TEMPERED OPERATORS AND THE HEAT KERNEL AND COMPLEX POWERS OF ELLIPTIC PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. The resolvent $(A-\lambda)^{-1}$ of an elliptic b-pseudodifferential operator on a compact manifold with corners (of arbitrary codimension) is shown to lie in a calculus of operators, tempered in the parameter λ in a special way. We show that the Laplace and Mellin transforms, with respect to λ, of these tempered operators can be defined and that they have Schwartz kernels which can be described geometrically. As a corollary, we obtain the structures of the kernels of the heat operator and complex powers of b-pseudodifferential operators, as the heat operator and complex powers are the Laplace and Mellin transforms, respectively, of the resolvent. The heat operator and complex powers are then used to generalize the index formula of Atiyah, Patodi, and Singer for Dirac operators on manifolds with boundary to Fredholm b-pseudodifferential operators on arbitrary compact manifolds with corners.

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CONTENTS

1. INTRODUCTION

In this paper, we continue the analysis initiated in [15] of the resolvent family $(A-\lambda)^{-1}$ of an elliptic b-pseudodifferential A of real positive order on a compact manifold with corners. The calculus of b-pseudodifferential operators is a natural extension of the algebra of pseudodifferential operators on closed manifolds to the category of manifolds with (asymptotically) cylindrical ends. In particular, the results described in this paper hold for pseudodifferential operators on closed manifolds. In [15], we captured the resolvent in a certain calculus of parameter-dependent operators that depend on the parameter λ in a special way. This calculus of operators was used to describe geometrically, the asymptotic behavior of the Schwartz kernel of the resolvent as $\lambda\to\infty$. In this paper, we define a finer calculus of parameter-dependent operators and show that this finer calculus captures the resolvent. Applications of this result include geometric descriptions of the heat operator and complex powers of A and a generalization of the index formula of Atiyah, Patodi, and Singer to compact manifolds with corners.

Calculi of parameter-dependent operators were originally considered by Agmon, cf. [1, 2]. Seeley [27] developed such a calculus to construct the complex powers of pseudodifferential operators on closed manifolds. They have also been developed for elliptic and pseudodifferential boundary value

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problems [29, 28, 9, 10]. These calculi are mainly used to analyze asymptotic expansions; for example, small-time heat trace expansions, and zeta and eta function expansions, cf. [11].

Through a simple example, we will describe the parameter-dependent symbol spaces that we will consider. Thus, given a positive real number m, let $a(\lambda,\xi) := (|\xi|^m - \lambda)^{-1}$; a 'local model' for the (conormal) symbol of the resolvent $(A-\lambda)^{-1}$ of an elliptic, positive, b-pseudodifferential operator A. The heat operator can be defined by the integral: $e^{-tA} = \frac{i}{2\pi}\int_\Gamma e^{-t\lambda}(A-\lambda)^{-1}d\lambda$, where Γ is a contour around the positive real axis. Thus, a 'local model' for the (conormal) symbol of the heat operator is given by $\frac{i}{2\pi}a(t,\xi)$, where $a(t,\xi) = \int_\Gamma e^{-t\lambda}a(\lambda,\xi)d\lambda$ is the Laplace transform of $a(\lambda,\xi)$. The main observation is that although $(A-\lambda)^{-1}$ may have spectrum along the entire positive real axis, the symbol $a(\lambda,\xi)$ has singularities on at most a finite part of the real axis. Of course, $a(\lambda,\xi)$ is holomorphic everywhere except at $\lambda = |\xi|^m$. In particular, $a(t,\xi) = \int_{\Gamma_\xi} e^{-t\lambda}a(\lambda,\xi)d\lambda$ where Γ_ξ is a closed *bounded* contour around the point $\lambda = |\xi|^m$ on the real axis. It follows that $a(t,\xi)$ is smooth down to $t = 0$. In fact, one can easily determine the growth properties of $a(t,\xi)$ in both t and ξ. Roughly speaking, in this paper we will study symbols $a(\lambda,\xi)$ that are holomorphic for all λ in a sector $\Lambda \subset \mathbb{C}$ and which are holomorphic for all $\lambda \in \mathbb{C}$ with $|\lambda| \le C_1|\xi|^m$ or $|\lambda| \ge C_2|\xi|^m$ for some $C_1, C_2 > 0$. Such symbols will be called 'holomorphically tempered symbols'. Such symbols have been implicitly studied in, for instance [9, Lemma 4.2.3]. In this paper, we define them formally and study their Laplace and Mellin transforms, which turn out to have very simple properties. In the end, we will apply these results to analyze the heat operator and complex powers of elliptic b-pseudodifferential operators, and their asymptotic expansions.

On closed manifolds, the polyhomogeneity of the heat kernel is well known, cf. [9, 8]. Thus, let A be an elliptic, positive, classical pseudodifferential operator of real positive order m on a closed compact manifold X. Then the heat operator of A, e^{-tA}, exists and for each $t > 0$ is a smoothing operator. In particular, for $t > 0$, we can restrict its Schwartz kernel on X^2 to the diagonal, $\mathrm{Diag} \subset X^2$: $e^{-tA}|_{\mathrm{Diag}} \in C^\infty(X,\Omega)$, where Ω is the density bundle. Here, we identify the diagonal with X. Moreover, we have the following asymptotic expansion as $t \downarrow 0$:

$$e^{-tA}|_{\mathrm{Diag}} \sim_{t\downarrow 0} \sum_{k=0}^\infty t^{\frac{k-n}{m}}\gamma_k + \sum_{k=1}^\infty t^k \log t\,\gamma'_k + \sum_{k=0}^\infty t^k\gamma''_k,\ \gamma_k,\gamma'_k,\gamma''_k \in C^\infty(X,\Omega). \tag{1}$$

In the case that A is the Laplacian on a compact Riemannian manifold, Minakshisundaram [24] used this expansion to study the asymptotic behavior of the Laplacian's eigenvalues and eigenfunctions. The asymptotic expansion (1) can also be used to study various geometric properties of the underlying manifold [19, 12], and can also used to prove various types of index theorems [4].

Using our calculus of parameter-dependent operators, we will improve (1) as follows. Let $A \in \Psi_b^m(X)$ be an elliptic, classical, b-pseudodifferential operator of positive real order m on a compact manifold with corners X (of any codimension). We will prove that if its principal symbol never takes values in a sector of the form $\{\lambda \in \mathbb{C}\,|\,\epsilon \le \arg(\lambda) \le 2\pi - \epsilon\}$, where $0 < \epsilon < \pi/2$, the heat operator e^{-tA} of A exists. Moreover, cf. Corollary 6.4, if $B \in \Psi_b^{m'}(X)$, $m' \in \mathbb{R}$, then

$$Be^{-tA}|_{\mathrm{Diag}} \sim_{t\downarrow 0} \sum_{k=0}^\infty t^{\frac{k-m'-n}{m}}\gamma_k + \sum_{k=0}^\infty t^{\frac{k-m'-n}{m}} \log t\,\gamma'_k + \sum_{k=0}^\infty t^k\gamma''_k, \tag{2}$$

where γ_k, γ'_k, $\gamma''_k \in C^\infty(X,\Omega_b)$ with Ω_b the b-density bundle, and where in the first sum, $\gamma_k = 0$ if $\frac{k-m'-n}{m} \in \mathbb{N}_0$ and in the second sum, $\gamma'_k = 0$ if $\frac{k-m'-n}{m} \notin \mathbb{N}_0$. Furthermore, if $l = \frac{k-m'-n}{m}$, then explicitly

$$\gamma_k = \frac{\Gamma(-l)}{m}\mathrm{Res}(BA^l) \text{ if } l \notin \mathbb{N}_0,\ \gamma'_k = \frac{(-1)^{l+1}}{m\cdot l!}\mathrm{Res}(BA^l) \text{ if } l \in \mathbb{N}_0,$$

where $\mathrm{Res}(BA^l)$ is the residue of BA^l in the sense of Wodzicki [31]. These residues are 'local' in the sense that they are determined from the expansions of the local symbols of A and B, while the coefficients γ''_k are in general 'global' in the sense that they are determined by the whole Schwartz kernels of A and B. If A and B are b-differential operators, then cf. Corollary 6.5,

$$Be^{-tA}|_{\mathrm{Diag}} \sim_{t\downarrow 0} \sum_{k=0}^\infty t^{\frac{k-m'-n}{m}}\gamma_k,\quad \gamma_k \in C^\infty(X,\Omega_b),$$

where if $l = \frac{k-m'-n}{m}$,

$$\gamma_k = \frac{\Gamma(-l)}{m}\mathrm{Res}(BA^l) \text{ if } l \notin \mathbb{N}_0\,, \quad \gamma_k = \frac{(-1)^{l+1}}{m \cdot l!}\mathrm{Res}(B \log A\, A^l) \text{ if } l \in \mathbb{N}_0\,.$$

Note that the operators BA^l and $B\log A\,A^l$ need not exist; however, the residue densities $\mathrm{Res}(BA^l)$ and $\mathrm{Res}(B\log A\,A^l)$ can still be defined, cf. Equations (49) and (50) in Section 6.

In [22], for the case of second order b-differential operators, Melrose not only describes the asymptotic behavior of the heat kernel e^{-tA} on the diagonal, but he also describes its uniform asymptotic behavior near it. The procedure he uses is both geometric and constructive. A manifold with corners, let us call it $X^2_{b,H}$, is defined by blowing up certain submanifolds of $[0,\infty)_s \times X \times X$, where $t = s^2$, and it is shown that the kernel of e^{-tA} defines a polyhomogeneous, or classical, function on $X^2_{b,H}$. The construction of the manifold $X^2_{b,H}$ is given in Section 6. In this paper, we will prove a similar result for the heat kernels of b-pseudodifferential operators on any compact manifold with corners, cf. Corollary 6.4.

As mentioned earlier, Seeley [27] utilized parameter-dependent operators in order to study the complex powers of pseudodifferential operators on closed manifolds. Under appropriate conditions, the complex powers, A^z, of a classical pseudodifferential operator A of order $m \in \mathbb{R}^+$ exists and his main results were that A^z defines an entire family of pseudodifferential operators of order $m\mathrm{Re}\, z$, and that the Schwartz kernel of A^z when restricted the diagonal, is meromorphic with equidistant simple poles on the real axis, regular at $z = 0$. We prove very similar results for b-pseudodifferential operators on manifolds with corners, cf. Corollary 7.13. Moreover, the result that A^z is a b-pseudodifferential operator of order $m\mathrm{Re}\, z$ does not require that A be classical (see Corollaries 7.8 and 7.7).

The Atiyah-Patodi-Singer index theorem [3] (APS theorem) relates analytic invariants of Dirac operators on manifolds with a cylindrical end with corresponding topological/geometric/spectral invariants of the underlying Riemannian manifold and its boundary. Let A be the positive part of a Dirac operator on a manifold Y with cylindrical end. Then the noted APS index theorem states that if A is Fredholm (on appropriate Sobolev spaces), then

$$\mathrm{ind}\, A = \int_Y AS - \frac{1}{2}\eta, \tag{3}$$

where AS is the Atiyah-Singer density of A, and where η is the eta invariant. The density AS is determined by the local symbol of the Dirac operator, while η is determined by spectrum of the induced boundary Dirac operator. In [25], Piazza extended the index formula (3) from Dirac operators to elliptic b-pseudodifferential operators on manifolds with boundary. In [17], Melrose and the author extend (3) to perturbed Dirac operators (that is, a Dirac operator modified by a b-pseudodifferential operator of order $-\infty$) on compact manifolds with corners of arbitrary codimension. In this paper, cf. Theorems 8.5 and 8.9, we show how the result of [17] extends to general Fredholm b-pseudodifferential operators. A heat kernel and a zeta function version of the index theorem are presented.

A review of the b-calculus is provided in Section 2. In Section 3, we review the various classes of parameter-dependent operators, or tempered operators, introduced in [15]. In Section 4, we present another tempered operator space called 'holomorphically tempered operators'. The resolvent of a b-pseudodifferential operator is such a tempered operator. In Sections 5 and 6, we study the Laplace transforms of tempered operators and we introduce the manifold with corners $X^2_{b,H}$. We show how the Schwartz kernels of these Laplace transforms, when lifted to $X^2_{b,H}$, are just polyhomogeneous functions. As the heat operator is the Laplace transform of its resolvent, the polyhomogeneity of the heat kernel then follows. We also prove an expansion similar to (2), but for general Laplace transforms. In Section 7, we study the Mellin transforms of tempered operators and study their meromorphic properties. Since complex powers are just Mellin transforms of resolvents, we obtain for free, the meromorphic properties of the complex powers of b-pseudodifferential operators. Finally, in Section 8, we present an index theorem for b-pseudodifferential operators.

2. MANIFOLDS WITH CORNERS AND THE b-CALCULUS

There are many references on the b-calculus, consider for instance, [23], [22], [21], [20], or [18]. We denote $\{0,1,2,\ldots\} = \mathbb{N}_0$ and $\{1,2,\ldots\} = \mathbb{N}$.

One definition of a manifold with corners is the following. A pair $(X, C^\infty(X))$ where X is a set and $C^\infty(X)$ is a set of functions on X is called an n-dimensional *manifold with corners* if the following hold: (1) There exists an n-dimensional manifold without boundary $\widetilde{X}$ containing X such that $C^\infty(X)$ is the restriction of the smooth functions on $\widetilde{X}$ to X; (2) There exists a finite set of smooth functions $\{\rho_i\}_{i=1}^r$ on $\widetilde{X}$ having the properties that $X = \{\rho_i \geq 0\}$; and if for some $p \in \widetilde{X}$, we have $\rho_{i_j}(p) = 0$ for some $i_1 < \cdots < i_k$, then $\{(d\rho_{i_j})(p)\}$ are mutually independent differentials.

The subset $H_i := \{\rho_i = 0\}$ is called a *boundary hypersurface* and ρ_i is called a *boundary defining function* for H_i. Note that the functions ρ_i are not unique, and that we may assume that each H_i is connected. If X is compact, then near any boundary hypersurface H_i, X can be decomposed as a product $X \cong [0,1) \times H_i$. The product $\rho = \prod_{i=1}^r \rho_i$ is called a *total boundary defining function* for X. Note that ρ is non-negative and vanishes only on ∂X. For each $k \in \mathbb{N}_0$, each connected component of $H_{i_1} \cap \cdots \cap H_{i_k}$, where $i_1 < \cdots < i_k$, is called a *codimension* k boundary face. Note that each boundary face is manifold with corners. The collection of such faces is denoted by $M_k(X)$. The largest k such that $M_k(X)$ is non-empty is called the *codimension of* X. Note that X is the only codimension 0 face. The union over all proper faces, the faces except X itself, is denoted by $M'(X)$.

Examples of manifold with corners include closed manifolds, manifolds with boundary, and finite products of such manifolds.

Any manifold with corners can be covered by coordinate patches of the form $\mathbb{R}^{n,k} = [0,\infty)^k \times \mathbb{R}^{n-k}$. In fact, one can define a manifold with corners as a topological space locally modeled by these $\mathbb{R}^{n,k}$ for fixed n and various k, cf. [23]. A subset $Y \subset X$ is called an l dimensional *p-submanifold* if given any point $y \in Y$, there exists a coordinate patch of the form $\mathcal{U} \cong \mathbb{R}^{l,k} \times \mathbb{R}^{j,i}$ centered at y such that $\mathcal{U} \cap Y \cong \mathbb{R}^{l,k} \times \{0\}$. If $i = 0$ on any such coordinate patch, then Y is called an *interior* p-submanifold.

For the rest of this paper, X will always denote a *compact* manifold with corners and n will always denote its dimension. The codimension of X will never be stated since it is not important for the constructions involved in this paper.

The space of *b-differential operators of order* $m \in \mathbb{N}_0$, $\operatorname{Diff}_b^m(X)$, is the space of operators P on $C^\infty(X)$ such that on any coordinate patch $\mathbb{R}^{n,k} = [0,\infty)_x^k \times \mathbb{R}_y^{n-k}$ of X, P takes the form

$$P = \sum_{|\alpha|+|\beta| \leq m} a_{\alpha,\beta}(x,y)(x\partial_x)^\alpha \partial_y^\beta, \text{ where } a_{\alpha,\beta}(x,y) \in C^\infty(\mathbb{R}^{n,k}).$$

We now consider various classes of conormal functions on X. The first space of conormal functions we define is the space of *symbols of order 0*, $S^0(X)$, as the space of functions $u \in L^\infty(X)$ such that all b-derivatives of u are also in $L^\infty(X)$; that is, $Pu \in L^\infty(X)$ for any $P \in \operatorname{Diff}_b^*(X)$. Symbols are smooth on the interior of X (by the Sobelev imbedding theorem) but may have 'logarithmic' singularities at ∂X. For instance, $u(x) = \cos(\log x) \in S^0([0,1))$ and $v(x) = 1/\log(\epsilon x) \in S^0([0,1))$ for any $0 < \epsilon < 1$. To allow for more general growth at ∂X, we first define a multi-index on X. A *multi-index* α is a map $\alpha : M_1(X) \to \mathbb{R}$, which can also be considered as an r-tuple of real numbers $(\alpha_1, \ldots, \alpha_r)$, each α_i corresponding to the hypersurface H_i. Given any multi-index α, we define $S^\alpha(X) := \rho^{-\alpha} S^0(X)$, where ρ is a total boundary defining function for X and where $\rho^{-\alpha} := \prod_{i=1}^r \rho_i^{-\alpha_i}$.

The symbol spaces $S^\alpha(X)$ are L^∞ based conormal spaces. In order to define L^2 based conormal spaces, we need to consider densities. Thus, given any $\alpha \in \mathbb{R}$, we denote by Ω_b^α the line bundle whose sections are of the form $\phi\,|\mu/\rho|^\alpha$, where $\phi \in C^\infty(X)$, ρ is a total boundary defining function for X, and where μ is a smooth, positive density on X. We will usually take $\alpha = \frac{1}{2}$. We denote by $L_b^2(X, \Omega_b^{\frac{1}{2}})$ the space of L^2-integrable b-half densities. Then the Sobelev space of b-half densities of order $s \in \mathbb{N}_0$, $H_b^s(X, \Omega_b^{\frac{1}{2}})$ is the space of functions $u \in L_b^2(X, \Omega_b^{\frac{1}{2}})$ such that $Pu \in L_b^2(X, \Omega_b^{\frac{1}{2}})$ for any $P \in \operatorname{Diff}_b^s(X, \Omega_b^{\frac{1}{2}})$. By interpolation and duality, one can define $H_b^s(X, \Omega_b^{\frac{1}{2}})$ for any $s \in \mathbb{R}$.

We now consider functions that have expansions at ∂X. The expansions are governed index sets. Here, a *(C^∞) index set* is a discrete subset $E \subset \mathbb{C} \times \mathbb{N}_0$ satisfying:

(1) For each $M \in \mathbb{R}$, the set $E \cap (\{z \in \mathbb{C} \,|\, \operatorname{Re} z \leq M\} \times \mathbb{N}_0)$ is finite;
(2) If $(z,k) \in E$, then $(z,l) \in E$ for all $0 \leq l \leq k$;
(3) If $(z,k) \in E$, then $(z+l,k) \in E$ for all $l \in \mathbb{N}$.

The following operation on index sets will be used quite often. Given two index sets E and F we define the *extended union* $E\overline{\cup}F$ as the index set $E \cup F \cup \{(z, k_1+k_2+1) | (z,k_1) \in E, \, (z,k_2) \in F\}$.

Let $H \in M_1(X)$. Given any multi-index α on X, we define the multi-index α_H on H as follows. Given $G \in M_1(H)$, we have $G \subset H \cap H'$ for some $H' \in M_1(X)$. We define $\alpha_H(G) := \alpha(H')$. Now let E be any index set and let $u \in S^\alpha(X)$. We say that u has a *classical or polyhomogeneous expansion* at H with index E if for any product decomposition $X \cong [0,1)_x \times H_y$ of X near H, the following holds: For each $(z,k) \in E$, there exists a $u_{(z,k)} \in S^{\alpha_H}(H)$ such that for each $N \in \mathbb{N}$, there exists an $M \in \mathbb{N}$ with

$$u(x,y) \;-\; \sum_{(z,k)\in E,\, \operatorname{Re} z \leq M} x^z (\log x)^k \, u_{(z,k)}(y) \in x^N S^{(0,\alpha_H)}([0,1) \times H). \tag{4}$$

We then write $u \sim \sum_{(z,k)\in E} x^z (\log x)^k u_{(z,k)}$. The space of all such functions is denoted by $\mathcal{A}^E(X)$ whenever H is fixed in a discussion. In the appendix of [15], it is shown that if the expansion (4) holds for one product decomposition of X near H, then it holds for any other one. A special case to note is if $E = \emptyset$, then u vanishes to infinite order at H.

For example, let $u(x) = \cos(x^z \log x)$ where $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. Then $u \in S^0([0,1))$. Now $\cos(y) \sim \sum_{j=0}^\infty y^{2j}/(2j)!$. Thus, it follows that $u \in \mathcal{A}^E([0,1))$ with $E = \{(2zj+k, l) \,|\, j,k \in \mathbb{N}_0, \, 0 \leq l \leq 2j\}$. In general, a polyhomogeneous expansion is just a generalized Taylor expansion, but one that admits complex powers and powers of log's.

The following theorem, due to R. Melrose, is the fundamental result used to prove the existence of expansions for functions. The proof can be found in the appendix of [15]. Here, Y denotes a compact manifold with corners.

Theorem 2.1. *A function $u \in S^\alpha([0,1)_x \times Y)$ has an asymptotic expansion at $Y \equiv \{0\}_x \times Y$ with index set E iff for each $N \in \mathbb{N}$, there exists an $M \in \mathbb{N}$ such that*

$$P^E_M u \in x^N S^{(0,\alpha_Y)}([0,1) \times Y),$$

where P^E_M is the differential operator $\prod_{(z,k)\in E,\, \operatorname{Re} z \leq M} (x\partial_x - z)$.

An *index family* $\mathcal{E}$ is a family of index sets $\mathcal{E} = \{E_i\}_{i=1}^r$, one index set for each hypersurface of X. Then we denote by $\mathcal{A}^{\mathcal{E}}_{phg}(X)$ the space of functions that can be expanded at each H_i with corresponding index set E_i.

Our final space of conormal functions that we will need are Hölder differentiable functions. Thus, let $H \in M_1(X)$ and let x be a boundary defining function for H. Then a symbol $u \in S^0(X)$ is said to be *η-Hölder differentiable* at H if u is of the form $u = v + x^\eta w$ where v can be expanded at H with index set $\mathbb{N}_0$, and where $w \in S^0(X)$ is continuous, with all b-derivatives, up to H. Given any collection $\mathcal{A} = \{H_{i_j}\}$ of boundary hypersurfaces and any non-negative multi-index $\eta = \{\eta_{i_j}\}$ associated to $\mathcal{A}$, we denote by $S^{0,\eta}_{\mathcal{A}}(X)$ the space of functions that are η_{i_j}-Hölder differentiable at each $H_{i_j} \in \mathcal{A}$.

We now define the class of b-pseudodifferential operators. We could define them as operators that act between the sections of any given vector bundles, but for simplicity we usually have them act only on b-half densities.

We denote by $\dot{C}^\infty(X, \Omega_b^{\frac12})$ the space of smooth functions on X that vanish to infinite order at ∂X. The *small calculus of b-pseudodifferential operators of order $m \in \mathbb{R}$* is the space operators on $\dot{C}^\infty(X, \Omega_b^{\frac12})$ that have Schwartz kernels given by

$$\Psi^m_b(X, \Omega_b^{\frac12}) = \{A \in I^m(X^2_b, \Delta_b, \Omega_b^{\frac12}) \,|\, A \equiv 0 \text{ at } lb \cup rb\}. \tag{5}$$

Here, X^2_b is called the *b-stretched product*, cf. [18, 21, 23], which is constructed from X^2 by blowing up the connected components of $\partial X \times \partial X$ that intersect the diagonal Δ in X^2. The collection of faces introduced in X^2_b through the blow-ups is denoted by *ff* (for *front face*); Δ_b is a p-submanifold called the *b-diagonal* and is the lift into X_b^2 of the diagonal Δ in X^2; finally, lb and rb are the lifts into X_b^2 of the left and right boundary hypersurfaces of $X^2 = X \times X$. The space $I^m(X_b^2, \Delta_b, \Omega_b^{\frac{1}{2}})$ is the space of distributional b-half densities on X_b^2 which are conormal to Δ_b of degree m ([21]), and the notation $\equiv 0$ in (5) means 'vanishing in Taylor series'. The *classical or one-step* subspace $\Psi^m_{b,os}(X, \Omega_b^{\frac{1}{2}}) \subset \Psi^m_b(X, \Omega_b^{\frac{1}{2}})$ consists of those operators with classical, one-step polyhomogeneous, symbols.

Basically, b-pseudodifferential operators are the 'usual' pseudodifferential operators, but with the Mellin transform replacing the Fourier transform in the direction normal to the boundary. Indeed, let $\mathbb{R}^{n,k} = [0,\infty)^k_x \times \mathbb{R}^{n-k}_y$ be a coordinate patch on X and let $A \in \Psi^m_b(X, \Omega_b^{\frac{1}{2}})$ have Schwartz kernel supported on the coordinate patch $\mathbb{R}^{n,k} \times \mathbb{R}^{n,k}$ of X^2. Then there exists a function $a(x, y, \xi)$, smooth in $(x, y) \in \mathbb{R}^{n,k}$ and a symbol of order m in ξ, such that given any $u = \phi(x, y)|\frac{dx}{x}dy|^{\frac{1}{2}} \in \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}})$ having support in the coordinate patch $\mathbb{R}^{n,k}$, we have

$$Au(x, y) = \frac{1}{(2\pi)^n} \int x^{i\xi'} e^{iy\cdot\xi''} \, a(x, y, \xi) \, \widetilde{\phi}(\xi) d\xi \cdot |\frac{dx}{x} dy|^{\frac{1}{2}}, \ \xi = (\xi', \xi''),$$

where $\widetilde{\phi}(\xi)$ is the Mellin transform in x and the Fourier transform in y of $\phi(x, y)$:

$$\widetilde{\phi}(\xi) = \int_{\mathbb{R}^{n,k}} x^{-i\xi'} e^{-iy\cdot\xi''} \, \phi(x, y) \frac{dx}{x} dy.$$

One can check that $Au \in \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}})$. The small calculus of b-pseudodifferential operators also define operators on $C^\infty(X, \Omega_b^{\frac{1}{2}})$.

The space of b-pseudodifferential operators share many properties with the usual pseudodifferential operators. For example, $\mathrm{Diff}^m_b(X, \Omega_b^{\frac{1}{2}}) \subset \Psi^m_b(X, \Omega_b^{\frac{1}{2}})$ for any $m \in \mathbb{N}_0$, and the space of b-pseudodifferential is a symbolically filtered $*$-algebra of operators. Thus, it is closed under taking adjoints and compositions, and there exists a (principal) symbol map, ${}^b\sigma_m(A) : \Psi^m_b(X, \Omega_b^{\frac{1}{2}}) \to S^{[m]}({}^bT^*X)$, where $S^{[m]}({}^bT^*X)$ is the quotient of the symbols of order m and $m-1$ on ${}^bT^*X$, the b-cotangent bundle of X, cf. [22]. This symbol map preserves adjoints and composition, and vanishes exactly on $\Psi^{m-1}_b(X, \Omega_b^{\frac{1}{2}})$. An operator A is called *elliptic* if ${}^b\sigma_m(A)$ is invertible. If $A \in \Psi^m_{b,os}(X, \Omega_b^{\frac{1}{2}})$, then ${}^b\sigma_m(A) \in C^\infty_{\mathrm{hom}(m)}({}^bT^*X)$, the space of smooth homogeneous functions of degree m on ${}^bT^*X \setminus \{0\}$.

Let α be a multi-index on X_b^2 with $\alpha|_{\mathit{ff}} \geq 0$. Then we define

$$\Psi_b^{-\infty,\alpha}(X, \Omega_b^{\frac{1}{2}}) := \rho_{lb}^{\alpha|_{lb}} \rho_{rb}^{\alpha|_{rb}} \bigcup_{\epsilon>0} \rho^\epsilon_{lb} \rho^\epsilon_{rb} S^{0,\alpha|_{\mathit{ff}}}_{\mathit{ff}}(X_b^2, \Omega_b^{\frac{1}{2}}). \tag{6}$$

Here, $S^{0,\alpha|_{\mathit{ff}}}_{\mathit{ff}}(X_b^2, \Omega_b^{\frac{1}{2}})$ is the space of symbols of order 0 on X_b^2 that are $\alpha|_{\mathit{ff}}$-Hölder differentiable at the front face *ff* of X_b^2. For any $m \in \mathbb{R}$, we define

$$\Psi_b^{m,\alpha}(X, \Omega_b^{\frac{1}{2}}) := \Psi^m_b(X, \Omega_b^{\frac{1}{2}}) + \Psi_b^{-\infty,\alpha}(X, \Omega_b^{\frac{1}{2}}).$$

The spaces $\Psi_b^{*,*}(X, \Omega_b^{\frac{1}{2}})$ form the 'calculus with bounds'. If $\alpha|_{lb} \geq \gamma'$, $\alpha|_{rb} \geq -\gamma$, and $\gamma \geq \gamma'$, then each $A \in \Psi_b^{m,\alpha}(X, \Omega_b^{\frac{1}{2}})$ defines a continuous linear map

$$A : \rho^\gamma H^s_b(X, \Omega_b^{\frac{1}{2}}) \to \rho^{\gamma'} H^{s-m}_b(X, \Omega_b^{\frac{1}{2}}), \ s \in \mathbb{R}. \tag{7}$$

Given a multi-index α for X^2, we define

$$\Psi^{-\infty,\alpha}(X, \Omega_b^{\frac{1}{2}}) := \rho^{\alpha|_{lb}} (\rho')^{\alpha|_{rb}} H^\infty_b(X^2, \Omega_b^{\frac{1}{2}}), \tag{8}$$

where ρ, respectively ρ', is the lift of a total boundary defining function ρ on X to the left, respectively right, factor of X^2.

We will now show how any b-pseudodifferential operator induces operators on the boundary hypersurfaces. Henceforth, we will fix a boundary defining function for each boundary hypersurface. Let M be a codimension k boundary face and denote by $x_1, \ldots, x_k$ the (fixed) boundary defining functions such that $M \subset \{x_i = 0\}$. Let $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$. Then the *normal operator of A at M* is the holomorphic family

$$\mathbb{C}^k \ni \tau \mapsto N_M(A)(\tau) := (x^{-i\tau} A x^{i\tau})|_M \in \Psi_b^m(M, \Omega_b^{\frac{1}{2}}), \tag{9}$$

where $x^{\pm i\tau} = x_1^{\pm i\tau_1} \cdots x_k^{\pm i\tau_k}$, and where $(x^{-i\tau} A x^{i\tau})|_M$ is the restriction of the operator $x^{-i\tau} A x^{i\tau}$ to $\dot{C}^\infty(M, \Omega_b^{\frac{1}{2}})$. Properties of the normal operator can be found in [13] or [23]. Then it follows that $N_M(A)(\tau)$ takes values in $\Psi_b^m(M, \Omega_b^{\frac{1}{2}})$, and the normal operator satisfies $N_M(AB)(\tau) = N_M(A)(\tau) N_M(B)(\tau)$ and $N_M(A^*)(\tau) = N_M(A)(\overline{\tau})^*$ for any $B \in \Psi_b^{m'}(X, \Omega_b^{\frac{1}{2}})$ and all $\tau \in \mathbb{C}^k$. For example, let $P \in \mathrm{Diff}_b^m(X, \Omega_b^{\frac{1}{2}})$. Assume that X is decomposed as a product $X \cong [0,1)^k \times M$ near M, where $x_1, \ldots, x_k$ are the coordinates on $[0,1)^k$. Then near M, we can write

$$P = \sum_{|\alpha| \le m} (xD_x)^\alpha P_\alpha(x) \cdot |\frac{dx}{x}|^{\frac{1}{2}}, \; D_{x_j} = \frac{1}{i}\partial_{x_j},$$

where $[0,1)^k \ni x \mapsto P_\alpha(x) \in \mathrm{Diff}_b^{m-|\alpha|}(M, \Omega_b^{\frac{1}{2}})$ are smoothly varying b-differential operators on M. Then as $(x_j D_{x_j}) x_j^{i\tau_j} = \tau_j \, x_j^{i\tau_j}$ for each j, by (9), it follows that

$$N_M(P)(\tau) = \sum_{|\alpha| \le m} \tau^\alpha P_\alpha(0).$$

Let $\{\rho_i\}$ be the fixed boundary defining functions corresponding to the hypersurfaces $M_1(X) = \{H_1, \ldots, H_r\}$. Observe that if $u \in C^\infty(X, \Omega_b)$, then $\rho^z u$, where $\rho^z = \rho_1^{z_1} \cdots \rho_r^{z_r}$, is integrable for all $z \in \mathbb{C}_+^r := \{(z_1, \ldots, z_r) \in \mathbb{C}^r \,|\, \mathrm{Re}\, z_i > 0 \text{ for all } i\}$. Thus, $\mathbb{C}_+^r \ni z \mapsto \int \rho^z u$ is holomorphic. In fact, cf. [17], it extends to be a meromorphic function on $\mathbb{C}^r$ with poles on the set $\{z \in \mathbb{C}^r \,|\, z_i \in -\mathbb{N}_0 \text{ for some } i\}$. The regular value of this function at $z = 0$ is called the *b-integral* of u and is denoted by ${}^b\!\int u$. If $A \in \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})$, then $A|_{\Delta_b} \in C^\infty(X, \Omega_b)$, where we identify Δ_b with X. The b-integral, ${}^b\!\int A|_{\Delta_b}$, is called the *regularized* or *b-trace* of A and is denoted by ${}^b\mathrm{Tr}(A)$.

3. TEMPERED OPERATORS

In this section we review the relevant material from [15] that we will need for this paper. In Section 4, we present another tempered operator space called 'holomorphically tempered operators'.

Throughout this paper, $\Lambda \subset \mathbb{C}$ will always denote a *proper* sector of the form $\Lambda = \{\lambda \in \mathbb{C} \,|\, \epsilon_1 \le \arg(\lambda) \le \epsilon_2\}$ for some ϵ_1, ϵ_2. We will use the notation $\dot{\le}$ from [9], which means "$\le$ a constant, independent of variables".

The ultimate goal of defining a parameter-dependent space of operators is to understand the properties of the resolvents of pseudodifferential operators; in particular, their asymptotic behavior as $\lambda \to \infty$.

The general procedure is as follows. If Λ is a sector of $\mathbb{C}$, then we would like to define a subspace of $C^\infty(\Lambda, \Psi_b^*(X, \Omega_b^{\frac{1}{2}}))$ of tempered operators, where we believe that the resolvents of at least certain pseudodifferential operators might possibly be found. We use the word 'tempered' because these parameter dependent operators should have some controlled growth as $\lambda \to \infty$. Our next step is to show that this tempered space does indeed capture resolvents. In order to give a good candidate for such a space, we need to fix the growth properties that the local symbols of these tempered operators should satisfy. To do so, consider the following simple example. Let $m \in \mathbb{R}^+$ and set $a(\lambda, \xi) := (|\xi|^m - \lambda)^{-1}$, where we choose Λ to be any sector not intersecting the positive real axis. The function $a(\lambda, \xi)$ should be a prototypical local model for the resolvent of pseudodifferential operators. Consider now the estimates on $a(\lambda, \xi)$. Fix $\alpha \in \mathbb{N}_0^2$ and $\beta \in \mathbb{N}_0^n$. Observe that

$$|\,(\partial_\lambda^\alpha \partial_\xi^\beta a)(\,\lambda, \xi/|\xi|\,)\,| \;\dot{\le}\; (1 + |\lambda|)^{-1-|\alpha|}. \tag{10}$$

Also, observe that $a(\delta^m \lambda, \delta\xi) = \delta^{-m} a(\lambda, \xi)$ for all $\delta > 0$. Hence, $(\partial_\lambda^\alpha \partial_\xi^\beta a)(\lambda, \xi)$ has the homogeneity property:

$$(\partial_\lambda^\alpha \partial_\xi^\beta a)(\delta^m \lambda, \delta\xi) = \delta^{-m-m|\alpha|-|\beta|} (\partial_\lambda^\alpha \partial_\xi^\beta a)(\lambda, \xi) \text{ for all } \delta > 0. \tag{11}$$

Combining this homogeneity property with inequality (10) yield

$$\begin{aligned} |(\partial_\lambda^\alpha \partial_\xi^\beta a)(\lambda,\xi)| &= |\xi|^{-m-m|\alpha|-|\beta|} \cdot |(\partial_\lambda^\alpha \partial_\xi^\beta a)(|\xi|^{-m}\lambda, \xi/|\xi|)| \\ &\dot\leq |\xi|^{-m-m|\alpha|-|\beta|} \cdot (1 + ||\xi|^{-m}\lambda|)^{-1-|\alpha|} \\ &\dot\leq (|\lambda| + |\xi|^m)^{-1-|\alpha|} |\xi|^{-|\beta|}. \end{aligned}$$

Thus, for each α and β,

$$|\partial_\lambda^\alpha \partial_\xi^\beta a(\lambda,\xi)| \dot\leq (|\lambda|^{1/m} + |\xi|)^{-m-m|\alpha|} |\xi|^{-|\beta|}. \tag{12}$$

Suppose now that $p(\xi)$ is a real valued, positive, homogeneous polynomial of degree $m \in \mathbb{N}$. Then $p(\lambda,\xi) := (p(\xi) - \lambda)^{-1}$ should be a prototypical local model for the resolvent of differential operators. We now consider its estimates. Fix α and β and define $C := \max\{|\partial_\lambda^\alpha \partial_\xi^\beta p(\lambda,\xi)| : |\lambda| + |\xi|^m = 1, (\lambda,\xi) \in \Lambda \times \mathbb{R}^n\}$. Since $p(\lambda,\xi)$ is smooth on $\Lambda \times \mathbb{R}^n \setminus \{(0,0)\}$, C is finite. Thus, as $p(\lambda,\xi)$ also has the homogeneous property (11), we have

$$\begin{aligned} |\partial_\lambda^\alpha \partial_\xi^\beta p(\lambda,\xi)| &= (|\lambda| + |\xi|^m)^{\frac{1}{m}(m-m|\alpha|-|\beta|)} \cdot |\partial_\lambda^\alpha \partial_\xi^\beta p(\frac{\lambda}{(|\lambda| + |\xi|^m)}, \frac{\xi}{(|\lambda| + |\xi|^m)^{\frac{1}{m}}})| \\ &\dot\leq (|\lambda|^{1/m} + |\xi|)^{m-m|\alpha|-|\beta|} \cdot C. \end{aligned}$$

Thus,

$$|\partial_\lambda^\alpha \partial_\xi^\beta p(\lambda,\xi)| \dot\leq (|\lambda|^{1/m} + |\xi|)^{m-m|\alpha|-|\beta|}. \tag{13}$$

Note that this computation does not apply to the function $a(\lambda,\xi)$ considered previously, since in that case, the constant C defined above would, in general, be infinite since high order derivatives of $|\xi|^m$ would be unbounded at $\xi = 0$. The estimates (12) and (13) show that the tempered operator spaces that should be used to capture the resolvent of pseudodifferential operators and differential operators need to be distinctly different. The estimates (12) and (13) are also the motivation for the following general definition.

Definition 3.1. Let $m, p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and let Λ be any sector of $\mathbb{C}$. Then the space of *tempered symbols*, $S_\Lambda^{m,p,d}(\mathbb{R}^n)$, consists of those functions $a(\lambda,\xi) \in C^\infty(\Lambda \times \mathbb{R}^n)$ satisfying the following estimates: for any α and β,

$$|\partial_\lambda^\alpha \partial_\xi^\beta a(\lambda,\xi)| \dot\leq (1 + |\lambda|^{1/d} + |\xi|)^{p-d|\alpha|} (1 + |\xi|)^{m-p-|\beta|}. \tag{14}$$

Here, $\partial_\lambda^\alpha = \partial_{\lambda_1}^{\alpha_1} \partial_{\lambda_2}^{\alpha_2}$ where $\lambda = \lambda_1 + i\lambda_2 = (\lambda_1,\lambda_2)$ are the standard coordinates on $\mathbb{C}$. The space of *strongly tempered symbols*, $S_\Lambda^{m,d}(\mathbb{R}^n)$, consists of those functions $a \in C^\infty(\Lambda \times \mathbb{R}^n)$ satisfying the following estimates: for each α, β,

$$|\partial_\lambda^\alpha \partial_\xi^\beta a(\lambda,\xi)| \dot\leq (1 + |\lambda|^{1/d} + |\xi|)^{m-d|\alpha|-|\beta|}. \tag{15}$$

The spaces $S_\Lambda^{m,p,d}(\mathbb{R}^n)$ and $S_\Lambda^{m,d}(\mathbb{R}^n)$ are the very basic symbol spaces that we will use to define our tempered operator spaces. They are modified versions of parameter-dependent symbols already found in the literature, cf. [30]. Some algebra (see [15]) shows that for any $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$,

$$S_\Lambda^{m,d}(\mathbb{R}^n) \subset S_\Lambda^{m,m,d}(\mathbb{R}^n). \tag{16}$$

We now give various refinements of the tempered symbol spaces $S_\Lambda^{*,*,*}(\mathbb{R}^n)$. Our first refinement are symbols that have more 'resolvent-like' behavior. Define $\Lambda_{cc} = \{\overline{\lambda} \mid \lambda \in \Lambda\}$ and set $\mu := 1/\lambda = \overline{\lambda}/|\lambda|^2 \in \Lambda_{cc}$, where $\lambda \in \Lambda \setminus \{0\}$. Consider the function $a(\lambda,\xi) = (|\xi|^m - \lambda)^{-1}$. Observe that if we set $\mu = 1/\lambda \in \Lambda_{cc}$, where in this case we assume that Λ does not intersect the positive real axis, then $a(1/\mu,\xi) = \mu(\mu|\xi|^m - 1)^{-1} = \mu\, \tilde a(\mu,\xi)$, where $\tilde a(\mu,\xi) = (\mu|\xi|^m - 1)^{-1}$ is *smooth* for $\mu \in \Lambda_{cc}$ (even at $\mu = 0$). Moreover, the homogeneity property: $\tilde a(\delta^{-m}\mu, \delta\xi) = \tilde a(\mu,\xi)$, plus an argument similar to how we proved (12) shows that $\tilde a(\mu,\xi)$ satisfies the estimates:

$$|\partial_\mu^\alpha \partial_\xi^\beta \tilde a(\mu,\xi)| \dot\leq (|\mu|\,|\xi|^m + 1)^{-1-|\alpha|} |\xi|^{m|\alpha|-|\beta|}.$$

This example is motivation for the following general definition.

Definition 3.2. Let m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, and let Λ be any sector of $\mathbb{C}$. The subspace $S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n) \subset S^{m,p,d}_{\Lambda}(\mathbb{R}^n)$ of *resolvent tempered symbols* consists of those $a(\lambda,\xi) \in S^{m,p,d}_{\Lambda}(\mathbb{R}^n)$ such that if we define

$$\tilde{a}(\mu,\xi) := \mu^{p/d}\, a(1/\mu,\xi) = \mu^{p/d}\, a(\overline{\mu}/|\mu|^2,\xi),$$

then $\tilde{a}(\mu,\xi) \in C^\infty(\Lambda_{cc} \times \mathbb{R}^n)$ (thus, $\tilde{a}(\mu,\xi)$ is assumed smooth even at $\mu = 0$) and it satisfies the estimates: for any α, β, and $K \in [0,\infty)$,

$$|\,\partial^\alpha_\mu \partial^\beta_\xi \tilde{a}(\mu,\xi)\,| \stackrel{.}{\leq} (1+|\mu|\,|\xi|^d)^{p/d-|\alpha|}\,(1+|\xi|)^{d|\alpha|+m-p-|\beta|},\ |\mu| \leq K. \tag{17}$$

Given sectors Λ, $\Lambda' \subset \mathbb{C}$, Λ is said to be a *proper subsector* of Λ' if $\Lambda \subset \mathrm{int}(\Lambda') \cup \{0\}$. The following lemma can be found in [15].

Lemma 3.3. *Let $\Lambda \subset \Lambda'$ be a proper subsector of a sector $\Lambda' \subset \mathbb{C}$. Let $a(\xi) \in S^m(\mathbb{R}^n)$, $m \in \mathbb{R}^+$, be elliptic and suppose that for some $R > 0$, $a(\xi) \notin \Lambda'$ for all $|\xi| \geq R$. Then for any $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ for $|\xi| \leq R$ and $\chi(\xi) \equiv 1$ for $|\xi| \geq R+1$, we have*

$$\chi(\xi)\,(\,a(\xi)-\lambda)^{-1} \in S^{-m,-m,m}_{\Lambda,r}(\mathbb{R}^n).$$

This lemma suggests that the symbols of resolvents for b-pseudodifferential operators are captured by these resolvent tempered symbols. We now describe the types of homogeneous symbols we will work with.

Definition 3.4. Let m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. Then the space of functions $C^{\infty,d}_{\mathrm{hom}(m)}(\Lambda \times \mathbb{R}^n)$ consists of those $a(\lambda,\xi) \in C^\infty(\Lambda \times \mathbb{R}^n \setminus \{(0,0)\})$, such that $a(\delta^d\lambda,\delta\xi) = \delta^m a(\lambda,\xi)$ for all $\delta > 0$. The space of functions $C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$ consists of those functions $a(\lambda,\xi) \in C^\infty(\Lambda \times (\mathbb{R}^n \setminus \{0\}))$ such that

(1) $a(\delta^d\lambda,\delta\xi) = \delta^m a(\lambda,\xi)$ for all $\delta > 0$;
(2) if we define $\tilde{a}(\mu,\xi) := \mu^{p/d}\, a(1/\mu,\xi) = \mu^{p/d}\, a(\overline{\mu}/|\mu|^2,\xi)$ for all $(\mu,\xi) \in \Lambda_{cc} \times (\mathbb{R}^n \setminus \{0\})$, then we have $\tilde{a}(\mu,\xi/|\xi|) \in C^\infty(\Lambda_{cc} \times \mathbb{S}^{n-1})$.

Lemma 3.5. *Let $a(\xi)$ be an elliptic homogeneous polynomial of degree $m \in \mathbb{N}$ that never takes values in a sector Λ for $\xi \neq 0$. Then, $(\,a(\xi)-\lambda)^{-1} \in C^{\infty,m}_{\mathrm{hom}(-m)}(\Lambda \times \mathbb{R}^n)$. If instead, $a(\xi) \in C^\infty_{\mathrm{hom}(m)}(\mathbb{R}^n)$, $m \in \mathbb{R}^+$ (the space of smooth functions on $\mathbb{R}^n \setminus \{0\}$ that are homogeneous of degree m), then $(\,a(\xi)-\lambda)^{-1} \in C^{\infty,-m,m}_{\Lambda,\mathrm{hom}(-m)}(\mathbb{R}^n)$.*

In [15] it is shown that if $\chi(\lambda,\xi) \in C^\infty(\Lambda \times \mathbb{R}^n)$ with $\chi(\lambda,\xi) \equiv 0$ near $(\lambda,\xi) = 0$ and $\chi(\lambda,\xi) \equiv 1$ outside a neighborhood of 0, then $\chi(\lambda,\xi)C^{\infty,d}_{\mathrm{hom}(m)}(\Lambda \times \mathbb{R}^n) \subset S^{m,d}_{\Lambda}(\mathbb{R}^n)$ and that if $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ near $\xi = 0$ and $\chi(\xi) \equiv 1$ outside a neighborhood of 0, then $\chi(\xi)C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n) \subset S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$.

We next define the notion of a 'polyhomogeneous tempered symbol'.

Definition 3.6. Let $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$. The space of *one-step, strongly polyhomogeneous symbols*, $S^{m,d}_{\Lambda,os}(\mathbb{R}^n)$, consists of those elements $a(\lambda,\xi) \in S^{m,d}_{\Lambda}(\mathbb{R}^n)$ such that

$$a(\lambda,\xi) \sim \sum_{j=0}^{\infty} \chi(\lambda,\xi)\, a_{m-j}(\lambda,\xi), \tag{18}$$

where $a_{m-j}(\lambda,\xi) \in C^{\infty,d}_{\mathrm{hom}(m-j)}(\Lambda \times \mathbb{R}^n)$, where $\chi(\lambda,\xi) \in C^\infty(\Lambda \times \mathbb{R}^n)$ with $\chi(\lambda,\xi) \equiv 0$ near $(\lambda,\xi) = 0$ and $\chi(\lambda,\xi) \equiv 1$ outside a neighborhood of 0, and where the asymptotic sum (18) is in $S^{m,d}_{\Lambda}(\mathbb{R}^n)$. Thus, for each $N \in \mathbb{N}$, we have $a(\lambda,\xi) - \sum_{j=0}^{N-1} \chi(\lambda,\xi)\, a_{m-j}(\lambda,\xi) \in S^{m-N,d}_{\Lambda}(\mathbb{R}^n)$. Let m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. The space of *resolvent one-step, polyhomogeneous symbols*, $S^{m,p,d}_{\Lambda,ros}(\mathbb{R}^n) \subset S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$ consists of those elements $a(\lambda,\xi) \in S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$ such that

$$a(\lambda,\xi) \sim \sum_{j=0}^{\infty} \chi(\xi)\, a_{m-j}(\lambda,\xi), \tag{19}$$

where for each j, $a_{m-j}(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m-j)}(\mathbb{R}^n)$, where $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ near $\xi = 0$ and $\chi(\xi) \equiv 1$ outside a neighborhood of 0, and where the asymptotic sum (19) means that for all $N \in \mathbb{N}$, $a(\lambda,\xi) - \sum_{j=0}^{N-1} a_{m-j}(\lambda,\xi) \in S^{m-N,p,d}_{\Lambda,r}(\mathbb{R}^n)$.

Lemma 3.5 implies the following.

Lemma 3.7. *Let $a(\xi)$ be an elliptic homogeneous polynomial of degree $m \in \mathbb{N}$ and suppose that $a(\xi)$ never takes values in a sector Λ for $\xi \neq 0$. Then, for any $\chi(\lambda,\xi) \in C^\infty(\Lambda \times \mathbb{R}^n)$ with $\chi(\lambda,\xi) \equiv 0$ near $(\lambda,\xi) = 0$ and $\chi(\lambda,\xi) \equiv 1$ outside a neighborhood of 0,*

$$\chi(\lambda,\xi)\,(\,a(\xi) - \lambda)^{-1} \in S^{-m,m}_{\Lambda,os}(\mathbb{R}^n).$$

If instead, $a(\xi) \in C^\infty_{\mathrm{hom}(m)}(\mathbb{R}^n)$, $m \in \mathbb{R}^+$, then for any $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ near $\xi = 0$ and $\chi(\xi) \equiv 1$ outside a neighborhood of 0,

$$\chi(\xi)\,(\,a(\xi) - \lambda)^{-1} \in S^{-m,-m,m}_{\Lambda,ros}(\mathbb{R}^n).$$

This lemma suggests that the symbols of the resolvents for b-differential operators are captured by the one-step, strongly tempered symbols and that the symbols of the resolvents for one-step, b-pseudodifferential operators are captured by the resolvent one-step symbols. These statements were proved true in [15]. Later, we will strengthen these results by considering slightly finer symbol spaces.

We now use our symbols to define a corresponding space of conormal distributions. Let $\Lambda \subset \mathbb{C}$ be a sector and let X be compact with corners. If $m \in \mathbb{Z}$ and $\mathcal{F}$ is any Fréchet space, we define

$$S^m_r(\Lambda;\mathcal{F}) := \{a(\lambda) \in S^m(\Lambda;\mathcal{F}) \,|\, \mu^m a(1/\mu) \text{ for } \mu \in \Lambda_{cc}, \text{ is smooth at } \mu = 0\}. \tag{20}$$

Definition 3.8. Let $Y \subset X$ be a closed interior p-submanifold and let $m, p \in \mathbb{R}$ and $d \in \mathbb{R}^+$. The space of *tempered conormal distributions to Y of degree m,p,d*, $I^{m,p,d}_\Lambda(X,Y)$, consists of those elements $u \in C^\infty(\Lambda; I^m(X,Y)\,)$ satisfying the following two conditions:

1. Given any $\phi \in C^\infty_c(X \setminus Y)$, we have $\phi u \in S^{p/d}(\Lambda; C^\infty(X)\,)$.
2. Given any coordinate patch $\mathbb{R}^{l,k}_y \times \mathbb{R}^q_z$ on X such that $Y \cong \mathbb{R}^{l,k} \times \{0\}$ and any compactly supported function ϕ on the coordinate patch, we have

$$\phi\, u = \frac{1}{(2\pi)^q} \int e^{iz\cdot\xi}\, a(\lambda,y,\xi)\, d\xi,$$

   where $y \mapsto a(\lambda,y,\xi) \in C^\infty(\mathbb{R}^{l,k};\, S^{m',p,d}_\Lambda(\mathbb{R}^q)\,)$, where $m' = m - n/4 + l/4$.

If $p/d \in \mathbb{Z}$, the space $I^{m,p,d}_{\Lambda,*}(X,Y)$ of *resolvent tempered* and *resolvent one-step* conormal distributions where $*$ is r and ros respectively, consists of those elements $u \in C^\infty(\Lambda; I^m(X,Y)\,)$ satisfying (1) with $S^{p/d}$ replaced by $S^{p/d}_r$ and satisfying (2) with $S^{m',p,d}_\Lambda$ replaced by $S^{m',p,d}_{\Lambda,r}$ and $S^{m',p,d}_{\Lambda,ros}$ respectively. The space of *strongly tempered* and *strongly tempered one-step* conormal distributions, $I^{m,d}_\Lambda(X,Y)$ and $I^{m,d}_{\Lambda,os}(X,Y)$ respectively, consists of those elements $u \in C^\infty(\Lambda; I^m(X,Y)\,)$ satisfying (1) with $S^{p/d}$ replaced by $S^{-\infty}$, and satisfying (2) with $S^{m',p,d}_\Lambda$ replaced by $S^{m',d}_\Lambda$, and $S^{m',d}_{\Lambda,os}$ respectively.

Remark 3.9. One can check, just as in the definition for the usual conormal distributions, that the condition (2) above is well-defined, independent of the (product) coordinates chosen.

In spirit of Melrose, we use these tempered distributions as the basis for defining the Schwartz kernels of our tempered operators.

Definition 3.10. Let $m, p \in \mathbb{R}$ and $d \in \mathbb{R}^+$. The *small calculus of tempered b-pseudodifferential operators of degree m,p,d* is the space of kernels

$$\Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}}) := \{K \in I^{m,p,d}_\Lambda(X^2_b,\Delta_b,\Omega_b^{\frac{1}{2}}) \,|\, K \equiv 0 \text{ at } lb \cup rb\}.$$

If $p/d \in \mathbb{Z}$, the *small calculus of resolvent tempered and resolvent one-step b-pseudodifferential operators of degree m,p,d* are the space of kernels

$$\Psi^{m,p,d}_{b,\Lambda,*}(X,\Omega_b^{\frac{1}{2}}) := \{K \in I^{m,p,d}_{\Lambda,*}(X^2_b,\Delta_b,\Omega_b^{\frac{1}{2}}) \,|\, K \equiv 0 \text{ at } lb \cup rb\},$$

where $*$ is r and ros respectively. The *small calculus of strongly tempered and strongly tempered one-step b-pseudodifferential operators of degree* m, d are the space of kernels

$$\Psi^{m,d}_{b,*}(X,\Omega_b^{\frac{1}{2}}) := \{K \in I^{m,d}_*(X_b^2,\Delta_b,\Omega_b^{\frac{1}{2}}) \mid K \equiv 0 \text{ at } lb \cup rb\},$$

where $*$ is Λ and Λ, os respectively.

Note that $\Psi^{m,p,d}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}}) \subset \Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}})$ (for $p/d \in \mathbb{Z}$) and by equation (16),

$$\Psi^{m,d}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}}) \subset \Psi^{m,m,d}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}}). \tag{21}$$

Some properties of these tempered operators are given in the following lemma. See [15] for the proof.

Lemma 3.11. *Let $m,p \in \mathbb{R}$ and $d \in \mathbb{R}^+$. Then,*

(1) *for any $m',p' \in \mathbb{R}$, $\Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}}) \subset \Psi^{m',p',d}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}})$, provided $m \le m'$ and $p \le p'$;*

(2) *for any α, $\partial^\alpha_\lambda \Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}}) \subset \Psi^{m-d|\alpha|,p-d|\alpha|,d}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}})$;*

(3)

$$\begin{aligned} \Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}}) &\subset S^{p/d}(\Lambda;\Psi^m_b(X,\Omega_b^{\frac{1}{2}})), && \text{if } p>0;\\ \Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}}) &\subset S^{p/d}(\Lambda;\Psi^{m-p}_b(X,\Omega_b^{\frac{1}{2}})), && \text{if } p\le 0. \end{aligned}$$

(4) *any operator $A \in \Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}})$ defines continuous linear maps*

$$\begin{aligned} A &: \dot{C}^\infty(X,\Omega_b^{\frac{1}{2}}) \to S^{p/d}(\Lambda;\dot{C}^\infty(X,\Omega_b^{\frac{1}{2}}));\\ A &: C^\infty(X,\Omega_b^{\frac{1}{2}}) \to S^{p/d}(\Lambda;C^\infty(X,\Omega_b^{\frac{1}{2}})). \end{aligned}$$

(5) *$A \circ B \in \Psi^{m+m',p+p',d}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}})$ if $A \in \Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}})$, $B \in \Psi^{m',p',d}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}})$.*

Similar statements hold for Lemma 3.11 for the resolvent, resolvent one-step, strongly tempered, and strongly tempered one-step spaces.

We now define the corresponding calculus with bounds, cf. (6) and (20). Let α be a multi-index on X_b^2 with $\alpha|_{ff} \ge 0$. We define

$$\begin{aligned} \Psi^{-\infty,p,d;\alpha}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}}) &:= \rho_{lb}^{\alpha|_{lb}}\rho_{rb}^{\alpha|_{rb}} \textstyle\bigcup_{\epsilon>0} \rho_{lb}^\epsilon \rho_{rb}^\epsilon S_r^{p/d}(\Lambda; S^{0,\alpha|_{ff}}_{ff}(X_b^2,\Omega_b^{\frac{1}{2}}));\\ \Psi^{-\infty,d;\alpha}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}}) &:= \rho_{lb}^{\alpha|_{lb}}\rho_{rb}^{\alpha|_{rb}} \textstyle\bigcup_{\epsilon>0} \rho_{lb}^\epsilon \rho_{rb}^\epsilon S^{-\infty}(\Lambda; S^{0,\alpha|_{ff}}_{ff}(X_b^2,\Omega_b^{\frac{1}{2}})), \end{aligned} \tag{22}$$

where $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. For any $m \in \mathbb{R}$, we define

$$\begin{aligned} \Psi^{m,p,d;\alpha}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}}) &:= \Psi^{m,p,d}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}}) + \Psi^{-\infty,p,d;\alpha}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}});\\ \Psi^{m,p,d;\alpha}_{b,\Lambda,ros}(X,\Omega_b^{\frac{1}{2}}) &:= \Psi^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac{1}{2}}) + \Psi^{-\infty,p,d;\alpha}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}});\\ \Psi^{m,d;\alpha}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}}) &:= \Psi^{m,d}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}}) + \Psi^{-\infty,d;\alpha}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}});\\ \Psi^{m,d;\alpha}_{b,\Lambda,os}(X,\Omega_b^{\frac{1}{2}}) &:= \Psi^{m,d}_{b,\Lambda,os}(X,\Omega_b^{\frac{1}{2}}) + \Psi^{-\infty,d;\alpha}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}}). \end{aligned} \tag{23}$$

4. HOLOMORPHICALLY TEMPERED OPERATORS

In this section, we introduce a new space of operators, already mentioned in the introduction, called 'holomorphically tempered operators'.

Definition 4.1. Let $m, p \in \mathbb{R}$, and let $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. Then a function $a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\hom(m)}(\mathbb{R}^n)$ or $a(\lambda,\xi) \in C^{\infty,d}_{\hom(m)}(\Lambda \times \mathbb{R}^n)$, is said to be *holomorphically tempered* if there exists an $\epsilon > 0$ such that $a(\lambda,\xi/|\xi|)$ extends to be a smooth function, holomorphic in λ, for $(\lambda,\xi/|\xi|)$ in

$$\{(\lambda,\xi/|\xi|) \in \mathbb{C} \times \mathbb{S}^{n-1} \mid \lambda \in \Lambda \text{ or } |\lambda| \le \epsilon \text{ or } \frac{1}{\epsilon} \le |\lambda|\}. \tag{24}$$

Lemma 4.2. *Let $m, p \in \mathbb{R}$, and let $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. Let $a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\hom(m)}(\mathbb{R}^n)$ or $a(\lambda,\xi) \in C^{\infty,d}_{\hom(m)}(\Lambda \times \mathbb{R}^n)$. Then $a(\lambda,\xi)$ is holomorphically tempered iff there exists an $\epsilon > 0$ such that $a(\lambda,\xi)$ extends to be a smooth function, holomorphic in λ, for (λ,ξ) in*

$$\{(\lambda,\xi) \in \mathbb{C} \times (\mathbb{R}^n \setminus \{0\}) \mid \lambda \in \Lambda \text{ or } |\lambda| \le \epsilon|\xi|^d \text{ or } \frac{1}{\epsilon}|\xi|^d \le |\lambda|\}. \tag{25}$$

Proof. We need only prove necessity. Thus, let $a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$ or $a(\lambda,\xi) \in C^{\infty,d}_{\mathrm{hom}(m)}(\Lambda\times\mathbb{R}^n)$ and suppose that $a(\lambda,\xi)$ is holomorphically tempered. Observe that for $(\lambda,\xi) \in \Lambda \times (\mathbb{R}^n \setminus \{0\})$, we have $a(\lambda,\xi) = |\xi|^m\, a(\,|\xi|^{-d}\lambda, \xi/|\xi|\,)$. It follows that $a(\lambda,\xi)$ extends to be a smooth function, holomorphic in λ, for (λ,ξ) in

$$\{(\lambda,\xi) \in \mathbb{C}\times(\mathbb{R}^n\setminus\{0\}) \,|\, \lambda\in\Lambda \ \text{ or } \ |\xi|^{-d}|\lambda| \le \epsilon \ \text{ or } \ \frac{1}{\epsilon} \le |\xi|^{-d}|\lambda|\};$$

that is, $\{(\lambda,\xi) \in \mathbb{C}\times(\mathbb{R}^n\setminus\{0\}) \,|\, \lambda\in\Lambda \ \text{ or } \ |\lambda| \le \epsilon\, |\xi|^d \ \text{ or } \ \frac{1}{\epsilon}\,|\xi|^d \le |\lambda|\}$. □

Of course, the model polyhomogeneous resolvent symbols are holomorphically tempered.

Lemma 4.3. *If $a(\xi)$ is an elliptic homogeneous polynomial of degree $m\in\mathbb{N}$ that never takes values in a sector Λ for $\xi\neq 0$, then $(a(\xi)-\lambda)^{-1} \in C^{\infty,m}_{\mathrm{hom}(-m)}(\Lambda\times\mathbb{R}^n)$ is holomorphically tempered. If $a(\xi) \in C^\infty_{\mathrm{hom}(m)}(\mathbb{R}^n)$, $m\in\mathbb{R}^+$ never takes values in a sector Λ for $\xi\neq 0$, then $(a(\xi)-\lambda)^{-1} \in C^{\infty,-m,m}_{\Lambda,\mathrm{hom}(-m)}(\mathbb{R}^n)$ is holomorphically tempered.*

Proof. Let $a(\xi)$ be either an elliptic homogeneous polynomial of degree $m\in\mathbb{N}$ or an element of $C^\infty_{\mathrm{hom}(m)}(\mathbb{R}^n)$ with $m\in\mathbb{R}^+$, that never takes values in a sector Λ for $\xi\neq 0$. Let $\epsilon>0$ be such that $\epsilon < \min_{|\xi|=1}|a(\xi)|$ and $\frac{1}{\epsilon} > \max_{|\xi|=1}|a(\xi)|$. Then it follows that $(a(\xi/|\xi|)-\lambda)^{-1}$ extends to be a smooth function, holomorphic in λ, for (λ,ξ) in the set (24). □

As usual, we let $\chi(\lambda,\xi)\in C^\infty(\Lambda\times\mathbb{R}^n)$ be such that $\chi(\lambda,\xi)\equiv 0$ near $(\lambda,\xi)=0$ and $\chi(\lambda,\xi)\equiv 1$ outside a neighborhood of 0, and $\chi(\xi)\in C^\infty(\mathbb{R}^n)$ with $\chi(\xi)\equiv 0$ near $\xi=0$ and $\chi(\xi)\equiv 1$ outside a neighborhood of 0.

Definition 4.4. Let m, $p\in\mathbb{R}$, and let $d\in\mathbb{R}^+$ with $p/d\in\mathbb{Z}$. Suppose that $a(\lambda,\xi)\in C^\infty(\Lambda\times\mathbb{R}^n)$ extends to be holomorphic for $\lambda\in\Lambda\cup\mathcal{U}$ for some neighborhood $\mathcal{U}$ of $\lambda=0$. If $a(\lambda,\xi)\in S^{m,d}_{\Lambda,os}(\mathbb{R}^n)$, then $a(\lambda,\xi)$ is *holomorphically tempered* if $a(\lambda,\xi)\sim\sum_{j=0}^\infty \chi(\lambda,\xi)\,a_{m-j}(\lambda,\xi)$, where $a_{m-j}(\lambda,\xi)\in C^{\infty,d}_{\mathrm{hom}(m-j)}(\Lambda\times\mathbb{R}^n)$ is holomorphically tempered (as defined in Definition 4.1). If $a(\lambda,\xi)\in S^{m,p,d}_{\Lambda,ros}(\mathbb{R}^n)$, then $a(\lambda,\xi)$ is *holomorphically tempered* if $a(\lambda,\xi)\sim\sum_{j=0}^\infty\chi(\xi)\,a_{m-j}(\lambda,\xi)$, where $a_{m-j}(\lambda,\xi)\in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m-j)}(\mathbb{R}^n)$ is holomorphically tempered. In both cases, the constant ϵ of (24) is assumed to hold for each of the $a_{m-j}(\lambda,\xi)$.

Recall that Λ is always assumed to be a proper sector of the form $\{\lambda\in\mathbb{C}\,|\,\epsilon_1\le\arg(\lambda)\le\epsilon_2\}$. Thus, we can always choose $v\in\Lambda$ such that $\lambda+v\in\Lambda$ for all $\lambda\in\Lambda$ by choosing, for example, $\arg(v)$ to be half-way between ϵ_1 and ϵ_2. The following lemma gives an easy way to define holomorphically tempered symbols.

Lemma 4.5. *Let $m\in\mathbb{R}$, $d\in\mathbb{R}^+$, and let $a(\lambda,\xi)\in C^{\infty,d}_{\mathrm{hom}(m)}(\Lambda\times\mathbb{R}^n)$ be holomorphically tempered. Let $v\in\mathrm{int}(\Lambda)$ be such that $\lambda+v\in\Lambda$ for all $\lambda\in\Lambda$, and define*

$$b(\lambda,\xi) := a(\lambda+v,\xi)\in S^{m,d}_\Lambda(\mathbb{R}^n).$$

Then, $b(\lambda,\xi)\in S^{m,d}_{\Lambda,os}(\mathbb{R}^n)$ and is holomorphically tempered, and for any $\chi(\lambda,\xi)\in C^\infty(\Lambda\times\mathbb{R}^n)$ with $\chi(\lambda,\xi)\equiv 0$ near 0 and with $\chi(\lambda,\xi)\equiv 1$ outside a neighborhood of 0, we have $b(\lambda,\xi) = \chi(\lambda,\xi)\,a(\lambda,\xi)$ modulo $S^{m-1,d}_{\Lambda,os}(\mathbb{R}^n)$.

Proof. Let $\chi(\lambda,\xi)\in C^\infty(\Lambda\times\mathbb{R}^n)$ with $\chi(\lambda,\xi)\equiv 0$ near $(\lambda,\xi)=0$, $\chi(\lambda,\xi)\equiv 1$ outside a neighborhood of 0, and with $\chi(\lambda+v,\xi)\equiv 1$ for all $\lambda\in\Lambda$. (We can find such a function since Λ is a proper sector of $\mathbb{C}$.) Since $\chi(\lambda+v,\xi)\equiv 1$ for all $\lambda\in\Lambda$,

$$b(\lambda,\xi)\equiv\chi(\lambda+v,\xi)\,a(\lambda+v,\xi) \text{ for all } (\lambda,\xi)\in\Lambda\times\mathbb{R}^n.$$

For each $t\in[0,\infty)$, set $b(t,\lambda,\xi) := \chi(\lambda+tv,\xi)\,a(\lambda+tv,\xi)$. Then, $b(t,\lambda,\xi)\in C^\infty([0,\infty);S^{m,d}_\Lambda(\mathbb{R}^n))$ and $b(0,\lambda,\xi)\equiv\chi(\lambda,\xi)\,a(\lambda,\xi)$. Now by Taylor's Theorem, for each $N\in\mathbb{N}$, we have $b(t,\lambda,\xi) = \sum_{j=0}^{N-1} t^j b_{m-j}(\lambda,\xi) + b_N(t,\lambda,\xi)$, where

$$b_{m-j}(\lambda,\xi) := (\partial_t^j b)(0,\lambda,\xi) = \sum_{k=0}^{j}\binom{j}{k}(v\cdot\partial_\lambda)^k\chi(\lambda,\xi)\,(v\cdot\partial_\lambda)^{j-k}a(\lambda,\xi), \tag{26}$$

with $v \cdot \partial_\lambda = v_1 \partial_{\lambda_1} + v_2 \partial_{\lambda_2}$, and where

$$b_N(t,\lambda,\xi) = \frac{t^N}{(N-1)!} \int_0^1 (1-s)^{N-1} \, (\partial_t^N b)(s\,t,\lambda,\xi) \, ds.$$

Set $a_{m-j}(\lambda,\xi) := (v \cdot \partial_\lambda)^j a(\lambda,\xi) \in C^{\infty,d}_{\mathrm{hom}(m-j)}(\Lambda \times \mathbb{R}^n)$. Then $a_{m-j}(\lambda,\xi)$ is holomorphically tempered and by (26), $b_{m-j}(\lambda,\xi) = \chi(\lambda,\xi)\, a_{m-j}(\lambda,\xi)$ modulo $S^{-\infty,d}_\Lambda(\mathbb{R}^n)$. Observe that $b_N(t,\lambda,\xi) \in C^\infty([0,\infty)_t; S^{m-N,d}_\Lambda(\mathbb{R}^n))$. Thus, for each $N \in \mathbb{N}$, we have

$$\begin{aligned} b(\lambda,\xi) = b(1,\lambda,\xi) &= \sum_{j=0}^{N-1} b_{m-j}(\lambda,\xi) + b_N(1,\lambda,\xi) \\ &= \sum_{j=0}^{N-1} \chi(\lambda,\xi)\, a_{m-j}(\lambda,\xi) + r_N(\lambda,\xi), \end{aligned}$$

where $r_N(\lambda,\xi) \in S^{m-N,d}_\Lambda(\mathbb{R}^n)$. □

We now define holomorphically tempered symbols that are not necessarily homogeneous.

Definition 4.6. Let $m,\, p \in \mathbb{R}$, and let $d \in \mathbb{R}^+$. Then a symbol $a(\lambda,\xi) \in S^{m,p,d}_\Lambda(\mathbb{R}^n)$, respectively $a(\lambda,\xi) \in S^{m,d}_\Lambda(\mathbb{R}^n)$, is said to be *holomorphically tempered* if there exists an $\epsilon > 0$ such that $a(\lambda,\xi)$ extends to be a smooth function, holomorphic in λ, for (λ,ξ) in

$$\{(\lambda,\xi) \in \mathbb{C} \times \mathbb{R}^n \mid \lambda \in \Lambda \ \text{ or } \ |\lambda| \le \epsilon\,(1+|\xi|)^d \ \text{ or } \ \frac{1}{\epsilon}\,(1+|\xi|)^d \le |\lambda|\}; \tag{27}$$

and moreover, $a(\lambda,\xi)$ continues to satisfy the symbol estimates (14), respectively (15), for (λ,ξ) in the set given by (27). If $a(\lambda,\xi) \in S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$, we say that $a(\lambda,\xi)$ is *holomorphically tempered* if it is holomorphically tempered as an element of $S^{m,p,d}_\Lambda(\mathbb{R}^n)$.

Of course, the model non-polyhomogeneous resolvent symbols are holomorphically tempered.

Lemma 4.7. *Let $a(\xi) \in S^m(\mathbb{R}^n)$, $m \in \mathbb{R}^+$ be elliptic and let $\Lambda \subset \mathbb{C}$ be a sector. Suppose that Λ is a proper subsector of a sector $\Lambda' \subset \mathbb{C}$ and suppose that for some $R > 0$, $a(\xi) \notin \Lambda'$ for all $|\xi| \ge R$. Let $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ for $|\xi| \le R$ and $\chi(\xi) \equiv 1$ for $|\xi| \ge R+1$. Then, $\chi(\xi)\,(a(\xi)-\lambda)^{-1} \in S^{-m,-m,m}_{\Lambda,r}(\mathbb{R}^n)$ is holomorphically tempered.*

Proof. By Lemma 3.3, $a(\lambda,\xi) := \chi(\xi)\,(a(\xi)-\lambda)^{-1} \in S^{-m,-m,m}_{\Lambda,r}(\mathbb{R}^n)$. Since $a(\xi)$ is elliptic, there exists a $\delta > 0$ such that $\delta(1+|\xi|)^m \le |a(\xi)| \le \frac{1}{\delta}(1+|\xi|)^m$ for all $|\xi| \ge R$. It follows that $a(\lambda,\xi)$ is a holomorphic function for all (λ,ξ) in the set (27) with $d = m$. The same argument used to prove Lemma 3.3, cf. [15], shows that $a(\lambda,\xi)$ continues to satisfy its same symbol estimates for (λ,ξ) in the set (27) as in the set $\Lambda \times \mathbb{R}^n$. □

We now define a corresponding space of holomorphically tempered operators.

Definition 4.8. Let $m,\, p \in \mathbb{R}$ and let $d \in \mathbb{R}^+$. Then $A \in \Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac12})$, respectively $A \in \Psi^{m,d}_{b,\Lambda}(X,\Omega_b^{\frac12})$, is said to be *holomorphically tempered* if

(1) $\lambda \mapsto A(\lambda) \in \Psi^m_b(X,\Omega_b^{\frac12})$ extends to be holomorphic for $\lambda \in \Lambda \cup \mathcal{U}$ for some neighborhood $\mathcal{U}$ of $\lambda = 0$; and
(2) for any coordinate patch $\mathbb{R}^{n,k}_y \times \mathbb{R}^n_z$ on X_b^2 such that $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}$ and any compactly supported function ϕ on the coordinate patch, we have

$$\phi\, A = \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, a(\lambda,y,\xi)\, d\xi \cdot \nu, \ \ \nu \in C^\infty(X_b^2, \Omega_b^{\frac12}), \tag{28}$$

where for each $y \in \mathbb{R}^{n,k}$, $a(\lambda,y,\xi) \in S^{m,p,d}_\Lambda(\mathbb{R}^n)$, respectively $a(\lambda,y,\xi) \in S^{m,d}_\Lambda(\mathbb{R}^n)$, is holomorphically tempered (where the constant ϵ in (27) is uniform in $y \in \mathbb{R}^{n,k}$).

If $A \in \Psi^{m,p,d}_{b,\Lambda,r}(X,\Omega_b^{\frac12})$ (for $p/d \in \mathbb{Z}$), we say that A is *holomorphically tempered* if it is holomorphically tempered as an element of $\Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac12})$.

If $A \in \Psi^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac12})$ (for $p/d \in \mathbb{Z}$), respectively $A \in \Psi^{m,d}_{b,\Lambda,os}(X,\Omega_b^{\frac12})$, then A is said to be *holomorphically tempered* if (1) holds and (2) holds with $S^{m,p,d}_{\Lambda}(\mathbb{R}^n)$ replaced with $S^{m,p,d}_{\Lambda,ros}(\mathbb{R}^n)$, respectively $S^{m,d}_{\Lambda,os}(\mathbb{R}^n)$, where the constant ϵ in Definition 4.4 (see also (24)) is uniform in $y \in \mathbb{R}^{n,k}$.

Finally, if A is in any of the tempered calculi with bounds as defined in (23), then A is *holomorphically tempered* if decomposing A as a sum of two operators as in the definition (23): $A = B + C$, the operator B is holomorphically tempered, and the operator C in the corresponding space (22) is holomorphic on $\Lambda \cup \mathcal{U}$ for some neighborhood $\mathcal{U}$ of $\lambda = 0$.

We now prove that the resolvent is a holomorphically tempered operator. The b-pseudodifferential operators that we will consider have symbols satisfying the following criteria.

Definition 4.9. Let $m \in \mathbb{R}^+$ and $\Lambda \subset \mathbb{C}$ be a sector. Then an elliptic operator $A \in \Psi^m_b(X,\Omega_b^{\frac12})$ is said to be *elliptic with respect to* Λ if the principal symbol ${}^b\sigma_m(A) \in S^{[m]}({}^bT^*X)$ can be represented by an element $a(x,\xi) \in S^m({}^bT^*X)$ having the property that for some sector $\Lambda' \subset \mathbb{C}$ with $\Lambda \subset \Lambda'$ a proper subsector and some compact set $K \subset {}^bT^*X$, we have $a(x,\xi) \notin \Lambda'$ for all $(x,\xi) \notin K$.

This definition makes sense for if $b(x,\xi) \in S^{m-1}({}^bT^*X)$, then $a(x,\xi) + b(x,\xi) \in S^m({}^bT^*X)$ has the same property as $a(x,\xi)$, cf. [15]. Note that if $A \in \Psi^m_{b,os}(X,\Omega_b^{\frac12})$, then A is elliptic with respect to Λ iff the principal symbol ${}^b\sigma_m(A)(\xi) \in C^\infty_{\mathrm{hom}(m)}({}^bT^*X)$ never takes values in Λ for $\xi \neq 0$.

The following theorem is the main result of [15].

Theorem 4.10. *If $A \in \Psi^m_b(X,\Omega_b^{\frac12})$, $m \in \mathbb{R}^+$, is elliptic with respect to Λ, then*

$$A - \lambda : H^s_b(X,\Omega_b^{\frac12}) \to H^{s-m}_b(X,\Omega_b^{\frac12}),\ \ s \in \mathbb{R},$$

is invertible for $\lambda \in \Lambda$ sufficiently large. Moreover, there is a continuous increasing function $r : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $a > 0$, $(A-\lambda)^{-1} \in \Psi^{-m,-m,m;a}_{b,\Lambda,r}(X,\Omega_b^{\frac12})$ for $|\lambda| \geq r(a)$. If $A \in \Psi^m_{b,os}(X,\Omega_b^{\frac12})$ or $A \in \mathrm{Diff}^m_b(X,\Omega_b^{\frac12})$, the same holds, but for $|\lambda| \geq r(a)$, the resolvent is in $\Psi^{-m,-m,m;a}_{b,\Lambda,ros}(X,\Omega_b^{\frac12})$ or $\Psi^{-m,m;a}_{b,\Lambda,os}(X,\Omega_b^{\frac12})$ respectively.

In the above theorem, we identify the number a with the multi-index that assigns to every hypersurface of X^2_b, the number a. In the following theorem, we improve the above result.

Theorem 4.11. *Let $A \in \Psi^m_b(X,\Omega_b^{\frac12})$, $m \in \mathbb{R}^+$, be elliptic with respect to Λ. Then there exists a holomorphically tempered operator $B(\lambda) \in \Psi^{-m,-m,m}_{b,\Lambda,r}(X,\Omega_b^{\frac12})$ such that for $\lambda \in \Lambda$ sufficiently large,*

$$(A-\lambda)^{-1} = B(\lambda) + R(\lambda), \tag{29}$$

where for some continuous increasing function $r : \mathbb{R}^+ \to \mathbb{R}^+$, for each $a > 0$, we have $R(\lambda) \in \Psi^{-\infty,-m,m;a}_{b,\Lambda,r}(X,\Omega_b^{\frac12})$ for $|\lambda| \geq r(a)$.

If $A \in \Psi^m_{b,os}(X,\Omega_b^{\frac12})$, then (29) holds but $B(\lambda) \in \Psi^{-m,-m,m}_{b,\Lambda,ros}(X,\Omega_b^{\frac12})$; and if $A \in \mathrm{Diff}^m_b(X,\Omega_b^{\frac12})$, then (29) holds but with $B(\lambda) \in \Psi^{-m,m}_{b,\Lambda,os}(X,\Omega_b^{\frac12})$ and with $R(\lambda) \in \Psi^{-\infty,m;a}_{b,\Lambda}(X,\Omega_b^{\frac12})$ for $|\lambda| \geq r(a)$.

Proof. As the proof of this theorem is almost identical to the proof of Theorem 4.10, we only give an outline the proof and leave the interested reader to check [15] for the details.

For simplicity, we will only prove this result for $A \in \Psi^m_b(X,\Omega_b^{\frac12})$ and only note at the end of the proof, the changes that must be made for $A \in \Psi^m_{b,os}(X,\Omega_b^{\frac12})$ or $A \in \mathrm{Diff}^m_b(X,\Omega_b^{\frac12})$. Let $a \in S^m({}^bT^*X)$ be a representative of the symbol of A as in Definition 4.9.

Let $\mathbb{R}^{n,k} \times \mathbb{R}^n_z$ be a coordinate patch on X^2_b near Δ_b such that $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}_z$. Note that since $\Delta_b \cong X$, the factor $\mathcal{U} = \mathbb{R}^{n,k} = [0,\infty)^k_x \times \mathbb{R}^{n-k}_y$ can be considered a coordinate patch on X. By Definition 4.9, there exists a $k > 0$ such that $a(x,y,\xi) \notin \Lambda'$ for some sector Λ' containing Λ as a proper subsector and for all $|\xi| > k$. Let $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ where $\chi(\xi) \equiv 0$ for all $|\xi| \leq k$ and $\chi(\xi) \equiv 1$

for $|\xi| \geq k+1$. Define

$$Q_{\mathcal{U}} := \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, q(\lambda,x,y,\xi)\, d\xi \cdot |\frac{dx}{x} dy dz|^{\frac{1}{2}}, \tag{30}$$

where $q(\lambda,x,y,\xi) := \chi(\xi)(a(x,y,\xi)-\lambda)^{-1}$. By Lemma 4.7, for any compactly supported smooth functions $\phi(x,y)$ on $\mathcal{U}$ and $\psi(z)$ on $\mathbb{R}^n$, $Q'_{\mathcal{U}} = \phi(x,y)\psi(z)Q_{\mathcal{U}} \in \Psi^{-m,-m,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$ is holomorphically tempered. Moreover, cf. the composition property (5) of Lemma 3.11, it follows that $(A-\lambda)Q'_{\mathcal{U}} = \phi - S_{\mathcal{U}}$, where $S_{\mathcal{U}} \in \Psi^{-1,0,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$ is holomorphically tempered. Following the standard procedure of patching together local models of the form (30) by taking the $\mathcal{U}$'s to be a coordinate cover of X and taking the ϕ's to be a partition of unity with respect to the cover, cf. [15, 30], produces a holomorphically tempered operator $Q \in \Psi^{-m,-m,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$ such that $(A-\lambda)Q = \mathrm{Id} - S$, where $S \in \Psi^{-1,0,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$ is holomorphically tempered.

The usual composition results, cf. (5) of Lemma 3.11, imply that for any j, $S^j \in \Psi^{-j,0,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$ is holomorphically tempered. Moreover, the usual asymptotic summation proof can be used to show that there exists a holomorphically tempered operator $S' \in \Psi^{-1,0,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$ such that $S' \sim \sum_{j=1}^{\infty} S^j$. Thus, $B := Q(\mathrm{Id}+S') \in \Psi^{-m,-m,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$ is holomorphically tempered and $(A-\lambda)B = \mathrm{Id} - S''$, where $S'' \in \Psi^{-\infty,0,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$ is holomorphically tempered. Multiplying both sides of the equation $(A-\lambda)B = \mathrm{Id} - S''$ by $(A-\lambda)^{-1}$ and using Theorem 4.10 proves that $(A-\lambda)^{-1}$ is of the form (29), where $R = (A-\lambda)^{-1}S''$.

If $A \in \Psi^m_{b,os}(X,\Omega_b^{\frac{1}{2}})$, the proof is unchanged except we understand that $a(x,y,\xi)$ used to define $Q_{\mathcal{U}}$ in (30) is homogeneous of degree m in ξ. If $A \in \mathrm{Diff}^m_b(X,\Omega_b^{\frac{1}{2}})$, then we set $q(\lambda,x,y,\xi) := (a(x,y,\xi)-v-\lambda)^{-1}$ in (30) where $a(x,y,\xi)$ is the homogeneous polynomial symbol of A, and where $v \in \mathrm{int}(\Lambda)$ is such that $\lambda + v \in \Lambda$ for all $\lambda \in \Lambda$, cf. Lemma 4.5. We then follow the same argument as we used above for $A \in \Psi^m_b(X,\Omega_b^{\frac{1}{2}})$, but staying within the framework of the strongly tempered calculus, to finish off the proof for $A \in \mathrm{Diff}^m_b(X,\Omega_b^{\frac{1}{2}})$. □

5. LAPLACE TRANSFORMS OF TEMPERED OPERATORS

For the rest of this paper, we will assume that Λ is of the form

$$\text{Assumption: } \Lambda = \{\lambda \in \mathbb{C} \,|\, \epsilon_0 \leq \arg(\lambda) \leq 2\pi - \epsilon_0\}, \text{ for some } 0 < \epsilon_0 < \pi/2. \tag{31}$$

Let $m, p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and let $A(\lambda) \in \Psi^{m,p,d;\alpha}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}})$ be holomorphic for $\lambda \in \Lambda$. Let Γ be any contour in Λ of the form

$$\Gamma = c + \{\lambda \in \mathbb{C} \,|\, \arg(\lambda) = \delta \text{ or } \arg(\lambda) = 2\pi - \delta\}, \quad c < 0,\ \epsilon_0 < \delta < \pi/2. \tag{32}$$

Let $u \in \rho^\gamma H^s_b(X,\Omega_b^{\frac{1}{2}})$. Then, if $\alpha|_{lb} \geq \gamma'$, $\alpha|_{rb} \geq -\gamma$, and $\gamma \geq \gamma'$, by (3) of Lemma 3.11 and the mapping property (7), $A(\lambda)u \in S^{p/d}(\Lambda; \rho^{\gamma'} H^{s'}_b(X,\Omega_b^{\frac{1}{2}}))$ where $s' = s-m$ if $p>0$ and $s' = s-(m-p)$ if $p \leq 0$. Thus, for each $t>0$, $e^{-t\lambda} A(\lambda)u$ is exponentially decreasing in $\rho^{\gamma'} H^{s'}_b(X,\Omega_b^{\frac{1}{2}})$ as $|\lambda| \to \infty$ for $\lambda \in \Gamma$. Hence, for each $t>0$, the integral

$$\mathcal{L}(A)(t)u := \int_\Gamma e^{-t\lambda} A(\lambda)u \, d\lambda \tag{33}$$

converges in $\rho^{\gamma'} H^{s'}_b(X,\Omega_b^{\frac{1}{2}})$. Observe that since $A(\lambda)$ is holomorphic, the integral (33) is defined *independent* of the contour Γ chosen (where Γ is of the form given in (32)). Since for any $k \in \mathbb{N}$ and $t>0$, $e^{-t\lambda}\lambda^k A(\lambda)u$ is still exponentially decreasing in $\rho^{\gamma'} H^{s'}_b(X,\Omega_b^{\frac{1}{2}})$ as $|\lambda| \to \infty$ for $\lambda \in \Gamma$, it follows that

$$\mathcal{L}(A)(t) : \rho^\gamma H^s_b(X,\Omega_b^{\frac{1}{2}}) \to C^\infty((0,\infty)_t; \rho^{\gamma'} H^{s'}_b(X,\Omega_b^{\frac{1}{2}})). \tag{34}$$

Definition 5.1. Let $m,\, p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and let $A(\lambda) \in \Psi^{m,p,d;\alpha}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$ be holomorphic for $\lambda \in \Lambda$. Then the *Laplace Transform* of A is the map (34) defined by equation (33) for any contour Γ of the form given in (32).

Of course, by (21), the Laplace transforms of strongly tempered holomorphic operators are defined in the same way. In the following theorem, we show that the Laplace transforms are always b-pseudodifferential operators of order $-\infty$ for positive time.

Theorem 5.2. *Let $m, p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and let $A(\lambda) \in \Psi^{m,p,d;\alpha}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$ be holomorphic. Then, $\mathcal{L}(A)(t) \in C^\infty(\,(0,\infty)_t;\, \Psi_b^{-\infty,\alpha}(X, \Omega_b^{\frac{1}{2}})\,)$. In fact, if $k \in \mathbb{N}_0$ and $M \in \mathbb{N}_0$ are such that $p/d - M < -k-1$, then for any $N \in \mathbb{N}_0$,*

$$t^{M+N}\mathcal{L}(A)(t) \in C^\infty((0,\infty)_t; \Psi_b^{-\infty,\alpha}(X, \Omega_b^{\frac{1}{2}})) \cap C^k([0,\infty)_t; \Psi_b^{m-p-dN,\alpha}(X, \Omega_b^{\frac{1}{2}})). \tag{35}$$

Moreover, if $A(\lambda)$ extends to be holomorphic on a neighborhood of $\lambda = 0$, then

$$\mathcal{L}(A)(t) \to 0 \text{ exponentially in } \Psi_b^{-\infty,\alpha}(X, \Omega_b^{\frac{1}{2}}) \text{ as } t \to \infty. \tag{36}$$

Proof. Let $k \in \mathbb{N}_0$ and let $M \in \mathbb{N}_0$ be such that $p/d - M < -k-1$. Then to prove (35), it suffices to show that for any $N \in \mathbb{N}_0$,

$$t^{M+N}\mathcal{L}(A)(t) \in C^k(\,[0,\infty)_t;\, \Psi_b^{m-p-dN,\alpha}(X, \Omega_b^{\frac{1}{2}})\,). \tag{37}$$

To see this, observe that

$$\begin{aligned} t^{M+N}\mathcal{L}(A)(t) &= (-1)^{M+N}\int_\Gamma (\partial_\lambda^{M+N} e^{-t\lambda})\, A(\lambda)\, d\lambda \\ &= \int_\Gamma e^{-t\lambda} (\partial_\lambda^{M+N} A(\lambda))\, d\lambda. \end{aligned} \tag{38}$$

By Lemma 3.11, and the fact that $p/d - M < 0$, we have

$$\begin{aligned} \partial_\lambda^{M+N} A(\lambda) &\in \Psi_{b,\Lambda}^{m-dM-dN,p-dM-dN,d;\alpha}(X, \Omega_b^{\frac{1}{2}}) \\ &\subset \Psi_{b,\Lambda}^{m-dM-dN,p-dM,d;\alpha}(X, \Omega_b^{\frac{1}{2}}) \\ &\subset S^{(p-dM)/d}(\Lambda; \Psi_b^{m-dM-dN-(p-dM),\alpha}(X, \Omega_b^{\frac{1}{2}})\,). \end{aligned}$$

Thus, $\partial_\lambda^{M+N} A(\lambda) \in S^{-k-1-\epsilon}(\Lambda; \Psi_b^{m-p-dN,\alpha}(X, \Omega_b^{\frac{1}{2}})\,)$ where $\epsilon = -k-1-p/d+M > 0$. Since λ^{-r} is integrable at $\lambda = \infty$ for $r > 1$, (37) follows from (38).

Now suppose that $A(\lambda)$ extends to be holomorphic on a neighborhood of $\lambda = 0$. Then it follows that we can write

$$\mathcal{L}(A)(t) = \int_{\Gamma'} e^{-t\lambda}\, A(\lambda)\, d\lambda, \tag{39}$$

where Γ' is a contour given in (32), but where we may assume that $c > 0$. Using equation (39) and the fact that $e^{-t\lambda} \to 0$ uniformly in $\lambda \in \Gamma'$ as $t \to \infty$, it follows that $\mathcal{L}(A)(t) \to 0$ exponentially in $\Psi_b^{-\infty,\alpha}(X, \Omega_b^{\frac{1}{2}})$ as $t \to \infty$. ☐

We now discuss the 'most important' example of a Laplace transform: the heat operator of a b-pseudodifferential operator. Thus, let $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be elliptic with respect to Λ. Then by Theorem 4.11, there exists a continuous increasing function $r : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $a > 0$,

$$(A-\lambda)^{-1} \in \Psi_{b,\Lambda,r}^{-m,-m,m;a}(X, \Omega_b^{\frac{1}{2}})$$

for $|\lambda| \geq r(a)$. Let Γ be any contour in Λ of the form given in (32), where $(A-\lambda)^{-1}$ defined on Γ (for example, choose Γ such that $|\lambda| > r(1)$ for all $\lambda \in \Gamma$). Then the *heat operator*, e^{-tA}, of A is the operator

$$e^{-tA} := \frac{i}{2\pi}\int_\Gamma e^{-t\lambda}\,(A-\lambda)^{-1}\, d\lambda = \frac{i}{2\pi}\mathcal{L}(\,(A-\lambda)^{-1})(t).$$

The following theorem shows that for holomorphically tempered operators, the Schwartz kernels of their Laplace transforms are quite simple, even down to $t=0$.

Theorem 5.3. *Let $A(\lambda) \in \Psi^{m,p,d;\alpha}_{b,\Lambda,r}(X,\Omega_b^{\frac12})$, where $m,p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, be a holomorphically tempered operator. Then for any $k \in \mathbb{N}_0$,*

$$\mathcal{L}(A)(t) \in C^\infty(\,(0,\infty)_t;\ \Psi_b^{-\infty,\alpha}(X,\Omega_b^{\frac12})\,) \cap C^k(\,[0,\infty)_t;\ \Psi_b^{dk+m+d,\alpha}(X,\Omega_b^{\frac12})\,).$$

Moreover, as $t \to \infty$, $\mathcal{L}(A)(t) \to 0$ exponentially in $\Psi_b^{-\infty,\alpha}(X,\Omega_b^{\frac12})$.

Before proving this theorem, we apply it to the heat operator.

Corollary 5.4. *Let $A \in \Psi_b^m(X,\Omega_b^{\frac12})$, $m \in \mathbb{R}^+$ be elliptic with respect to Λ. Then for any $k \in \mathbb{N}_0$,*

$$e^{-tA} \in C^\infty(\,(0,\infty)_t;\ \Psi_b^{-\infty}(X,\Omega_b^{\frac12})\,) \cap C^k(\,[0,\infty)_t;\ \Psi_b^{mk}(X,\Omega_b^{\frac12})\,).$$

Proof. By Theorem 4.11, there is a holomorphically tempered operator $B(\lambda) \in \Psi^{-m,-m,m}_{b,\Lambda,r}(X,\Omega_b^{\frac12})$ such that for $\lambda \in \Lambda$ with $|\lambda|$ sufficiently large, we have $(A-\lambda)^{-1} = B(\lambda) + R(\lambda)$, where for some continuous increasing function $r : \mathbb{R}^+ \to \mathbb{R}^+$, for each $a>0$, we have $R(\lambda) \in \Psi^{-\infty,-m,m;a}_{b,\Lambda,r}(X,\Omega_b^{\frac12})$ for $|\lambda| \geq r(a)$. Hence,

$$e^{-tA} := \frac{i}{2\pi}\mathcal{L}(\,(A-\lambda)^{-1}\,)(t) = \frac{i}{2\pi}\mathcal{L}(B)(t) + \frac{i}{2\pi}\mathcal{L}(R)(t). \tag{40}$$

By Theorem 5.3, for any $k \in \mathbb{N}_0$,

$$\mathcal{L}(B)(t) \in C^\infty(\,(0,\infty)_t;\ \Psi_b^{-\infty}(X,\Omega_b^{\frac12})\,) \cap C^k(\,[0,\infty)_t;\ \Psi_b^{mk}(X,\Omega_b^{\frac12})\,).$$

If $a>0$ is given, we can choose the contour Γ as in (32) such that $|\lambda| \geq r(a)$ for all $\lambda \in \Gamma$; in which case, by the remark after Lemma 5.6 below, we have $\mathcal{L}(R)(t) \in C^\infty(\,[0,\infty)_t;\ \Psi_b^{-\infty,a}(X,\Omega_b^{\frac12})\,)$. Since $a>0$ is arbitrary, it follows that

$$\mathcal{L}(R)(t) \in C^\infty(\,[0,\infty)_t;\ \Psi_b^{-\infty}(X,\Omega_b^{\frac12})\,). \tag{41}$$

The corollary now follows. ◻

The proof of Theorem 5.3 uses the following two lemmas.

Lemma 5.5. *Let $m,p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and let $a(\lambda,\xi) \in S^{m,p,d}_\Lambda(\mathbb{R}^n)$ be holomorphically tempered. Let Γ be any contour in Λ of the form given in* (32) *and define*

$$\mathcal{L}(a)(t,\xi) := \int_\Gamma e^{-t\lambda}\, a(\lambda,\xi)\, d\lambda.$$

Then, $\mathcal{L}(a)(t,\xi) \in t^b C^\infty(\,[0,\infty) \times \mathbb{R}^n)$ where $b = \max\{-p/d-1, 0\}$ and moreover,

1. *for any $k \in \mathbb{N}_0$,*
$$\mathcal{L}(a)(t) \in C^\infty(\,(0,\infty)_t;\ S^{-\infty}(\mathbb{R}^n)\,) \cap C^k(\,[0,\infty)_t;\ S^{dk+m+d}(\mathbb{R}^n)\,);$$
2. *there exists a constant $C>0$ such that $\mathcal{L}(a)(t,\xi)$ satisfies the estimates: for any k and β,*
$$|\,\partial_t^k \partial_\xi^\beta \mathcal{L}(a)(t,\xi)\,| \ \dot{\leq}\ (1+|\xi|)^{dk+m+d-|\beta|}\, e^{-tC(1+|\xi|)^d}.$$

Proof. In this proof, we follow [9, Lemma 4.2.3]. Since $a(\lambda,\xi)$ is holomorphically tempered, there exists an $\epsilon>0$ such that $a(\lambda,\xi)$ extends to be a smooth function, holomorphic in λ, for (λ,ξ) in

$$\{(\lambda,\xi) \in \mathbb{C} \times \mathbb{R}^n \mid \lambda \in \Lambda \ \text{ or } \ |\lambda| \leq \epsilon\,(1+|\xi|)^d \ \text{ or } \ \frac{1}{\epsilon}\,(1+|\xi|)^d \leq |\lambda|\}; \tag{42}$$

and moreover, $a(\lambda,\xi)$ continues to satisfy similar symbol estimates for (λ,ξ) in the set given by (42) as for (λ,ξ) in $\Lambda \times \mathbb{R}^n$. Let $\xi \in \mathbb{R}^n$. Then, as $a(\lambda,\xi)$ is holomorphic for λ in the set (42), it follows that $\mathcal{L}(a)(t,\xi) = \int_{\Gamma_\xi} e^{-t\lambda}\, a(\lambda,\xi)\, d\lambda$, where Γ_ξ is the contour shown in Figure 1, where the radius of the inner arc of Γ_ξ is $\frac{\epsilon}{2}(1+|\xi|)^d$ and the radius of the outer arc of Γ_ξ is $\frac{2}{\epsilon}(1+|\xi|)^d$. It follows that $\mathcal{L}(a)(t,\xi) \in C^\infty([0,\infty) \times \mathbb{R}^n)$. Moreover, since $a(\lambda,\xi)$ decays like $\lambda^{p/d}$ as $|\lambda| \to \infty$, by Cauchy's Theorem it follows that $\mathcal{L}(a)(t,\xi)$ vanishes like t^b where $b = \max\{-p/d-1, 0\}$ at $t=0$.

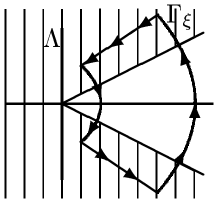

FIGURE 1. Deformation of the contour Γ into the contour Γ_ξ.

We will prove that $\mathcal{L}(a)(t,\xi)$ satisfies property (2). These estimates will automatically prove (1) of this lemma. Let $k \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^n$. Then, $\partial_t^k \partial_\xi^\beta \mathcal{L}(a)(t,\xi) = \int_{\Gamma_\xi} e^{-t\lambda}(-\lambda)^k \, \partial_\xi^\beta a(\lambda,\xi)\, d\lambda$. Now observe that

(1) the length of $\Gamma_\xi \,\dot{\leq}\, (1+|\xi|)^d$ (independent of ξ);
(2) $|e^{-t\lambda}| \leq e^{-tC(1+|\xi|)^d}$ for all $\lambda \in \Gamma_\xi$, for some $C > 0$ (independent of ξ);
(3) $|\lambda|^k \leq \frac{2^k}{\epsilon^k}(1+|\xi|)^{dk}$ for all $\lambda \in \Gamma_\xi$;
(4) for all $\lambda \in \Gamma_\xi$,

$$|\,\partial_\xi^\beta a(\lambda,\xi)\,| \,\dot{\leq}\, (1+|\lambda|^{1/d}+|\xi|)^p\,(1+|\xi|)^{m-p-|\beta|} \,\dot{\leq}\, (1+|\xi|)^{m-|\beta|}.$$

Hence,

$$\begin{aligned} |\,\partial_t^k \partial_\xi^\beta \mathcal{L}(a)(t,\xi)\,| &\leq \int_{\Gamma_\xi} |e^{-t\lambda}| \cdot |\lambda|^k \cdot |\,\partial_\xi^\beta a(\lambda,\xi)\,|\, d\lambda \\ &\dot{\leq} (1+|\xi|)^d \cdot e^{-tC(1+|\xi|)^d} \cdot \frac{2^k}{\epsilon^k}(1+|\xi|)^{dk} \cdot (1+|\xi|)^{m-|\beta|} \\ &\dot{\leq} (1+|\xi|)^{dk+m+d-|\beta|}\, e^{-tC(1+|\xi|)^d}. \end{aligned}$$

□

Lemma 5.6. *Let $R \in \Psi_{b,\Lambda,r}^{-\infty,p,d;\alpha}(X,\Omega_b^{\frac12})$, where $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, be holomorphic. Then, $\mathcal{L}(R)(t) \in t^b C^\infty(\,[0,\infty)_t;\, \Psi_b^{-\infty,\alpha}(X,\Omega_b^{\frac12})\,)$ where $b = \max\{-p/d-1, 0\}$.*

Proof. Fix any $N >> 0$. Then by definition, we can write, away from $\lambda = 0$, $R = \sum_{j=0}^{N-1} \lambda^{p/d-j} R_j + \lambda^{p/d-N} R_N(\lambda)$, where $R_j \in \Psi_b^{-\infty,\alpha}(X,\Omega_b^{\frac12})$ and, away from $\lambda = 0$, $R_N(\lambda) \in S^0(\Lambda;\, \Psi_b^{-\infty,\alpha}(X,\Omega_b^{\frac12})\,)$. Hence,

$$\mathcal{L}(R)(t) = \sum_{j=0}^{N-1} \int_\Gamma e^{-t\lambda}\lambda^{p/d-j} d\lambda \cdot R_j + \int_\Gamma e^{-t\lambda}\lambda^{p/d-N} R_N(\lambda) d\lambda,$$

where Γ is any contour in Λ of the form given in (32). Observe that for any $j \in \mathbb{N}_0$,

$$\int_\Gamma e^{-t\lambda}\lambda^{p/d-j} d\lambda = t^{j-p/d-1}\int_\Gamma e^{-\lambda}\lambda^{p/d-j} d\lambda = \begin{cases} 0, & j - p/d \notin \mathbb{N}; \\ (-t)^{j-p/d-1}, & j - p/d \in \mathbb{N}. \end{cases} \tag{43}$$

Also, observe that Cauchy's Theorem implies $\int_\Gamma \lambda^{-k} R_N(\lambda)\, d\lambda \equiv 0$ for all $k \geq 2$. Hence, for any $M \in \mathbb{N}$ such that $p/d - N + M \leq -2$,

$$S_N(t) := \int_\Gamma e^{-t\lambda}\lambda^{p/d-N} R_N(\lambda)\, d\lambda \in C^M(\,[0,\infty)_t;\, \Psi_b^{-\infty,\alpha}(X,\Omega_b^{\frac12})\,), \tag{44}$$

with $\partial_t^j S_N(0) = 0$ for $j = 0, \ldots, M$. Now by (43) and (44), we can write

$$\mathcal{L}(R)(t) = \sum_{0 \leq j \leq N-1,\, j-p/d-1 \geq 0} (-t)^{j-p/d-1} R_j + S_N(t);$$

and thus, since N is arbitrary, our lemma is proved. □

Remark 5.7. Observe that this result still holds if we only assume that $R(\lambda)$ is holomorphic for all $|\lambda| \geq r$ for some $r > 0$ (in which case, to define $\mathcal{L}(R)(t)$, we take the contour Γ of the form given in (32) such that $|\lambda| \geq r$ for all $\lambda \in \Gamma$).

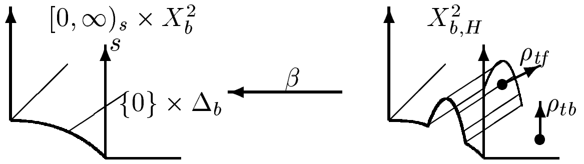

FIGURE 2. The manifold $X_{b,H}^2$. The map β is the blow-down map. The functions ρ_{tf} and ρ_{tb} represent boundary defining functions.

Proof of Theorem 5.3: By Theorem 5.2, $\mathcal{L}(A)(t) \to 0$ exponentially as an element of the space $\Psi_b^{-\infty,\alpha}(X,\Omega_b^{\frac12})$ as $t \to \infty$. By the definition (23), A is the sum of two parts: $A = B + R$, where $B \in \Psi_{b,\Lambda,r}^{m,p,d}(X,\Omega_b^{\frac12})$ and $R \in \Psi_{b,\Lambda,r}^{-\infty,p,d;\alpha}(X,\Omega_b^{\frac12})$. By Lemma 5.6, we have $\mathcal{L}(R)(t) \in t^b C^\infty(\,[0,\infty)_t;\, \Psi_b^{-\infty,\alpha}(X,\Omega_b^{\frac12})\,)$, where $b = \max\{-p/d - 1, 0\}$. Now by Definition 3.8, away from the b-diagonal, $B \in \Psi_{b,\Lambda,r}^{-\infty,p,d}(X,\Omega_b^{\frac12})$. Again by Lemma 5.6, away from the b-diagonal, $\mathcal{L}(B)(t) \in t^b C^\infty(\,[0,\infty)_t;\, \Psi_b^{-\infty}(X,\Omega_b^{\frac12})\,)$. Thus, it suffices to assume that $A \in \Psi_{b,\Lambda,r}^{m,p,d}(X,\Omega_b^{\frac12})$ and that A is supported near the b-diagonal. Let $\mathbb{R}_y^{n,k} \times \mathbb{R}_z^n$ be a coordinate patch on X_b^2 such that $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}$ and let ϕ be any compactly supported function on the coordinate patch. Then by definition, we can write

$$\phi\, A(\lambda) = \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, a(\lambda,y,\xi)\, d\xi \cdot \nu,$$

where $\nu \in C^\infty(X_b^2, \Omega_b^{\frac12})$, and where for each $y \in \mathbb{R}^{n,k}$, $(\lambda,\xi) \mapsto a(\lambda,y,\xi) \in S_\Lambda^{m,p,d}(\mathbb{R}^n)$ is holomorphically tempered. Hence,

$$\phi\, \mathcal{L}(A)(t) = \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, \mathcal{L}(a)(t,y,\xi)\, d\xi \cdot \nu,$$

where by Lemma 5.5, for any $k \in \mathbb{N}_0$,

$$\begin{aligned}&\mathcal{L}(a)(t,y,\xi) \in \\ &\qquad C^\infty(\,(0,\infty)_t \times \mathbb{R}_y^{n,k};\, S^{-\infty}(\mathbb{R}_\xi^n)\,) \cap C^k(\,[0,\infty)_t \times \mathbb{R}_y^{n,k};\, S^{dk+m+d}(\mathbb{R}_\xi^n)\,).\end{aligned}$$

Thus, for any $k \in \mathbb{N}_0$,

$$\phi\, \mathcal{L}(A)(t) \in C^\infty(\,(0,\infty)_t;\, \Psi_b^{-\infty}(X,\Omega_b^{\frac12})\,) \cap C^k(\,[0,\infty)_t;\, \Psi_b^{dk+m+d}(X,\Omega_b^{\frac12})\,).$$

The proof of Theorem 5.3 is now complete.

6. ASYMPTOTIC EXPANSIONS

The goal of this section is to realize the Schwartz kernels of the Laplace transforms of tempered operators (in particular, the heat operator), as polyhomogeneous *functions* on a blown-up manifold with corners $X_{b,H}^2$, cf. Theorems 6.2 and 6.3 and Corollaries 6.4 and 6.5. At the same time, we will determine the asymptotics of the kernels restricted to the diagonal. For a detailed account of the mathematical process of blow-up, please see [18, 21]. Let $[0,\infty)_s$ be the half-line with variable s. We define

$$X_{b,H}^2 := [\,[0,\infty)_s \times X_b^2; \{0\} \times \Delta_b\,].$$

This notation means that $X_{b,H}^2$ is $[0,\infty)_s \times X_b^2$ blown-up along $\{0\} \times \Delta_b$. Figure 2 gives a pictorial representation of $X_{b,H}^2$.

We define $tf := \beta^{-1}(\{0\} \times \Delta_b)$ and $tb := \overline{\beta^{-1}(\{0\} \times X_b^2 \setminus \{0\} \times \Delta_b)}$, and we call tf the 'temporal face' and tb the 'temporal boundary' respectively.

We now fix the notation for local coordinates on $X_{b,H}^2$. Let $\mathcal{U} = \mathbb{R}_y^{n,k} \times \mathbb{R}_z^n$ be a coordinate patch on X_b^2 with $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}$. Then in these coordinates,

$$X_{b,H}^2 \cong H^n \times \mathbb{R}_y^{n,k}, \quad \text{where } H^n := [\,[0,\infty)_s \times \mathbb{R}_z^n; \{0\} \times \{0\}]. \tag{45}$$

$$\left.\begin{array}{l} r := |z| \\ \gamma_0 := s/|z| \\ \gamma' := z/|z| \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} s = r\,\gamma_0 \\ z = r\,\gamma' \end{array}\right.$$

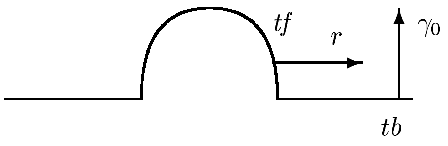

FIGURE 3. Coordinates on H^n near tb

Now by definition of blow-up, $H^n \equiv [0,\infty)_\rho \times \mathbb{S}^{n,1}_{(\omega_0,\omega')}$, where $\mathbb{S}^{n,1} := \mathbb{R}^{n+1,1} \cap \mathbb{S}^n$, and where

$$\left.\begin{array}{l} \rho := (|z|^2 + s^2)^{1/2} \\ \omega_0 := s/(|z|^2 + s^2)^{1/2} \\ \omega' := z/(|z|^2 + s^2)^{1/2} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} s = \rho\,\omega_0 \\ z = \rho\,\omega'. \end{array}\right. \tag{46}$$

In H^n, we define $\mathit{tf} := \{\rho = 0\}$ and $tb := \{\omega_0 = 0\}$. Some useful coordinates near tb in H^n are given in Figure 3. These coordinates give a product decomposition

$$H^n \setminus \{\omega_0 = 1\} \cong [0,\infty)_r \times [0,\infty)_{\gamma_0} \times \mathbb{S}^{n-1}_{\gamma'}.$$

We now touch on the subject of densities. We claim that if $t = s^d$, then when lifted to H^n, $|dz\,\frac{dt}{t}|^{\frac{1}{2}}$ is of the form $|dz\,\frac{dt}{t}|^{\frac{1}{2}} = \rho^{\frac{n}{2}}\,\mu$, where $0 < \mu \in C^\infty(H^n, \Omega_b^{\frac{1}{2}})$. Indeed, observe that since $t = \rho^d \omega_0^d$ and $z = \rho\omega'$,

$$\frac{dt}{t} = \frac{d\rho}{\rho}\cdot d + \frac{d\omega_0}{\omega_0}\cdot d$$

and for each $i = 1,\ldots,n$, $dz_i = \omega_i'\,d\rho + \rho\,d\omega_i'$. Thus, $|dz\,\frac{dt}{t}|^{\frac{1}{2}} = \rho^n \frac{d\rho}{\rho}\nu$, where $\nu \in C^\infty(\mathbb{S}^{n,1}_{(\omega_0,\omega')}, \Omega_b^{\frac{1}{2}})$. Since $|dz\,\frac{dt}{t}|^{\frac{1}{2}} > 0$, we must have $\nu > 0$. Thus, $|dz\,\frac{dt}{t}|^{\frac{1}{2}} = \rho^{\frac{n}{2}}\,\mu$, where $0 < \mu \in C^\infty(H^n, \Omega_b^{\frac{1}{2}})$. More generally, we have the following lemma.

Lemma 6.1. *Let* $0 < \nu \in C^\infty(X_b^2, \Omega_b^{\frac{1}{2}})$. *Then, if* $t = s^d$, $\beta^*(\nu\,|\frac{dt}{t}|^{\frac{1}{2}}) = \rho_{\mathit{tf}}^{\frac{n}{2}}\,\mu$, *where* $0 < \mu \in C^\infty(X_{b,H}^2, \Omega_b^{\frac{1}{2}})$.

Example: The heat operator on $[0,\infty)$**.** We will now look at the polyhomogeneity for the heat kernel of the b-Laplacian $\Delta = -(x\partial_x)^2$ on $[0,\infty)$. Observe that under the diffeomorphism $(0,\infty) \ni x \mapsto w = \log x \in \mathbb{R}$, the b-Laplacian Δ becomes the usual Laplacian $-\partial_w^2$. Thus, as the heat kernel for $-\partial_w^2$ is of the form $\frac{1}{\sqrt{4\pi t}} e^{-z^2/4t}$ with $z = w - w'$ on $\mathbb{R}_w \times \mathbb{R}_{w'}$, it follows that for the b-Laplacian, $e^{-t\Delta} = \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t}$ with $z = \log v$, where $v = x/x'$. Setting $t = s^2$, we have

$$e^{-s^2\Delta} = \frac{1}{\sqrt{4\pi}} s^{-1} e^{-(\log v)^2/4s^2}. \tag{47}$$

Now by definition, $v = x/x'$ and $1/v = x'/x$ are projective coordinates that define lb and rb respectively of $[0,\infty)_b^2$. Thus, (47) implies that *away* from $\Delta_b = \{v = 1\}$, $e^{-s^2\Delta}$ vanishes to infinite order at lb, rb, and tf of $[0,\infty)_{b,H}^2$. Near $\Delta_b = \{v = 1\} = \{z = \log v = 0\}$, in the coordinates $(\rho, \omega_0, \omega')$ in (46), we have

$$e^{-s^2\Delta} = \frac{1}{\sqrt{4\pi}} \rho^{-1} \omega_0^{-1} e^{-(\omega'/\omega_0)^2/4}.$$

It follows that $e^{-s^2\Delta}$ vanishes to infinite order at $tb = \{\omega_0 = 0\}$ and blows up at $\mathit{tf} = \{\rho = 0\}$ like ρ^{-1}. Hence,

$$e^{-s^2\Delta} \in \mathcal{A}^{\mathcal{E}}_{phg}([0,\infty)_{b,H}^2),$$

where $\mathcal{E}(\mathit{ff}) := \mathbb{N}_0$; $\mathcal{E}(lb) = \mathcal{E}(rb) = \mathcal{E}(tb) := \emptyset$; $\mathcal{E}(\mathit{tf}) := \mathbb{N}_0 - 1$. Here, given any $a, b \in \mathbb{R}$, $a + b\mathbb{N}_0 := \{(a + bk, 0) \,|\, k \in \mathbb{N}_0\}$. We will generalize this result to b-pseudodifferential operators on manifolds with corners.

Recall that Λ is always assumed to be of the form given in (31). Let $A \in \Psi^{m,p,d;\alpha}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$ or $A \in \Psi^{m,d;\alpha}_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$ be holomorphically tempered. We define the *residue density* $\mathrm{Res}(\mathcal{M}(A)(\tau))$ at $\tau = il$ where $l = \frac{k-m-n}{d} - 1$ for $k \in \mathbb{N}_0$, as follows (if τ is not of this form, the residue is defined to be 0). The notation $\mathcal{M}(A)$ denotes the Mellin transform of A to be studied in Section 7. Thus, write A locally as in (28) of Definition 4.8, and let $a_{m-k}(\lambda, y, \xi)$ be the $m-k$ degree homogeneous term in the expansion of the local symbol of A, where $y \in \mathbb{R}^{n,k}$ is a coordinate patch on $\Delta_b \cong X$. By Lemma 4.2, for some $\epsilon > 0$ (independent of y) $a_{m-k}(\lambda, y, \xi)$ extends to be a smooth function, holomorphic in λ, for (λ, ξ) in

$$\{(\lambda, \xi) \in \mathbb{C} \times (\mathbb{R}^n \setminus \{0\}) \mid \lambda \in \Lambda \text{ or } |\lambda| \leq \epsilon |\xi|^d \text{ or } \frac{1}{\epsilon}|\xi|^d \leq |\lambda|\}.$$

Given any $\xi \in \mathbb{R}^n \setminus \{0\}$, we define

(48) Γ_ξ to be the contour shown in Figure 1 where the radius of the inner arc of Γ_ξ is $\frac{\epsilon}{2}|\xi|^d$ and the radius of the outer arc is $\frac{2}{\epsilon}|\xi|^d$.

Then we locally define

$$\operatorname{Res}(\mathcal{M}(A)(il)) := \frac{1}{(2\pi)^n} \int_{|\xi|=1} \mathcal{M}(a_{m-k})(il, y, \xi) d\xi \cdot \nu|_{\Delta_b}, \tag{49}$$

where for any $\xi \in \mathbb{R}^n \setminus \{0\}$, $\mathcal{M}(a_{m-k})(il, y, \xi) := \int_{\Gamma_\xi} \lambda^l a_{m-k}(\lambda, y, \xi) d\lambda$. By (2) of Lemma 6.7, observe that $\mathcal{M}(a_{m-k})(il, y, \xi)$ is homogeneous of degree $-n$ in ξ.

Note that $\nu|_{\Delta_b} \in C^\infty(X, \Omega_b)$ by the usual identification of Δ_b with X and the fact that $\Omega_b^{\frac{1}{2}}(X_b^2)|_{\Delta_b} \equiv \Omega_b(X)$, cf. [22, Section 4.18]. We also locally define

$$D_\tau \operatorname{Res}(\mathcal{M}(A))(il) := \frac{1}{(2\pi)^n} \int_{|\xi|=1} D_\tau \mathcal{M}(a_{m-k})(il, y, \xi) d\xi \cdot \nu|_{\Delta_b}, \tag{50}$$

where $D_\tau \mathcal{M}(a_{m-k})(il, y, \xi) := -\int_{\Gamma_\xi} \lambda^l \log(\lambda) a_{m-k}(\lambda, y, \xi) d\lambda$. Residue densities of the sort described in (49) and (50) are studied in [14], where such residue densities are shown to be coordinate invariant. The residues $\operatorname{Res}(\mathcal{M}(A))(il)$ and $D_\tau \operatorname{Res}(\mathcal{M}(A))(il)$ thus give well-defined elements of $C^\infty(X, \Omega_b)$.

Let $m \in \mathbb{R}$. Define the index set $\mathcal{E}_{m,p,d}$ on $X_{b,H}^2$ by

$$\begin{gathered}\mathcal{E}_{m,p,d}(\mathit{ff}) := \mathbb{N}_0; \mathcal{E}_{m,p,d}(lb) = \mathcal{E}_{m,p,d}(rb) := \emptyset; \mathcal{E}_{m,p,d}(tb) := (-p-d+d\mathbb{N}_0) \cap d\mathbb{N}_0;\\ \mathcal{E}_{m,p,d}(tf) := \tfrac{n}{2} + (-m-n-d+\mathbb{N}_0) \overline{\cup} [(-p-d+d\mathbb{N}_0) \cap d\mathbb{N}_0].\end{gathered}$$

Here, for any $a, b \in \mathbb{R}$, $a + b\mathbb{N}_0 := \{(a+bk, 0) \mid k \in \mathbb{N}_0\}$. The first structure theorem is the following.

Theorem 6.2. *Let $A(\lambda) \in \Psi_{b,\Lambda,ros}^{m,p,d}(X, \Omega_b^{\frac{1}{2}})$, where m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, be holomorphically tempered. Then if $t = s^d$,*

$$\mathcal{L}(A)(t) \, |\frac{dt}{t}|^{\frac{1}{2}} \in \mathcal{A}_{phg}^{\mathcal{E}_{m,p,d}}(X_{b,H}^2, \Omega_b^{\frac{1}{2}}); \tag{51}$$

and, as $t \downarrow 0$,

$$\mathcal{L}(A)(t)|_{\Delta_b} \sim \sum_{k \in \mathbb{N}_0} t^{\frac{k-m-n}{d}-1} \gamma_k + \sum_{k \in \mathbb{N}_0} t^{\frac{k-m-n}{d}-1} \log t \, \gamma'_k + \sum_{k, k+p/d+1 \in \mathbb{N}_0} t^k \gamma''_k, \tag{52}$$

where γ_k, γ'_k, $\gamma''_k \in C^\infty(X, \Omega_b)$, and where in the first sum, $\gamma_k = 0$ if $\frac{k-m-n}{d} - 1 \in \mathbb{N}_0$, and in the second sum, $\gamma'_k = 0$ if $\frac{k-m-n+p}{d} \notin \mathbb{N}_0$ or $\frac{k-m-n}{d} - 1 \notin \mathbb{N}_0$. Furthermore, if $l = \frac{k-m-n}{d} - 1$, then γ_k and γ'_k are given by

$$\gamma_k = \frac{\Gamma(-l)}{d} \operatorname{Res}(\mathcal{M}(A)(il)) \text{ if } l \notin \mathbb{N}_0, \; \gamma'_k = \frac{(-1)^{l+1}}{d \cdot l!} \operatorname{Res}(\mathcal{M}(A)(il)) \text{ if } l \in \mathbb{N}_0,$$

where $\Gamma(-l)$ is the Gamma function at $-l$.

We will prove both (51) and (52) at the same time. We now state the corresponding theorem for the strongly tempered space. For each $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$, define the index set $\mathcal{F}_{m,d}$ on $X_{b,H}^2$ by

$$\begin{gathered}\mathcal{F}_{m,d}(\mathit{ff}) := \mathbb{N}_0; \; \mathcal{F}_{m,d}(lb) = \mathcal{F}_{m,d}(rb) = \mathcal{F}_{m,d}(tb) := \emptyset;\\ \mathcal{F}_{m,d}(tf) := \tfrac{n}{2} + (-m-n-d+\mathbb{N}_0).\end{gathered}$$

The second structure theorem is the following.

Theorem 6.3. *Let $A(\lambda) \in \Psi^{m,d}_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$, where $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$, be holomorphically tempered. Then if $t = s^d$,*

$$\mathcal{L}(A)(t)\, |\frac{dt}{t}|^{\frac{1}{2}} \in \mathcal{A}^{\mathcal{F}_{m,d}}_{phg}(X^2_{b,H}, \Omega_b^{\frac{1}{2}});$$

and, as $t \downarrow 0$,

$$\mathcal{L}(A)(t)|_{\Delta_b} \sim \sum_{k=0}^{\infty} t^{\frac{k-m-n}{d}-1}\, \gamma_k, \quad \gamma_k \in C^\infty(X, \Omega_b), \tag{53}$$

where if $l = \frac{k-m-n}{d} - 1$,

$$\gamma_k = \frac{\Gamma(-l)}{d} \mathrm{Res}(\mathcal{M}(A)(il)), \ \textit{if } l \notin \mathbb{N}_0, \ \gamma_k = \frac{(-1)^l}{d \cdot l!} D_\tau \mathrm{Res}(\mathcal{M}(A))(il) \ \textit{if } l \in \mathbb{N}_0.$$

Before going into the (long) proofs of Theorems 6.2 and 6.3, we apply them to the heat operator. For each $m \in \mathbb{R}^+$, define the index set $\mathcal{E}_m$ on $X^2_{b,H}$ by

$$\mathcal{E}_m(ff) := \mathbb{N}_0; \ \mathcal{E}_m(lb) = \mathcal{E}_m(rb) := \emptyset; \ \mathcal{E}_m(tb) := m\mathbb{N};$$
$$\mathcal{E}_m(tf) := \tfrac{n}{2} + (-n + \mathbb{N}_0) \overline{\cup} (\mathbb{N}_0 + m\mathbb{N}_0).$$

Corollary 6.4. *Let $A \in \Psi^m_{b,os}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be elliptic with respect to Λ, where Λ is of the form given in (31). Then if $t = s^m$,*

$$e^{-tA}\, |\frac{dt}{t}|^{\frac{1}{2}} \in \mathcal{A}^{\mathcal{E}_m}_{phg}(X^2_{b,H}, \Omega_b^{\frac{1}{2}});$$

and given any $B \in \Psi^{m'}_{b,os}(X, \Omega_b^{\frac{1}{2}})$, $m' \in \mathbb{R}$, as $t \downarrow 0$,

$$Be^{-tA}|_{\Delta_b} \sim \sum_{k \in \mathbb{N}_0} t^{\frac{k-m'-n}{m}}\, \gamma_k + \sum_{k \in \mathbb{N}_0} t^{\frac{k-m'-n}{m}} \log t\, \gamma'_k + \sum_{k \in \mathbb{N}_0} t^k \gamma''_k, \tag{54}$$

where γ_k, γ'_k, $\gamma''_k \in C^\infty(X, \Omega_b)$, and where in the first sum, $\gamma_k = 0$ if $\frac{k-m'-n}{m} \in \mathbb{N}_0$ and in the second sum, $\gamma'_k = 0$ if $\frac{k-m'-n}{m} \notin \mathbb{N}_0$. Furthermore, if $l = \frac{k-m'-n}{m}$, then

$$\gamma_k = \frac{\Gamma(-l)}{m} \mathrm{Res}(BA^l) \ \textit{if } l \notin \mathbb{N}_0, \ \gamma'_k = \frac{(-1)^{l+1}}{m \cdot l!} \mathrm{Res}(BA^l) \ \textit{if } l \in \mathbb{N}_0,$$

where $\mathrm{Res}(BA^l) := \frac{i}{2\pi} \mathrm{Res}(\mathcal{M}(B(A-\lambda)^{-1})(il))$.

In particular, as $\mathrm{Res}(\mathrm{Id}) = 0$, we have

$$e^{-tA}|_{\Delta_b} \sim \sum_{k \in \mathbb{N}_0} t^{\frac{k-n}{m}}\, \gamma_k + \sum_{k \in \mathbb{N}_0} t^{\frac{k-n}{m}} \log t\, \gamma'_k + \sum_{k \in \mathbb{N}_0} t^k \gamma''_k, \tag{55}$$

where γ_k, γ'_k, $\gamma''_k \in C^\infty(X, \Omega_b)$, and where in the first sum, $\gamma_k = 0$ if $\frac{k-n}{m} \in \mathbb{N}_0$ and in the second sum, $\gamma'_k = 0$ if $\frac{k-n}{m} \notin \mathbb{N}$; and, if $l = \frac{k-n}{m}$, then

$$\gamma_k = \frac{\Gamma(-l)}{m} \mathrm{Res}(A^l) \ \textit{if } l \notin \mathbb{N}_0, \ \gamma'_k = \frac{(-1)^{l+1}}{m \cdot l!} \mathrm{Res}(A^l) \ \textit{if } l \in \mathbb{N}_0.$$

Proof. As we showed in (40) and (41), one can show that

$$e^{-tA} = \frac{i}{2\pi} \mathcal{L}(B)(t) + \frac{i}{2\pi} \mathcal{L}(R)(t),$$

where $\mathcal{L}(R)(t) \in C^\infty([0,\infty)_t; \Psi^{-\infty}(X, \Omega_b^{\frac{1}{2}}))$, and where $B \in \Psi^{-m,-m,m}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$ is holomorphically tempered. By Theorem 6.2, $\mathcal{L}(B)(t)\, |\frac{dt}{t}|^{\frac{1}{2}} \in \mathcal{A}^{\mathcal{E}'_m}_{phg}(X^2_{b,H}, \Omega_b^{\frac{1}{2}})$, where $\mathcal{E}'_m(G) = \mathcal{E}_m(G)$ if $G \neq tb$ and $\mathcal{E}'_m(tb) = m\mathbb{N}_0$. Thus, as $\mathcal{L}(R)(t)\, |\frac{dt}{t}|^{\frac{1}{2}} \in \mathcal{A}^{\mathcal{E}'_m}_{phg}(X^2_{b,H}, \Omega_b^{\frac{1}{2}})$, we have, $e^{-tA}\, |\frac{dt}{t}|^{\frac{1}{2}} \in \mathcal{A}^{\mathcal{E}'_m}_{phg}(X^2_{b,H}, \Omega_b^{\frac{1}{2}})$. But note that as operators, $e^{-tA}|_{t=0} = \mathrm{Id}$, and the identity is zero off of Δ_b. It follows that $e^{-tA}\, |\frac{dt}{t}|^{\frac{1}{2}}$ must vanish at tb. Hence, $e^{-tA}\, |\frac{dt}{t}|^{\frac{1}{2}} \in \mathcal{A}^{\mathcal{E}_m}_{phg}(X^2_{b,H}, \Omega_b^{\frac{1}{2}})$.

Observe that if $C \in \Psi^{m'}_{b,ros}(X,\Omega_b^{\frac{1}{2}}) \subset \Psi^{m',0,m}_{b,\Lambda,os}(X,\Omega_b^{\frac{1}{2}})$, $m' \in \mathbb{R}$, then

$$Ce^{-tA} = \frac{i}{2\pi}\mathcal{L}(CB)(t) + \frac{i}{2\pi}C\mathcal{L}(R)(t),$$

where $C\mathcal{L}(R)(t) \in C^\infty([0,\infty)_t;\Psi^{-\infty}(X,\Omega_b^{\frac{1}{2}}))$, and where (see (5) of Lemma 3.11) $C \circ B(\lambda) \in \Psi^{-m+m',-m,m}_{b,\Lambda,ros}(X,\Omega_b^{\frac{1}{2}})$ is holomorphically tempered. Thus, the expansion for $Ce^{-tA}|_{\Delta_b}$ follows from Theorem 6.2. □

For each $m \in \mathbb{R}^+$, define the index set $\mathcal{F}_m$ on $X^2_{b,H}$ by

$$\mathcal{F}_m(\mathit{ff}) := \mathbb{N}_0;\ \mathcal{F}_m(lb) = \mathcal{F}_m(rb) = \mathcal{F}_m(tb) := \emptyset;\ \mathcal{F}_m(tf) := \frac{n}{2} + (-n+\mathbb{N}_0).$$

The following is proved much like Corollary 6.4, but here we use Theorem 6.3.

Corollary 6.5. *Let $A \in \mathrm{Diff}^m_b(X,\Omega_b^{\frac{1}{2}})$, $m \in \mathbb{N}$ be elliptic with respect to Λ, where Λ is of the form given in* (31)*. Then if $t = s^m$,*

$$e^{-tA}\,|\frac{dt}{t}|^{\frac{1}{2}} \in \mathcal{A}^{\mathcal{F}_m}_{phg}(X^2_{b,H},\Omega_b^{\frac{1}{2}});$$

and given any $B \in \mathrm{Diff}^{m'}_b(X,\Omega_b^{\frac{1}{2}})$, $m' \in \mathbb{N}_0$, as $t \downarrow 0$,

$$Be^{-tA}|_{\Delta_b} \sim \sum_{k=0}^\infty t^{\frac{k-m'-n}{m}}\gamma_k,\quad \gamma_k \in C^\infty(X,\Omega_b), \tag{56}$$

where if $l = \frac{k-m'-n}{m}$,

$$\gamma_k = \frac{\Gamma(-l)}{m}\mathrm{Res}(BA^l)\ \text{if}\ l \notin \mathbb{N}_0,\ \gamma_k = \frac{(-1)^{l+1}}{m\cdot l!}\mathrm{Res}(B\log A\,A^l)\ \text{if}\ l \in \mathbb{N}_0,$$

where $\mathrm{Res}(B\log A\,A^l) := -\frac{i}{2\pi}D_\tau\mathrm{Res}(\mathcal{M}(B(A-\lambda)^{-1}))(il)$.

The proofs of Theorems 6.2 and 6.3 consist of a long sequence of lemmas, in which we prove various properties of the Laplace transforms of holomorphically tempered symbols. Before we start, we note the following fact that will be used in the proofs that follow. Let $a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\hom(m)}(\mathbb{R}^n)$ or $a(\lambda,\xi) \in C^{\infty,d}_{\hom(m)}(\Lambda\times\mathbb{R}^n)$ be holomorphically tempered, where m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. Then by Lemma 4.2, there exists an $\epsilon > 0$ such that for any $\xi \in \mathbb{R}^n \setminus \{0\}$, $a(\lambda,\xi)$ is holomorphic in λ, for λ in

$$\{\lambda \in \mathbb{C} \,|\, \lambda \in \Lambda\ \text{ or }\ |\lambda| \le \epsilon\,|\xi|^d\ \text{ or }\ \frac{1}{\epsilon}\,|\xi|^d \le |\lambda|\}. \tag{57}$$

Let Γ be any contour in Λ of the form given in (32). Then by Cauchy's Theorem, for any $\xi \in \mathbb{R}^n\setminus\{0\}$,

$$\mathcal{L}(a)(t,\xi) := \int_\Gamma e^{-t\lambda}\,a(\lambda,\xi)\,d\lambda = \int_{C_\xi} e^{-t\lambda}\,a(\lambda,\xi)\,d\lambda, \tag{58}$$

where C_ξ is *any* closed contour in the set (57) homotopic to the curve Γ, as contours in the set (57) on the Riemann sphere. The function $\mathcal{L}(a)(t,\xi)$ is the Laplace transform of $a(\lambda,\xi)$. We begin with the following lemma which gives the homogeneous properties and decay properties of these Laplace transforms.

Lemma 6.6. *Let $a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\hom(m)}(\mathbb{R}^n)$, or $a(\lambda,\xi) \in C^{\infty,d}_{\hom(m)}(\Lambda\times\mathbb{R}^n)$, be a holomorphically tempered symbol, where $m, p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. If $a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\hom(m)}(\mathbb{R}^n)$, then $\mathcal{L}(a)(t,\xi) \in t^bC^\infty(\,[0,\infty)\times(\mathbb{R}^n\setminus\{0\})\,)$ where $b = \max\{-p/d-1,0\}$; and if $a(\lambda,\xi) \in C^{\infty,d}_{\hom(m)}(\Lambda\times\mathbb{R}^n)$, then we have $\mathcal{L}(a)(t,\xi) \in t^{b'}C^\infty(\,[0,\infty)\times\mathbb{R}^n\,)$ where $b' = \max\{-\lfloor m/d\rfloor - 1, 0\}$. In either case,*

1. *$\mathcal{L}(a)(\delta^{-d}t,\delta\xi) = \delta^{m+d}\mathcal{L}(a)(t,\xi)$ for all $\delta > 0$;*
2. *there exists a constant $C > 0$ such that $\mathcal{L}(a)(t,\xi)$ satisfies the estimates: for any k and β,*

$$|\,\partial_t^k\partial_\xi^\beta\mathcal{L}(a)(t,\xi)\,| \ \dot{\le}\ |\xi|^{dk+m+d-|\beta|}\,e^{-tC|\xi|^d}.$$

Proof. For simplicity, we will only prove this lemma for $a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$; the proof for $a(\lambda,\xi) \in C^{\infty,d}_{\mathrm{hom}(m)}(\Lambda\times\mathbb{R}^n)$ follows the same exact argument. By (58), $\mathcal{L}(a)(t,\xi) = \int_{S_\xi} e^{-t\lambda}\, a(\lambda,\xi)\, d\lambda$, where S_ξ is the circle of radius $\frac{2}{\epsilon}(1+|\xi|)^d$. It follows that $\mathcal{L}(a)(t,\xi) \in C^\infty(\,[0,\infty)\times(\mathbb{R}^n\setminus\{0\})\,)$. The fact that $\mathcal{L}(a)(t,\xi)$ vanishes like t^b at $t=0$, where $b=\max\{-p/d-1,0\}$, follows from the fact that $a(\lambda,\xi)$ grows like $\lambda^{p/d}$ (for fixed ξ) as $\lambda\to\infty$. That $\mathcal{L}(a)(t,\xi)$ satisfies the estimates in (2) is proved by writing $\mathcal{L}(a)(t,\xi) = \int_{\Gamma_\xi} e^{-t\lambda}\, a(\lambda,\xi)\, d\lambda$, where Γ_ξ is the contour given in (48), and then following the argument of Lemma 5.5. We now prove (1). Thus, making the change of variables $\lambda\mapsto\delta^d\lambda$ and using the homogeneous properties of $a(\lambda,\xi)$, we find that

$$\begin{aligned}\mathcal{L}(a)(\delta^{-d}t,\delta\xi) &= \int_\Gamma e^{-t\delta^{-d}\lambda}\, a(\lambda,\delta\xi)\, d\lambda = \delta^d\int_{\delta^{-d}\Gamma} e^{-t\lambda}\, a(\delta^d\lambda,\delta\xi)\, d\lambda\\ &= \delta^{m+d}\int_\Gamma e^{-t\lambda}\, a(\lambda,\xi)\, d\lambda\\ &= \delta^{m+d}\mathcal{L}(a)(t,\xi).\end{aligned}$$

□

The following lemma relates Laplace transforms with Mellin transforms.

Lemma 6.7. *Let $a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$ or $a(\lambda,\xi) \in C^{\infty,d}_{\mathrm{hom}(m)}(\Lambda\times\mathbb{R}^n)$ be holomorphically tempered, where $m,\,p\in\mathbb{R}$ and $d\in\mathbb{R}^+$ with $p/d\in\mathbb{Z}$. We define*

$$\mathcal{M}(a)(\tau,\xi) := \int_{\Gamma_\xi} \lambda^{-i\tau} a(\lambda,\xi)d\lambda,\ \ \tau\in\mathbb{C},$$

where Γ_ξ is the contour given in (48). *Then,*

(1) $\mathcal{M}(\mathcal{L}(a)(\cdot,\xi))(\tau) = \Gamma(-i\tau)\mathcal{M}(a)(-\tau,\xi)$; *that is,*

$$\int_0^\infty t^{-i\tau-1}\mathcal{L}(a)(t,\xi)dt = \Gamma(-i\tau)\mathcal{M}(a)(-\tau,\xi);$$

(2) *for any $\delta>0$, $\mathcal{M}(a)(\tau,\delta\xi) = \delta^{-id\tau+m+d}\mathcal{M}(a)(\tau,\xi)$.*

Proof. By (58), $\mathcal{L}(a)(t,\xi) = \int_{\Gamma_\xi} e^{-t\lambda}\, a(\lambda,\xi)d\lambda$. Thus,

$$\int_0^\infty t^{-i\tau-1}\mathcal{L}(a)(t,\xi)dt = \int_{\Gamma_\xi}\int_0^\infty t^{-i\tau-1}e^{-t\lambda}dt\, a(\lambda,\xi)d\lambda = \Gamma(-i\tau)\int_{\Gamma_\xi}\lambda^{i\tau}a(\lambda,\xi)d\lambda.$$

The proof of (2) follows the proof of property (1) of Lemma 6.6. □

We now begin showing that the Laplace transforms of holomorphically tempered homogeneous symbols have classical expansions on the manifold H^n discussed around (45). We start by showing that they have expansions at the face *tf* of H^n.

Lemma 6.8. *Let $a(\lambda,\xi)\in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$, where $m,\,p\in\mathbb{R}$ and $d\in\mathbb{R}^+$ with $p/d\in\mathbb{Z}$, be holomorphically tempered and let $\chi(\xi)\in C^\infty(\mathbb{R}^n)$ with $\chi(\xi)\equiv 0$ near 0 and $\chi(\xi)\equiv 1$ outside a neighborhood of 0. Let $t=s^d$. Then,*

$$A(t) := \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}\,\chi(\xi)\,\mathcal{L}(a)(t,\xi)\,d\xi \in \mathcal{A}^{\mathcal{E}_{tf}}(H^n),$$

where $\mathcal{E}_{tf} := (-m-n-d+\mathbb{N}_0)\overline{\cup}[(-p-d+\mathbb{N}_0+d\mathbb{N}_0)\cap(\mathbb{N}_0+d\mathbb{N}_0)]$ is an index set on H^n associated to tf. Also, we can write $A(t)|_{z=0} = t^l a(t) + b(t)$, where $l=\frac{-m-n}{d}-1$, $b(t)\in t^bC^\infty([0,\infty)_t)$ where $b=\max\{-p/d-1,0\}$, and where

$$a(t) = \begin{cases}\frac{\Gamma(-l)}{d}\mathrm{Res}(\mathcal{M}(A)(il)) & \text{if } l\notin\mathbb{N}_0;\\ \log t\cdot\frac{(-1)^{l+1}}{d\cdot l!}\mathrm{Res}(\mathcal{M}(A)(il)) & \text{if } l,\, l+p/d+1\in\mathbb{N}_0;\\ 0 & \text{otherwise,}\end{cases} \tag{59}$$

where $\mathrm{Res}(\mathcal{M}(A)(il)) = \frac{1}{(2\pi)^n}\int_{|\xi|=1}\mathcal{M}(a)(il,\xi)d\xi$.

Proof. Let $đ\xi = \frac{1}{(2\pi)^n} d\xi$. Then in the coordinates $(\rho, \omega_0, \omega')$ on H^n given in (46), we can write $A(s^d) := \int e^{i\rho\omega'\cdot\xi}\, \chi(\xi)\, \mathcal{L}(a)(\rho^d \omega_0^d, \xi)\, đ\xi$. Thus, making the change of variables $\xi \mapsto \rho^{-1}\xi$ and using the homogeneity property of $\mathcal{L}(a)(t,\xi)$ found in Lemma 6.6 yield

$$A(s^d) = \rho^{-m-n-d} \int e^{i\omega'\cdot\xi}\, \chi(\xi/\rho)\, \mathcal{L}(a)(\omega_0^d, \xi)\, đ\xi.$$

As $(\rho\partial_\rho - (-m-n-d))\rho^{-m-n-d} = 0$ and $\rho\partial_\rho \chi(\xi/\rho) = -(\xi \cdot \partial_\xi \chi)(\xi/\rho)$, we have

$$\begin{aligned}(\rho\partial_\rho - (-m-n-d))A(s^d) &= -\rho^{-m-n-d} \int e^{i\omega'\cdot\xi}\, (\xi \cdot \partial_\xi \chi)(\xi/\rho)\, \mathcal{L}(a)(\omega_0^d, \xi)\, đ\xi \\ &= -\int e^{i\rho\omega'\cdot\xi}\, (\xi \cdot \partial_\xi \chi)(\xi)\, \mathcal{L}(a)(\rho^d\omega_0^d, \xi)\, đ\xi, \end{aligned} \tag{60}$$

where from the first line to the second line, we made the change of variables $\xi \mapsto \rho\xi$ and used the homogeneity property of $\mathcal{L}(a)(t,\xi)$. Since $(\xi \cdot \partial_\xi \chi)(\xi) \equiv 0$ near $\xi = 0$ and outside a neighborhood of 0, it follows that the function $f(u,v,\omega') := \int e^{iu\omega'\cdot\xi}\, (\xi \cdot \partial_\xi \chi)(\xi)\, \mathcal{L}(a)(v,\xi)\, đ\xi$ is smooth in ω' and for $(u,v) \in [0,\infty)^2$. Thus, since for ξ bounded, $\mathcal{L}(a)(t,\xi) = O(t^b)$ where $b = \max\{-p/d - 1, 0\}$ by Lemma 6.6, we can expand the right hand side of (60) as

$$(\rho\partial_\rho - (-m-n-d))A(s^d) \sim \sum_{i,j=0}^{\infty} \rho^{db+i+dj} A_{i,j}(\omega_0, \omega'), \tag{61}$$

where for each i and j, $A_{i,j}(\omega_0,\omega') \in S^0(\mathbb{S}^{n,1}_{(\omega_0,\omega')})$. Since $(\rho\partial_\rho - z)\rho^z = 0$ for any $z \in \mathbb{C}$, for any $N \in \mathbb{N}$, we have

$$\prod_{db+i+dj \le N} (\rho\partial_\rho - (db+i+dj))(\rho\partial_\rho - (-m-n-d))A(s^d) = O(\rho^N). \tag{62}$$

Thus by Theorem 2.1, it follows that $A(s^d) \in \mathcal{A}^{\mathcal{E}_{tf}}(H^n)$.

Changing back to the t variable in (60), we have

$$(t\partial_t - l)A(t)|_{z=0} = \alpha(t), \tag{63}$$

where $l = \frac{-m-n}{d} - 1$ and $\alpha(t) = -\frac{1}{d}\int (\xi \cdot \partial_\xi \chi)(\xi)\, \mathcal{L}(a)(t,\xi)\, đ\xi$. Since $(\xi \cdot \partial_\xi \chi)(\xi) \equiv 0$ near $\xi = 0$ and outside a neighborhood of 0, $\alpha(t)$ is a smooth function of t. As $\mathcal{L}(a)(t,\xi)$ vanishes like t^b at $t = 0$, $\alpha(t)$ does as well. Now (63) implies that $t\partial_t(t^{-l}A(t)|_{z=0}) = t^{-l}\alpha(t)$. Dividing this equation by t and then integrating both sides from t to 1 implies that $A(1)|_{z=0} - t^{-l}A(t)|_{z=0} = \int_t^1 s^{-l-1}\alpha(s)ds$, or

$$A(t)|_{z=0} = t^l C - t^l \int_t^1 s^{-l-1}\alpha(s)ds, \quad C = A(1)|_{z=0} \in \mathbb{C}. \tag{64}$$

Expanding out $\alpha(s)$ at $s = 0$, using the fact that $\alpha(s) = O(s^b)$, and integrating the resulting expansion from t to 1 shows that $A(t)|_{z=0}$ must have the form $t^l a(t) + b(t)$, where $a(t) \in \mathbb{C}$ for $l \notin \mathbb{N}_0$, $a(t) \in \log t \cdot \mathbb{C}$ for $l,\, l + p/d + 1 \in \mathbb{N}_0$, and where $b(t) \in t^b C^\infty([0,\infty)_t)$ where $b = \max\{-p/d - 1, 0\}$. It remains to prove that $a(t)$ has the forms given in (59).

First, assume that $l,\, l + p/d + 1 \in \mathbb{N}_0$. Then, by (64), it follows that the coefficient of the $\log t$ term of $A(t)|_{z=0}$ is just $\frac{1}{l!}\partial_t^l \alpha(0)$. Now we can compute $\frac{1}{l!}\partial_t^l\alpha(0)$ via polar coordinates $\xi = \varrho\gamma$ and the homogeneous properties of $\mathcal{L}(a)(t,\xi)$:

$$\begin{aligned}\frac{1}{l!}\partial_t^l \alpha(0) &= -\frac{1}{d \cdot l!}\int (\xi \cdot \partial_\xi \chi)(\xi)\, (\partial_t^l \mathcal{L}(a))(0,\xi)\, d\xi \\ &= -\frac{1}{d \cdot l!(2\pi)^n}\int_{\mathbb{S}^{n-1}}\int_0^\infty \varrho\partial_\varrho \chi(\varrho\gamma)\, (\partial_t^l \mathcal{L}(a))(0,\varrho\gamma)\varrho^{n-1} d\varrho d\gamma \\ &= -\frac{1}{d \cdot l!(2\pi)^n}\int_{\mathbb{S}^{n-1}}\int_0^\infty \partial_\varrho \chi(\varrho\gamma) d\varrho\, (\partial_t^l \mathcal{L}(a))(0,\gamma) d\gamma \\ &= -\frac{1}{d \cdot l!(2\pi)^n}\int_{\mathbb{S}^{n-1}} (\partial_t^l \mathcal{L}(a))(0,\gamma) d\gamma \\ &= -\frac{(-1)^l}{d \cdot l!(2\pi)^n}\int_{\mathbb{S}^{n-1}} \mathcal{M}(a)(il,\gamma) d\gamma,\end{aligned}$$

where we used that $\int_0^\infty \partial_\varrho \chi(\varrho\gamma)d\varrho = 1$ since $\chi(\xi) \equiv 0$ on a neighborhood of 0 and $\chi(\xi) \equiv 1$ outside a neighborhood of 0, and that

$$(\partial_t^l \mathcal{L}(a))(0,\xi) = \partial_t^l \int_{\Gamma_\xi} e^{-t\lambda} a(\lambda,\xi) d\lambda|_{t=0} = \int_{\Gamma_\xi} (-\lambda)^l a(\lambda,\xi)d\lambda = (-1)^l \mathcal{M}(a)(il,\xi).$$

Now, taking the Mellin transform of (63) gives $(i\tau - l)\mathcal{M}(A)(\tau) = \mathcal{M}(\alpha)(\tau)$, or $\mathcal{M}(A)(\tau) = \frac{1}{i}\frac{\mathcal{M}(\alpha)(\tau)}{\tau+il}$. Thus, the Mellin inversion formula and Cauchy's Theorem imply that $a(t) = -\mathcal{M}(\alpha)(-il)$ for $l \notin \mathbb{N}_0$. Using this fact, we will compute $a(t)$ for $l \notin \mathbb{N}_0$. By Lemma 6.7, $\mathcal{M}(\mathcal{L}(a)(\cdot,\xi))(-il) = \Gamma(-l)\mathcal{M}(a)(il,\xi)$ and for any $\delta > 0$, $\mathcal{M}(a)(il,\delta\xi) = \delta^{-n}\mathcal{M}(a)(il,\xi)$; hence, using polar coordinates $\xi = \varrho\gamma$,

$$\begin{aligned}\mathcal{M}(\alpha)(-il) &= -\frac{\Gamma(-l)}{d}\int (\xi\cdot\partial_\xi\chi)(\xi)\mathcal{M}(a)(il,\xi)\,d\!\!\!{}^-\xi \\ &= -\frac{\Gamma(-l)}{d(2\pi)^n}\int_{\mathbb{S}^{n-1}}\int_0^\infty \partial_\varrho\chi(\varrho\gamma)\,d\varrho\,\mathcal{M}(a)(il,\gamma)d\gamma \\ &= -\frac{\Gamma(-l)}{d(2\pi)^n}\int_{\mathbb{S}^{n-1}} \mathcal{M}(a)(il,\gamma)d\gamma.\end{aligned}$$

□

We prove a similar result for elements of $C^{\infty,d}_{\mathrm{hom}(m)}(\Lambda\times\mathbb{R}^n)$.

Lemma 6.9. *Let $a(\lambda,\xi) \in C^{\infty,d}_{\mathrm{hom}(m)}(\Lambda\times\mathbb{R}^n)$, where $m\in\mathbb{R}$ and $d\in\mathbb{R}^+$, be holomorphically tempered and define*

$$A(t) := \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}\,\mathcal{L}(a)(t,\xi)\,d\xi.$$

Then if $t = s^d$, $A(t) \in \mathcal{A}^{\mathcal{F}_{tf}}(H^n)$ where $\mathcal{F}_{tf} := -m-n-d+\mathbb{N}_0$ is an index set on H^n associated to tf. Also, $A(t)|_{z=0} = t^l a$, where $l = \frac{-m-n}{d} - 1$ and

$$a = \begin{cases} \frac{\Gamma(-l)}{d}\mathrm{Res}(\mathcal{M}(A)(il)) & \text{if } l\notin\mathbb{N}_0\,; \\ \frac{(-1)^l}{d\cdot l!}D_\tau\mathrm{Res}(\mathcal{M}(A))(il) & \text{if } l\in\mathbb{N}_0\,, \end{cases} \tag{65}$$

where $D_\tau\mathrm{Res}(\mathcal{M}(A))(il) = \frac{1}{(2\pi)^n}\int_{|\xi|=1} D_\tau\mathcal{M}(a)(\tau,\xi)|_{\tau=il}d\xi$.

Proof. Making the change of variables $\xi\mapsto\rho^{-1}\xi$ and using the homogeneity property of $\mathcal{L}(a)(t,\xi)$ found in Lemma 6.6 yield

$$A(s^d) = \frac{\rho^{-m-n-d}}{(2\pi)^n}\int e^{i\omega'\cdot\xi}\,\mathcal{L}(a)(\omega_0^d,\xi)\,d\xi.$$

This proves that $A(t)\in\mathcal{A}^{\mathcal{F}_{tf}}(H^n)$. Switching to polar coordinates $\xi = \varrho\gamma$, by Lemmas 6.6 and 6.7, we have in terms of t,

$$\begin{aligned}A(t)|_{z=0} = \frac{t^l}{(2\pi)^n}\int\mathcal{L}(a)(1,\xi)\,d\xi &= \frac{t^l}{(2\pi)^n}\int_{\mathbb{S}^{n-1}}\int_0^\infty \mathcal{L}(a)(1,\varrho\gamma)\varrho^{n-1}d\varrho d\gamma \\ &= \frac{t^l}{d(2\pi)^n}\int_{\mathbb{S}^{n-1}}\int_0^\infty \varrho^{\frac{n}{d}-1}\mathcal{L}(a)(1,\varrho^{\frac{1}{d}}\gamma)d\varrho d\gamma\ (\varrho\mapsto\varrho^{\frac{1}{d}}) \\ &= \frac{t^l}{d(2\pi)^n}\int_{\mathbb{S}^{n-1}} B(0)d\gamma,\end{aligned}$$

where $B(\tau) = \int_0^\infty \varrho^{i\tau+\frac{n}{d}-1}\mathcal{L}(a)(1,\varrho^{\frac{1}{d}}\gamma)d\varrho$, which is holomorphic for $\mathrm{Im}\,\tau < \frac{n}{d}$. If $l = \frac{-m-n}{d} - 1$, the homogeneous properties of $\mathcal{L}(a)$ and Lemma 6.7 imply that

$$B(\tau) = \int_0^\infty \varrho^{i\tau-l-1}\mathcal{L}(a)(\varrho,\gamma)d\varrho = \Gamma(i\tau-l)\mathcal{M}(a)(\tau+il,\gamma).$$

If $l\notin\mathbb{N}_0$, then $B(0) = \Gamma(-l)\mathcal{M}(a)(il,\gamma)$; and if $l\in\mathbb{N}_0$, then since $\Gamma(z)$ has a pole at $-l$ with residue $\frac{(-1)^l}{l!}$, we have, $B(0) = \frac{(-1)^l}{l!}\cdot D_\tau\mathcal{M}(a)(il,\gamma)$. □

The following lemma is proved in [15].

Lemma 6.10. *For each $N \in \mathbb{N}_0$, define $\mathcal{R}_N(\mathbb{S}^{n-1}_\omega \times \mathbb{R}^n_\xi)$ as the space of functions $a \in C^\infty(\mathbb{S}^{n-1} \times (\mathbb{R}^n \setminus \{0\}))$ satisfying the following estimates: for each $P \in \mathrm{Diff}^*(\mathbb{S}^{n-1})$ and β,*

$$|\, P\partial_\xi^\beta a(\omega,\xi)\,| \mathrel{\dot{\le}} \begin{cases} |\xi|^{-n+N+1-|\beta|}, & |\xi| \le 1; \\ |\xi|^{-n-2-|\beta|}, & |\xi| \ge 1. \end{cases} \tag{66}$$

Then, given $a \in \mathcal{R}_N(\mathbb{S}^{n-1} \times \mathbb{R}^n)$, the function $A(r,\omega) := \int e^{i\frac{\omega}{r}\cdot\xi}\, a(\omega,\xi)\, d\xi$ is an element of the space $r^N S^0([0,\infty)_r \times \mathbb{S}^{n-1}_\omega)$.

We now work on the expansion at the face tb of H^n.

Lemma 6.11. *Let $a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$, where m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, be holomorphically tempered and let $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ on a neighborhood of 0 and $\chi(\xi) \equiv 1$ outside a neighborhood of 0. Let $t = s^d$. Then,*

$$A(t) := \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, \chi(\xi)\, \mathcal{L}(a)(t,\xi)\, d\xi \in \mathcal{A}^{\mathcal{E}_{tb}}(H^n),$$

where $\mathcal{E}_{tb} := (-p-d+d\mathbb{N}_0) \cap d\mathbb{N}_0$ is an index set on H^n associated to tb.

Proof. Let $đ\xi = \frac{1}{(2\pi)^n} d\xi$ and let (r,γ_0,γ') be coordinates found in Figure 3. Fix any $N >> 0$ and choose $M >> 0$ such that $dM + m + d \ge -n + N + 1$. Then by Lemma 6.6, for any k and $|\xi| \le 1$,

$$|\, \partial_t^{M+k} \mathcal{L}(a)(t,\xi)\,| \mathrel{\dot{\le}} |\xi|^{d(M+k)+m+d} \le |\xi|^{-n+1}, \tag{67}$$

where we used that $dM + m + d \ge -n + N + 1 \ge -n + 1$. Consider the function

$$g(u,v,\gamma_0,\gamma') := u^M \int e^{iv\gamma'\cdot\xi}\, (\chi(\xi) - 1)\, (\partial_t^M \mathcal{L}(a))(u\gamma_0,\xi)\, đ\xi,$$

where $(u,v) \in [0,\infty)^2$. Then by (67), it follows that we can differentiate g as many times as we want, with respect to any of its variables, and its integrand still remains integrable. Thus, g is smooth in all its variables. Now, we can write

$$A(s^d) = \int e^{ir\gamma'\cdot\xi}\, \chi(\xi)\, \mathcal{L}(a)(r^d\gamma_0^d,\xi)\, đ\xi. \tag{68}$$

Set $w := \gamma_0^d$. By (67), $(\partial_t^M \mathcal{L}(a))(r^d\gamma_0^d,\xi)$ is integrable near $\xi = 0$. Thus, we can write

$$\begin{aligned} \partial_w^M A &= r^{dM} \int e^{ir\gamma'\cdot\xi}\, \chi(\xi)\, (\partial_t^M \mathcal{L}(a))(r^d\gamma_0^d,\xi)\, đ\xi \\ &= r^{dM} \int e^{ir\gamma'\cdot\xi}\, (\partial_t^M \mathcal{L}(a))(r^d\gamma_0^d,\xi)\, đ\xi + g(r^d, r, \gamma_0^d, \gamma'). \end{aligned}$$

Making the change of coordinates $\xi \mapsto r^{-1}\gamma_0^{-1}\xi$ in the first integral and using the homogeneity property of $\mathcal{L}(a)(t,\xi)$ found in Lemma 6.6 yield

$$\partial_w^M A = r^{-m-n-d} \gamma_0^{-dM-m-n-d} \int e^{i\frac{\gamma'}{\gamma_0}\cdot\xi}\, (\partial_t^M \mathcal{L}(a))(1,\xi)\, đ\xi + g(r^d, r, \gamma_0^d, \gamma').$$

Hence,

$$w^M \partial_w^M A = r^{-m-n-d} \gamma_0^{-m-n-d} f + \gamma_0^{dM} g(r^d, r, \gamma_0^d, \gamma'), \tag{69}$$

where $f := \int e^{i\frac{\gamma'}{\gamma_0}\cdot\xi}\, (\partial_t^M \mathcal{L}(a))(1,\xi)\, đ\xi$. Now by Lemma 6.6, there is a $C > 0$ such that for any β,

$$|\, \partial_\xi^\beta (\partial_t^M \mathcal{L}(a))(1,\xi)\,| \mathrel{\dot{\le}} |\xi|^{dM+m+d-|\beta|}\, e^{-C|\xi|^d}.$$

Observe that since $dM + m + d \ge -n + N + 1$,

$$|\xi|^{dM+m+d-|\beta|}\, e^{-C|\xi|^d} \mathrel{\dot{\le}} \begin{cases} |\xi|^{-n+N+1-|\beta|}, & |\xi| \le 1; \\ |\xi|^{-n-2-|\beta|}, & |\xi| \ge 1. \end{cases}$$

Thus, for any β,

$$|\, \partial_\xi^\beta (\partial_t^M \mathcal{L}(a))(1,\xi)\,| \mathrel{\dot{\le}} \begin{cases} |\xi|^{-n+N+1-|\beta|}, & |\xi| \le 1; \\ |\xi|^{-n-2-|\beta|}, & |\xi| \ge 1. \end{cases}$$

Hence, by Lemma 6.10, $f \in \gamma_0^N S^0([0,\infty)_{\gamma_0} \times \mathbb{S}^{n-1}_{\gamma'})$. Since

$$w^M \partial_w^M = (w\partial_w - M + 1)(w\partial_w - M + 2) \cdots (w\partial_w - 1)(w\partial_w)$$

and $\gamma_0 \partial_{\gamma_0} = d\, w\partial_w$, (69) implies that

$$\begin{aligned}(\gamma_0\partial_{\gamma_0} - d(M+1))(\gamma_0\partial_{\gamma_0} - d(M+2)) \cdots (\gamma_0\partial_{\gamma_0} - d)(\gamma_0\partial_{\gamma_0})A \in \\ r^{-m-n-d}\gamma_0^{N-m-n-d} S^0([0,\infty)_{\gamma_0} \times \mathbb{S}^{n-1}_{\gamma'}) + \gamma_0^{dM} S^0_{loc}(H^n).\end{aligned}$$

Since N was arbitrary, by Theorem 2.1, $A \in \mathcal{A}^{d\mathbb{N}_0}(H^n)$. We now show that in fact, $A \in \mathcal{A}^{\mathcal{E}_{tb}}(H^n)$ where $\mathcal{E}_{tb} = (-p-d+d\mathbb{N}_0) \cap d\mathbb{N}_0$. Indeed, note that *away* from $r = 0$ in (68), by the usual technique of regularizing oscillatory integrals (via integration by parts), we can replace $\chi(\xi)\mathcal{L}(a)(t,\xi)$ with $\mathcal{L}(a')(t,\xi)$, where $a'(\lambda,\xi) \in S^{m',p,d}_{\Lambda,r}(\mathbb{R}^n)$ where m' can be made as negative as we wish. Hence, by Lemma 6.13, it follows that $A(t) = O(t^b)$, where $b = \max\{-p/d - 1, 0\}$, away from $r = 0$. Thus, A can be expanded at *tb* with index set $\mathcal{E}_{tb}$ away from $r = 0$. Since the expansion of A to *tb* must be uniform, A can be expanded at *tb* with index set $\mathcal{E}_{tb}$. □

We prove a similar result for elements of $C^{\infty,d}_{\hom(m)}(\Lambda \times \mathbb{R}^n)$.

Lemma 6.12. *Let $a(\lambda,\xi) \in C^{\infty,d}_{\hom(m)}(\Lambda \times \mathbb{R}^n)$, where $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$, be holomorphically tempered and define*

$$A(t) := \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, \mathcal{L}(a)(t,\xi)\, d\xi.$$

Then if $t = s^d$, $A(t) \in \mathcal{A}^{\mathcal{F}_{tb}}(H^n)$, where $\mathcal{F}_{tb} := \emptyset$ is an index set on H^n associated to tb.

Proof. Making the change of variables $\xi \mapsto s^{-1}\xi$ and using the homogeneity property of $\mathcal{L}(a)(t,\xi)$ found in Lemma 6.6 yield

$$\begin{aligned} A(s^d) &= \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, \mathcal{L}(a)(s^d,\xi)\, d\xi \\ &= \frac{s^{-m-n-d}}{(2\pi)^n} \int e^{i\frac{z}{s}\cdot\xi}\, \mathcal{L}(a)(1,\xi)\, d\xi \\ &= \frac{r^{-m-n-d}\gamma_0^{-m-n-d}}{(2\pi)^n} \int e^{i\frac{\gamma'}{\gamma_0}\cdot\xi}\, \mathcal{L}(a)(1,\xi)\, d\xi. \end{aligned}$$

Since $\mathcal{L}(a)(1,\xi)$ is a Schwartz function on $\mathbb{R}^n$, $\int e^{iz\cdot\xi}\, \mathcal{L}(a)(1,\xi)\, d\xi \equiv 0$ at $|z| = \infty$. It follows that $\int e^{i\frac{\gamma'}{\gamma_0}\cdot\xi}\, \mathcal{L}(a)(1,\xi)\, d\xi \equiv 0$ at $\gamma_0 = 0$. □

The following two lemmas deal with the always annoying 'remainder terms' in an asymptotic expansion.

Lemma 6.13. *Let $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. Let $r(\lambda,\xi) \in S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$ and suppose that $\Lambda \ni \lambda \mapsto r(\lambda,\xi) \in S^m(\mathbb{R}^n)$ is holomorphic. Then given any $N \in \mathbb{N}$, for $m \in \mathbb{R}$ sufficiently negative, we have*

$$R(t) := \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}\, \mathcal{L}(r)(t,\xi)\, d\xi \in C^\infty(\,(0,\infty)_t; C^\infty(\mathbb{R}^n_z)\,) \cap t^b C^N([0,\infty)_t \times \mathbb{R}^n_z),$$

where $b = \max\{-p/d - 1, 0\}$.

Proof. Note that $R(t) \in C^\infty(\,(0,\infty)_t; C^\infty(\mathbb{R}^n)\,)$ follows from the estimates (2) of Lemma 5.5. Thus, we just need to prove that $R(t)$ has the stated smoothness near $t = 0$ for $m \in \mathbb{R}$ sufficiently negative.

Let $N \in \mathbb{N}$ and choose $m \in \mathbb{R}$ with $m + |p| \le -(d+1)N - n - d$. Since $r \in S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$, if we define $\tilde{r}(\mu,\xi) := \mu^{p/d}\, r(1/\mu,\xi)$, then $\tilde{r}(\mu,\xi) \in C^\infty(\Lambda_{cc} \times \mathbb{R}^n)$ and it satisfies the estimates: for any α, β, and K,

$$|\,\partial_\mu^\alpha \partial_\xi^\beta \tilde{r}(\mu,\xi)\,| \,\dot{\le}\, (1 + |\mu|\,|\xi|^d)^{p/d - |\alpha|}\,(1 + |\xi|)^{d|\alpha| + m - p - |\beta|},\ |\mu| \le K.$$

Observe that $(1+|\mu|\,|\xi|^d)^{p/d}(1+|\xi|)^{-p} \dot{\leq} (1+|\xi|)^{|p|}$ for $|\mu| \leq K$. Thus, for any l and β, $|\,\partial_\mu^l \partial_\xi^\beta \tilde{r}(\mu,\xi)\,| \dot{\leq} (1+|\xi|)^{dl+m+|p|-|\beta|}$; or, as $m+|p| \leq -(d+1)N-n-d$, we have

$$|\,\partial_\mu^l \partial_\xi^\beta \tilde{r}(\mu,\xi)\,| \dot{\leq} (1+|\xi|)^{dl-dN-N-n-d-|\beta|}. \tag{70}$$

In particular, $|\,\partial_\mu^l \partial_\xi^\beta \tilde{r}(0,\xi)\,| \dot{\leq} (1+|\xi|)^{dl-dN-N-n-d-|\beta|}$. Thus, for any $0 \leq l \leq N$, we have

$$\tilde{r}_l(\xi) := \partial_\mu^l \tilde{r}(0,\xi) \in S^{-N-n-d}(\mathbb{R}^n) \subset S^{-N-n}(\mathbb{R}^n).$$

Now by Taylor's Theorem,

$$\tilde{r}(\mu,\xi) = \sum_{l=0}^{N} \mu^l\, \tilde{r}_l(\xi) + \mu^{N+1}\, \tilde{r}_{N+1}(\mu,\xi),$$

where $\tilde{r}_{N+1}(\mu,\xi) := \frac{1}{N!}\int_0^1 (1-s)^N (\partial_\mu^{N+1}\tilde{r})(s\,\mu,\xi)\,ds$. Note that the estimates (70) imply that $\tilde{r}_{N+1}(\mu,\xi) \in C^0(\Lambda_{cc}; S^{-N-n}(\mathbb{R}^n))$. Since $r(\lambda,\xi) = \lambda^{p/d}\tilde{r}(1/\lambda,\xi)$, we have, away from $\lambda = 0$,

$$r(\lambda,\xi) = \sum_{l=0}^{N} \lambda^{p/d-l}\, \tilde{r}_l(\xi) + \lambda^{-N-1}\, \tilde{r}_{N+1}(\lambda^{-1},\xi).$$

Thus,

$$\mathcal{L}(r)(t,\xi) = \sum_{l=0}^{N} \int_\Gamma e^{-t\lambda}\lambda^{p/d-l} d\lambda \cdot \tilde{r}_l(\xi) + \int_\Gamma e^{-t\lambda}\lambda^{-N-1}\, \tilde{r}_{N+1}(\lambda^{-1},\xi)\,d\lambda, \tag{71}$$

where Γ is any contour in Λ of the form given in (32). Observe that

$$\int_\Gamma e^{-t\lambda}\lambda^{p/d-l} d\lambda = t^{l-p/d-1}\int_\Gamma e^{-\lambda}\lambda^{p/d-l} d\lambda = \begin{cases} 0, & l-p/d \notin \mathbb{N}; \\ (-t)^{l-p/d-1}, & l-p/d \in \mathbb{N}. \end{cases}$$

Thus, we can write the first term on the right hand side of (71) as $\sum_{l=0}^N t^{l-p/d-1}\, r_l(\xi)$, where $r_l(\xi) \in S^{-N-n}(\mathbb{R}^n)$ and where $r_l(\xi) = 0$ if $l - p/d - 1 \leq 0$. Also, by Cauchy's Theorem, $\int_\Gamma \lambda^{-k}\tilde{r}_{N+1}(\lambda^{-1},\xi)\,d\lambda = 0$ for all $k \geq 2$. Hence, $r_{N+1}(t,\xi) := \int_\Gamma e^{-t\lambda}\lambda^{-N-1}\tilde{r}_{N+1}(\lambda^{-1},\xi)\,d\lambda \in C^{N-1}([0,\infty)_t; S^{-N-n}(\mathbb{R}^n))$ with $\partial_t^j r_{N+1}(0,\xi) = 0$ for $j = 0,\dots,N-1$. Since the inverse Fourier transform of a symbol in $S^{-N-n}(\mathbb{R}^n)$ is in $C^{N-1}(\mathbb{R}^n)$, and N is arbitrary, our lemma follows. $\square$

Lemma 6.14. *Let $r(\lambda,\xi) \in S^{m,d}_\Lambda(\mathbb{R}^n)$ and suppose that $\Lambda \ni \lambda \mapsto r(\lambda,\xi) \in S^m(\mathbb{R}^n)$ is holomorphic. Then given any $N \in \mathbb{N}$, for $m \in \mathbb{R}$ sufficiently negative, we have*

$$R(t) := \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}\,\mathcal{L}(r)(t,\xi)\,d\xi \in t^N C^N([0,\infty)_t \times \mathbb{R}^n_z).$$

Proof. Let $m \leq -d(2N+2) - N - n - 1$. Then by the symbol estimates for $S^{m,d}_\Lambda(\mathbb{R}^n)$ found in (15), it follows that $s(\lambda,\xi) := \lambda^{2N+2} r(\lambda,\xi) \in S^0(\Lambda; S^{-N-n-1}(\mathbb{R}^n))$. Thus we can write

$$\mathcal{L}(r)(t,\xi) = \int_\Gamma e^{-t\lambda}\,\lambda^{-2N-2} s(\lambda,\xi)\,d\lambda.$$

By Cauchy's Theorem, for all $k \geq 2$, we have $\int_\Gamma \lambda^{-k} s(\lambda,\xi)\,d\lambda = 0$, and thus by Taylor's Theorem, $\mathcal{L}(r)(t,\xi) = t^N r_N(t,\xi)$, where $r_N(t,\xi) \in C^N([0,\infty)_t; S^{-N-n-1}(\mathbb{R}^n))$. Since the inverse Fourier transform of a symbol in $S^{-N-n-1}(\mathbb{R}^n)$ is in $C^N(\mathbb{R}^n)$, our lemma follows. $\square$

We now prove local results corresponding to Theorems 6.2 and 6.3. From these lemmas, Theorems 6.2 and 6.3 follow easily. For each $m, p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, define the index set $\mathcal{E}_{m,p,d}$ on H^n by

$$\mathcal{E}_{m,p,d}(tb) := (-p-d+d\mathbb{N}_0) \cap d\mathbb{N}_0\,;$$
$$\mathcal{E}_{m,p,d}(tf) := \tfrac{n}{2} + (-m-n-d+\mathbb{N}_0)\overline{\cup}[(-p-d+d\mathbb{N}_0)\cap d\mathbb{N}_0].$$

Lemma 6.15. *Let $a(\lambda,\xi) \in S^{m,p,d}_{\Lambda,ros}(\mathbb{R}^n)$, where m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, be holomorphically tempered and define*
$$A(t) := \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, \mathcal{L}(a)(t,\xi)\, d\xi.$$
Then if $t = s^d$, $A(t)\,|dz\frac{dt}{t}|^{\frac12} \in \mathcal{A}^{\mathcal{E}_{m,p,d}}_{phg}(H^n, \Omega_b^{\frac12})$; and, as $t \downarrow 0$,
$$A(t)|_{z=0} \sim \sum_{k\in\mathbb{N}_0} t^{\frac{k-m-n}{d}-1}\,\gamma_k + \sum_{k\in\mathbb{N}_0} t^{\frac{k-m-n}{d}-1}\log t\,\gamma_k' + \sum_{k,k+p/d+1\in\mathbb{N}_0} t^k\gamma_k'',$$
where $\gamma_k, \gamma_k', \gamma_k'' \in \mathbb{C}$, and where in the first sum, $\gamma_k = 0$ if $\frac{k-m-n}{d} - 1 \in \mathbb{N}_0$, and in the second sum, $\gamma_k' = 0$ if $\frac{k-m-n+p}{d} \notin \mathbb{N}_0$ or $\frac{k-m-n}{d} - 1 \notin \mathbb{N}_0$. Furthermore, if $l = \frac{k-m-n}{d} - 1$, then γ_k and γ_k' are given by
$$\gamma_k = \frac{\Gamma(-l)}{d}\mathrm{Res}(\mathcal{M}(A)(il)) \text{ if } l \notin \mathbb{N}_0,\ \ \gamma_k' = \frac{(-1)^{l+1}}{d\cdot l!}\mathrm{Res}(\mathcal{M}(A)(il)) \text{ if } l \in \mathbb{N}_0,$$
where $\Gamma(-l)$ is the Gamma function at $-l$.

Proof. By Lemma 6.1, $|dz\frac{dt}{t}|^{\frac12} = \rho^{\frac{n}{2}}\mu$, where $0 < \mu \in C^\infty(H^n, \Omega_b^{\frac12})$. Hence, as $a(\lambda,\xi)$ is an asymptotic sum of elements of $C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m-j)}(\mathbb{R}^n)$ for each $j \in \mathbb{N}_0$, Theorem 2.1 applied to the asymptotic expansion of A, where we use Lemma 6.8, Lemma 6.11, and Lemma 6.13 on the remainder term (with the fact that the number N in Lemma 6.13 can be chosen as large as we wish), proves the lemma. □

For each $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$, define the index set $\mathcal{F}_{m,d}$ on H^n by
$$\mathcal{F}_{m,d}(tb) := \emptyset;\ \mathcal{F}_{m,d}(tf) := \frac{n}{2} + (-m-n-d+\mathbb{N}_0).$$

Lemma 6.16. *Let $a(\lambda,\xi) \in S^{m,d}_{\Lambda,os}(\mathbb{R}^n)$, where $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$, be holomorphically tempered and define*
$$A(t) := \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, \mathcal{L}(a)(t,\xi)\, d\xi.$$
Then if $t = s^d$, $A(t)\,|dz\frac{dt}{t}|^{\frac12} \in \mathcal{A}^{\mathcal{F}_{m,d}}_{phg}(H^n, \Omega_b^{\frac12})$. Moreover, as $t \downarrow 0$,
$$A(t)|_{z=0} \sim \sum_{k=0}^{\infty} t^{\frac{k-m-n}{d}-1}\,\gamma_k, \quad \gamma_k \in \mathbb{C},$$
where if $l = \frac{k-m-n}{d} - 1$,
$$\gamma_k = \frac{\Gamma(-l)}{d}\mathrm{Res}(\mathcal{M}(A)(il)),\ \text{ if } l \notin \mathbb{N}_0,\ \ \gamma_k = \frac{(-1)^l}{d\cdot l!}D_\tau\mathrm{Res}(\mathcal{M}(A))(il) \text{ if } l \in \mathbb{N}_0.$$

Proof. This lemma is proved similar to Lemma 6.15, except here, we use Lemma 6.9, Lemma 6.12, and Lemma 6.14. □

Proof of Theorems 6.2 and 6.3: Let $A \in \Psi^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac12})$ be holomorphically tempered. Then by definition, off of Δ_b, $A \in \Psi^{-\infty,p,d}_{b,\Lambda,r}(X, \Omega_b^{\frac12})$, and in any decomposition $X_b^2 \cong \mathbb{R}^{n,k} \times \mathbb{R}^n_z$, where $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}_z$, A has the form given in (28). Thus, Theorem 6.2 follows from Lemmas 5.6 and 6.15. Similarly, Theorem 6.3 follows from Lemmas 5.6 and 6.16.

We now improve Theorem 6.3 and Theorem 6.5 under an additional assumption on the symbols.

Definition 6.17. Let $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$. Then an element $a(\lambda,\xi) \in S^{m,d}_{\Lambda,os}(\mathbb{R}^n)$ is said to be of *alternating parity* if
$$a(\lambda,\xi) \sim \sum_{j=0}^{\infty} \chi(\lambda,\xi)\, a_{m-j}(\lambda,\xi),$$
where $\chi(\lambda,\xi) \in C^\infty(\Lambda \times \mathbb{R}^n)$ with $\chi(\lambda,\xi) \equiv 0$ near 0 and $\chi(\lambda,\xi) \equiv 1$ outside a neighborhood of 0, and where for each j, $a_{m-j}(\lambda,\xi) \in C^{\infty,d}_{\mathrm{hom}(m-j)}(\Lambda \times \mathbb{R}^n)$ is such that $a_{m-j}(\lambda,-\xi) = (-1)^j a_{m-j}(\lambda,\xi)$.

The name 'alternating parity' was suggested by G. Grubb. For example, let $a(\xi)$ be a homogeneous polynomial of *even degree* m such that $a(\xi) \notin \Lambda$ for $\xi \neq 0$. Let $v \in \operatorname{int}(\Lambda)$ be such that $\lambda + v \in \Lambda$ for all $\lambda \in \Lambda$. Then,

$$(a(\xi) - \lambda - v)^{-1} \in S^{-m,m}_{\Lambda,os}(\mathbb{R}^n) \tag{72}$$

is of alternating parity. (See Lemma 4.5.)

One can check that the alternating parity condition is coordinate independent. Hence, the following definition makes sense.

Definition 6.18. An operator $A \in \Psi^{m,d}_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$, is of *alternating parity* if given any coordinate patch $\mathbb{R}^{n,k}_y \times \mathbb{R}^n_z$ on X_b^2 such that $\Delta_b \cong \mathbb{R}^{n,k}_y \times \{0\}_z$ and any compactly supported function ϕ on the coordinate patch, we can write

$$\phi A = \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi} a(\lambda, y, \xi)\, d\xi \cdot \nu, \;\; \nu \in C^\infty(X_b^2, \Omega_b^{\frac{1}{2}})$$

where $a(\lambda, y, \xi) \in C^\infty(\mathbb{R}^{n,k}_y ; S^{m,d}_{\Lambda,os}(\mathbb{R}^n))$ is such that for each y, $(\lambda, \xi) \mapsto a(\lambda, y, \xi) \in S^{m,d}_{\Lambda,os}(\mathbb{R}^n)$ is of alternating parity.

One can check that for any $d \in \mathbb{R}^+$, if $A \in \Psi^{m,d}_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}$, and $B \in \Psi^{m',d}_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$, $m' \in \mathbb{R}$ are both of alternating parity, then $AB \in \Psi^{m+m',d}_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$ is also of alternating parity.

The same proof used in Theorem 4.11 can be used to prove the following.

Proposition 6.19. *Let $A \in \operatorname{Diff}^m_b(X, \Omega_b^{\frac{1}{2}})$, where m is even, be elliptic with respect to Λ, where Λ is of the form given in* (31)*. Then, there exists a holomorphically tempered operator $B(\lambda) \in \Psi^{-m,m}_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$ of alternating parity such that for $\lambda \in \Lambda$ with $|\lambda|$ sufficiently large,*

$$(A - \lambda)^{-1} = B(\lambda) + R(\lambda),$$

where for some continuous increasing function $r : \mathbb{R}^+ \to \mathbb{R}^+$, for each $a > 0$, we have $R(\lambda) \in \Psi^{-\infty,m;a}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r(a)$.

Corollary 6.20. *Let $A(\lambda) \in \Psi^{m,d}_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$, where $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$, be holomorphically tempered and be of alternating parity. Then as $t \downarrow 0$,*

$$\mathcal{L}(A)(t)|_{\Delta_b} \sim \sum_{k=0}^{\infty} t^{\frac{2k-m-n}{d}-1}\, \gamma_{2k}, \;\; \textit{where } \gamma_{2k} \in C^\infty(X, \Omega_b).$$

Proof. The proof is a consequence of the following two observations:

(1) Observe that if $a(\lambda, \xi)$ satisfies $a(\lambda, -\xi) = -a(\lambda, \xi)$, then the number a in (65) is zero as the integral of an odd function of ξ over $\mathbb{S}^{n-1}$ is zero.

(2) Using (1), it follows that in the expansion of the function $A(t)|_{z=0}$ in Lemma 6.16, the terms with k *odd* are zero. □

This corollary plus Proposition 6.19 yield the following.

Corollary 6.21. *Let $A \in \operatorname{Diff}^m_b(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{N}$ be elliptic with respect to Λ, where Λ is of the form given in* (31)*, and let $B \in \operatorname{Diff}^{m'}_b(X, \Omega_b^{\frac{1}{2}})$, $m' \in \mathbb{N}_0$, where both m and m' are even. Then as $t \downarrow 0$,*

$$Be^{-tA}|_{\Delta_b} \sim \sum_{k=0}^{\infty} t^{\frac{2k-m'-n}{m}}\, \gamma_{2k}, \;\; \textit{where } \gamma_{2k} \in C^\infty(X, \Omega_b).$$

7. MELLIN TRANSFORMS OF TEMPERED OPERATORS

Let $m, p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and let $A(\lambda) \in \Psi^{m,p,d;\alpha}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$ be holomorphic for $\lambda \in \Lambda$ and suppose that it extends to be holomorphic on a neighborhood $\mathcal{U}$ of $\lambda = 0$. Recall that Λ is assumed to be of the form given in (31). Let $u \in \rho^\gamma H^s_b(X, \Omega_b^{\frac{1}{2}})$. Then, if $\alpha|_{lb} \geq \gamma'$, $\alpha|_{rb} \geq -\gamma$, and $\gamma \geq \gamma'$, by Lemma 3.11 and the mapping property (7), $A(\lambda)u \in S^{p/d}(\Lambda; \rho^{\gamma'} H^{s'}_b(X, \Omega_b^{\frac{1}{2}}))$ where $s' = s - m$ if $p > 0$ and

$s' = s - (m-p)$ if $p \leq 0$. Let Γ be any contour in $\Lambda \cup \mathcal{U}$ of the form given in (32) with $c > 0$. Then for each $\tau \in \mathbb{C}$ with $\operatorname{Im} \tau < -p/d - 1$, the integral

$$\mathcal{M}(A)(\tau)u := \int_{\Gamma} \lambda^{-i\tau} A(\lambda) u \, d\lambda \tag{73}$$

converges in $\rho^{\gamma'} H_b^{s'}(X, \Omega_b^{\frac{1}{2}})$, where $\lambda^{-i\tau}$ is defined by using the *principal branch* of the logarithm. Observe that since $A(\lambda)$ is holomorphic on $\Lambda \cup \mathcal{U}$, the integral (73) is defined independent of the contour Γ chosen (where Γ is of the form given in (32) with $c > 0$). Since for any $k \in \mathbb{N}$ and $\operatorname{Im} \tau < -p/d - 1$, $\lambda^{-i\tau} (\log \lambda)^k A(\lambda) u$ is still integrable in $\rho^{\gamma'} H_b^{s'}(X, \Omega_b^{\frac{1}{2}})$, it follows that $\mathcal{M}(A)(\tau)u$ is holomorphic for $\operatorname{Im} \tau < -p/d - 1$. Thus, $\mathcal{M}(A)(\tau)u \in \mathcal{H}ol(\{\operatorname{Im} \tau < -p/d - 1\}; \rho^{\gamma'} H_b^{s'}(X, \Omega_b^{\frac{1}{2}}))$, the space of holomorphic functions on $\{\operatorname{Im} \tau < -p/d - 1\}$ taking values in $\rho^{\gamma'} H_b^{s'}(X, \Omega_b^{\frac{1}{2}})$.

Definition 7.1. Let $m, p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and let $A(\lambda) \in \Psi_{b,\Lambda}^{m,p,d;\alpha}(X, \Omega_b^{\frac{1}{2}})$ be holomorphic for $\lambda \in \Lambda$ and suppose that $A(\lambda)$ extends to be holomorphic on a neighborhood of $\lambda = 0$. Then the operator

$$\mathcal{M}(A)(\tau) : \rho^{\gamma} H_b^{s}(X, \Omega_b^{\frac{1}{2}}) \to \mathcal{H}ol(\{\operatorname{Im} \tau < -p/d - 1\}; \rho^{\gamma'} H_b^{s'}(X, \Omega_b^{\frac{1}{2}})), \tag{74}$$

where $s' = s - m$ if $p > 0$ and $s' = s - (m - p)$ if $p \leq 0$, is called the *Mellin Transform* of A.

Remark 7.2. Since holomorphically tempered operators are holomorphic on a neighborhood of 0 (by definition), their Mellin transforms are always defined. The Mellin transforms of strongly tempered operators are defined similarly.

The 'most important' examples of Mellin transforms are complex powers of elliptic operators. Thus, let $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be elliptic with respect to Λ and suppose that the resolvent $(A - \lambda)^{-1}$ extends to be holomorphic on $\Lambda \cup \mathcal{U}$ for some neighborhood $\mathcal{U}$ of $\lambda = 0$. Then by Theorem 4.11, there exists an $\delta > 0$ such that $(A - \lambda)^{-1} \in \Psi_{b,\Lambda,r}^{-m,-m,m;\delta}(X, \Omega_b^{\frac{1}{2}})$ is holomorphically tempered. For $\operatorname{Re} z < 0$, we define A^z by

$$A^z := \frac{i}{2\pi} \mathcal{M}((A - \lambda)^{-1})(iz) = \frac{i}{2\pi} \int_{\Gamma} \lambda^z (A - \lambda)^{-1} \, d\lambda,$$

where $\Gamma \subset \Lambda \cup \mathcal{U}$ is any contour of the form given (32) where $c > 0$. By (74),

$$A^z : \rho^{\gamma} H_b^{s}(X, \Omega_b^{\frac{1}{2}}) \to \mathcal{H}ol(\{\operatorname{Re} z < 0\}; \rho^{\gamma} H_b^{s}(X, \Omega_b^{\frac{1}{2}})), \; s \in \mathbb{R}, \; \gamma \geq 0.$$

Now let $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, be formally self adjoint and positive. Then by oscillatory testing, it follows that A is elliptic with respect to Λ (for any Λ of the form (31)). Suppose also that A is Fredholm. Then by Theorem 4.11 and Analytic Fredholm Theory, there exists a neighborhood $\mathcal{V} \subset \mathbb{C}$ containing 0 and a $\delta > 0$ such that on $\Lambda \cup \mathcal{V}$, $(A - \lambda)^{-1} \in \Psi_{b,\Lambda,r}^{-m,-m,m;\delta}(X, \Omega_b^{\frac{1}{2}})$ is meromorphic with only a simple pole at $\lambda = 0$ given by $-\pi$, where $\pi \in \Psi^{-\infty,\delta}(X, \Omega_b^{\frac{1}{2}})$ is the orthogonal projection onto the null space of A (for the definition of $\Psi^{-\infty,\delta}(X, \Omega_b^{\frac{1}{2}})$, please see (8)). Here, we identity the number δ with the multi-index that assigns to each hypersurface of X^2, the number δ.

Lemma 7.3. *Let $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, be formally self adjoint, positive, and Fredholm. Let π be the orthogonal projection onto the null space of A and define $A_0 := A + \pi$. Then,*

$$(A_0 - \lambda)^{-1} = (A - \lambda)^{-1} + \lambda^{-1}(1 - \lambda)^{-1}\pi. \tag{75}$$

It follows that $(A_0 - \lambda)^{-1} \in \Psi_{b,\Lambda,r}^{-m,-m,m;\delta}(X, \Omega_b^{\frac{1}{2}})$ for some $\delta > 0$, and is holomorphically tempered. In particular, the complex power A_0^z is well defined, and the heat operator e^{-tA_0} vanishes exponentially as $t \to \infty$. Moreover, $e^{-tA_0} = (e^{-t} - 1)\pi + e^{-tA}$.

Proof. An elementary verification proves the identity (75). Then the heat operator identity $e^{-tA_0} = (e^{-t} - 1)\pi + e^{-tA}$ follows by taking the Laplace transform of (75). ∎

Before we move onto describing the structure of the Schwartz kernels of Mellin transforms, we first relate them to Laplace transforms.

Lemma 7.4. *Let $m, p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and let $A(\lambda) \in \Psi^{m,p,d;\alpha}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$ be holomorphic for $\lambda \in \Lambda$ and suppose that $A(\lambda)$ extends to be holomorphic on a neighborhood of $\lambda = 0$. Then, for $\operatorname{Im}\tau << 0$, we have*

$$\mathcal{M}(A)(\tau) = \frac{1}{\Gamma(i\tau)} \int_0^\infty t^{i\tau-1} \mathcal{L}(A)(t)\, dt.$$

Proof. By Theorem 5.2, for any $k, M \in \mathbb{N}_0$ with $p/d - M < -k-1$, we have $t^{M+N}\mathcal{L}(A)(t) \in C^\infty((0,\infty)_t; \Psi_b^{-\infty,\alpha}(X, \Omega_b^{\frac{1}{2}})) \cap C^k([0,\infty)_t; \Psi_b^{m-p-dN,\alpha}(X, \Omega_b^{\frac{1}{2}}))$ for any $N \in \mathbb{N}_0$; and moreover, $\mathcal{L}(A)(t) \to 0$ to exponential order in $\Psi_b^{-\infty,\alpha}(X, \Omega_b^{\frac{1}{2}})$ as $t \to \infty$. Thus, the following interchanging of integrals is allowed for $\operatorname{Im}\tau << 0$:

$$\begin{aligned}\int_\Gamma \lambda^{-i\tau} A(\lambda) d\lambda &= \frac{1}{\Gamma(i\tau)} \int_\Gamma \int_0^\infty t^{i\tau-1}\, e^{-t\lambda} dt\, A(\lambda) d\lambda = \frac{1}{\Gamma(i\tau)} \int_0^\infty t^{i\tau-1} \int_\Gamma e^{-t\lambda} A(\lambda) d\lambda\, dt \\ &= \frac{1}{\Gamma(i\tau)} \int_0^\infty t^{i\tau-1} \mathcal{L}(A)(t)\, dt.\end{aligned}$$

□

We will prove the following theorem shortly.

Theorem 7.5. *Let $A(\lambda) \in \Psi^{m,p,d;\alpha}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$, where $m, p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, be a holomorphically tempered operator. Then, the Mellin transform $\mathcal{M}(A)(\tau)$ defined in (74) defines an entire family of b-pseudodifferential operators: $\mathcal{M}(A)(\tau) \in \Psi_b^{d\operatorname{Im}\tau+m+d,\alpha}(X, \Omega_b^{\frac{1}{2}})$ in the sense that for each $R \in \mathbb{R}$, the map*

$$\{\tau \in \mathbb{C} : \operatorname{Im}\tau < R\} \ni \tau \mapsto \mathcal{M}(A)(\tau) \in \Psi_b^{d\operatorname{Im}\tau+m+d,\alpha}(X, \Omega_b^{\frac{1}{2}})$$

is an element of $\mathcal{H}ol(\{\operatorname{Im}\tau < R\}; \Psi_b^{dR+m+d,\alpha}(X, \Omega_b^{\frac{1}{2}}))$, the space of holomorphic functions on $\{\operatorname{Im}\tau < R\}$ taking values in $\Psi_b^{dR+m+d,\alpha}(X, \Omega_b^{\frac{1}{2}})$.

Remark 7.6. The same conclusion of Theorem 7.5 holds for $A \in \Psi^{m,d;\alpha}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$ holomorphically tempered (see (21)). Moreover, by (2) of Lemma 6.7, if $A \in \Psi^{m,p,d;\alpha}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$ or $A \in \Psi^{m,d;\alpha}_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$, one can show that $\mathcal{M}(A)(\tau)$ is a classical b-pseudodifferential operator of (complex) degree $d\tau+m+d$.

As a consequence of this theorem, we get for free, the analytic behavior of the complex powers. Note that in Corollaries 7.7 and 7.8 below, the pseudodifferential operators are *not* assumed to be classical. In previous treatments of the complex powers, cf. [27], [30], [26], or [9], the pseudodifferential operators were always assumed to be classical.

Corollary 7.7. *Let $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, be elliptic with respect to Λ and suppose that the resolvent $(A-\lambda)^{-1}$ is holomorphic on $\Lambda \cup \mathcal{U}$, where $\mathcal{U}$ is a neighborhood of $\lambda = 0$. Then for some $\delta > 0$, for each $z \in \mathbb{C}$, $A^z \in \Psi_b^{m\operatorname{Re} z,\delta}(X, \Omega_b^{\frac{1}{2}})$, and moreover, it is an entire family of b-pseudodifferential operators in the sense that for each $R \in \mathbb{R}$, the map*

$$\{z \in \mathbb{C} : \operatorname{Re} z < R\} \ni z \mapsto A^z \in \Psi_b^{m\operatorname{Re} z,\delta}(X, \Omega_b^{\frac{1}{2}})$$

is an element of $\mathcal{H}ol(\{\operatorname{Re} z < R\}; \Psi_b^{mR,\delta}(X, \Omega_b^{\frac{1}{2}}))$.

The following corollary is a consequence of Lemma 7.3.

Corollary 7.8. *Let $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, be formally self adjoint, positive, and Fredholm and let $A_0 = A + \pi$, where π is the orthogonal projection onto the null space of A. Then for some $\delta > 0$, for each $z \in \mathbb{C}$, $A_0^z \in \Psi_b^{m\operatorname{Re} z,\delta}(X, \Omega_b^{\frac{1}{2}})$, and moreover, it is an entire family of b-pseudodifferential operators in the sense that for each $R \in \mathbb{R}$, the map*

$$\{z \in \mathbb{C} : \operatorname{Re} z < R\} \ni z \mapsto A_0^z \in \Psi_b^{m\operatorname{Re} z,\delta}(X, \Omega_b^{\frac{1}{2}})$$

is an element of $\mathcal{H}ol(\{\operatorname{Re} z < R\}; \Psi_b^{mR,\delta}(X, \Omega_b^{\frac{1}{2}}))$.

By Remark 7.6, if A is classical, then the complex powers described in the above two corollaries are classical of complex degree mz.

To prove Theorem 7.5, we start with the following lemma.

Lemma 7.9. *Let $m, p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and let $a(\lambda,\xi) \in S^{m,p,d}_{\Lambda}(\mathbb{R}^n)$ be a holomorphically tempered symbol. Let Γ be any contour in Λ of the form given in (32) with $c > 0$ and define*

$$\mathcal{M}(a)(\tau,\xi) := \int_{\Gamma} \lambda^{-i\tau}\, a(\lambda,\xi)\, d\lambda$$

for $\operatorname{Im}\tau << 0$. Then, $\mathcal{M}(a)(\tau,\xi)$ extends to be entire in τ. Moreover, for each $\tau \in \mathbb{C}$, $\mathcal{M}(a)(\tau,\xi) \in S^{d\operatorname{Im}\tau+m+d}(\mathbb{R}^n)$ and is an entire family of symbols in the sense that for each $R \in \mathbb{R}$, the map

$$\{\tau \in \mathbb{C} : \operatorname{Im}\tau < R\} \ni \tau \mapsto \mathcal{M}(a)(\tau,\xi) \in S^{d\operatorname{Im}\tau+m+d}(\mathbb{R}^n)$$

is an element of $\mathcal{H}ol(\{\operatorname{Im}\tau < R\};\, S^{dR+m+d}(\mathbb{R}^n))$.

Proof. We follow the proof given in [9, Lemma 4.2.3]. Since $a(\lambda,\xi)$ is holomorphically tempered, there exists an $\epsilon > 0$ such that $a(\lambda,\xi)$ extends to be a smooth function, holomorphic in λ, for (λ,ξ) in

$$(76)\qquad \{(\lambda,\xi) \in \mathbb{C}\times\mathbb{R}^n \mid \lambda \in \Lambda \text{ or } |\lambda| \le \epsilon\,(1+|\xi|)^d \text{ or } \frac{1}{\epsilon}\,(1+|\xi|)^d \le |\lambda|\};$$

and moreover, $a(\lambda,\xi)$ continues to satisfy similar symbol estimates for (λ,ξ) in the set given by (76) as for (λ,ξ) in $\Lambda\times\mathbb{R}^n$. Let $\xi \in \mathbb{R}^n$. Then, as $a(\lambda,\xi)$ is holomorphic for λ in the set (76), it follows that $\mathcal{M}(a)(t,\xi) = \int_{\Gamma_\xi} \lambda^{-i\tau}\, a(\lambda,\xi)\, d\lambda$, where Γ_ξ is the contour shown in Figure 1, where the radius of the inner arc of Γ_ξ is $\frac{\epsilon}{2}(1+|\xi|)^d$ and the radius of the outer arc of Γ_ξ is $\frac{2}{\epsilon}(1+|\xi|)^d$. Since Γ_ξ is a finite contour, it follows that $\mathcal{M}(a)(\tau,\xi)$ extends to be entire in τ. Now let $k \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^n$. Then,

$$\partial_\tau^k \partial_\xi^\beta \mathcal{M}(a)(\tau,\xi) = \int_{\Gamma_\xi} \lambda^{-i\tau}(-i\log\lambda)^k\, \partial_\xi^\beta a(\lambda,\xi)\, d\lambda.$$

Observe that

(1) the length of $\Gamma_\xi \,\dot{\le}\, (1+|\xi|)^d$ (independent of ξ);
(2) $|\lambda^{-i\tau}| \,\dot{\le}\, (1+|\xi|)^{d\operatorname{Im}\tau}$ for all $\lambda \in \Gamma_\xi$, (independent of ξ);
(3) $|\log\lambda|^k \,\dot{\le}\, |\log((1+|\xi|)^d)|^k$ for all $\lambda \in \Gamma_\xi$ (independent of ξ);
(4) for all $\lambda \in \Gamma_\xi$,

$$|\,\partial_\xi^\beta a(\lambda,\xi)\,| \,\dot{\le}\, (1+|\lambda|^{1/d}+|\xi|)^p\,(1+|\xi|)^{m-p-|\beta|} \,\dot{\le}\, (1+|\xi|)^{m-|\beta|}.$$

Hence,

$$\begin{aligned}
|\,\partial_\tau^k\partial_\xi^\beta \mathcal{M}(a)(\tau,\xi)\,| &\le \int_{\Gamma_\xi} |\lambda^{-i\tau}|\,|\log\lambda|^k\,|\,\partial_\xi^\beta a(\lambda,\xi)\,|\, d\lambda \\
&\dot{\le} (1+|\xi|)^d \cdot (1+|\xi|)^{d\operatorname{Im}\tau} \cdot |\log((1+|\xi|)^d)|^k \cdot (1+|\xi|)^{m-|\beta|} \\
&\dot{\le} (1+|\xi|)^{d\operatorname{Im}\tau+m+d-|\beta|}\,|\log(1+|\xi|)|^k.
\end{aligned}$$

Thus, for any $k \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^n$,

$$|\,\partial_\tau^k\partial_\xi^\beta \mathcal{M}(a)(\tau,\xi)\,| \,\dot{\le}\, (1+|\xi|)^{d\operatorname{Im}\tau+m+d-|\beta|}\,|\log(1+|\xi|)|^k.$$

This estimate implies that for fixed τ, $\mathcal{M}(a)(\tau,\xi) \in S^{d\operatorname{Im}\tau+m+d}(\mathbb{R}^n_\xi)$, and for each $R \in \mathbb{R}$, the map $\{\tau \in \mathbb{C} : \operatorname{Im}\tau < R\} \ni \tau \mapsto \mathcal{M}(a)(\tau,\xi) \in S^{d\operatorname{Im}\tau+m+d}(\mathbb{R}^n)$ is an element of $\mathcal{H}ol(\{\operatorname{Im}\tau < R\};\, S^{dR+m+d}(\mathbb{R}^n))$. □

Lemma 7.10. *Let $R \in \Psi^{-\infty,p,d;\alpha}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$, where $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, be holomorphic for $\lambda \in \Lambda$ and suppose that $R(\lambda)$ extends to be holomorphic on a neighborhood of $\lambda = 0$. Then, $\mathcal{M}(R)(\tau) \in \mathcal{H}ol(\mathbb{C};\, \Psi_b^{-\infty,\alpha}(X,\Omega_b^{\frac{1}{2}}))$.*

Proof. By Lemma 7.4, $\mathcal{M}(R)(\tau) = \frac{1}{\Gamma(i\tau)}\int_0^\infty t^{i\tau-1}\mathcal{L}(R)(t)\,dt$ for $\operatorname{Im}\tau << 0$. By Lemma 5.6, $\mathcal{L}(R)(t) \in C^\infty([0,\infty)_t;\, \Psi_b^{-\infty,\alpha}(X,\Omega_b^{\frac{1}{2}}))$ and $\mathcal{L}(R)(t) \to 0$ exponentially in $\Psi_b^{-\infty,\alpha}(X,\Omega_b^{\frac{1}{2}})$ as $t \to \infty$. It follows that $\frac{1}{\Gamma(i\tau)}\int_1^\infty t^{i\tau-1}\mathcal{L}(R)(t)\,dt$ extends to be an *entire* function of $\tau \in \mathbb{C}$. Thus, we just have to show that the function $\tau \mapsto \frac{1}{\Gamma(i\tau)}\int_0^1 t^{i\tau-1}\mathcal{L}(R)(t)\,dt$ extends to be entire. Since, $\mathcal{L}(R)(t) \in C^\infty([0,\infty)_t;\, \Psi_b^{-\infty,\alpha}(X,\Omega_b^{\frac{1}{2}}))$, for any $M \in \mathbb{N}$, we can write

$$\mathcal{L}(R)(t) = \sum_{l=0}^{M-1} t^l R_l + t^M R_M(t),$$

where $R_l \in \Psi_b^{-\infty,\alpha}(X,\Omega_b^{\frac{1}{2}})$, and $R_M(t) \in C^\infty([0,\infty)_t;\, \Psi_b^{-\infty,\alpha}(X,\Omega_b^{\frac{1}{2}}))$. Hence, for any $M \in \mathbb{N}$, for $\operatorname{Im}\tau << 0$,

$$\begin{aligned}\frac{1}{\Gamma(i\tau)}\int_0^1 t^{i\tau-1}\mathcal{L}(R)(t)\,dt &= \sum_{l=0}^{M-1}\frac{1}{\Gamma(i\tau)}\int_0^1 t^{i\tau+l-1}dt\cdot R_l + \frac{1}{\Gamma(i\tau)}\int_0^1 t^{i\tau+M-1}R_M(t)\,dt \\ &= \sum_{l=0}^{M-1}\frac{1}{\Gamma(i\tau)}\,\frac{1}{i\tau+l}\,R_l + \frac{1}{\Gamma(i\tau)}\int_0^1 t^{i\tau+M-1}R_M(t)\,dt.\end{aligned}$$

Observe that since $1/\Gamma(i\tau)$ is entire and it vanishes whenever $i\tau \in -\mathbb{N}_0$, for any $l \in \mathbb{N}_0$, $\frac{1}{\Gamma(i\tau)}\,\frac{1}{i\tau+l}$ is entire. Also, observe that $\int_0^1 t^{i\tau+M-1}R_M(t)\,dt$ is holomorphic for $\operatorname{Im}\tau < M$. Thus, as $M \in \mathbb{N}$ is arbitrary, $\frac{1}{\Gamma(i\tau)}\int_0^1 t^{i\tau-1}\mathcal{L}(R)(t)\,dt$ extends to be an entire function of τ. □

Proof of Theorem 7.5: We proceed just as we did in the proof of Theorem 5.3. By the definition (23), A is the sum of two parts: $A = B + R$, where $B \in \Psi_{b,\Lambda,r}^{m,p,d}(X,\Omega_b^{\frac{1}{2}})$ and $R \in \Psi_{b,\Lambda,r}^{-\infty,p,d;\alpha}(X,\Omega_b^{\frac{1}{2}})$. By Lemma 7.10, we have $\mathcal{M}(R)(\tau) \in \mathcal{H}ol(\mathbb{C};\, \Psi_b^{-\infty,\alpha}(X,\Omega_b^{\frac{1}{2}}))$. Now by Definition 3.8, away from the b-diagonal, we have $B \in \Psi_{b,\Lambda,r}^{-\infty,p,d}(X,\Omega_b^{\frac{1}{2}})$. Thus, again by Lemma 7.10, we have $\mathcal{M}(B)(\tau) \in \mathcal{H}ol(\mathbb{C};\, \Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}}))$ away from the b-diagonal. Thus, it suffices to assume that $A \in \Psi_{b,\Lambda,r}^{m,p,d}(X,\Omega_b^{\frac{1}{2}})$ and that A is supported near the b-diagonal.

Now let $\mathbb{R}_y^{n,k} \times \mathbb{R}_z^n$ be a coordinate patch on X_b^2 such that $\Delta_b \cong \mathbb{R}_y^{n,k} \times \{0\}_z$ and let ϕ be any compactly supported function on the coordinate patch. Then, we can write

$$\phi\, A(\lambda) = \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}\, a(\lambda,y,\xi)\,d\xi\cdot\nu,\ \ \nu \in C^\infty(X_b^2,\Omega_b^{\frac{1}{2}}),$$

where for each $y \in \mathbb{R}^{n,k}$, $(\lambda,\xi) \mapsto a(\lambda,y,\xi) \in S_\Lambda^{m,p,d}(\mathbb{R}^n)$ is holomorphically tempered. Hence,

$$\mathcal{M}(\phi\, A)(\tau) = \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}\,\mathcal{M}(a)(\tau,y,\xi)\,d\xi\cdot\nu.$$

By Lemma 7.9, $\mathcal{M}(a)(\tau,y,\xi)$ extends to be entire in τ and for each $\tau \in \mathbb{C}$, $\mathcal{M}(a)(\tau,y,\xi) \in C^\infty(\mathbb{R}_y^{n,k}; S^{d\operatorname{Im}\tau+m+d}(\mathbb{R}^n))$ and is an entire family of symbols in the sense that for each $R \in \mathbb{R}$, the map

$$\{\tau \in \mathbb{C} : \operatorname{Im}\tau < R\} \ni \tau \mapsto \mathcal{M}(a)(\tau,y,\xi) \in C^\infty(\mathbb{R}_y^{n,k}; S^{d\operatorname{Im}\tau+m+d}(\mathbb{R}^n))$$

is in $\mathcal{H}ol(\{\operatorname{Im}\tau < R\};\, C^\infty(\mathbb{R}_y^{n,k}; S^{d\operatorname{Im}\tau+m+d}(\mathbb{R}^n)))$. Hence, $\mathcal{M}(\phi\, A)(\tau)$ extends to be an entire function of τ such that $\mathcal{M}(\phi\, A)(\tau) \in \Psi_b^{d\operatorname{Im}\tau+m+d}(X,\Omega_b^{\frac{1}{2}})$ for each $\tau \in \mathbb{C}$, and it is an entire family of b-pseudodifferential operators in the sense that for each $R \in \mathbb{R}$, the map

$$\{\tau \in \mathbb{C} : \operatorname{Im}\tau < R\} \ni \tau \mapsto \mathcal{M}(\phi\, A)(\tau) \in \Psi_b^{d\operatorname{Im}\tau+m+d}(X,\Omega_b^{\frac{1}{2}})$$

is an element of $\mathcal{H}ol(\{\operatorname{Im}\tau < R\};\, \Psi_b^{dR+m+d}(X,\Omega_b^{\frac{1}{2}}))$. The proof of Theorem 7.5 is now complete.

If A is a classical pseudodifferential operator on a closed compact manifold, Seeley [27] showed that the Schwartz kernel of A^z when restricted the diagonal, is meromorphic with equidistant simple poles on the real axis, regular at $z = 0$. We prove a very similar result for b-pseudodifferential operators. The proof is based on the following lemma.

Lemma 7.11. *Let E be any index set such that $E \subset \mathbb{C}\times\{0,1\}$ and such that $E\cap(\mathbb{C}\times\{1\}) \subset \mathbb{N}_0\times\{1\}$. Let $u(t) \in \mathcal{A}^E_{phg}([0,\infty)_t)$ and suppose that $u(t)$ is Schwartz at ∞; thus, for any $k \in \mathbb{N}$, $t^k u(t)$ and all its derivatives are bounded outside a neighborhood of 0. Denote the coefficients of the expansion of $u(t)$ by $u_{(z,k)}$, where $(z,k) \in E$. Define for $\operatorname{Im}\tau << 0$,*

$$v(\tau) := \frac{1}{\Gamma(i\tau)} \int_0^\infty t^{i\tau-1} u(t)\, dt.$$

Then, $v(\tau)$ extends to be a meromorphic function on all of $\mathbb{C}$ having only simple poles at the points

$$i \cdot \{z \,|\, (z,0) \in E \text{ and } z \notin \mathbb{N}_0,\ \text{or } (z,1) \in E\}.$$

The residue of $v(\tau)$ at $\tau = iz$, when $z \notin \mathbb{N}_0$, is $-i\,\frac{u_{(z,0)}}{\Gamma(-z)}$, and when $z \in \mathbb{N}_0$, the residue is $i(-1)^z\Gamma(z+1)\, u_{(z,1)}$. Moreover, the regular value of $v(\tau)$ at $\tau = 0$ is $u_{(0,0)} - \gamma \cdot u_{(0,1)}$, where $\gamma := \lim_{k\to\infty}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} - \log k)$ is Euler's constant.

Proof. For $\operatorname{Im}\tau << 0$, we can write

$$v(\tau) = \frac{1}{\Gamma(i\tau)} \int_0^1 t^{i\tau-1} u(t)\, dt + \frac{1}{\Gamma(i\tau)} \int_1^\infty t^{i\tau-1} u(t)\, dt.$$

Since $u(t)$ is Schwartz at $t = \infty$ and since $1/\Gamma(i\tau)$ is an entire function vanishing at $\tau = 0$, the term $\frac{1}{\Gamma(i\tau)} \int_1^\infty t^{i\tau-1} u(t)\, dt$ is an entire function vanishing at $\tau = 0$. Thus, we need only concern ourselves with the term $\frac{1}{\Gamma(i\tau)} \int_0^1 t^{i\tau-1} u(t)\, dt$. Fix $N >> 0$. Then there is an $M >> 0$ such that

$$u(t) = \sum_{z\notin\mathbb{N}_0, z\le N} t^z u_{(z,0)} + \sum_{z\in\mathbb{N}_0, z\le N} t^z \log t\, u_{(z,1)} + \sum_{k\in\mathbb{N}_0, k\le N} t^k u_{(k,0)} + t^M u_M(t),$$

where $u_M(t) \in S^0_{loc}([0,\infty)_t)$. Observe that for any $z \in \mathbb{C}$ with $\operatorname{Re} z >> 0$, we have

$$\int_0^1 t^{z-1}\, dt = \frac{1}{z} \text{ and } \int_0^1 t^{z-1} \log t\, dt = \partial_z \int_0^1 t^{z-1}\, dt = -\frac{1}{z^2}.$$

Thus, for $\operatorname{Im}\tau << 0$,

$$\frac{1}{\Gamma(i\tau)} \int_0^1 t^{i\tau-1} u(t)\, dt = \sum_{z\notin\mathbb{N}_0, z\le N} \frac{u_{(z,0)}}{\Gamma(i\tau)(i\tau+z)} - \sum_{z\in\mathbb{N}_0, z\le N} \frac{u_{(z,1)}}{\Gamma(i\tau)(i\tau+z)^2}$$
$$+ \sum_{k\in\mathbb{N}_0, k\le N} \frac{1}{\Gamma(i\tau)(i\tau+k)}\, u_{(k,0)} + \frac{1}{\Gamma(i\tau)} \int_0^1 t^{i\tau+M-1} u_M(t)\, dt. \tag{77}$$

Now observe that since $\Gamma(z)$ has simple poles only on the set $-\mathbb{N}_0$, with residue at $-j \in -\mathbb{N}_0$ given by $(-1)^j/j! = (-1)^j/\Gamma(j+1)$, we have

(1) if $z \notin \mathbb{N}_0$, $1/[\Gamma(i\tau)(i\tau+z)]$ is meromorphic having only a simple pole at $i\tau = -z$, with residue $-i/\Gamma(-z)$, and it vanishes at $\tau = 0$;
(2) if $z \in \mathbb{N}_0$, $1/[\Gamma(i\tau)(i\tau+z)^2]$ is meromorphic having only a simple pole at $i\tau = -z$, with residue $i(-1)^{z+1}\Gamma(z+1)$;
(3) if $z \in \mathbb{N}_0$, $1/[\Gamma(i\tau)(i\tau+z)]$ is entire and vanishes at $\tau = 0$, unless $z = 0$, in which case it has the value 1 at $\tau = 0$.

Also, observe that $\frac{1}{\Gamma(i\tau)} \int_0^1 t^{i\tau+M-1} u_M(t)\, dt$ is holomorphic on $\{\operatorname{Im}\tau < M-1\}$ and it vanishes at $\tau = 0$ for any $M > 1$ since $\frac{1}{\Gamma(i\tau)}$ vanishes at $\tau = 0$.

Thus, from (77), it follows that $v(\tau)$ extends to be meromorphic on all of $\mathbb{C}$, having only simple poles at the points

$$i \cdot \{z \,|\, (z,0) \in E \text{ and } z \notin \mathbb{N}_0,\ \text{or } (z,1) \in E\}$$

where the residue of $v(\tau)$ at $\tau = iz$, when $z \notin \mathbb{N}_0$, is $-i\,\frac{u_{(z,0)}}{\Gamma(-z)}$; and when $z \in \mathbb{N}_0$, the residue is $i(-1)^z\Gamma(z+1)\, u_{(z,1)}$. It also follows from (1)–(3) that the regular value of (77) at $\tau = 0$ is

$$u_{(0,0)} - [\text{the regular value of } \frac{1}{\Gamma(i\tau)(i\tau)^2} \text{ at } \tau = 0] \cdot u_{(0,1)}.$$

Since $\Gamma(z)z^2 = z\Gamma(z+1)$, the regular value of $1/[\Gamma(z)z^2]$ at $z=0$ is

$$\partial_z(z\cdot\frac{1}{z\Gamma(z+1)})\Big|_{z=0} = -\frac{\Gamma'(z+1)}{\Gamma(z+1)^2}\Big|_{z=0} = -\Gamma'(1).$$

Since (see [7, page 179]), $\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma - \sum_{k=1}^{\infty}(\frac{1}{z+k} - \frac{1}{k})$, it follows that $\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma$; and hence, the regular value of $1/[\Gamma(z)z^2]$ at $z=0$ is γ. $\square$

In the following theorem, we use the following convention: given a multi-index α on X^2_b, we consider $\alpha|_{ff}$ as an index set on X given by $\alpha|_{ff}(H) := \alpha(ff(H))$ for all $H \in M_1(X)$, where $ff(H)$ is the face in X^2_b coming from the blow-up of $H\times H$ in X^2_b.

Theorem 7.12. *Let $A(\lambda) \in \Psi^{m,p,d;\alpha}_{b,\Lambda,ros}(X,\Omega_b^{\frac{1}{2}})$, where $m,p\in\mathbb{R}$ and $d\in\mathbb{R}^+$ with $p/d\in\mathbb{Z}$, be holomorphically tempered. Then the Schwartz kernel of the Mellin transform $\mathcal{M}(A)(\tau)$, when restricted to Δ_b, extends from $\operatorname{Im}\tau < \frac{-m-n}{d} - 1$ to be a meromorphic function on all of $\mathbb{C}$, with values in $S^{0,\alpha|_{ff}}(X,\Omega_b)$, having only simple poles at the points*

$$i\cdot\{z_k = \frac{k-m-n}{d} - 1 \,:\, k\in\mathbb{N}_0\}.$$

The residue of $\mathcal{M}(A)(\tau)|_{\Delta_b}$ at $\tau = iz_k$, when $z_k\notin\mathbb{N}_0$, is $-i\frac{\gamma_k}{\Gamma(-z_k)}$, and when $z_k\in\mathbb{N}_0$, the residue is $i(-1)^{z_k}\Gamma(z_k+1)\,\gamma'_k$, where γ_k and γ'_k are given in the expansion (52) of $\mathcal{L}(A)(t)|_{\Delta_b}$. Moreover, the regular value of $\mathcal{M}(A)(\tau)|_{\Delta_b}$ at $\tau=0$ is equal to

$$\begin{aligned}&\text{the constant term in the expansion of } \mathcal{L}(A)(t)|_{\Delta_b} \text{ as } t\downarrow 0\\ &-\gamma\cdot(\text{the coefficient of } \log t \text{ in the expansion of } \mathcal{L}(A)(t)|_{\Delta_b} \text{ as } t\downarrow 0),\end{aligned}$$

where γ is Euler's constant.

Proof. By Lemma 7.4, $\mathcal{M}(A)(\tau) = \frac{1}{\Gamma(i\tau)}\int_0^\infty t^{i\tau-1}\mathcal{L}(A)(t)\,dt$ for $\operatorname{Im}\tau << 0$. Thus, by properties (35) and (36) of Theorem 5.2, it follows that for $\operatorname{Im}\tau << 0$,

$$\mathcal{M}(A)(\tau)|_{\Delta_b} = \frac{1}{\Gamma(i\tau)}\int_0^\infty t^{i\tau-1}(\mathcal{L}(A)(t)|_{\Delta_b})\,dt.$$

Now applying Lemma 7.11 to the asymptotic expansion of $\mathcal{L}(A)(t)|_{\Delta_b}$ given by (52) proves this theorem. $\square$

Corollary 7.13. *Let $A\in\Psi^m_{b,os}(X,\Omega_b^{\frac{1}{2}})$, $m\in\mathbb{R}^+$ be elliptic, formally self adjoint, positive, and Fredholm and let $A_0 = A+\pi$, where π is the orthogonal projection onto the null space of A. Then the Schwartz kernel of the complex power A_0^z, when restricted to Δ_b, extends from $\operatorname{Re}z < -n/m$ to be a meromorphic function on all of $\mathbb{C}$, with values in $S^{0,\delta}(X,\Omega_b)$ for some $\delta>0$, having only simple poles at the points*

$$\{z_k = \frac{k-n}{m} \,:\, k\in\mathbb{N}_0,\, z_k\neq 0\}.$$

The residue of $A_0^z|_{\Delta_b}$ at $z=z_k$, when $z_k\notin\mathbb{N}$, is $-\frac{\gamma_k}{\Gamma(-z_k)}$, and when $z_k\in\mathbb{N}$, the residue is $(-1)^{z_k}\Gamma(z_k+1)\,\gamma'_k$, where γ_k and γ'_k are given in the expansion (54). Moreover, the value of $A_0^z|_{\Delta_b}$ at $z=0$ is equal to the constant term in the expansion of $e^{-tA}|_{\Delta_b}$ as $t\downarrow 0$.

Proof. By Lemma 7.3, $e^{-tA_0} = (e^{-t}-1)\pi + e^{-tA}$. In particular, $e^{-tA_0}|_{\Delta_b}$ has an expansion of the same form as in (54). Now, if $B(\lambda) = \frac{i}{2\pi}(A_0-\lambda)^{-1}$, then as $A_0^z = \mathcal{M}(B)(iz)$ and $e^{-tA_0} = \mathcal{L}(B)(t)$, this corollary follows Theorem 7.12 applied to $\mathcal{M}(B)(\tau)$ and substituting in $\tau = iz$. $\square$

Since for a differential operator, the expansion (56) for $e^{-tA}|_{\Delta_b}$ has no log terms, Corollary 7.13 simplifies as follows.

Corollary 7.14. *Let $A\in\operatorname{Diff}^m_b(X,\Omega_b^{\frac{1}{2}})$, $m\in\mathbb{N}$ be elliptic, formally self adjoint, positive, and Fredholm and let $A_0 = A+\pi$, where π is the orthogonal projection onto the null space of A. Then the Schwartz kernel of the complex power A_0^z, when restricted to Δ_b, extends from $\operatorname{Re}z < -n/m$*

to be a meromorphic function on all of $\mathbb{C}$*, with values in* $S^{0,\delta}(X,\Omega_b)$ *for some* $\delta>0$*, having only simple poles at the points*

$$\{z_k=\frac{k-n}{m}\ :\ k\in\mathbb{N}_0\,,\ z_k\notin\mathbb{N}_0\,\}.$$

The residue of $A_0^z|_{\Delta_b}$ *at* $z=z_k$*, is* $-\frac{\gamma_k}{\Gamma(-z_k)}$*, where* γ_k *is given in the expansion (56). Moreover, the value of* $A_0^z|_{\Delta_b}$ *at* $z=0$ *is equal to the constant term in the expansion of* $e^{-tA}|_{\Delta_b}$ *as* $t\downarrow 0$*.*

Using the complex powers and the b-Trace, we generalize the zeta function for pseudodifferential operators on closed manifolds, originally studied by Seeley [27], to b-pseudodifferential operators on manifolds with corners. Let $A\in\Psi^m_{b,os}(X,\Omega_b^{\frac12})$, $m\in\mathbb{R}^+$ be elliptic, formally self adjoint and positive, and suppose that A is Fredholm. Then by Corollary 7.13, $A_0^{-z}|_{\Delta_b}$ is holomorphic for $\operatorname{Re} z>n/m$. In particular, for $\operatorname{Re} z>n/m$, ${}^b\mathrm{Tr}(A_0^{-z})={}^b\!\!\int A_0^{-z}|_{\Delta_b}$ is well defined.

Definition 7.15. Let $A\in\Psi^m_{b,os}(X,\Omega_b^{\frac12})$, $m\in\mathbb{R}^+$ be elliptic, formally self adjoint and positive, and suppose that A is Fredholm. Then the *b-zeta function*, ${}^b\zeta_A(z)$, defined for $z\in\mathbb{C}$ with $\operatorname{Re} z>n/m$, is the function

$${}^b\zeta_A(z):={}^b\mathrm{Tr}(A_0^{-z}).$$

Then Corollary 7.13 implies the following.

Theorem 7.16. *Let* $A\in\Psi^m_{b,os}(X,\Omega_b^{\frac12})$*,* $m\in\mathbb{R}^+$ *be elliptic, formally self adjoint and positive, and suppose that A is Fredholm. Then the b-zeta function,* ${}^b\zeta_A(z)$*, extends from* $\operatorname{Re} z>n/m$ *to be a meromorphic function on all of* $\mathbb{C}$*, having only simple poles at the points*

$$\{z_k=\frac{n-k}{m}\ :\ k\in\mathbb{N}_0\,,\ z_k\neq 0\,\}.$$

The residue of ${}^b\zeta_A(z)$ *at* $z=z_k$*, when* $z_k\notin\mathbb{N}$*, is* $\frac{{}^b\!\int\gamma_k}{\Gamma(-z_k)}$*, and when* $z_k\in\mathbb{N}$*, the residue is* $(-1)^{z_k+1}\Gamma(z_k+1)\,{}^b\!\!\int\gamma_k'$*, where* γ_k *and* γ_k' *are given in the expansion (54). Moreover, the value of* ${}^b\zeta_A(z)$ *at* $z=0$ *is the same as the constant term in expansion of* ${}^b\mathrm{Tr}(e^{-tA})$ *as* $t\downarrow 0$*.*

8. APPLICATIONS TO INDEX THEORY

In [17], the following theorem is proved.

Theorem 8.1. *Let* $A\in\Psi^m_{b,os}(X,\Omega_b^{\frac12})$*,* $m\in\mathbb{R}$*. Then the following are equivalent:*

(1) *A is Fredholm as a map from* $H_b^m(X,\Omega_b^{\frac12})$ *into* $L_b^2(X,\Omega_b^{\frac12})$*.*
(2) *A is elliptic and for each* $M\in M'(X)$*,* $N_M(A)(\tau)$ *is invertible from* $H_b^m(M,\Omega_b^{\frac12})$ *into* $L_b^2(M,\Omega_b^{\frac12})$ *for all real* τ*.*
(3) *A is elliptic and for each* $H\in M_1(X)$*,* $N_H(A)(\tau)$ *is invertible from* $H_b^m(H,\Omega_b^{\frac12})$ *into* $L_b^2(H,\Omega_b^{\frac12})$ *for all real* τ*.*

Let $A\in\Psi^m_{b,os}(X,\Omega_b^{\frac12})$, $m\in\mathbb{R}^+$ be Fredholm. We will find formulas for the index of A in terms of the local invariants of the heat kernel and zeta function expansions. Later in Theorem 8.11, we specialize to the case when A is a Dirac operator.

Definition 8.2. The *b-eta invariant*, ${}^b\eta_A$, is defined to be the constant term in the expansion, as $t\downarrow 0$, of

$$(78)\qquad {}^b\eta_A(t)=\sum_{M\in M'(X)}{}^b\eta_M(t),$$

where for $M\in M_k(X)$, if $N_M(A)(\tau)$ is the normal operator of A at M as defined in (9), then

$$(79)\qquad \begin{aligned}{}^b\eta_M(t)&=\frac{2}{(2\pi)^k}\int_t^\infty\!\!\int_{\mathbb{R}^k}{}^b\mathrm{Tr}(\,\mathbf{D}_\tau^k N_M(A)(\tau)\,N_M(A^*)(\tau)\,N_M(e^{-sAA^*})(\tau)\,)\,d\tau ds\\ &=-\frac{2}{(2\pi)^k}\int_{\mathbb{R}^k}{}^b\mathrm{Tr}(\,\mathbf{D}_\tau^k N_M(A)(\tau)\,N_M(A)(\tau)^{-1}\,N_M(e^{-tAA^*})(\tau)\,)\,d\tau,\end{aligned}$$

where $\mathbf{D}_\tau^k=D_{\tau_1}\cdots D_{\tau_k}$ with $D_{\tau_j}=\frac{1}{i}\partial_{\tau_j}$.

Remark 8.3. The properties of normal operators, cf. [17], imply that the integrals defining each ${}^b\eta_M(t)$ are well-defined; and also imply the equivalence of the two forms given for ${}^b\eta_M(t)$, the equivalence following from the fact that

$$\begin{aligned}
&\int_t^\infty {}^b\mathrm{Tr}(\,\mathbf{D}_\tau^k N_M(A)(\tau)\, N_M(A^*)(\tau)\, N_M(e^{-sAA^*})(\tau)\,)\, ds \\
&\quad = -\int_t^\infty {}^b\mathrm{Tr}(\,\mathbf{D}_\tau^k N_M(A)(\tau)\, N_M(A)(\tau)^{-1}\, \partial_s N_M(e^{-sAA^*})(\tau)\,)\, ds \\
&\quad = -{}^b\mathrm{Tr}(\,\mathbf{D}_\tau^k N_M(A)(\tau)\, N_M(A)(\tau)^{-1}\, [\, N_M(e^{-sAA^*})(\tau)\,]_{s=t}^{s=\infty}\,) \\
&\quad = {}^b\mathrm{Tr}(\,\mathbf{D}_\tau^k N_M(A)(\tau)\, N_M(A)(\tau)^{-1}\, N_M(e^{-tAA^*})(\tau)\,),
\end{aligned}$$

where we used that $N_M(e^{-sAA^*})(\tau) = e^{-sN_M(A)(\tau)N_M(A)(\tau)^*}$ vanishes as $s \to \infty$ since $N_M(A)(\tau)$ is invertible for all real τ by assumption. In [16, Lemma 7.4], it is proved that for any $M \in M_k(X)$, $k \in \mathbb{N}$, as $t \downarrow 0$, we have

$$(80)\qquad {}^b\eta_M(t) \sim \sum_{j=0}^\infty t^{\frac{j-k-n}{2m}}\,\eta_j + \sum_{j,\,\frac{j-k-n}{2m}\in\mathbb{N}_0} t^{\frac{j-k-n}{2m}}\,\log t\,\eta_j' + \sum_{j=0}^\infty t^j\eta_j'',\ \ \eta_j,\, \eta_j',\, \eta_j'' \in \mathbb{C}.$$

Thus, ${}^b\eta_A$ is well defined. Moreover, in [17] it is shown that if A is a Dirac operator on a manifold with corners, then our definition of the (b-)eta invariant is the same as the original definition given in [3]; see also Lemma 8.10.

Lemma 8.4. *Let $A \in \Psi_{b,os}^m(X, \Omega_b^{\frac12})$, $m \in \mathbb{R}^+$ be Fredholm. Then for any $t > 0$,*

$$(81)\qquad \operatorname{ind} A = h(t) - \frac{1}{2}\,{}^b\eta_A(t),$$

where $h(t) = {}^b\mathrm{Tr}(e^{-tA^*A}) - {}^b\mathrm{Tr}(e^{-tAA^*})$ and where ${}^b\eta_A(t)$ is given in (78).

Proof. For the proof of this lemma, we refer the reader to [17]. We remark that in [17], (81) is proved under the assumption that A is of the form $\eth + R$, where $\eth$ is a Dirac operator and where R is a b-pseudodifferential operator of order $-\infty$. But the proof only used the fact that $\eth + R$ was a Fredholm b-pseudodifferential operator and nothing more. ☐

Taking $t \downarrow 0$ in (81) and using Theorem 6.4 proves the following theorem.

Theorem 8.5. *Let $A \in \Psi_{b,os}^m(X, \Omega_b^{\frac12})$, $m \in \mathbb{R}^+$ be Fredholm. Then,*

$$\operatorname{ind} A = {}^b\zeta_{A^*A} - {}^b\zeta_{AA^*} - \frac{1}{2}\,{}^b\eta_A,$$

where ${}^b\zeta_{A^*A}$ and ${}^b\zeta_{AA^*}$ are the constant terms in the expansions, as $t \downarrow 0$, of ${}^b\mathrm{Tr}(e^{-tA^*A})$ and ${}^b\mathrm{Tr}(e^{-tAA^*})$ respectively.

Corollary 8.6. *Let $A \in \mathrm{Diff}_b^m(X, \Omega_b^{\frac12})$, $m \in \mathbb{N}$ be Fredholm. Suppose that X is odd dimensional. Then,* $\operatorname{ind} A = -\frac12\,{}^b\eta_A$.

Proof. By Corollary 6.21, if n is *odd*, then ${}^b\mathrm{Tr}(e^{-tA^*A})$ and ${}^b\mathrm{Tr}(e^{-tAA^*})$ have no constant terms in their expansions as $t \downarrow 0$. ☐

We will now find a formula for the index of A in terms of zeta functions. Define $(A^*A)_0 := A^*A + \pi_A$, where π_A is the orthogonal projection onto the null space of A and define $(AA^*)_0 := AA^* + \pi_{A^*}$, where π_{A^*} is the orthogonal projection onto the null space of A^* . Then by Lemma 7.3,

$$(82)\qquad e^{-t(A^*A)_0} = (e^{-t} - 1)\pi_A + e^{-tA^*A} \text{ and } e^{-t(AA^*)_0} = (e^{-t} - 1)\pi_{A^*} + e^{-tAA^*}.$$

For each $t > 0$, define $h_0(t) = {}^b\mathrm{Tr}(e^{-t(A^*A)_0}) - {}^b\mathrm{Tr}(e^{-t(AA^*)_0})$. Then, the equations in (82) imply that $h_0(t) = e^{-t}\operatorname{ind} A - \operatorname{ind} A + h(t)$ where $h(t) = {}^b\mathrm{Tr}(e^{-tA^*A}) - {}^b\mathrm{Tr}(e^{-tAA^*})$; and hence, by (81), for all $t > 0$,

$$(83)\qquad e^{-t}\operatorname{ind} A = h_0(t) - \frac{1}{2}\,{}^b\eta_A(t).$$

Multiplying both sides of equation (83) by $t^{z-1}/\Gamma(z)$ and integrating the resulting equation from $t=0$ to $t=\infty$ yield

$$\operatorname{ind} A = \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}h_0(t)dt - \frac{1}{2}\cdot\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\,{}^b\eta_A(t)dt.$$

We will work out the right-hand side of this equation. Observe that properties (35) and (36) of Theorem 5.2 justify the following manipulations for $\operatorname{Re} z >> 0$:

$$\begin{aligned}\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}{}^b\mathrm{Tr}(\,e^{-t(A^*A)_0}\,)\,dt &= \frac{1}{\Gamma(z)}\int_0^\infty {}^b\!\!\int t^{z-1}e^{-t(A^*A)_0}|_{\Delta_b}\,dt\\ &= {}^b\!\!\int \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}e^{-t(A^*A)_0}|_{\Delta_b}\,dt\\ &= {}^b\!\!\int (A^*A)_0^{-z}|_{\Delta_b}\\ &= {}^b\zeta_{A^*A}(z),\end{aligned}$$

where we used Lemma 7.4. Similarly, $\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}{}^b\mathrm{Tr}(\,e^{-t(AA^*)_0}\,)\,dt = {}^b\zeta_{AA^*}(z)$. Since for any $M\in M_k(X)$ and $\tau\in\mathbb{R}^k$, $N_M(e^{-tAA^*})(\tau) = e^{-tN_M(AA^*)(\tau)}$ and by Lemma 7.4, $N_M(AA^*)(\tau)^{-z} = \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}e^{-tN_M(AA^*)(\tau)}dt$, the properties of normal operators, cf. [17], justify the following manipulations for $\operatorname{Re} z >> 0$:

$$\begin{aligned}&\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\int_{\mathbb{R}^k}{}^b\mathrm{Tr}(\,\mathbf{D}^k_\tau N_M(A)(\tau)\,N_M(A)(\tau)^{-1}\,e^{-tN_M(AA^*)(\tau)}\,)\,d\tau\,dt\\ &= \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\int_{\mathbb{R}^k}{}^b\!\!\int_M \{\mathbf{D}^k_\tau N_M(A)(\tau)\,N_M(A)(\tau)^{-1}e^{-tN_M(AA^*)(\tau)}\}|_{\Delta_b}d\tau\,dt\\ &= \int_{\mathbb{R}^k}{}^b\!\!\int_M \{\mathbf{D}^k_\tau N_M(A)(\tau)\,N_M(A)(\tau)^{-1}(\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}e^{-tN_M(AA^*)(\tau)}dt)\}|_{\Delta_b}d\tau\\ &= \int_{\mathbb{R}^k}{}^b\!\!\int_M \{\,\mathbf{D}^k_\tau N_M(A)(\tau)\,N_M(A)(\tau)^{-1}\,N_M(AA^*)(\tau)^{-z}\,\}|_{\Delta_b}\,d\tau\\ &= \int_{\mathbb{R}^k}{}^b\mathrm{Tr}(\,\mathbf{D}^k_\tau N_M(A)(\tau)\,N_M(A)(\tau)^{-1}\,N_M(AA^*)(\tau)^{-z}\,)\,d\tau\\ &= \int_{\mathbb{R}^k}{}^b\mathrm{Tr}(\,\mathbf{D}^k_\tau N_M(A)(\tau)\,N_M(A^*)(\tau)\,N_M(AA^*)(\tau)^{-z-1}\,)\,d\tau,\end{aligned}$$

where we used that $N_M(A)(\tau)^{-1} = N_M(A^*)(\tau)N_M(AA^*)(\tau)^{-1}$. Hence,

$$\operatorname{ind} A = {}^b\zeta_{A^*A}(z) - {}^b\zeta_{AA^*}(z) - \frac{1}{2}{}^b\eta_A(z), \tag{84}$$

where ${}^b\eta_A(z) = \sum_{M\in M'(X)}{}^b\eta_M(z)$, where if $M\in M_k(X)$,

$$\begin{aligned}{}^b\eta_M(z) &= -\frac{2}{(2\pi)^k}\int_{\mathbb{R}^k}{}^b\mathrm{Tr}(\mathbf{D}^k_\tau N_M(A)(\tau)\,N_M(A)(\tau)^{-1}\,N_M(AA^*)(\tau)^{-z})\,d\tau\\ &= -\frac{2}{(2\pi)^k}\int_{\mathbb{R}^k}{}^b\mathrm{Tr}(\mathbf{D}^k_\tau N_M(A)(\tau)\,N_M(A^*)(\tau)\,N_M(AA^*)(\tau)^{-z-1})\,d\tau.\end{aligned} \tag{85}$$

Definition 8.7. The function ${}^b\eta_A(z) = \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\,{}^b\eta_A(t)\,dt$ is called the *b-eta function.*

Lemma 8.8. *For $M\in M_k(X)$, ${}^b\eta_M(z)$ extends to be a meromorphic function on all of $\mathbb{C}$, having only simple poles at the points $\{z_j = \frac{n-k-j}{2m} : j\in\mathbb{N}_0\}$. In particular, ${}^b\eta_A(z) = \sum_{M\in M'(X)}{}^b\eta_M(z)$ extends to be a meromorphic function on all of $\mathbb{C}$, having only simple poles at the points $\{z_j = \frac{n-j}{2m} : j\in\mathbb{N}_0\}$. Moreover, ${}^b\eta_A(z)$ is holomorphic at $z=0$, with value ${}^b\eta_A$, where ${}^b\eta_A$ is the b-eta invariant.*

Proof. Since ${}^b\eta_M(z) = \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}{}^b\eta_M(t)dt$, the asymptotic expansion (80) together with the results of [6, Section 4.3], imply that ${}^b\eta_M(z)$ extends to be a meromorphic function on all of $\mathbb{C}$, having only simple poles at the points $\{z_j = \frac{n-k-j}{2m} : j\in\mathbb{N}_0\}$. By (55) of Corollary 6.4, the right hand side of the equation $\frac{1}{2}{}^b\eta_A(t) = \operatorname{ind} A - h(t)$ has an expansion as $t\downarrow 0$ with *no* $\log t$ term. On the other hand, by definition of ${}^b\eta_A$, the expansion of the left hand side has constant term equal to $\frac{1}{2}\,{}^b\eta_A$. Hence, by Lemma 7.11, ${}^b\eta_A(z) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1}\, {}^b\eta_A(t)\, dt$ is holomorphic at $z = 0$ with value ${}^b\eta_A$. $\square$

Formula (84) yields the following version of the index theorem in terms of zeta functions. It is a generalization of the index theorem of Piazza [25] for manifolds with boundary.

Theorem 8.9. *Let $A \in \Psi^m_{b,os}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be Fredholm. Then,*

$$\operatorname{ind} A = {}^b\zeta_{A^*A}(z) - {}^b\zeta_{AA^*}(z) - \frac{1}{2}{}^b\eta_A(z),$$

where ${}^b\zeta_{A^*A}(z)$ and ${}^b\zeta_{AA^*}(z)$ are the b-zeta functions of A^*A and AA^* respectively, and where ${}^b\eta_A(z) = \sum_{M \in M'(X)} {}^b\eta_M(z)$, where ${}^b\eta_M(z)$ is given in (85), is the b-eta function.

Note that Corollary 7.13 implies that the values of ${}^b\zeta_{A^*A}(z)$ and ${}^b\zeta_{AA^*}(z)$ at $z = 0$ are ${}^b\zeta_{A^*A}$ and ${}^b\zeta_{AA^*}$ respectively, where ${}^b\zeta_{A^*A}$ and ${}^b\zeta_{AA^*}$ are the constant terms in the expansions, as $t \downarrow 0$, of ${}^b\mathrm{Tr}(e^{-tA^*A})$ and ${}^b\mathrm{Tr}(e^{-tAA^*})$ respectively. In particular, at $z = 0$, we recover the index formula of Theorem 8.5.

We remark that although Theorem 8.5 and Theorem 8.9 were proved only for the vector bundle of b-half densities (to simplify notation in the analysis), they also hold for b-pseudodifferential operators acting on sections of Hermitian vector bundles. We will apply these theorems to Dirac operators. Assume that X is an even-dimensional compact manifold with corners and, using notation from Section 2, let $\{\rho_i\}_{i=1}^r$ be the fixed boundary defining functions for X. Then a Riemannian metric g on the interior of X is said to be an *exact b-metric* (with respect to $\{\rho_i\}$) if g is of the form

$$g = \sum_{i=1}^r (\frac{d\rho_i}{\rho_i})^2 + g', \tag{86}$$

where $g' \in C^\infty(X, T^*X \otimes T^*X)$. Henceforth, we will assume that an exact b-metric g is fixed on X.

Let $E = E^+ \oplus E^-$ be a $\mathbb{Z}_2$-graded, Hermitian, Clifford module over X associated to the exact b-metric g, cf. [22], [4]; and let $\eth \in \mathrm{Diff}_b^1(X, E)$ be the (generalized) Dirac operator associated to E and our fixed exact b-metric. Denote the positive part of the Dirac operator by $\eth^+ : C^\infty(X, E^+) \to C^\infty(X, E^-)$. Then for each $H \in M_1(X)$, the normal operator of $\eth^+$ satisfies (cf. [22], [17])

$$N_H(\eth^+)(\tau) = \frac{1}{i}\sigma_H\{i\tau + \eth_H\}, \tag{87}$$

where $\eth_H$ is the induced Dirac operator on $E_H := E^+|_H$, and where σ_H is multiplication by the Clifford action in the normal variable to H. Since $\eth_H$ is formally self-adjoint, $i\tau + \eth_H$ is invertible for all $\tau \in \mathbb{R} \setminus \{0\}$. Thus, by Theorem 8.1,

$$\eth^+ : H_b^1(X, E^+) \to L_b^2(X, E^-)$$

is Fredholm iff $\eth_H : H_b^1(H, E_H) \to L_b^2(H, E_H)$ is invertible for each $H \in M_1(X)$.

In [17], the function $\eta_{\eth^+}(t)$ given in (78) for $A = \eth^+$ is worked out.

Lemma 8.10. *Suppose that $\eth^+ \in \mathrm{Diff}_b^1(X, E^+, E^-)$ is Fredholm, and let $\eta_{\eth^+}(t)$ be the function given in (78) for $A = \eth^+$. Then, ${}^b\eta_{\eth^+}(t) = \sum_{H \in M_1(X)} {}^b\eta_H(t)$, where*

$${}^b\eta_H(t) = \frac{1}{\sqrt{\pi}} \int_t^\infty s^{-1/2}\, {}^b\mathrm{Tr}(\, \eth_H e^{-s\eth_H^2}\,)\, ds. \tag{88}$$

Theorem 8.11. *Suppose that $\eth^+ \in \mathrm{Diff}_b^1(X, E^+, E^-)$ is Fredholm. Then,*

$$\operatorname{ind} \eth^+ = {}^b\!\!\int_X \mathrm{AS} - \frac{1}{2} \sum_{H \in M_1(X)} {}^b\eta_H, \tag{89}$$

where AS *is the Atiyah-Singer characteristic class associated to E, and where*

$${}^b\eta_H = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2}\, {}^b\mathrm{Tr}(\, \eth_H e^{-t\eth_H^2}\,)\, dt. \tag{90}$$

Proof. The same formal arguments of [22, Chapter 8] used to prove the local index theorem in the manifold with boundary case (the fact that ${}^b\zeta_{\eth^-\eth^+} - {}^b\zeta_{\eth^+\eth^-} = {}^b\!\int_X \mathrm{AS}$, where $\eth^- = (\eth^+)^*$) can be carried over to the manifolds with corners case without any essential changes. In fact, the rescaling construction given in [22, Chapter 8] was presented in such a way so that its adaptation to b-geometry would be automatic. Also, the same argument used to prove [22, Theorem 8.36], see also [5, Theorem 2.4], shows that $s^{-1/2}\,{}^b\mathrm{Tr}(\,\eth_H e^{-s\eth_H^2}\,) = O(s^{1/2})$ near $s = 0$. Thus, the function ${}^b\eta_H(t)$ in (88) is well-defined at $t = 0$ and is given by (90). Thus, Theorem 8.5 yields formula (89). $\square$

Assume that $\eth^+ \in \mathrm{Diff}_b^1(X, E^+, E^-)$ is Fredholm. Consider now Theorem 8.9 for Dirac operators. We need to analyze ${}^b\eta_{\eth^+}(z) = \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\,{}^b\eta_{\eth^+}(t)\,dt$. Observe that for $\operatorname{Re} z >> 0$ and $H \in M_1(X)$,

$$
\begin{aligned}
\frac{\sqrt{\pi}}{\Gamma(z)}\int_0^\infty t^{z-1}\,{}^b\eta_H(t)\,dt &= \frac{1}{\Gamma(z)}\int_0^\infty\int_t^\infty t^{z-1}s^{-1/2}\,{}^b\mathrm{Tr}(\,\eth_H e^{-s\eth_H^2}\,)\,ds\,dt\\
&= \frac{1}{\Gamma(z)}\int_0^\infty\int_0^s t^{z-1}s^{-1/2}\,{}^b\mathrm{Tr}(\,\eth_H e^{-s\eth_H^2}\,)\,dt\,ds\\
&= \frac{1}{\Gamma(z)}\int_0^\infty\Big(\int_0^s t^{z-1}\,dt\Big)\,s^{-1/2}\,{}^b\mathrm{Tr}(\,\eth_H e^{-s\eth_H^2}\,)\,ds\\
&= \frac{1}{z\Gamma(z)}\int_0^\infty s^{z-1/2}\,{}^b\mathrm{Tr}(\,\eth_H e^{-s\eth_H^2}\,)\,ds\\
&= \frac{\Gamma(z+1/2)}{\Gamma(z+1)}\cdot\frac{1}{\Gamma(z+1/2)}\int_0^\infty s^{(z+1/2)-1}\,{}^b\mathrm{Tr}(\,\eth_H e^{-s\eth_H^2}\,)\,ds\\
&= \frac{\Gamma(z+1/2)}{\Gamma(z+1)}\,{}^b\mathrm{Tr}(\,\eth_H\,(\eth_H^2)^{-z-1/2}\,)\\
&= \frac{\Gamma(z+1/2)}{\Gamma(z+1)}\,{}^b\mathrm{Tr}(\,\operatorname{sign}\eth_H\cdot|\eth_H|^{-2z}\,),
\end{aligned}
$$

where we used Lemma 7.4, and where we set $|\eth_H| := (\eth_H^2)^{1/2}$ and $\operatorname{sign}\eth_H := \eth_H\cdot|\eth_H|^{-1}$. Thus, we have proved the following.

Theorem 8.12. *Let* $\eth^+ \in \mathrm{Diff}_b^1(X, E^+, E^-)$ *be Fredholm. Then,*

$$
\operatorname{ind}\eth^+ = {}^b\zeta_{\eth^-\eth^+}(z) - {}^b\zeta_{\eth^+\eth^-}(z) - \frac{1}{2}\sum_{H\in M_1(X)}{}^b\eta_H(z),
$$

where $\eth^- = (\eth^+)^*$ *and where* ${}^b\zeta_{\eth^-\eth^+}(z)$ *and* ${}^b\zeta_{\eth^+\eth^-}(z)$ *are the b-zeta functions of* $\eth^-\eth^+$ *and* $\eth^+\eth^-$ *respectively, and where*

$$
{}^b\eta_H(z) = \frac{\Gamma(z+1/2)}{\sqrt{\pi}\,\Gamma(z+1)}\,{}^b\mathrm{Tr}(\,\operatorname{sign}\eth_H\cdot|\eth_H|^{-2z}\,).
$$

The following corollary is (modulo slightly different constants) the original APS index theorem found in [3].

Corollary 8.13. *Let* $\eth^+ \in \mathrm{Diff}_b^1(X, E^+, E^-)$ *be Fredholm. Suppose that* $\operatorname{codim} X = 1$. *Then,*

$$
\operatorname{ind}\eth^+ = {}^b\zeta_{\eth^-\eth^+}(z) - {}^b\zeta_{\eth^+\eth^-}(z) - \frac{1}{2}\sum_{H\in M_1(X)}{}^b\eta_H(z),
$$

where $\eth^- = (\eth^+)^*$ *and where* ${}^b\zeta_{\eth^-\eth^+}(z)$ *and* ${}^b\zeta_{\eth^+\eth^-}(z)$ *are the b-zeta functions of* $\eth^-\eth^+$ *and* $\eth^+\eth^-$ *respectively, and where*

$$
{}^b\eta_H(z) = \frac{\Gamma(z+1/2)}{\sqrt{\pi}\,\Gamma(z+1)}\sum_j(\operatorname{sign}\lambda_j)\,|\lambda_j|^{-2z},
$$

where $\{\lambda_j\}$ *are the eigenvalues of* $\eth_H$.

References

1. S. Agmon, *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure Appl. Math. **15** (1962), 119–147.
2. S. Agmon and Y. Kannai, *On the asymptotic behavior of spectral functions and resolvent kernels of elliptic operators*, Israel J. Math. **5** (1967), 1–30.
3. M.F. Atiyah, V.K. Patodi, and I.M. Singer, *Spectral asymmetry and Riemannian geometry I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.
4. N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Springer-Verlag, New York, 1992.
5. J.-M. Bismut and D.S. Freed, *The analysis of elliptic families. II. Dirac operators, eta invariants, and the holonomy theorem*, Comm. Math. Phys. **107** (1986), no. 1, 103–163.
6. N. Bleistein and R. A. Handelsman, *Asymptotic expansions of integrals*, Dover, New York, 1986.
7. J. Conway, *Functions of One Complex Variable*, Springer-Verlag, New York, 1978.
8. P. Greiner, *An asymptotic expansion for the heat equation*, Arch. Rational Mech. Anal. **41** (1971), 163–218.
9. G. Grubb, *Functional calculus of pseudodifferential boundary problems*, second ed., Progress in Math., Birkhäuser, Boston, 1996.
10. G. Grubb and R.T. Seeley, *Weakly parametric pseudodifferential operators and Atiyah-Patoti-Singer operators*, Invent. Math. **121** (1995), 481–529.
11. ———, *Zeta and eta functions for Atiyah-Patodi-Singer operators*, J. Geom. Anal. **6** (1996), no. 1, 31–77.
12. M. Kac, *Can one hear the shape of a drum?*, Amer. Math. Monthly **73** (1966), 1–23.
13. R. Lauter, *On representations of Ψ^*-algebras and C^*-algebras of b-pseudo-differential operators on manifolds with corners*, J. Math. Sci. (New York) **98** (2000), no. 6, 684–705.
14. M. Lesch, *On the noncommutative residue for pseudodifferential operators with log-poly-homogeneous symbols*, Ann. Global Anal. Geom. **17** (1999), no. 2, 151–187.
15. P. Loya, *The structure of the resolvent of elliptic pseudodifferential operators*, to appear in J. Funct. Anal.
16. ———, *On the b-pseudodifferential calculus on manifolds with corners*, Ph.D. thesis, MIT, 1998.
17. P. Loya and R.B. Melrose, *The index of perturbed Dirac operators on manifolds with corners*, in preparation.
18. R. Mazzeo, *Elliptic theory of differential edge operators. I*, Comm. Partial Differential Equations **16** (1991), no. 10, 1615–1664.
19. H.P. McKean and I.M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Diff. Geometry **1** (1967), 43–69.
20. R.B. Melrose, *Differential Analysis on Manifolds with Corners*, in preparation.
21. ———, *Calculus of conormal distributions on manifolds with corners*, International Math. Research Notices **3** (1992), 51–61.
22. ———, *The Atiyah-Patodi-Singer Index Theorem*, A.K. Peters, Wellesley, 1993.
23. R.B. Melrose and P. Piazza, *Analytic K-theory on manifolds with corners*, Adv. Math. **92** (1992), no. 1, 1–26.
24. S. Minakshisundaram, *Eigenfunctions on Riemannian Manifolds*, J. Indian Math. Soc. **17** (1953), 158–165.
25. P. Piazza, *On the index of elliptic operators on manifolds with boundary*, J. of Func. Anal. **117** (1993), 308–359.
26. Elmar Schrohe, *Complex powers of elliptic pseudodifferential operators*, Integral Equations Operator Theory **9** (1986), no. 3, 337–354.
27. R.T. Seeley, *Complex powers of an elliptic operator*, A.M.S. Symp. Pure Math. **10** (1967), 288–307.
28. ———, *Analytic extension of the trace associated with elliptic boundary problems*, Amer. J. Math. **91** (1969), 963–983.
29. ———, *The resolvent of an elliptic boundary problem*, Amer. J. Math. **91** (1969), 889–920.
30. M. A. Shubin, *Pseudodifferential operators and spectral theory*, Springer-Verlag, Berlin, 1987, Translated by Stig I. Andersson.
31. M. Wodzicki, *Noncommutative residue. I. Fundamentals*, K-theory, arithmetic and geometry (Moscow, 1984–1986), Springer, Berlin, 1987, pp. 320–399.

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