Math 455 Class 22 March 16

Midterm 1 has been graded:

mean $\approx$ 108 std. deviation $\approx$ 20.

Check the arithmetic on the front of your test!

Regrades: requests for regrades must be IN WRITING. If you request a regrade I will look at your <u>entire</u> <u>test</u> and your score may go up OR DOWN.

Today: Review of Independence.

Remember: independence is an ordinary word: you can look it up in the dictionary.

This is NOT the meaning of independence in probability theory.

Independence was defined in Wackerly 5.4 (p. 247).

The definition is that X and Y are independent if the cdfs $F_X(x)$ and $F_Y(y)$ and $F_{X,Y}(x,y)$ have the property:

$$F_{X,Y}(x,y) = F_X(x)\, F_Y(y).$$

This is a very strong property and many things follow from this.

For example, if X and Y are independent, then

$$E[XY] = E[X]\,E[Y].$$

$$\mathrm{Cov}(X,Y) = 0 = \mathrm{Cor}(X,Y)$$

$$E[Y \mid X] = E[Y]$$

None of these things imply independence.

Also, statements that are ~~grammatical~~ grammatically correct in English may not make sense in probability.

Example: "X is independent but Y is dependendent on X."

This doesn't make sense in probability because there is no concept of a single random variable being independent.

We can speak of a collection of 2 (or more) RVs being independent, but not a single RV.

More subtle errors: confusing the consequences of independence with the definition of independence.

To see the non-equivalence of the consequences with the definition, we consider an example.

From Math 447: Joint dist of X and Y

(Y) \ (X)	-1	0	1	marginal dist of Y
-1	$1/16$	$3/16$	$1/16$	$5/16$
0	$3/16$	0	$3/16$	$6/16$
1	$1/16$	$3/16$	$1/16$	$5/16$
marginal dist: X	$5/16$	$6/16$	$5/16$	

In this example: $p(x,y) = 0$ if $x=0$ and $y=0$.

But $p_X(x) \neq 0$ and $p_Y(y)$ ~~is not~~ $\neq 0$

if $x=0$ and $y=0$.

Thus $p(x,y) \neq p_X(x) \cdot p_Y(y)$

so X and Y are NOT independent.

In this example we have:

$$E[X] = E[Y] = 0.$$

Also $E[XY] = 0$ because

$$E[XY] = \frac{1}{16} + \frac{1}{16} + \frac{1}{16}(-1) + \frac{1}{16}(-1) = 0.$$

Thus $Cov(X,Y) = E[XY] - E[X]E[Y]$

$$= 0.$$

Hence $Cor(X,Y) = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y} = 0.$

Furthermore $E[Y|X] = 0 = E[Y]$

and $E[X|Y] = 0 = E[X].$

This is a calculation; example

$$E[X|Y=-1] = \frac{\left(\frac{1}{16}(-1) + \frac{3}{16}\cdot 0 + \frac{1}{16}\,1\right)}{(5/16)} = 0.$$

The point: none of these consequences of independence imply independence.

You might be remembering a result of the form: if, for any measurable functions g, h we have

$$E[g(X)\,h(Y)] = E[g(X)] \cdot E[h(Y)]$$

then X and Y are independent.

Such a result is beyond the scope of this course. If you don't know the word "measurable," think "measurable = reasonable".

For more on this: Math 501 or 505.

Wackerly Thm. 7.3 tells us (among other things) that ~~the~~ if $X_1, \dots, X_n$ are iid normal with given mean μ and var. σ^2.

then $\bar{X}$ (sample mean) and S^2 (sample variance) ~~are~~ are indep. RVs. 7

This result <u>requires</u> that the RVs be normal. and is not true for other distributions.

Intuition for why this independence requires normality: think of, for example, the Exponential Dist. with param λ has mean λ and var. λ^2. So if we know the mean, we have a good guess at the variance.

This is not a proof, but gives a sense that the result might not be true for other distributions.

Remarks on midterm 1:

this is only 15% of your grade.

any result is still possible.

Grade guidance: avg score $\cong$ B−.

avg score + 1 sd $\cong$ A−

− 1 sd $\cong$ C−.

Note the correspondence between the sample test and the actual test.

In particular, problem 10, which people generally had trouble with, was close to what was on the sample test.

LLMs can make up many problems similar to a given problem, so one sample test can be used to generate many sample tests.

Midterm 2: After Spring Break: mid April.

~ End of Ch. 9. (?).

Notes: I will catch up and post notes today.

One final note on independence:

Pairwise Independence $\not\Rightarrow$ Independence.

Let X, Y be indep. RVs which take values ± 1 (with prob $\frac{1}{2}$)

	$X = -1$	$X = 1$
$Y = -1$	$1/4$	$1/4$
$Y = 1$	$1/4$	$1/4$

Let Z by $Z = \begin{cases} 1 & \text{if } X = Y \\ -1 & \text{if } X \neq Y. \end{cases}$

Then X, Y are indep. Y, Z indep. X, Z indep, but X, Y, Z are NOT indep.