Towards Minimalistic N-U Theorems

Work in progress / with Adam Topaz

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I) Introduction: Notation/Motivation

- $K_0, L_0, \dots$ global fields, $\mathbb{P}(K_0)$ places of $K_0, \dots$
- $G_{K_0} = \mathrm{Aut}(K_0^{\mathrm{sep}}|K_0)$ abs. Galois group, and its several can quotients, e.g.:

$$G_{K_0}^{\mathrm{sol}} = \mathrm{Aut}(K_0^{\mathrm{sol}}|K_0),\ G_{K_0}^{\mathrm{ab}} = \mathrm{Aut}(K_0^{\mathrm{ab}}|K_0), \dots$$

- For Galois extensions $K|K_0$, $L|L_0$, consider isoms of field extensions

$$\mathrm{Isom}^{\mathrm{F}}(L|L_0, K|K_0) \text{ isoms up to Frobenius twists.}$$

- Recall the famous result by Neukirch and Uchida:

THM (Classical Neukirch & Uchida Thm)

In the above notation, on has a canonical bijection

$$\mathrm{Isom}^{\mathrm{F}}(L_0^{\mathrm{sep}}|L_0, K_0^{\mathrm{sep}}|K_0) \to \mathrm{Isom}(G_{K_0}, G_{L_0})$$

Comments:

- Not co-authored (!); further contributions by Ikeda, Komatsu, Iwasawa, …
- Neukirch developed the "local theory" (1960's): *"what looks like, it is indeed."* Get:

$\Phi \in \mathrm{Isom}(G_{K_0}, G_{L_0})$ defines can bij. dec. groups $\mathcal{D}_{\mathbb{P}(K_0)} \to \mathcal{D}_{\mathbb{P}(L_0)}$, etc.

- Uchida (1976+): *constructive* proof for $\mathrm{char}(K_0) > 0$, hom-form over $\mathbb{Q}$, etc.
- The "local theory" and all assertions hold correspondingly for $G_\bullet^{\mathrm{sol}}$, hence:

$$\mathrm{Isom}^{\mathrm{F}}(L_0^{\mathrm{sol}}|L_0, K_0^{\mathrm{sol}}|K_0) \to \mathrm{Isom}(G_{K_0}^{\mathrm{sol}}, G_{L_0}^{\mathrm{sol}})$$

I) Introduction: Notation/Motivation

Comments (cont): The modern/present times (& some *anab. geometry*)

- P (1995): For all K_0, L_0 infinite finitely generated fields, one has:

$$\mathrm{Isom}^{\mathrm{F}}(L_0^{\mathrm{sep}}|L_0, K_0^{\mathrm{sep}}|K_0) \to \mathrm{Isom}(G_{K_0}, G_{L_0})$$

- Tamagawa (1997), Mochizuki (2007): Uchida's Thm holds at the level of *fundam groups of hyperbolic $\mathbb{F}_p$-curves*, say $K_0 = \mathbb{F}_p(X_0)$, $L_0 = \mathbb{F}_q(Y_0)$:

$$\mathrm{Isom}^{\mathrm{F}}(X_0, Y_0) \cong \mathrm{Out}\big(\pi_1(X_0), \pi_1(Y_0)\big).$$

- Saidi–Tamagawa (2017): Uchida's Thm for $G_{K_0}^{\Sigma_{K_0}}$, $\mathrm{char}(K_0) > 0$, Σ_{K_0} non-slim:

$$\mathrm{Isom}^{\mathrm{F}}(L_0^{\Sigma_{L_0}}|L_0, K_0^{\Sigma_{K_0}}|K_0) \to \mathrm{Isom}\left(G_{K_0}^{\Sigma_{K_0}}, G_{L_0}^{\Sigma_{L_0}}\right) \text{ is bijective.}$$

(Here $\Sigma_\bullet$ are sets of primes, and $G_\bullet^{\Sigma_\bullet}$ is the pro-$\Sigma_\bullet$ completion.)

- Hoshi (2019): Effective reconstruction of *number fields* K_0 from $G_{K_0}^{\mathrm{sol}}$.
- **Unfortunately**, no such result for $X_0 \subset \mathcal{O}_{K_0}$ Zariski open, and $\pi_1(X_0)\dots$
- Saidi–Tamagawa (2019+ϵ): *m-step solvable **N&U Thm** for number fields*:

$$\mathrm{Isom}(L_0^m|L_0, K_0^m|K_0) \to \mathrm{Isom}\left(G_{K_0}^{m+3}, G_{L_0}^{m+3}\right)/\!\sim \text{ is a bijection for } m > 0.$$

(Here $\Phi^{m+3} \sim \Phi'^{\,m+3}$ if their "restrictions" to $G_{K_0}^m$ are equal.)

- "*Non-anabelian type results*" by Cornelissen, de Smit, Marcolli, H. Smit…
- One recovers $\mathrm{Isom}(L_0, K_0)$ from "decorated" $\mathrm{Isom}^*(\widehat{G}_{K_0}, \widehat{G}_{L_0})$.

II) The work in progress/Notations

- **Fundamental Q: Which/how much Galois information detects K?**

- Let $\Lambda = \mathbb{Z}/\ell$, or $\Lambda = \mathbb{Z}_\ell$ (with $\ell > 2$), or $\Lambda = \prod_{\ell' \in S} \mathbb{Z}/\ell'$ with S infinite
- For fields F, let $F^c|F^a|F$ be the Λ *abelian-by-central*, resp. *abelian* extensions
- If $F|F_0$ Galois, $F^c|F^a|F|F_0$ are Galois, and one has the exact sequence:

$$1 \to \mathcal{G}_F^c \to \mathcal{G}_{F^c|F_0} := \mathrm{Gal}(F^c|F_0) \to \mathcal{G}_{F|F_0} := \mathrm{Gal}(F|F_0) \to 1$$

- For $F|F_0$, $E|E_0$ Galois, consider $\mathcal{G}_{F^c|F_0} \to \mathcal{G}_{F|F_0}$, $\mathcal{G}_{E^c|E_0} \to \mathcal{G}_{E|E_0}$. Denote:

$$\mathrm{Isom}^c(\mathcal{G}_{F|F_0}, \mathcal{G}_{E|E_0}) = \left\{\Phi \in \mathrm{Isom}(\mathcal{G}_{F|F_0}, \mathcal{G}_{E|E_0}) \,\middle|\, \Phi \text{ prolongs to } \mathcal{G}_{F^c|F_0} \xrightarrow{\Phi^c} \mathcal{G}_{E^c|E_0}\right\}$$

- For $\Sigma \subset$ **Primes**, $\delta(\Sigma) = 1$, let $\Sigma_{F_0} \subset \Sigma$ be the totally split part of Σ in F_0.
- Given Σ, consider Galois extensions $F|F_0$ satisfying the following:

- **Hypothesis** $(\mathrm{H})_\Sigma$: One has $\mu_\Lambda \subset F$, and for almost all $p \in \Sigma_{F_0}$, all the prolongations $v|p$ of p to F satisfy $\mathbb{Q}_p^{\mathrm{ab}} \subset F_v$.

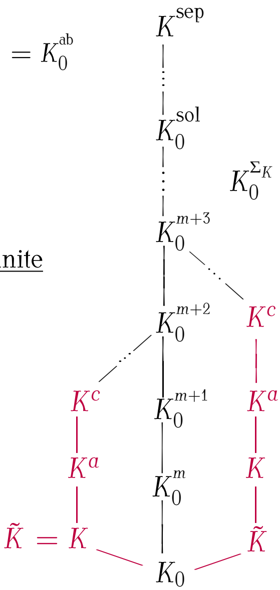
$(*)$ Further, if $\Lambda = \mathbb{Z}/\ell$, then $\sqrt[n_p]{p} \in F_v$, where $n_p = \ell(p-1)$.

Examples. Suppose that $\mu_{p^\infty} \subset F$ for almost all $p \in \Sigma$, e.g. $F_0^{\mathrm{ab}} \subset F$.
- If $\Lambda \neq \mathbb{Z}/\ell$, then $F|F_0$ satisfies $(\mathrm{H})_\Sigma$.
- If $\sqrt[n_p]{p} \in F$ for almost all $p \in \Sigma_{F_0}$, then $F|F_0$ satisfies $(\mathrm{H})_\Sigma$.

$\underline{m=1}$.

i.e., $K_0^m = K_0^{\mathrm{ab}}$

$\underline{m>1}$. $K := K_0^m$ satisfies $(\mathrm{H})_\Sigma$.



provided $2 \notin \Sigma_K$

$\underline{\Lambda \text{ infinite}}$

$\underline{\Lambda = \mathbb{Z}/\ell}$

Main Result (P-Topaz/work in progress)

For number fields K_0, L_0 and Galois extensions $K|K_0$, $L|L_0$ satisfying Hypothesis $(\mathrm{H})_\Sigma$, *one has a canonical bijection:*

$$\mathrm{Isom}(L|L_0, K|K_0) \to \mathrm{Isom}^c(\mathcal{G}_{K|K_0}, \mathcal{G}_{L|L_0}).$$

Comments:

- Let $K = K_0^{\mathrm{ab}}$, $L = L_0^{\mathrm{ab}}$. Then $\Phi^c(\mathcal{G}_K^{\mathrm{c}}) = \mathcal{G}_L^{\mathrm{c}}\ \forall\ \Phi^c \in \mathrm{Isom}(\mathcal{G}_{K^c|K_0}, \mathcal{G}_{L^c|L_0})$.
  Hence $\mathrm{Isom}^c(\mathcal{G}_{K|K_0}, \mathcal{G}_{L|L_0}) = \mathrm{Isom}(\mathcal{G}_{K^c|K_0}, \mathcal{G}_{L^c|L_0})/\sim$
- Similar holds for any "*verbal*" Galois extension $K|K_0$ satisfying $(\mathrm{H})_\Sigma$.

- **In particular**:

(1) $K_0^{\mathrm{ab}}|K_0$ *is functorially encoded in its* Λ *abelian-by-central extension,* $\Lambda \neq \mathbb{Z}/\ell$.

(2) **Minimalistic N & U Thm.** *Let* $K|K_0^{\mathrm{ab}}$ *be the* $\mathbb{Z}/\ell$ *elementary abelian extension. Then* $K|K_0$ *is functorially encoded in its* $\mathbb{Z}/\ell$ *abelian-by-central extension.*

(3) The above results hold in the same form for $K_0^{\mathrm{cycl}}|K_0$ in stead of $K_0^{\mathrm{ab}}|K_0$.

(4) The result (1) holds for finitely generated fields F_0 with $\mathrm{char}(F_0) = 0$.

- **Unfortunately**, we cannot **(yet?)** prove (2) for all fin. gen. F_0, $\mathrm{char}(F_0) = 0$.

III) Work in progress/Methods

- Two main Steps: (1) **Local Theory**(LT); (2) **Global Theory** (GT)

- **To (2)**: This is inspired by / uses work of **Harry Smit**. In a nutshell: For every finite Galois subextension $K_0'|K_0$ of $K|K_0$, consider $\Sigma_{K_0'} \subset \Sigma_{K_0}$. TFH:

- $\Sigma_{K_0'}$ and its prolongation $\Sigma(K_0')$ can be *effectively* described via the (LT).
- In particular, for $p \in \Sigma_{K_0'}$ one can detect $\Sigma(K_0') \cap \mathbb{P}_p(K_0')$, hence $[K_0' : \mathbb{Q}]$
- For $v'|p$, $p \in \Sigma_{K_0'}$, one can *effectively* detect $\mathrm{Frob}_{v',\ell}$ via the (LT).
- Hence one can effectively detect the ℓ-character group $\widehat{G}_{K_0'}$, etc.
- By (LT), everything is invariant under $\Phi \in \mathrm{Isom}^c(\mathcal{G}_{K|K_0}, \mathcal{G}_{L|L_0})$.

- **To (1)**: The main technical tool is the (long) ***c.l.p. strory*** : Ware, followed by Jacob, Arason, Bogomolov, B-Tschinkel, Koenigsmann, Efrat, Minac, **Topaz**, ...

Thm. *Let $\mu_\ell \subset F$, and $\sigma, \tau \in \mathcal{G}_F^a$ be independent. If σ, τ have commuting liftings to $\mathcal{G}_F^c$ under $\mathcal{G}_F^c \twoheadrightarrow \mathcal{G}_F^a$, there is $w \in \mathrm{Val}(F)$ such that $\sigma, \tau \in D_w^a$.*

Fact. *If $w \in \mathrm{Val}(F)$ has $D_w^a \neq 1$, then $D_{w|w_0} = \mathrm{Stab}_{\mathcal{G}_{F|F_0}}(D_w^a)$.*

- Note that the above "recipies" are invariant under isoms. Hence get:
- $\Phi \in \mathrm{Isom}^c(\mathcal{G}_{K|K_0}, \mathcal{G}_{L|L_0})$ gives rise to $\varphi_\Phi : \Sigma_{K_0'} \to \Sigma_{L_0'}$ + extra info ...

- Prove the *Minimalistic N&U Thm* in higher dimensions.
- Is the condition $\Phi(\mathcal{G}_K^c) = \mathcal{G}_L^c$ automatically satisfied under Hypo $(\mathrm{H})_\Sigma$?
- What about "much smaller " sets Σ, i.e., not necessarily $\delta(\Sigma) = 1$?
- Is there a 3-step *Minimalistic N&U Thm* with S finite ?
- Can one prove the above in higher dimensions as well ?
- **BigBigBig Q:** Does the 3-step *Minimalistic N&U Thm* hold ?

- What about positive characteristic ?

THANK YOU !

…and

STAY SAFE !