Worksheet 1: Metric Spaces and Continuity

Math 461, Fall 2021

Thursday, September 2

Submit any progress toward solutions to the following exercises at end of class. Write the names of your group members at the top of your sheet. You'll be marked for submitting thoughtful work on solutions, not correctness or whether you have completed the work.

1. **Teleporter metric.** If you have any lingering doubt, check that the teleporter metric on a set X is, in fact, a metric.

2. **Bounded functions.** Consider the metric space $X = \mathrm{Bdd}([0,1],\mathbb{R})$, the set of bounded functions $f : [0,1] \to \mathbb{R}$, with the supremum metric d.

   (a) Find a pair of bounded functions $f, g : [0,1] \to \mathbb{R}$ such that $|f(1/2) - g(1/2)| = d(f,g)$.

   (b) Find a pair of bounded functions $f, g : [0,1] \to \mathbb{R}$ such that there is no value $x \in [0,1]$ such that $|f(x) - g(x)| = d(f,g)$.

   (c) What property can be imposed on f and g to guarantee that the behavior of the previous part does not happen?

3. **Metric subspaces.** Suppose (X, d_X) is a metric space. For any subset $Y \subset X$, the restriction of the function d defines a metric d_Y on Y. The metric is defined by the formula

$$d_Y(y_1, y_2) := d_X(y_1, y_2).$$

   We call d_Y the *subspace metric* on Y, and say Y is a *metric subspace* of X.

   (a) Convince yourself that d_Y is a metric on Y. What do you need to check? Write down details of any step that doesn't seem obvious.

   (b) Consider the sphere $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. On the second day of class we described a way to put a metric d_c on S by minimizing lengths of piecewise differentiable curves. (You can think of d_c as the "airplane metric"; it is the length of the flight path an airplane would take between airports on the sphere.) The sphere also has the subspace metric d_S inherited from the Euclidean metric d on $\mathbb{R}^3$. Are the metrics d_c and d_S the same? If not, can you qualitatively say how they are different? For example, how do open balls look in each metric?

4. **Coordinate functions.** Suppose (X, d) is a metric space, and consider $\mathbb{R}^n$ with the usual Euclidean metric. Any function $f : X \to \mathbb{R}^n$ is determined by n coordinate functions $f_k : X \to \mathbb{R}$, for $k = 1, \ldots, n$. For example, the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(x,y) = (x^2 + y, \cos(x), ye^x)$ has coordinate functions $f_1(x,y) = x^2 + y$, $f_2(x,y) = \cos(x)$, and $f_3(x,y) = ye^x$.

(a) Prove that if $f : X \to \mathbb{R}^n$ is continuous then each of its coordinate functions $f_k : X \to \mathbb{R}$ is continuous, with respect to the standard metric on $\mathbb{R}$. (You may use any definition of continuity that is helpful! It may be helpful to start by considering special cases, such as letting $X = \mathbb{R}$ and $n = 2$.)

(b) Prove the converse: If $f : X \to \mathbb{R}^n$ is a function whose coordinate functions $f_k : X \to \mathbb{R}$ are each continuous, then f is continuous.

5. **The Hilbert Cube.** Let X be the set of sequences of real numbers $\{x_n\}_{n=1}^{\infty}$ such that $0 \leq x_k \leq 1$ for all k. To simplify notation, write $\mathbf{x} = \{x_n\}_{n=1}^{\infty}$ for elements of X.

(a) Show that the formula

$$d(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{\infty} \frac{1}{2^k} |x_k - y_k|$$

defines a metric on the space X. (First make sure to explain why the sum converges!)

(b) There are infinitely many "coordinate functions" $p_i : X \to [0, 1]$ defined by $p_i(\mathbf{x}) = x_i$ for $i = 1, 2, \ldots$. Prove that each function p_i is continuous, where $[0, 1]$ is the unit interval with the standard metric.

(c) Is it true that, given any metric space (Y, d_Y), a function $f : Y \to X$ is continuous if and only if $p_i \circ f : Y \to [0, 1]$ is continuous for each i? (This is an infinite dimensional analogue of the previous problem.)