP(\pi_0) = 2$ depending on the fact that (M_0, ϕ_0) has or has not slope 1 with positive multiplicity. If we are in case b) with $UP(\pi_0) = 3$, than we can define T_0 to be subtorus of $GL(M_0)$ fixing the following tensors:

– a_i for $i = \overline{1, q}$;

– all the possible endomorphism of M_0;

– $a_{q+s} \otimes b_{r+s} \otimes b_j^*$, with $s \in \{1, ..., n-q\}$ and with $j \in \{1, ..., r\}$ (here b_j^* is the tensor of M_0^* naturally defined by b_j and $\mathcal{B}$).

3.1.5. Definition. We refer to the element t of the Claim of 3.1 D) or to its image in $T_0(\mathcal{K})$ as a correction factor. We refer to the element $\tilde{t}$ constructed at the end of Case 4 of 3.1, as the standard correction factor.

We invite the reader to compute explicitely more (than the simplest case of 3.1.4) such standard correction factors. Warning: it is much more easier to compute such correction factors while being over $\mathcal{V}$ instead of being over $B^+(W(k))$.

3.2. Variants of 3.1. There are many variants of 3.1 (cf. also [Va3, 2.2]). Here we list some of them.

V1 (the principally quasi-polarized context of 3.1). We assume that D has a principal quasi-polarization p_D such that $\mathcal{DP} := (D, (t_\alpha)_{\alpha\in\mathcal{J}}, p_D)$ is a principally quasi-polarized Shimura p-divisible group over V. Then we can assume that ρ_0 of 3.1 takes also p_D into p_D. The same proof applies.

There are principally quasi-polarized variants of the variants V2 and V4-7 below; we do not stop to rephrase them.

V2 (the étale form of Milne's conjecture for p-divisible groups over DVR's). Let D be a p-divisible group over V, and let H^1 have the same significance as in 2.1.4. Let $(v_\alpha)_{\alpha\in\mathcal{J}}$ be a family of étale cycles of $\mathcal{T}(H^1[\frac{1}{p}])$ such that the subgroup $G_{\mathbb{Z}_p}$ of $GL(H^1)$, defined as in 2.1.4 is reductive. We assume that $\forall\alpha\in\mathcal{J}$ we have:

$$\deg(v_\alpha) \le 2p-2.$$

We define similarly $H^1_{dR}(D)$; using Fontaine's comparison theory we define $\forall\alpha\in\mathcal{J}$, the de Rham counterpart $t_\alpha\in\mathcal{T}(H^1_{dR}(D)[\frac{1}{p}])$ of v_α.

Theorem. *An isomorphism ρ_0 as in 3.1 does exist in this context.*

Proof: This is a consequence of [Va3, 2.4]: loc. cit. shows that $(D, (t_\alpha)_{\alpha\in\mathcal{J}})$ is a Shimura p-divisible group; so 3.1 applies.

V3 (Milne's conjecture for a SHS). Let $(f, L_{(p)}, v)$ be a SHS. Let V be as in the beginning of §3. Let $K_V := V[\frac{1}{p}]$. Let $z_V\colon \mathrm{Spec}(V)\to\mathcal{N}$ be a morphism; let $y_V\colon\mathrm{Spec}(k)\to\mathcal{N}$ be the resulting morphism. From the proof of [Va2, 3.2.9] we deduce that it forces k to contain $\mathbb{F}$. Let $(A_V, p_{A_V}) := z_V^*((\mathcal{A},\mathcal{P}_{\mathcal{A}}))$, and let $(t_\alpha)_{\alpha\in\mathcal{J}'}$ be the family of de Rham components of the Hodge cycles with which the abelian variety A_V is naturally endowed. We have:

Theorem. *We assume that either $k=\bar{k}$ or y_V is a G-ordinary point (see def. [Va2, 4.2.9]). Then there is an isomorphism*

$$L_{(p)}\otimes_{\mathbb{Z}_{(p)}} V = H^1_{\text{ét}}(A_{\overline{K_V}},\mathbb{Z}_p)\otimes_{\mathbb{Z}_p} V \xrightarrow{\sim} H^1_{dR}(A_V/V)$$

respecting the symplectic forms defined by p_{A_V} and taking v_α into t_α, $\forall\alpha\in\mathcal{J}'$.

Proof: We first recall that the identification $L_{(p)}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Z}_pH^1_{\text{ét}}(A_{\overline{K_V}},\mathbb{Z}_p)$ is unique up to an isomorphism of $L_{(p)}\otimes\mathbb{Z}_p$ defined by an element of $G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})$ (cf. [Va1, 5.1.2]). If $k=\bar{k}$ then V1 applies (cf. [Va2, 2.3.10]). If y_V is G-ordinary point, then the local deformation theory of [Va1, 5.4] (cf. [Va2, 2.3.11]) allows us to assume that $V=W(k)$, and that (cf. [Va2, 4.4.1]) z_V is a G-canonical lift. So we have $M_0 := H^1_{dR}(A_V/V) = H^1_{\text{crys}}(A_k/W(k))$. We consider the Shimura filtered σ-crystal $(M_0, F^1_0, \phi_0, G_{W(k)}, (t_\alpha)_{\alpha\in\mathcal{J}'})$ attached to z_V (cf. [Va2, 2.3.9.2 and 2.3.10]). From [Va2, a) 4.4.1 3) and 3.11.1], as in the proof of [Va2, 4.1.2], we deduce that there is a split subtorus T_0 of $G_{W(k)}$ such that we still have a Shimura filtered σ-crystal $(M_0, F_0^1, \phi_0, T_0)$. As $H^1_{\text{ét}}(\text{Spec}(W(k), T_0) = 0$, V1 applies. This ends the proof of the theorem.

V4 (the étale form of Milne's conjecture for families of p-divisible groups). We come back to the context of V2, but we do not impose any condition on the degrees of v_α's (this is suggested by V3 and [Va2, 2.3.8 2)]). We assume that there is a global deformation of $(D, (v_\alpha)_{\alpha \in \mathcal{J}})$ over a regular formally smooth scheme Y over $W(k)$ having a connected special fibre, such that:

– there is a morphism $z \colon \text{Spec}(W(\bar{k})) \to Y$ with the property that its attached Galois representation $\rho_z \colon \text{Gal}(\overline{B(\bar{k})}/B(\bar{k})) \to G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ factors through the $\mathbb{Z}_p$-valued points of a subtorus of $G_{\mathbb{Z}_p}$.

We refer to this assumption as: $(D, (v_\alpha)_{\alpha \in \mathcal{J}})$ can be deformed to a quasi CM-type situation. Following the proof of 3.1 we conclude:

Corollary. *An isomorphism ρ_0 as in 3.1 does exist.*

V5 (Milne's conjecture for piano rings). Let Re and $\widetilde{Re}$ be as in 2.1.1 c). Let $Y = \text{Spec}(S)$ be the spectrum of one of these two $W(k)$-algebras. Let $(D, (t_\alpha)_{\alpha \in \mathcal{J}}))$ be a Shimura p-divisible group over Y. Let the quadruple (M, F^1, Φ_M, ∇) have the same significance as in 2.1.1 c). We have:

Theorem. *We assume $k = \bar{k}$. Then there is an isomorphism*

$$\rho : H^1 \otimes_{\mathbb{Z}_p} S \tilde{\to} M$$

taking v_α into t_α, $\forall \alpha \in \mathcal{J}$.

Proof: As in [Va3, 2.4] we can replace S by $W(k)[[T]]$. So the proof of 3.1 applies.

V6 (the case $p = 2$ of Milne's conjecture). For simplicity we restrict to the context of a $p = 2$ SHS $(f, L_{(2)}, v)$ (see [Va2, 2.3.18]). We also assume that one of the following two conditions is satisfied:

a) $k(v) = \mathbb{F}_p$;

b) we have in fact a $p = 2$ standard PEL situation $(f, L_{(2)}, v, \mathcal{B})$ and (see [Ko]) we are in case A or in case C.

Corollary. *Then the statement of [Va2, d) of 4.4.1 3)] remains true in this context.*

Proof: This is so mainly due to [Va2, 4.14.3]. As we assumed that we are dealing with a $p = 2$ SHS, 3.1.2 applies as well. In [Va2, 5.2.10] and [Va2, 5.4] we assumed that $p \geq 3$; but over Witt rings they are still valid for $p = 2$, as the arguments of loc. cit. depend on [Fa2, Sketch of Ch. 6, and Th. 10] which are results for any prime.

So, if a) above is satisfied, then Case 0 of the proof of 3.1 applies directly (cf. [Va2, 4.14.3] for the density of ordinary points of $\mathcal{N}_{k(v)}$ and of the existence of their canonical lifts). If b) above is satisfied, then the principally quasi-polarized context of Case 1 of the proof of 3.1 applies directly, cf. V1 of 3.2 (the fact that we have assumed that we are in case A or in case C, guarantees that G is the subgroup of $GSp(W, \psi)$ fixing the elements of $\mathcal{B}$, cf. [Ko, first paragraph of §7]).

It is worth remarking that for the part b) of this corollary, we do not need at all to use the deformation theory, as it is a standard result on representations of division algebras (cf. also [Mi1, (0.2) c)]). We do expect that the Corollary is still valid, without assuming that one of the above two conditions is satisfied: to see this, one just needs to adapt Case 4 of 3.1 to the $p = 2$ situation (we recall that [Fa2, th. 5] is worked out for odd primes). As the general results on the existence of $p = 2$ SHS will be presented only in part 2C and in [Va4], we do not stop to handle this problem here. However we just point out that the above conditions a) and b) can be considerably weaken: see rm. 4) of 4.2.1 as a more general form of a), and see [Va1, 4.3.16] for an incipient relative version of b) (see also [Va4] for the most general setting).

V7 (the case of a finite field). We refer to the situation of 3.1 with k a finite field. Combining 3.1 with Lang' s theorem on connected algebraic groups over finite fields (see [La] and [Bo, ch. 16]) we deduce:

Corollary. *There is an isomorphism*

$$\rho: H^1 \otimes_{\mathbb{Z}_p} V \xrightarrow{\sim} H^1_{dR}(D)$$

taking v_α *into* t_α, $\forall \alpha \in \mathcal{J}$.

As usual the same remains true for k a C_1 field.

V8 (the strictly henselian DVR case). We refer to the principally polarized context of 3.1.2, but with V a strictly henselian DVR $\mathbb{Z}_p$-algebra, having k as its residue field. We assume that the extension of the pair $(D, (t_\alpha)_{\alpha\in\mathcal{J}})$ to the completion $V^\wedge$ of V defines a Shimura p-divisible group over $V^\wedge$. From V1 and [BLR, th. 12, p. 83] we deduce directly (we can assume that $\mathcal{J}$ is a finite set):

Corollary. *There is an isomorphism*

$$H^1 \otimes_{\mathbb{Z}_p} V \xrightarrow{\sim} H^1_{dR}(A/V)$$

taking v_α *to* t_α, $\forall \alpha \in \mathcal{J}$, *and taking* p_D *into* p_D.

Variants V7-8 can be combined: in this last corollary it is enough to assume that V is a henselian DVR $\mathbb{Z}_{(p)}$-algebra, having a residue field which is a C_1 field. We will not restate the non principally polarized case of V8.

V9. It is worth stating explicitely what V3 and V7 become when combined. We consider the context of a SHS $(f, L_{(p)}, v)$. Let H_0 be as in the standard notations of [Va2, 3.2.3]. Let H_1 be an open subgroup of H_0 such that all the Hodge cycles of $\mathcal{A}$ are defined over $\mathcal{N}/H_1$, i.e. are also Hodge cycles of $\mathcal{A}_{H_1}$ (the pull back of $\mathcal{A}_{H_0}$ to $\mathcal{N}/H_1$). Let $z_V : \mathrm{Spec}(V) \to \mathcal{N}_{H_0}$. We assume that k is finite. Let V_1 be the minimal connected étale cover of V, such that the extension of z_V to V_1, lifts to a morphism $z_{V_1}\mathrm{Spec}(V_1) \to \mathcal{N}/H_1$. Let k_1 be the residue field of V_1. We have $[k_1 : k] \leq [H_0 : H_1]$. Let $(A_V, p_{A_V}) = z_V^*((\mathcal{A}_{H_0}, \mathcal{P}_{\mathcal{A}_{H_0}}))$, and let $(t^1_\alpha)_{\alpha\in\mathcal{J}'}$ be the family of Hodge cycles with which A_{V_1} is naturally endowed. We have (cf. V3 and V7):

Corollary. *There is an isomorphism*

$$L_{(p)} \otimes_{\mathbb{Z}_{(p)}} V_1 = H^1_{\text{ét}}(A_{\overline{K_V}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} V_1 \tilde{\to} H^1_{dR}(A_{V_1}/V_1)$$

respecting the symplectic forms defined by $p_{A_{V_1}}$ and taking v_α into t^1_α, $\forall \alpha \in \mathfrak{J}'$.

§4. Direct applications of the basic result

Milne came across his conjecture (see [Mi1, 0.1]), by working on motives of abelian varieties over finite fields. So we regard 3.1 and its variants of 3.2 as essentially motivic results. For applications of §3 in Milne's direction see the third part; here we will just mention that V3 of 3.2 is a fundamental step in the proof of Langlands-Rapoport conjecture (see [Va1, 1.7.1]), in Milne's approach (see [Mi1]) to its proof. Moreover V3 of 3.2 is the main tool for the parts 2B and 2C (cf. the last paragraphs of [Va2, 1.16] and [Va2, 1.7]). In what follows we mainly restrict our attention to complementing [Fa2, cor. 9].

4.1. Theorem. *We refer to the context of the beginning paragraph of §3. We have:*

a) **(the integrality principle for Shimura p-divisible groups over DVR's)** *For any $\alpha \in \mathfrak{J}$, t_α is an integral de Rham cycle (i.e. we have $t_\alpha \in \mathcal{T}(H^1_{dR}(D))$, iff v_α is an integral étale cycle (i.e. iff we have $v_\alpha \in \mathcal{T}(H^1)$).*

b) **(the non-vanishing principle for Shimura p-divisible groups over DVR's)** *For any $\alpha \in \mathfrak{J}$, t_α is an integral de Rham cycle having a non zero image in $\mathcal{T}(H^1_{dR}(D) \otimes k)$), iff v_α is an integral étale cycle having a non zero image in $\mathcal{T}(H^1/pH^1)$.*

Proof: This is a consequence of 3.1.

As in §3, there are many variants of the above theorem. We list some of them in an abbreviated form, without extra comment (these variants are obtained as 4.1, from the variants of 3.2). For not being too long, we do not restate the principally quasi-polarized context.

4.1.1. Theorem (the non-vanishing principle for Shimura p-divisible groups over piano rings). *We refer to the context of V5 of 3.2. Then the statement of 4.1 remains true.*

4.1.2. Theorem (the integrality and non-vanishing principle for Shimura varieties of Hodge type). *Let $(f, L_{(p)}, v)$ be a SHS. We use the standard notations of [Va2, 2.3.1-3]. Let $\alpha \in \mathfrak{J}'$. Let R be a local integral faithfully flat $O_{(v)}$-algebra, and let z_R: $Spec(R) \to \mathcal{N}$ be a morphism. Then v_α is integral (resp. is integral and non zero mod p) with respect to $L_{(p)}$, iff t^R_α is integral (resp. is integral and non zero modulo the maximal ideal of R) with respect to the H^1_{dR} cohomology group of the abelian scheme over R obtained from $\mathcal{A}$ by pull back via z_R.*

Proof: We can assume that z_R is the natural morphism defined by the completion of a local ring of $\mathcal{N}_{W(k)}$ in a k-valued point of it; here $k = \bar{k}$ contains $k(v)$. So [Va2, 2.3.5 and 2.3.11] reduces the statement of the theorem to 4.1.

From this theorem and the standard identification of H^1's (see [Va1, 5.1.2]) we get:

4.1.2.1. Corollary (the non zero specialization property). *If for a morphism z_R as in 4.1.2, t_α^R is integral (resp. is integral and non zero modulo the maximal ideal of R), then for any other morphism $z_{R_1}\colon Spec(R_1) \to \mathcal{N}$, with R_1 a ring of the same type as R, $t_\alpha^{R_1}$ is integral (resp. is integral and non zero modulo the maximal ideal of R_1).*

4.2. Comments. It is worth commenting how 4.1 complements [Fa2, cor. 9]. The new things worth pointing out are:

a) In 4.1 and 4.1.1-2 we have no restriction on the degrees of the cycles involved;

b) 4.1 and 4.1.1 work out for the case $p = 3$ as well (we recall that [Fa2, cor. 9] does not bring any new information for $p = 3$);

c) The whole of 4.1 makes sense for tensors of arbitrary high odd degree (of course the odd degrees do not show up in the case of Shimura varieties of Hodge type; see [Va1, 4.1]);

d) About the even degree tensors we do not assume that they have equal partial degrees (i.e. we do not assume, for instance in the context of 4.1, that $t_\alpha \in (H^1_{dR}(D) \oplus H^1_{dR}(D)^*)^{\otimes m_\alpha}[\frac{1}{p}]$, for some $m_\alpha \in \mathbb{N}$);

e) The proofs of 3.1 and 4.1 can be adapted to many other situations. In particular, we have in mind:

– the "H^2"-situation for K3-surfaces, see [Va5];

– all the situations which can be deformed in a way to a quasi CM-type situation (cf. V4 of 3.2);

– situations involving other Fontaine's categories $\mathcal{MF}_{[a,b]}(*)$ (cf. [Va2, 3.4]).

What is the weak point of 4.1: we are always in a context of reductive groups and some extra conditions hold, cf. [Va2, def. 1-2) of 2.2.8]; we recall that in [Fa2, th. 5 or cor. 9] such a reductive context is not required. However, in [Va4] we will see that the same methods and ideas apply to a larger (usually still smooth) context (cf. [Va2, 4.6.8 2) and 3) of the Appendix]). For most geometrical applications, this larger context suffices.

4.2.1. Remarks. 1) The whole of §3 and of 4.1 can be adapted easily to the more general context of generalized Shimura p-divisible objects of the category $\mathcal{MF}(W(k))$ (see def. [Va2, 2.2.8 3)]), cf. [Va2, 3.1.7] and the principle underlying V4 of 3.2 (see [Va4] and part 2D for details). Of course, as the proof of Case 4 of 3.1 reveals, some precautions are in order. One usually just simplifies the things by assuming that $p-1$ is bigger than the degrees of the cycles involved; however in many cases such assumption is not needed, cf. [Va2, 3.4.1.6].

2) There are no Shimura σ-crystals (M, ϕ, G) over k, with G a semisimple group (cf. the assumptions of [Va2, 2.2.8 c)]). However there are plenty of generalized Shimura p-divisible objects (M, ϕ, G) over k (see def. [Va2, 2.2.8 4)]), with G a semisimple group (most common with G an adjoint group). So for a generalized Shimura p-divisible object $\mathcal{C}$ over k we can have (for some $\alpha \in \mathcal{J}$) $\deg(t_\alpha)$ equal to 1 or 3 (or more generally to any other positive odd number). We would like to point out that the tensors t_α of degree 3 are particularly important, as they are related to some algebra structure on $\mathcal{C}$ (associated to some morphism $M \otimes M \to M$) or to some coalgebra structure on $\mathcal{C}$ (associated to some morphism $M \to M \otimes M$). So we can speak about generalized Shimura algebra (or Hopf algebra) p-divisible objects over k or of $\mathcal{MF}(W(k))$. We will not pursue these ideas here.

3) We think it is possible to prove Tate conjecture for abelian varieties over finite fields in many cases, starting from V7 (or V9) of 3.2.

4) Let $(f, L_{(p)}, v)$ be a SHS. Often there is an injective map $(T, \{x_T\}, H_T, v_T) \hookrightarrow (G, X, H, v)$ of Shimura quadruples, with T a torus, such that the closure of T in $GSp(L_{(p)}, \psi)$ is a torus equal to the center of its centralizer in $GSp(L_{(p)}, \psi)$. In this case we can get a simpler proof of V3 of 3.2: Case 1 of 3.1 suffices, as it can be checked easily starting from the techniques of [Va2, 3.4.2]; in other words the situation can be deformed to a quasi CM-type situation (cf. V4 of 3.2), for which Case 1 of 3.1 applies. The same applies for a $p = 2$ SHS.

5) It is easy to see that in the context of Faltings' example of the remark following [Fa2, cor. 9], one can not put any structure of a Shimura filtered σ-crystal.

6) For a generic form of 3.1 (resp. of V3 of 3.2) with effectively computable estimates, [Va3, 4.3.4.8 b)] would have been enough, cf. Case 3 (resp. Case 2) of 3.1. So Case 4 of 3.1 was the needed part for handling the case of small primes.

7) V3 of 3.2 fulfils the promise of [Va2, d) of 4.4.1 3)]. So accordingly we get simplifications in [Va2, 4.5.4, 4.9.2, 4.11.2, 4.12.5 and 4.12.12]. Moreover, when combined with [Va2, 2.3.6], it fulfils the promise of [Va1, 5.6.5 h)], and so also the one of [Va1, 5.8.8].

8) It seems to us that the assumption of the Theorem of V3 is not needed. This is so at least in the case when y_V is a toric point (see def. [Va2, 4.5.12.2]), provided we can choose a lift $z'_V\colon \mathrm{Spec}(W(k)) \to \mathcal{N}$ of it such that its associated Shimura filtered σ-crystal is a quasi CM type lift (in the sense of [Va2, 2.2.17]): the same arguments apply.

9) V9 of 3.2 is very much related to [Va2, 4.12.11]. It can be used to answer [Va2, 4.12.11.1] (see part three of the paper for details).

4.3. The connection to [Va3, 2.2]. For the terminology: an abelian variety over a DVR of mixted characteristic has the MC property, we refer to [Va3, 2.2.0].

We consider a SHS $(f, L_{(p)}, v)$ and a morphism $\mathrm{Spec}(R) \to \mathcal{N}$, with R a DVR of mixed characteristic. We assume that the abelian scheme A we get over $R[\frac{1}{p}]$ via it, has G as its Mumford-Tate group (cf. [Va3, 2.2.1-2]). From V3 of 3.2 we get directly that A has the MC property. A similar result can be stated for a $p = 2$ SHS, provided we are in the context of V6 of 3.2.

Combining these facts with [Va2, 2.3.6] and with [Va1, 5.8.6] we get:

4.3.1. Corollary. *We refer to the situation of [Va3, 2.2.2]. The number $M(f, L) \in \mathbb{N}$ introduced in [Va3, 2.2.2.1] is effectively computable.*

4.3.1.1. Example. We assume that $E(G, X) = \mathbb{Q}$ and that G is unramified over $\mathbb{Q}_q$, for any odd prime q. We also assume that the closure of G in $GL(L \otimes \mathbb{Z}_{(q)})$ is a reductive group over $\mathbb{Z}_{(q)}$ for any odd prime q. Plenty of such examples can be constructed (see [Va1, 5.7.5]). Then we can take

$$M(f, L) = \max(5, \frac{\dim(W)}{2}),$$

cf. [Va1, 5.8.6] and [Va2, 2.3.6]. In fact, assuming [Va2, 2.3.8 2)], we can take $M(f, L) = 3$.

4.3.2. Corollary. *Let A be an abelian variety over a number field E. Then there is an effectively computable number $M(A) \in \mathbb{N}$ such that:*

– A has good reduction with respect to any odd prime equal to or greater than $M(A)$;

– the Néron model of A with respect to any prime of E dividing a prime $p \geq M(A)$ has the MC property.

Any concrete sharp computations of the above numbers $M(f, L)$ and $M(A)$ are based on [Va1, 5.8.4-6] and on [Va2, 2.3.6 and 2.3.8 2)]. We refer to part 2C for the proof of [Va2, 2.3.8 2)] and for these computations.

4.4. Exercises. 1) We refer to the context of b) and b') of 2.1.4. Show that the triple $\mathcal{D}_{\text{ét}} = (H^1, G_{\mathbb{Z}_p}, (v_\alpha)_{\alpha \in \mathcal{J}})$ does not depend on the choice of f_V. Hint: use [Va3, 2.4] to achieve the replacement of Y by $\mathrm{Spec}(W(k))[[T]])$ (with T an independent variable); then copy the argument of B) of 3.1 to achieve the desired "movement" in families (to be compared with V5 of 3.2).

2) Show that the convention [Va2, 2.3.9.2] is superfluous, i.e. what we assumed in loc. cit. is always satisfied, under no restriction. Hint: with the notations of [Va2, 2.3.4], it is enough (argument using specializations, cf. [Va2, 4.2.1]) to check that $\tilde{G}$ is split, for $z \in \mathcal{N}(W(k))$ a G-canonical lift; but in this case V3 can be entirely adapted to give us that $\tilde{G}$ is a split group.

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