Math 525 **Alternate Midterm** **October 16, 2023**

Name: _____

Question	Points	Score
1	4	
2	4	
3	4	
4	14	
5	8	
6	12	
7	12	
8	6	
9	18	
10	15	
11	8	
Total:	105	

1. (4 points) Is the group ring $\mathbb{Z}[D_8]$ an integral domain? Why or why not? (Here D_8 is the dihedral group of order 8.)

2. (4 points) Let $I \subset \mathbb{Z}[x]$ be the set of all polynomials whose coefficient of x^2 is a multiple of 3. Is I an ideal of $Z[x]$? Why or why not?

3. (4 points) Is the ideal $(x^5 - 2x^2 + 3, x^3 - 7x + 1)$ in the ring $\mathbb{R}[x]$ a principal ideal? Why or why not?

4. Let $R = \mathbb{Z}[x]$, and define a norm $N\colon R - \{0\} \to \mathbb{Z}^+ \cup \{0\}$ by $N = \deg$, the degree of the polynomial.

   (a) (6 points) With this norm, is R a Euclidean Domain? Why or why not?

   (b) (8 points) If your answer in part (a) was "yes", show that R (with the norm N) satisfies the definition of "Euclidean Domain". If your answer was "no", give a specific example to that says part of the definition of "Euclidean Domain" is not satisfied.

5. (8 points) Chapter 8 discussed 3 major types of rings, and proved that there were certain relationships between them. What are the three types of rings, and what are the main relationships that were proved? (Hint: Every _____ is a _____ and every _____ is a _____. Also, look at the rest of the test if you're having trouble coming up with the right phrases.)

6. Answer the following questions about the ring $\mathbb{Z}[i]$.

   (a) (4 points) Is $\mathbb{Z}[i]$ a unique factorization domain? Why or why not?

   (b) (8 points) Give a factorization of $26 \in \mathbb{Z}[i]$ into irreducibles in this ring.

7. (12 points) What is the order and structure of the group of units of the ring $\mathbb{F}_2[x]/(x^4+x^2+1)$? Justify your answer.

8. (6 points) *How many* natural numbers simultaneously satisfy $1 \leq n \leq 735$ and the congruences below? What is the theorem that tells you the answer? *Hint:* $3 \cdot 5 \cdot 7^2 = 735$.

$$n \equiv 8 \mod 49 \qquad n \equiv 3 \mod 5 \qquad n \equiv 2 \mod 3$$

9. Fill in the blanks in the statement (a) below of Gauss' Lemma and answer the questions about this result.

(a) (4 points) (Gauss' Lemma): Let R be a unique factorization domain and F its field of fractions. If $p(x) \in R[x]$ is a polynomial which is reducible in $F[x]$, then _____.

(b) (6 points) Gauss' Lemma can be used to prove that if R is a unique factorization domain, then $R[x]$ is also a unique factorization domain. How does this proof use Gauss' Lemma?

(c) (8 points) Show that $\mathbb{Z}[2\sqrt{2}]$ is not a unique factorization domain. *Hint: You may assume without proof that* $\mathbb{Q}(\sqrt{2})$ *is the field of fractions of* $\mathbb{Z}[2\sqrt{2}]$.

10. (15 points) Let $f(x) = x^3 + 7x + 2 \in \mathbb{Z}[x]$. Show that this polynomial is irreducible. You may use the theorems we have discussed in class—just be specific in your reference to these results.

11. Let R be a Noetherian ring.

    (a) (2 points) What does the Hilbert Basis Theorem say about the polynomial ring $R[x]$?

    (b) (6 points) In order to prove the Hilbert Basis Theorem, there must be some connection between the ideals of R and the ideals of $R[x]$. Why does any such connection exist? *You need not prove the Hilbert Basis Theorem to answer this question. A short explanation of the connection between ideals of the two rings is sufficient.*