REGULARITY OF SOLUTIONS OF THE HEAT EQUATION ON A CONE

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ABSTRACT. We study differentiability properties of solutions to the heat equation on a cone manifold. The heat operator considered here is that which corresponds to the Friedrichs extension of the Laplacian. For initial data in appropriately defined Hölder spaces, the estimates are good enough to prove short time existence of solutions of certain semilinear parabolic equations.

1. INTRODUCTION

In this paper we study the differentiability properties of solutions to the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u &= 0 \\ u|_{t=0} &= f \end{cases}$$

on a compact surface M with cone singularities, and apply our results to short time existence of solutions of some nonlinear equations. A Riemannian metric g which is smooth away from a finite number of points $\{p_1, \ldots, p_m\}$ is said to have a cone singularity at a point p_i if in a neighborhood of p_i it is equivalent to $g(x,\theta) = dx^2 + \beta^2 x^2 d\theta^2$ for a parameter $\beta \in (0,1]$. The real number $\alpha = 1 - \beta$ is called the cone angle because a cone of radius one has a base of circumference $2\pi(1-\alpha)$. The case $\alpha = 0$ is just the Euclidean metric written in polar coordinates, that is to say, the punctured disk.

Each self-adjoint extension of the Laplacian gives rise to a heat operator, by the spectral theorem. Moreover, the extensions are fixed by requiring elements in the domain to have certain asymptotics at the singular set. For this reason, we assume without loss of generality that M has only one singular point. Note that because of the low dimension, the Laplacian on this cone possesses more than one self-adjoint extension. We take the Friedrichs extension.

Our estimates are for the heat operator of this extension between certain spaces which are compatible with the geometry and the analysis. Let us here establish some definitions. $C^0(M)$ consists of functions which are continuous on all of M. The space $C_b^k(M)$ contains those functions with up to k bounded b-derivatives. A b-derivative is a derivative by a vector field tangent to the boundary, that is, by $x\partial/\partial x$ or by $\partial/\partial\theta$. We also use the geometric Hölder space $C_g^\gamma(M)$ of functions for which

$$\sup_{p \neq q} \frac{|\, u(p) - u(q) \,|}{(d_g(p,q))^\gamma} < \infty.$$

In Section Four and Five these definitions will be written out in more detail. The main result of the paper is:

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Theorem 1.1. *The heat operator $e^{t\Delta}$ corresponding to the Friedrichs extension of the Laplacian is a bounded operator with estimates of the form*

$$e^{t\Delta} : C^0(M) \to C^k_b(M); \parallel u \parallel_k \leq Ct^{-k/2} \parallel f \parallel_0$$

and

$$e^{t\Delta} : C^\gamma_g(M) \to C^2_b(M); \parallel u \parallel_2 \leq Ct^{-1+\gamma/2} \parallel f \parallel_\gamma .$$

According to the first estimate, taking continuous initial data into twice-differentiable solutions produces a bound containing the factor t^{-1}. Asking for slightly less, taking C^γ into C^2 improves the estimate to $t^{-1+\gamma/2}$. The importance of this improvement is that it is good enough to reformulate certain nonlinear problems as contraction mappings. In fact it is for the applications to nonlinear problems that we choose to work in Hölder spaces rather than L^p or Sobolev spaces. Boundedness properties in those spaces have been given by Li [Li], Loya [Lo], and by Coriasco, Schrohe, and Seiler [CSS]. We remark that the results of our paper generalize to higher dimensional manifolds with conic singularities, see Section Six.

The heat operator, mapping initial data to solutions, can also be expressed as an integral operator. Once the singular structure of an integral kernel is understood, the mapping properties of the corresponding operator can be determined. In [Mo1] and [Mo2], Mooers provided a description of the singular stucture of these kernels for all self-adjoint extensions of the Laplacian on a cone manifold. These two papers are part of the continuing exploration of the b-calculus initiated by Melrose. A number of authors have contributed to the development of this theory, but excellent exposition is found in Grieser [G], Mazzeo [Ma], and Melrose [Me] to select a few. A central idea of the b-calculus is that it is equivalent in theory, but easier in practice, to discover a "blown-up" space that these integral kernels naturally inhabit. This space is more singular than $M \times M$, while the lifted kernel is more regular. The singularities of the original kernel are now encoded in the singularities of the space itself and in the asymptotic behavior of the kernel on approach to the various faces, edges, and corners. Finally, we mention that the study of the heat kernel on cone manifolds was initiated by Cheeger [C] and further developed by many other authors, a list of references can be found in [Li].

We begin with the determination of a self-adjoint extension. We choose the Friedrichs extension as being more natural for future applications.

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2. Friedrichs Extension for the Laplacian

Being an unbounded operator, the Laplacian has two natural extensions, its maximal extension and its minimal extension, and if these do not coincide one searches for a domain intermediate to these on which the operator is self-adjoint. The domains of the maximal and minimal extensions of the Laplacian on $(M; g)$, where g is the metric with a cone singularity at a point p are as follows:

$$D_{\min} = \{u \in L^2(M; g) \mid \exists u_j \in C^\infty_0(M \setminus p), u_j \to u,\ \Delta u_j \to \Delta u\}$$

where the convergence is in L^2; in effect, $D_{\min}$ is the domain of the graph closure of Δ. Also,

$$D_{\max} = \{u \in L^2(M; g) \mid \Delta u \in L^2(M; g)\}.$$

The Laplacian Δ is not self-adjoint on $D_{\min}$. Working out the integration by parts using the invariant formulation of Greens formula,

$$\int_{\partial\Omega} i_{v\nabla u}\omega_g = \int_{\Omega} < \nabla u, \nabla v > \omega_g + \int_{\Omega} v\Delta u\, \omega_g$$

and determining the free boundary conditions for v reveal that the domain of the adjoint operator is strictly larger than $D_{\min}$.

Determination of the possible self-adjoint extensions begins with an investigation of $D_{\max}/D_{\min}$. Elements of L^2 which are in the kernel of Δ are automatically in $D_{\max}$ and in fact the difference $D_{\max}/D_{\min}$ is spanned by homogeneous solutions (see [Mo1]). Whether such a solution is also in $D_{\min}$ depends on its behavior near the cone tip. A small enough neighborhood of the cone tip is approximated by a product, so that the Laplacian can be modeled by

$$\frac{\partial^2}{\partial x^2} + \frac{1}{x}\cdot\frac{\partial}{\partial x} + \frac{1}{\beta^2}\cdot\frac{\partial^2}{\partial \theta^2}$$

and a function u supported in this neighborhood can be written using separation of variables as the product $u(x,\theta) = x^\gamma v(\theta)$. Nontrivial elements of the kernel occur when $\gamma = k/\beta$, where $1-\beta$ is the cone angle with $\beta \in (0,1]$, $k \in \mathbb{Z}$, and $v(\theta)$ is the corresponding eigenfunction, a linear combination of $\cos(k\theta)$ and $\sin(k\theta)$.

Let us denote such a solution by u_k, noting that since $k=0$ is a double indicial root, there are two corresponding elements of the kernel, $u_0^+ = x^0 \cdot b$ and $u_0^- = x^0 \log x \cdot b$. If $k \leq -1$ then the corresponding solution is not even in L^2, whereas if $k \geq 2$, then $u_k(x,\theta) \in D_{\min}$, so these cases may be discarded. Now if $k=1$, and $\beta < 1$ then u_1 is in L^2; if $\beta = 1$ we can show that $u_1(x,\theta) = x^1 v(\theta)$ is in $D_{\min}$ also. As explained in the paper of Grieser and Lesch, [GL], one needs to find a sequence $\phi_\varepsilon \in C_0^\infty(M \setminus p)$ such that $\phi_\varepsilon u_1 \to u_1$ in $L^2(M,g)$ and $\| \Delta(\phi_\varepsilon u_1) \|_{L^2} \leq C$. Let $\phi(x) \in C^\infty([0,\infty))$ be such that $\phi(x) = 0$ for $x \leq 1/2$ and $\phi(x) = 1$ for $x \geq 1$. Then $\phi_\varepsilon = \phi(x/\varepsilon)$ fulfills the conditions we need.

These calculations show that $D_{\max}/D_{\min}$ is spanned by $\{u_0^+,\, u_0^-\}$ and by choosing a subspace of half the dimension, we obtain a self-adjoint extension. Choosing the bounded function u_0^+ gives the Friedrichs extension.

3. Heat Space and the Lifted Heat Kernel

In this section, we recall Mooers' theorem [Th. 3.1, Mo2] which describes the heat kernel as a polyhomogeneous function on a blown-up manifold with corners. Here, by an n-dimensional manifold with corners, we mean a paracompact Hausdorff topological space with local charts of the form $[0,\infty)^k \times \mathbb{R}^{n-k}$ where k can run anywhere between 0 and n depending on where the chart is located on the manifold. Of course, the transition maps between two charts are required to be smooth. The main examples of manifolds with corners come from taking products of manifolds with boundary and from the process of blow-up. See [G] for an exposition on manifolds with corners and the blow-up procedure.

The blown-up space $\tilde{M}_h^2$ on which the heat kernel lives is constructed through an intermediate space defined as follows. Let $M \cong [0,\varepsilon)_x \times N$ near the cone tip $N \cong \mathbb{S}^1$. Denote the radial variable on the left and right (first and second) factors of $M \times M$ by x and y respectively. In the top left picture in Figure 1, $lb = \{x = 0\}$ is the left boundary, $rb = \{y=0\}$ is the right boundary, and $tf = \{t = 0\}$. We define the manifold with corners M_1^2 by blowing-up the set $\{x = y = 0\} \times \{0\}_t$ in

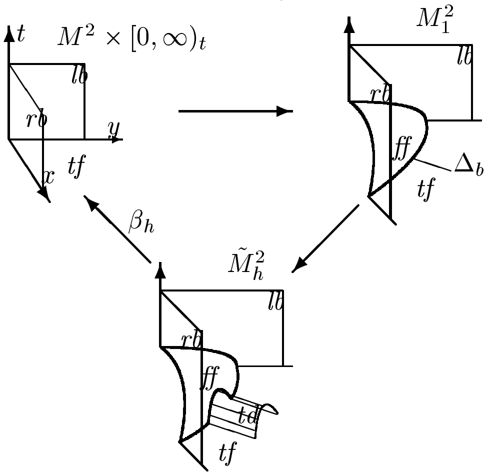

FIGURE 1. How $\tilde{M}_h^2$ is formed. The map β_h denotes the blow-down map.

$M^2 \times [0,\infty)_t$ parabolically in t. (See the top right picture in Figure 1.) This means that M_1^2 has an atlas consisting of the usual coordinate patches on $M^2 \times [0,\infty)$ away from $\{x = y = 0\} \times \{0\}$ together with "parabolic" polar coordinate charts around $\{x = y = 0\} \times \{0\}$ of the form $(\rho, \omega, \theta, \theta')$, where θ and θ' are coordinates on the left and right factor of $N \times N$ respectively, and

$$\rho = \sqrt{|(x,y)|^4 + t^2}, \quad \omega = (\frac{x}{\rho}, \frac{y}{\rho}, \frac{t}{\rho^2}). \tag{3.1}$$

The set where $\rho = 0$ is called the front face and is denoted *ff*. Instead of using these global polar coordinates near *ff*, it is often practical to use projective coordinates defined locally near *ff*. For example, near the top corner of *ff* away from *tf*,

$$s = t^{1/2}, \;\; w = x/\sqrt{t}, \;\; z = y/\sqrt{t},$$

together with θ and θ', can be used as coordinates on M_1^2 instead of ρ and ω. Here, s defines *ff*, w defines *lb*, and z defines *rb*. Near the bottom left-hand corner of *ff* away from *rb* (looking at $\tilde{M}_h^2$ from above), the projective coordinates

$$\tau = t/y^2, \;\; \xi = x/y, \;\; \eta = y \tag{3.2}$$

can be used. Here, τ defines *tf*, ξ defines *lb*, and η defines *ff*. Reversing the roles of x and y give coordinates on the bottom right-hand corner of *ff* away from *lb*.

Let Δ_b denote the lift of $\Delta \times \{0\}$ into M_1^2. The manifold $\tilde{M}_h^2$ is defined by blowing up Δ_b in M_1^2 parabolically in t. This means that $\tilde{M}_h^2$ has an atlas consisting of the usual coordinate patches on M_1^2 away from Δ_b together with "parabolic" polar coordinate charts around Δ_b defined in a similar manner as in (3.1). For instance, consider the coordinates $(\tau, \xi, \eta, \theta, \theta')$ defined in (3.2) above, where θ and θ' are identical sets of coordinates on the left and right factors of N^2. Then Δ_b is just the set $\{\tau = 0, \, \xi - 1 = 0, \, \theta - \theta' = 0\}$, and projective coordinates on $\tilde{M}_h^2$ defined near the top of *td* and away from *tf* are given by

$$z_1 = \tau^{1/2}, \;\; z_2 = (\xi - 1)/\tau^{1/2}, \;\; z_3 = (\theta - \theta')/\tau^{1/2}, z_4 = \theta', \;\; z_5 = \eta. \tag{3.3}$$

In these coordinates, the face *tf* is the face in the limit $|\,(z_2, z_3)\,| \to \infty$. The manifold $\tilde{M}_h^2$ is equipped with a blow-down map β_h from $\tilde{M}_h^2$ onto $M^2 \times [0,\infty)$, by simply writing the original coordinates on $M^2 \times [0,\infty)$ in terms of the parabolic polar coordinates on $\tilde{M}_h^2$.

We now describe the heat kernel. We denote the heat operator $e^{t\Delta}$ by H. Then on a coordinate chart, the action of the heat operator H on f (representing initial data) is given by

$$Hf(x,\theta,t) = \int \kappa_H(t,x,y,\theta,\theta') f(y,\theta')\, \beta\, y dy d\theta'$$

where κ_H is the heat kernel on $M^2 \times [0,\infty)$. On the interior of $M^2 \times [0,\infty)$, the kernel κ_H has the same asymptotics as the heat kernel on a manifold without boundary. Mooers gave a complete description of the asymptotics of κ_H near the boundary of $M^2 \times [0,\infty)$.

Theorem 3.1 (Mo2, Th. 3.1)**.** *The heat kernel lifted to $\tilde{M}_h^2$ under the blow-down map $\beta_h : \tilde{M}_h^2 \to M^2 \times [0,\infty)$ defines a polyhomogeneous function with leading powers -1 at ff, -2 at td, and vanishing to infinite order at tf.*

We denote the lift of the heat kernel to $\tilde{M}_h^2$ by H, it will always be clear from context whether H represents the heat operator or heat kernel. By polyhomogeneous, we mean that H defines a function on $\tilde{M}_h^2$ with certain asymptotics at each face of $\tilde{M}_h^2$. For instance, in the coordinates $(z_1,\ldots,z_5)$ defined in (3.3), *ff* is defined by z_5, *td* by z_1, and *tf* corresponds to the face in the limit $\mid (z_2,z_3) \mid \to \infty$. Mooers' theorem states that the heat kernel H written in these coordinates has the following asymptotics:

$$H \sim \sum_{j=0}^{\infty} z_5^{-1+j} a_j(z_1,z_2,z_3,z_4) \quad \text{as } z_5 \to 0,$$

$$H \sim \sum_{j=0}^{\infty} z_1^{-2+j} b_j(z_2,z_3,z_4,z_5) \quad \text{as } z_1 \to 0,$$

and as $\mid (z_2,z_3) \mid \to \infty$, $H \to 0$ with all derivatives.

The theorem of Mooers shows us that the asymptotic behavior of the heat kernel at the faces *ff*, *tf*, and *td* are features shared by all the heat operators. The asymptotic behavior at the left and right boundaries is behavior individual to the Laplacian and the metric of M, although we note that the *leading* asymptotics agree for all choices of self-adjoint extension. Her paper explains how the asymptotics are derived from a parametrix whose construction is carried out in stages, first inverting the operator at the front face and then solving away the errors created at the other faces. Because this correction of errors is an iterative process, the possible terms appearing in the asymptotic expansion are revealed step by step.

The asymptotics at the left and right boundaries are described as follows. Near the cone tip, the Laplacian is approximated by the model

$$A_\beta(x) = \frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} + \frac{1}{\beta^2 x^2}\frac{\partial^2}{\partial \theta^2},$$

with restriction to the boundary $x = 0$ given by

$$x^2 A_\beta(x) \mid_{x=0} = \frac{1}{\beta^2}\frac{\partial^2}{\partial \theta^2}.$$

The eigenvalues of this boundary Laplacian are given by $-k^2/\beta^2$ for integers k. For higher dimensional manifolds, the circle $N = \mathbb{S}^1$ could be replaced by another compact manifold and the eigenvalues would still be discrete; we will elaborate on this generalization in Section Six. Away from the singular set $t = 0$, $x = 0$, and $y = 0$, by definition $\tilde{M}^2_h$ is just $M^2 \times [0, \infty)$, and omitting angular variables the heat kernel $H(t, x, y)$ is just an integral operator with asymptotics of the form

$$H \sim \sum_{j,k=0}^{\infty} c_{jk}(t)\, x^{j/\beta} y^{k/\beta},$$

and vanishes to infinite order as $t \to 0$ for $x > 0$ or $y > 0$ with $x \neq y$. The asymptotics of the heat kernel at the left and right boundaries near the singular set are described as follows. Consider the coordinates in (3.2), where τ is a boundary defining function for *tf*, ξ for *lb*, and η for *ff*. Then the asymptotics of H can be found applying the algorithm described in [Mo2]:

$$H \sim \sum_{j,k,\ell=0}^{\infty} c_{jk\ell}(\tau)\, \eta^{-1+j} \xi^{k/\beta+\ell}$$

and vanishes to infinite order as $\tau \to 0$. Regularity is determined by the lowest order terms in the expansion. Putting $j, k, \ell = 0$ gives $\eta^{-1} \cdot \xi^0$. There is a similar expansion to *rb*. Note that the leading term is the same for all cone angles. This is why we are able to treat all of them, including the punctured disk, simultaneously.

4. Local Calculations of Derivatives

Definition 4.1. The space $C^k_b(M)$ consists of functions u defined on $M \setminus p$ for which

$$\sup \mid u \mid + \sup \mid V_1 u \mid + \ldots + \sup \mid V_1 \cdots V_k u \mid < \infty$$

for every choice of k b-vector fields.

The b-vector fields are those that are tangent to the boundary $\{x = 0\}$ and are locally generated by $\{x\partial/\partial x,\, \partial/\partial\theta\}$.

In Section Three, we explained in detail the geometry of the blown-up space. For our purposes here, however, we emphasize that locally, blow up is nothing more than the introduction of singular coordinates, and that these coordinates are motivated by the geometry of the space and the symmetries of the operator. In this section, we use these local, singular coordinates to prove the first assertion of the main theorem.

Recall that in the previous section we showed that the leading terms in the asymptotics are the same for all the cone angles $\alpha \in [0, 1)$. Therefore for simplicity we will now set $\beta = 1$. The only effect of a different choice of cone angle would be to insert a constant β in the measures and change the asymptotics of the heat kernel as shown in the previous two displayed equations. Also recall that on the interior of $M^2 \times [0, \infty)$ the heat kernel has the same asymptotics as on a manifold without boundary near the diagonal of M^2 and is an integral operator vanishing to infinite order at $t = 0$ away from the diagonal. Thus, in the calculations that follow, we may assume that the heat kernel is supported near the front face in $\tilde{M}^2_h$.

Estimates near the left-hand corner

Near the intersection of *ff*, *lb*, and *tf* a local coordinate system is given by

$$\tau = t/y^2,\ \xi = x/y,\ \eta = y,\ \theta = \theta,\ \theta' = \theta'.$$

Regularity is a local issue and in the interior regions regularity is as in the smooth case, so by taking a cut-off function we may assume that the kernel H is supported compactly in a cube near the corner. The relevant asymptotics of the heat kernel at the intersection of these faces are $H \sim \xi^0\eta^{-1}$, and vanishing to infinite order as $\tau \to 0$. Dropping the angular variables which play no role here, we have

$$Hf(x,t) = \int H(t/y^2, x/y, y) f(y)\, y dy. \tag{4.1}$$

Cancelling a factor of $\eta^{-1} = y^{-1}$ from the kernel with the y^1 in the measure, we rewrite the kernel as $\tilde{H}$ with new asymptotics $\tilde{H} \sim \xi^0\eta^0$. The integrand is now integrable. Estimating,

$$\mid Hf(x,t) \mid \leq C \parallel f \parallel_0 .$$

If we consider this in more detail, it will become clear that the bounds on the b-derivatives are of the same form. If the kernel above is written out in more detail, what it means to have a joint expansion is that first expanding in powers of ξ and then expanding the coefficients in powers of η, we have

$$\begin{aligned} H(\tau,\xi,\eta) &= \xi^0 a_0(\tau,\eta) + \xi^1 a_1(\tau,\eta) + \ldots \\ &= \xi^0(\eta^{-1}a_{0-1}(\tau) + \eta^0 a_{00}(\tau) + \ldots) \\ &+ \xi^1(\eta^{-1}a_{1-1}(\tau) + \eta^0 a_{10}(\tau) + \ldots) \\ &+ \ldots . \end{aligned}$$

The $a_{ij}(\tau)$ are smooth functions that vanish to all orders as $\tau \to 0$. As above, applying $x\partial/\partial x$ to the operator, the derivative passes inside the integral and falls on the kernel, so we calculate

$$\begin{aligned} x\partial H/\partial x &= x(\partial H/\partial\xi)(\partial\xi/\partial x) \\ &= x\left[(\eta^{-1}a_{1-1}(\tau) + \eta^0 a_{10}(\tau) + \ldots)\right. \\ &+ \left. 2\xi^1(\eta^{-1}a_{2-1}(\tau) + \eta^0 a_{20}(\tau) + \ldots) + \ldots\right] \cdot \frac{1}{y} \\ &= \frac{x}{y}\left[\eta^{-1}a_{1-1}(\tau) + \eta^0 a_{10}(\tau) + \ldots + 2\xi^1(\eta^{-1}a_{2-1}(\tau) + \ldots)\right]. \end{aligned}$$

The factor x/y in front is again ξ, so $x(\partial H/\partial\xi)(\partial\xi/\partial x) = \xi(\partial H/\partial\xi)$. In other words, the effect of the derivative in ξ is to reduce the powers in the expansion by one, but then multiplying by ξ replaces this power, so the new kernel has an expansion of exactly the same form as before. It follows that the same argument that was applied above can be applied here, and moreover, that one can take as many $x\partial/\partial x$ derivatives as one wishes and still get bounds of the form $C \parallel f \parallel_0$. Because the coefficients in the expansion are smooth in the angular variables, derivatives $\partial/\partial\theta$ produce the same kinds of bounds.

Estimates near the right-hand corner

Because Δ is self-adjoint, the asymptotics of the heat kernel at the right face are the same as at the left face. By symmetry, the calculation is the same.

Estimates near the top corner

Local coordinates, eliminating the θ and θ' coordinates which don't play any role at the moment, are

$$s = t^{1/2},\ w = x/\sqrt{t},\ z = y/\sqrt{t},$$

and locally the asymptotics of the kernel are $H \sim w^0 z^0 s^{-1}$. Again we multiply by a cut-off function in order to assume that the kernel has support in a box. Then

$$Hf(x,t) = \int H(s,w,z) f(y)\, y dy.$$

This time the fiber over which we integrate is already a coordinate direction, so we can integrate directly in these coordinates (geometrically, upstairs in the blown-up space). Then

$$Hf(x,t) = \int H(s,w,z) f(sz)\, s^2 z dz.$$

The asymptotics are $H \sim w^0$ as $w \to 0$, $H \sim z^0$ as $z \to 0$, and $H \sim s^{-1}$ as $s \to 0$. Cancelling the s^{-1} with one of the positive powers of s in the measure allows us to write

$$Hf(x,t) = \int \tilde{H}(s,w,z) f(sz)\, s^1 z dz, \tag{4.2}$$

with $\tilde{H}$ having new asymptotics $\tilde{H} \sim s^0 w^0 z^0$. Then it is possible to bound

$$\mid Hf(x,t) \mid \leq C \parallel f \parallel_0 .$$

For the derivative estimate, applying the operator $x\partial/\partial x$ to the kernel gives

$$x \cdot \frac{\partial}{\partial x}\tilde{H} = x \cdot \frac{\partial \tilde{H}}{\partial w}\frac{\partial w}{\partial x} = w\sqrt{t}\frac{\partial \tilde{H}}{\partial w} t^{-1/2} = w\frac{\partial \tilde{H}}{\partial w}.$$

Similar to the calculation done in detail above for the left-hand corner, applying the $w-$ derivative to the expansion reduces the powers of w by one, but then everything gets multiplied by w, so that the kernel for the derivative operator has an expansion with the same index sets as the original kernel, and a bound of the same sort can be calculated. In fact, one can apply as many $x\partial/\partial x$ or $\partial/\partial\theta$ derivatives as desired and still obtain bounds of the form $C \parallel f \parallel_0$.

Estimates where the diagonal hits the front face

Recall that in this region the heat space was obtained by parabolic blow up first at the corner of $X \times X$ and then at the diagonal. The first stage is accomplished by introduction of the local coordinates

$$\tau = t/y^2,\ \xi = x/y,\ \eta = y,\ \gamma = \theta - \theta',\ \theta' = \theta',$$

and the second stage by

$$z_1 = \tau^{1/2},\ z_2 = (\xi - 1)/\tau^{1/2},\ z_3 = \gamma/\tau^{1/2},\ z_5 = \eta.$$

The asymptotic behavior of the kernel is $H \sim z_1^{-2} z_5^{-1}$, and vanishing to infinite order as $\mid (z_2, z_3) \mid \to \infty$. We have

$$Hf(x,\theta,t) = \int H(\frac{t^{1/2}}{y}, \frac{x-y}{t^{1/2}}, y\frac{\theta-\theta'}{t^{1/2}}, \theta', y) f(y,\theta')\, y dy d\theta',$$

and borrowing a factor of $z_5 = y$ from the kernel to cancel with the y^1 from the measure gives

$$Hf(x,\theta,t) = \int G(\frac{t^{1/2}}{y}, \frac{x-y}{t^{1/2}}, y\frac{\theta-\theta'}{t^{1/2}}, \theta', y) f(y,\theta')\, dy d\theta',$$

with new asymptotics $G \sim z_1^{-2} z_5^0$. We now write the integration variables in terms of the blow-up coordinates z_2 and z_3. So, first making the substitution $z_2 = (x-y)/t^{1/2}$ for y and then the substitution $z_3 = (x - t^{1/2}z_2)(\theta-\theta')/t^{1/2}$ for θ', gives

$$Hf(x,\theta,t) = \int G(\frac{t^{1/2}}{x-t^{1/2}z_2}, z_2, z_3, \theta - \frac{t^{1/2}z_3}{x-t^{1/2}z_2}, x - t^{1/2}z_2)\cdot$$

$$f(x-t^{1/2}z_2, \theta - \frac{t^{1/2}z_3}{x-t^{1/2}z_2})\, \frac{t}{x-t^{1/2}z_2}\, dz_2 dz_3.$$

Since $z_1^{-2} = t^{-1}(x-t^{1/2}z_2)^2$, it follows that

$$G(\frac{t^{1/2}}{x-t^{1/2}z_2}, z_2, z_3, \theta - \frac{t^{1/2}z_3}{x-t^{1/2}z_2}, x - t^{1/2}z_2)\frac{t}{x-t^{1/2}z_2} =$$

$$\tilde{H}(\frac{t^{1/2}}{x-t^{1/2}z_2}, z_2, z_3, \theta - \frac{t^{1/2}z_3}{x-t^{1/2}z_2}, x - t^{1/2}z_2),$$

where $\tilde{H}(z_1,z_2,z_3,z_4,z_5)$ has the asymptotics $\tilde{H} \sim z_1^0 z_5^0$ and vanishes to infinite order as $\mid (z_2,z_3) \mid \to \infty$. Thus, we can write

$$Hf(x,\theta,t) = \int \tilde{H}(\frac{t^{1/2}}{x-t^{1/2}z_2}, z_2, z_3, \theta - \frac{t^{1/2}z_3}{x-t^{1/2}z_2}, x - t^{1/2}z_2)\cdot \tag{4.3}$$

$$f(x-t^{1/2}z_2, \theta - \frac{t^{1/2}z_3}{x-t^{1/2}z_2})\, dz_2 dz_3.$$

Since $\tilde{H}(z_1,z_2,z_3,z_4,z_5)$ is bounded in all variables and vanishes to infinite order as $\mid (z_2,z_3) \mid \to \infty$, it follows that

$$\mid Hf(x,\theta,t) \mid \leq Ct^0 \parallel f \parallel_0 .$$

To estimate the first derivative, we calculate

$$x\frac{\partial}{\partial x}Hf(x,\theta,t) = \int x\frac{\partial}{\partial x}H(\frac{t^{1/2}}{y}, \frac{x-y}{t^{1/2}}, y\frac{\theta-\theta'}{t^{1/2}}, \theta', y) f(y,\theta')\, y dy d\theta'.$$

Differentiating the kernel,

$$x\frac{\partial H}{\partial x} = x\frac{\partial H}{\partial z_2}\frac{\partial z_2}{\partial x} = x\frac{\partial H}{\partial z_2}\cdot t^{-1/2},$$

and

$$\frac{\partial H}{\partial \theta} = \frac{\partial H}{\partial z_3}\frac{\partial z_3}{\partial \theta} = \frac{\partial H}{\partial z_3}\cdot y t^{-1/2}.$$

Since the asymptotics of $\partial H/\partial z_2$ and $\partial H/\partial z_3$ are the same as those of H, the same argument as above produces bounds of the form

$$\mid x\frac{\partial}{\partial x}Hf(x,\theta,t) \mid \leq Ct^{-1/2} \parallel f \parallel_0$$

and

$$\mid \frac{\partial}{\partial \theta}Hf(x,\theta,t) \mid \leq Ct^{-1/2} \parallel f \parallel_0 .$$

Similarly, for any j and k we have

$$\mid \left(x\frac{\partial}{\partial x}\right)^j\left(\frac{\partial}{\partial\theta}\right)^k Hf(x,\theta,t)\mid\leq Ct^{-(j+k)/2}\parallel f\parallel_0 .$$

Taken together, these local computations prove the first assertion of the main theorem.

5. Hölder Regularity $C^\gamma_g \to C^2_b$.

Definition 5.1. The Hölder space $C^\gamma_g(M)$ for any fixed cone metric as defined in the introduction, consists of continuous functions with

$$\parallel u\parallel_\gamma = \sup\mid u\mid + \sup_{(x,\theta)\neq(y,\theta')}\frac{\mid u(y,\theta')-u(x,\theta)\mid}{\mid y-x\mid^\gamma+\beta^\gamma y^\gamma\mid\theta'-\theta\mid^\gamma}<\infty.$$

We already noted that in most places of the blown-up space, one can take any number of derivatives without the cost of worsening powers of t. Only at the diagonal did successive differentiations produce factors of $t^{-1/2}$. Restricting therefore our attention to this region,

$$\begin{aligned}x^2\frac{\partial^2}{\partial x^2}Hf(x,\theta,t) &= \int x^2\frac{\partial^2 H}{\partial x^2}f(y,\theta')\,ydyd\theta'\\ &= \int\frac{\partial^2\tilde H}{\partial x^2}\left[f(y,\theta')-f(x,\theta)\right]dyd\theta' + f(x,\theta)\int x^2\frac{\partial^2\tilde H}{\partial x^2}\,dyd\theta',\end{aligned}$$

where, as in Section Four, the factor of y in the measure has been cancelled with a factor of y^{-1} in the kernel, and the kernel with this change rewritten as $\tilde H$.

The first integral can be estimated by inserting the Hölder condition:

$$\begin{aligned}\mid\int x^2\frac{\partial^2\tilde H}{\partial x^2}\cdot\left[f(y,\theta')-f(x,\theta)\right]dyd\theta'\mid &\leq \int x^2\frac{\partial^2}{\partial x^2}\tilde H\Big(\frac{t^{1/2}}{y},\frac{x-y}{t^{1/2}},y\frac{\theta-\theta'}{t^{1/2}},\theta',y\Big)\cdot\\ &\qquad\left[\,\mid y-x\mid^\gamma+y^\gamma\mid\theta-\theta'\mid^\gamma\,\right]dyd\theta'\\ &= \int x^2\frac{\partial^2}{\partial x^2}\tilde H\Big(\frac{t^{1/2}}{y},\frac{x-y}{t^{1/2}},y\frac{\theta-\theta'}{t^{1/2}},\theta',y\Big)\cdot\\ &\qquad\frac{\mid y-x\mid^\gamma+y^\gamma\mid\theta-\theta'\mid^\gamma}{t^{\gamma/2}}t^{\gamma/2}\,dyd\theta'\\ &\leq Ct^{-1+\gamma/2}\parallel f\parallel_\gamma .\end{aligned}$$

In the last steps, the factors coming from the Hölder estimate of f are absorbed into the derivative of the heat kernel. This does not affect the asymptotics of the integrand, because the heat kernel was vanishing to infinite order as $\mid(z_2,z_3)\mid\to\infty$.

The second integral can be converted by the divergence theorem into a first derivative (already estimated) integrated around the boundary. Or, an easier estimate is provided by borrowing an observation which will be explained in Section Six. There it is shown that $H1=1$. Written as an integral operator, this says

$$1=\int H\cdot ydyd\theta'.$$

Taking partial derivatives, it follows that

$$0=\int x^2\frac{\partial^2 H}{\partial x^2}\,ydyd\theta'.$$

Hence,

$$\| x^2 \frac{\partial^2}{\partial x^2} Hf(\cdot,t) \|_0 \leq Ct^{-1+\gamma/2} \| f \|_\gamma .$$

Similar calculations show that applying any combination of partial derivatives to Hf up to order two in $x\partial/\partial x$ and $\partial/\partial\theta$ still obtains bounds of the form $Ct^{-1+\gamma/2}$ $\| f \|_\gamma$. The second assertion of the main theorem is now proved.

6. Applications

A straightforward generalization yields similar results in higher dimensions. Suppose that M is a compact manifold of arbitrary dimension such that near a finite number of points it looks metrically like a cone over a compact manifold. That is to say, near each of these finite number of points the metric can be written locally as $dx^2 + x^2 g_N$, where g_N is a smooth metric on the cross section N, a smooth compact manifold without boundary. The heat kernel associated to a conic metric of this sort corresponding to the Friedrichs extension is described in [Mo2]. The heat kernel defines a polyhomogeneous function on a heat space that is a higher dimensional analog of the heat space explained in Section Three. Moreover, using the description of the asymptotics found in Theorem 3.1 of [Mo2], arguments similar to those presented in Section Four and Five can be used to prove the following theorem.

Theorem 6.1. *The heat operator of the Friedrichs extension of the Laplacian on a cone manifold M is a bounded operator with estimates of the form*

$$e^{t\Delta} : C^0(M) \to C^k_b(M);\ \| u \|_k \leq Ct^{-k/2} \| f \|_0$$

and

$$e^{t\Delta} : C^\gamma_g(M) \to C^2_b(M);\ \| u \|_2 \leq Ct^{-1+\gamma/2} \| f \|_\gamma .$$

We now turn to a applying these results to obtain solutions of semilinear parabolic equations. The book of Taylor [T] explains how to reformulate a nonlinear problem as a contraction mapping. Rewrite the equation by separating out the higher order terms so that they can be considered as an inhomogeneous term, with the equation now taking the form:

$$\frac{\partial u}{\partial t} - \Delta u - Q(u, Vu) = 0,$$

where V is a b-vector field, and with initial condition $u(0) = f$. Applications of semilinear equations of this sort can be found in [CH]. The reaction-diffusion equation is one such equation. Applying Duhamel's principle, Taylor proves the following result.

Theorem 6.2 (T, page 273)**.** *Suppose that X and Y are Banach spaces such that the following four conditions are satisfied:*

1. *$e^{t\Delta} : X \to X$ is a strongly continuous semigroup, for $t \geq 0$.*
2. *$Q : X \to Y$ is locally Lipschitz .*
3. *$e^{t\Delta} : Y \to X$, for $t > 0$.*
4. *For some $-\gamma > -1$,*

$$\| e^{t\Delta} \|_{L(Y,X)} \leq Ct^{-\gamma}.$$

Then the initial value problem with initial data $f \in Y$ has a unique solution $u \in C([0,T], X)$ where $T > 0$ and is estimable from below in terms of $\| f \|_X$.

Interpreting this in the context of a cone manifold, we have the following.

Theorem 6.3. *On a cone manifold M, let $Y = C^0(M)$ and $X = C_b^1(M)$ or $Y = C_g^\gamma(M)$ and $X = C_b^2(M)$, and let $Q : X \to Y$ be locally Lipschitz. Then given any initial data in Y, the above semilinear initial value problem has a solution in X existing on some time interval $[0,T]$ for some $T > 0$.*

This result follows as an immediate corollary to our estimates and to Taylor's theorem once we prove that $e^{t\Delta} : X \to X$ is a strongly continuous semigroup for $t \geq 0$. We prove this for $X = C_b^k(M)$ for any $k \geq 0$. By the group property of the heat operator, it suffices to prove the semicontinuity at $t = 0$. Let $f \in X$. We show that

$$\| e^{t\Delta} f - f \|_k \to 0 \quad \text{as} \quad t \to 0.$$

Since on the interior of $M^2 \times [0,\infty)$ the heat kernel has the same asymptotics as on a manifold without boundary near the diagonal of M^2 and is an integral operator vanishing to infinite order at $t = 0$ away from the diagonal, we may assume that the heat kernel is supported near the front face of $\tilde{M}_h^2$. We now repeat the arguments in Section Four.

Estimates near the left-hand corner

We assume that the heat kernel is supported in a cube near the left-hand corner. Then as in Equation (4.1) we can write

$$Hf(x,t) = \int H(t/y^2, x/y, y) f(y)\, y dy,$$

where the asymptotics of the heat kernel are $H(\tau,\xi,\eta) \sim \xi^1\eta^{-1}$, and vanishing to infinite order as $\tau \to 0$. Since H vanishes to infinite order at $\tau = 0$, and $(x\partial_x)^j H(t/y^2, x/y, y)$ has the same asymptotics as $H(t/y^2, x/y, y)$, this formula shows that

$$\| Hf(\cdot,t) \|_k \to 0 \quad \text{as} \quad t \to 0.$$

Estimates near the top corner

Assuming that the kernel has support in a box near the top corner, as in Equation (4.2) we can write

$$Hf(x,t) = t^{1/2} \int_0^1 \tilde{H}(t^{1/2}, x/t^{1/2}, z) f(t^{1/2}z)\, zdz,$$

where the kernel has asymptotics $\tilde{H}(s,w,z) \sim s^0 w^0 z^0$. In particular, the kernel is bounded. As already noted in Section Four, for any j, $(x\partial/\partial x)^j \tilde{H}$ has the same asymptotics as $\tilde{H}$ does. It follows that

$$\| Hf(\cdot,t) \|_k \to 0 \quad \text{as} \quad t \to 0.$$

By this estimate and the previous one, we may assume that the heat kernel is supported near the diagonal where it hits the front face.

Estimates where the diagonal hits the front face.

We first note that $e^{t\Delta}1 = 1$. In other words, the heat operator applied to the constant function $f(x) = 1$ is the constant function 1. This follows from the fact that 1 is in the null space of Δ so that the function $u(x,t) = 1$ satisfies the heat equation:

$$(\partial_t - \Delta)u(x,t) = 0, \quad u(x,0) = 1.$$

By uniqueness of the heat equation, $e^{t\Delta}1 = 1$. Thus, setting $f = 1$ in Equation (4.3), we have

$$1 = \int \tilde{H}(\frac{t^{1/2}}{x - t^{1/2}z_2}, z_2, z_3, \theta - \frac{t^{1/2}z_3}{x - t^{1/2}z_2}, x - t^{1/2}z_2)\, dz_2 dz_3,$$

where $\tilde{H}(z_1, z_2, z_3, z_4, z_5)$ has the asymptotics $\tilde{H} \sim z_1^0 z_5^0$ and vanishes to infinite order as $\mid (z_2, z_3) \mid \to \infty$. Multiplying by $f(x,\theta)$ gives

$$f(x,\theta) = \int \tilde{H}(\frac{t^{1/2}}{x - t^{1/2}z_2}, z_2, z_3, \theta - \frac{t^{1/2}z_3}{x - t^{1/2}z_2}, x - t^{1/2}z_2)\, f(x,\theta)\, dz_2 dz_3,$$

and subtracting this equation from the Equation (4.3) yields

$$\begin{aligned}(6.1) \quad Hf(x,\theta,t) - f(x,\theta) = \int \tilde{H}(\frac{t^{1/2}}{x - t^{1/2}z_2}, z_2, z_3, \theta - \frac{t^{1/2}z_3}{x - t^{1/2}z_2}, x - t^{1/2}z_2)\cdot \\ \big[f(x - t^{1/2}z_2, \theta - t^{1/2}z_3) - f(x,\theta)\big]\, dz_2 dz_3.\end{aligned}$$

Note that

$$\mid f(x - t^{1/2}z_2, \theta - t^{1/2}z_3) - f(x,\theta) \mid \to 0 \quad \text{as} \quad t \to 0$$

for z_2 and z_3 finite. This fact plus the fact that $\tilde{H}(z_1, z_2, z_3, z_4, z_5)$ is bounded in all variables and vanishes to infinite order as $\mid (z_2, z_3) \mid \to \infty$ imply that

$$\| Hf(\cdot,t) - f \|_0 \to 0 \quad \text{as} \quad t \to 0.$$

Applying b-derivatives to (6.1) shows that for any $i + j \leq k$,

$$\| \Big(x\frac{\partial}{\partial x}\Big)^i \Big(\frac{\partial}{\partial \theta}\Big)^j \big[Hf(\cdot,t) - f\big] \|_0 \to 0 \quad \text{as} \quad t \to 0.$$

Thus,

$$\| Hf(\cdot,t) - f \|_k \to 0 \quad \text{as} \quad t \to 0.$$

Our proof of semicontinuity of the heat operator at $t = 0$ is now complete.

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