COMPLEX POWERS OF DIFFERENTIAL OPERATORS ON MANIFOLDS WITH CONICAL SINGULARITIES

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ABSTRACT. We construct the complex powers A_z for an elliptic cone (or Fuchs type) differential operator A on a manifold with boundary. We show that A_z exists as an entire family of b-pseudodifferential operators. We also examine the analytic structure of the Schwartz kernel of A_z, both on and off the diagonal. Finally, we study the meromorphic behavior of the zeta function $\mathrm{Tr}(A_z)$.

1. INTRODUCTION

The purpose of this paper is to construct the complex powers of an elliptic cone differential operator on a compact manifold X with (connected) boundary. Let x denote a fixed boundary defining function for ∂X. A cone differential operator of order m is an operator $A \in x^{-m}\mathrm{Diff}_b^m(X)$, where $\mathrm{Diff}_b^m(X)$ is the space of b-differential operators of order m. Thus, A is a usual differential operator of order m on the interior of X such that in any collar neighborhood $X \cong [0,\varepsilon)_x \times \partial X$ of the boundary, we can write

$$A = x^{-m} \sum_{k=0}^{m} A_{m-k}(x)(x\partial_x)^k, \tag{1.1}$$

where $A_{m-k}(x)$ are differential operators on ∂X of order $m-k$ depending smoothly on x. The primary examples of such operators are geometric operators, for instance the Laplace-Beltrami operator, associated to a conic metric.

If $A \in x^{-m}\mathrm{Diff}_b^m(X)$ is "fully elliptic" with respect to a closed angle $\Lambda \subset \mathbb{C}$ and to a number $\alpha \in \mathbb{R}$ (see Definition 3.1), Gil [4] shows that

$$A - \lambda \colon x^{\alpha} H_b^s(X) \longrightarrow x^{\alpha-m} H_b^{s-m}(X) \quad \text{for any } s \in \mathbb{R},$$

is invertible for all $\lambda \in \Lambda$ sufficiently large. Here, $H_b^s(X)$ is the b-Sobolev space of order s (see Appendix). We remark that this full-ellipticity condition comprises a general class of operators, including self-adjoint ones. If Λ is the closure of the complement of an angle in the right-half plane, and if $(A-\lambda)^{-1}$ exists on a neighborhood of Λ, then, cf. [3], the complex power A_z for $\mathrm{Re}\, z < 0$ exists as a map

$$A_z \colon x^{\alpha-m} H_b^s(X) \longrightarrow x^{\alpha-m} H_b^s(X) \quad \text{for any } s \in \mathbb{R},\ \mathrm{Re}\, z < 0. \tag{1.2}$$

In Theorem 1.2 below, we describe the extension of A_z from $\mathrm{Re}\, z < 0$ as an entire family of b-pseudodifferential operators. To simplify exposition, we state our result for operators acting on functions; however, every result in this paper has a corresponding statement for cone operators acting on sections of a vector bundle.

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We remark that the study of complex powers originated with Seeley's famous paper [21], and his methods have been used to construct complex powers for operators on certain types of non-compact manifolds [19], manifolds with boundary [5, 16, 17], and manifolds with corners [11]. Complex powers can also be constructed by other methods; see [6, 2, 23], and for general types of pseudodifferential operators [7, 18]. In Seeley's original paper [21], his principal result was that the Schwartz kernel $K_z(p,q)$ of the complex powers extends to be entire on the whole complex plane for $(p,q) \in X^2$ away from the diagonal, and when restricted to the diagonal, it extends to be meromorphic with simple poles on the real axis. We prove a similar result for the conic case in Theorem 1.1 below. Recently, Coriasco, Schrohe, and Seiler [3] analyzed the boundedness properties of the imaginary powers of cone operators. Their methods are quite distinct from ours, as they rely on Schulze's cone calculus [20, 4], while the methods featured here are the geometric "blow-up" techniques of Melrose. See [8, 9] for a comparison of the b-calculus and Schulze's calculus. Finally, we mention that our analysis of the complex powers is based on the heat kernel. This is why we assume that Λ is given by the closure of the complement of an angle lying in the right-half plane. The analysis can also be carried out using the resolvent in [12], in which case, Λ can be any sector of the complex plane.

There is a peculiarity present in the Schwartz kernel of A_z due to the colliding of exponents in asymptotic expansions. To orient the reader to this phenomenon, before describing our main results we look at an example. Fix $\varepsilon > 0$, and let

$$u(z,x) \;=\; \frac{x^{-z+\varepsilon}}{z} - \frac{x^{\varepsilon}}{z} \;=\; x^{-z}\Big\{\frac{x^{\varepsilon} - x^{z+\varepsilon}}{z}\Big\}. \tag{1.3}$$

We can consider $u(z,x)$ as being meromorphic or entire in z depending on where we view $u(z,x)$ as residing. As an element of $x^{-z+\varepsilon}C^\infty([0,1)_x) + x^{\varepsilon}C^\infty([0,1)_x)$, we say that $u(z,x)$ is meromorphic because in the middle expression of (1.3), $1/z$ is meromorphic with values in $C^\infty([0,1)_x)$ as a constant function. Note that expressing $u(z,x)$ as the far right function in (1.3), it is clear that $u(0,x)$ can still be defined, and is given by $u(0,x) = -x^{\varepsilon}\log x$. We can also consider $u(z,x)$ as an entire function as follows. We denote by $S^{0,0}([0,1))$, the space of continuous functions $f(x)$ on $[0,1)$ such that for any k, $(x\partial_x)^k f(x)$ is also continuous on $[0,1)$. (One can define the space $S^{0,0}(X)$ on X similarly.) Then, note that the middle expression of (1.3) implies that $u(z,x) \in S^{0,0}([0,1))$ for $\operatorname{Re} z < 0$. The far right expression of (1.3), plus the fact that $u(0,x) = -x^{\varepsilon}\log x \in S^{0,0}([0,1))$, imply that $u(z,x) \in x^{-z}S^{0,0}([0,1))$ for $\operatorname{Re} z \geq 0$. Now for each $z \in \mathbb{C}$, define $S_z := S^{0,0}([0,1))$ if $\operatorname{Re} z < 0$; and, $S_z := x^{-z}S^{0,0}([0,1))$ if $\operatorname{Re} z \geq 0$. Note that $u(z,x) \in S_z$ for each $z \in \mathbb{C}$, and $S_z \subset S_w$ if $\operatorname{Re} z < \operatorname{Re} w$. Then one can check that $u(z,x)$ is *entire* as a map into S_z in the sense that for each $r \in \mathbb{R}$, the map

$$\{z \in \mathbb{C} \,:\, \operatorname{Re} z < r\} \ni z \mapsto u(z,x) \in S_z \subset S_r,$$

is holomorphic as a map into S_r. Note that viewing $u(z,x)$ as either meromorphic or entire is caused by the exponents, $-z+\varepsilon$ and ε, colliding at $z = 0$ in the middle expression of (1.3). If the exponents never coincided, then we could only consider $u(z,x)$ as being meromorphic.

With this example in mind, we are now ready to describe our main results. We begin by describing the Schwartz kernel of the complex powers. See the appendix for detailed explanations of the notations that follow. If E is an index set, we denote by $\mathcal{A}^E_{phg}(X)$ the space of smooth functions on the interior of X that are of "asymptotic type" E at ∂X; that is, that can be expanded at ∂X in terms of powers of x and powers of logs of x, where E determines the powers that can occur. Given an index family $\mathcal{E} = (E_{\mathrm{lb}}, E_{\mathrm{rb}})$, we denote by $\mathcal{A}^{\mathcal{E}}_{phg}(X^2)$ the space of smooth functions on the interior of X^2 that are of asymptotic type E_{lb} at the left boundary, $\partial X \times X$, and of asymptotic type E_{rb} at the right boundary, $X \times \partial X$. By fixing a smooth b-measure on X, that is, a density of the form $x^{-1}\times$ a smooth positive density on X, we will omit the density factor inherent in the Schwartz kernel K_z of the complex power A_z. Let $n = \dim X$. We assume that $\alpha \in \mathbb{R}$ is fixed such that complex power A_z in (1.2) exists. We remark that "full ellipticity" relies on the value of the weight parameter α. The explicit dependence of the complex powers on the weight parameter is reflected in certain index sets $\mathcal{F}(\alpha)$ and $E(\alpha)$ appearing in Theorems 1.1 and 1.2 that govern the asymptotics of the Schwartz kernel.

Theorem 1.1 (Analyticity of the Schwartz Kernel)**.**

(1) If $C \subset X^2$ is closed and disjoint to the diagonal, then the restriction of K_z to C extends to be entire with values in $\mathcal{A}^{\mathcal{F}(\alpha)}_{phg}(X^2)$. Here $\mathcal{F}(\alpha)$ is the index family (3.12) *given in terms of the indicial roots (or boundary spectrum) of A.*

(2) The restriction of K_z to the diagonal $\Delta \cong X$ is defined for $\operatorname{Re} z < -n/m$, *and can be written in the form*

$$K_z|_\Delta = b(z) + c(z),$$

where $b(z) \in x^{-mz}C^\infty(X)$ is meromorphic on $\mathbb{C}$ with (possible) simple poles only on the discrete set $\{\frac{k-n}{m} : k \in \mathbb{N}_0, \frac{k-n}{m} \notin \mathbb{N}_0\}$; and where

$$c(z) \in \begin{cases} x^m S^{0,0}(X), & \operatorname{Re} z < -1; \\ x^{-mz} S^{0,0}(X), & \operatorname{Re} z \geq -1, \end{cases} \quad \textit{is entire}, \tag{1.4}$$

and vanishes for $z \in \mathbb{N}_0$. Finally, $c(z)$ can be considered a meromorphic function on $\mathbb{C}$, but with values in the polyhomogeneous space

$$c(z) \in x^{-mz}C^\infty(X) + \mathcal{A}^{E(\alpha)}_{phg}(X), \tag{1.5}$$

where $E(\alpha)$ is given explicitly in (3.13) *in terms of the indicial roots of A, and where $c(z)$ has (possible) poles occurring only on the discrete set*

$$\{z \in \mathbb{C} : (\mathbb{N}_0 - mz) \cap E(\alpha) \neq \varnothing\}. \tag{1.6}$$

Due to the colliding of the set of exponents $\mathbb{N}_0 - mz$ and $E(\alpha)$, occurring whenever the set (1.6) is nonempty, for similar reasons encountered with the function $u(z, x)$ in (1.3), we can consider $c(z)$ as being entire with values in the function spaces in (1.4), or as meromorphic with values in the polyhomogeneous space in (1.5). The precise sense in which the map (1.5) is meromorphic, including the orders of the (possible) poles, is provided in Lemma 3.10, see also Lemma 3.8.

We now describe our second result, Theorem 1.2 below. Readers not familiar with the b-calculus should see the appendix for complete definitions of the notations that follow. The b-calculus consists of operators whose Schwartz kernels define certain classes of distributions on a manifold, denoted by X^2_b, which looks like X^2 but with the corner $(\partial X)^2$ replaced by a quarter circle called the "front face", denoted by ff (please see Figure 1 in the appendix). Given $s \in \mathbb{R}$, we denote by $\Psi^s_b(X)$ the "small" calculus of b-pseudodifferential operators of order s. The "full" b-calculus also incorporates operators whose Schwartz kernels are functions on X^2_b. These classes are denoted by $\Psi^{-\infty,\mathcal{F}}_{b,\infty}(X)$ and $\Psi^{-\infty,\mathcal{F},E}_b(X)$, where $\mathcal{F}$ is an index family and E is an index set. Here, $\mathcal{F}$ determines the asymptotic type of the Schwartz kernels at the sides of X_b^2 while E determines the asymptotic type at the front face of X_b^2. The subscript "∞" in $\Psi_{b,\infty}^{-\infty,\mathcal{F}}(X)$ means, roughly speaking, that operators in $\Psi_{b,\infty}^{-\infty,\mathcal{F}}(X)$ have Schwartz kernels that are continuous up to ff. We denote by ϱ, a boundary defining function for ff.

Theorem 1.2 (Analyticity in the b-calculus). *The complex power A_z in* (1.2) *can be written in the form $A_z = B(z) + C(z)$, where $B(z)$ and $C(z)$ are entire families of operators within the following spaces of b-pseudodifferential operators:*

$$\begin{aligned} &B(z) \in \varrho^{-mz}\Psi_b^{m\mathrm{Re}\, z}(X); \\ &C(z) \in \begin{cases} \varrho^{m}\Psi_{b,\infty}^{-\infty,\mathcal{F}(\alpha)}(X), & \mathrm{Re}\, z < -1; \\ \varrho^{-mz}\Psi_{b,\infty}^{-\infty,\mathcal{F}(\alpha)}(X), & \mathrm{Re}\, z \geq -1. \end{cases} \end{aligned} \tag{1.7}$$

The complex powers satisfy the usual group properties:

$$A_k = A^k, \; A_z A_w = A_{z+w} \quad \text{for all } k \in \mathbb{Z}, \text{ and } \; z, w \in \mathbb{C}.$$

Finally, $C(z)$ vanishes for $z \in \mathbb{N}_0$, and can be considered a meromorphic function on $\mathbb{C}$, but with values in the polyhomogeneous b-calculus space

$$C(z) \in \varrho^{-mz}\Psi_b^{-\infty,\mathcal{F}(\alpha),\mathbb{N}_0}(X) + \Psi_b^{-\infty,\mathcal{F}(\alpha),E(\alpha)}(X), \tag{1.8}$$

where $C(z)$ has (possible) poles occurring only on the discrete set

$$\{z \in \mathbb{C} \,:\, (\mathbb{N}_0 - mz) \cap E(\alpha) \neq \varnothing\}.$$

The same phenomena related to the colliding of the set of exponents $\mathbb{N}_0 - mz$ and $E(\alpha)$ accounts for the meromorphic properties of $C(z)$. The precise sense in which the map (1.8) is meromorphic, including the orders of the (possible) poles, is provided in Lemma 3.10. Theorem 1.2 plus the mapping properties of b-calculus, see [14, Ch. 5] or [13, Sec. 3], imply the following.

Corollary 1.3. *For each $z \in \mathbb{C}$ and $s \in \mathbb{R}$, we have*

$$A_z \colon x^{\alpha-m}H_b^s(X) \longrightarrow \begin{cases} x^{\alpha}H_b^{s-m\mathrm{Re}\, z}(X), & \mathrm{Re}\, z < -1; \\ x^{\alpha-m-m\mathrm{Re}\, z}H_b^{s-m\mathrm{Re}\, z}(X), & \mathrm{Re}\, z \geq -1. \end{cases}$$

Our final theorem concerns the poles of the zeta function of A. By (1.7) of Theorem 1.2 and mapping properties of the b-calculus, it follows that if $\mathrm{Re}\, z < -n/m$ where $n = \dim X$, then A_z is trace class on $x^{\alpha-m}H_b^s(X)$ for each $s \in \mathbb{R}$.

Theorem 1.4 (Analyticity of the Zeta Function). *The zeta function of A, $\zeta_A(z) := \mathrm{Tr}\, A_z$, is holomorphic for $\mathrm{Re}\, z < -n/m$; and extends to be meromorphic on the whole complex plane, with (possible) simple poles on the set $\{\frac{k-n}{m} \,:\, k \in \mathbb{N}_0\}$ and with (possible) double poles on the set $\{\frac{k}{m} \,:\, k \in \mathbb{N}_0,\, \frac{k}{m} \notin \mathbb{N}_0\}$.*

In Section 2, we describe some properties of the heat operator and complex powers on closed compact manifolds. In particular, we prove the analytic properties of the complex powers using the heat operator. These results are needed later in the conic manifold case. In Section 3.1, we describe cone differential operators. In Section 3.2, we describe the heat operator for conic manifolds, which will be used in Section 3.3 to prove the analytic properties of the complex powers. Finally, in the appendix we give a short description of Melrose's b-calculus.

Lastly, I thank the referee for helpful comments in improving this paper.

2. Complex powers on closed compact manifolds

2.1. **The heat operator.** In this section, X will denote a closed compact n-dimensional manifold on which there is a fixed smooth measure. Let $\Lambda = \{\lambda \in \mathbb{C} : \varepsilon \leq \arg(\lambda) \leq 2\pi - \varepsilon\}$ where $0 < \varepsilon < \pi/2$. Let $A \in \mathrm{Diff}^m(X)$, $m \in \mathbb{Z}^+$, be elliptic and assume that the principal symbol of A never takes values in $\Lambda \setminus \{0\}$, and that $(A - \lambda)^{-1}$ exists on a neighborhood of Λ. Now it is well-known (see for example, [5, Ch. 4]) that the heat operator e^{-tA} exists and can be defined by a Dunford integral of the resolvent $(A-\lambda)^{-1}$ around the spectrum of A. Since X has a fixed smooth measure, we can consider the Schwartz kernel of e^{-tA} a function on X^2. Then, please see [5, Ch. 4] or [11, Sec. 5], we have the following.

Theorem 2.1. *Let K_t denote the Schwartz kernel of e^{-tA}. Then,*

(A) $K_t \in C^\infty((0,\infty)_t; C^\infty(X^2))$ and it vanishes to infinite order as $t \to \infty$ (that is, with all derivatives, faster than any polynomial in t).

(B) If $\varphi \in C^\infty(X^2)$ has support disjoint to the diagonal Δ, then

$$\varphi K_t \in C^\infty([0,\infty)_t;\ C^\infty(X^2))$$

and φK_t vanishes to infinite order at $t = 0$ and as $t \to \infty$.

(C) Given a coordinate patch of X^2 overlapping Δ of the form $\mathcal{U}_y \times \mathbb{R}^n_w$ such that $\Delta \cong \mathcal{U} \times \{0\}$, and given $\varphi \in C_c^\infty(\mathcal{U} \times \mathbb{R}^n)$, we have

$$\varphi K_t = \int e^{iw\cdot\xi}\, a(t,y,\xi)\, đ\xi, \quad đ\xi = \frac{1}{(2\pi)^n} d\xi,$$

where $a(t,y,\xi) \in C^\infty([0,\infty)_t \times \mathcal{U}_y \times \mathbb{R}^n_\xi)$ and has the following properties:

(a) There exists a constant $C' > 0$ such that $a(t,y,\xi)$ satisfies the estimates: for any k, α, and β, there exists a C such that

$$| \partial_t^k \partial_y^\alpha \partial_\xi^\beta a(t,y,\xi) | \leq C\,(1 + |\xi|)^{mk-|\beta|}\, e^{-tC'(1+|\xi|^m)}. \tag{2.1}$$

(b) If $N \in \mathbb{Z}^+$, then we can write

$$a(t,y,\xi) = \sum_{j=0}^{N-1} a_j(t,y,\xi) + r_N(t,y,\xi), \tag{2.2}$$

where $a_j, r_N \in C^\infty([0,\infty)_t \times \mathcal{U}_y \times \mathbb{R}^n_\xi)$, and satisfy

(i) $a_j(\delta^{-m}t, y, \delta\xi) = \delta^{-j} a_j(t,y,\xi)$ for all $\delta > 0$;

(ii) $a_j(t,y,\xi)$ satisfies estimates of the form (2.1) but with mk replaced with $mk - j$ and with $e^{-tC'(1+|\xi|^m)}$ replaced with $e^{-tC'|\xi|^m}$;

(iii) r_N satisfies estimates of the form (2.1) but with mk replaced with $mk - N$ and with $e^{-tC'(1+|\xi|^m)}$ replaced with $e^{-tC'|\xi|^m}$.

The complex power of A, A_z, is defined for $z \in \mathbb{C}$ with $\mathrm{Re}\, z < 0$ via the contour integral [22, Sec. 10]

$$A_z = \frac{i}{2\pi} \int_\Gamma \lambda^z (A - \lambda)^{-1} d\lambda, \tag{2.3}$$

where λ^z is defined by its standard branch, and where Γ is a contour lying in the right-half plane around the spectrum of A. The complex power defines a map between Sobolev spaces:

$$A_z \colon H^s(X) \longrightarrow H^s(X), \quad \text{for any } s \in \mathbb{R},\ \mathrm{Re}\, z < 0.$$

Note that for $\operatorname{Re} z < 0$,

$$\lambda^z = \frac{1}{\Gamma(-z)} \int_0^\infty t^{-z} e^{-t\lambda} \frac{dt}{t},$$

where $\Gamma(z)$ is the Gamma function. Also note that properties (A) – (C) of Theorem 2.1 imply that e^{-tA} defines a map on $H^s(X)$ which is continuous down to $t = 0$ and vanishes to infinite order as $t \to \infty$. Using this fact, we can plug the equation for λ^z into (2.3) and freely interchange integrals, obtaining

$$A_z = \frac{1}{\Gamma(-z)} \int_0^\infty t^{-z} e^{-tA} \frac{dt}{t}, \quad \operatorname{Re} z < 0. \tag{2.4}$$

2.2. **Analyticity of the complex power.** The following theorem is well-known (see, e.g., [21, 22, 5]). Surprisingly, the author has never seen the "heat kernel" proof of the theorem in the literature. The proof usually employs the resolvent; however, if one knows the properties (A) – (C) of Theorem 2.1 of the heat operator, the proof is quite straightforward. Since the following theorem is key to proving Theorems 1.1 and 1.2, we give a full proof. We denote by $\Psi^s(X)$ the space of pseudodifferential operators on X of order s.

Theorem 2.2. *Let K_z denote the Schwartz kernel of A_z.*

(1) The complex power A_z extends from $\operatorname{Re} z < 0$ to an entire function of $z \in \mathbb{C}$, with values in $\Psi^{m \operatorname{Re} z}(X)$, in the sense that for each $r \in \mathbb{R}$, the map

$$\{z \in \mathbb{C} \,:\, \operatorname{Re} z < r\} \ni z \;\mapsto\; A_z \in \Psi^{m \operatorname{Re} z}(X)$$

is holomorphic as a map into $\Psi^{mr}(X)$.

(2) If $C \subset X^2$ is closed and disjoint to the diagonal Δ, then the restriction $K_z|_C$ of the Schwartz kernel of A_z to C is an entire family, with values in $C^\infty(X^2)$.

(3) The restriction $K_z|_\Delta$ of the Schwartz kernel of A_z to the diagonal $\Delta \cong X$ is meromorphic on $\mathbb{C}$ taking values in $C^\infty(X)$ with (possible) simple poles only on the discrete set $\{\frac{k-n}{m} \,:\, k \in \mathbb{N}_0,\, \frac{k-n}{m} \notin \mathbb{N}_0\}$.

Proof. We first prove (2). Thus, let $\varphi \in C^\infty(X^2)$ have support disjoint to Δ. Then by (B) of Theorem 2.1, Equation (2.4), and properties of the Mellin transform (see [1, Ch. 4]), it follows that φK_z is an entire family, with values in $C^\infty(X^2)$. Thus, (2) holds.

Now we work near Δ. Thus, let $\varphi \in C^\infty(X^2)$ have support in a coordinate patch $\mathcal{U}_y \times \mathbb{R}^n_w$ of X^2 such that $\Delta \cong \mathcal{U} \times \{0\}$. Then using the notation in (C) of Theorem 2.1, it follows that for $\operatorname{Re} z < 0$ we have

$$\varphi K_z = \int e^{iw\cdot\xi}\, a_z(y,\xi)\, đ\xi, \tag{2.5}$$

where

$$a_z(y,\xi) = \frac{1}{\Gamma(-z)} \int_0^\infty t^{-z} a(t,y,\xi) \frac{dt}{t}. \tag{2.6}$$

Let $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) \equiv 0$ for $|\xi| \le 1$ and $\chi(\xi) \equiv 1$ for $|\xi| \ge 2$. Let $N >> 0$. Then the expansion (2.2) of $a(t,y,\xi)$ implies that we can write

$$a(t,y,\xi) = \sum_{j=0}^{N-1} \chi(\xi)\, a_j(t,y,\xi) + s_N(t,y,\xi), \tag{2.7}$$

where a_j is already given in (2.2), and where $s_N \in C^\infty([0,\infty)_t \times \mathcal{U}_y \times \mathbb{R}^n_\xi)$ satisfies estimates of the form (2.1) but with mk replaced with $mk-N$. The cut-off $\chi(\xi)$ has the effect of making each term on the right-hand side of (2.7) vanish individually to exponential order as $t \to \infty$ for all $\xi \in \mathbb{R}^n$ (rather than for only $\xi \neq 0$ when $\chi(\xi)$ is not present). Hence, for $\operatorname{Re} z < 0$,

$$a_z(y,\xi) = \sum_{j=0}^{N-1} \chi(\xi)\, a_{z,j}(y,\xi) + s_{z,N}(y,\xi), \tag{2.8}$$

where $a_{z,j}(y,\xi)$ and $s_{z,N}(y,\xi)$ are given by the integral (2.6) with $a(t,y,\xi)$ replaced with $a_j(t,y,\xi)$ and $s_N(t,y,\xi)$, respectively.

Since $a_j(\delta^{-m}t, y, \delta\xi) = \delta^{-j} a_j(t,y,\xi)$, making the change of variables $t \mapsto \delta^{-m}t$ in the integral (2.6) defining $a_{z,j}(y,\delta\xi)$ shows that

$$a_{z,j}(y,\delta\xi) = \delta^{mz-j} a_{z,j}(y,\xi) \quad \text{for all } \delta > 0 \text{ and } \xi \neq 0,\ \operatorname{Re} z < 0. \tag{2.9}$$

Thus, $a_{z,j}(y,\xi)$ is homogeneous of degree $mz - j$. Also, one can show that the estimates on $s_N(t,y,\xi)$ imply that $s_{z,N}(y,\xi)$ is a symbol of order $-N$ for $\operatorname{Re} z < 0$.

We now show that $a_{z,j}(y,\xi)$ and $s_{z,N}(y,\xi)$ extend from $\operatorname{Re} z < 0$ to be entire. Note that for $\xi \neq 0$,

$$a_{z,j}(y,\xi) = \frac{1}{\Gamma(-z)} \int_0^1 t^{-z} a_j(t,y,\xi) \frac{dt}{t} + \frac{1}{\Gamma(-z)} \int_1^\infty t^{-z} a_j(t,y,\xi) \frac{dt}{t}, \quad \xi \neq 0. \tag{2.10}$$

Since $a_j(t,y,\xi) \to 0$ to exponential order as $t \to \infty$ for $\xi \neq 0$, the second integral in (2.10) is entire in $z \in \mathbb{C}$. The following facts will be used in the sequel. For $\ell \in \mathbb{R}$,

$$\int_0^1 t^{-z+\ell} \frac{dt}{t} = \frac{1}{-z+\ell} \text{ for } \operatorname{Re} z < \ell; \ \ \frac{1}{\Gamma(-z)} \text{ is entire, vanishing at } z \in \mathbb{N}_0. \tag{2.11}$$

Expanding $a_j(t,y,\xi)$ in Taylor series at $t = 0$, and using (2.11), we find that the first integral in (2.10) extends to be entire in $z \in \mathbb{C}$. Thus, $a_{z,j}(y,\xi)$ extends to be an entire function of $z \in \mathbb{C}$. The identity (2.9) continues to hold for all $z \in \mathbb{C}$; taking the derivative of (2.9) shows that for any $\ell \in \mathbb{N}_0$, $\partial_z^\ell a_{z,j}(y,\xi)$ is "log-polyhomogeneous" of order $mz - j$ with at most ℓ powers of $\log|\xi|$, for all $z \in \mathbb{C}$. A similar argument shows that $s_{z,N}(y,\xi)$ extends from $\operatorname{Re} z < 0$ to be entire. Moreover, given any $r \in \mathbb{N}_0$, expanding $s_N(t,y,\xi)$ in Taylor series at $t = 0$ up to order t^r, using (2.11) and the estimate (2.1) with mk replaced with $mk - N$ shows that for $\operatorname{Re} z < r$, $s_{z,N}(y,\xi)$ is holomorphic with values in the symbols of order $mr - N$. Since N is arbitrary, the expression (2.8) and the properties of $a_{z,j}(y,\xi)$ and $s_{z,N}(y,\xi)$ just examined imply that for all $z \in \mathbb{C}$, $a_z(y,\xi)$ is a symbol of order $m\operatorname{Re} z$ and is an entire function of $z \in \mathbb{C}$ in the sense that for any $r \in \mathbb{R}$,

$$\{z \in \mathbb{C} : \operatorname{Re} z < r\} \ni z \mapsto a_z(y,\xi)$$

is holomorphic as a map into the symbols of order mr. This, together with statement (2), prove statement (1).

To prove statement (3), we analyze $\varphi K_z|_\Delta$ in (2.5). Observe that for $\operatorname{Re} z << 0$, we can write

$$\varphi K_z|_\Delta = \frac{1}{\Gamma(-z)} \int_0^1 \!\! \int t^{-z} a(t,y,\xi)\, đ\xi \frac{dt}{t} + \frac{1}{\Gamma(-z)} \int_1^\infty \!\! \int t^{-z} a(t,y,\xi)\, đ\xi \frac{dt}{t}. \tag{2.12}$$

By the estimate (2.1), the second integral in (2.12) is entire in $z \in \mathbb{C}$. Consider now the first integral. Let $N >> 0$. Then by the expansion (2.2) of $a(t, y, \xi)$,

$$\int_0^1 \!\! \int t^{-z} a(t, y, \xi)\, đ\xi \frac{dt}{t} = \int_0^1 \Big\{ \sum_{j=0}^{N-1} \int t^{-z} a_j(t, y, \xi)\, đ\xi + \int t^{-z} r_N(t, y, \xi)\, đ\xi \Big\} \frac{dt}{t}.$$

Changing variables $\xi \mapsto t^{-1/m}\xi$ and using the homogeneity of $a_j(t, y, \xi)$, we obtain

$$\frac{1}{\Gamma(-z)} \int_0^1 \!\! \int t^{-z} a_j(t, y, \xi)\, đ\xi \frac{dt}{t} = \frac{1}{\Gamma(-z)} \int_0^1 t^{-z + \frac{j-n}{m}} \frac{dt}{t} \cdot \int a_j(1, y, \xi)\, đ\xi.$$

By (2.11), it follows that

$$\frac{1}{\Gamma(-z)} \int_0^1 \!\! \int t^{-z} a_j(t, y, \xi)\, đ\xi \frac{dt}{t} \in C^\infty(X)$$

and is meromorphic on $\mathbb{C}$ with only a (possible) simple pole at $z = \frac{j-n}{m}$ for $\frac{j-n}{m} \notin \mathbb{N}_0$. Let $r >> 0$, $r \in \mathbb{N}_0$, and assume that $N \in \mathbb{N}_0$ is such that $mr - N \leq -n - 1$. Then as $(1 + |\xi|)^{-n-1}$ is integrable on $\mathbb{R}^n$, expanding $r_N(t, y, \xi)$ in Taylor series at $t = 0$ up to order t^r, and using (2.11) and the derivative estimates on r_N, see (2.1) with mk replaced with $mk - N$ and with $e^{-tC'(1+|\xi|^m)}$ replaced with $e^{-tC'|\xi|^m}$, show that for $\operatorname{Re} z < r$,

$$\frac{1}{\Gamma(-z)} \int_0^1 \!\! \int t^{-z} r_N(t, y, \xi)\, đ\xi \frac{dt}{t}$$

takes values in $C^\infty(X)$, where it is holomorphic. Thus, statement (3) is proved. □

Remark 2.3. Note that the properties of the kernel of A_z near the diagonal can be determined directly from the properties of the symbol $a(t, y, \xi)$ listed in (a) and (b) in Part (C) of Theorem 2.1. This fact will be used later in Lemma 3.8.

3. Complex powers of cone differential operators

3.1. **Cone differential operators.** Now let X be a compact n-dimensional manifold with connected boundary on which there is a fixed boundary defining function x and a fixed b-measure (a density of the form $x^{-1}\times$ a smooth positive density on X). This b-measure allows us to omit density factors in the Schwartz kernels of operators, such as the kernels of the heat operator and complex powers in Theorem 3.4 and Lemma 3.8 below.

We begin this section by describing "full-ellipticity". We will continue to use the same notation as in the Introduction (see also the Appendix). If $A = x^{-m}P \in x^{-m}\mathrm{Diff}_b^m(X)$, then the b-principal symbol of P, ${}^b\sigma_m(P)$, is just the usual principal symbol of P in the interior of X and, in the decomposition (1.1) of A, is defined by

$${}^b\sigma_m(P)(\xi) = \sum_{k=0}^{m} \sigma_{m-k}(A_{m-k}(x))(\xi') \cdot (i\xi_1)^k, \quad \xi = (\xi_1, \xi') \in \mathbb{R} \times T^*(\partial X),$$

where $\sigma_{m-k}(A_{m-k}(x))$ is the principal symbol of the operator $A_{m-k}(x)$ as a differential operator on ∂X. The symbol ${}^b\sigma_m(P)$ is actually a function on the b-cotangent bundle ${}^bT^*X$ (see [14, Ch. 4]). Writing A in the representation (1.1) and freezing the coefficients at the boundary defines the operator

$$I(A) := r^{-m} \sum_{k=0}^{m} A_{m-k}(0)(r\partial_r)^k. \tag{3.1}$$

The *boundary spectrum* or set of *indicial roots* of A (see [14, Sec. 5.1]), $\operatorname{spec}_c(A) \subset \mathbb{C}$, is the set of points $\tau \in \mathbb{C}$ where the Mellin transform of the operator $r^m I(A)$:

$$\widehat{P}(\tau) := \sum_{k=0}^{m} A_{m-k}(0)(i\tau)^k\colon H^m(\partial X) \longrightarrow L^2(\partial X), \tag{3.2}$$

fails to be invertible. Before defining full-ellipticity, we need to define some "cone" Sobolev spaces. First, let $(\mathbb{S}^{n-1})^\wedge = [0,\infty)_r \times \mathbb{S}^{n-1}$ be the usual polar coordinates decomposition of $\mathbb{R}^n$, where $(\mathbb{S}^{n-1})^\wedge \ni (r,\omega) \mapsto r\omega \in \mathbb{R}^n$ is the polar coordinates map. Let $\chi(r) \in C_c^\infty([0,\infty))$ be such that $\chi(r) \equiv 1$ near $r = 0$. Let $s \in \mathbb{R}$. Then the space $H_c^s((\mathbb{S}^{n-1})^\wedge)$ consists of those distributions u on $(\mathbb{S}^{n-1})^\wedge$ such that $\chi(r)u \in H_b^s((\mathbb{S}^{n-1})^\wedge)$ and such that $(1-\chi(r))u \in H^s(\mathbb{R}^n)$. Here, we use the identification of the interior of $(\mathbb{S}^{n-1})^\wedge$ with $\mathbb{R}^n \setminus \{0\}$ provided by the polar coordinates map. By reference to local coordinates on ∂X, we can define $H_c^s(\partial X^\wedge)$, where $\partial X^\wedge := [0,\infty)_r \times \partial X$. Finally, let $\rho \in C^\infty([0,\infty))$ be such that $\rho(r) = r$ for $r \le 1$, $\rho > 0$ for $1 \le r \le 2$, and $\rho(r) \equiv 1$ for $r \ge 2$. Then observe that for any $\alpha, s \in \mathbb{R}$, $I(A)$ defines a continuous map between $\rho^\alpha H_c^s(\partial X^\wedge)$ and $\rho^{\alpha-m} H_c^{s-m}(\partial X^\wedge)$.

Definition 3.1. Let $\Lambda \subset \mathbb{C}$ be a closed sector. Then $A \in x^{-m}\mathrm{Diff}_b^m(X)$ is said to be *fully elliptic on* Λ *with respect to* $\alpha \in \mathbb{R}$ if

(1) ${}^b\sigma_m(x^m A)(\xi) - \lambda$ is invertible for all $\xi \neq 0$ and $\lambda \in \Lambda$;
(2) $\alpha \notin -\operatorname{Im}\operatorname{spec}_c(A)$;
(3) $I(A) - \lambda\colon \rho^\alpha H_c^s(\partial X^\wedge) \longrightarrow \rho^{\alpha-m} H_c^{s-m}(\partial X^\wedge)$ for any $s \in \mathbb{R}$, is invertible for all $\lambda \in \Lambda$ sufficiently large.

Remark 3.2. In [4], full-ellipticity is called "parameter ellipticity". Originally in [4] and later in [12], it is shown that if $A \in x^{-m}\mathrm{Diff}_b^m(X)$ is fully elliptic on Λ with respect to $\alpha \in \mathbb{R}$, then for each $s \in \mathbb{R}$,

$$A - \lambda\colon x^\alpha H_b^s(X) \longrightarrow x^{\alpha-m} H_b^{s-m}(X) \tag{3.3}$$

is invertible for all $\lambda \in \Lambda$ sufficiently large.

Example 3.3 (Laplace-Beltrami operator)**.** A Riemannian metric g on the interior of X is a *conic metric* if on a collar $[0,\varepsilon)_x \times \partial X$ of ∂X, we have $g = dx^2 + x^2 h(x)$, where $h(x)$ is a smooth family of metrics on ∂X. The Laplace-Beltrami operator of g, denoted by Δ_c, is in $x^{-2}\mathrm{Diff}_b^2(X)$. In fact, on $[0,\varepsilon)_x \times \partial X$ we have

$$\Delta_c = x^{-2}[-(x\partial_x)^2 - (n-2)x\partial_x + \Delta_h - \frac{x^2}{\sqrt{\det h(x)}}(\partial_x \sqrt{\det h(x)}) \cdot \partial_x], \tag{3.4}$$

where $n = \dim X$ and $\Delta_h = \Delta_{h(x)}$ is the Laplacian on Y associated to the metric $h(x)$. Fix a closed sector Λ that does not intersect the positive real axis and assume that $n = \dim X \ge 5$. We will show that Δ_c is fully elliptic on Λ with respect to $-n/2+2$. Condition (1) of Definition 3.1 is straightforward to verify. Condition (2) is verified as follows. Let $0 \le \lambda_1 < \lambda_2 < \cdots$ be the distinct eigenvalues of $\Delta_{h(0)}$. Then by (3.4), $\operatorname{spec}_c(\Delta_c)$ consists of the roots of the equation $\tau^2 - i(n-2)\tau + \lambda_j$ for each j. Hence, $\operatorname{spec}_c(\Delta_c) = i \cdot \{\mu_j\}_{j=\pm1,\pm2,\ldots}$, where

$$\mu_{\pm j} = \frac{1}{2}\Big(n - 2 \pm \sqrt{(n-2)^2 + 4\lambda_j}\Big), \quad j = 1,2,3,\ldots. \tag{3.5}$$

It follows that $\alpha := -n/2 + 2 \notin -\operatorname{Im}\operatorname{spec}_c(\Delta_c)$ for $n \ge 5$. (This is where we need $n \ge 5$). To verify condition (3), we first work out the indicial operator,

$$I(\Delta_c) = x^{-2}\big\{-(x\partial_x)^2 - (n-2)x\partial_x + \Delta_{h(0)}\big\}, \tag{3.6}$$

where $\Delta_{h(0)}$ is the Laplacian on ∂X fixed by the metric $h(0)$. Thus, $I(\Delta_c)$ is just the Laplacian on the model cone $\partial X^\wedge$ associated to the metric $dx^2+x^2h(0)$. Using the fact that $\alpha = -n/2+2 \notin -\mathrm{Im}\,\mathrm{spec}_c(\Delta_c)$, a little more analysis (see [3]) shows that $I(\Delta_c)\colon \rho^\alpha H^2_c(\partial X^\wedge) \longrightarrow \rho^{\alpha-2}H^0_c(\partial X^\wedge)$ is self-adjoint and non-negative. Thus, condition (3) is satisfied.

3.2. **The heat operator.** Let $\Lambda = \{\lambda \in \mathbb{C} : \varepsilon \le \arg(\lambda) \le 2\pi - \varepsilon\}$ where $0 < \varepsilon < \pi/2$. For the rest of the paper, we assume:

(AZ) $A \in x^{-m}\mathrm{Diff}^m_b(X)$ is fully elliptic on Λ with respect to $\alpha \in \mathbb{R}$ and $(A-\lambda)^{-1}$, the inverse of (3.3), exists on a neighborhood of Λ.

Gil [4] (see also [10]) shows that the heat operator e^{-tA} exists and can be defined by the Dunford integral for $t > 0$:

$$e^{-tA} = \frac{i}{2\pi}\int_\Gamma e^{-t\lambda}(A-\lambda)^{-1}d\lambda,$$

where Γ is a curve in Λ going around the spectrum of A. We remark that the heat operator e^{-tA} exists even without the assumption that the resolvent exists on a neighborhood of Λ.

Given Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, we denote the bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ by $\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$. The following mapping properties are a consequence of results found in [10] and the fact that, since $(A-\lambda)^{-1}$ exists on a neighborhood of Λ, the contour Γ can be shifted entirely to the right of the imaginary axis. For all $s, s' \in \mathbb{R}$,

$$e^{-tA} \in C^\infty(\,(0,\infty)_t;\, \mathcal{B}(x^{\alpha-m}H^s_b(X), x^\alpha H^{s'}_b(X))\,), \tag{3.7}$$

where $e^{-tA} \to 0$ to infinite order as $t \to \infty$. Down to $t = 0$, we have for each $k \in \mathbb{N}_0$ and $s \in \mathbb{R}$,

$$e^{-tA} \in C^k([0,\infty)_t;\, \mathcal{B}(x^{\alpha-m}H^s_b(X), x^{\alpha-m-mk}H^{s-mk}_b(X))\,). \tag{3.8}$$

Moreover, $e^{-tA}|_{t=0} = \mathrm{Id}$ on $x^{\alpha-m}H^s_b(X)$, and e^{-tA} satisfies the heat equation

$$(\partial_t + A)e^{-tA} = 0, \quad t > 0. \tag{3.9}$$

Before stating Theorem 3.4, which describes the Schwartz kernel of e^{-tA}, we give some intuition for the splitting of e^{-tA} in (3.10) below. Note that $A = x^{-m}P$, where $P \in \mathrm{Diff}^m_b(X)$. Thus, formally speaking, $e^{-tA} = H(t/x^m)$, where $H(t) = e^{-tP}$. The term $B(t)$ in (3.10) is, roughly speaking, $H(t/x^m)$. Actually, for symmetry reasons, when describing Schwartz kernels, we replace x with, say, $\varrho = x+x'$, where "x" here, respectively "x'", is the boundary defining function x on X considered on the left, respectively right, factor of X^2. Then, ϱ is a boundary defining function for the front face ff of X^2_b. Now, $B(t)$ does not satisfy the heat equation (3.9). Solving away the error term to make this equation hold exactly on weighted Sobolev spaces involves the weight parameter α and the boundary spectrum of A. The term $C(t)$ in (3.10) is what must be added to $B(t)$ so that the heat equation holds exactly. The weight α is manifested in the asymptotics of $C(t)$ as described in (D)$'$ below.

We now state the precise structure of e^{-tA}. Throughout this paper, we always let ϱ denote a boundary defining function for ff (e.g., $\varrho = x+x'$ works). For the definition of the "lifted" diagonal Δ_b, and of the various spaces of b-pseudodifferential operators appearing in the following theorem, see the Appendix. As always, using the fixed b-measure on X, we will consider the Schwartz kernel of e^{-tA} as a function. For the proof of the following result, see [10].

Theorem 3.4. ([10]) *Under the assumption (AZ), we can write*

$$e^{-tA} = B(t) + C(t), \tag{3.10}$$

where the following properties hold.

(A)′ $e^{-tA} \in C^\infty((0,\infty)_t;\, \Psi_b^{-\infty,\mathcal{F}(\alpha),E(\alpha)}(X))$*, where* $\mathcal{F}(\alpha)$ *and* $E(\alpha)$ *are given in (3.12) and (3.13) below, and where* $e^{-tA} \to 0$ *to infinite order as* $t \to \infty$*.*

(B)′ Let $\varphi \in C_c^\infty(X_b^2 \setminus \Delta_b)$*. Then,* $\varphi K_{B(t)}$ *is of the form* $k(t/\varrho^m, p)$*, where* $k(t,p)$ *is a smooth function of* $(t,p) \in [0,\infty) \times X_b^2$ *and* $k(t,p)$ *vanishes to infinite order at* $t = 0$*,* $t = \infty$*,* $p \in \mathrm{lb}$*, and at* $p \in \mathrm{rb}$*.*

(C)′ Given a coordinate patch of X_b^2 *overlapping* Δ_b *of the form* $\mathcal{U}_y \times \mathbb{R}^n_w$ *such that* $\Delta_b \cong \mathcal{U} \times \{0\}$*, and given* $\varphi \in C_c^\infty(\mathcal{U} \times \mathbb{R}^n)$*, we have*

$$\varphi K_{B(t)} = \int e^{iw\cdot\xi}\, b(\frac{t}{\varrho^m}, y, \xi)\, đ\xi, \tag{3.11}$$

where $b(t,y,\xi) \in C^\infty([0,\infty)_t \times \mathcal{U}_y \times \mathbb{R}^n_\xi)$ *and has the properties listed in Part (C) of Theorem 2.1 for the closed manifold case.*

(D)′ If $t = s^m$*, then the kernel of* $C(t)$ *is of the form*

$$K_{C(t)} = c(s, \frac{\varrho}{s}, p), \quad p \in X_b^2,$$

where $c(s,v,p)$ *has expansions at* $p \in \mathrm{lb}$ *and* $p \in \mathrm{rb}$ *of* X_b^2 *with index sets* $F_{\mathrm{lb}}(\alpha)$ *and* $F_{\mathrm{rb}}(\alpha)$ *respectively, and is otherwise smooth in* $p \in X_b^2$*; smooth in* $s \in [0,\infty)$*; and smooth in* $v \in (0,\infty)$*, vanishes to infinite order as* $v \to \infty$*, and can be expanded at* $v = 0$ *with the index set* $E(\alpha)$*. Here, the index sets* $F_{\mathrm{lb}}(\alpha)$*,* $F_{\mathrm{rb}}(\alpha)$*, and* $E(\alpha)$ *appear in (3.12) and (3.13) below.*

As mentioned already, the explicit dependence of the heat kernel on the weight parameter α is reflected in the index sets $F_{\mathrm{lb}}(\alpha)$, $F_{\mathrm{rb}}(\alpha)$, and $E(\alpha)$, which are defined as follows. We first define

$$\begin{aligned}\widehat{E}^{\pm}(\alpha) := \{(z+r,k) \,:\, & r \in \mathbb{N}_0,\ \tau = \mp iz \in \operatorname{spec}_c(A) + im, \\ & 1 \le k+1 \le \sum_{\ell=0}^{r} \operatorname{ord}(\tau - im \mp i\ell), \text{ and } \operatorname{Re} z > \pm(\alpha - m)\},\end{aligned}$$

where $\operatorname{ord}(\tau)$ denotes the order of the pole at $\tau \in \operatorname{spec}_c(A)$ of $\widehat{P}(\tau)^{-1}$ and $\widehat{P}(\tau)$ is given in (3.2). Then

$$\mathcal{F}(\alpha) = (F_{\mathrm{lb}}(\alpha), F_{\mathrm{rb}}(\alpha)) := (\hat{E}^+(\alpha) + m, \hat{E}^-(\alpha)), \tag{3.12}$$

and

$$E(\alpha) := \mathbb{Z}^+ \,\overline{\cup}\, (\widehat{E}^+(\alpha) + \widehat{E}^-(\alpha)) + m, \tag{3.13}$$

where, for any two index sets G and H,

$$G \overline{\cup} H := G \cup H \cup \{(\zeta, k+\ell+1) \,:\, (\zeta,k) \in G,\ (\zeta,\ell) \in H\}. \tag{3.14}$$

The index sets $\mathcal{F}(\alpha)$ and $E(\alpha)$ are the ones that appear in Theorems 1.1 and 1.2.

3.3. **Complex powers.** Under the assumption (AZ), for $\mathrm{Re}\, z < 0$, we can define the complex power

$$A_z \colon x^{\alpha-m} H_b^s(X) \longrightarrow x^{\alpha-m} H_b^s(X) \quad \text{for any } s \in \mathbb{R},$$

directly by (2.4). The mapping property (3.8) shows that the integral defining A_z is well-defined in the topology of $\mathcal{B}(x^{\alpha-m} H_b^s(X), x^{\alpha-m} H_b^s(X))$ for $\mathrm{Re}\, z < 0$.

Proposition 3.5. *For all $k \in \mathbb{Z}^+$ and $z, w \in \mathbb{C}$ with $\mathrm{Re}\, z, \mathrm{Re}\, w, \mathrm{Re}\,(z+w) < 0$, we have*

$$A_{-k} = A^{-k}, \quad A_z A_w = A_{z+w}. \tag{3.15}$$

Proof. Let $u \in x^{\alpha-m} H_b^s(X)$. Using the group property of the heat operator $e^{-sA} e^{-tA} = e^{-(t+s)A}$, then changing variables $t \mapsto r = s + t$, and using that $e^{-tA} \to 0$ to infinite order as $t \to \infty$ to justify interchanging integrals, we obtain

$$\begin{aligned} A_z A_w u &= \frac{1}{\Gamma(-z)\Gamma(-w)} \int_0^\infty \int_0^\infty s^{-z} t^{-w} e^{-(s+t)A} u \frac{dt}{t} \frac{ds}{s} \\ &= \frac{1}{\Gamma(-z)\Gamma(-w)} \int_0^\infty \int_s^\infty s^{-z} (r-s)^{-w} e^{-rA} u \frac{dr}{r} \frac{ds}{s} \\ &= \frac{1}{\Gamma(-z)\Gamma(-w)} \int_0^\infty \int_0^r s^{-z} (r-s)^{-w} \frac{ds}{s}\, e^{-rA} u \frac{dr}{r} \\ &= \frac{B(-z,-w)}{\Gamma(-z)\Gamma(-w)} \int_0^\infty r^{-z-w} e^{-rA} u \frac{dr}{r}, \end{aligned} \tag{3.16}$$

where

$$B(-z,-w) = \int_0^1 t^{-z} (1-t)^{-w} \frac{dt}{t}$$

is the beta function. Since $B(-z,-w) = \Gamma(-z)\Gamma(-w)/\Gamma(-z-w)$ is another expression for the beta function, (3.16) implies that $A_z A_w = A_{z+w}$.

Let $B := A A_{-1}$, which a priori maps $x^{\alpha-m} H_b^s(X)$ into $x^{\alpha-2m} H_b^{s-m}(X)$. However, by Corollary 1.3, which is proved independent of this proposition, A_{-1} maps $x^{\alpha-m} H_b^s(X)$ into $x^{\alpha} H_b^{s+m}(X)$. Thus, B is a map on $x^{\alpha-m} H_b^s(X)$. We show that $B = \mathrm{Id}$. Indeed, the heat equation (3.9) implies that for $u \in x^{\alpha-m} H_b^s(X)$, in the topology of $x^{\alpha-2m} H_b^{s-m}(X)$ we have

$$Bu = \int_0^\infty A e^{-tA} u\, dt = -\int_0^\infty \partial_t e^{-tA} u\, dt = u,$$

where we have used that $e^{-tA} \to 0$ as $t \to \infty$, and that $e^{-tA}|_{t=0} = \mathrm{Id}$. Thus, $A_{-1} = A^{-1}$. The composition property $A_z A_w = A_{z+w}$ then implies that $A_{-k} = A^{-k}$ for each $k \in \mathbb{Z}^+$. □

Now we can proceed to define the complex power A_z for $\mathrm{Re}\, z \geq 0$ by the usual method. Let $z \in \mathbb{C}$ and choose $\ell \in \mathbb{N}_0$ such that $\mathrm{Re}\, z - \ell < 0$; then define

$$A_z := A^\ell A_{z-\ell} \colon x^{\alpha-m} H_b^s(X) \longrightarrow x^{\alpha-m-m\ell} H_b^{s-m\ell}(X). \tag{3.17}$$

The identities (3.15) imply the following.

Corollary 3.6. *The complex power A_z is well-defined, independent of the ℓ chosen in* (3.17) *(in particular, the image of A_z in* (3.17) *is actually independent of ℓ), and $A_k = A^k$ for all $k \in \mathbb{Z}$.*

Remark 3.7. Once we know the structure of the complex power as an entire family of b-pseudodifferential operators as in Theorem 1.2, the image space in (3.17) can be identified precisely (see Corollary 1.3) by using the mapping properties of the b-calculus described in [13, Sec. 3]. The composition property $A_z A_w = A_{z+w}$ would also hold as b-pseudodifferential operators for all $z, w \in \mathbb{C}$ by using the composition properties of the b-calculus found in [14].

We now analyze the structure of the complex power as a b-pseudodifferential operator. The following lemma proves both Theorem 1.1 and Theorem 1.2. The proof uses Lemma 3.10, which we prove shortly.

Lemma 3.8. *For* $\operatorname{Re} z < 0$, *we can write*

$$A_z = B(z) + C(z), \quad \text{where } D(z) = \frac{1}{\Gamma(-z)} \int_0^\infty t^{-z} D(t) \frac{dt}{t},\ D = B, C,$$

where $B(t)$ *and* $C(t)$ *are given in* (3.10) *of Theorem 3.4.*

(1) The operator $B(z)$ *extends from* $\operatorname{Re} z < 0$ *to be an entire function of* $z \in \mathbb{C}$, *with values in* $\varrho^{-mz}\Psi_b^{m \operatorname{Re} z}(X)$. *That is, for each* $r \in \mathbb{R}$, *the map*

$$\{z \in \mathbb{C} \, : \, \operatorname{Re} z < r\} \ni z \,\mapsto\, \varrho^{mz} B(z) \in \Psi_b^{m \operatorname{Re} z}(X)$$

is holomorphic as a map into $\Psi_b^{mr}(X)$. *Furthermore,* $b(z) := K_{B(z)}|_\Delta$, *where* Δ *is the diagonal* $\Delta \cong X$ *in* X^2, *has the properties given in Theorem 1.1.*

(2) The operator $C(z)$ *extends from* $\operatorname{Re} z < 0$ *to be an entire function of* z, *vanishing for* $z \in \mathbb{N}_0$, *with values in*

$$C(z) \in \Psi_z := \begin{cases} \varrho^m \Psi_{b,\infty}^{-\infty,\mathcal{F}(\alpha)}(X), & \operatorname{Re} z < -1; \\ \varrho^{-mz} \Psi_{b,\infty}^{-\infty,\mathcal{F}(\alpha)}(X), & \operatorname{Re} z \geq -1. \end{cases}$$

That is, for each $r \in \mathbb{R}$, *the map* $\{z \in \mathbb{C} \, : \, \operatorname{Re} z < r\} \ni z \,\mapsto\, C(z) \in \Psi_z \subset \Psi_r$ *is holomorphic as a map into* Ψ_r. *Moreover, for each* $z \in \mathbb{C}$,

$$C(z) \in \Psi_b^{-\infty,\mathcal{F}(\alpha),E_z(\alpha)}(X), \text{ where } E_z(\alpha) = (\mathbb{N}_0 - mz)\overline{\cup} E(\alpha),$$

and can be considered a meromorphic function on $\mathbb{C}$ *with values in the polyhomogeneous space* $\varrho^{-mz}\Psi_b^{-\infty,\mathcal{F}(\alpha),\mathbb{N}_0}(X) + \Psi_b^{-\infty,\mathcal{F}(\alpha),E(\alpha)}(X)$, *with (possible) poles occurring only on the discrete set* $\{z \in \mathbb{C} \, : \, (\mathbb{N}_0 - mz) \cap E(\alpha) \neq \varnothing\}$. *The poles only occur in the expansion up to* ff, *where the meaning of meromorphic, along with the order of the poles, are explained in Part (*3*) of Lemma 3.10 below. Furthermore,* $c(z) := K_{C(z)}|_\Delta$, *where* Δ *is the diagonal* $\Delta \cong X$ *in* X^2, *has the properties given in Theorem 1.1.*

(3) The Schwartz kernel of A_z *restricted to any compact subset of* X^2 *disjoint to the diagonal extends to be entire with values in* $\mathcal{A}_{phg}^{\mathcal{F}(\alpha)}(X^2)$.

Proof. To prove (1), we need to analyze the Schwartz kernel of $B(z)$. First consider the kernel away from Δ_b. Let $\varphi \in C_c^\infty(X_b^2 \setminus \Delta_b)$. Then, making the change of variables $t \mapsto t\varrho^m$, we can write

$$\varphi K_{B(z)} \;=\; \frac{1}{\Gamma(-z)} \int_0^\infty t^{-z} k(\frac{t}{\varrho^m}, p) \frac{dt}{t} \;=\; \frac{\varrho^{-mz}}{\Gamma(-z)} \int_0^\infty t^{-z} k(t,p) \frac{dt}{t},$$

where $k(t,p)$ is given in (B)$'$ of Theorem 3.4. The properties of $k(t,p)$ plus standard facts on the Mellin transform (see [1, Ch. 4]), imply that $\varphi K_{B(z)}$ extends from $\operatorname{Re} z < 0$ to be an entire function of $z \in \mathbb{C}$, with values in $\varrho^{-mz} C^\infty(X_b^2)$ vanishing in Taylor series at lb and rb. Now let $\varphi \in C^\infty(X_b^2)$ be supported in a patch $\mathcal{U}_y \times \mathbb{R}^n_w$ on X_b^2 such that $\Delta_b \cong \mathcal{U}_y \times \{0\}$. Then, writing $\varphi K_{B(t)}$ as in (3.11), we have

$$\varphi K_{B(z)} = \int e^{iw\cdot\xi}\, b_z(\varrho, y, \xi)\, đ\xi,$$

where making the change of variables $t \mapsto t\varrho^m$ yields

$$b_z(\varrho, y, \xi) \ := \ \frac{1}{\Gamma(-z)} \int_0^\infty t^{-z} b(\frac{t}{\varrho^m}, y, \xi) \frac{dt}{t} \ = \ \frac{\varrho^{-mz}}{\Gamma(-z)} \int_0^\infty t^{-z} b(t, y, \xi) \frac{dt}{t}.$$

Set

$$b_z(y, \xi) := \frac{1}{\Gamma(-z)} \int_0^\infty t^{-z} b(t, y, \xi) \frac{dt}{t}.$$

Since $\Delta \cong \Delta_b \cong X$, we may assume that $\varrho|_\Delta = x$. Then

$$\varphi K_{B(z)} = \varrho^{-mz} \int e^{iw\cdot\xi}\, b_z(y, \xi)\, đ\xi, \quad \varphi K_{B(z)}|_\Delta = x^{-mz} \int b_z(y, \xi)\, đ\xi.$$

The properties of $b_z(y, \xi)$ were analyzed in Theorem 2.2; see also Remark 2.3. This completes the proof of statement (1).

Setting $t = s^m$, we can write

$$K_{C(z)} = \frac{1}{\Gamma(-z)} \int_0^\infty s^{-zm} c(s, \frac{\varrho}{s}, p) \frac{ds}{s}, \quad c(z) = \frac{1}{\Gamma(-z)} \int_0^\infty s^{-zm} c(s, \frac{\varrho}{s}, p)|_{\Delta_b} \frac{ds}{s},$$

where the properties of $c(s, v, p)$ are explained in (D)$'$ of Theorem 3.4. Now statement (2) follows from properties (1)–(3) in Lemma 3.10 to be proved shortly.

Finally, statement (3) is a direct consequence of (1) and (2). $\square$

The following lemma proves Theorem 1.4. Its proof also uses Lemma 3.10, which we prove afterwards.

Lemma 3.9. *Write $A_z = B(z) + C(z)$ as in Lemma 3.8.*

(1) The operator $B(z)$ is trace class on $x^{\alpha-m}H^s_b(X)$ for $\operatorname{Re} z < -n/m$. Its trace $\operatorname{Tr} B(z)$ is holomorphic for $\operatorname{Re} z < -n/m$ and extends to be meromorphic on the complex plane with (possible) simple poles on the set $\{\frac{k-n}{m} : k \in \mathbb{N}_0\}$ and with (possible) double poles on the set $\{\frac{k}{m} : k \in \mathbb{N}_0, \frac{k}{m} \notin \mathbb{N}_0\}$.

(2) The operator $C(z)$ is trace class on $x^{\alpha-m}H^s_b(X)$ for $\operatorname{Re} z < 0$. Its trace $\operatorname{Tr} C(z)$ is holomorphic for $\operatorname{Re} z < 0$ and extends to be meromorphic on the complex plane with (possible) simple poles on the set $\{\frac{k}{m} : k \in \mathbb{N}_0, \frac{k}{m} \notin \mathbb{N}_0\}$.

Proof. By Lemma 3.8, $B(z)$ is holomorphic with values in $\varrho^{-mz}\Psi_b^{m\operatorname{Re} z}(X)$. Hence, as operators in $\varrho^\varepsilon \Psi_b^{-n-\varepsilon}(X)$ for any $\varepsilon > 0$ are trace class on $x^{\alpha-m}H^s_b(X)$ (see [14, Sec. 4.18]), it follows that $B(z)$ is trace class for $\operatorname{Re} z < -n/m$. Let $d\mathfrak{m}$ denote the fixed b-measure on X. Then, by Lemma 3.8, $K_{B(z)}|_\Delta = x^{-mz} f(z) d\mathfrak{m}$, where $f(z)$ takes values in $C^\infty(X)$ and is meromorphic on $\mathbb{C}$ with (possible) simple poles only on the discrete set $\{\frac{k-n}{m} : k \in \mathbb{N}_0, \frac{k-n}{m} \notin \mathbb{N}_0\}$. Expanding $f(z)$ in a Taylor series at $x = 0$ near the boundary and using the fact that

$$\int_0^1 x^{-mz+k} \frac{dx}{x} = \frac{1}{-mz+k}, \quad k \in \mathbb{N}_0,$$

we see that $\int_X x^{-mz} f(z) d\mathfrak{m}$ is holomorphic for $\operatorname{Re} z < -n/m$ and extends to be meromorphic on the whole complex plane with (possible) simple poles on the set $\{\frac{k-n}{m} : k \in \mathbb{N}_0\}$ and with (possible) double poles on the set $\{\frac{k}{m} : k \in \mathbb{N}_0, \frac{k}{m} \notin \mathbb{N}_0\}$. Thus, statement (1) is proved.

Arguing as we did for $B(z)$ shows that $C(z)$ is trace class on $x^{\alpha-m}H_b^s(X)$ for $\operatorname{Re} z < 0$, where $\operatorname{Tr} C(z) = \int_X K_{C(z)}|_\Delta$. Since $\Delta \cong \Delta_b \cong X$, we may assume that $\varrho|_\Delta = x$. Then we can write

$$\operatorname{Tr} C(z) = \frac{1}{\Gamma(-z)} \int_0^\infty \int_X s^{-zm} c(s, \frac{x(p)}{s}, p)|_\Delta \cdot d\mathfrak{m}(p) \frac{ds}{s},$$

where $t = s^m$ and the properties of $c(s, v, p)$ are explained in (D)$'$ of Theorem 3.4. Thus, (2) follows from statement (4) of Lemma 3.10 below. $\square$

The following technical lemma finishes off the proofs of the previous two lemmas.

Lemma 3.10. *Let $u(s,t)$ be a smooth function of $s \in [0,\infty)$ and $t \in (0,\infty)$, vanishing to infinite order as $|(s,t)| \to \infty$ and having an expansion at $t = 0$ with an index set E such that $\operatorname{Re} E > m$, where $m \in \mathbb{Z}^+$. For $\operatorname{Re} z < 0$, define*

$$u_z(\varrho) := \frac{1}{\Gamma(-z)} \int_0^\infty s^{-mz} u(s, \frac{\varrho}{s}) \frac{ds}{s}, \quad \varrho \in [0,1).$$

(1) The function $u_z(\varrho)$ extends from $\operatorname{Re} z < 0$ to be an entire function of $z \in \mathbb{C}$, vanishing for $z \in \mathbb{N}_0$, with values in

$$u_z(\varrho) \in S_z := \begin{cases} \varrho^m S^{0,0}([0,1)), & \operatorname{Re} z < -1; \\ \varrho^{-mz} S^{0,0}([0,1)), & \operatorname{Re} z \geq -1, \end{cases}$$

in the sense that for each $r \in \mathbb{R}$, the map $\{z \in \mathbb{C} : \operatorname{Re} z < r\} \ni z \mapsto u_z \in S_z \subset S_r$ is holomorphic as a map into S_r.

(2) For each $z \in \mathbb{C}$ (see (3.14) for the definition of the extended union $\overline{\cup}$),

$$u_z(\varrho) \in \mathcal{A}^{E_z}_{phg}([0,1)), \quad \text{where } E_z = (\mathbb{N}_0 - mz)\overline{\cup}E.$$

(3) The function $u_z(\varrho)$ can be considered a meromorphic function on $\mathbb{C}$ with values in the polyhomogeneous space

$$u_z(\varrho) \in \varrho^{-mz} C^\infty([0,1)) + \mathcal{A}^E_{phg}([0,1)),$$

with (possible) poles occurring only on the discrete set

$$\mathcal{D} = \{z \in \mathbb{C} : (\mathbb{N}_0 - mz) \cap E \neq \varnothing\}.$$

Here, "meromorphic" means the following.

(a) If $z_0 \notin \mathcal{D}$, then given any $N >> 0$, for $z \in \mathbb{C}$ sufficiently near z_0, we can write

(3.18)
$$u_z(\varrho) = \varrho^{-mz} \sum_{\ell \in \mathbb{N}_0, \ell - m\operatorname{Re} z_0 \leq N} \varrho^\ell u_\ell(z) + \sum_{(\zeta,k) \in E, \operatorname{Re}\zeta \leq N} \varrho^\zeta (\log \varrho)^k u_{(\zeta,k)}(z) + \varrho^N u_N(z,\varrho),$$

where $u_\ell(z), u_{(\zeta,k)}(z)$, and $u_N(z,\varrho)$ are holomorphic in z for z near z_0, and where $u_N(z,\varrho)$ takes values in $S^{0,0}([0,1)_\varrho)$.

(b) If $z_0 \in \mathcal{D}$, then given any $N >> 0$, for $z \in \mathbb{C}$ sufficiently near z_0, we can write

$$\begin{aligned} u_z(\varrho) \;=\; & \varrho^{-mz} \sum_{\substack{\ell \in \mathbb{N}_0, \ell - m\operatorname{Re} z_0 \leq N \\ \ell - mz_0 \notin E}} \varrho^\ell u_\ell(z) + \sum_{\substack{(\zeta,k) \in E, \operatorname{Re}\zeta \leq N \\ \zeta \notin \mathbb{N}_0 - mz_0}} \varrho^\zeta (\log\varrho)^k u_{(\zeta,k)}(z) \\ & + \varrho^{-mz} \sum_{\substack{\ell \in \mathbb{N}_0, \ell - m\operatorname{Re} z_0 \leq N \\ \ell - mz_0 \in E}} \varrho^\ell u'_\ell(z) + \sum_{\substack{(\zeta,k) \in E, \operatorname{Re}\zeta \leq N \\ \zeta \in \mathbb{N}_0 - mz_0}} \varrho^\zeta (\log\varrho)^k u'_{(\zeta,k)}(z) \;+\; \varrho^N u_N(z,\varrho), \end{aligned} \tag{3.19}$$

where $u_\ell(z), u_{(\zeta,k)}(z)$, and $u_N(z,\varrho)$ are holomorphic near $z = z_0$, $u_N(z,\varrho)$ takes values in $S^{0,0}([0,1)_\varrho)$, and $u'_\ell(z)$ and $u'_{(\zeta,k)}(z)$ have a pole at $z = z_0$. If $\ell - mz_0 = \zeta \in E$ with $(\zeta, k_0) \in E$ but $(\zeta, k_0+1) \notin E$ for some k_0, then $u'_\ell(z)$ has a pole of order k_0+1 at $z = z_0$ and $u'_{(\zeta,k)}(z)$ has a pole of order k_0+1-k at $z = z_0$; if $z_0 \in \mathbb{N}_0$, then the poles are of order k_0 and $k_0 - k$, respectively.

(4) *Define $u(z) := \int_0^\infty u_z(\varrho)\frac{d\varrho}{\varrho}$. Then $u(z)$ is holomorphic for $\operatorname{Re} z < 0$, and extends to be meromorphic on the whole complex plane with (possible) simple poles on the set $\{\frac{k}{m} : k \in \mathbb{N}_0,\ \frac{k}{m} \notin \mathbb{N}_0\}$.*

Proof. We first prove (1). Observe that we can write $u(s,t) = t^m v(s,t)$, where $v(s,t) \in t^\varepsilon S^{0,0}([0,\infty)_s \times [0,\infty)_t)$ for some $\varepsilon > 0$, and where $v(s,t)$ vanishes to infinite order as $|(s,t)| \to \infty$. Then

$$u_z(\varrho) = \frac{\varrho^m}{\Gamma(-z)} \int_0^\infty s^{-mz-m} v(s, \frac{\varrho}{s}) \frac{ds}{s}, \quad \varrho > 0.$$

Since $v(s,t) \in t^\varepsilon S^{0,0}([0,\infty)_s \times [0,\infty)_t)$ and $s^{-mz-m-1}$ is integrable in a neighborhood of $s = 0$ for $\operatorname{Re} z < -1$, it follows that $u_z(\varrho) \in \varrho^m S^{0,0}([0,1))$ for $\operatorname{Re} z < -1$ and is holomorphic in z. Now make the change of variables $s \mapsto \varrho/s$ (for $\varrho > 0$). We have

$$u_z(\varrho) = \frac{\varrho^{-mz}}{\Gamma(-z)} \int_0^\infty s^{mz+m} v(\frac{\varrho}{s}, s) \frac{ds}{s}.$$

Since $v(s,t) \in t^\varepsilon S^{0,0}([0,\infty)_s \times [0,\infty)_t)$ and $s^{mz+m+\varepsilon-1}$ is integrable in a neighborhood of $s = 0$ for $\operatorname{Re} z \geq -1$, it follows that $u_z(\varrho) \in \varrho^{-mz} S^{0,0}([0,1))$ for $\operatorname{Re} z \geq -1$ and is holomorphic in z. Since $1/\Gamma(-z)$ vanishes at $z \in \mathbb{N}_0$, it follows that $u_z(\varrho)$ vanishes for $z \in \mathbb{N}_0$. Thus, (1) is proved.

We now prove statement (2). Let $\mathcal{M}(u_z)(\xi)$ denote the Mellin transform of $u_z(\varrho)$

$$\begin{aligned} \mathcal{M}(u_z)(\xi) &:= \int_0^\infty \varrho^{-\xi} u_z(\varrho) \frac{d\varrho}{\varrho} \\ &= \frac{1}{\Gamma(-z)} \int_0^\infty \varrho^{-\xi} \int_0^\infty s^{-mz} u(s, \frac{\varrho}{s}) \frac{ds}{s} \frac{d\varrho}{\varrho} \\ &= \frac{1}{\Gamma(-z)} \int_0^\infty t^{-\xi} \int_0^\infty s^{-\xi-mz} u(s,t) \frac{ds}{s} \frac{dt}{t}, \quad \varrho \mapsto t = \varrho/s \\ &= \frac{1}{\Gamma(-z)} \mathcal{M}(u)(\xi + mz, \xi), \end{aligned} \tag{3.20}$$

where

$$\mathcal{M}(u)(\xi,\eta) := \int_0^\infty \int_0^\infty s^{-\xi} t^{-\eta} u(s,t) \frac{ds}{s} \frac{dt}{t}$$

is the Mellin transform of $u(s,t)$. Taking the inverse Mellin transform of $\mathcal{M}(u_z)(\xi)$, we have for any $r >> 0$

$$u_z(\varrho) = \frac{1}{2\pi i \Gamma(-z)} \int_{\operatorname{Re}\xi = -r} \varrho^\xi \mathcal{M}(u)(\xi + mz, \xi)\, d\xi. \tag{3.21}$$

Since $u(s,t)$ is smooth at $s = 0$ and has an expansion at $t = 0$ with an index set E, by [1, Sec. 4.3], $\mathcal{M}(u)(\xi,\eta)$ is meromorphic on $\mathbb{C}$ with simple poles at $\xi \in \mathbb{N}_0$ and a pole of order $k+1$ at $\eta = \zeta$, where $(\zeta,k) \in E$ but $(\zeta, k+1) \notin E$.

In what follows, we use the following fact. If $\ell \in \mathbb{N}_0$, $\psi(\xi)$ is holomorphic in a neighborhood of $\xi_0 \in \mathbb{C}$, and C is a closed counter-clockwise curve sufficiently near ξ_0, then by Cauchy's theorem,

$$\frac{1}{2\pi i}\int_C \varrho^\xi \frac{\psi(\xi)}{(\xi-\xi_0)^{\ell+1}}\,d\xi = \partial_\xi^\ell(\varrho^\xi\psi(\xi))|_{\xi=\xi_0} = \sum_{k=0}^{\ell}\varrho^{\xi_0}(\log\varrho)^k a_k, \tag{3.22}$$

where $a_k = \binom{\ell}{k}(\partial_\xi^{\ell-k}\psi)(\xi_0)$. Now shifting the contour $\operatorname{Re}\xi = -r$ in (3.21) to the right, all the way to $\operatorname{Re}\xi = \infty$, introduces contour integrals around the poles of $\mathcal{M}(u)(\xi + mz, \xi)$. One can check that, for fixed z, $\mathcal{M}(u)(\xi+mz,\xi)$ has a pole of order $\ell+1$ at $\xi = \zeta$, where $(\zeta,\ell) \in E_z = (\mathbb{N}_0 - mz)\overline{\cup}E$ but $(\zeta,\ell+1) \notin E_z$. Thus, by (3.22) and the structure of the poles of $\mathcal{M}(u)(\xi+mz,\xi)$, it follows that $u_z(\varrho)$ has an expansion at $\varrho = 0$ with index set E_z. This proves statement (2).

Consider now statement (3). Let $z_0 \notin \mathcal{D}$ and let $N >> 0$. Then for z sufficiently near z_0, we have

$$(\mathbb{N}_0 - mz) \cap \{(\zeta,k) \in E \,:\, \operatorname{Re}\zeta \le N\} = \varnothing.$$

Thus, if z is sufficiently near $z_0 \notin \mathcal{D}$, then for $\operatorname{Re}\xi \le N$, $\mathcal{M}(u)(\xi+mz,\xi)$ has simple poles at $\xi \in \mathbb{N}_0 - mz$ and a pole of order $k+1$ at $\xi = \zeta$, where $(\zeta,k) \in E$ but $(\zeta,k+1) \notin E$. Thus, by Cauchy's theorem and (3.22), in the integral (3.21), we can push the contour $\operatorname{Re}\xi = -r$ to the right until we get just past the contour $\operatorname{Re}\xi = N$, picking up the expansion (3.18) along the way.

Now let $z_0 \in \mathcal{D}$ and let $N >> 0$. Then for z sufficiently near z_0 but not equal to z_0, again by Cauchy's theorem and (3.22), in the integral (3.21), we can push the contour $\operatorname{Re}\xi = -r$ to the right until we get just past the contour $\operatorname{Re}\xi = N$, this time picking up the expansion

$$\begin{aligned} u_z(\varrho) \;&=\; \varrho^{-mz}\sum_{\substack{\ell\in\mathbb{N}_0,\,\ell-m\operatorname{Re}z_0\le N\\ \ell-mz_0\notin E}}\varrho^\ell u_\ell(z) + \sum_{\substack{(\zeta,k)\in E,\operatorname{Re}\zeta\le N\\ \zeta\notin\mathbb{N}_0-mz_0}}\varrho^\zeta(\log\varrho)^k u_{(\zeta,k)}(z)\\ &\qquad + v(z,\varrho) + \varrho^N u_N(z,\varrho), \end{aligned}$$

where $u_\ell(z), u_{(\zeta,k)}(z)$, and $u_N(z,\varrho)$ are holomorphic near $z = z_0$, $u_N(z,\varrho)$ takes values in $S^{0,0}([0,1)_\varrho)$, and

$$v(z,\varrho) = \sum_{\substack{(\zeta,k)\in E,\operatorname{Re}\zeta\le N\\ \zeta\in\mathbb{N}_0-mz_0}}\frac{1}{2\pi i\Gamma(-z)}\int_{C_\zeta}\varrho^\xi\mathcal{M}(u)(\xi+mz,\xi)\,d\xi. \tag{3.23}$$

Here, if $\zeta = \ell - mz_0$ for some $\ell \in \mathbb{N}_0$, then C_ζ is a small counter-clockwise curve about ζ that contains both $\ell - mz$ and ζ, but no other $\ell' - mz$ or ζ' where $(\zeta',k) \in E$ for some k.

Let $\zeta = \ell - mz_0$, with $\ell \in \mathbb{N}_0$ and $\operatorname{Re}\zeta \le N$, and suppose that $(\zeta,k_0) \in E$ but $(\zeta,k_0+1) \notin E$. Then by [1, Sec. 4.3], $\mathcal{M}(u)(\xi+mz,\eta)$ has a simple pole at $\xi = \ell - mz$ and a pole of order k_0+1 at $\eta = \zeta$. Thus, on and inside the curve C_ζ we can write

$$\mathcal{M}(u)(\xi+mz,\xi) = \frac{a(\xi)}{(\ell-mz-\xi)(\zeta-\xi)^{k_0+1}}, \tag{3.24}$$

where $a(\xi)$ is holomorphic on and inside C_ζ. Plugging (3.24) into (3.23) and using (3.22) then shows that

$$\frac{1}{2\pi i\Gamma(-z)}\int_{C_\zeta}\varrho^{\xi}\mathcal{M}(u)(\xi+mz,\xi)\,d\xi=\varrho^{-mz+\ell}b_\ell(z)+\sum_{k=0}^{k_0}\varrho^{\zeta}(\log\varrho)^k c_k(z), \tag{3.25}$$

where

$$b_\ell(z)=\frac{a(\ell-mz)}{\Gamma(-z)(\zeta-\ell+mz)^{k_0+1}}$$

and

$$c_k(z)=\frac{1}{\Gamma(-z)}\binom{k_0}{k}\partial_\xi^{k_0-k}\Big(\frac{a(\xi)}{\ell-mz-\xi}\Big)\Big|_{\xi=\zeta}.$$

Thus, plugging (3.25) into (3.23) proves the second line of (3.19) with the required singularities. Note that since $1/\Gamma(-z)$ vanishes at $z\in\mathbb{N}_0$, the singularities are one less when $z\in\mathbb{N}_0$. Statement (3) is now proved.

It remains to prove statement (4). The calculation given in (3.20) shows that

$$u(z):=\int_0^\infty u_z(\varrho)\frac{d\varrho}{\varrho}=\frac{1}{\Gamma(-z)}\mathcal{M}(u)(mz,0),$$

where $\mathcal{M}(u)(\xi,\eta)$ is the Mellin transform of $u(s,t)$. As already discussed after (3.21), $\mathcal{M}(u)(\xi,\eta)$ has simple poles at $\xi\in\mathbb{N}_0$ and a pole of order $k+1$ at $\eta=\zeta$, where $(\zeta,k)\in E$ but $(\zeta,k+1)\notin E$. Thus, as $\operatorname{Re}E>m$, $\mathcal{M}(u)(mz,0)$ has only simple poles at $mz\in\mathbb{N}_0$. Since $\frac{1}{\Gamma(-z)}$ vanishes for $z\in\mathbb{N}_0$, (4) is proved. $\square$

Appendix A. The category of b-objects

In this appendix, we give an informal presentation of the b-calculus. There is now a wealth of resources containing information about the b-calculus; see, for example, [14], [13], [11], or [12]. Let X be a manifold with boundary on which there is a fixed b-measure $d\mathfrak{m}$. We begin by describing various classes of conormal functions. First, the space $S^{0,0}(X)$ of Hölder differentiable symbols of order 0 is the space of continuous functions u on X such that all b-derivatives of u are also continuous, that is, such that $Bu\in C^0(X)$ for all $B\in\operatorname{Diff}_b^*(X)$. The space of functions that are square integrable with respect to $d\mathfrak{m}$ is denoted by $L^2_b(X)$. Given $m\in\mathbb{N}_0$, the b-Sobolev space $H^m_b(X)$ consists of those $u\in L^2_b(X)$ such that all b-derivatives of u up to order m are also square integrable. By interpolation and duality, $H^m_b(X)$ can be defined for any $m\in\mathbb{R}$.

An index set E is just a discrete subset of $\mathbb{C}\times\mathbb{N}_0$ such that if $(\zeta,k)\in E$, then $(\zeta+\ell,j)\in E$ for $\ell\in\mathbb{N}_0$ and $0\le j\le k$ and such that given any $N\in\mathbb{N}_0$, $\{(\zeta,k)\in E\,:\,\operatorname{Re}\zeta\le N\}$ is a finite set. Given an index set E, a smooth function u on the interior of X is said to have an (asymptotic) expansion at ∂X with index E if given any collar $X\cong[0,\varepsilon)_x\times\partial X$, for each $N\in\mathbb{N}_0$ we can write

$$u(x,y)=\sum_{(\zeta,k)\in E,\operatorname{Re}\zeta\le N}x^\zeta(\log x)^k u_{(\zeta,k)}(y)\;+\;x^N u_N(x,y), \tag{A.1}$$

where $u_{(\zeta,k)}(y)\in C^\infty(\partial X)$, and where $u_N(x,y)$ is bounded, with all b-derivatives, at $x=0$. The space of such functions is denoted by $\mathcal{A}^E_{phg}(X)$. One can also define the space of polyhomogeneous functions for any manifold with corners using (A.1) plus induction on the codimension of the manifold; see [14].

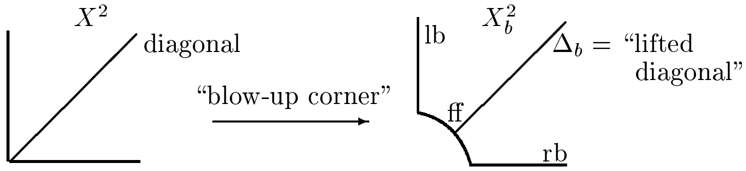

FIGURE 1. One of the "standard" pictures of X_b^2.

We now describe the calculus of b-pseudodifferential operators. The Schwartz kernels of these operators live on the manifold with corners X_b^2. This manifold is constructed by radially blowing-up the corner $(\partial X)^2$ in X^2; or in other words, is defined by replacing the usual coordinates about $(\partial X)^2$ in X^2 with polar coordinates. The analytical process of blow-up can be found in [14], [13]. The new face created in the blow-up is denoted by ff and is called the front face. The familiar picture of X_b^2 along with its various boundary faces is found in Figure 1.

Let $d\mathfrak{m}'$ be the b-measure $d\mathfrak{m}$ lifted to X^2 under the projection onto the right factor $X^2 \ni (p,q) \mapsto q \in X$. Given $m \in \mathbb{R}$, the space of b-pseudodifferential operators of order m, $\Psi_b^m(X)$, is the space of operators B on smooth functions that have a Schwartz kernel K_B satisfying the following two conditions:

(1) Given $\varphi \in C_c^\infty(X_b^2 \setminus \Delta_b)$, the kernel φK_B is of the form $k\, d\mathfrak{m}'$, where $k \in C^\infty(X_b^2)$ and vanishes in Taylor series at the sets lb and rb.
(2) Given a coordinate patch of X_b^2 overlapping Δ_b of the form $\mathcal{U}_y \times \mathbb{R}^n_w$ such that $\Delta_b \cong \mathcal{U} \times \{0\}$, and given $\varphi \in C_c^\infty(\mathcal{U} \times \mathbb{R}^n)$, we have
$$\varphi K_B = \int e^{iw\cdot\xi}\, b(y,\xi) đ\xi \cdot d\mathfrak{m}',$$
where $b(y,\xi)$ is a symbol of order m.

Given an index family $\mathcal{E} = (E_{\mathrm{lb}}, E_{\mathrm{rb}})$, we denote by $\Psi_{b,\infty}^{-\infty,\mathcal{E}}(X)$, the space of operators that have a Schwartz kernel of the form $k\, d\mathfrak{m}'$, where k is a function on X_b^2 that has expansions at lb and rb of X_b^2 with index sets E_{lb} and E_{rb} respectively, and that is continuous, with all b-derivatives, up to the front face ff of X_b^2. The subscript "∞" refers to the "calculus with bounds". If F is an additional index set, then $\Psi_b^{-\infty,\mathcal{E},F}(X)$ consists of those operators that have a Schwartz kernel of the form $k\, d\mathfrak{m}'$, where k is a function on X_b^2 that has expansions at lb, rb, and ff with index sets E_{lb}, E_{rb}, and F respectively.

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