The subleading term of p-adic L-functions

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§1 The subleading term

$E: y^2 = x^3 + ax + b$; $a, b \in \mathbb{Q}$. Conductor N

$E(\mathbb{Q}) \cup \infty \cong \mathbb{Z}^{\text{rank}} \times E_{tor}(\mathbb{Q})$ all day: $(p, N) = 1$

$a_p := p + 1 - |E(\mathbb{F}_p)|$

$$L(E,s) = \prod_{p \nmid N} (1 - a_p p^{-s} + p^{1-2s})^{-1} \times \prod_{\ell | N} (\dots)$$

Known: $= a(s-1)^r + b(s-1)^{r+1} + \dots$

$(a \neq 0)$

Conj. (Birch, Swinnerton-Dyer)

- $r \overset{?}{=} \text{rank}$
- $a = \#Ш(E) \dfrac{\Omega \operatorname{Reg} \prod_{\ell | N} c_\ell}{r! \, |E_{tor}(\mathbb{Q})|^2}$

a : "leading term". b : "subleading term".

Q What does b encode?

Thm (Wuthrich, 2016)

$$b = a \times \Big(-\frac{1}{2} \log(N) + \log(2\pi) + \underset{\approx 0.577216}{\text{Euler's constant}} \Big)$$

Pf Put $\Lambda(s) := \dfrac{\sqrt{N}}{2\pi} \Gamma(s) L(E,s)$.

Fn'al eq'n : $\Lambda(s) = (-1)^r \Lambda(2-s) \Rightarrow \dfrac{d^i}{ds^i} \Lambda(s)\Big|_{s=1} = 0$ if $i \not\equiv r \pmod 2$

Q: p-adic version? What else does the functional equation give us?

§2 p-adic L-functions.

	Complex-analytic	p-adic analytic
$\mathbb{Z}$	$\zeta(s)$	$\zeta_p(X) \in \mathbb{Z}_p[[X]]$
$\mathbb{Z}, \psi$ ψ: Dirichlet char.	$L(\psi, s)$	$L_p(\psi, X)$.
	interpolate at $s = 1-2n$	
E	$L(E, s)$	$L_p(E, X)$
E, ψ	$L(E, \psi, s)$	$L_p(E, \psi, X)$

Idea: $L_p(E, X)$ interpolates $\sum \frac{L(E, \chi_{p^n}, 1)}{\alpha^n} x(\ldots)$ χ_{p^n}: Diri-char. of cond. p^n.

(Mazur + Swinnerton-Dyer 1970) $\alpha =$ sol'n of $Y^2 - a_p Y + p = 0$.
$(Y - \alpha)(Y - \beta)$

Two scenarios:

- $p \nmid a_p$; $\mathrm{ord}_p(\alpha) = 0 \Rightarrow L_p(E, X)$ and $L_p(E, \psi, X)$ "$\in$" $\mathbb{Z}_p[[X]] \cancel{\otimes \mathbb{Q}}$
- $p \mid a_p$; $\mathrm{ord}_p(\alpha) > 0$
  $\Rightarrow L_p(E, X)$ and $L_p(E, \psi, X) \in \mathbb{C}_p[[X]]$

(Amice + Vélu, Višik, 1970's)
Вишик

$\underline{\text{Thm}}$ (Pollack, $a_p = 0$ [2000] ; S. plap [2012])

Gave explicit construction of $L_\#(E,\psi,X)$, $L_b(E,\psi,X) \in \mathbb{Z}_p[[X]]$

from $L_p(E,\psi,X) \notin \mathbb{Z}_p[[X]]$.

Important fact: L_p, $L_\#$, L_b all satisfy functional eq'n.

§3 Theorems about p-adic L-functions

Write $L_p(E,X) = aX^r + bX^{r+1} + \dots$

p-adic BSD still possible (Mazur, Tate, Teitelbaum 1980's)

Let $p > 2$ from now on.

$\underline{\text{Thm}}$ (Bianchi, 2017)

$$b = -\frac{a}{2}\left(\log_\gamma(N) + r\right)$$; $\log_\gamma$: p-adic version of log that knows relationship btw. s and X.

p-adic Weierstraß Preparation Thm

$$f \in \mathbb{Z}_p[[X]] \implies f = p^\mu\left(X^\lambda + c_{\lambda-1}X^{\lambda-1} + \dots + c_0\right)U.$$

$\mu, \lambda \in \mathbb{N}^{\geq 0}$; $c_i \in p\mathbb{Z}_p$; $U \in \mathbb{Z}_p[[X]]^\times$

$\mu(f)$: "μ- invariant of f"

$\lambda(f)$: "λ- invariant of f".

Let $p \nmid a_p$. Let ψ be a Dirichlet character with conductor prime to p.
$\bar{\psi}$ be complex-conjugate of ψ.

Thm (Bianchi)

$$\mu(L_p(E, \psi, X)) = \mu(L_p(E, \bar{\psi}, X)).$$

Pfs use fnal eq'ns.

§ 4 Results.

Let $p \mid a_p$. Write $L_\sharp(E, X) = a_\sharp X^{r_\sharp} + b_\sharp X^{r_\sharp + 2} + \ldots$

and $L_\flat(E, X) = a_\flat X^{r_\flat} + b_\flat X^{r_\flat + 1} + \ldots.$

Thm (Dion + S.)

$$b_{\sharp/\flat} = -\frac{a_{\sharp/\flat}}{2}(\log_\gamma(N) + r_{\sharp/\flat})$$

Thm (Dion + S.)

- $\mu(L_{\sharp/\flat}(E, \psi, X)) = \mu(L_{\sharp/\flat}(E, \bar{\psi}, X))$.
- $\lambda(L_{\sharp/\flat}(E, \psi, X)) = \lambda(L_{\sharp/\flat}(E, \bar{\psi}, X))$
- (when $p \nmid a_p$, have $\lambda(L_p(E, \psi, X)) = \lambda(L_p(E, \bar{\psi}, X))$.

Conj. (S., 2015) $r_\sharp \stackrel{?}{=} r_\flat \stackrel{?}{=} \text{rank}$.

$\Longleftrightarrow$ (p-adic BSD of $L_p(E, X)$
by Bernardi + Perrin-Riou (+MTT)).

Thm (Dion + S.) $r_\# \equiv r_b \pmod 2$

Pf. Functional Equations state

$$L_\#(E, X) = \pm (1+X)^{-\log_\gamma(N)} L_\#\left(E, \frac{1}{1+X} - 1\right)$$
$$L_b(E, X) = \pm (1+X)^{-\log_\gamma(N)} L_b\left(E, \frac{1}{1+X} - 1\right)$$

same sign!

Differentiate the above $r_\#$ resp. r_b times and evaluate

at $X = 0$. Get $(-1)^{r_\#} = (-1)^{r_b} \implies r_\# \equiv r_b \pmod 2$

Q (Dinesh Thakur/दिनेश ठाकुर)

What about sub-subleading terms?

A No idea in the # field case.

fn field case $\longrightarrow$ Zhiwei Yun + Wei Zhang / 惲之瑋 + 張偉
恽之玮 + 张伟

Q Dirichlet L-fns?

A See Colmez's 1993 Annals paper.

Q (Ravi Ramakrishna /(not sure which script))

Relation with Greenberg - Stevens?

A Maybe — spec. value at 0 vs at 1.