Bo-Hae Im

$E/\mathbb{Q}$ an elliptic curve

$\sigma_1, \dots, \sigma_n \in G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

$\bar{\mathbb{Q}}(\sigma_1, \dots, \sigma_n) = \bar{\mathbb{Q}}^{\langle \sigma_1, \dots, \sigma_n \rangle}$

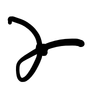
What is the rank of E over $\bar{\mathbb{Q}}(\sigma_1, \dots, \sigma_n)$?

<u>Conj</u>: The rank is always infinite.

<u>Def</u> A field K is anti-Mordell-Weil (AMW) if every non-isotrivial abelian variety A/K has infinite rank.

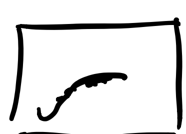
<u>Conj</u> G_K f.g. $\Rightarrow$ K is AMW.

<u>Def</u> A field K is <u>ample</u> if every pointed non-singular curve over K has infinitely many K-points.



$\mathbb{C}((t))$

<u>Th</u> (Fehm-Peterson) ample $\Rightarrow$ AMW



A

<u>Conj</u>: G_K f.g. $\Rightarrow$ K ample.

$$\begin{array}{c} E \\ {\scriptstyle 2:1}\downarrow \\ \mathbb{P}^1(\mathbb{Q}) \ni \alpha \end{array} \qquad n=1 \qquad \bar{\mathbb{Q}}(\sigma_1)$$

$y^2 = (x-a)(x-b)(x-c)(x-d)$

$y^2 = \qquad (x-c)(x-d)(x-e)(x-f)$

$y^2 = (x-a)(x-b) \qquad (x-e)(x-f)$

E.g. $(x-a)(x-b)(x-c)(x-d) \in (\overline{\mathbb{Q}}(\sigma_i)^K)^2$.

E^n/G $\qquad$ G is a finite group

Suppose $X/\mathbb{Q}$ is a $\mathbb{Q}$-rational curve in E^m/G.

This gives points in E^m which are defined over G-extensions.

$\sigma_1, \ldots, \sigma_n$ will map to G.

$E^{2k-1}/A_{2k} \supset X \cong \mathbb{CP}^1$

$K = \mathbb{Q}(\sigma)$

$x \in X(K)$

x comes from $(P_1, P_2, \ldots, P_{2k}) \in X(\overline{\mathbb{Q}})$.

$P_1 + \cdots + P_{2k} = 0$.

σ must have at least two cycles in its action on the P_i;

$P_1, \ldots, P_m$ could be one cycle.

$P_1 + \cdots + P_m \in E(K)$

Th If K is of char. 0 and G_K is cyclic, then K is AMW.

Suppose $E/\mathbb{Q}$ and suppose $G \subset \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ finite.
Suppose for all $g \in G \setminus \{1\}$
E_g has positive rank.
If $|G| > 2^m$, then

$$\mathrm{rk}\, E(\mathbb{Q}(\sigma_1, \dots, \sigma_m)) > 0.$$

E_2, E_3, E_6 all have positive rank.

Want $E_y, E_{2y}, E_{3y}, E_{6y}$ all have positive rank.

Iwaniec, Perelli-Pomykala,
$$L'(1, E_y) \neq 0$$

$E^n/$ elementary abelian 2-group

$E_1 \times \dots \times E_n$ / elementary ab. 2-group
Are there rational curves at all?

Does $E_1 \times \dots \times E_n / \pm 1$ have diagonal rational curves.

Th (I, –, Sailun Zhan) for $n \geq 4$, generically there are no rational curves.

<u>Additive</u> <u>combinatorics</u>

<u>Th</u> (van der Waerden)
If $\mathbb{N}$ are partitioned into finitely many subsets, some subset must contain an arbitrarily long arith prog.

<u>Ex</u> $y^2 = x(x-1)(x-2)(x-4)$

In $\overline{\mathbb{Q}}(\sigma_1, \ldots, \sigma_m)$, by Kummer theory there are only 2^m classes modulo squares.

$\dim \overline{\mathbb{Q}}(\sigma_1, \ldots, \sigma_m)^\times \otimes \mathbb{F}_2 \leq m$.

For all $n \in \mathbb{N}$ consider the class of n modulo squares in $\overline{\mathbb{Q}}(\sigma_1, \ldots, \sigma_n)$

Suppose we have an arithmetic progression

$$m,\ m+d,\ m+2d,\ m+3d,\ m+4d \in \alpha(\overline{\mathbb{Q}}(\sigma_1, \ldots, \sigma_n)^\times)^2 .$$

$$m(m+d)(m+2d)(m+4d) \in (\overline{\mathbb{Q}}(\sigma_1, \ldots, \sigma_n)^\times)^2$$

$$\Rightarrow \frac{m}{d}\left(\frac{m}{d}+1\right)\left(\frac{m}{d}+2\right)\left(\frac{m}{d}+4\right) \in \square .$$

That gives a source of rational points.

Th If K has char. 0, G_K is f.g. and E/K has rational 2-torsion, then rank $E(K) = \infty$.

Heegner point arguments.

Th (Breuer - Im) proved using Heegner point methods that for elliptic curves over $\mathbb{Q}$ and for $n=1$, rank has to be infinite.

Th (Dokchitser and Dokchitser) proved that the parity conjecture implies for elliptic curves over number fields that rk $E(\overline{\mathbb{Q}}(\sigma_1, \ldots, \sigma_n)) = \infty$.

Geyer - Jarden

rk $E(\overline{\mathbb{Q}}(\sigma)) = \infty$
holds with prob. 1 in Haar meas.

$A/\mathbb{Q}$ $\quad A(\overline{\mathbb{Q}}) \otimes \mathbb{C}$

is a rep. of the free profinite group on n generators.

$F_n = \langle x_1, \ldots, x_n \rangle$

$\sigma_1, \ldots, \sigma_n$ defines an action of F_n on this infinite-dimensional space.