(joint with Shuddhodan)

I. Formulation of result

Notation: let X scheme of f.t$/\overline{\mathbb{F}_q}$ defined$/\mathbb{F}_q$

$\phi_{q^n}: X \to X$ geometric Frobenius

$\Gamma_{q^n} = \{(x, \phi_{q^n}(x))\} \subset X \times X$ graph of ϕ_{q^n}.

let $c: C \to X \times X$ of schemes f.t$/\overline{\mathbb{F}_q}$
$\quad\;\; \overset{\shortparallel}{(c_1, c_2)}$

Question: $|c^{-1}(\Gamma_{q^n})| = ?$

Particular case: $c: C \hookrightarrow X \times X$, then
$c^{-1}(\Gamma_{q^n}) = C \cap \Gamma_{q^n}$.

Thm 1: (1) Assume that either c_1 or c_2 is quasifinite. Then $\forall n \gg 0$, $|c^{-1}(\Gamma_{q^n})| < \infty$

Moreover, $|c^{-1}(\Gamma_{q^n})| = O(q^{n \dim C})$ $n \gg 0$.

(2) Assume moreover, that C, X irreducible, and c_1, c_2 are dominant, then $\forall n \gg 0$

$$\boxed{|c^{-1}(\Gamma_{q^n})| = \frac{\deg(c_1)}{\deg(c_2)_{insep}} q^{n \dim X} + O\left(q^{n(\dim X - \frac{1}{2})}\right)}$$

Thm 2 (uniform version) k - alg. closed field of char p, $\sigma_q: k \xrightarrow{\sim} k$ arith Frobenius

X - f.t/k; $\Phi_q : X \to {}^{\sigma_q}X$, $\Gamma_q \subset X \times {}^{\sigma_q}X$.

Then $\forall\, c: C \hookrightarrow X \times {}^{\sigma_q}X$, the $|c^{-1}(\Gamma_q)|$ "has the same assymptotic as in Thm 1"

(1) $|c^{-1}(\Gamma_q)| = O(q^{\dim C})$ $q \gg 0$

"depends only on degrees of X and C, C'"

$\forall n, d\ \exists M\ \forall X, C$, $\deg X, C \le d$, $\dim X, C \le n$ $\forall q > M$

II. Corollaries: Cor 1: let $c: C \to X \times X$ morphism of f.t/$\mathbb{F}_q$, C, X - irreducible, c_1, c_2 - dominant, X is defined /$\mathbb{F}_q$.

Then $\forall n \gg 0$ $c^{-1}(\Gamma_{q^n}) \neq \emptyset$.

Cor 2: In the situation of Cor I, $\bigcup c^{-1}(\Gamma_{q^n}) \subseteq C$ is Zariski dense.

pf: set $Z := \overline{\bigcup c^{-1}(\Gamma_{q^n})}$, if $Z \neq C$

Apply Cor 1 to $C - Z \to X \times X$ ☒

Cor 3: let X scheme f.t/$\mathbb{F}_q$; $f: X \to X$ dominant.

Then the set of f-periodic points of X is Zariski dense (Fakhruddin)

pf: use Cor 2 for $(Id, f): X \to X \times X$. ☒

III. Strategy of the proof: 2 methods

(A) of Thm 1+2 uses intersection theory, de Jong theorem on alterations, Pink's trick.

(B) of Thm 1 (Shuddhodan) sheaf-theoretic does not use de Jong.

Remarks: (a) By Noeth induction (1) $\Leftarrow$ (2)

(b) we can replace X, C by open subschemes thus assuming X, C affine, X-smooth $C \subset X \times X$ closed embedding.

(c) (*) $\Leftrightarrow$ $|c^{-1}(\Gamma_{q^n})| \deg(c_1)_{insep} = \deg(c_1) q^{n \dim X} + O(\quad)$

Goal to give cohomological interpretation of the LHS + use purity.

IV Particular case: Assume X is smooth projective, $c_1 : C \to X$ is étale

Not: $\forall$ alg var X $\forall i \leadsto A_i(X) = $ i-cycles/rat equiv

Grothendieck-Lefschetz trace formula

$\forall$ smooth proper X of dim d

$\forall [C] \in A_d(X \times X)$ defined

$H^i([C]) \in \mathrm{End}(H^i(X, \mathbb{Q}_\ell))$ $\forall i$ and

$\forall f: X \to X$ we have.

$$\boxed{[C]\cdot[\Gamma_f] = \sum_{i=0}^{2d} (-1)^i \operatorname{Tr}\left(f^* \circ H^i([C]), H^i(X, \overline{\mathbb{Q}}_\ell)\right)}$$

<u>Deligne purity Theorem (= Weil I)</u>

$\forall X$ as before $\forall \tau: \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$ $\forall$ e.v λ of $Fr^* \big|_{H^i(X, \overline{\mathbb{Q}}_\ell)}$ we have $|\tau(\lambda)| = q^{\frac{i}{2}}$.

<u>proof of Thm 1 in part case 1:</u>

$$\forall n \gg 0 \quad |C \cap \Gamma_{g^n}| \underset{\substack{\uparrow \\ \text{transversal intersection} \\ c_2: C \to X \text{ is étale}}}{=} [C]\cdot[\Gamma_{g^n}] = \sum_{i=0}^{2d} (-1)^i \operatorname{Tr}\left(\Phi_{g^n}^* \circ H^i([C]), H^i(X, \overline{\mathbb{Q}}_\ell)\right)$$

use purity + $H^{2d}(X, \overline{\mathbb{Q}}_\ell)$ is 1-dimension

$H^{2d}([C]) = \deg(c_1)\cdot \mathrm{Id}$, ∎

<u>IV. Particular case 2:</u> Assume that

(1) $\exists$ smooth compact. $\overline{X} \supseteq X$ s.t $\overline{X} - X$ is $\cup X_i$ — of smooth divisors with normal crossing.

(2) $\overline{X} - X$ is "locally c-invariant"

<u>Pink's construction:</u> $\overline{X}$ - smooth, $\{X_i\}_{i\in I}$ smooth div with normal. crossing.

Set $\widetilde{Y} = Bl_{\bigcup(X_i \times X_i)}(\overline{X} \times \overline{X}) \leftarrow$ smooth proper of dim $2d$.

<u>Ex</u>: $\overline{X} = \mathbb{A}^n$, $X_i = Z(x_i)$. Then $\widetilde{Y} = (\underline{Bl_0(\mathbb{A}^2)})^n$

$\overline{X} \times \overline{X} = (\mathbb{A}^2)^n$

<u>Notation</u>: $\forall J \subseteq I$ set $\bigcap_{i \in J} X_i := X_J \subseteq \overline{X}$

X_J - smooth of dimension $d - |J|$ (or $\emptyset$)

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\pi"] & \overline{X} \times \overline{X} \\ E_J \arrow[u, "i_J"] \arrow[r, "\pi_J"] \arrow[ur, phantom, "\square"] & X_J \times X_J \arrow[u, phantom, "\cup"] \end{tikzcd}$$

; $\forall J \subseteq I$ set $E_J := \pi^{-1}(X_J \times X_J)$

<u>Exercise</u>: E_J is smooth of dimension $2d - |J|$

<u>Notation</u>: $\forall [\widetilde{C}] \in A_d(\widetilde{Y})$ set.

$[\widetilde{C}]_J := (\pi_J)_* i_J^*([\widetilde{C}]) \in \underline{A_{d-|J|}(X_J \times X_J)}$

$\uparrow$ Gysin pullback

<u>lemma 1</u>: let $\widetilde{\Gamma}_{g^n} \subseteq \widetilde{Y}$ be the strict preimage of $\Gamma_{g^n} \subseteq \overline{X} \times \overline{X}$. Then $\forall [\widetilde{C}] \in A_d(X \times X)$,

$$[\widetilde{C}] \cdot [\widetilde{\Gamma}_{g^n}] = \sum_J (-1)^{|J|} [\widetilde{C}]_J \cdot [\Gamma_{X_J, g^n}]$$

<u>lemma 2</u>: If $\overline{X} - X$ is locally g-invariant, then $\widetilde{C} \cap \widetilde{\Gamma}_{g^n} \subseteq \pi^{-1}(X \times X) \Rightarrow [\widetilde{C}] \cdot [\widetilde{\Gamma}_{g^n}] = |C \cap \Gamma_{g^n}|$ $\forall n \gg 0$.

<u>Pf of Thm 1 in case 2:</u>

$$|C \cap \Gamma_{gu}| \underset{\substack{\uparrow \\ \text{Lemma 2}}}{=} [\tilde{C}] \cdot [\tilde{\Gamma}_{gu}] = \sum (-1)^{|J|} [\tilde{C}]_J \cdot [\Gamma_{X_J}, gu]$$

Finish as in case 1.

(VI) Locally invariant subsets:

Def: Let $c = (c_1, c_2) : C \to X \times X$, $Z \subseteq X$ closed

(a) Z is called c-invariant if

$c_1(c_2^{-1}(Z)) \subseteq Z$ set-theoretically.

$\iff c_2^{-1}(Z) \subseteq c_1^{-1}(Z)$

(b) $Z \subseteq X$ is called locally c-invariant

if $\forall x \in Z$ $\exists$ open $x \in U \subseteq X$ s.t.

$Z \cap U \subseteq U$ is c_U-invariant.

$c_U : c^{-1}(U \times U) \to U \times U$.

(c) Let $c : C \hookrightarrow X \times X$ closed embedding

$\bar{X} \supseteq X$ compactification, $\bar{c} : \bar{C} \hookrightarrow \bar{X} \times \bar{X}$

we say that $\bar{X} - X$ is locally c-invariant

if it is — " — $\bar{c}$-invariant

Prop: Let $c : C \hookrightarrow X \times X$ closed embedding,

s.t. X, C - irreducible, c_2 - dominant gen. finite,

Then $\exists$ open $U \subseteq X$ and compactification

$\bar{X} \supseteq U$ s.t. $\bar{X} - U$ is locally c_U-invariant.