AN(OTHER) ELEMENTARY PROOF THE FUNDAMENTAL THEOREM OF ALGEBRA

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In this note we prove the Fundamental Theorem of Algebra without using any topology or analytic function theory. The only requirements are familiarity with complex numbers (including the complex exponential for pure imaginary arguments), properties of differentiation and integration of complex-valued functions with respect to real variables, and Leibniz's rule for integrals [1, p. 222]: If $f(t,x)$ and $\partial_t f(t,x)$ are continuous functions on $[a,b]\times[c,d]$, then $F(t)=\int_c^d f(t,x)\,dx$ is differentiable on $[a,b]$, and

$$F'(t)=\int_c^d \partial_t f(t,x)\,dx.$$

See the beautiful book [2] for other proofs of the fundamental theorem.

Let $p(z)=z^n+a_{n-1}z^{n-1}+\cdots+a_0$ be a polynomial with complex coefficients, $n\geq 1$, and suppose, as usual, that p has no roots. For real numbers t and x, define

$$f(t,x)=\frac{1}{p(te^{ix})}=\frac{1}{t^n\,e^{inx}+\cdots+a_1t\,e^{ix}+a_0}. \tag{1}$$

Since, by assumption, the bottom is never zero and is a continuously differentiable function of t and x in $\mathbb{R}\times\mathbb{R}$, the function $f(t,x)$ is also a continuously differentiable function on $\mathbb{R}\times\mathbb{R}$. Define

$$F(t)=\int_0^{2\pi} f(t,x)\,dx.$$

By Leibniz's rule for integrals, $F(t)$ is a differentiable function of t, and

$$F'(t)=\int_0^{2\pi}\partial_t f(t,x)\,dx.$$

We claim that $F'=0$, which shows that F is constant. Indeed, using the quotient (or reciprocal) rule, we obtain

$$\partial_t f(t,x)=-\frac{nt^{n-1}\,e^{inx}+\cdots+a_1e^{ix}}{(t^n\,e^{inx}+\cdots+a_1t\,e^{ix}+a_0)^2}.$$

On the other hand, differentiating with respect to x, we find

$$\partial_x f(t,x)=-\frac{t^n\,e^{inx}\cdot in+\cdots+a_1t\,e^{ix}\cdot i}{(t^n\,e^{inx}+\cdots+a_1t\,e^{ix}+a_0)^2}=it\,\partial_t f(t,x).$$

Thus,

$$F'(t)=\frac{1}{it}\int_0^{2\pi}\partial_x f(t,x)\,dx=\frac{1}{it}f(t,x)\Big|_{x=0}^{x=2\pi}=\frac{1}{it}\left[\frac{1}{p(te^{i2\pi})}-\frac{1}{p(te^{i0})}\right]=0,$$

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since $e^{2\pi i} = e^{i0} = 1$. Hence, $F(t)$ is constant. Setting $t = 0$, we see that

$$F(0) = \int_0^{2\pi} f(0, x)\, dx = \int_0^{2\pi} \frac{1}{p(0)}\, dx = \frac{2\pi}{p(0)},$$

so F is some nonzero constant. However, the expression (1) for $f(t, x)$ implies that $F(t) \to 0$ as $t \to \infty$. This shows that F must be the constant zero, which is impossible because we just said that F is equal to a nonzero constant. Thus, our assumption that p has roots must have been false.

References

[1] K. Davidson and A. Donsig. *Real analysis with real applications.* Prentice Hall, Upper Saddle River, NJ, 2002.

[2] B. Fine and G. Rosenberger, *The Fundamental Theorem of Algebra*, Springer-Verlag, New York, 1997.

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