# THE STRUCTURE OF THE RESOLVENT OF ELLIPTIC PSEUDODIFFERENTIAL OPERATORS

PAUL LOYA

Abstract. We show that the resolvent kernel of an elliptic $b$-pseudodifferential operator on a compact manifold with corners (of arbitrary codimension) is a polyhomogeneous, or classical, function on a certain manifold with corners. The singularities of the resolvent kernel are shown to localize near the diagonal as the resolvent parameter goes to infinity. Explicit descriptions of the expansions, including logarithmic terms, are given. In particular, the asymptotics of the resolvent restricted to the diagonal follows as a corollary. Applications to the asymptotic behavior of the heat kernel and to the analysis of the poles of complex powers are also given.

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## Contents



## 1. Introduction

In this paper, we study various classes of $b$-pseudodifferential operators on a compact manifold with corners $X$ (of arbitrary codimension) that depend on a parameter $\lambda \in \mathbb{C}$. The algebra of $b$-pseudodifferential operators is a natural extension of the usual pseudodifferential operators on closed manifolds to manifolds with cylindrical ends. In particular, if $X$ has no boundary, then these operators reduce to the usual pseudodifferential operators. The systematic study of parameter dependent pseudodifferential operators was initiated by Agmon, cf. [1, 2], and further developed by Seeley [15]

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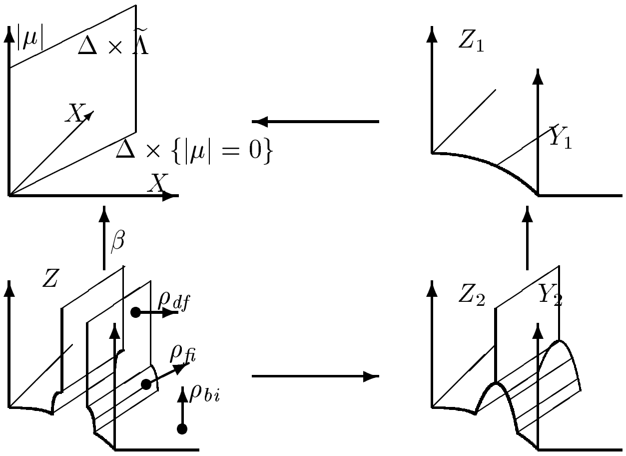


FIGURE 1. The map $\beta : Z \to X^2 \times \widetilde{\Lambda}$ is the composition of the arrows going around. The hypersurface *df* in $Z$ comes from the submanifold $\Delta \times \widetilde{\Lambda}$ in $X^2 \times \widetilde{\Lambda}$, *fi* from $\Delta \times \{|\mu| = 0\}$, and *bi* from $X^2 \times \{|\mu| = 0\}$.

in the context of pseudodifferential operators on closed manifolds. They have also been used in the case of elliptic and pseudodifferential boundary problems [17, 16, 6]. The purpose of studying such classes of operators is to provide a natural framework in which the diagonal asymptotics of the resolvent, as in (1) below, can be more readily investigated. For example, in [7], a class of 'weakly parametric' operators were used to study the resolvent of a Dirac-type operator with APS boundary conditions.

Let $A$ be a classical, elliptic pseudodifferential operator of order $m \in \mathbb{N}$ on a closed compact manifold $X$. Assume that the principal symbol of $A$ never takes values in some closed sector (a closed angle with vertex at the origin) $\Lambda \subseteq \mathbb{C}$. Then, the resolvent $(A - \lambda)^{-1}$ exists for sufficiently large $\lambda \in \Lambda$ and for each such $\lambda$, is a pseudodifferential operator of order $-m$. In particular, if we choose $N \in \mathbb{N}$ such that $-Nm < -n$, where $n = \dim X$, then for fixed $\lambda$, we can restrict the Schwartz kernel of $(A - \lambda)^{-N}$ on $X^2$, to the diagonal $\Delta \subseteq X^2$: $(A - \lambda)^{-N}|_\Delta \in C^\infty(X, \Omega)$. Here, $\Omega$ is the density bundle on $X$, and we identify $\Delta = \{(x, x) \,|\, x \in X\}$ with $X$. Moreover, as $|\lambda| \to \infty$ in $\Lambda$ we have, cf. [7],

$$(1) \qquad (A - \lambda)^{-N}|_\Delta \sim \sum_{k=0}^{\infty} \lambda^{\frac{n-k}{m} - N} \, \alpha_k(x) + \sum_{k=0}^{\infty} \lambda^{-k-N} \, \log \lambda \, \alpha'_k(x) + \sum_{k=0}^{\infty} \lambda^{-k-N} \alpha''_k(x), \text{ where } \alpha_k,\, \alpha'_k,\, \alpha''_k \in C^\infty(X, \Omega).$$

This expansion can be used to analyze the poles of the zeta function, $\mathrm{Tr}(A^{-z})$, of $A$ [8]. The zeta function can then be used to obtain index formulas [3]. One can also use the asymptotic behavior of $(A - \lambda)^{-N}|_\Delta$ to analyze the asymptotics of the spectral function [2].

Let $A$ be a classical, elliptic $b$-pseudodifferential operator of order $m \in \mathbb{R}^+$ on an arbitrary compact manifold with corners. Then our parameter dependent spaces will be used to show that if $\Lambda \subseteq \mathbb{C}$ is a sector such that the principal symbol of $A$ never takes values in it, then for any $N \in \mathbb{N}$, $(A - \lambda)^{-N}$ exists for all $\lambda \in \Lambda$ sufficiently large and is a $b$-pseudodifferential operator of order $-Nm$. Moreover, we will prove that its Schwartz kernel not only has an asymptotic expansion when restricted to the diagonal as in (1), but also that it has uniform expansions near the diagonal. In fact, the singularities the are shown to localize near the diagonal as the resolvent parameter $\lambda$ goes to infinity, as explained in the following paragraphs. The procedure to prove this is both geometric and constructive.

The general idea is as follows. Since $A$ is a $b$-pseudodifferential operator of order $m$, to study the resolvent for large $\lambda$, we will instead analyze the following operator: $R_N(\mu) := (A - \lambda)^{-N}$, where $\lambda = 1/\mu^m$. Note that the kernel of $R_N(\mu)$ is a distribution on $X^2$ depending on the parameter $\mu \in \widetilde{\Lambda}$, where $\widetilde{\Lambda} = \{\mu \,|\, 1/\mu^m \in \Lambda\}$; thus, we will consider $R_N(\mu)$ as a distribution living on $X^2 \times \widetilde{\Lambda}$.

The upper left-hand corner of Figure 1 gives a representation of $X^2 \times \widetilde{\Lambda}$ in the simple case that $X$ is a 1-dimensional manifold with boundary. The rest of the manifolds in Figure 1 are blown-up manifolds and the arrows on the top, bottom, and the right-hand side represent blow-down maps (a review of blow-ups and their blow-down maps is provided in Subsection 2.3).

If $X$ is a 1-dimensional manifold with boundary, the manifold with corners $Z$ at the bottom left-hand corner is constructed as follows. First, the manifold with corners $Z_1$ is constructed by blowing up the manifold $\partial X \times \partial X \times \widetilde{\Lambda}$ in $X^2 \times \widetilde{\Lambda}$. One can think of 'blowing up' a submanifold as the process of carving out a spherical area around the submanifold. This can be seen in blowing up $\partial X \times \partial X \times \widetilde{\Lambda}$ in $X^2 \times \widetilde{\Lambda}$. In the blown-up space $Z_1$, the manifold $\Delta \times \{|\mu| = 0\}$ becomes the manifold $Y_1$. Blowing up the manifold $Y_1$ in $Z_1$ constructs the manifold with corners $Z_2$. In this space, the manifold $\Delta \times \widetilde{\Lambda}$ becomes the manifold $Y_2$. Finally, the manifold with corners $Z$ is constructed by blowing up the manifold $Y_2$ in $Z_2$, which in the case that $X$ is 1-dimensional, just has the effect of separating $Z_2$ at the submanifold $Y_2$. The general procedure to construct $Z$ for any manifold with corners $X$ is similar (see Subsection 4.1).

We show that for any $N \in \mathbb{N}$, the distributional kernel of $R_N(\mu)$ lifted under $\beta$ to $Z$, $\beta^* R_N(\mu)$, can be realized as a polyhomogeneous *function* on the space $Z$ (Theorem 52).[1] That is, $\beta^* R_N(\mu)$ is smooth on the interior of $Z$ and has expansions at each of the faces *df*, *fi*, and *bi* of $Z$. Moreover, any $b$-pseudodifferential factor of arbitrary real order can be allowed. At the face *bi*, the kernel $\beta^* R_N(\mu)$ has no log terms in its asymptotic expansion, while at both the boundary faces *df* and *fi*, it does pick up log terms in its expansions to these faces. We also reproduce an expansion similar to (1): if $B$ a classical, $b$-pseudodifferential operator of order $m' \in \mathbb{R}$, then

$$B(A-\lambda)^{-N}|_\Delta \sim \sum_{k=0}^{\infty} \lambda^{\frac{n+m'-k}{m}-N}\, \alpha_k(x) + \sum_{k,\, \frac{k-m'-n}{m} \in \mathbb{N}_0} \lambda^{\frac{n+m'-k}{m}-N} \log\lambda\, \alpha'_k(x)$$

$$+ \sum_{k=0}^{\infty} \lambda^{-k-N} \alpha''_k(x), \text{ where } \alpha_k,\, \alpha'_k,\, \alpha''_k \in C^\infty(X, \Omega_b). \tag{2}$$

Here, $\Omega_b$ is the $b$-density bundle on $X$. This result extends the result of [7, Theorem 2.7] to operators of real order rather than just of integer order. The main applications of this result are complete descriptions of the poles of the complex power and the small time asymptotic behavior of the heat kernel.

A simpler result holds for $b$-differential operators (see Theorem 53). Here, $\beta^* R_N(\mu)$ vanishes to infinite order at the face *bi*, and has no log terms in its expansion to the face *fi*. (In its expansion to *df*, the same log terms occur as for the resolvent of $b$-pseudodifferential operators.) Also, in the case that $A$ and $B$ are $b$-differential operators, the expansion (2) now takes the form:

$$B(A-\lambda)^{-N}|_\Delta \sim \sum_{k=0}^{\infty} \lambda^{\frac{n+m'-k}{m}-N}\, \alpha_k(x), \text{ where } \alpha_k \in C^\infty(X, \Omega_b).$$

A review of the $b$-calculus is provided in Section 2. In Section 3, we introduce various classes of parameter dependent operators. We show that the resolvent of an elliptic $b$-pseudodifferential operator is such a parameter dependent operator. In Section 4, we introduce the manifold with corners $Z$ and show how the Schwartz kernels of parameter dependent operators, when lifted to it, are just polyhomogeneous functions. The polyhomogeneity of the resolvent (with a $b$-pseudodifferential factor) then follows. In Subsection 4.2, we give an example showing explicitly how the resolvent of the $b$-Laplacian on $[0, \infty)$ is polyhomogeneous on $Z$. In Section 5, we present various applications.

## 2. $b$-PSEUDODIFFERENTIAL OPERATORS

The main references for this section are [14], [13], [12], [11], and [10]. We denote $\{0, 1, 2, \ldots\} = \mathbb{N}_0$ and $\{1, 2, \ldots\} = \mathbb{N}$.

[1]Note that the kernel is à priori only a distribution, we show that it can in fact, be described as a polyhomogeneous function.

2.1. **Manifolds with corners and the $b$-category.** An $n$ dimensional *manifold with corners* is a *set* $X$ together with a *set* of functions $C^\infty(X)$ on $X$ (called the $C^\infty$ structure of $X$) such that there exists a smooth $n$ dimensional manifold without boundary $\widetilde{X}$ and an injection $\iota : X \hookrightarrow \widetilde{X}$ with $C^\infty(X) = \iota^*(C^\infty(\widetilde{X}))$ (henceforth, we will consider $X \subseteq \widetilde{X}$) and such that there exists a finite set of smooth functions $\{\rho_i\}_{i=1}^r \subseteq C^\infty(\widetilde{X})$ having the property that 1. $X = \{p \in \widetilde{X} \,|\, \rho_i(p) \geq 0 \text{ for all } i\}$; 2. for each $i$, $\{\rho_i = 0\}$ is connected; 3. if $p \in \widetilde{X}$ and $\rho_{i_j}(p) = 0$ for some $1 \leq i_1 < \ldots < i_k \leq r$, then $\{(d\rho_{i_j})(p) \,|\, j = 1, \ldots k\}$ is a set of independent differentials. For each $i = 1, \ldots, r$, the subset $H_i := \{\rho_i = 0\} \subseteq X$ is called a *boundary hypersurface* of $X$ and $\rho_i$ is called a *boundary defining function* for $H_i$.

The model non-compact manifold with corners is a product of lines and half-lines: $\mathbb{R}^{n,k} := [0,\infty)^k_x \times \mathbb{R}^{n-k}_y$, where $0 \leq k \leq n$. Here, we may choose $C^\infty(\mathbb{R}^{n,k})$ to be the restriction of $C^\infty(\mathbb{R}^n)$ to $\mathbb{R}^{n,k}$, $\widetilde{\mathbb{R}^{n,k}} = \mathbb{R}^n$, and $\{\rho_i\} = \{x_i\}_{i=1}^k$. The model compact manifold with corners is $\mathbb{S}^{n-1,k} := \mathbb{S}^{n-1} \cap \mathbb{R}^{n,k}$. Thus, in short, a manifold with corners is a topological space locally modeled by $\mathbb{R}^{n,k}$ for various $0 \leq k \leq n$, such that it has only finitely many boundary hypersurfaces, each of which, is an embedded submanifold. Examples of manifolds with corners include closed manifolds, manifolds with boundaries, and finitely many products of manifolds with boundaries.

By the definition above, it follows that each boundary hypersurface $H_i$, is itself a manifold with corners and that if $X$ is compact, then near $H_i$, there exists a local diffeomorphism $X \cong [0,1) \times H_i$. The set of boundary hypersurfaces is denoted by $M_1(X)$. The connected components of the intersection of $k$ boundary hypersurfaces are each called a *codimension $k$ boundary face*. The collection of such faces is denoted by $M_k(X)$. A *total boundary defining function* for $X$ is a function of the form $\rho = \prod_{i=1}^r \rho_i$.

A subset $Y \subseteq X$ is said to be an $l$ dimensional *p-submanifold* (p for 'product') if $Y$ is embedded in $X$ in the following sense: for each $y \in Y$, there exists a coordinate patch $\mathbb{R}^{l,k} \times \mathbb{R}^{j,i}$ on $X$ centered at $y$ such that $Y \cong \mathbb{R}^{l,k} \times \{0\}$; $Y$ is said to be an *interior p-submanifold* if on any such coordinate patch, $i = 0$.

The smooth vector fields on $X$ is denoted by $\mathcal{V}(X) = C^\infty(X, TX)$. The intrinsic subspace of vector fields on $X$ are those vector fields which are tangent to the boundary of $X$. These are called the *b-vector fields on $X$* ($b$ for 'boundary'), $\mathcal{V}_b(X)$:

$$\mathcal{V}_b(X) := \{v \in \mathcal{V}(X) \,|\, v \text{ is tangent to each } H \in M_1(X)\}.$$

These are intrinsic to $X$ in the sense that on the boundary of $X$, they never point 'outward'. For example, consider the model manifold $\mathbb{R}^{n,k} = [0,\infty)^k_x \times \mathbb{R}^{n-k}_y$. Observe that a vector field of the form $a(x,y)\partial_{x_i}$ is tangent to each hypersurface $\{x_j = 0\}$ with $j \neq i$, and is tangent to $\{x_i = 0\}$ iff $a(x,y)$ vanishes at $x_i = 0$. Thus, a basis for $\mathcal{V}_b(\mathbb{R}^{n,k})$ is given by $\{x_1\partial_{x_1}, \ldots, x_k\partial_{x_k}, \partial_{y_1}, \ldots, \partial_{y_{n-k}}\}$. The fundamental properties of $b$-vector fields are that they form a Lie Algebra: if $v, w \in \mathcal{V}_b(X)$, then $[v,w] \in \mathcal{V}_b(X)$; and that they can be realized as the sections of a vector bundle ${}^bTX$ over $X$, called the *b-tangent bundle.* If $p \in X$, then the fibre of ${}^bTX$ at $p$ is

$${}^bT_pX := \mathcal{V}_b(X)/(\,\mathcal{I}_p(X) \cdot \mathcal{V}_b(X)\,),$$

where $\mathcal{I}_p(X) \subseteq C^\infty(X)$ is the space of smooth functions which vanish at $p$, and $\mathcal{I}_p(X) \cdot \mathcal{V}_b(X)$ consists of finite sums of products of an element of $\mathcal{I}_p(X)$ and $\mathcal{V}_b(X)$. Consider the model manifold $\mathbb{R}^{n,k} = [0,\infty)^k_x \times \mathbb{R}^{n-k}_y$. Then, as $\{x_1\partial_{x_1}, \ldots, x_k\partial_{x_k}, \partial_{y_1}, \ldots, \partial_{y_{n-k}}\}$ is a basis for $\mathcal{V}_b(\mathbb{R}^{n,k})$, it follows that at any $p \in \mathbb{R}^{n,k}$, $\{[x_1\partial_{x_1}], \ldots, [x_k\partial_{x_k}], [\partial_{y_1}], \ldots, [\partial_{y_{n-k}}]\}$, where [ ] denotes equivalence class in $\mathcal{V}_b(\mathbb{R}^{n,k})/(\mathcal{I}_p(\mathbb{R}^{n,k}) \cdot \mathcal{V}_b(\mathbb{R}^{n,k}))$, form a basis for ${}^bT_p\mathbb{R}^{n,k}$. Thus, ${}^bTX$ is an $n$ dimensional vector bundle over $X$. The *b-cotangent bundle* is the dual bundle ${}^bT^*X := ({}^bTX)^*$.

If $\alpha \in \mathbb{R}$, the *b-alpha density bundle*, $\Omega^\alpha_b X$, is the line bundle whose smooth sections are of the form $\rho^{-\alpha} f \mu^\alpha$, where $f$ is a smooth function and $\mu$ is a smooth, non-vanishing density. Thus, locally in a coordinate patch $\mathbb{R}^{n,k}$, a $b$-alpha density $u$ is of the form $u = f(x,y)|\frac{dx_1}{x_1} \cdots \frac{dx_k}{x_k} dy|^\alpha$, where $f(x,y)$ is smooth.

2.2. **Conormal functions.** Henceforth, $X$ will always be compact with corners. The space of smooth functions which vanish to infinite order at the boundary of $X$ is denoted by $\dot{C}^\infty(X)$. The space of *(extendible) distributions* on $X$, $C^{-\infty}(X)$, is the topological dual space of $\dot{C}^\infty(X,\Omega_b)$: $C^{-\infty}(X) := (\dot{C}^\infty(X,\Omega_b))^*$.

The space of $m$-th order *b-differential operators*, $\mathrm{Diff}_b^m(X)$, is the space of operators

$$\mathrm{Diff}_b^m(X) := \mathrm{span}_{0\le k\le m}\mathcal{V}_b(X)^k,$$

where the span is over $C^\infty(X)$. Thus, an operator $P \in \mathrm{Diff}_b^m(X)$ locally on a model $\mathbb{R}^{n,k} = [0,\infty)_x^k \times \mathbb{R}_y^{n-k}$ has the form

$$P = \sum_{|\alpha|+|\beta|\le m} a_{\alpha,\beta}(x,y)(x\partial_x)^\alpha\partial_y^\beta, \text{ where } a_{\alpha,\beta}(x,y) \text{ are smooth.}$$

The *symbols of order 0*, $S^0(X)$, is the space of those $u \in C^{-\infty}(X)$ such that $\mathrm{Diff}_b^*(X)u \subseteq L^\infty(X)$. By the Sobolev Lemma, symbols are smooth on the interior of $X$; it's only at the boundary of $X$ where they may be singular. For example, $u(x) = \cos(\log x) \in S^0([0,1))$. Note that $u(x)$ is not even continuous at $x = 0$, but yet, it and all its $b$-derivatives are bounded. If $\alpha$ is a multi-index on $X$ (that is, a map $\alpha : M_1(X) \to \mathbb{R}$, or equivalently, an $r$-tuple $(\alpha_1,\dots,\alpha_r)$ of real numbers) and $\rho$ is a total boundary defining function for $X$, then the space of *symbols of order* $\alpha$, $S^\alpha(X)$, is given by

$$S^\alpha(X) := \rho^{-\alpha}S^0(X) = \{\rho^{-\alpha}u \mid u \in S^0(X)\},$$

where $\rho^{-\alpha} := \prod_{i=1}^r \rho_i^{-\alpha_i}$. Observe that $S^{-\infty}(X) := \cap_\alpha S^\alpha(X) \equiv \dot{C}^\infty(X)$. The topology on $S^0(X)$ is the Fréchet topology generated by $\{S_m^0(X)\}_{m\in\mathbb{N}_0}$, where $S_m^0(X) := \{u \in C^{-\infty}(X) \mid \mathrm{Diff}_b^m(X)u \subseteq L^\infty(X)\}$. The topology on $S^\alpha(X)$ is the one generated by the spaces $S_m^\alpha(X) := \rho^{-\alpha}S_m^0(X)$, $m \in \mathbb{N}_0$.

There are two refinements of the symbol spaces that we will present; the Hölder differentiable symbols and the polyhomogeneous symbols.

Let $H \in M_1(X)$ and let $\eta$ be a non-negative real number. Then a symbol $u \in S^0(X)$ is said to be *$\eta$-Hölder differentiable* at $H$, if for any product decomposition $X \cong [0,1)_x \times H_y$ of $X$ near $H$, if we write $u = u(x,y)$ with respect to this product decomposition, the function $u(x,y)$ has the form

$$u(x,y) = u_0(y) + xu_1(y) + \cdots + x^k u_k(y) + x^\eta u_{k+1}(x,y), \text{ where} \tag{3}$$

$0 \le \eta - k < 1$, $u_i(y) \in S^0(H)$ for each $i$, and $u_{k+1}(x,y) \in S^0([0,1)\times H)$ is such that for any $P \in \mathrm{Diff}_b^*(X)$, $Pu_{k+1}$ is continuous up to $H$. Roughly speaking, an $\eta$-Hölder differentiable function has a Taylor expansion up to order $\eta$. Observe that $u(x) = \cos(\log x)$ is not $\eta$-Hölder differentiable for any $\eta \ge 0$. If $v \in C^\infty([0,1))$ and $\eta \ge 0$, then $u(x) = v(x) + x^\eta\cos(x\log x)$ is $\eta$-Hölder differentiable. Although it looks like the definition (3) depends on the product decomposition of $X$ near $H$, it is in fact coordinate independent (the proof is similar to the proof of Proposition 60 in the appendix). Let $\mathcal{A} \subseteq M_1(X)$ and let $\eta$ be a non-negative multi-index associated to $\mathcal{A}$; that is, a map $\eta : \mathcal{A} \to [0,\infty)$. Then a function $u \in S^0(X)$ is said to be an *$\eta$-Hölder differentiable function on $\mathcal{A}$*, if for each $H \in \mathcal{A}$, $u$ is $\eta(H)$-Hölder differentiable at $H$. The space of such functions is denoted by $S_{\mathcal{A}}^{0,\eta}(X)$.

A $(C^\infty)$ *index set* is a discrete subset $E \subseteq \mathbb{C}\times\mathbb{N}_0$ satisfying the following conditions: 1. For each $M \in \mathbb{R}$, the set $E \cap (\{z \in \mathbb{C} \mid \mathrm{Re}\, z \le M\} \times \mathbb{N}_0)$ is finite; 2. If $(z,k) \in E$, then $(z,l) \in E$ for all $0 \le l \le k$; 3. If $(z,k) \in E$, then $(z+l,k) \in E$ for all $l \in \mathbb{N}$.

We identify a number $a \in \mathbb{C}$ with the index set $\{(a+k,0) | k \in \mathbb{N}_0\}$ and given two index sets $E$ and $F$, we define the extended union $E\overline{\cup}F$ as the index set $E\cup F\cup\{(z,k_1+k_2+1) | (z,k_1) \in E,\ (z,k_2) \in F\}$.

Let $\alpha$ be an index set. Then given $H \in M_1(X)$, we define the index set $\alpha_H$ on $H$ as follows. If $F \in M_1(H)$, then $F \subseteq G \cap H$ for some $G \in M_1(X)$. We define $\alpha_H(F) := \alpha(G)$. Let $E$ be an index set and let $H \in M_1(X)$. A function $u \in S^\alpha(X)$ is said to have an *asymptotic (or classical) expansion at $H$ with index set $E$*, if for any product decomposition $X \cong [0,1)_x \times H_y$ of $X$ near $H$, the following holds. For each $(z,k) \in E$, there exists a $u_{(z,k)} \in S^{\alpha_H}(H)$ such that for each $N \in \mathbb{N}$, there exists an $M \in \mathbb{N}$ with

$$u(x,y) - \sum_{(z,k)\in E,\,\mathrm{Re}\, z\le M} x^z(\log x)^k\, u_{(z,k)}(y) \in x^N S^{(0,\alpha_H)}([0,1)\times H). \tag{4}$$

We then write $u \sim \sum_{(z,k)\in E} x^z(\log x)^k u_{(z,k)}$. In the appendix, it is shown that the expansion (4) is defined independent of the product decomposition of $X$ near $H$. Basically, an asymptotic expansion is just a generalized Taylor series; but instead of allowing just integer powers in an expansion, we allow complex powers and log terms. We denote the space of all functions having expansions at $H$

with index set $E$, by $\mathcal{A}^E(X)$ (where $E$ is understood to be associated to $H$). Note that if $E = \emptyset$, then (4) holds for all $N$ iff $u$ vanishes to infinite order at $H$. Also, observe that if $E = \mathbb{N}_0$, then $u \in \mathcal{A}^E(X)$ iff $u$ is smooth up to $H$. For example, $u(x) = \sin(x \log x) \in S^0([0,1))$. By Taylor's Theorem, $\sin(y) \sim \sum_{j=0}^{\infty} \frac{y^{2j+1}}{(2j+1)!}$, and so, $u \in \mathcal{A}^E([0,1))$ with $E = \{(2j+k, l) \,|\, j \in \mathbb{N}_0,\, k \in \mathbb{N},\, 0 \leq l \leq 2j+1\}$. If $\mathcal{A} \subseteq M_1(X)$, then an *index family associated to* $\mathcal{A}$ is a set $\mathcal{E} = \{\mathcal{E}(H) \,|\, H \in \mathcal{A}\}$, where each $\mathcal{E}(H)$ is an index set. We define $\mathcal{A}^{\mathcal{E}}(X)$ to be the space of those functions $u$ such that for each $H \in \mathcal{A}$, $u$ has an expansion at $H$ with index set $\mathcal{E}(H)$. If $\mathcal{A} = M_1(X)$, we denote $\mathcal{A}^{\mathcal{E}}(X)$ by $\mathcal{A}^{\mathcal{E}}_{phg}(X)$.

Let $Y$ be compact with corners. Then any index set $\alpha$ on $[0,1) \times Y$ can be written as a pair $\alpha = (a, \alpha_Y)$. The following results, due to Melrose, are fundamental to prove the existence of expansions. The proofs are given in the appendix.

**Theorem 1.** *A function $u \in S^{\alpha}([0,1)_x \times Y)$ has an asymptotic expansion at $Y$ with index set $E$ iff for each $N \in \mathbb{N}$, there exists an $M \in \mathbb{N}$ such that*

$$P^E_M u \in x^N S^{(0,\alpha_Y)}([0,1) \times Y), \tag{5}$$

*where $P^E_M$ is the differential operator $\prod_{(z,k)\in E,\, \mathrm{Re}\, z \leq M} (x\partial_x - z)$.*

**Proposition 2.** *Let $u \in S^{\beta}([0,1)_x \times Y)$ (for some $\beta$) and let $E$ be an index set. Suppose that for each $N \in \mathbb{N}$, there exists an $M \in \mathbb{N}$ with*

$$u(x,y) = \sum_{(z,k)\in E,\, \mathrm{Re}\, z \leq M} x^z (\log x)^k u_{(z,k,N)} + u_N(x,y), \tag{6}$$

*for some $u_{(z,k,N)} \in S^{\beta_Y}_N(Y)$ and $u_N(x,y) \in x^N S^{(0,\beta_Y)}_N([0,1) \times Y)$. Then $u$ has an asymptotic expansion at $Y$ with index set $E$.*

2.3. **Blow-ups.** Given any p-submanifold $Z \subseteq X$, the *inward pointing tangent vectors of* $Z$, $T^+Z$ has fibres

$$T^+_p Z := \begin{cases} T_p Z, & \text{if } p \in \operatorname{int} Z; \\ \{v \in T_p Z \,|\, d\rho_i(v) \geq 0 \text{ for all } H_i \in M_1(X) \text{ with } p \in H_i\}, & \text{if } p \in \partial Z. \end{cases}$$

Let $Y \subseteq X$ be a *closed* p-submanifold of $X$. Then the *inward pointing normals to* $Y$, $N^+Y$, is the space $N^+Y := T^+X / T^+Y$. We define $X$ *blown-up along* $Y$, $[X;Y]$, as a *set*, by

$$[X;Y] := (N^+Y \setminus \{0\})/\mathbb{R}^+ \sqcup (X \setminus Y). \tag{7}$$

The set $(N^+Y \setminus \{0\})/\mathbb{R}^+$ is called the *front face* of $[X;Y]$ and is denoted by *ff*$([X;Y])$. The $C^{\infty}$ structure on $[X;Y]$ is taken to be the set of functions on $[X;Y]$ which define smooth functions in any local coordinates in $X \setminus Y$, and which also define smooth functions in any local polar coordinates about $Y$. This abstract definition is actually very easy to describe in terms of coordinate patches: the charts on $[X;Y]$ are just the local polar coordinate patches about $Y$. For example, if $X = \mathbb{R}^{2,2} = [0,\infty)_x \times [0,\infty)_{x'}$ and $Y = \{0\}$, then $[\mathbb{R}^{2,2};\{0\}] \cong [0,\infty)_r \times \mathbb{S}^{1,2}_{\theta}$ (recall that $\mathbb{S}^{1,2} = \mathbb{S}^1 \cap \mathbb{R}^{2,2}$), where $r = |(x,x')|$ and $\theta = \tan^{-1}(x'/x)$ are polar coordinates about $\{0\}$. See Figure 2 for a picture of $[\mathbb{R}^{2,2};\{0\}]$. The *left boundary, lb*, is where $\theta = \pi/2$, the *right boundary, rb*, is where $\theta = 0$, and the *front face, ff*, is where $r = 0$. The *b-diagonal*, $\Delta_b$, is the set where $\theta = \pi/4$. Other useful coordinates are the coordinates $(x,z)$, where $z = \log(x/x')$. These coordinates give the following decomposition:

$$[\mathbb{R}^{2,2};\{0\}] \setminus \{lb,\, rb\} \cong [0,\infty)_x \times \mathbb{R}_z, \text{ where } \Delta_b = [0,\infty)_x \times \{0\}_z. \tag{8}$$

Projective polar coordinates are also very useful:

$$[\mathbb{R}^{2,2};\{0\}] \setminus \{rb\} \cong [0,\infty)_{x'} \times [0,\infty)_s, \text{ where } s = x/x'. \tag{9}$$

Here, *lb*, *rb*, *ff*, and $\Delta_b$ are the sets where $s = 0$, $s = +\infty$, $x' = 0$, and $s = 1$ respectively. Reversing the roles of $x$ and $x'$ give another set of projective coordinates.

Back to the general case of $[X;Y]$, the *blow-down map* $\beta : [X;Y] \to X$ is the map defined as follows: if $x \in X \setminus Y$, then we define $\beta(x) := x$; if $[v] \in (N^+_p Y \setminus \{0\})/\mathbb{R}^+$, then we define $\beta([v]) := p \in Y$. For example, in the case that $X = \mathbb{R}^{2,2}$ and $Y = \{0\}$, $\beta(r,\theta) = (r\cos\theta, r\sin\theta)$ is the usual polar coordinates map.

If $Z \subseteq X$ is a closed subset of $X$, then the *lift* of $Z$ into $[X;Y]$, $\beta^* Z \subseteq [X;Y]$, is defined under the following conditions:

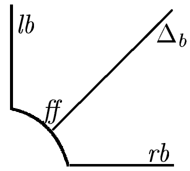


FIGURE 2. The manifold $[\mathbb{R}^{2,2};\{0\}]$. The $b$-streched product $X_b^2$ is a local product of such manifolds and products of $\mathbb{R}$.

(1) if $Z \subseteq Y$, we define $\beta^* Z := \beta^{-1}(Z)$;
(2) if $Z = \overline{Z \setminus Y}$, we define $\beta^* Z := \overline{\beta^{-1}(Y \setminus Z)}$.

For example, if $X = \mathbb{R}^{2,2}$ and $Y = \{0\}$, then $lb$, $rb$, $ff$, and $\Delta_b$ shown in Figure 2 are the lifts of $\{0\} \times [0,\infty)$, $[0,\infty) \times \{0\}$, $\{0\}$, and $\{x = x'\}$ respectively. If $Z$ satisfies either (1) or (2) above, and if in addition, $\beta^* Z$ is a p-submanifold of $[X;Y]$, then $[X;Y]$ blown-up along $\beta^* Z$ is defined, and we denote it by $[X;Y;Z] \equiv [\,[X;Y];\beta^* Z]$.

A family $\mathcal{Y} = \{Y_1,\ldots,Y_N\}$ of p-submanifolds of $X$ are said to intersect *normally* if their conormal bundles are independent at intersections. For any such family, the iterated blow-up $[X;Y_{i_1};\cdots;Y_{i_N}]$ is defined independent of the ordering. This manifold is denoted by $[X;\mathcal{Y}]$.

$b$-pseudodifferential operators are associated to the following blown-up manifold. Let $\mathcal{B} := \{H \times H \,|\, H \in M_1(X)\}$. Then, as $\mathcal{B}$ is a normally intersecting family, $X_b^2 := [X^2;\mathcal{B}]$ is defined. The $b$-diagonal is the p-submanifold $\Delta_b := \beta^*(\Delta)$, where $\Delta$ is the diagonal in $X^2$. Given $H \in M_1(X)$, we define

$$lb(H) := \beta^*(H \times X);\; rb(H) := \beta^*(X \times H);\; ff(H) := \beta^*(H \times H).$$

Also, we define $lb := \cup_i lb(H_i)$, $rb := \cup_i rb(H_i)$, and $ff := \cup_i ff(H_i)$. We now consider $X_b^2$ locally. Thus, let $\mathbb{R}^{n,k} = [0,\infty)_x^k \times \mathbb{R}_y^{n-k}$ be a coordinate patch on $X$. Then, locally, $X^2 \cong \mathbb{R}^{2,2}_{(x_1,x_1')} \times \ldots \times \mathbb{R}^{2,2}_{(x_k,x_k')} \times \mathbb{R}_y^{n-k} \times \mathbb{R}_{y'}^{n-k}$, where $(x',y')$ are the coordinates on the right factor of $(\mathbb{R}^{n,k})^2$. If $H_i$ is the hypersurface $\{x_i = 0\} \subseteq \mathbb{R}^{n,k}$, then $H_i \times H_i \cong \{x_i, x_i' = 0\}$. Thus, it follows that locally

$$X_b^2 \cong \underbrace{[\mathbb{R}^{2,2};\{0\}] \times \cdots \times [\mathbb{R}^{2,2};\{0\}]}_{k \text{ times}} \times \mathbb{R}_y^{n-k} \times \mathbb{R}_{y'}^{n-k}.$$

Thus, $X_b^2$ is a local product of the manifold in Figure 2 and products of $\mathbb{R}$. Observe that by (8), coordinates on $X_b^2$, away from $lb$ and $rb$, are given by

$$X_b^2 \cong \mathbb{R}^{n,k}_{(x,y)} \times \mathbb{R}^n_z, \text{ where } z = (\log(x_1/x_1'),\ldots,\log(x_k/x_k'), y - y'). \tag{10}$$

Moreover, in these coordinates, $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}$.

2.4. **$b$-pseudodifferential operators.** For each $m \in \mathbb{R}$, the space of *$b$-pseudodifferential operators of order $m$* is the space of distributional kernels

$$\Psi_b^m(X,\Omega_b^{\frac{1}{2}}) := \{A \in I^m(X_b^2,\Delta_b,\Omega_b^{\frac{1}{2}}) \,|\, A \equiv 0 \text{ at } lb \cup rb\}.^2 \tag{11}$$

Here, $I^m(X_b^2,\Delta_b,\Omega_b^{\frac{1}{2}})$ is the space of distributional $b$-half densities on $X_b^2$ which are conormal to $\Delta_b$ of degree $m$. The space of *one-step $b$-pseudodifferential operators of order $m$*, $\Psi^m_{b,os}(X,\Omega_b^{\frac{1}{2}})$ is the subspace of $\Psi_b^m(X,\Omega_b^{\frac{1}{2}})$ that have classical (one-step) symbols. Locally, these operators are easy to describe. Thus, let $\mathbb{R}^{n,k}_{(x,y)}$ be a coordinate patch on $X$. Then by (10), $\mathbb{R}^{n,k}_{(x,y)} \times \mathbb{R}^n_z$, where $z = (\log(x_1/x_1'),\ldots,\log(x_k/x_k'), y-y')$, is a coordinate patch on $X_b^2$ near $\Delta_b$, with $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}_z$. (Recall that $(x',y')$ are the coordinates on the right factor of $(\mathbb{R}^{n,k})^2$.) Hence, if $A \in \Psi_b^m(X,\Omega_b^{\frac{1}{2}})$, then by definition of $I^m(X_b^2,\Delta_b,\Omega_b^{\frac{1}{2}})$, on this coordinate patch, we can write

$$A = \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}\, a(x,y,\xi) d\xi \cdot |\frac{dx}{x} dy dz|^{\frac{1}{2}}, \tag{12}$$

[2] In general, $\equiv 0$ at a submanifold, means that the Taylor series vanishes at the submanifold.

where $a(x,y,\xi)$ is a symbol of order $m$. The condition that $A \equiv 0$ at $lb \cup rb$ translates to the local model (12) vanishing to exponential order as $z_i \to \pm\infty$ for any $0 \le i \le k$. We now show how $A$ defines a map on $\dot{C}^\infty(X, \Omega_b^{\frac12})$. Thus, observe that since $z = (\log(x_1/x_1'), \dots, \log(x_k/x_k'), y - y')$, we can write

$$\frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, a(x,y,\xi) d\xi \cdot |\frac{dx}{x} dy dz|^{\frac12} = \frac{1}{(2\pi)^n} \int x^{i\xi'} e^{iy\cdot\xi''} (x')^{-i\xi'} e^{-iy'\cdot\xi''}\, a(x,y,\xi) d\xi \cdot |\frac{dx}{x} dy \frac{dx'}{x'} dy'|^{\frac12},$$

where $\xi' = (\xi_1, \dots, \xi_k)$ and $\xi'' = (\xi_{k+1}, \dots, \xi_n)$. If $u \in \dot{C}^\infty(X, \Omega_b^{\frac12})$, $Au$ is defined as follows. Thus, assuming that $u = u(x,y)|\frac{dx}{x} dy|^{\frac12}$, where $u(x,y)$ has support in the coordinate patch $\mathbb{R}^{n,k}$, we define,

$$Au(x,y) := \frac{1}{(2\pi)^n} \int x^{i\xi'} e^{iy\cdot\xi''}\, a(x,y,\xi) \widetilde{u}(\xi) d\xi\, |\frac{dx}{x} dy|^{\frac12},$$

where $\widetilde{u}$ is the Mellin transform in $x$ and the Fourier transform in $y$ of $u(x,y)$:

$$\widetilde{u}(\xi) = \int_{\mathbb{R}^{n,k}} x^{-i\xi'} e^{-iy\cdot\xi''}\, u(x,y) \frac{dx}{x} dy.$$

One can check that $Au \in \dot{C}^\infty(X, \Omega_b^{\frac12})$.

The (principal) symbol, ${}^b\sigma_m(A)$, of $A \in \Psi_b^m(X, \Omega_b^{\frac12})$, is an element of the quotient space

$$S^{[m]}({}^bT^*X) = S^m({}^bT^*X)/S^{m-1}({}^bT^*X),$$

where $S^m({}^bT^*X)$ consists of the symbols of order $m$ on ${}^bT^*X$. If $A \in \Psi_{b,os}^m(X, \Omega_b^{\frac12})$, then ${}^b\sigma_m(A)$ is an element of $C^\infty_{\hom(m)}({}^bT^*X)$, the space of smooth functions on ${}^bT^*X \setminus \{0\}$ which are homogeneous of degree $m$. $A$ is called *elliptic* if ${}^b\sigma_m(A)$ is invertible. The symbol map gives a short exact sequence

$$0 \hookrightarrow \Psi_b^{m-1}(X, \Omega_b^{\frac12}) \hookrightarrow \Psi_b^m(X, \Omega_b^{\frac12}) \xrightarrow{{}^b\sigma_m} S^{[m]}({}^bT^*X) \to 0;$$
$${}^b\sigma_{m+m'}(A \circ B) = {}^b\sigma_m(A) \cdot {}^b\sigma_{m'}(B) \text{ if } A \in \Psi_b^m(X, \Omega_b^{\frac12}),\ B \in \Psi_b^{m'}(X, \Omega_b^{\frac12}).$$

As mentioned before, $\Psi_b^m(X, \Omega_b^{\frac12})$ maps continuously on $\dot{C}^\infty(X, \Omega_b^{\frac12})$. It also maps continuously on $C^\infty(X, \Omega_b^{\frac12})$. We define $L_b^2(X, \Omega_b^{\frac12})$ to be the completion of $\dot{C}^\infty(X, \Omega_b^{\frac12})$ with respect to the norm $\dot{C}^\infty(X, \Omega_b^{\frac12}) \ni u \mapsto \sqrt{\int_X |u|^2}$. For each $m \in \mathbb{R}$, we define the *Sobolev space of order* $m$ by

$$H_b^m(X, \Omega_b^{\frac12}) := \{u \in C^{-\infty}(X, \Omega_b^{\frac12}) \mid \Psi_b^m(X, \Omega_b^{\frac12}) u \subseteq L_b^2(X, \Omega_b^{\frac12})\}.$$

Note that $H_b^0(X, \Omega_b^{\frac12}) \equiv L_b^2(X, \Omega_b^{\frac12})$. If $A \in \Psi_b^m(X, \Omega_b^{\frac12})$, then for any $s \in \mathbb{R}$, $A$ defines a continuous linear map $A : H_b^s(X, \Omega_b^{\frac12}) \to H_b^{s-m}(X, \Omega_b^{\frac12})$.

If $\alpha$ is a multi-index for $X_b^2$ with $\alpha|_{ff} \ge 0$, then we define

$$\Psi_b^{-\infty,\alpha}(X, \Omega_b^{\frac12}) := \rho_{lb}^{\alpha|_{lb}} \rho_{rb}^{\alpha|_{rb}} \bigcup_{\epsilon > 0} \rho_{lb}^\epsilon \rho_{rb}^\epsilon S_{ff}^{0,\alpha|_{ff}}(X_b^2, \Omega_b^{\frac12}), \tag{13}$$

where $\rho_{lb}$ and $\rho_{rb}$ are total boundary defining functions for $lb$ and $rb$ respectively, and where $S_{ff}^{0,\alpha|_{ff}}(X_b^2, \Omega_b^{\frac12})$ consists of the $\alpha|_{ff}$-Hölder differentiable $b$-half densities at $ff$. For any $m \in \mathbb{R}$, we define

$$\Psi_b^{m,\alpha}(X, \Omega_b^{\frac12}) := \Psi_b^m(X, \Omega_b^{\frac12}) + \Psi_b^{-\infty,\alpha}(X, \Omega_b^{\frac12}).$$

Let $\rho_{ff}$ be a boundary defining function for $ff$ and assume that $\alpha|_{rb} + \alpha'|_{lb} \ge 0$; that is, $\alpha(rb(H)) + \alpha'(lb(H)) \ge 0$ for each $H \in M_1(X)$. Then the following composition result holds: Provided that $\alpha|_{lb} + \alpha'|_{rb} \ge \gamma + \gamma' + \alpha''|_{ff}$, where $\alpha''|_{ff} := \min\{\alpha|_{ff} + \alpha'|_{ff}\}$,

$$\rho_{ff}^\gamma \Psi_b^{m,\alpha}(X, \Omega_b^{\frac12}) \circ \rho_{ff}^{\gamma'} \Psi_b^{m',\alpha'}(X, \Omega_b^{\frac12}) \subseteq \rho_{ff}^{\gamma+\gamma'} \Psi_b^{m+m',\alpha''}(X, \Omega_b^{\frac12}), \tag{14}$$

where $\alpha''|_{lb} := \min\{\alpha|_{lb}, \alpha'|_{lb} + \gamma\}$, $\alpha''|_{rb} := \min\{\alpha|_{rb} + \gamma',\ \alpha'|_{rb}\}$. These spaces $\Psi_b^{*,*}(X, \Omega_b^{\frac12})$ form the 'calculus with bounds'.

Assume that each boundary hypersurface of $X$ has a *fixed* boundary defining function. Let $M \in M_l(X)$, $l \in \mathbb{N}$, and let $X \cong [0,1)^l \times M$ near $M$, where the coordinates on $[0,1)^l$ are the fixed boundary defining functions for the hypersurfaces that define $M$. Let $x_1, \ldots, x_l$ and $x_1', \ldots, x_l'$ be the left and right coordinate functions on $([0,1)^l)^2$. Then, as $X^2 \cong ([0,1)^2)^l \times M^2$, it follows that $X_b^2 \cong ([0,1)_b^2)^l \times M_b^2$, and $(x_1, \ldots, x_l, z_1, \ldots, z_l)$, where $z_i = \log(x_i/x_i')$, define coordinates on the $([0,1)_b^2)^l$ factor of $X_b^2$ (cf. (8) and (10)). Let $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$. Then the *normal operator of $A$ at $M$*, $N_M(A)$, is the Fourier transform:

$$\mathbb{C}^l \ni \tau \mapsto N_M(A)(\tau) := < A|_{ff(M)}, e^{-iz_1\tau_1} \cdots e^{-iz_l\tau_l} |dz_1 \cdots dz_l|^{\frac{1}{2}} >,$$

where $<,>$ denotes the distributional pairing. (It can also be defined as the Mellin transform $< A|_{ff(M)}, s_1^{-i\tau_1} \cdots s_l^{-i\tau_l} |\frac{ds_1}{s_1} \cdots \frac{ds_l}{s_l}|^{\frac{1}{2}} >$, where $s_i = x_i/x_i'$, using projective coordinates.) Observe that $N_M(A)(\tau)$ is holomorphic in $\tau$. Consider now $N_M(A)$ in local coordinates. Thus, let $\mathbb{R}^{n-l,k-l} = [0,\infty)_{\tilde{x}}^{k-l} \times R_y^{n-k}$, where $\tilde{x} = (x_{l+1}, \ldots, x_k)$, be a coordinate patch on $M$ and let $A$ be given locally by formula (12). Then, as $ff(M) = \{x_1, \ldots, x_l = 0\}$, omitting the $|\frac{dx_1}{x_1} \cdots \frac{dx_l}{x_l}|^{\frac{1}{2}}$ factor, we have

$$A|_{ff(M)} = \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, a(0, \tilde{x}, y, \xi) d\xi \cdot |\frac{d\tilde{x}}{\tilde{x}} dy dz|^{\frac{1}{2}}.$$

Hence, $N_M(A)(\tau)$ is given by

$$N_M(A)(\tau) = \frac{1}{(2\pi)^{n-l}} \int e^{i\tilde{z}\cdot\tilde{\xi}}\, a(0, \tilde{x}, y, \tau, \tilde{\xi}) d\tilde{\xi} \cdot |\frac{d\tilde{x}}{\tilde{x}} dy d\tilde{z}|^{\frac{1}{2}},$$

where $\tilde{z} = (z_{l+1}, \ldots, z_n)$ and $\tilde{\xi} = (\xi_{l+1}, \ldots, \xi_n)$ with $\xi = (\tau, \tilde{\xi})$. It follows that for each $\tau$, $N_M(A)(\tau) \in \Psi_b^m(M, \Omega_b^{\frac{1}{2}})$, and moreover, if $m = -\infty$ and the imaginary part of $\tau$ is bounded, then $N_M(A)(\tau)$ vanishes, with all its derivatives, to infinite order in $\Psi_b^{-\infty}(M, \Omega_b^{\frac{1}{2}})$ whenever $|\mathrm{Re}\,\tau_i| \to \infty$ for any $1 \le i \le l$.

## 3. Tempered operators

3.1. **Tempered symbols.** In this subsection, we describe our two basic symbol spaces, which are modified versions of parameter-dependent symbols that are already found in the literature, cf. [18, 6]. Throughout this paper, $\Lambda \subseteq \mathbb{C}$ will always denote a (closed) sector of $\mathbb{C}$; that is, an angle of the form $\Lambda = \{\lambda \in \mathbb{C} \,|\, \epsilon_1 \le \arg(\lambda) \le \epsilon_2\}$ for some $\epsilon_1, \epsilon_2$. Taking notation from [6], we define $\dot{\le}$ to mean "$\le$ a constant, independent of variables", $\dot{\ge}$ to mean "$\ge$ a constant, independent of variables", and $\doteq$ to mean both $\dot{\le}$ and $\dot{\ge}$.

**Definition 3.** Let $m, p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and $\Lambda \subseteq \mathbb{C}$ be a sector. Then the space of *tempered symbols*, $S_\Lambda^{m,p,d}(\mathbb{R}^n)$, consists of those functions $a(\lambda, \xi) \in C^\infty(\Lambda \times \mathbb{R}^n)$ satisfying the following estimates: for any $\alpha$ and $\beta$,

$$|\, \partial_\lambda^\alpha \partial_\xi^\beta a(\lambda, \xi)\,| \;\dot{\le}\; (1 + |\lambda|^{1/d} + |\xi|)^{p - d|\alpha|}\, (1 + |\xi|)^{m-p-|\beta|}. \tag{15}$$

Here, the index set $\alpha$ is a pair, $\alpha = (\alpha_1, \alpha_2)$, and $\partial_\lambda^\alpha := \partial_{\lambda_1}^{\alpha_1} \partial_{\lambda_2}^{\alpha_2}$, where $\lambda = (\lambda_1, \lambda_2) \equiv \lambda_1 + i\lambda_2$ are the usual coordinates on $\mathbb{C}$.

Given sectors $\Lambda$, $\Lambda' \subseteq \mathbb{C}$, $\Lambda$ is said to be a *proper subsector* of $\Lambda'$ if $\Lambda \subseteq \mathrm{int}(\Lambda') \cup \{0\}$.

**Lemma 4.** *Let $\Lambda \subseteq \Lambda'$ be a proper subsector of a sector $\Lambda' \subseteq \mathbb{C}$. Let $a(\xi) \in S^m(\mathbb{R}^n)$, $m \in \mathbb{R}^+$ be elliptic and suppose that for some $r > 0$, $a(\xi) \notin \Lambda'$ for all $|\xi| \ge r$. Then for any $N \in \mathbb{Z}$ and $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ for $|\xi| \le r$ and $\chi(\xi) \equiv 1$ for $|\xi| \ge r+1$, we have*

$$\chi(\xi)\,(\,a(\xi) - \lambda)^N \in S_\Lambda^{Nm,Nm,m}(\mathbb{R}^n).$$

*Proof.* If for any $N \in \mathbb{Z}$, $a_N(\lambda, \xi) = (\,a(\xi) - \lambda)^N$, then since $\partial_\lambda^\alpha a_N = c_\alpha a_{N-|\alpha|}$ for some $c_\alpha \in \mathbb{C}$, it suffices to prove the estimates (15) assuming that $\alpha = 0$. The following estimate will be proved by induction on $\beta$:

$$|\, \partial_\xi^\beta a_N(\lambda, \xi)\,| \;\dot{\le}\; (1 + |\lambda| + |\xi|^m)^N\, (1 + |\xi|)^{-|\beta|}, \;\; |\xi| \ge r. \tag{16}$$

Assume that $\beta = 0$. Then, since $a(\xi)$ is elliptic, it follows that there exists an $\epsilon > 0$ such that $\epsilon|\xi|^m \le |a(\xi)| \le \frac{1}{\epsilon}|\xi|^m$ for all $|\xi| \ge r$. Thus,

$$|\,(a(\xi) - \lambda)^N\,| \,\dot{\le}\, (1 + |\lambda| + |\xi|^m)^N \text{ for all } \lambda \in \Lambda \text{ and } |\xi| \ge r.$$

Assume now that (16) is true for $\beta$. Then observe that

$$\partial_{\xi_i}\partial_\xi^\beta a_N = N\partial_\xi^\beta[a_{N-1}\cdot\partial_{\xi_i}a(\xi)] = N\sum_{|\gamma|\le|\beta|}\binom{\beta}{\gamma}\partial_\xi^\gamma a_{N-1}(\lambda,\xi)\cdot\partial_\xi^{\beta-\gamma}\partial_{\xi_i}a(\xi).$$

Applying the induction hypothesis on $\partial_\xi^\gamma a_{N-1}$ and using the fact that $a(\xi)$ is a symbol of order $m$, proves (16) for the induction step. Our lemma now follows from the estimate (16). ☐

This lemma suggests that the symbols of resolvents for $b$-pseudodifferential operators are captured by these tempered symbols. In the next subsection we will show that a stronger result holds, cf. Lemma 13.

Let $\Lambda \subseteq \mathbb{C}$ be a sector. Then we define $S^m(\Lambda)$ to be the space of functions $a(\lambda) \in C^\infty(\Lambda)$ such that for each $\alpha$,

$$|\partial_\lambda^\alpha a(\lambda)\,| \,\dot{\le}\, (1 + |\lambda|)^{m-|\alpha|}.$$

If $\mathcal{F}$ is any Fréchet space, the space $S^m(\Lambda;\mathcal{F})$ of $\mathcal{F}$ valued symbols of order $m$ on $\Lambda$ is defined similarly.

The following lemma follows directly from the definition of tempered symbols.

**Lemma 5.** *Let $\Lambda \subseteq \mathbb{C}$ be a sector and $d \in \mathbb{R}^+$. Then,*

1. $S^m(\mathbb{R}^n) \subseteq S_\Lambda^{m,p,d}(\mathbb{R}^n)$ *for all $m \in \mathbb{R}$ and $p \ge 0$;*
2. $S_\Lambda^{m,p,d}(\mathbb{R}^n)\cdot S_\Lambda^{m',p',d}(\mathbb{R}^n) \subseteq S_\Lambda^{m+m',p+p',d}(\mathbb{R}^n)$ *for any $m$, $m'$, $p$, $p' \in \mathbb{R}$;*
3. $S_\Lambda^{m,p,d}(\mathbb{R}^n) \subseteq S_\Lambda^{m',p',d}(\mathbb{R}^n)$ *if $m \le m'$ and $p \le p'$;*
4. $\partial_\lambda^\alpha\partial_\xi^\beta S_\Lambda^{m,p,d}(\mathbb{R}^n) \subseteq S_\Lambda^{m-d|\alpha|-|\beta|,p-d|\alpha|,d}(\mathbb{R}^n)$ *for any $m$, $p \in \mathbb{R}$ and for any $\alpha$, $\beta$.*
5. $$\begin{aligned} S_\Lambda^{m,p,d}(\mathbb{R}^n) &\subseteq S^{p/d}(\Lambda;S^m(\mathbb{R}^n)), && \textit{if } p > 0;\\ S_\Lambda^{m,p,d}(\mathbb{R}^n) &\subseteq S^{p/d}(\Lambda;S^{m-p}(\mathbb{R}^n)), && \textit{if } p \le 0.\end{aligned}$$

   *In particular,* $S_\Lambda^{-\infty,p,d}(\mathbb{R}^n) \equiv S^{p/d}(\Lambda, S^{-\infty}(\mathbb{R}^n))$.

**Definition 6.** Let $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$. The space of *Strongly tempered symbols*, $S_\Lambda^{m,d}(\mathbb{R}^n)$, consists of those functions $a \in C^\infty(\Lambda\times\mathbb{R}^n)$ satisfying the following estimates: for each $\alpha,\beta$,

$$|\,\partial_\lambda^\alpha\partial_\xi^\beta a(\lambda,\xi)\,| \,\dot{\le}\, (1 + |\lambda|^{1/d} + |\xi|)^{m-d|\alpha|-|\beta|}. \tag{17}$$

**Lemma 7.** *For any $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$, $S_\Lambda^{m,d}(\mathbb{R}^n) \subseteq S_\Lambda^{m,m,d}(\mathbb{R}^n)$.*

*Proof.* Observe that for any $(\lambda,\xi)$,

$$\begin{aligned}(1 + |\lambda|^{1/d} + |\xi|)^{m-d|\alpha|-|\beta|} &\le (1 + |\lambda|^{1/d} + |\xi|)^{m-d|\alpha|}(1 + |\xi|)^{-|\beta|}\\ &= (1 + |\lambda|^{1/d} + |\xi|)^{m-d|\alpha|}(1 + |\xi|)^{m-m-|\beta|}.\end{aligned}$$

This estimate, together with (15) and (17), prove the lemma. ☐

**Definition 8.** Let $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$. The space $C^{\infty,d}_{\mathrm{hom}(m)}(\Lambda\times\mathbb{R}^n)$ consists of those functions $a(\lambda,\xi) \in C^\infty((\Lambda\times\mathbb{R}^n)\setminus\{(0,0)\})$, such that

$$a(\delta^d\lambda,\delta\xi) = \delta^m a(\lambda,\xi) \text{ for all } \delta > 0.$$

For example, if $\Lambda\cap(-\infty,0] = \{0\}$, $m \in \mathbb{R}$, and $d \in \mathbb{N}$ is even, then

$$a(\lambda,\xi) := (\lambda + |\xi|^d)^{m/d} \in C^{\infty,d}_{\mathrm{hom}(m)}(\Lambda\times\mathbb{R}^n), \tag{18}$$

where the principal branch of log is used to define the power function $(\ )^{m/d}$. Also, if $a(\xi)$ is a homogeneous polynomial of degree $m$, for any $d \in \mathbb{R}^+$, $a(\xi) \in C^{\infty,d}_{\mathrm{hom}(m)}(\Lambda\times\mathbb{R}^n)$.

**Lemma 9.** *Let $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$ and let $a(\lambda,\xi) \in C^{\infty,d}_{\text{hom}(m)}(\Lambda \times \mathbb{R}^n)$. Then for each $\alpha$ and $\beta$,*

$$|\,\partial^\alpha_\lambda \partial^\beta_\xi a(\lambda,\xi)\,| \dot{\leq} (\,|\lambda|^{1/d} + |\xi|\,)^{m-d|\alpha|-|\beta|}. \tag{19}$$

*In particular, if $\chi(\lambda,\xi) \in C^\infty(\Lambda \times \mathbb{R}^n)$ with $\chi(\lambda,\xi) \equiv 0$ near $(\lambda,\xi) = 0$ and $\chi(\lambda,\xi) \equiv 1$ outside a neighborhood of 0, then $\chi(\lambda,\xi)a(\lambda,\xi) \in S^{m,d}_\Lambda(\mathbb{R}^n)$.*

*Proof.* Fix $\alpha$ and $\beta$ and define $C := \max_{|\lambda|^{1/d}+|\xi|=1} |\partial^\alpha_\lambda \partial^\beta_\xi a(\lambda,\xi)|$. Observe that the homogeneous property $a(\delta^d\lambda, \delta\xi) = \delta^m a(\lambda,\xi)$ for all $\delta > 0$, implies that $(\partial^\alpha_\lambda \partial^\beta_\xi a)(\delta^d\lambda,\delta\xi) = \delta^{m-d|\alpha|-|\beta|}$ $(\partial^\alpha_\lambda \partial^\beta_\xi a)(\lambda,\xi)$. Hence,

$$\begin{aligned}
&|\,\partial^\alpha_\lambda \partial^\beta_\xi a(\lambda,\xi)\,| \\
&= |\,\partial^\alpha_\lambda \partial^\beta_\xi a(\,(\,|\lambda|^{1/d} + |\xi|\,)^d \frac{\lambda}{(\,|\lambda|^{1/d} + |\xi|\,)^d}, (\,|\lambda|^{1/d} + |\xi|\,) \frac{\xi}{(\,|\lambda|^{1/d} + |\xi|\,)})\,| \\
&= (\,|\lambda|^{1/d} + |\xi|\,)^{m-d|\alpha|-|\beta|} |\,\partial^\alpha_\lambda \partial^\beta_\xi a(\frac{\lambda}{(\,|\lambda|^{1/d} + |\xi|\,)^d}, \frac{\xi}{(\,|\lambda|^{1/d} + |\xi|\,)})\,| \\
&\leq C\,(\,|\lambda|^{1/d} + |\xi|\,)^{m-d|\alpha|-|\beta|}.
\end{aligned}$$

Thus, $|\,\partial^\alpha_\lambda \partial^\beta_\xi a(\lambda,\xi)\,| \leq C\,(\,|\lambda|^{1/d} + |\xi|\,)^{m-d|\alpha|-|\beta|}$. □

**Definition 10.** Let $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$. The space of *one-step, strongly polyhomogeneous symbols*, $S^{m,d}_{\Lambda,os}(\mathbb{R}^n)$, consists of those $a(\lambda,\xi) \in S^{m,d}_\Lambda(\mathbb{R}^n)$ such that

$$a(\lambda,\xi) \sim \sum_{j=0}^\infty \chi(\lambda,\xi)\, a_{m-j}(\lambda,\xi), \tag{20}$$

where $a_{m-j}(\lambda,\xi) \in C^{\infty,d}_{\text{hom}(m-j)}(\Lambda \times \mathbb{R}^n)$, where $\chi(\lambda,\xi) \in C^\infty(\Lambda \times \mathbb{R}^n)$ with $\chi(\lambda,\xi) \equiv 0$ near $(\lambda,\xi) = 0$ and $\chi(\lambda,\xi) \equiv 1$ outside a neighborhood of 0, and where the asymptotic sum (20) is in $S^{m,d}_\Lambda(\mathbb{R}^n)$. Thus, for each $N \in \mathbb{N}$, we have $a(\lambda,\xi) - \sum_{j=0}^{N-1} \chi(\lambda,\xi)\, a_{m-j}(\lambda,\xi) \in S^{m-N,d}_\Lambda(\mathbb{R}^n)$.

**Lemma 11.** *Let $a(\xi)$ be an elliptic homogeneous polynomial of degree $m \in \mathbb{N}$ and suppose that $a(\xi)$ never takes values in the sector $\Lambda$ for $\xi \neq 0$. Then for all $N \in \mathbb{Z}$,*

$$(\,a(\xi) - \lambda)^N \in C^{\infty,m}_{\text{hom}(Nm)}(\Lambda \times \mathbb{R}^n).$$

*In particular, for any $\chi(\lambda,\xi) \in C^\infty(\Lambda \times \mathbb{R}^n)$ with $\chi(\lambda,\xi) \equiv 0$ near $(\lambda,\xi) = 0$ and $\chi(\lambda,\xi) \equiv 1$ outside a neighborhood of 0, $\chi(\lambda,\xi)\,(\,a(\xi) - \lambda)^N \in S^{Nm,m}_{\Lambda,os}(\mathbb{R}^n)$.*

This lemma suggests that the symbols of resolvents for $b$-differential operators are captured by these strongly tempered symbols.

3.2. **Resolvent tempered symbols.** In the following subsections, we give various refinements of the tempered symbol spaces $S^{*,*,*}_\Lambda(\mathbb{R}^n)$. Our first refinement are symbols that have more 'resolventlike' behavior. These symbols will capture the local symbols of the resolvents of general elliptic $b$-pseudodifferential operators. Define $\Lambda_{cc} = \{\overline{\lambda} \,|\, \lambda \in \Lambda\}$ and set $\mu := 1/\lambda = \overline{\lambda}/|\lambda|^2 \in \Lambda_{cc}$, where $\lambda \in \Lambda \setminus \{0\}$.

**Definition 12.** Let $m$, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. The subspace $S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n) \subseteq S^{m,p,d}_\Lambda(\mathbb{R}^n)$ of *resolvent tempered symbols* consists of those elements $a(\lambda,\xi) \in S^{m,p,d}_\Lambda(\mathbb{R}^n)$ such that if we define

$$\tilde{a}(\mu,\xi) := \mu^{p/d}\, a(1/\mu,\xi) = \mu^{p/d}\, a(\overline{\mu}/|\mu|^2,\xi),$$

then $\tilde{a}(\mu,\xi) \in C^\infty(\Lambda_{cc} \times \mathbb{R}^n)$ (thus, $\tilde{a}(\mu,\xi)$ is assumed smooth even at $\mu = 0$) and it satisfies the estimates: for any multi-indices $\alpha$ and $\beta$ and for $R \in \mathbb{R}^+$, for all $(\mu,\xi) \in \Lambda_{cc} \times \mathbb{R}^n$ with $|\mu| \leq R$,

$$|\,\partial^\alpha_\mu \partial^\beta_\xi \tilde{a}(\mu,\xi)\,| \dot{\leq} (1 + |\mu|\,|\xi|^d)^{p/d-|\alpha|}\,(1 + |\xi|)^{d|\alpha|+m-p-|\beta|}. \tag{21}$$

**Lemma 13.** *Let $\Lambda \subseteq \Lambda'$ be a proper subsector of a sector $\Lambda' \subseteq \mathbb{C}$. Let $a(\xi) \in S^m(\mathbb{R}^n)$, $m \in \mathbb{R}^+$ be elliptic and suppose that for some $r > 0$, $a(\xi) \notin \Lambda'$ for all $|\xi| \geq r$. Then for any $N \in \mathbb{Z}$ and $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ for $|\xi| \leq r$ and $\chi(\xi) \equiv 1$ for $|\xi| \geq r+1$, we have*

$$\chi(\xi)\,(\,a(\xi) - \lambda)^N \in S^{Nm,Nm,m}_{\Lambda,r}(\mathbb{R}^n).$$

*Proof.* If $a_N(\lambda,\xi) = \chi(\xi)\,(\,a(\xi)-\lambda)^N$, then $\tilde{a}_N(\mu,\xi) = \chi(\xi)\,(\,\mu\, a(\xi) - 1)^N$. Since $a(\xi) \notin \Lambda'$ for all $|\xi| \geq r$ and $\chi(\xi) \equiv 0$ for $|\xi| \leq r$, it follows that $\tilde{a}_N(\mu,\xi) \in C^\infty(\Lambda_{cc} \times \mathbb{R}^n)$. Since for any $\alpha$, $\partial^\alpha_\mu \tilde{a}_N = c_\alpha\, a(\xi)^{|\alpha|}\, a_{N-|\alpha|}$ for some $c_\alpha \in \mathbb{C}$, it suffices to prove the estimates (21) assuming that $\alpha = 0$. The following estimate will be proved by induction on $\beta$:

$$|\,\partial^\beta_\xi \tilde{a}_N(\mu,\xi)\,| \overset{\cdot}{\leq} (1 + |\mu||\xi|^m)^N\,(1+|\xi|)^{-|\beta|}. \tag{22}$$

Assume that $\beta = 0$. Since $a(\xi)$ is elliptic, there exists a constant $\epsilon > 0$ such that $\epsilon|\xi|^m \leq |a(\xi)| \leq \frac{1}{\epsilon}|\xi|^m$ for all $|\xi| \geq r$. As $\tilde{a}_N(\mu,\xi) = \chi(\xi)(\mu a(\xi) - 1)^N$, it follows that the estimate (22) holds when $\beta$ is 0. Assume now that (22) is true for $\beta$. Then modulo a term with a factor of $\partial_{\xi_i}\chi(\xi)$, which vanishes for $|\xi|$ sufficiently large, we have

$$\partial_{\xi_i}\partial^\beta_\xi \tilde{a}_N = N\partial^\beta_\xi[\tilde{a}_{N-1}\cdot \mu\,\partial_{\xi_i} a(\xi)] = N\sum_{|\gamma|\leq|\beta|}\binom{\beta}{\gamma}\partial^\gamma_\xi \tilde{a}_{N-1}\cdot \mu\,\partial^{\beta-\gamma}_\xi \partial_{\xi_i} a(\xi).$$

Applying the induction hypothesis on $\partial^\gamma_\xi \tilde{a}_{N-1}$ and using the fact that $a(\xi)$ is a symbol of order $m$, proves (22) for the induction step. □

The following lemma is proved using the same line of reasoning used to prove the usual 'Asymptotic Summation Lemma' for the usual symbols.

**Lemma 14.** *Given a sequence $a_j(\lambda,\xi) \in S^{m-j,p,d}_{\Lambda,r}(\mathbb{R}^n)$, $j \in \mathbb{N}_0$, there exists an $a(\lambda,\xi) \in S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$ such that for all $N \in \mathbb{N}$, $a(\lambda,\xi) - \sum_{j=0}^{N-1} a_j(\lambda,\xi) \in S^{m-N,p,d}_{\Lambda,r}(\mathbb{R}^n)$.*

3.3. **Polyhomogeneous tempered symbols.** We will now describe a subset

$$S^{m,p,d}_{\Lambda,ros}(\mathbb{R}^n) \subseteq S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$$

of one-step, polyhomogeneous symbols. These symbols will capture the local symbols of the resolvents of classical $b$-pseudodifferential operators.

**Definition 15.** Let $m$, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. Then the space of functions $C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$ consists of those $a(\lambda,\xi) \in C^\infty(\Lambda \times (\mathbb{R}^n \setminus \{0\}))$ such that

(1) $a(\delta^d\lambda, \delta\xi) = \delta^m a(\lambda,\xi)$ for all $\delta > 0$;
(2) if we define $\tilde{a}(\mu,\xi) := \mu^{p/d}\, a(1/\mu,\xi) = \mu^{p/d}\, a(\overline{\mu}/|\mu|^2,\xi)$, then we have $\tilde{a}(\mu,\xi/|\xi|) \in C^\infty(\Lambda_{cc} \times \mathbb{S}^{n-1})$ (thus, $\tilde{a}(\mu,\xi/|\xi|)$ is smooth even at $\mu = 0$).

For example, the function $a(\lambda,\xi)$ given in (18) is an element of $C^{\infty,m,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$ if $m/d \in \mathbb{Z}$. If $m \in \mathbb{R}$, $p \geq 0$, and $d \in \mathbb{R}^+$ is such that $p/d \in \mathbb{N}_0$, then for any $\Lambda$, $C^\infty_{\mathrm{hom}(m)}(\mathbb{R}^n) \subseteq C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$. Finally, if $a(\xi) \in C^\infty_{\mathrm{hom}(1)}(\mathbb{R}^n)$ is non-negative and if $\lambda^2 \notin (-\infty,0)$ for $\lambda \in \Lambda \setminus \{0\}$, then[3] for any $m \in \mathbb{Z}$,

$$a(\lambda,\xi) = [(|\xi|^2 + \lambda^2)^{\frac{1}{2}} + a(\xi)]^m \in C^{\infty,m,1}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n).$$

**Lemma 16.** *Let $a(\lambda,\xi) \in C^\infty(\,(\Lambda \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\})\,)$ be such that for some $m$, $e \in \mathbb{R}$, $a(\delta^e\lambda, \delta\xi) = \delta^m a(\lambda,\xi)$ for all $\delta > 0$ and such that $a(\lambda,\xi/|\xi|) \in C^\infty(\Lambda \times \mathbb{S}^{n-1})$. Then, $a(\lambda,\xi) \in C^\infty(\Lambda \times (\mathbb{R}^n \setminus \{0\})\,)$ and there exists an $\epsilon > 0$ such that if $B_\epsilon = \{\lambda \in \mathbb{C}\,|\,|\lambda| \leq \epsilon\}$,*

(1) *for any $\beta$, $(\partial^\beta_\xi a)(\lambda,\xi/|\xi|) \in C^\infty(\,(\Lambda \cup B_\epsilon) \times \mathbb{S}^{n-1})$;*
(2) *for any $r > 0$,*

$$a(\lambda,\xi) \in \begin{cases} C^\infty(\,(\Lambda \cup B_{\epsilon r^e}) \times \{|\xi| \geq r\}\,), & \text{if } e \geq 0; \\ C^\infty(\,(\Lambda \cup B_{\epsilon r^e}) \times \{0 < |\xi| \leq r\}\,), & \text{if } e < 0. \end{cases}$$

[3]This example was provided by Gerd Grubb.

*Proof.* Since $a(\lambda, \xi/|\xi|) \in C^\infty(\Lambda \times \mathbb{S}^{n-1})$, by definition of $C^\infty(\Lambda \times \mathbb{S}^{n-1})$, there exists an $\epsilon > 0$ such that

$$a(\lambda, \xi/|\xi|) \in C^\infty(\,(\Lambda \cup B_\epsilon) \times \mathbb{S}^{n-1}). \tag{23}$$

Observe that

$$a(\lambda, \xi) = a(\,|\xi|^e |\xi|^{-e}\lambda, |\xi|\,\xi/|\xi|\,) = |\xi|^m a(\,|\xi|^{-e}\lambda, \xi/|\xi|\,). \tag{24}$$

Then (23) and (24) imply property (1). To see (2), observe that (23) and (24) imply that $a(\lambda, \xi)$ is smooth in $\lambda$ and $\xi$ if $|\xi|^{-e}|\lambda| \le \epsilon$ and $|\xi| > 0$; that is, $|\lambda| \le \epsilon|\xi|^e$ and $|\xi| > 0$. If $e \ge 0$, this holds when $|\lambda| \le \epsilon r^e$ and $|\xi| \ge r$; and if $e < 0$, this holds when $|\lambda| \le \epsilon r^e$ and $0 < |\xi| \le r$. ☐

**Proposition 17.** *Let $a(\lambda, \xi) \in C^{\infty,p,d}_{\mathrm{hom}(m)}(\mathbb{R}^n)$ where $m,\, p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. Then,*

(1) *For any $\alpha$ and $\beta$,*

$$|\,\partial_\lambda^\alpha \partial_\xi^\beta a(\lambda, \xi)\,| \,\dot{\le}\, (\,|\lambda|^{1/d} + |\xi|\,)^{p-d|\alpha|}\,|\xi|^{m-p-|\beta|}.$$

(2) *If $\tilde{a}(\mu, \xi) := \mu^{p/d}\,a(1/\mu, \xi)$, then $\tilde{a}(\mu, \xi) \in C^\infty(\Lambda_{cc} \times \mathbb{R}^n)$ and for any $\alpha$ and $\beta$,*

$$|\,\partial_\mu^\alpha \partial_\xi^\beta \tilde{a}(\mu, \xi)\,| \,\dot{\le}\, (1 + |\mu|\,|\xi|^d)^{p/d-|\alpha|}\,|\xi|^{d|\alpha|+m-p-|\beta|}.$$

*In particular, for any $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ near $\xi = 0$ and $\chi(\xi) \equiv 1$ outside a neighborhood of 0, we have $\chi(\xi)\,a(\lambda, \xi) \in S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$.*

*Proof.* First observe that $\tilde{a}(\mu, \xi) \in C^\infty(\Lambda_{cc} \times (\mathbb{R}^n \setminus \{0\}))$ by Lemma 16. We will now prove (1). Since $\tilde{a}(\mu, \xi/|\xi|) \in C^\infty(\Lambda_{cc} \times \mathbb{S}^{n-1})$, by Lemma 16, $(\partial_\xi^\beta \tilde{a})(\mu, \xi/|\xi|) \in C^\infty(\Lambda_{cc} \times \mathbb{S}^{n-1})$ for any $\beta$. Thus, if $k \in \mathbb{N}_0$ and $\beta$ are given, for all $|\lambda| \ge 1$ (that is, $|\mu| \le 1$),

$$\begin{aligned} |\,(\lambda \cdot \partial_\lambda)^k (\partial_\xi^\beta a)(\lambda, \xi/|\xi|)\,| &= |\,(-\mu \cdot \partial_\mu)^k (\partial_\xi^\beta \mu^{-p/d}\tilde{a})(\mu, \xi/|\xi|)\,| \\ &\dot{\le} |\mu|^{-p/d} \\ &\doteq |\lambda|^{p/d}. \end{aligned}$$

Hence, as $a(\lambda, \xi/|\xi|) \in C^\infty(\Lambda \times \mathbb{S}^{n-1})$,

$$|\,(\partial_\lambda^\alpha \partial_\xi^\beta a)(\lambda, \xi/|\xi|)\,| \,\dot{\le}\, (1 + |\lambda|)^{p/d-|\alpha|} \tag{25}$$

for all $(\lambda, \xi) \in \Lambda \times (\mathbb{R}^n \setminus \{0\})$. Since $a(\delta^d\lambda, \delta\xi) = \delta^m a(\lambda, \xi)$ for all $\delta > 0$, we have $(\partial_\lambda^\alpha \partial_\xi^\beta a)(\delta^d\lambda, \delta\xi)$ $= \delta^{m-d|\alpha|-|\beta|}(\partial_\lambda^\alpha \partial_\xi^\beta a)(\lambda, \xi)$ for all $\delta > 0$. This, and (25), imply that

$$\begin{aligned} |\,\partial_\lambda^\alpha \partial_\xi^\beta a(\lambda, \xi)\,| &= |\,(\partial_\lambda^\alpha \partial_\xi^\beta a)(\,|\xi|^d|\xi|^{-d}\lambda, |\xi|\;\xi/|\xi|\,)\,| \\ &= |\xi|^{m-d|\alpha|-|\beta|}\,|\,(\partial_\lambda^\alpha \partial_\xi^\beta a)(\,|\xi|^{-d}\lambda, \xi/|\xi|\,)| \\ &\dot{\le} |\xi|^{m-d|\alpha|-|\beta|}\,(1 + |\xi|^{-d}|\lambda|\,)^{p/d-|\alpha|} \\ &\doteq (\,|\lambda| + |\xi|^d\,)^{p/d-|\alpha|}\,|\xi|^{m-p-|\beta|} \\ &\dot{\le} (\,|\lambda|^{1/d} + |\xi|\,)^{p-d|\alpha|}\,|\xi|^{m-p-|\beta|}. \end{aligned}$$

Exchanging the roles of $a$ and $\tilde{a}$, one can use a similar type argument to prove (2). ☐

**Definition 18.** Let $m,\, p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. The space of *resolvent one-step, polyhomogeneous symbols*, $S^{m,p,d}_{\Lambda,ros}(\mathbb{R}^n) \subseteq S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$ consists of those elements $a(\lambda, \xi) \in S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$ such that

$$a(\lambda, \xi) \sim \sum_{j=0}^\infty \chi(\xi)\,a_{m-j}(\lambda, \xi), \tag{26}$$

where for each $j$, $a_{m-j}(\lambda, \xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m-j)}(\mathbb{R}^n)$, where $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ near $\xi = 0$ and $\chi(\xi) \equiv 1$ outside a neighborhood of 0, and where the asymptotic sum (26) is in the sense of Lemma 14.

Observe that by definition, for any $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ near $\xi = 0$ and $\chi(\xi) \equiv 1$ outside a neighborhood of 0, we have

$$\chi(\xi)\, C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n) \subseteq S^{m,p,d}_{\Lambda,ros}(\mathbb{R}^n). \tag{27}$$

**Lemma 19.** *Let $a(\xi) \in C^\infty_{\mathrm{hom}(m)}(\mathbb{R}^n)$, $m \in \mathbb{R}^+$ be elliptic and suppose that $a(\xi)$ never takes values in the sector $\Lambda$ for $\xi \neq 0$. Then for all $N \in \mathbb{Z}$,*

$$(\,a(\xi) - \lambda)^N \in C^{\infty,Nm,m}_{\Lambda,\mathrm{hom}(Nm)}(\mathbb{R}^n).$$

*In particular, for any $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ near $\xi = 0$ and $\chi(\xi) \equiv 1$ outside a neighborhood of 0, $\chi(\xi)\,(\,a(\xi) - \lambda)^N \in S^{Nm,Nm,m}_{\Lambda,ros}(\mathbb{R}^n)$.*

*Proof.* If $a(\lambda,\xi) := (\,a(\xi) - \lambda)^N$, then $a(\delta^d\lambda, \delta\xi) = \delta^{Nm} a(\lambda,\xi)$ for all $\delta > 0$. Also, if $\tilde{a}(\mu,\xi) := \mu^N a(1/\mu,\xi) = (\mu a(\xi) - 1)^N$, then since $a(\xi)$ never takes values in the sector $\Lambda$, it follows that $\tilde{a}(\mu, \xi/|\xi|) \in C^\infty(\Lambda_{cc} \times \mathbb{S}^{n-1})$. Hence, by Definition 15, $a(\lambda,\xi) \in C^{\infty,Nm,m}_{\Lambda,\mathrm{hom}(Nm)}(\mathbb{R}^n)$. The second statement follows from the inclusion (27). □

This lemma suggests that the space $S^{-m,-m,m}_{\Lambda,ros}(\mathbb{R}^n)$ describes the local symbols for the resolvents of elliptic, one-step, $b$-pseudodifferential operators.

**Definition 20.** If $m \in \mathbb{Z}$ and $\mathcal{F}$ is any Fréchet space, $S^m_r(\Lambda;\mathcal{F})$ consists of those $a(\lambda) \in C^\infty(\Lambda,\mathcal{F})$ that are symbols of order $m$, $a(\lambda) \in S^m(\Lambda;\mathcal{F})$, and are such that $\mu^m a(1/\mu)$, where $\mu \in \Lambda_{cc}$, is smooth at $\mu = 0$.

**Lemma 21.** *Let $\Lambda \subseteq \mathbb{C}$ be a sector. Then*

$$S^{-\infty,p,d}_{\Lambda,ros}(\mathbb{R}^n) \equiv S^{-\infty,p,d}_{\Lambda,r}(\mathbb{R}^n) \equiv S^{p/d}_r(\Lambda, S^{-\infty}(\mathbb{R}^n)\,).$$

*Proof.* Note that $S^{-\infty,p,d}_{\Lambda,ros}(\mathbb{R}^n) \equiv S^{-\infty,p,d}_{\Lambda,r}(\mathbb{R}^n)$ by definition. By (5) of Lemma 5, $S^{-\infty,p,d}_{\Lambda,ros}(\mathbb{R}^n) \equiv S^{-\infty,p,d}_{\Lambda,r}(\mathbb{R}^n) \subseteq S^{p/d}_r(\Lambda, S^{-\infty}(\mathbb{R}^n)\,)$. Now given any multi-index $\alpha$, we can choose $N \in \mathbb{N}$ with $N \geq |p| + |\alpha|$. Then observe that given any $N' \in \mathbb{N}$ and multi-index $\beta$, for $|\mu|$ less than a fixed bound,

$$(1+|\xi|)^{-N-N'-|\beta|} \,\dot{\leq}\, (1 + |\mu|\,|\xi|^d)^{p/d-|\alpha|}\,(1+|\xi|)^{d|\alpha|-N'-p-|\beta|}.$$

Thus, an element of $S^{p/d}_r(\Lambda, S^{-\infty}(\mathbb{R}^n)\,)$ satisfies the estimate (21) for any $m \in \mathbb{R}$. Thus,

$$S^{p/d}_r(\Lambda, S^{-\infty}(\mathbb{R}^n)\,) \subseteq S^{-\infty,p,d}_{\Lambda,ros}(\mathbb{R}^n) \equiv S^{-\infty,p,d}_{\Lambda,r}(\mathbb{R}^n).$$

□

3.4. **Tempered distributions on manifolds.** Let $\Lambda \subseteq \mathbb{C}$ be a sector and let $X$ be compact with corners.

**Definition 22.** Let $Y \subseteq X$ be a closed interior p-submanifold and let $m, p \in \mathbb{R}$ and $d \in \mathbb{R}^+$. The space of *tempered conormal distributions to $Y$ of degree $m,p,d$*, $I^{m,p,d}_\Lambda(X,Y)$, consists of those elements $u \in C^\infty(\Lambda; I^m(X,Y)\,)$ satisfying the following two conditions:

1. Given any $\phi \in C^\infty_c(X \setminus Y)$, we have $\phi u \in S^{p/d}(\Lambda; C^\infty(X)\,)$.
2. Given any coordinate patch $\mathbb{R}^{l,k}_y \times \mathbb{R}^q_z$ on $X$ such that $Y \cong \mathbb{R}^{l,k} \times \{0\}$ and any compactly supported function $\phi$ on the coordinate patch, we have
$$\phi\, u = \frac{1}{(2\pi)^q} \int e^{iz\cdot\xi}\, a(\lambda,y,\xi)\, d\xi,$$
where $y \mapsto a(\lambda,y,\xi) \in C^\infty(\mathbb{R}^{l,k};\, S^{m',p,d}_\Lambda(\mathbb{R}^q)\,)$, where $m' = m - n/4 + l/4$.

If $p/d \in \mathbb{Z}$, the space of *resolvent tempered* and *resolvent one-step* conormal distributions, $I^{m,p,d}_{\Lambda,*}(X,Y)$, where $*$ is $r$ and $ros$ respectively, consists of those elements $u \in C^\infty(\Lambda; I^m(X,Y)\,)$ satisfying (1) with $S^{p/d}$ replaced by $S^{p/d}_r$ and satisfying (2) with $S^{m',p,d}_\Lambda$ replaced by $S^{m',p,d}_{\Lambda,r}$ and $S^{m',p,d}_{\Lambda,ros}$ respectively. The space of *strongly tempered* and *strongly tempered one-step* conormal distributions, $I^{m,d}_*(X,Y)$, where $*$ is $\Lambda$ and $\Lambda, os$ respectively, consists of those elements $u \in C^\infty(\Lambda; I^m(X,Y)\,)$ satisfying (1) with $S^{p/d}$ replaced by $S^{-\infty}$, and satisfying (2) with $S^{m',p,d}_\Lambda$ replaced by $S^{m',d}_\Lambda$, and $S^{m',d}_{\Lambda,os}$ respectively.

**Remark 23.** One can check, just as in the definition for the usual (non-parameter dependent) conormal distributions, that the condition (2) above is well-defined, independent of the (product) coordinates chosen.

As in the case for $b$-pseudodifferential operators, cf. (11), we define tempered $b$-pseudodifferential operators in terms of their Schwartz kernels.

**Definition 24.** Let $m,\, p \in \mathbb{R}$ and $d \in \mathbb{R}^+$. The *small calculus of tempered $b$-pseudodifferential operators of degree $m, p, d$* is the space of kernels

$$\Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac12}) := \{K \in I^{m,p,d}_{\Lambda}(X_b^2,\Delta_b,\Omega_b^{\frac12}) \,|\, K \equiv 0 \text{ at } lb \cup rb\}.[4]$$

If $p/d \in \mathbb{Z}$, the *small calculus of resolvent tempered and resolvent one-step $b$-pseudodifferential operators of degree $m, p, d$* are the space of kernels

$$\Psi^{m,p,d}_{b,\Lambda,r}(X,\Omega_b^{\frac12}) := \{K \in I^{m,p,d}_{\Lambda,r}(X_b^2,\Delta_b,\Omega_b^{\frac12}) \,|\, K \equiv 0 \text{ at } lb \cup rb\},$$
$$\Psi^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac12}) := \{K \in I^{m,p,d}_{\Lambda,ros}(X_b^2,\Delta_b,\Omega_b^{\frac12}) \,|\, K \equiv 0 \text{ at } lb \cup rb\}.$$

The *small calculus of strongly tempered and strongly tempered, one-step $b$-pseudo-differential operators of degree $m, d$* are the space of kernels

$$\Psi^{m,d}_{b,\Lambda}(X,\Omega_b^{\frac12}) := \{K \in I^{m,d}_{\Lambda}(X_b^2,\Delta_b,\Omega_b^{\frac12}) \,|\, K \equiv 0 \text{ at } lb \cup rb\},$$
$$\Psi^{m,d}_{b,\Lambda,os}(X,\Omega_b^{\frac12}) := \{K \in I^{m,d}_{\Lambda,os}(X_b^2,\Delta_b,\Omega_b^{\frac12}) \,|\, K \equiv 0 \text{ at } lb \cup rb\}.$$

Lemma 5 implies the following.

**Lemma 25.** *Let $m,p \in \mathbb{R}$ and $d \in \mathbb{R}^+$. Then,*

(1) *for any $m', p' \in \mathbb{R}$, $\Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac12}) \subseteq \Psi^{m',p',d}_{b,\Lambda}(X,\Omega_b^{\frac12})$, provided $m \le m'$ and $p \le p'$;*
(2) *for any $\alpha$, $\partial^\alpha_\lambda \Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac12}) \subseteq \Psi^{m-d|\alpha|,p-d|\alpha|,d}_{b,\Lambda}(X,\Omega_b^{\frac12})$;*
(3)

$$\Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac12}) \subseteq S^{p/d}(\Lambda;\Psi^m_b(X,\Omega_b^{\frac12})), \qquad \text{if } p > 0;$$
$$\Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac12}) \subseteq S^{p/d}(\Lambda;\Psi^{m-p}_b(X,\Omega_b^{\frac12})), \quad \text{if } p \le 0.$$

The next theorem follows from property (3) of the previous theorem and the fact that the small calculus acts continuously on $\dot{C}^\infty(X,\Omega_b^{\frac12})$ and on $C^\infty(X,\Omega_b^{\frac12})$.

**Theorem 26.** *Let $m,\, p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ and let $A \in \Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac12})$. Then $A$ defines continuous linear maps*

$$A : \dot{C}^\infty(X,\Omega_b^{\frac12}) \to S^{p/d}(\Lambda;\dot{C}^\infty(X,\Omega_b^{\frac12}));$$
$$A : C^\infty(X,\Omega_b^{\frac12}) \to S^{p/d}(\Lambda;C^\infty(X,\Omega_b^{\frac12})).$$

Since $\Psi^{m,p,d}_{b,\Lambda,r}(X,\Omega_b^{\frac12}) \subseteq \Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac12})$ and, by Lemma 7, $\Psi^{m,d}_{b,\Lambda}(X,\Omega_b^{\frac12}) \subseteq \Psi^{m,m,d}_{b,\Lambda}(X,\Omega_b^{\frac12})$, a similar statement holds for this theorem for the resolvent, resolvent one-step, strongly tempered, and strongly tempered one-step spaces.

The following results are proved just like their corresponding results in the usual non-parameter dependent small calculus.

**Theorem 27.** *If $m, m', p, p' \in \mathbb{R}$ and $d \in \mathbb{R}^+$, then*

$$\Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac12}) \circ \Psi^{m',p',d}_{b,\Lambda}(X,\Omega_b^{\frac12}) \subseteq \Psi^{m+m',p+p',d}_{b,\Lambda}(X,\Omega_b^{\frac12}).$$

*If $p/d, p'/d \in \mathbb{Z}$, the same statement holds for the resolvent and the resolvent one-step spaces. If $m, m' \in \mathbb{R}$ and $d \in \mathbb{R}^+$, then*

$$\Psi^{m,d}_{b,\Lambda}(X,\Omega_b^{\frac12}) \circ \Psi^{m',d}_{b,\Lambda}(X,\Omega_b^{\frac12}) \subseteq \Psi^{m+m',d}_{b,\Lambda}(X,\Omega_b^{\frac12})$$

*The same result holds for the strongly tempered one-step spaces.*

[4]Here, $\equiv 0$ at a submanifold means that the Taylor series vanishes at the submanifold.

**Lemma 28.** *Let $m,p \in \mathbb{R}$ and $d \in \mathbb{R}^+$. Then given a sequence $A_j \in \Psi^{m-j,p,d}_{b,\Lambda}(X,\Omega_b^{\frac12})$, $j \in \mathbb{N}_0$, there exists an $A \in \Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac12})$ such that for all $N \in \mathbb{N}$, $A - \sum_{j=0}^{N-1} A_j \in \Psi^{m-N,p,d}_{b,\Lambda}(X,\Omega_b^{\frac12})$; in which case, we write $A \sim \sum_{j=0}^\infty A_j$. If $p/d \in \mathbb{Z}$, the same statement holds for the resolvent and the resolvent one-step spaces.*

*Similarly, given a sequence $A_j \in \Psi^{m-j,d}_{b,\Lambda}(X,\Omega_b^{\frac12})$, $j \in \mathbb{N}_0$, there exists an $A \in \Psi^{m,d}_{b,\Lambda}(X,\Omega_b^{\frac12})$ such that for all $N \in \mathbb{N}$, $A - \sum_{j=0}^{N-1} A_j \in \Psi^{m-N,d}_{b,\Lambda}(X,\Omega_b^{\frac12})$; in which case, we write $A \sim \sum_{j=0}^\infty A_j$. The same statement holds for the strongly tempered one-step spaces.*

3.5. **Tempered calculus with bounds.** In this subsection, we define the tempered calculus with bounds, and will prove Lemma 30, which will in turn, be used to prove that our tempered operator spaces capture the resolvents of $b$-pseudodifferential operators, cf. Theorem 33. Let $\alpha$ be a multi-index on $X_b^2$. Let $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. We define (cf. Definitions (13) and (20))

$$\begin{aligned}
\Psi^{-\infty,p,d;\alpha}_{b,\Lambda,r}(X,\Omega_b^{\frac12}) &:= \rho_{lb}^{\alpha|_{lb}}\rho_{rb}^{\alpha|_{rb}} \textstyle\bigcup_{\epsilon>0} \rho_{lb}^\epsilon \rho_{rb}^\epsilon S_r^{p/d}(\Lambda; S_{ff}^{0,\alpha|_{ff}}(X_b^2,\Omega_b^{\frac12}));\\
\Psi^{-\infty,d;\alpha}_{b,\Lambda}(X,\Omega_b^{\frac12}) &:= \rho_{lb}^{\alpha|_{lb}}\rho_{rb}^{\alpha|_{rb}} \textstyle\bigcup_{\epsilon>0} \rho_{lb}^\epsilon \rho_{rb}^\epsilon S^{-\infty}(\Lambda; S_{ff}^{0,\alpha|_{ff}}(X_b^2,\Omega_b^{\frac12})).
\end{aligned}$$

For any $m \in \mathbb{R}$, we define

$$\begin{aligned}
\Psi^{m,p,d;\alpha}_{b,\Lambda,r}(X,\Omega_b^{\frac12}) &:= \Psi^{m,p,d}_{b,\Lambda,r}(X,\Omega_b^{\frac12}) + \Psi^{-\infty,p,d;\alpha}_{b,\Lambda,r}(X,\Omega_b^{\frac12});\\
\Psi^{m,p,d;\alpha}_{b,\Lambda,ros}(X,\Omega_b^{\frac12}) &:= \Psi^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac12}) + \Psi^{-\infty,p,d;\alpha}_{b,\Lambda,r}(X,\Omega_b^{\frac12});\\
\Psi^{m,d;\alpha}_{b,\Lambda}(X,\Omega_b^{\frac12}) &:= \Psi^{m,d}_{b,\Lambda}(X,\Omega_b^{\frac12}) + \Psi^{-\infty,d;\alpha}_{b,\Lambda}(X,\Omega_b^{\frac12});\\
\Psi^{m,d;\alpha}_{b,\Lambda,os}(X,\Omega_b^{\frac12}) &:= \Psi^{m,d}_{b,\Lambda,os}(X,\Omega_b^{\frac12}) + \Psi^{-\infty,d;\alpha}_{b,\Lambda}(X,\Omega_b^{\frac12}).
\end{aligned}$$

Let $\mathcal{B}$ be a Banach algebra with norm $\|\ \|$, and let $\mathcal{I}$ be a subspace of $\mathcal{B}$. Suppose that $\mathcal{I}$ has a Fréchet topology in addition to the relative topology that it inherits from $\mathcal{B}$. Let $\{\|\ \|_m \,|\, m \in \mathbb{N}_0\}$ be its family of norms. We will say that $\mathcal{I}$ is an *ideal* in $\mathcal{B}$ if for any $A, B \in \mathcal{I}$ and $C \in \mathcal{B}$, $ACB \in \mathcal{I}$, and for any $m \in \mathbb{N}_0$ there exists a $K_m > 0$ such that

$$\|ACB\|_m \le K_m \|A\|_m \|C\| \|B\|_m. \tag{28}$$

**Lemma 29.** *Let $\mathcal{B}$ be a Banach algebra and let $T(\lambda) \in S^m(\Lambda;\mathcal{B})$ where $m \in -\mathbb{N}$. Suppose that $\|T(\lambda)\| \le 1/2$ for each $\lambda \in \Lambda$. Then for each $\lambda \in \Lambda$, $S(\lambda) := \sum_{j=1}^\infty T(\lambda)^j \in \mathcal{B}$ exists, and $S \in S^m(\Lambda;\mathcal{B})$. If $T \in S_r^m(\Lambda;\mathcal{B})$, then so is $S$. Moreover, if a Fréchet space $\mathcal{I}$ is an ideal of $\mathcal{B}$ and $T \in S^m(\Lambda;\mathcal{I})$ or $T \in S_r^m(\Lambda;\mathcal{I})$ (symbols with respect to the Fréchet topology on $\mathcal{I}$), then $S$ is in the same space.*

*Proof.* Since $\|T(\lambda)\| \le 1/2 < 1$ for each $\lambda \in \Lambda$, the geometric series defining $S(\lambda)$ exists for each $\lambda \in \Lambda$. In fact, since the series for $S(\lambda)$ is dominated by $\sum_{j=1}^\infty (\frac12)^j$, $S(\lambda)$ is continuous and uniformly bounded on $\Lambda$. Observe that for each $\lambda \in \Lambda$,

$$(\mathrm{Id} - T(\lambda))^{-1} = \mathrm{Id} + S(\lambda). \tag{29}$$

Using this fact, we will prove that $S(\lambda)$ is smooth. Indeed, by algebraic manipulations using (29), one can show that for any $\lambda$, $v \in \Lambda$ and $t \in \mathbb{R}$ small,

$$S(\lambda+tv) - S(\lambda) = (\mathrm{Id} + S(\lambda+tv))(T(\lambda+tv) - T(\lambda))(\mathrm{Id} + S(\lambda)).$$

Dividing by $t$ and taking $t \to 0$ shows that

$$D_v S(\lambda) = (\mathrm{Id} + S(\lambda)) D_v T(\lambda)(\mathrm{Id} + S(\lambda)). \tag{30}$$

Hence, $D_v S(\lambda)$ exists and is uniformly bounded on $\Lambda$. Using equation (30) together with induction and the fact that $T \in S^m(\Lambda;\mathcal{B})$ with $m \le 0$ shows that $S$ is smooth and all its derivatives are uniformly bounded on $\Lambda$.

To see that $S$ is a symbol of order $m$, observe that for any $N \in \mathbb{N}$, we can write

$$S(\lambda) = \sum_{j=1}^{2N} T(\lambda)^j + R_N(\lambda), \text{ where } R_N(\lambda) = T(\lambda)^N S(\lambda) T(\lambda)^N. \tag{31}$$

Since $T \in S^m(\Lambda;\mathcal{B})$, $\sum_{j=1}^{2N} T(\lambda)^j \in S^m(\Lambda;\mathcal{B})$ and since $S(\lambda)$ is uniformly bounded with all its derivatives, $R_N(\lambda)$ is $O(|\lambda|^{2Nm})$ with all its derivatives. Since $m<0$, $R_N(\lambda)$ vanishes to high order as $|\lambda|\to\infty$. It follows that $S \in S^m(\Lambda;\mathcal{B})$. If $T\in S^m_r(\Lambda;\mathcal{B})$, then observe that $\sum_{j=1}^{2N} T(\lambda)^j \in S^m_r(\Lambda;\mathcal{B})$. Indeed,

$$\mu^m \sum_{j=1}^{2N} T(1/\mu)^j = \sum_{j=1}^{2N} \mu^{(1-j)m}(\mu^m T(1/\mu))^j.$$

Since $\mu^{(1-j)m}$ is smooth at $\mu=0$ (as $m\in-\mathbb{N}$) and since $(\mu^m T(1/\mu))^j$ is smooth at $\mu=0$ as $T\in S^m_r(\Lambda;\mathcal{B})$, $\mu^m\sum_{j=1}^{2N} T(1/\mu)^j$ is smooth at $\mu=0$. Thus, as $R_N(\lambda)$ vanishes to high order as $|\lambda|\to\infty$ and $N$ is arbitrary, it follows that $S\in S^m_r(\Lambda;\mathcal{B})$. If $T\in S^m(\Lambda;\mathcal{I})$ or $T\in S^m_r(\Lambda;\mathcal{I})$, similar arguments work to show that $S$ is in the same space, but here we must use (28), and the fact that in (31), the remainder term $R_N(\lambda)$ is written as $T(\lambda)^N S(\lambda) T(\lambda)^N \in \mathcal{I}$. ☐

**Lemma 30.** *Let $R\in\Psi^{-\infty,p,d}_{b,\Lambda,r}(X,\Omega_b^{\frac12})$, where $p<0$ and $d>0$ with $p/d\in\mathbb{Z}$. Then there exists a continuous increasing function $r:\mathbb{R}^+\to\mathbb{R}^+$ such that for each $a>0$ and $|\lambda|\geq r(a)$, $\mathrm{Id}-R(\lambda): L^2_b(X,\Omega_b^{\frac12})\to L^2_b(X,\Omega_b^{\frac12})$ is invertible. Moreover, its inverse is of the form*

$$(\mathrm{Id}-R(\lambda))^{-1} = \mathrm{Id}+S(\lambda),$$

*where $S(\lambda)\in\Psi^{-\infty,p,d;a}_{b,\Lambda,r}(X,\Omega_b^{\frac12})$ for $|\lambda|\geq r(a)$.*

*Proof.* For each $M\in M_k(X)$ and $B\in\Psi_b^{-\infty}(X,\Omega_b^{\frac12})$, we denote $N_M(B)(\tau)$ by $B_M(\tau)$. Also, if $S^k_a=\{\tau\in\mathbb{C}^k \mid |\mathrm{Im}\,\tau|\leq a\}$, we define $\mathcal{A}_a(M)$ as those functions $B(\lambda,\tau)\in S^{(p/d,-\infty)}(\Lambda\times S^k_a;\Psi_b^{-\infty,a}(M,\Omega_b^{\frac12}))$, holomorphic in $\tau$, such that $\mu^{p/d}B(1/\mu,\tau)$ is smooth at $\mu=0$. Thus, an element $B(\lambda,\tau)\in\mathcal{A}_a(M)$ takes values in the space $\Psi_b^{-\infty,a}(M,\Omega_b^{\frac12})$, is holomorphic in $\tau$ vanishing to infinite order as $|\tau|\to\infty$ in $S^k_a$, is a symbol of order $p/d$ in $\lambda$, and is such that $\mu^{p/d}B(1/\mu,\tau)$ is smooth at $\mu=0$. Let $N=\operatorname{codim} X$. Then we will prove the following statement by induction on $k=N,\,N-1,\,N-2,\ldots,2,\,1,\,0$:

(32) Given $M\in M_k(X)$, there exists a continuous increasing function $r:\mathbb{R}^+\to\mathbb{R}^+$ such that for each $a>0$ and $|\lambda|\geq r(a)$, $\mathrm{Id}-R_M(\lambda,\tau)$ is invertible on $L^2_b(M,\Omega_b^{\frac12})$ for $\tau\in S^k_a$, with inverse of the form $\mathrm{Id}+S(\lambda,\tau)$, where $S(\lambda,\tau)\in\mathcal{A}_a(M)$ for all $|\lambda|\geq r(a)$.

Setting $k=0$ proves our lemma. Thus, assume that $M\in M_N(X)$. Observe that for any $a>0$, $R_M(\lambda,\tau)$ takes values in the smoothing operators on the compact manifold without boundary $M$. Thus, as $p/d<0$ and $R(\lambda,\tau)$ vanishes to infinite order as $|\tau|\to\infty$ in $S^N_a$ for any $a$, there exists a continuous increasing function $r:\mathbb{R}^+\to\mathbb{R}^+$ such that for any $a>0$, $\|R_M(\lambda,\tau)\|_{L^2(M,\Omega^{\frac12})}\leq 1/2$ for all $|\lambda|\geq r(a)$ and $\tau\in S^N_a$. Hence, by a slight extension of Lemma 29 and the fact that the smoothing operators are an ideal in the bounded operators on $L^2(M,\Omega^{\frac12})$, it follows that for $|\lambda|\geq r(a)$ and $\tau\in S^N_a$, we have

$$(\mathrm{Id}-R_M(\lambda,\tau))^{-1}=\mathrm{Id}+S_M(\lambda,\tau),\ S_M(\lambda,\tau):=\sum_{j=1}^{\infty}R_M(\lambda,\tau)^j,$$

where $S_M(\lambda,\tau)\in S^{(p/d,-\infty)}(\Lambda\times S^N_a;\Psi^{-\infty}(M,\Omega^{\frac12}))$ for $|\lambda|\geq r(a)$ and is such that $\mu^{p/d}B(1/\mu,\tau)$ is smooth at $\mu=0$. By Morera's Theorem ([5, p. 86]), $S_M(\lambda,\tau)$ is holomorphic in $\tau$. Thus, $S_M(\lambda,\tau)\in\mathcal{A}_a(M)$ for $|\lambda|\geq r(a)$. Hence, (32) is proved for $k=N$.

Assume that (32) holds for $k+1$; we will prove it for $k$. Fix $M\in M_k(X)$ and let $\rho$ be a total boundary defining function for $M$. Let $M_1(M)=\{F_1,\ldots,F_l\}$.

To prove (32), we first claim that for each $j=1,\ldots,l$, there exists a continuous increasing function $r_j:\mathbb{R}^+\to\mathbb{R}^+$, such that

$$(\mathrm{Id}-R_M)(\mathrm{Id}+S_j)=\mathrm{Id}-T_j, \tag{33}$$

where for each $a > 0$, $S_j \in \mathcal{A}_a(M)$ and $T_j \in \rho_{ff,1} \cdots \rho_{ff,j} \cdot \mathcal{A}_a(M)$ for all $|\lambda| \geq r_j(a)$, where $\rho_{ff,i}$ is the boundary defining function for the front face of $M_b^2$ coming from $F_i \times F_i$. To see this, we use induction on $j$. Thus, assume that $j = 1$. Then, as $F_1 \in M_{k+1}(X)$ and $N_{F_1}(R_M) = R_{F_1}$, by replacing $r : \mathbb{R}^+ \to \mathbb{R}^+$ in the induction hypothesis (32) with $r_1(a) := r(a+1)$, it follows that for all $a > 0$, $(\mathrm{Id} - N_{F_1}(R_M))^{-1} \in \mathrm{Id} + \mathcal{A}_{a+1}(F_1)$ for all $|\lambda| \geq r_1(a)$. Let $x$ be the fixed boundary defining function for $F_1$ and let $\tau$ be its dual variable. Then we define (cf. the last part of Subsection 2.4)

$$S_1 := \phi(x)\frac{1}{2\pi}\int_{\mathbb{R}} e^{iz\tau}(\mathrm{Id} - N_{F_1}(R_M)(\tau))^{-1} N_{F_1}(R_M)(\tau) d\tau \cdot |\frac{dx}{x} dz|^{\frac{1}{2}},$$

where $\phi(x) \equiv 1$ near $x = 0$ and vanishes outside of $x = 0$, and where $z = \log(x/x')$. Then it follows that $S_1 \in \mathcal{A}_a(M)$ for all $|\lambda| \geq r_1(a)$. Now since $(\mathrm{Id} - R_M)(\mathrm{Id} + S_1) = \mathrm{Id} - (R_M - (\mathrm{Id} - R_M)S_1)$, by the definition of $S_1$, we have $N_{F_1}(R_M - (\mathrm{Id} - R_M)S_1) = 0$ and so (33) is proved for $j = 1$. Assuming (33) holds for $j$, the $j+1$ case is proved just like the $j = 1$ case, but we take $S_{j+1}$ to be the inverse Fourier transform of $(\mathrm{Id} - N_{F_{j+1}}(R_M))^{-1} N_{F_{j+1}}(T_j)$. Hence, (33) holds for $j + 1$.

Now, setting $j = l$ in (33) and $r(a) := r_l(2a + 1)$ implies that $(\mathrm{Id} - R_M)(\mathrm{Id} + S) = \mathrm{Id} - \rho_{ff} T$, where for each $a > 0$, $S \in \mathcal{A}_{2a+1}(M)$ and $T \in \mathcal{A}_{2a+1}(M)$ for all $|\lambda| \geq r(a)$. Let $n_a$ be the smallest integer strictly greater than $2a$. Then,

$$(\mathrm{Id} - R_M)(\mathrm{Id} + S)(\mathrm{Id} + S'_a) = \mathrm{Id} - S''_a,$$

where $S'_a = \sum_{j=1}^{n_a - 1} (\rho_{ff} T)^j$ and $S''_a = (\rho_{ff} T)^{n_a}$. By (14), $S'_a \in \rho_{ff} \mathcal{A}_{2a+1}(M)$ and $S''_a \in \rho_{ff}^{n_a} \mathcal{A}_{2a+1}(M)$. In particular, $S''_a$ takes values in $\rho_{lb}^{a+\epsilon} \rho_{rb}^{a+\epsilon} S^0(M^2, \Omega_b^{\frac{1}{2}})$ for some $\epsilon > 0$. Since $p/d < 0$ and since $\rho_{lb}^{a+\epsilon} \rho_{rb}^{a+\epsilon} S^0(M^2, \Omega_b^{\frac{1}{2}})$ is an ideal in the bounded operators on $L_b^2(M, \Omega_b^{\frac{1}{2}})$, we can argue just as in the $k = N$ case to invert $\mathrm{Id} - S''_a$ proving (32) for $k$. $\square$

3.6. **The Resolvent as a Tempered operator.** Recall that given sectors $\Lambda$, $\Lambda' \subseteq \mathbb{C}$, $\Lambda$ is a proper subsector of $\Lambda'$ if $\Lambda \subseteq \mathrm{int}(\Lambda') \cup \{0\}$. We will study the resolvents for the following class of operators.

**Definition 31.** Let $m \in \mathbb{R}^+$ and $\Lambda \subseteq \mathbb{C}$ be a sector. Then an elliptic operator $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$ is said to be *elliptic with respect to* $\Lambda$ if the principal symbol ${}^b\sigma_m(A) \in S^{[m]}({}^bT^*X)$ can be represented by an element $a(x,\xi) \in S^m({}^bT^*X)$ such that there exists a sector $\Lambda' \subseteq \mathbb{C}$ with $\Lambda \subseteq \Lambda'$ a proper subsector and a compact set $K \subseteq {}^bT^*X$, such that $a(x,\xi) \notin \Lambda'$ for all $(x,\xi) \notin K$.

This definition makes sense because of the following lemma.

**Lemma 32.** *Let $a(x,\xi) \in S^m({}^bT^*X)$, $m \in \mathbb{R}^+$ be elliptic and let $\Lambda \subseteq \mathbb{C}$ be a sector. Suppose that $a(x,\xi)$ has the property that there exists a sector $\Lambda' \subseteq \mathbb{C}$ with $\Lambda \subseteq \Lambda'$ a proper subsector, and a compact set $K \subseteq {}^bT^*X$ such that $a(x,\xi) \notin \Lambda'$ for all $(x,\xi) \notin K$. Then for any $b(x,\xi) \in S^{m-1}({}^bT^*X)$, $a(x,\xi) + b(x,\xi)$ has the same property.*

*Proof.* Let $\chi(x,\xi) \in C^\infty({}^bT^*X)$ with $\chi(x,\xi) \equiv 0$ for $(x,\xi) \in K$ and with $\chi(x,\xi) \equiv 1$ outside a neighborhood of $K$. Let $b(x,\xi) \in S^{m-1}({}^bT^*X)$. Then, $a(x,\xi) + b(x,\xi) = a(x,\xi) + \chi(x,\xi)\, b(x,\xi) + (1 - \chi(x,\xi))\, b(x,\xi)$. Since $1 - \chi(x,\xi) \equiv 0$ outside of $K$, we just need $a(x,\xi) + \chi(x,\xi)\, b(x,\xi)$ to have the same property as $a(x,\xi)$. Since $a$ is elliptic and non-zero outside of $K$, we can write

$$a(x,\xi) + \chi(x,\xi)\, b(x,\xi) = a(x,\xi)\,(1 + \frac{\chi(x,\xi)\, b(x,\xi)}{a(x,\xi)}).$$

Thus, we just have to show that for any $c(x,\xi) \in S^{-1}({}^bT^*X)$, $a(x,\xi)\,(1 + c(x,\xi))$ has the same property as $a(x,\xi)$. But, this follows from the fact that since $a(x,\xi)$ is elliptic, it grows like $|\xi|^m$ as $|\xi| \to \infty$, where $|\ |$ is the norm of any metric on ${}^bT^*X$, and $c(x,\xi)$ decays like $|\xi|^{-1}$ as $|\xi| \to \infty$. $\square$

Observe that if $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, is any elliptic operator, then $A^*A$ is elliptic with respect to any sector $\Lambda \subseteq \mathbb{C}$ with $\Lambda \cap (0,\infty) = \emptyset$. Also, observe that if $A \in \Psi_{b,os}^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, then $A$ is elliptic with respect to a sector $\Lambda$ iff ${}^b\sigma_m(A) \in C^\infty_{\mathrm{hom}(m)}({}^bT^*X)$ is such that ${}^b\sigma_m(A)(x,\xi) \notin \Lambda$ for all $\xi \neq 0$.

**Theorem 33.** *Let $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be elliptic with respect to $\Lambda$. Then for any $s \in \mathbb{R}$,*

$$A - \lambda : H_b^s(X, \Omega_b^{\frac{1}{2}}) \to H_b^{s-m}(X, \Omega_b^{\frac{1}{2}}),$$

*is invertible for $\lambda \in \Lambda$ sufficiently large, and there is a continuous increasing function $r : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $a > 0$, $(A-\lambda)^{-1} \in \Psi^{-m,-m,m;a}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r(a)$. If $A \in \Psi^m_{b,os}(X, \Omega_b^{\frac{1}{2}})$ or if $A \in \mathrm{Diff}^m_b(X, \Omega_b^{\frac{1}{2}})$, the same holds, but the resolvent is resolvent one-step or strongly tempered one-step respectively.*

*Proof.* For simplicity of notation, we will only prove this theorem for $A \in \Psi^m_b(X, \Omega_b^{\frac{1}{2}})$; the other cases are proved basically the same.[5] Thus, let $A \in \Psi^m_b(X, \Omega_b^{\frac{1}{2}})$. Let $a_m(x,\xi) \in S^m({}^bT^*X)$ be a representative of the principal symbol ${}^b\sigma_m(A) \in S^{[m]}({}^bT^*X)$ such that there exists a sector $\Lambda' \subseteq \mathbb{C}$ with $\Lambda \subseteq \Lambda'$ a proper subsector, and a compact set $K \subseteq {}^bT^*X$ such that $a_m(x,\xi) \notin \Lambda'$ for all $(x,\xi) \notin K$. Define $r := 1 + \max_{(x,\xi)\in K} |a_m(x,\xi)|$.

Let $\mathcal{U} = \mathbb{R}^{n,k}_w \times \mathbb{R}^n_z$ be a coordinate patch on $X^2_b$ such that $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}$. Since $X \cong \Delta_b$, $\mathbb{R}^{n,k}_w$ is coordinate patch on $X$. Let $\phi \in C^\infty_c(\mathbb{R}^{n,k})$ and let $\psi \in C^\infty_c(\mathcal{U})$ be such that $\psi(w,0) \equiv 1$ on $\operatorname{supp}\phi$. Since $\psi A$ is supported on $\mathcal{U}$, we can write $\psi A = \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, a(w,\xi)\, d\xi \cdot \nu$, where $0 < \nu \in C^\infty(\mathcal{U}, \Omega_b^{\frac{1}{2}})$, and where

$$a(w,\xi) = \psi(w,0)\, a_m(w,\xi) \text{ modulo } S^{m-1}({}^bT^*X). \tag{34}$$

Now observe that $\phi(w)(\,a_m(w,\xi) - \lambda\,)^{-1}$ is smooth for $\lambda \in \Lambda$ with $|\lambda| \geq r$. In fact, by Lemma 13, one can show that $w \mapsto \phi(w)(\,a_m(w,\xi)-\lambda\,)^{-1} \in C^\infty(\mathbb{R}^{n,k}; S^{-m,-m,m}_{\Lambda,r}(\mathbb{R}^n)\,)$ for all $\lambda \in \Lambda$ with $|\lambda| \geq r$. Let $\psi' \in C^\infty_c(\mathcal{U})$ be such that $\psi'(w,0) \equiv 1$ on $\operatorname{supp}\phi$ and $\psi \equiv 1$ on $\operatorname{supp}\psi'$. For $\lambda \in \Lambda$ with $|\lambda| \geq r$, define

$$B(\lambda) := \psi'(w,z)\,\phi(w)\frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}\,(\,a_m(w,\xi)-\lambda\,)^{-1}\,d\xi\cdot\nu. \tag{35}$$

Then, $B(\lambda) \in \Psi^{-m,-m,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r$. Since $1-\psi \equiv 0$ on $\operatorname{supp}\psi'$, it follows that $[(1-\psi)A]B(\lambda) \in \Psi^{-\infty,-m,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r$. Hence, for $|\lambda| \geq r$, $AB(\lambda) = [\psi A]B(\lambda) + [(1-\psi)A]B(\lambda) = [\psi A]B(\lambda)$ modulo $\Psi^{-\infty,-m,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$. Also, observe that (34) implies that for $|\lambda| \geq r$, we have $(\psi A - \lambda)B(\lambda) = \phi$ modulo $\Psi^{-1,-m,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$. Hence, for $\lambda \in \Lambda$ with $|\lambda| \geq r$,

$$(A-\lambda)B(\lambda) = \phi - R(\lambda), \text{ where } R(\lambda) \in \Psi^{-1,-m,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}}) \text{ for } |\lambda| \geq r. \tag{36}$$

Let $\{\mathcal{U}_i\}_{i=1}^N$ be coordinate patches of $X^2_b$ giving product decompositions of $X^2_b$ with respect to the p-submanifold $\Delta_b$, such that $\{\mathcal{U}_i \cap \Delta_b\}$ covers $\Delta_b$. Let $\{\phi_i\}$ be a partition of unity of $\Delta_b$ with respect to the cover $\{\mathcal{U}_i \cap \Delta_b\}$. Then for each $i$, by (36), there exists a $B_i(\lambda) \in \Psi^{-m,-m,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$ for $\lambda \in \Lambda$ with $|\lambda| \geq r$, such that $(A-\lambda)B_i(\lambda) = \phi_i - R_i(\lambda)$, where $R_i(\lambda) \in \Psi^{-1,-m,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r$. Setting $B(\lambda) := \sum_{i=1}^N B_i(\lambda)$, we then have $(A-\lambda)B(\lambda) = \mathrm{Id} - R(\lambda)$, where $R(\lambda) \in \Psi^{-1,-m,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r$. Since $R(\lambda)^j \in \Psi^{-j,-jm,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}}) \subseteq \Psi^{-j,-m,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$ for each $j$, we can choose $R'(\lambda) \in \Psi^{-1,-m,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$ such that $R'(\lambda) \sim \sum_{j=1}^\infty R(\lambda)^j$. Thus, if $B'(\lambda) := B(\lambda)\circ(\mathrm{Id} + R'(\lambda)) \in \Psi^{-m,-m,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r$, it follows that

$$(A-\lambda)B'(\lambda) = \mathrm{Id} - R''(\lambda), \tag{37}$$

where $R''(\lambda) \in \Psi^{-\infty,-m,m}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r$. Thus, by Lemma 30, there exists a continuous increasing function $r : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $a > 0$ and $|\lambda| \geq r(a)$, $(\mathrm{Id} - R''(\lambda))^{-1} = \mathrm{Id} + S(\lambda)$, where for each $a > 0$, $S(\lambda) \in \Psi^{-\infty,-m,m;a}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r(a)$. Hence, for each $a > 0$ and $|\lambda| \geq r(a)$,

$$(A-\lambda)^{-1} = B'(\lambda)\circ(\mathrm{Id} + S(\lambda)) \in \Psi^{-m,-m,m;a}_{b,\Lambda,r}(X,\Omega_b^{\frac{1}{2}})$$

for all $\lambda \in \Lambda$ with $|\lambda| \geq r(a)$. □

[5] If $A \in \Psi^m_{b,os}(X,\Omega_b^{\frac{1}{2}})$, the only difference occurs in equation (35), where we should set $a_m(y,\xi) := \chi(\xi)\,{}^b\sigma_m(A)(y,\xi)$, where $\chi(\xi) \equiv 0$ near 0 and $\chi(\xi) \equiv 1$ outside of a neighborhood of 0.

Coordinates on $[X_b^2 \times \overline{\Lambda}; \Delta_b \times \partial_\infty \overline{\Lambda}]$:

$$\left.\begin{array}{l} \rho := (|z|^2 + r^2)^{1/2} \\ \omega_0 := r/(|z|^2 + r^2)^{1/2} \\ \omega' := z/(|z|^2 + r^2)^{1/2} \end{array}\right\} \leftrightarrow \left\{ \begin{array}{l} r = \rho\, \omega_0 \\ z = \rho\, \omega' \end{array}\right.$$

Coordinates near $bi$:

$$\left.\begin{array}{l} s := |z| \\ \gamma_0 := r/|z| \\ \gamma' := z/|z| \end{array}\right\} \leftrightarrow \left\{ \begin{array}{l} r = s\, \gamma_0 \\ z = s\, \gamma' \end{array}\right.$$

Coordinates near $\{\omega' \equiv 0\}$:

$$\left.\begin{array}{l} t := r \\ w := z/r \end{array}\right\} \leftrightarrow \left\{ \begin{array}{l} r = t \\ z = t\, w \end{array}\right.$$

Coordinates on $X^2_{b,\Lambda}$ near $df$:

$$\left.\begin{array}{l} t := t \\ u := |w| \\ \gamma := w/|w| \end{array}\right\} \leftrightarrow \left\{ \begin{array}{l} t = t \\ w = u\, \gamma \end{array}\right\} \leftrightarrow \left\{ \begin{array}{l} r = t \\ z = t\, u\, \gamma \end{array}\right.$$

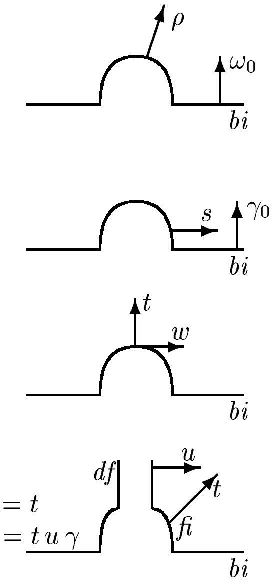


FIGURE 3. Various coordinates on $X^2_{b,\Lambda}$

## 4. THE STRUCTURE THEOREMS

4.1. **The blown-up tempered space.** In this section, we realize the kernels of the operators in $\Psi^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$ and $\Psi^{m,d}_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$ as classical conormal functions on a certain manifold with corners.

Let $\Lambda \subseteq \mathbb{C}$ be a sector. Denote by $\overline{\Lambda}$, the manifold $\Lambda$ radially compactified. We denote by $\partial_\infty\overline{\Lambda}$, the boundary 'at infinity'. Some convenient coordinates near $\partial_\infty\overline{\Lambda}$ can be obtained as follows. Let $\Lambda_{cc} := \{\overline{\lambda} \,|\, \lambda \in \Lambda\}$ be the 'complex conjugate' sector of $\Lambda$. Observe that $\Lambda_{cc} \ni \mu \mapsto \lambda = 1/\mu = \overline{\mu}/|\mu|^2 \in \Lambda$ is a biholomorphism of $\Lambda_{cc} \setminus \{0\}$ onto $\Lambda \setminus \{0\}$. Then by the definition of $\overline{\Lambda}$, $[\Lambda_{cc}; \{0\}] \equiv \overline{\Lambda} \setminus \{0\}$, where $\mathit{ff}[\Lambda_{cc}; \{0\}] \equiv \partial_\infty\overline{\Lambda}$.

Let $X$ be compact with corners. Then we define

$$X^2_{b,\Lambda} := [X_b^2 \times \overline{\Lambda}; \Delta_b \times \partial_\infty\overline{\Lambda}; \Delta_b \times \overline{\Lambda}].$$

The manifold $Z$ in Figure 1 of the introduction gives pictorial representation of $X^2_{b,\Lambda}$. We define

$$\begin{aligned} bi &:= \overline{\beta^{-1}(X_b^2 \times \partial_\infty\overline{\Lambda} \setminus (\Delta_b \times \partial_\infty\overline{\Lambda}))}, \text{ boundary at infinity;} \\ \mathit{fi} &:= \beta^{-1}(\Delta_b \times \partial_\infty\overline{\Lambda}), \text{ face at infinity;} \\ df &:= \overline{\beta^{-1}(\Delta_b \times \overline{\Lambda} \setminus (\Delta_b \times \partial_\infty\overline{\Lambda}))}, \text{ diagonal face.} \end{aligned}$$

We will now fix the notation for local coordinates on $X^2_{b,\Lambda}$. We will use the identification $[\Lambda_{cc}; \{0\}] \equiv \overline{\Lambda} \setminus \{0\}$, with $\mathit{ff}[\Lambda_{cc}; \{0\}] \equiv \partial_\infty\overline{\Lambda}$. We define $r := |\mu|$, $\omega := \mu/|\mu|$, where $\mu = 1/\lambda \in \Lambda_{cc}$. Let $\mathcal{U} = \mathbb{R}^{n,k}_y \times \mathbb{R}^n_z$ be a coordinate patch on $X_b^2$ with $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}$. Then,

$$X^2_{b,\Lambda} \cong \mathbb{R}^{n,k}_y \times T^n_\Lambda, \text{ with } T^n_\Lambda := [\mathbb{R}^n_z \times \overline{\Lambda}; Y_1; Y_2], \text{ where} \tag{38}$$

$Y_1 := \{0\}_z \times \{0\}_r \times (\mathbb{S}^1 \cap \Lambda_{cc})_\omega$ and $Y_2 := \{0\}_z \times [0,\infty)_r \times (\mathbb{S}^1 \cap \Lambda_{cc})_\omega$. The coordinates $(y, \omega)$, together with each of the first three coordinates shown in Figure 3, give various coordinate systems on $[X_b^2 \times \overline{\Lambda}; \Delta_b \times \partial_\infty\overline{\Lambda}]$ and the coordinates $(y, \omega)$, together with the last set of coordinates shown in Figure 3, give coordinates on $X^2_{b,\Lambda} = [X_b^2 \times \overline{\Lambda}; \Delta_b \times \partial_\infty\overline{\Lambda}; \Delta_b \times \overline{\Lambda}]$ near $df$. On $T^n_\Lambda$, we define $bi := \{\omega_0 = 0\}$, $\mathit{fi} := \{t = 0\}$, and $df := \{u = 0\}$.

Consider now densities. We claim that $|dz\, \frac{d\lambda \wedge d\overline{\lambda}}{\lambda^2}|^{\frac{1}{2}} \in C^\infty(\mathbb{R}^n_z \times \overline{\Lambda}, \Omega_b^{\frac{1}{2}})$, when lifted to $T^n_\Lambda$, is of the form $|dz\, \frac{d\lambda \wedge d\overline{\lambda}}{\lambda^2}|^{\frac{1}{2}} = \rho_{df}^{\frac{n}{2}} \rho_{\mathit{fi}}^{\frac{n}{2}}\, \mu$, where $\rho_{df}$ and $\rho_{\mathit{fi}}$ are boundary defining functions for the hypersurfaces $df$ and $\mathit{fi}$ respectively, and where $\mu \in C^\infty(T^n_\Lambda, \Omega_b^{\frac{1}{2}})$. Indeed, observe that in the coordinates $r = s\gamma_0$

and $z = s\gamma'$ near $bi$, we have

$$|dz\,\frac{dr}{r}|^{\frac{1}{2}} = s^{\frac{n}{2}}|\frac{ds}{s}\frac{d\gamma_0}{\gamma_0}|^{\frac{1}{2}}\,\mu_1, \tag{39}$$

where $\mu_1 \in C^\infty(\mathbb{S}^{n-1}_{\gamma'}, \Omega^{\frac{1}{2}})$. In the coordinates $r = t$ and $z = t\,u\,\gamma$ near $df$, observe that $|dz\,\frac{dr}{r}|^{\frac{1}{2}} = t^{\frac{n}{2}}u^{\frac{n}{2}}\,|\frac{dt}{t}\frac{du}{u}|^{\frac{1}{2}}\,\mu_2$, where $\mu_2 \in C^\infty(\mathbb{S}^{n-1}_{\gamma}, \Omega^{\frac{1}{2}})$. This expression, together with (39) implies that $|dz\,\frac{d\lambda\wedge d\bar\lambda}{\lambda^2}|^{\frac{1}{2}} = \rho_{df}^{\frac{n}{2}}\rho_{fi}^{\frac{n}{2}}\,\mu$, where $\mu \in C^\infty(T^n_\Lambda, \Omega_b^{\frac{1}{2}})$. More generally, we have the following.

**Lemma 34.** *If $\nu \in C^\infty(X_b^2, \Omega_b^{\frac{1}{2}})$, then $\beta^*(\nu\,|\frac{d\lambda\wedge d\bar\lambda}{\lambda^2}|^{\frac{1}{2}}) = \rho_{df}^{\frac{n}{2}}\rho_{fi}^{\frac{n}{2}}\,\mu$, where $\mu \in C^\infty(X^2_{b,\Lambda}, \Omega_b^{\frac{1}{2}})$.*

4.2. **The Resolvent on $[0,\infty)$.** In this subsection, we will investigate the polyhomogeneity of the resolvent of the $b$-Laplacian for $X = [0,\infty)$. Let $\Lambda \subseteq \mathbb{C}$ be a sector such that $\operatorname{Im}\lambda > 0$ for all $\lambda \in \Lambda$, $\lambda \neq 0$; in particular, $\lambda^2 \notin \mathbb{R}^+$ for all $\lambda \in \Lambda$. Let $\Delta = -(x\partial_x)^2$ be the $b$-Laplacian. Then observe that the kernel of $\Delta$ can be written as

$$\Delta = \frac{1}{2\pi}\int_{\mathbb{R}} v^{i\xi}\,\xi^2 d\xi, \text{ where } v = x/x' \in \mathbb{R}^+,$$

where $x$ and $x'$ are the usual coordinates on the left and right factors of $X \times X$. Hence, the resolvent kernel at $\lambda^2$, $R(\lambda^2)$, is given explicitly by the inverse Mellin transform

$$R(\lambda^2) = \frac{1}{2\pi}\int_{\mathbb{R}} v^{i\xi}\,(\xi^2 - \lambda^2)^{-1} d\xi.$$

We will show that $R(\lambda^2)$, when lifted to $X^2_{b,\Lambda}$, is a polyhomogeneous function. We proceed as follows.

Since $\xi^2 - \lambda^2 = (\xi - \lambda)(\xi + \lambda)$ and $\operatorname{Im}\lambda > 0$, by moving the contour $\mathbb{R} = \{\xi \in \mathbb{C}\,|\operatorname{Im}\xi = 0\}$ up to $\{\operatorname{Im}\xi = +\infty\}$, Cauchy's theorem implies that if $v > 1$

$$R(\lambda^2) = \frac{1}{2\pi}\int_{\operatorname{Im}\xi=0} v^{i\xi}\,(\xi-\lambda)^{-1}(\xi+\lambda)^{-1}\,d\xi = \frac{1}{2\lambda i}v^{i\lambda}.$$

Similarly, moving the contour $\mathbb{R}$ to $\{\operatorname{Im}\xi = -\infty\}$, Cauchy's theorem implies that if $v < 1$,

$$R(\lambda^2) = \frac{1}{2\pi}\int_{\operatorname{Im}\xi=0} v^{i\xi}\,(\xi-\lambda)^{-1}(\xi+\lambda)^{-1}\,d\xi = \frac{1}{2\lambda i}v^{-i\lambda}.$$

Thus, we conclude that

$$R(\lambda^2) = \frac{1}{2\lambda i}\begin{cases} v^{i\lambda}, & v > 1; \\ v^{-i\lambda}, & v < 1, \end{cases} \tag{40}$$

$$= \frac{1}{2\lambda i}\begin{cases} e^{i\lambda z}, & z > 0; \\ e^{-i\lambda z}, & z < 0, \end{cases} \quad \text{where } z = \log v. \tag{41}$$

Now we'll see what type of function $R(\lambda^2)$ is on $X^2_{b,\Lambda}$. First, note that $v = x/x'$ and $1/v = x'/x$ are boundary defining functions for $lb$ and $rb$ respectively (these are just projective polar coordinates, see (9)). Thus, the first expression (40) implies that for any $a > 0$,

$$R(\lambda^2) \in S^{-\infty}(\Lambda;\, \Psi_b^{-\infty,a}(X)\,) \text{ for } \operatorname{Im}\lambda \geq 2a \tag{42}$$

*away* from $\Delta_b$ (the set where $v = 1$ or $z = 0$).

Near $\Delta_b$, we can use the coordinate $z \in \mathbb{R}$ which identifies $\Delta_b$ with $\{z = 0\}$. Let $\mu = 1/\lambda$ and write $\mu$ in polar coordinates, $\mu = r\omega$, where $r = |\mu|$ and $\omega = \mu/|\mu|$.

Consider the second set of coordinates $(s, \gamma_0, \gamma)$ shown in Figure 3, where $r = s\gamma_0$, $z = s\gamma'$, and where $s = |z|$, $\gamma_0 = r/|z|$, and $\gamma' = z/|z| = 1$ if $z > 0$ and $\gamma' = z/|z| = -1$ if $z < 0$. Thus, since $\lambda z = \gamma'/(\gamma_0\omega)$, by (41),

$$R(\lambda^2) = \frac{s\gamma_0\omega}{2i}\begin{cases} e^{i/(\gamma_0\omega)}, & z > 0; \\ e^{i/(\gamma_0\omega)}, & z < 0. \end{cases} \tag{43}$$

In the fourth set of coordinates $(t, u, \gamma)$ in Figure 3, we have $r = t$, $z = tu\gamma$, where $t = r$, $u = |z|/r$, and $\gamma = z/|z| = 1$ if $z > 0$ and $\gamma = z/|z| = -1$ if $z < 0$. Thus, since $\lambda z = u\gamma/\omega$, by (41),

$$R(\lambda^2) = \frac{t\omega}{2i}\begin{cases} e^{iu/\omega}, & z > 0; \\ e^{iu/\omega}, & z < 0. \end{cases} \tag{44}$$

From (42), (43), and (44), it follows that $R(\lambda^2) \in \mathcal{A}^{\mathcal{E}}(X^2_{b,\Lambda})$, where $\mathcal{E}(f\!f) = \mathbb{N}_0$, $\mathcal{E}(bi) = \emptyset$, $\mathcal{E}(df) = \mathbb{N}_0$, and $\mathcal{E}(f\!i) = \mathbb{N}$.

4.3. **The Structure Theorems.** For the rest of this paper, we will assume that $\Lambda \neq \mathbb{C}$. Thus, we can *fix* a branch of the log function so that $\log : \Lambda \setminus \{0\} \to \mathbb{C}$ is holomorphic. Thus, for any $a \in \mathbb{C}$, $\lambda^a := e^{a \log \lambda}$, where $\lambda \in \Lambda \setminus \{0\}$, is well defined. If $d \in \mathbb{R}^+$, we define $\Lambda^{1/d} := \{\lambda^{1/d} \mid \lambda \in \Lambda \setminus \{0\}\} \cup \{0\}$. Then $\Lambda^{1/d}$ is also a sector and if $\zeta \in \Lambda^{1/d}$, then $\zeta^d \in \Lambda$. Since $\Lambda^{1/d}$ is a sector, the manifold $X^2_{b,\Lambda^{1/d}}$ is defined.

Recall that $\Psi^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac{1}{2}}) \subseteq C^\infty(\Lambda; \Psi_b^m(X,\Omega_b^{\frac{1}{2}}))$. If $m \in \mathbb{N}_0$, we define

$$\mathrm{Diff}^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac{1}{2}}) := \Psi^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac{1}{2}}) \cap C^\infty(\Lambda; \mathrm{Diff}_b^m(X,\Omega_b^{\frac{1}{2}})).$$

If $m \notin \mathbb{N}_0$, then we define $\mathrm{Diff}^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac{1}{2}}) := 0$.

Let $m$, $p \in \mathbb{R}$. Define the index set $\mathcal{E}_{m,p,d}$ on $X^2_{b,\Lambda^{1/d}}$ by

$$\begin{gathered}
\mathcal{E}_{m,p,d}(f\!f) = \mathcal{E}_{m,p,d}(\text{edges of } \Lambda) := \mathbb{N}_0;\\
\mathcal{E}_{m,p,d}(lb) = \mathcal{E}_{m,p,d}(rb) := \emptyset;\ \mathcal{E}_{m,p,d}(bi) = -p + d\mathbb{N}_0;\\
\mathcal{E}_{m,p,d}(df) := \tfrac{n}{2} + \{(k-m-n+dl, 0) \mid k,l \in \mathbb{N}_0\} \overline{\cup} \mathbb{N}_0;\\
\mathcal{E}_{m,p,d}(f\!i) := \tfrac{n}{2} + \{(k-m-n, 0) \mid k \in \mathbb{N}_0\} \overline{\cup} (-p + \mathbb{N}_0 + d\mathbb{N}_0);
\end{gathered}$$

and define the index set $\mathcal{F}_{m,p,d}$ on $\overline{\Lambda^{1/d}}$, associated to $\partial_\infty \overline{\Lambda^{1/d}}$, by

$$\mathcal{F}_{m,p,d} := \{(k-m-n, 0) \mid k \in \mathbb{N}_0\} \overline{\cup} (-p + d\mathbb{N}_0).$$

The first structure theorem is the following.

**Theorem 35.** *Let $m$, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. Then away from $\lambda = 0$, there exists an exact sequence of continuous linear maps*

$$\begin{gathered}
0 \hookrightarrow \mathrm{Diff}^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac{1}{2}}) \hookrightarrow \Psi^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac{1}{2}}) \to \mathcal{A}^{\mathcal{E}_{m,p,d}}_{phg}(X^2_{b,\Lambda^{1/d}}, \Omega_b^{\frac{1}{2}});\\
A(\lambda) \mapsto A(\lambda)\, |\tfrac{d\lambda \wedge d\bar{\lambda}}{\lambda^2}|^{\frac{1}{2}};
\end{gathered}$$

*and if $m < -n$, there exists a continuous linear map*

$$\Psi^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac{1}{2}}) \to \mathcal{A}^{\mathcal{F}_{m,p,d}}_{phg}(\overline{\Lambda^{1/d}}; C^\infty(X,\Omega_b));\ A(\lambda) \mapsto A(\lambda)|_{\Delta_b}. \tag{45}$$

We have a similar theorem for the strongly tempered spaces. If $m \in \mathbb{N}_0$, we define $\mathrm{Diff}^{m,d}_{b,\Lambda,os}(X,\Omega_b^{\frac{1}{2}}) := \Psi^{m,d}_{b,\Lambda,os}(X,\Omega_b^{\frac{1}{2}}) \cap C^\infty(\Lambda; \mathrm{Diff}_b^m(X,\Omega_b^{\frac{1}{2}}))$. If $m \notin \mathbb{N}_0$, then we define $\mathrm{Diff}^{m,d}_{b,\Lambda,os}(X,\Omega_b^{\frac{1}{2}}) := 0$.

For each $m \in \mathbb{R}$, we define the index set $\mathcal{E}_{m,d}$ on $X^2_{b,\Lambda^{1/d}}$ by

$$\begin{gathered}
\mathcal{E}_{m,d}(f\!f) = \mathcal{E}_{m,d}(\text{edges of } \Lambda) := \mathbb{N}_0;\\
\mathcal{E}_{m,d}(lb) = \mathcal{E}_{m,d}(rb) = \mathcal{E}_{m,d}(bi) = \emptyset;\\
\mathcal{E}_{m,d}(df) := \tfrac{n}{2} + \{(k-m-n+dl, 0) \mid k,l \in \mathbb{N}_0\} \overline{\cup} \mathbb{N}_0;\\
\mathcal{E}_{m,d}(f\!i) := \tfrac{n}{2} + \{(k-m-n, 0) \mid k \in \mathbb{N}_0\};
\end{gathered}$$

and we define the index set $\mathcal{F}_{m,d}$ on $\overline{\Lambda^{1/d}}$, associated to $\partial_\infty \overline{\Lambda^{1/d}}$, by

$$\mathcal{F}_{m,d} := \{(k-m-n, 0) \mid k \in \mathbb{N}_0\}.$$

**Theorem 36.** *Let $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$. Then away from $\lambda = 0$, there exists an exact sequence of continuous linear maps*

$$\begin{gathered}
0 \hookrightarrow \mathrm{Diff}^{m,d}_{b,\Lambda,os}(X,\Omega_b^{\frac{1}{2}}) \hookrightarrow \Psi^{m,d}_{b,\Lambda,os}(X,\Omega_b^{\frac{1}{2}}) \to \mathcal{A}^{\mathcal{E}_{m,d}}_{phg}(X^2_{b,\Lambda^{1/d}}, \Omega_b^{\frac{1}{2}});\\
A(\lambda) \mapsto A(\lambda)\, |\tfrac{d\lambda \wedge d\bar{\lambda}}{\lambda^2}|^{\frac{1}{2}};
\end{gathered}$$

*and if $m < -n$, there exists a continuous linear map*

$$\Psi^{m,d}_{b,\Lambda,os}(X,\Omega_b^{\frac{1}{2}}) \to \mathcal{A}^{\mathcal{F}_{m,d}}_{phg}(\overline{\Lambda^{1/d}}; C^\infty(X,\Omega_b));\ A(\lambda) \mapsto A(\lambda)|_{\Delta_b}. \tag{46}$$

The following two subsections provide the very long proofs of the these structure theorems. The next section applies these theorems to analyze the resolvents of $b$-pseudodifferential operators.

4.4. **Preliminaries to the Proof of the Structure Theorems.** We will use the following lemma various times.

**Lemma 37.** *For each $N \in \mathbb{N}_0$, define $\mathcal{R}_N(\mathbb{S}^{n-1}_\omega \times \mathbb{R}^n_\xi)$ as the space of functions $a \in C^\infty(\mathbb{S}^{n-1} \times (\mathbb{R}^n \setminus \{0\}))$ satisfying the following estimates: for each $P \in \mathrm{Diff}^*(\mathbb{S}^{n-1})$ and $\beta$,*

$$|\, P\partial_\xi^\beta a(\omega,\xi)\,| \,\dot{\le}\, \begin{cases} |\xi|^{-n+N+1-|\beta|}, & |\xi| \le 1; \\ |\xi|^{-n-2-|\beta|}, & |\xi| \ge 1. \end{cases} \tag{47}$$

*Then, given $a \in \mathcal{R}_N(\mathbb{S}^{n-1} \times \mathbb{R}^n)$, the function*

$$A(r,\omega) := \int e^{i\frac{\omega}{r}\cdot\xi}\, a(\omega,\xi)\, d\xi$$

*is an element of $r^N S^0([0,\infty)_r \times \mathbb{S}^{n-1}_\omega)$.*

*Proof.* Let $N \in \mathbb{N}$ and let $a \in \mathcal{R}_N(\mathbb{S}^{n-1} \times \mathbb{R}^n)$. Observe that the inequalities (47) imply that the integral defining $A(r,\omega)$ converges. We need to show that

$$r^{-N} A(r,\omega) = r^{-N} \int e^{i\frac{\omega}{r}\cdot\xi}\, a(\omega,\xi)\, d\xi \in S^0([0,\infty)_r \times \mathbb{S}^{n-1}_\omega).$$

To see this, define $L_\omega := -\omega \cdot D_\xi = -\sum_{i=1}^n \omega_i D_{\xi_i}$ for each $\omega \in \mathbb{S}^{n-1}$. Then observe that $L_\omega e^{i\frac{\omega}{r}\cdot\xi} = -r^{-1} e^{i\frac{\omega}{r}\cdot\xi}$. Hence, using the estimates (47) to justify integrating by parts, we have

$$\begin{aligned} r^{-N} A(r,\omega) = (-1)^N \int (L_\omega^N e^{i\frac{\omega}{r}\cdot\xi})\, a(\omega,\xi)\, d\xi \quad &= \int e^{i\frac{\omega}{r}\cdot\xi}\, L_\omega^N a(\omega,\xi)\, d\xi \\ &= \int e^{i\frac{\omega}{r}\cdot\xi}\, a_0(\omega,\xi)\, d\xi, \end{aligned}$$

where $a_0(\omega,\xi) := L_\omega^N a(\omega,\xi) \in \mathcal{R}_0(\mathbb{S}^{n-1} \times \mathbb{R}^n)$. Thus, we are reduced to the case when $N = 0$. Now observe that $A$ is bounded since $a \in \mathcal{R}_0(\mathbb{S}^{n-1} \times \mathbb{R}^n)$ is integrable. We claim that

$$r\partial_r A(r,\omega) = \int e^{i\frac{\omega}{r}\cdot\xi}\, a_1(\omega,\xi)\, d\xi, \text{ where } a_1 \in \mathcal{R}_0(\mathbb{S}^{n-1} \times \mathbb{R}^n) \tag{48}$$

and that given any smooth vector field $v$ on $\mathbb{S}^{n-1}$,

$$vA(r,\omega) = \int e^{i\frac{\omega}{r}\cdot\xi}\, a_v(\omega,\xi)\, d\xi, \text{ where } a_v \in \mathcal{R}_0(\mathbb{S}^{n-1} \times \mathbb{R}^n). \tag{49}$$

Note that once we prove these two statements, given any $k \in \mathbb{N}$ and $P \in \mathrm{Diff}^*(\mathbb{S}^{n-1})$, we have $(r\partial_r)^k PA(r,\omega) = \int e^{i\frac{\omega}{r}\cdot\xi}\, a_{k,P}(\omega,\xi)\, d\xi$, where $a_{k,P} \in \mathcal{R}_0(\mathbb{S}^{n-1} \times \mathbb{R}^n)$. Since $a_{k,P}$ is integrable, $(r\partial_r)^k PA(r,\omega)$ is bounded. Thus, $A(r,\omega) \in S^0([0,\infty)_r \times \mathbb{S}^{n-1}_\omega)$. So, it remains to prove (48) and (49). We first prove (48). Thus, using the estimates (47) to justify integrating by parts, we find that

$$\begin{aligned} r\partial_r A(r,\omega) &= -r^{-1} \int e^{i\frac{\omega}{r}\cdot\xi}\, (i\omega\cdot\xi)\, a(\omega,\xi)\, d\xi \\ &= \int (L_\omega e^{i\frac{\omega}{r}\cdot\xi})\, (i\omega\cdot\xi)\, a(\omega,\xi)\, d\xi \\ &= -\int e^{i\frac{\omega}{r}\cdot\xi}\, L_\omega[\,(i\omega\cdot\xi)\, a(\omega,\xi)\,]\, d\xi. \end{aligned} \tag{50}$$

Note that $L_\omega[\,(i\omega\cdot\xi)\, a(\omega,\xi)\,] \in \mathcal{R}_0(\mathbb{S}^{n-1} \times \mathbb{R}^n)$. Thus, (48) holds. Now let $v \in \mathrm{Diff}^1(\mathbb{S}^{n-1})$. Then,

$$vA(r,\omega) = \int e^{i\frac{\omega}{r}\cdot\xi}\, va(\omega,\xi)\, d\xi + r^{-1} \int e^{i\frac{\omega}{r}\cdot\xi}\, p_v(\omega,\xi)\, a(\omega,\xi)\, d\xi, \tag{51}$$

where, for each $\omega$, $p_v(\omega,\xi)$ is a homogeneous polynomial in $\xi$ of degree 1. Observe that $va \in \mathcal{R}_0(\mathbb{S}^{n-1} \times \mathbb{R}^n)$. Hence, by using a similar integration by parts argument on the second term of (51) as we used in (50) above, it follows that (49) holds. ☐

Throughout this section, fix $d \in \mathbb{R}^+$. We use the identification $[\Lambda_{cc}^{1/d}; \{0\}] \equiv \overline{\Lambda^{1/d}} \setminus \{0\}$, where $\mathit{ff}[\Lambda_{cc}^{1/d}; \{0\}] \equiv \partial_\infty \overline{\Lambda^{1/d}}$. We define

$$\nu := \mu^{1/d} \in \Lambda_{cc}^{1/d}, \text{ where } \mu = 1/\lambda \in \Lambda_{cc}. \tag{52}$$

Then we set $r := |\nu|$ and $\omega := \nu/|\nu|$. Coordinates on $T^n_{\Lambda^{1/d}}$ (cf. the definition (38)) are shown in Figure 3. Recall that $bi = \{\omega_0 = 0\}$, $\mathit{fi} = \{t = 0\}$, and $\mathit{df} = \{u = 0\}$.

We first work on the expansion at $bi$.

**Lemma 38.** *Let $m, p \in \mathbb{R}$ with $p/d \in \mathbb{Z}$ and let $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ near $0$ and $\chi(\xi) \equiv 1$ outside a neighborhood of $0$. Let*

$$A(\lambda) = \int e^{iz\cdot\xi}\, \chi(\xi)\, a(\lambda,\xi)\, d\xi, \text{ where } a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n).$$

*Then away from $\lambda = 0$, we have $A(\lambda) \in \mathcal{A}^{\mathcal{E}_{bi}}(T^n_{\Lambda^{1/d}})$, where $\mathcal{E}_{bi} := -p + d\mathbb{N}_0$ is an index set on $T^n_{\Lambda^{1/d}}$ associated to $bi$.*

*Proof.* We will use the second coordinates $(s, \gamma_0, \gamma')$ given in Figure 3. Let $a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$. Then $a(\lambda,\xi) \in C^\infty(\Lambda \times (\mathbb{R}^n \setminus \{0\}))$ and is such that $a(\delta^d\lambda, \delta\xi) = \delta^m a(\lambda,\xi)$ for all $\delta > 0$ and if we define $\tilde{a}(\mu,\xi) := \mu^{p/d} a(1/\mu, \xi)$ for all $(\mu,\xi) \in \Lambda_{cc} \times \mathbb{R}^n$, then $\tilde{a}(\mu,\xi) \in C^\infty(\Lambda_{cc} \times (\mathbb{R}^n \setminus \{0\}))$. Observe that since $\nu = r\omega = s\gamma_0\omega$, cf. the coordinates (52), we have

$$\begin{aligned} A(\lambda) &= \int e^{is\gamma'\cdot\xi}\, \chi(\xi)\, a(\,(s\gamma_0\omega)^{-d}, \xi)\, d\xi \\ &= s^{-p}\gamma_0^{-p}\omega^{-p} \int e^{is\gamma'\cdot\xi}\, \chi(\xi)\, \tilde{a}(s^d\gamma_0^d\omega^d, \xi)\, d\xi. \end{aligned}$$

Thus, it suffices to prove that $\int e^{is\gamma'\cdot\xi}\, \chi(\xi)\, \tilde{a}(s^d\gamma_0^d\omega^d,\xi)\, d\xi \in \mathcal{A}^{\mathcal{E}'_{bi}}(T^n_{\Lambda^{1/d}})$, where $\mathcal{E}'_{bi} := d\mathbb{N}_0$. To see this, set $L_{\gamma'} := -\gamma' \cdot D_\xi$ and choose $N \in \mathbb{N}$ with $m - N \le -n - 2$. Then observe that

$$\begin{aligned} &\int e^{is\gamma'\cdot\xi}\, \chi(\xi)\, \tilde{a}(s^d\gamma_0^d\omega^d, \xi)\, d\xi \\ &\qquad = s^{-N}(-1)^N \int (\, L^N_{\gamma'} e^{is\gamma'\cdot\xi}\,)\, \chi(\xi)\, \tilde{a}(s^d\gamma_0^d\omega^d,\xi)\, d\xi \\ &\qquad = s^{-N} \int e^{is\gamma'\cdot\xi}\, \chi(\xi)\, (L^N_{\gamma'}\tilde{a})(s^d\gamma_0^d\omega^d,\xi)\, d\xi \\ &\qquad\qquad + s^{-N} A_N(s, s^d\gamma_0^d\omega^d, \gamma'), \end{aligned}$$

where $A_N(s,\mu,\gamma') = \int e^{is\gamma'\cdot\xi}[\sum_{l=1}^N \binom{N}{l} (L^l_{\gamma'}\chi(\xi))(L^{N-l}_{\gamma'}\tilde{a})(\mu,\xi)]\, d\xi$. Now since for $l \ge 1$, $L^l_{\gamma'}\chi(\xi) \equiv 0$ for $\xi$ near $0$ and for $\xi$ outside a neighborhood of $0$, it follows that $A_N(s,\mu,\gamma')$ is $C^\infty$ in all variables $(s,\mu,\gamma')$. In particular, $A_N(s, s^d\gamma_0^d\omega^d, \gamma')$ can be expanded in powers of $\gamma_0^d$. Hence,

$$s^{-N} A_N(s, s^d\gamma_0^d\omega^d, \gamma') \in \mathcal{A}^{\mathcal{E}'_{bi}}(T^n_{\Lambda^{1/d}}).$$

Thus, it remains to show that

$$\int e^{is\gamma'\cdot\xi}\, \chi(\xi)\, (L^N_{\gamma'}\tilde{a})(s^d\gamma_0^d\omega^d,\xi)\, d\xi \in \mathcal{A}^{\mathcal{E}'_{bi}}(T^n_{\Lambda^{1/d}}).$$

Since $m - N \le -n - 2$, $(L^N_{\gamma'}a)(\lambda,\xi) \in C^\infty(\mathbb{S}^{n-1}_{\gamma'}; C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m')}(\mathbb{R}^n))$, where $m' \le -n-2$. Thus, it suffices to prove the following statement: given $a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$, where $m \le -n-2$,

$$A := \int e^{is\gamma'\cdot\xi}\, \chi(\xi)\, \tilde{a}(s^d\gamma_0^d\omega^d,\xi)\, d\xi \in \mathcal{A}^{\mathcal{E}'_{bi}}(T^n_{\Lambda^{1/d}}).$$

To see this, set $v := \gamma_0^d$ and if $\omega = e^{i\theta}$, set $D_\omega := \cos(d\theta)D_{\mu_1} + \sin(d\theta)D_{\mu_2}$. Then, for any $M \in \mathbb{N}$,

$$D_v^M A = s^{dM} \int e^{is\gamma'\cdot\xi}\, \chi(\xi)\, (D^M_\omega \tilde{a})(s^d\gamma_0^d\omega^d,\xi)\, d\xi.$$

By Proposition 17, it follows that $\tilde{a}(\mu,\xi)$ satisfies the estimates: for each $\alpha$ and $\beta$ and $M$,

$$|\,\partial_\mu^\alpha \partial_\xi^\beta D^M_\omega \tilde{a}(\mu,\xi)\,| \;\dot{\le}\; (1 + |\mu|\,|\xi|^d)^{p/d - (|\alpha|+M)}\, |\xi|^{d(|\alpha|+M)+m-p-|\beta|}. \tag{53}$$

Observe that this estimate implies that if $M$ is sufficiently large (for example, if $dM+m-p>-n$), then $(D^M_\omega\tilde{a})(w\omega^d,\xi)$ is integrable near $\xi=0$ for all $w\in[0,\infty)$, and moreover, it remains integrable near $\xi=0$ even after we differentiate it with respect to $w$ or $\omega$. Thus, if we define

$$B_M(s,w,\gamma',\omega):=\int e^{is\gamma'\cdot\xi}\,(\chi(\xi)-1)\,(D^M_\omega\tilde{a})(w\omega^d,\xi)\,d\xi,\ \ w\in[0,\infty),$$

then $B_M$ is smooth in all variables. Also, since $m\le -n-2$, observe that for any $M$, the estimate (53) implies that for any fixed $w>0$, $(D^M_\omega\tilde{a})(w\omega^d,\xi)$ is integrable near $\xi=\infty$. Thus, for $M$ sufficiently large and $s^d\gamma_0^d>0$,

$$\int e^{is\gamma'\cdot\xi}\ (D^M_\omega\tilde{a})(s^d\gamma_0^d\omega^d,\xi)\,d\xi$$

converges. Thus, for $M$ sufficiently large, we can write

$$D^M_vA \;=\; s^{dM}\int e^{is\gamma'\cdot\xi}\,(D^M_\omega\tilde{a})(s^d\gamma_0^d\omega^d,\xi)\,d\xi + s^{dM}B_M(s,s^d\gamma_0^d,\gamma',\omega). \tag{54}$$

Since $(D^M_\omega\tilde{a})(s^d\gamma_0^d\omega^d,(s\gamma_0)^{-1}\xi)=(s\gamma_0)^{-dM+p-m}(D^M_\omega\tilde{a})(\omega^d,\xi)$, the change of variables $\xi\mapsto(s\gamma_0)^{-1}\xi$ in the integral (54) yields

$$D^M_vA=s^{dM}(s\gamma_0)^{-dM+p-m-n}A_M+s^{dM}B_M(s,s^d\gamma_0^d,\gamma',\omega),$$

where $A_M:=\int e^{i\frac{\gamma'}{\gamma_0}\cdot\xi}\,(D^M_\omega\tilde{a})(\omega^d,\xi)\,d\xi$. Since $v=\gamma_0^d$, we thus have

$$v^MD^M_vA=(s\gamma_0)^{p-m-n}A_M+s^{dM}\gamma_0^{dM}B_M(s,s^d\gamma_0^d,\gamma',\omega).$$

Let $N>>0$. Since $m\le -n-2$, choosing $M$ such that $dM+m-p\ge -n+N+1$, the estimate (53) implies that given $\beta$,

$$\begin{aligned}|\,\partial_\xi^\beta D^M_\omega\tilde{a}(\omega^d,\xi)\,| \;&\dot{\le}\; (1+|\xi|^d)^{p/d-M}\,|\xi|^{dM+m-p-|\beta|}\\ &\dot{\le}\; \begin{cases}|\xi|^{-n+N+1-|\beta|}, & |\xi|\le 1;\\ |\xi|^{-n-2-|\beta|}, & |\xi|\ge 1.\end{cases}\end{aligned}$$

Thus, Lemma 37 implies that

$$A_M\in\gamma_0^NS^0([0,\infty)_{\gamma_0}\times\mathbb{S}^{n-1}_{\gamma'}\times(\mathbb{S}^1_\omega\cap\Lambda^{1/d}_{cc})\,).$$

Now observe that $v^M\partial_v^M=(v\partial_v-M+1)(v\partial_v-M+2)\cdots(v\partial_v-1)(v\partial_v)$ and that $\gamma_0\partial_{\gamma_0}=d\,v\partial_v$. Thus,

$$\begin{aligned}&(\gamma_0\partial_{\gamma_0}-d(M+1))\cdots(\gamma_0\partial_{\gamma_0}-d)(\gamma_0\partial_{\gamma_0})A\in\\ &\quad(s\gamma_0)^{p-m-n}\gamma_0^NS^0([0,\infty)_{\gamma_0}\times\mathbb{S}^{n-1}_{\gamma'}\times(\mathbb{S}^1_\omega\cap\Lambda^{1/d}_{cc})\,)\\ &\quad\quad+s^{dM}\gamma_0^{dM}C^\infty([0,\infty)_s\times[0,\infty)_{s^d\gamma_0^d}\times\mathbb{S}^{n-1}_{\gamma'}\times(\mathbb{S}^1_\omega\cap\Lambda^{1/d}_{cc})\,).\end{aligned}$$

Thus, by Theorem 1, $A\in\mathcal{A}^{\mathcal{E}'_{bi}}(T^n_{\Lambda^{1/d}})$ since $N$ and $M$ can be arbitrarily large. □

**Lemma 39.** *Let $m\in\mathbb{R}$ and let $\chi(\lambda,\xi)\in C^\infty(\Lambda\times\mathbb{R}^n)$ with $\chi(\lambda,\xi)\equiv 0$ on a neighborhood of $(\lambda,\xi)=0$ and $\chi(\lambda,\xi)\equiv 1$ outside a neighborhood of $0$. Let*

$$A(\lambda)=\int e^{iz\cdot\xi}\,\chi(\lambda,\xi)\,a(\lambda,\xi)\,d\xi,\ \ \textit{where } a(\lambda,\xi)\in C^{\infty,d}_{\hom(m)}(\Lambda\times\mathbb{R}^n).$$

*Then away from $\lambda=0$, we have $A(\lambda)\in\mathcal{A}^{\mathcal{E}_{bi}}(T^n_{\Lambda^{1/d}})$, where $\mathcal{E}_{bi}:=\emptyset$ is an index set on $T^n_{\Lambda^{1/d}}$ associated to bi.*

*Proof.* We must show that

$$A=\int e^{is\gamma'\cdot\xi}\,\chi(s^{-d}\gamma_0^{-d}\omega^{-d},\xi)\,a(s^{-d}\gamma_0^{-d}\omega^{-d},\xi)\,d\xi\equiv 0$$

at $\gamma_0=0$. Since $\chi(\lambda,\xi)\equiv 1$ outside a neighborhood of $(\lambda,\xi)=0$, for $s$ bounded and for $\gamma_0$ sufficiently small, $\chi(s^{-d}\gamma_0^{-d}\omega^{-d},\xi)\equiv 1$. Hence, it suffices to show that $A:=\int e^{is\gamma'\cdot\xi}\,a(s^{-d}\gamma_0^{-d}\omega^{-d},\xi)\,d\xi\equiv 0$ at

$\gamma_0 = 0$. Making the change of variables $\xi \mapsto (s\gamma_0)^{-1}\xi$ and using the homogeneous properties of $a$, we find that

$$A = (s\gamma_0)^{m-n} \int e^{i\frac{\gamma'}{\gamma_0}\cdot\xi}\, a(\omega^{-d},\xi)\, d\xi.$$

By integration by parts, $\int e^{iz\cdot\xi}\, a(\omega^{-d},\xi)\, d\xi \equiv 0$ at $|z| = \infty$. Thus, since $|\gamma'| \equiv 1$, $A \equiv 0$ at $\gamma_0 = 0$. □

We now work on the expansion at $df$.

**Lemma 40.** *Let $m, p \in \mathbb{R}$ with $p/d \in \mathbb{Z}$ and let $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ near $0$ and $\chi(\xi) \equiv 1$ outside a neighborhood of $0$. Let*

$$A(\lambda) = \int e^{iz\cdot\xi}\, \chi(\xi)\, a(\lambda,\xi)\, d\xi, \ \text{ where } a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n).$$

*Then away from $\lambda = 0$, we have $A(\lambda) \in \mathcal{A}^{\mathcal{E}_{df}}(T^n_{\Lambda^{1/d}})$, where $\mathcal{E}_{df}$ is an index set on $T^n_{\Lambda^{1/d}}$ associated to $df$ given by $\{(-m-n+dl, 0) \,|\, l \in \mathbb{N}_0\} \overline{\cup} \mathbb{N}_0$.*

*Proof.* We will use the coordinates (52) and the last set of coordinates $(t, u, \gamma)$ given in Figure 3. Then, $A(\lambda) = \int e^{itu\gamma\cdot\xi}\, \chi(\xi)\, a(t^{-d}\omega^{-d},\xi)\, d\xi$. Making the change of variables $\xi \mapsto t^{-1}\xi$ and using the homogeneous properties of $a$, we find that $A = t^{-m-n} \int e^{iu\gamma\cdot\xi}\, \chi(\xi/t)\, a(\omega^{-d},\xi)\, d\xi$. Thus,

$$\begin{aligned}
(t\partial_t - (-m-n))A &= -t^{-m-n} \int e^{iu\gamma\cdot\xi}\, (\xi\cdot\partial_\xi\chi)(\xi/t)\, a(\omega^{-d},\xi)\, d\xi \\
&= -\int e^{itu\gamma\cdot\xi}\, (\xi\cdot\partial_\xi\chi)(\xi)\, a(t^{-d}\omega^{-d},\xi)\, d\xi \\
&=: B(t,u,\gamma,\omega).
\end{aligned}$$

Since $\xi\cdot\partial_\xi\chi(\xi) \equiv 0$ near $\xi = 0$ and outside a neighborhood of $0$, it follows that $B$ can be expanded at $u = 0$ with index set $\mathbb{N}_0$. Since $(t\partial_t - (-m-n))A = B$, we have $t\partial_t(t^{m+n}A) = t^{m+n}B$. Integrating this equation from $1$ to $t$ yields

$$A = t^{-m-n}A(1,u,\gamma,\omega) - t^{-m-n}\int_1^t \tau^{m+n} B(\tau,u,\gamma,\omega)\frac{d\tau}{\tau}.$$

The second term in this equation can be expanded at $u = 0$ with index set $\mathbb{N}_0$, and so we are left to show that

$$A' := A(1,u,\gamma,\omega) = \int e^{iu\gamma\cdot\xi}\, \chi(\xi)\, a(\omega^{-d},\xi)\, d\xi \in \mathcal{A}^{\mathcal{E}_{df}}(T^n_{\Lambda^{1/d}}).$$

Note that for any $\alpha$, we have $(\partial^\alpha_\lambda a)(\delta^d\lambda, \delta\xi) = \delta^{m-d|\alpha|}(\partial^\alpha_\lambda a)(\lambda,\xi)$ for all $\delta > 0$. In particular, setting $\lambda = 0$ yields

$$(\partial^\alpha_\lambda a)(0,\delta\xi) = \delta^{m-d|\alpha|}(\partial^\alpha_\lambda a)(0,\xi) \text{ for all } \delta > 0.$$

Thus, expanding $[0,\infty) \ni v \mapsto a(v\omega^{-d},\xi)$ in Taylor Series at $v = 0$, we find that for each $N \in \mathbb{N}$,

$$a(v\omega^{-d},\xi) = \sum_{l=0}^{N-1} v^l a_{m-dl}(\omega,\xi) + b_N(v,\omega,\xi),$$

where for each $l$ and $\omega$, $a_{m-dl}(\omega,\xi) \in C^\infty_{\mathrm{hom}(m-dl)}(\mathbb{R}^n)$, and where

$$b_N(v,\omega,\xi) = \frac{v^N}{(N-1)!}\int_0^1 (1-s)^{N-1}\, \partial^N_w a(w\omega^{-d},\xi)|_{w=sv}\, ds.$$

Let $N >> 0$. Then, setting $v = 1$, it follows that if we define $A_{m-dl} := \int e^{iu\gamma\cdot\xi}\, \chi(\xi)\, a_{m-dl}(\omega,\xi)\, d\xi$ and $B_N := \int e^{iu\gamma\cdot\xi}\, \chi(\xi)\, b_N(1,\omega,\xi)\, d\xi$, then we have $A' = \sum_{l=0}^{N-1} A_{m-dl} + B_N$. By the estimates (1) of Proposition 17, for each $\omega$, $\chi(\xi)\, b_N(1,\omega,\xi) \in S^{m-dN}(\mathbb{R}^n)$. Thus, $B_N \in C^M([0,\infty)_u \times \mathbb{S}^{n-1}_\gamma \times (\mathbb{S}^1_\omega \cap$

$\Lambda_{cc}^{1/d})$ ) for all $M$ with $M+m-dN<-n$. Since $N$ is arbitrary, by Proposition 2, $A'\in\mathcal{A}^{\mathcal{E}_{df}}(T^n_{\Lambda^{1/d}})$ if for each $l$, $A_{m-dl}\in\mathcal{A}^{\mathcal{E}_{df}}(T^n_{\Lambda^{1/d}})$. Now,

$$\begin{aligned} A_{m-dl} &= \int e^{iu\gamma\cdot\xi}\,\chi(\xi)\,a_{m-dl}(\omega,\xi)\,d\xi \\ &= u^{-m-n+dl}\int e^{i\gamma\cdot\xi}\,\chi(\xi/u)\,a_{m-dl}(\omega,\xi)\,d\xi. \end{aligned}$$

where we made the change of variables $\xi\mapsto u^{-1}\xi$. Hence,

$$\begin{aligned} &(u\partial_u-(-m-n+dl))A_{m-dl} \\ &\qquad = -u^{-m-n+dl}\int e^{i\gamma\cdot\xi}\,(\xi\cdot\partial_\xi\chi)(\xi/u)\,a_{m-dl}(\omega,\xi)\,d\xi \\ &\qquad = \int e^{iu\gamma\cdot\xi}\,(\xi\cdot\partial_\xi\chi)(\xi)\,a_{m-dl}(\omega,\xi)\,d\xi. \end{aligned}$$

Since $(\xi\cdot\partial_\xi\chi)(\xi)\equiv 0$ near $\xi=0$ and outside a neighborhood of 0, it follows that $f(u):=\int e^{iu\gamma\cdot\xi}\,(\xi\cdot\partial_\xi\chi)(\xi)\,a_{m-dl}(\omega,\xi)\,d\xi$ is a smooth function of $u\in[0,\infty)$. Hence, $(u\partial_u-(-m-n+dl))A_{m-dl}\sim\sum_i^\infty u^iA_i(\gamma,\omega)$, where $A_i(\gamma,\omega):=(\partial_u^i f)(0)\in C^\infty(\mathbb{S}^{n-1}_\gamma\times(\mathbb{S}^1_\omega\cap\Lambda_{cc}^{1/d}))$. Since $(u\partial_u-i)u^i=0$, for any $M\in\mathbb{N}$,

$$(u\partial_u-M)\cdots(u\partial_u-1)(u\partial_u)(u\partial_u-(-m-n+dl))A_{m-dl}=O(u^{M+1}).$$

Thus by Theorem 1, $A_{m-dl}\in\mathcal{A}^{\mathcal{E}_{df}}(T^n_{\Lambda^{1/d}})$. □

**Lemma 41.** *Let $m\in\mathbb{R}$ and let $\chi(\lambda,\xi)\in C^\infty(\Lambda\times\mathbb{R}^n)$ with $\chi(\lambda,\xi)\equiv 0$ on a neighborhood of $(\lambda,\xi)=0$ and $\chi(\lambda,\xi)\equiv 1$ outside a neighborhood of* 0. *Let*

$$A(\lambda)=\int e^{iz\cdot\xi}\,\chi(\lambda,\xi)\,a(\lambda,\xi)\,d\xi,\ \text{where } a(\lambda,\xi)\in C^{\infty,d}_{\mathrm{hom}(m)}(\Lambda\times\mathbb{R}^n).$$

*Then away from $\lambda=0$, we have $A(\lambda)\in\mathcal{A}^{\mathcal{E}_{df}}(T^n_{\Lambda^{1/d}})$, where $\mathcal{E}_{df}$ is an index set on $T^n_{\Lambda^{1/d}}$ associated to df given by $\{(-m-n+dl,0)\,|\,l\in\mathbb{N}_0\}\overline{\cup}\mathbb{N}_0$.*

*Proof.* This lemma is proved similar to Lemma 40. □

We now work on the expansion at *fi*.

**Lemma 42.** *Let $m,p\in\mathbb{R}$ with $p/d\in\mathbb{Z}$ and let $\chi(\xi)\in C^\infty(\mathbb{R}^n)$ with $\chi(\xi)\equiv 0$ near* 0 *and $\chi(\xi)\equiv 1$ outside a neighborhood of* 0. *Let*

$$A(\lambda)=\int e^{iz\cdot\xi}\,\chi(\xi)\,a(\lambda,\xi)\,d\xi,\ \text{where } a(\lambda,\xi)\in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n).$$

*Then away from $\lambda=0$, we have $A(\lambda)\in\mathcal{A}^{\mathcal{E}_{fi}}(T^n_{\Lambda^{1/d}})$, where $\mathcal{E}_{fi}$ is an index set on $T^n_{\Lambda^{1/d}}$ associated to fi given by $\{(-m-n,0)\}\overline{\cup}(-p+\mathbb{N}_0+d\mathbb{N}_0)$. Moreover, if $m<-n$, $A(\lambda)|_{z=0}\in\mathcal{A}^{\mathcal{F}}(\overline{\Lambda^{1/d}})$, where $\mathcal{F}:=\{(-m-n,0)\}\overline{\cup}(-p+d\mathbb{N}_0)$ is an index set on $\overline{\Lambda^{1/d}}$ associated to $\partial_\infty\overline{\Lambda^{1/d}}$.*

*Proof.* We use the coordinates (52) and the first set of coordinates $(\rho,\omega_0,\omega)$ in Figure 3. Then, since $\nu=r\omega=\rho\omega_0\omega$ and $z=r\omega'$, we have

$$\begin{aligned} A(\lambda) &= \int e^{i\rho\omega'\cdot\xi}\,\chi(\xi)\,a(\,(\rho\omega_0\omega)^{-d},\xi)\,d\xi \\ &= (\rho\omega_0\omega)^{-p}\int e^{i\rho\omega'\cdot\xi}\,\chi(\xi)\,\tilde{a}(\rho^d\omega_0^d\omega^d,\xi)\,d\xi \\ &= \rho^{-m-n}(\omega_0\omega)^{-p}\int e^{i\omega'\cdot\xi}\,\chi(\xi/\rho)\,\tilde{a}((\omega_0\omega)^d,\xi)\,d\xi. \end{aligned}$$

where we made the change of variables $\xi \mapsto \rho^{-1}\xi$. Hence,

$$\begin{aligned}&(\rho\partial_\rho - (-m-n))A \\ &\qquad = -\rho^{-m-n}(\omega_0\omega)^{-p}\int e^{i\omega'\cdot\xi}\,(\xi\cdot\partial_\xi\chi)(\xi/\rho)\,\tilde a((\omega_0\omega)^d,\xi)\,d\xi \\ &\qquad = -\rho^{-p}(\omega_0\omega)^{-p}\int e^{i\rho\omega'\cdot\xi}\,(\xi\cdot\partial_\xi\chi)(\xi)\,\tilde a(\rho^d(\omega_0\omega)^d,\xi)\,d\xi.\end{aligned}$$

Since $(\xi\cdot\partial_\xi\chi)(\xi)\equiv 0$ near $\xi=0$ and outside a neighborhood of 0, it follows that $f(u,v) := -(\omega_0\omega)^{-p}\int e^{iu\omega'\cdot\xi}\,(\xi\cdot\partial_\xi\chi)(\xi)\,\tilde a(v(\omega_0\omega)^d,\xi)\,d\xi$ is smooth for $(u,v)\in[0,\infty)^2$. Hence, $(\rho\partial_\rho-(-m-n))A \sim \sum_{i,j=0}^\infty \rho^{-p+i+dj}A_{i,j}(\omega_0,\omega',\omega)$, where $A_{i,j}(\omega_0,\omega',\omega) := (\partial_u^i\partial_v^j f)(0,0)\in\omega_0^{-p}\omega^{-p}S^0(\mathbb{S}^{n,1}_{(\omega_0,\omega')}\times(\mathbb{S}^1_\omega\cap\Lambda_{cc}^{1/d}))$. As $(\rho\partial_\rho-(-p+i+dj))\rho^{-p+i+dj}=0$, for any $M\in\mathbb{N}$, we have

$$\prod_{-p+i+dj\le M}(\rho\partial_\rho-(-p+i+dj))(\rho\partial_\rho-(-m-n))A = O(\rho^M).$$

Note that $A_{i,j}(\omega_0,\omega',\omega)\equiv 0$ at $\omega'=0$ for $i>0$. Thus, by Theorem 1, it follows that $A\in\mathcal{A}^{\mathcal{E}_{fi}}(T^n_{\Lambda^{1/d}})$, and if $m<-n$, then $A(\lambda)|_{z=0}\in\mathcal{A}^{\mathcal{F}}(\overline{\Lambda^{1/d}})$. □

**Lemma 43.** *Let $m\in\mathbb{R}$ and let $\chi(\lambda,\xi)\in C^\infty(\Lambda\times\mathbb{R}^n)$ with $\chi(\lambda,\xi)\equiv 0$ on a neighborhood of $(\lambda,\xi)=0$ and $\chi(\lambda,\xi)\equiv 1$ outside a neighborhood of 0. Let*

$$A(\lambda)=\int e^{iz\cdot\xi}\,\chi(\lambda,\xi)\,a(\lambda,\xi)\,d\xi,\ \text{ where } a(\lambda,\xi)\in C^{\infty,d}_{\mathrm{hom}(m)}(\Lambda\times\mathbb{R}^n).$$

*Then away from $\lambda=0$, we have $A(\lambda)\in\mathcal{A}^{\mathcal{E}_{fi}}(T^n_{\Lambda^{1/d}})$, where $\mathcal{E}_{fi} := \{(-m-n,0)\}$ is an index set on $T^n_{\Lambda^{1/d}}$ associated to fi. Moreover, if $m<-n$, then $A(\lambda)|_{z=0}\in\mathcal{A}^{\mathcal{F}}(\overline{\Lambda^{1/d}})$, where $\mathcal{F}:=\{(-m-n,0)\}$ is an index set on $\overline{\Lambda^{1/d}}$ associated to $\partial_\infty\overline{\Lambda^{1/d}}$.*

*Proof.* Let $a(\lambda,\xi)\in C^{\infty,d}_{\mathrm{hom}(m)}(\Lambda\times\mathbb{R}^n)$. Then,

$$A(\lambda)=\int e^{i\rho\omega'\cdot\xi}\,\chi(\,(\rho\omega_0\omega)^{-d},\xi)\,a(\,(\rho\omega_0\omega)^{-d},\xi)\,d\xi.$$

Since $\chi(\lambda,\xi)\equiv 1$ outside of $(\lambda,\xi)=0$, for $\rho$ small, $\chi(\rho^{-d}\omega_0^{-d}\omega^{-d},\xi)\equiv 1$. Hence, it suffices to prove our lemma for $A=\int e^{i\rho\omega'\cdot\xi}\,a(\,(\rho\omega_0\omega)^{-d},\xi)\,d\xi$. Now changing variables $\xi\mapsto\rho^{-1}\xi$ gives $A=\rho^{-m-n}\int e^{i\omega'\cdot\xi}a((\omega_0\omega)^{-d},\xi)\,d\xi$, which proves our lemma. □

Near $bi$, we can write

$$T^n_{\Lambda^{1/d}}\cong[0,\infty)_s\times[0,\infty)_{\gamma_0}\times F_{bi}\ \text{where}\ F_{bi}:=\mathbb{S}^{n-1}_{\gamma'}\times(\mathbb{S}^1_\omega\cap\Lambda^{1/d}_{cc});$$

and near $df$, we can write

$$T^n_{\Lambda^{1/d}}\cong[0,\infty)_u\times[0,\infty)_t\times F_{df}\ \text{where}\ F_{df}:=\mathbb{S}^{n-1}_{\gamma}\times(\mathbb{S}^1_\omega\cap\Lambda^{1/d}_{cc}).$$

**Lemma 44.** *Suppose that $m+|p|\le -N-n-1$, where $N\in\mathbb{N}$, $m,\,p\in\mathbb{R}$ with $p/d\in\mathbb{Z}$. Then given any $R(\lambda)=\int e^{iz\cdot\xi}\,r(\lambda,\xi)\,d\xi$, where $r(\lambda,\xi)\in S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$, for any $M\in\mathbb{N}$ with $M+dM\le N$, we have*

$$\begin{aligned}R&\in(s\gamma_0)^{-p}C^M(\,[0,\infty)_s\times[0,\infty)_{s^d\gamma_0^d}\times F_{bi}\,)\ \textit{near } bi;\\ R&\in t^{-p}C^M(\,[0,\infty)_u\times[0,\infty)_t\times[0,\infty)_{t^d}\times F_{df}\,)\ \textit{near } df.\end{aligned}$$

*Proof.* We must show that if $R_{bi}:=\int e^{is\gamma'\cdot\xi}\,r((s\gamma_0\omega)^{-d},\xi)\,d\xi$ and $R_{df}:=\int e^{itu\gamma\cdot\xi}\,r((t\omega)^{-d},\xi)\,d\xi$, then

$$\begin{aligned}R_{bi}&\in(s\gamma_0)^{-p}C^M(\,[0,\infty)_s\times[0,\infty)_{s^d\gamma_0^d}\times F_{bi}\,),\\ R_{df}&\in t^{-p}C^M(\,[0,\infty)_u\times[0,\infty)_t\times[0,\infty)_{t^d}\times F_{df}\,).\end{aligned}$$

Since $r\in S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$, if we define $\tilde r(\mu,\xi):=\mu^{p/d}r(1/\mu,\xi)$, then $\tilde r(\mu,\xi)\in C^\infty(\Lambda_{cc}\times\mathbb{R}^n)$. Let $K\in\mathbb{R}^+$. Then, $\tilde r(\mu,\xi)$ satisfies the estimates: for any $\alpha$ and $\beta$,

$$|\,\partial_\mu^\alpha\partial_\xi^\beta\tilde r(\mu,\xi)\,|\ \dot\le\ (1+|\mu|\,|\xi|^d)^{p/d-|\alpha|}\,(1+|\xi|)^{d|\alpha|+m-p-|\beta|},\ |\mu|\le K.$$

Observe that for $|\mu| \leq K$, $(1 + |\mu|\,|\xi|^d)^{p/d}\,(1 + |\xi|)^{-p} \;\dot{\leq}\; (1 + |\xi|)^{|p|}$. Thus, for $|\mu| \leq K$, for any $\alpha$, $|\,\partial_\mu^\alpha \tilde{r}(\mu,\xi)\,| \;\dot{\leq}\; (1 + |\xi|)^{d|\alpha|+m+|p|}$; or, as $m + |p| \leq -N - n - 1$ we have

$$|\,\partial_\mu^\alpha \partial_\xi^\beta \tilde{r}(\mu,\xi)\,| \;\dot{\leq}\; (1 + |\xi|)^{d|\alpha|-N-n-1},\; |\mu| \leq K. \tag{55}$$

Rewriting $R_{bi}$ and $R_{df}$ in terms of $\tilde{r}$ gives

$$\begin{aligned} R_{bi} &= (s\gamma_0\omega)^{-p} \int e^{is\gamma'\cdot\xi}\, \tilde{r}((s\gamma_0)^d\omega^d,\xi)\,d\xi, \\ R_{df} &= (t\omega)^{-p} \int e^{itu\gamma\cdot\xi}\, \tilde{r}(t^d\omega^d,\xi)\,d\xi. \end{aligned}$$

Thus, to prove our lemma, it suffices to show that if

$$\begin{aligned} R_{bi}(s,v,\gamma',\omega) &:= \int e^{is\gamma'\cdot\xi}\, \tilde{r}(v\omega^d,\xi)\,d\xi, \\ R_{df}(u,t,v,\gamma,\omega) &:= \int e^{itu\gamma\cdot\xi}\, \tilde{r}(v\omega^d,\xi)\,d\xi, \end{aligned}$$

then for all $M \in \mathbb{N}$ with $M + dM \leq N$,

$$\begin{aligned} R_{bi}(s,v,\gamma',\omega) &\in C^M(\,[0,\infty)_s \times [0,\infty)_v \times F_{bi}\,), \\ R_{df}(u,t,v,\gamma,\omega) &\in C^M(\,[0,\infty)_u \times [0,\infty)_t \times [0,\infty)_v \times F_{df}\,). \end{aligned}$$

But these inclusions follow from the estimate (55). $\square$

**Lemma 45.** *Let $N \in \mathbb{N}$ and let $m \in \mathbb{R}$ with $m \leq -2N - n - 1$. Then given any $R(\lambda) = \int e^{iz\cdot\xi}\, r(\lambda,\xi)\,d\xi$, where $r(\lambda,\xi) \in S_\Lambda^{m,d}(\mathbb{R}^n)$, we have*

$$\begin{aligned} R &\in (s\gamma_0)^N S^0(\,[0,\infty)_{s^d\gamma_0^d}; C^N(T^n_{\Lambda^{1/d}})\,) \text{ near } bi; \\ R &\in t^N S^0(\,[0,\infty)_{t^d}; C^N(T^n_{\Lambda^{1/d}})\,) \text{ near } df. \end{aligned}$$

*Proof.* We must show that if $R_{bi} := \int e^{is\gamma'\cdot\xi}\, r((s\gamma_0\omega)^{-d},\xi)\,d\xi$ and $R_{df} := \int e^{itu\gamma\cdot\xi}\, r((t\omega)^{-d},\xi)\,d\xi$, then

$$\begin{aligned} R_{bi} &\in (s\gamma_0)^N S^0(\,[0,\infty)_{s^d\gamma_0^d}; C^N(T^n_{\Lambda^{1/d}})\,); \\ R_{df} &\in t^N S^0(\,[0,\infty)_{t^d}; C^N(T^n_{\Lambda^{1/d}})\,). \end{aligned} \tag{56}$$

Since $m \leq -N - (N + n + 1)$, by the symbol estimates for $S_\Lambda^{-2N-n-1,d}(\mathbb{R}^n)$, it follows that if $r'(\mu,\xi) := r(1/\mu,\xi)$, where $\mu = 1/\lambda \in \Lambda_{cc}$, then

$$r'(\mu,\xi) \in |\mu|^{N/d} S^0(\overline{\Lambda}; S^{-N-n-1}(\mathbb{R}^n)\,). \tag{57}$$

Since $R_{bi} = \int e^{is\gamma'\cdot\xi}\, r'((s\gamma_0)^d\omega^d,\xi)\,d\xi$ and $R_{df} = \int e^{itu\gamma\cdot\xi}\, r'(t^d\omega^d,\xi)\,d\xi$ in terms of $r'$, (57) implies that (56) holds. $\square$

For each $m$, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$, we define the index set $\mathcal{E}_{m,p,d}$ on $T^n_{\Lambda^{1/d}}$ by

$$\begin{aligned} \mathcal{E}_{m,p,d}(\text{edges of } \Lambda) &:= \mathbb{N}_0;\;\; \mathcal{E}_{m,p,d}(bi) = -p + d\mathbb{N}_0; \\ \mathcal{E}_{m,p,d}(df) &:= \tfrac{n}{2} + \{(k - m - n + dl, 0) \,|\, k,l \in \mathbb{N}_0\}\overline{\cup}\mathbb{N}_0; \\ \mathcal{E}_{m,p,d}(fi) &:= \tfrac{n}{2} + \{(k - m - n, 0) \,|\, k \in \mathbb{N}_0\}\overline{\cup}(-p + \mathbb{N}_0 + d\mathbb{N}_0) \end{aligned}$$

and we define $\mathcal{F}_{m,p,d} := \{(k - m - n, 0) \,|\, k \in \mathbb{N}_0\}\overline{\cup}(-p + d\mathbb{N}_0)$ associated to $\partial_\infty\overline{\Lambda^{1/d}}$ in $\overline{\Lambda^{1/d}}$.

**Lemma 46.** *Let $m, p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. Then away from $\lambda = 0$, there exists a continuous linear map*

$$I^{m+n/4,p,d}_{\Lambda,ros}(\mathbb{R}^n,\{0\},\Omega^{\frac12}) \;\to\; \mathcal{A}^{\mathcal{E}_{m,p,d}}_{phg}(\,T^n_{\Lambda^{1/d}},\Omega_b^{\frac12});\;\; A \;\mapsto\; A(\lambda)|\frac{d\lambda\wedge d\overline{\lambda}}{\lambda^2}|^{\frac12},$$

*such that if $m \notin \mathbb{N}_0$, then this map is injective and if $m \in \mathbb{N}_0$, then this map has kernel*

$$I^{m+n/4,p,d}_{\Lambda,ros}(\mathbb{R}^n,\{0\},\Omega^{\frac12}) \cap C^\infty(\,\Lambda; \mathrm{span}_{\mathbb{C}}\{\delta^k \cdot |dz|^{\frac12} \,|\, 0 \leq k \leq m\}\,).$$

*Moreover, if $m < -n$, then there exists a continuous linear map*

$$I^{m+n/4,p,d}_{\Lambda,ros}(\mathbb{R}^n,\{0\},\Omega^{\frac12})|_{z=0} \;\hookrightarrow\; \mathcal{A}^{\mathcal{F}_{m,p,d}}_{phg}(\,\overline{\Lambda^{1/d}});\;\; A \;\mapsto\; A(\lambda)|_{z=0}.$$

*Proof.* Let $\mathcal{E}'_{m,p,d}$ be the same index set as $\mathcal{E}_{m,p,d}$ except without the $\frac{n}{2}$ term on the right hand side of $\mathcal{E}_{m,p,d}(df)$ and $\mathcal{E}_{m,p,d}(fi)$. Then by Lemma 34, it suffices to show that there exists a continuous linear map

$$I^{m+n/4,p,d}_{\Lambda,ros}(\mathbb{R}^n,\{0\}) \to \mathcal{A}^{\mathcal{E}'_{m,p,d}}_{phg}(T^n_{\Lambda^{1/d}});\;\; A \mapsto A(\lambda),$$

such that if $m \notin \mathbb{N}_0$, then this map is injective and if $m \in \mathbb{N}_0$, then this map has kernel

$$I^{m+n/4,p,d}_{\Lambda,ros}(\mathbb{R}^n,\{0\}) \cap C^\infty(\Lambda;\operatorname{span}_{\mathbb{C}}\{\delta^k \,|\, 0 \le k \le m\}); \tag{58}$$

and if $m < -n$, then there exists a continuous linear map

$$I^{m+n/4,p,d}_{\Lambda,ros}(\mathbb{R}^n,\{0\})|_{z=0} \hookrightarrow \mathcal{A}^{\mathcal{F}_{m,p,d}}_{phg}(\overline{\Lambda^{1/d}});\;\; A \mapsto A(\lambda)|_{z=0}.$$

Now, the symbol of an element $A$ of $I^{m+n/4,p,d}_{\Lambda,ros}(\mathbb{R}^n,\{0\})$ is an asymptotic sum of elements of $C^{\infty,p,d}_{\Lambda,\hom(m-k)}(\mathbb{R}^n)$ for each $k \in \mathbb{N}_0$. Thus, Proposition 2 applied to the asymptotic expansion of $A$, where we use Lemma 38, Lemma 40, Lemma 42, and Lemma 44 on the remainder term, shows that $A(\lambda) \in \mathcal{A}^{\mathcal{E}'_{m,p,d}}_{phg}(T^n_{\Lambda^{1/d}})$, and if $m < -n$, then $A(\lambda)|_{z=0} \in \mathcal{A}^{\mathcal{F}_{m,p,d}}_{phg}(\overline{\Lambda^{1/d}})$. Note that for each $\lambda \in \Lambda$, $\operatorname{supp} A(\lambda) \subseteq \{0\}$ iff $A(\lambda) \equiv 0$ in $\mathcal{A}^{\mathcal{E}'_{m,p,d}}_{phg}(T^n_{\Lambda^{1/d}})$. Since the only distributions having support at 0 are linear combinations of derivatives of the delta distribution at 0, $A(\lambda) \equiv 0$ in $\mathcal{A}^{\mathcal{E}'_{m,p,d}}_{phg}(T^n_{\Lambda^{1/d}})$ iff (58) holds. □

For each $m \in \mathbb{R}$, we define the index set $\mathcal{E}_{m,d}$ on $T^n_{\Lambda^{1/d}}$ by

$$\begin{gathered}\mathcal{E}_{m,d}(\text{edges of }\Lambda) := \mathbb{N}_0;\; \mathcal{E}_{m,d}(bi) = \emptyset;\\ \mathcal{E}_{m,d}(df) := \tfrac{n}{2} + \{(k-m-n+dl,0)\,|\,k,l \in \mathbb{N}_0\}\overline{\cup}\mathbb{N}_0;\\ \mathcal{E}_{m,d}(fi) := \tfrac{n}{2} + \{(k-m-n,0)\,|\,k \in \mathbb{N}_0\};\end{gathered}$$

and we define $\mathcal{F}_{m,d} := \{(k-m-n,0)\,|\,k \in \mathbb{N}_0\}$ associated to $\partial_\infty\overline{\Lambda^{1/d}}$ in $\overline{\Lambda^{1/d}}$.

**Lemma 47.** *Let $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$. Then away from $\lambda = 0$, there exists a continuous linear map*

$$I^{m+n/4,d}_{\Lambda,os}(\mathbb{R}^n,\{0\},\Omega^{\frac12}) \to \mathcal{A}^{\mathcal{E}_{m,d}}_{phg}(T^n_{\Lambda^{1/d}},\Omega_b^{\frac12});\;\; A \mapsto A(\lambda)\Big|\frac{d\lambda \wedge d\overline{\lambda}}{\lambda^2}\Big|^{\frac12},$$

*such that if $m \notin \mathbb{N}_0$, then this map is injective and if $m \in \mathbb{N}_0$, then this map has kernel*

$$I^{m+n/4,d}_{\Lambda,os}(\mathbb{R}^n,\{0\},\Omega^{\frac12}) \cap C^\infty(\Lambda;\operatorname{span}_{\mathbb{C}}\{\delta^k \cdot |dz|^{\frac12} \,|\, 0 \le k \le m\}).$$

*Moreover, if $m < -n$, then there exists a continuous linear map*

$$I^{m+n/4,d}_{\Lambda,os}(\mathbb{R}^n,\{0\},\Omega^{\frac12})|_{z=0} \hookrightarrow \mathcal{A}^{\mathcal{F}_{m,d}}_{phg}(\overline{\Lambda^{1/d}});\;\; A \mapsto A(\lambda)|_{\Delta_b}.$$

*Proof.* This lemma is proved similar to Lemma 46, except here, we use Lemma 39, Lemma 41, Lemma 43, and Lemma 45. □

4.5. **Proof of Theorems 35 and 36.** Let $m,\,p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. Then, by the definition of the space $\Psi^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac12})$, we have $\Psi^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac12}) \subseteq S^{p/d}_r(\Lambda;\Psi_b^{-\infty}(X,\Omega_b^{\frac12}))$ off of $\Delta_b$, and in any product decomposition $X_b^2 \cong \mathbb{R}^{n,k} \times \mathbb{R}^n_z$, where $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}_z$, we can identify

$$\Psi^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac12}) \equiv I^{m,p,d}_{\Lambda,ros}(\mathbb{R}^{n,k} \times \mathbb{R}^n, \mathbb{R}^{n,k} \times \{0\},\Omega_b^{\frac12}),$$

such that if $m \in \mathbb{N}_0$,

$$\operatorname{Diff}^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac12}) \subseteq I^{m,p,d}_{\Lambda,ros}(\mathbb{R}^{n,k} \times \mathbb{R}^n, \mathbb{R}^{n,k} \times \{0\},\Omega_b^{\frac12})$$

consists of those elements with polynomial symbols (in $\xi$). Thus, Theorem 35 follows from Lemma 46. Similarly, Theorem 36 follows from Lemma 47.

## 5. Applications

Recall that $\Lambda$ always represents a proper sector of $\mathbb{C}$.

**Lemma 48.** *Let $u(\lambda)$ be a holomorphic function on $\Lambda \setminus \{0\}$. Suppose that in polar coordinates, $(r,\theta)$, $u$ is of the form $u(r,\theta) = r^z(\sum_{k=0}^N u_k(\theta)(\log r)^k)$, where $z \in \mathbb{C}$, $N \in \mathbb{N}_0$, and for each $k$, $u_k \in C^\infty(\mathbb{S}^1 \cap \Lambda)$. Then, we can write $u(\lambda) = \lambda^z(\sum_{k=0}^N a_k (\log \lambda)^k)$, for some constants $a_k \in \mathbb{C}$.*

*Proof.* [6] By replacing $u_k(\theta)$ with $e^{iz\theta}u_k(\theta)$, we may assume that $u(r,\theta) = \lambda^z(\sum_{k=0}^N u_k(\theta)(\log r)^k)$. Consider the change of variables, $w = \log \lambda = \rho + i\theta$, where $\rho = \log r$. Then observe that

$$v(w) := \lambda^{-z}u(\lambda) = \sum_{k=0}^N u_k(\theta)\rho^k. \tag{59}$$

Since $w$ is a holomorphic change of variables, $v(w)$ is holomorphic. Thus, as $\partial_w = \partial_\rho$ on holomorphic functions, $\partial_w^{N+1}v(w) = \partial_\rho^{N+1}v(w) = 0$ by (59). Thus, $v(w)$ is a polynomial in $w = \log \lambda$ of order $N$. □

**Proposition 49.** *Let $E$ be any index set associated to $\partial_\infty\overline{\Lambda}$. Let $u \in \mathcal{A}^E(\overline{\Lambda})$ and suppose that $u$ is holomorphic on $\Lambda$. Then, as $|\lambda| \to \infty$ in $\Lambda$,*

$$u(\lambda) \sim \sum_{(z,k)\in E} \lambda^{-z}(\log \lambda)^k u_{(z,k)}, \; u_{(z,k)} \in \mathbb{C}. \tag{60}$$

*Proof.* Let $F := \{z \in \mathbb{C} \,|\, (z,k) \in E \text{ for some } k\}$. Then by definition of $\mathcal{A}^E(\overline{\Lambda})$, if $\lambda = re^{i\theta}$,

$$u(\lambda) \sim \sum_{z\in F} u_z(r,\theta), \tag{61}$$

where $u_z(r,\theta) = \sum_{\{k\,|\,(z,k)\in E\}} r^{-z}(\log r)^k u_{(z,k)}(\theta)$, $u_{(z,k)}(\theta) \in C^\infty(\mathbb{S}^1\cap\Lambda)$. Since $\overline{\lambda}\,\overline{\partial}_\lambda = \frac{1}{2}r\partial_r + \frac{i}{2}\partial_\theta$, $\overline{\lambda}\,\overline{\partial}_\lambda$ preserves the order of decay in $r$; hence, as

$$0 \equiv \overline{\lambda}\,\overline{\partial}_\lambda u \sim \sum_{z\in F} \overline{\lambda}\,\overline{\partial}_\lambda u_z(r,\theta),$$

we must have $\overline{\lambda}\,\overline{\partial}_\lambda u_z(r,\theta) \equiv 0$ for each $z$. Thus, using Lemma 48 on $u_z(r,\theta)$ together with the expression (61), gives the expansion (60). □

**Theorem 50.** *Let $m$, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$ and $m < -n$. Then for any holomorphic $A(\lambda) \in \Psi^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac{1}{2}})$, as $|\lambda| \to \infty$ in $\Lambda$, we have*

$$\begin{aligned} A(\lambda)|_{\Delta_b} \;\sim\; & \sum_{k=0}^\infty \lambda^{\frac{m+n-k}{d}}\,\alpha_k(x) + \sum_{k,\,\frac{k-m-n+p}{d}\in\mathbb{N}_0} \lambda^{\frac{m+n-k}{d}} \log\lambda\,\alpha'_k(x) \\ & + \sum_{k=0}^\infty \lambda^{\frac{p}{d}-k}\alpha''_k(x), \text{ where } \alpha_k,\, \alpha'_k,\, \alpha''_k \in C^\infty(X,\Omega_b). \end{aligned}$$

*Proof.* We have $A(\lambda)|_{\Delta_b} \in \mathcal{A}^{\mathcal{F}_{m,p,d}}_{phg}(\overline{\Lambda^{1/d}}; C^\infty(X,\Omega_b))$ by (45) of Theorem 35, where $\mathcal{F}_{m,p,d}$ is the index set

$$\begin{aligned} \mathcal{F}_{m,p,d} = \{(k-m-n,0)\,|\,k\in\mathbb{N}_0\} \cup (-p+d\mathbb{N}_0) \cup \\ \{(k-m-n,1)\,|\,k\in\mathbb{N}_0,\, k-m-n \in -p+d\mathbb{N}_0\}. \end{aligned}$$

Since $A$ is holomorphic, Proposition 49 implies that

$$\begin{aligned} A(\lambda)|_{\Delta_b} \;\sim\; & \sum_{k=0}^\infty (\lambda^{1/d})^{m+n-k}\,\alpha_k(x) + \sum_{k\in\mathbb{N}_0,\,k-m-n+p\in d\mathbb{N}_0} (\lambda^{1/d})^{m+n-k} \log\lambda\,\alpha'_k(x) \\ & + \sum_{k=0}^\infty (\lambda^{1/d})^{p-dk}\alpha''_k(x), \text{ where } \alpha_k,\, \alpha'_k,\, \alpha''_k \in C^\infty(X,\Omega_b). \end{aligned}$$

[6] This proof was suggested by referee.

□

Similarly, Theorem 36 and Proposition 49 give the following theorem.

**Theorem 51.** *Let $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $m < -n$. Then for any holomorphic $A(\lambda) \in \Psi^{m,d}_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$, as $|\lambda| \to \infty$ in $\Lambda$, we have*

$$A(\lambda)|_{\Delta_b} \sim \sum_{k=0}^{\infty} \lambda^{\frac{m+n-k}{d}} \alpha_k(x), \text{ where } \alpha_k \in C^\infty(X, \Omega_b).$$

For each $m, m' \in \mathbb{R}$ and $N \in \mathbb{N}$, define the index set $\mathcal{E}_{m,m',N}$, associated to the boundary faces *bi*, *fi*, and *df* of $X^2_{b,\Lambda^{1/m}}$, by

$$\begin{aligned} \mathcal{E}_{m,m',N}(bi) &= mN + m\mathbb{N}_0; \\ \mathcal{E}_{m,m',N}(df) &:= \tfrac{n}{2} + \{(k - m' - n + m(N+l), 0) \mid k, l \in \mathbb{N}_0\} \overline{\cup} \mathbb{N}_0; \\ \mathcal{E}_{m,m',N}(fi) &:= \tfrac{n}{2} + \{(k - m' - n + mN, 0) \mid k \in \mathbb{N}_0\} \overline{\cup} (mN + \mathbb{N}_0 + m\mathbb{N}_0). \end{aligned}$$

**Theorem 52.** *Let $A \in \Psi^m_{b,os}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be elliptic with respect to $\Lambda$, and let $N \in \mathbb{N}$. Then, for any $B \in \Psi^{m'}_{b,os}(X, \Omega_b^{\frac{1}{2}})$ and $\lambda$ sufficiently large,*

$$B(A-\lambda)^{-N} \left|\frac{d\lambda \wedge d\overline{\lambda}}{\lambda^2}\right|^{\frac{1}{2}} \in \mathcal{A}^{\mathcal{E}_{m,m',N}}(X^2_{b,\Lambda^{1/m}}, \Omega_b^{\frac{1}{2}}); \tag{62}$$

*and, if $Nm - m' > n$, then as $|\lambda| \to \infty$ in $\Lambda$, we have*

$$\begin{aligned} B(A-\lambda)^{-N}|_{\Delta_b} \sim & \sum_{k=0}^{\infty} \lambda^{\frac{n+m'-k}{m} - N} \alpha_k(x) + \sum_{k,\, \frac{k-m'-n}{m} \in \mathbb{N}_0} \lambda^{\frac{n+m'-k}{m} - N} \log \lambda\, \alpha'_k(x) \\ & + \sum_{k=0}^{\infty} \lambda^{-k-N} \alpha''_k(x), \text{ where } \alpha_k,\, \alpha'_k,\, \alpha''_k \in C^\infty(X, \Omega_b). \end{aligned} \tag{63}$$

*Proof.* By Theorem 33, there exists a function $r : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $a > 0$ and $N \in \mathbb{N}$, $(A-\lambda)^{-N} \in \Psi^{-Nm,-Nm,m;a}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r(a)$. Thus, $B(A-\lambda)^{-N} \in \Psi^{m'-Nm,-Nm,m;a}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$ for all $a > 0$ and $|\lambda| \geq r(a)$. Thus, Theorem 35 implies (62). As $a \to \infty$, $B(A-\lambda)^{-N}$ becomes more and more an element of $\Psi^{m'-Nm,-Nm,m}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$. Thus, the expansion (63) follows from Theorem 50. □

For each $m, m' \in \mathbb{N}_0$ and $N \in \mathbb{N}$, define the index set $\mathcal{F}_{m,m',N}$, associated to the boundary faces *bi*, *fi*, and *df* of $X^2_{b,\Lambda^{1/m}}$, by

$$\begin{aligned} \mathcal{F}_{m,m',N}(bi) &= \emptyset; \\ \mathcal{F}_{m,m',N}(df) &:= \tfrac{n}{2} + \{(k - m' - n + m(N+l), 0) \mid k, l \in \mathbb{N}_0\} \overline{\cup} \mathbb{N}_0; \\ \mathcal{F}_{m,m',N}(fi) &:= \tfrac{n}{2} + \{(k - m' + mN, 0) \mid k \in \mathbb{N}_0\}. \end{aligned}$$

**Theorem 53.** *Let $A \in \mathrm{Diff}^m_b(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{N}$ be elliptic with respect to $\Lambda$, and let $N \in \mathbb{N}$. Then, for any $B \in \mathrm{Diff}^{m'}_b(X, \Omega_b^{\frac{1}{2}})$ and $\lambda$ sufficiently large,*

$$B(A-\lambda)^{-N} \left|\frac{d\lambda \wedge d\overline{\lambda}}{\lambda^2}\right|^{\frac{1}{2}} \in \mathcal{A}^{\mathcal{F}_{m,m',N}}_{phg}(X^2_{b,\Lambda^{1/m}}, \Omega_b^{\frac{1}{2}}). \tag{64}$$

*Moreover, if $Nm - m' > n$, then as $|\lambda| \to \infty$ in $\Lambda$, we have*

$$B(A-\lambda)^{-N}|_{\Delta_b} \sim \sum_{k=0}^{\infty} \lambda^{\frac{n+m'-k}{m} - N} \alpha_k(x), \text{ where } \alpha_k \in C^\infty(X, \Omega_b). \tag{65}$$

*Proof.* By Theorem 33, there exists a function $r : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $a > 0$ and $N \in \mathbb{N}$, $(A-\lambda)^{-N} \in \Psi^{-Nm,m;a}_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r(a)$. Thus, (64) follows from Theorem 36. Observe that as $a \to \infty$, $B(A-\lambda)^{-N}$ becomes more and more an element of $\Psi^{m'-Nm,m}_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$. Hence, the expansion (65) follows from Theorem 51. □

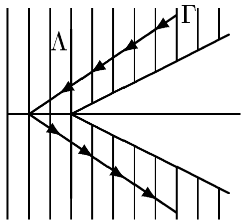


FIGURE 4. The contour $\Gamma$.

Henceforth, we will assume that $\Lambda$ is of the form

$$\Lambda = \{\lambda \in \mathbb{C} \,|\, \epsilon \le \arg(\lambda) \le 2\pi - \epsilon\},\ 0 < \epsilon < \pi/2. \tag{66}$$

Let $\Lambda$ be such a sector and let $A \in \Psi^m_{b,os}(X, \Omega_b^{\frac12})$, $m \in \mathbb{R}^+$ be elliptic with respect to $\Lambda$. Then by Theorem 33 and (1) of Lemma 25, there exists a continuous increasing function $r : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $a > 0$ and $N \in \mathbb{N}$,

$$(A-\lambda)^{-N} \in \Psi^{-Nm,-Nm,m;a}_{b,\Lambda,ros}(X, \Omega_b^{\frac12}) \subseteq \Psi^{-Nm,0,m;a}_{b,\Lambda,ros}(X, \Omega_b^{\frac12})$$

for $|\lambda| \ge r(a)$. Hence, by (3) of Lemma 25, for $|\lambda| \ge r(a)$,

$$(A-\lambda)^{-N} \subseteq S^0(\Lambda; \Psi_b^{-Nm,a}(X, \Omega_b^{\frac12})\,). \tag{67}$$

The *heat operator of* $A$, $e^{-tA}$, is defined by the Cauchy integral

$$e^{-tA} := \frac{i}{2\pi} \int_\Gamma e^{-t\lambda}\,(A-\lambda)^{-1}\,d\lambda, \tag{68}$$

where $\Gamma$, shown in Figure 4, is such that $|\lambda| \ge r(1)$ for all $\lambda \in \Gamma$. By (67), this integral converges absolutely in $\Psi_b^{-m,a}(X, \Omega_b^{\frac12})$ for all $a > 1$. By integration by parts $N-1$ times, we can rewrite (68) as

$$e^{-tA} = \frac{i}{2\pi} \frac{(-t)^{-N+1}}{(N-1)!} \int_\Gamma e^{-t\lambda}\,(A-\lambda)^{-N}\,d\lambda; \tag{69}$$

and hence, by (67), $e^{-tA} \in \Psi_b^{-Nm,a}(X, \Omega_b^{\frac12})$ for all $a > 1$. Thus, in fact,

$$e^{-tA} \in C^\infty((0,\infty)_t;\ \Psi_b^{-\infty}(X, \Omega_b^{\frac12})\,).$$

Hence, if $B$ is any $b$-pseudodifferential operator, then $Be^{-tA} \in C^\infty((0,\infty)_t;\ \Psi_b^{-\infty}(X, \Omega_b^{\frac12})\,)$ as well; in particular, the kernel of $Be^{-tA}$ restricted to $\Delta_b$ is defined. The proof of the following corollary follows from the results of [4, Section 4.6] applied to $B$ times the Laplace transform (69), together with the asymptotic expansions (63) and (65).

**Corollary 54.** *Let $\Lambda \subseteq \mathbb{C}$ be of the form (66). If $A \in \Psi^m_{b,os}(X, \Omega_b^{\frac12})$, $m \in \mathbb{R}^+$ is elliptic with respect to $\Lambda$, then given $B \in \Psi^{m'}_{b,os}(X, \Omega_b^{\frac12})$, $m' \in \mathbb{R}$, as $t \downarrow 0$,*

$$\begin{aligned} Be^{-tA}|_{\Delta_b} \ \sim\ & \sum_{k \in \mathbb{N}_0} t^{\frac{k-m'-n}{m}}\,\gamma_k(x) + \sum_{k,\ \frac{k-m'-n}{m} \in \mathbb{N}_0} t^{\frac{k-m'-n}{m}} \log t\, \gamma'_k(x) \\ & + \sum_{k \in \mathbb{N}_0} t^k \gamma''_k(x),\ \text{where } \gamma_k,\ \gamma'_k,\ \gamma''_k \in C^\infty(X, \Omega_b); \end{aligned} \tag{70}$$

*and if $A \in \mathrm{Diff}^m_b(X, \Omega_b^{\frac12})$, $m \in \mathbb{N}$, then given a $b$-differential operator $B \in \mathrm{Diff}^{m'}_b(X, \Omega_b^{\frac12})$, $m' \in \mathbb{N}_0$, as $t \downarrow 0$,*

$$Be^{-tA}|_{\Delta_b} \sim \sum_{k=0}^\infty t^{\frac{k-m'-n}{m}}\,\gamma_k(x),\ \text{where } \gamma_k \in C^\infty(X, \Omega_b). \tag{71}$$

Let $\Lambda$ be of the form (66) and let $A \in \Psi^m_{b,os}(X,\Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be elliptic with respect to $\Lambda$. Assume that $(A-\lambda)^{-1}$ is defined on all of $\Lambda$ and on a neighborhood of the origin in $\mathbb{C}$. Then, if $\Gamma$ is a contour similar to the one shown in Figure 4, but completely in the right-half plane, in [9] it is shown that the *complex power*, $A^z$, defined by the Cauchy integral

$$A^z := \frac{i}{2\pi} \int_\Gamma \lambda^z \, (A-\lambda)^{-1} \, d\lambda, \text{ for } \operatorname{Re} z < 0, \tag{72}$$

exists as an entire family of $b$-pseudodifferential operators, where for some fixed $\delta > 0$, for each $z$, $A^z \in \Psi_b^{m\operatorname{Re} z,\delta}(X,\Omega_b^{\frac{1}{2}})$. Note that by integration by parts, for any $N \geq 2$, we can rewrite (72) as

$$A^z = \frac{i}{2\pi} \frac{(-1)^{N-1} z(z+1)\cdots(z+N-2)}{(N-1)!} \int_\Gamma \lambda^{z+N-1} \, (A-\lambda)^{-N} \, d\lambda. \tag{73}$$

Observe that if $B \in \Psi^{m'}_{b,os}(X,\Omega_b^{\frac{1}{2}})$, $m' \in \mathbb{R}$, then for each $z \in \mathbb{C}$, $BA^z \in \Psi_b^{m\operatorname{Re} z+m',\delta}(X,\Omega_b^{\frac{1}{2}})$; in particular, for $\operatorname{Re} z$ sufficiently negative, the kernel of $BA^z$ restricted to $\Delta_b$ is defined. The proof of the following corollary follows from the results of [4, Section 4.3] applied to $B$ times the Mellin transform (73), together with the asymptotic expansions (63) and (65).

**Corollary 55.** *Let $\Lambda \subseteq \mathbb{C}$ be of the form (66) and let $A \in \Psi^m_{b,os}(X,\Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be elliptic with respect to $\Lambda$. Assume that $(A-\lambda)^{-1}$ is defined on all of $\Lambda$ and on a neighborhood of the origin in $\mathbb{C}$. Then given any $B \in \Psi^{m'}_{b,os}(X,\Omega_b^{\frac{1}{2}})$, $m' \in \mathbb{R}$, the Schwartz kernel of $BA^z$, when restricted to $\Delta_b$, is a meromorphic function on all of $\mathbb{C}$, with values in $S^{0,\delta}(X,\Omega_b)$ for some $\delta > 0$, having only simple poles at the points*

$$\{z_k = \frac{k-m'-n}{m} \; : \; k \in \mathbb{N}_0\}.$$

In this appendix, we will prove Theorem 1 and Proposition 2. We will also show that the definition of asymptotic expansion is coordinate independent. Throughout this appendix, let $E$ be a fixed $C^\infty$ index set. We first start with the proof of Theorem 1. The proof we give is due to Melrose.

By definition, for each $(z,k) \in E$, there exists a $u_{(z,k)} \in S^{\alpha_Y}(Y)$ such that for each $N \in \mathbb{N}$, there exists an $M \in \mathbb{N}$ with

$$u(x,y) - \sum_{(z,k)\in E,\, \operatorname{Re} z \leq M} x^z (\log x)^k \, u_{(z,k)}(y) \in x^N S^{(0,\alpha_Y)}([0,1) \times Y). \tag{74}$$

Since $(x\partial_x - z)^{k+1} x^z (\log x)^k \equiv 0$, by applying $P_N^E$ to (74), it follows that the condition (5) is necessary. The following four lemmas prove the sufficiency. For the rest of this appendix, we will denote

$$X = [0,1) \times Y.$$

**Lemma 56.** *Let $u \in S^\alpha(X)$ and suppose that for some $N \in \mathbb{N}$, $P_N^E u \equiv 0$. Then,*

$$u = \sum_{(z,k)\in E,\, \operatorname{Re} z \leq N} x^z (\log x)^k u_{(z,k)},$$

*for some $u_{(z,k)} \in S^{\alpha_Y}(Y)$.*

*Proof.* It suffices to show that $u = \sum_{(z,k)\in E,\, \operatorname{Re} z\leq N} e^{zt} t^k u_{(z,k)}$, where $u_{(z,k)} \in S^{\alpha_Y}(Y)$ and $t = \log x$. We can write $P_N^E = (\partial_t - z_1)^{k_1+1} \cdots (\partial_t - z_l)^{k_l+1}$, where $\operatorname{Re} z_1 < \operatorname{Re} z_2 < \cdots < \operatorname{Re} z_l \leq N$ and where $k_i \in \mathbb{N}_0$ for each $i$. For $m = 1,\ldots,l$, define $u_m := (\partial_t - z_m)^{k_m+1} \cdots (\partial_t - z_l)^{k_l+1} u$ and define $u_{l+1} := u$. We will prove the following statement by induction on $1 \leq m \leq l+1$: For each $m = 1,\ldots,l+1$,

$$u_m = \sum_{i=1}^{m-1} \sum_{j=0}^{k_i} e^{z_i t} t^j u^m_{ij}, \text{ where } u^m_{ij} \in S^{\alpha_Y}(Y). \tag{75}$$

Then setting $m = l+1$ proves our lemma.

Since $u_1 = P_N^E u \equiv 0$, (75) holds for $m = 1$. Thus, assume that (75) holds for some $2 \leq m \leq l$; we'll show that it holds for $m+1$. By definition of $u_{m+1}$, we have

$$(\partial_t - z_m)^{k_m+1} u_{m+1} = u_m. \tag{76}$$

Now observe that as operators, $e^{-z_m t} \circ (\partial_t - z_m) = \partial_t \circ e^{-z_m t}$. Hence, multiplying both sides of (76) on the left by $e^{-z_m t}$ yields

$$\partial_t^{k_m+1}(e^{-z_m t} u_{m+1}) = e^{-z_m t} u_m = \sum_{i=1}^{m-1} \sum_{j=0}^{k_{i-1}} e^{(z_i - z_m)t} t^j u_{ij}^m.$$

By integrating both sides of this equation from 0 to $t$, $k_m + 1$ times, we get an equation of the form

$$e^{-z_m t} u_{m+1} = \sum_{j=0}^{k_m} t^j u_j + \sum_{i=1}^{m-1} \sum_{j=0}^{k_{i-1}} e^{(z_i - z_m)t} t^j u_{ij}^{m+1},$$

where $u_j, u_{ij}^{m+1} \in S^{\alpha_Y}(Y)$. Multiplying both sides of this equation by $e^{z_m t}$ finishes our induction step. □

**Lemma 57.** *Let $u \in x^a S^{(0,\beta)}(X)$. Then,*

(1) *if $a < 0$, $v(x,y) := \int_1^x u(s,y)\, \frac{ds}{s} \in x^a S^{(0,\beta)}(X)$;*
(2) *if $a > 0$, $v(x,y) := \int_0^x u(s,y)\, \frac{ds}{s} \in x^a S^{(0,\beta)}(X)$.*

*Proof.* Let $\rho$ be a total boundary defining function for $Y$ and write $u = x^a \rho^{-\beta} w$, where $w \in S^0(X)$. We will prove (1). Observe that since $w \in S^0(X)$ and $a < 0$, if $C = \sup |w|$, then

$$\begin{aligned} |x^{-a} \rho^\beta v| &= |x^{-a} \int_1^x s^{a-1} w(s,y)\, ds| \leq C |x^{-a} \int_1^x s^{a-1}\, ds| \\ &= C|1 - x^{-a}| \\ &\leq C. \end{aligned}$$

Thus, $v \in x^a \rho^{-\beta} L^\infty(X)$. Since $x\partial_x v(x,y) = u(x,y)$, $x\partial_x v \in x^a S^{(0,\beta)}(X)$, and so $v \in x^a S^{(0,\beta)}(X)$. To prove (2) is similar. □

**Lemma 58.** *Let $a \in \mathbb{R}$ with $a < E$ and suppose that for some $N \in \mathbb{N}_0$, $u \in S^\beta(X)$ satisfies $P_N^E u \in x^a S^{(0,\beta_Y)}(X)$. Then $u \in x^a S^{(0,\beta_Y)}(X)$.*

*Proof.* Step 1. We prove the following statement: If $z \in \mathbb{C}$ with $\operatorname{Re} z > a$ and $u \in S^\beta(X)$ satisfies

$$(x\partial_x - z)^{k+1} u = v \in x^a S^{(0,\beta_Y)}(X) \tag{77}$$

for some $k \in \mathbb{N}_0$, then $u \in x^a S^{(0,\beta_Y)}(X)$.

To see this, observe that as operators, $x^{-z} \circ (x\partial_x - z) = x\partial_x \circ x^{-z}$. Hence, multiplying both sides of (77) on the left by $x^{-z}$ yields

$$(x\partial_x)^{k+1}(x^{-z} u) = x^{-z} v. \tag{78}$$

Observe that $x^{-z} v \in x^{a'} S^{(0,\beta_Y)}(X)$, where $a' = a - \operatorname{Re} z < 0$. Hence, integrating both sides of (78) from 1 to $x$ with respect to the measure $\frac{dx}{x}$, $k+1$ times, using (1) of Lemma 57 at each step, yields $x^{-z} u \in x^{a'} S^{(0,\beta_Y)}(X)$. Hence, $u \in x^z x^{a'} S^{(0,\beta_Y)}(X) = x^a S^{(0,\beta_Y)}(X)$, and Step 1 is proved.

Step 2. Finish the proof. We can write $P_N^E = (x\partial_x - z_1)^{k_1+1} \cdots (x\partial_x - z_l)^{k_l+1}$, where $a < \operatorname{Re} z_1 < \operatorname{Re} z_2 < \cdots < \operatorname{Re} z_l \leq N$ and where $k_i \in \mathbb{N}_0$ for each $i$. We are given $(x\partial_x - z_1)^{k_1+1} \cdots (x\partial_x - z_l)^{k_l+1} u \in x^a S^{(0,\beta_Y)}(X)$. For $m = 1, \ldots, l$, define $u_m := (x\partial_x - z_m)^{k_m+1} \cdots (x\partial_x - z_l)^{k_l+1} u$, and define $u_{l+1} := u$. We will prove that $u_m \in x^a S^{(0,\beta_Y)}(X)$ by induction on $m$. The case $m = 1$ is true by assumption. Thus, assume that $u_m \in x^a S^{(0,\beta_Y)}(X)$ for some $2 \leq m \leq l$; we'll show that $u_{m+1} \in x^a S^{(0,\beta_Y)}(X)$. Observe that $(x\partial_x - z_m)^{k_m+1} u_{m+1} = u_m \in x^a S^{(0,\beta_Y)}(X)$. Hence by Step 1, $u_{m+1} \in x^a S^{(0,\beta_Y)}(X)$. Thus, our induction step is proved and so in particular $u = u_{l+1} \in x^a S^{(0,\beta_Y)}(X)$. □

**Lemma 59.** *Let $v \in x^a S^{(0,\beta)}(X)$ and let $N \in \mathbb{N}$ with $N < a$. Then there exists a $u \in x^a S^{(0,\beta)}(X)$ such that $P_N^E u = v$.*

*Proof.* We first prove the following statement: If $z \in \mathbb{C}$ with $\operatorname{Re} z < a$, then given any $k \in \mathbb{N}_0$, there exists a $u \in x^a S^{(0,\beta)}(X)$ such that

$$(x\partial_x - z)^{k+1} u = v. \tag{79}$$

To see this, let us assume that $u$ exists and work *formally* to try to find a formula for it. First, we multiply both sides of (79) on the left by $x^{-z}$ to get

$$(x\partial_x)^{k+1}(x^{-z}u) = x^{-z}v. \tag{80}$$

Observe that $x^{-z}v \in x^{a'} S^{(0,\beta)}(X)$, where $a' = a - \operatorname{Re} z > 0$. Second, we integrate both sides of (80) from 0 to $x$ with respect to the measure $\frac{dx}{x}$, $k+1$ times, to get $x^{-z}u = w$, where

$$w = \int_0^x \int_0^{x_1} \cdots \int_0^{x_k} x_{k+1}^{-z}\, v(x_{k+1}, y) \frac{dx_{k+1}}{x_{k+1}} \cdots \frac{dx_1}{x_1}. \tag{81}$$

Since $a' > 0$, Part (2) of Lemma 57 implies that each iterated integral in (81) is an element of $x^{a'} S^{(0,\beta)}(X)$. In particular, $w \in x^{a'} S^{(0,\beta)}(X)$. Now we can just simply *define* $u := x^z w \in x^a S^{(0,\beta)}(X)$, then one can check that (79) holds.

To finish up the proof of this lemma, one can use a similar induction scheme as was used in Step 2 of Lemma 58. $\square$

PROOF OF THEOREM 1:

*Proof.* Let $u \in S^{\alpha}(X)$ and suppose that for each $N \in \mathbb{N}$, there exists an $M \in \mathbb{N}$ with $P^E_M u \in x^N S^{(0,\alpha_Y)}(X)$. We will show that $u \in \mathcal{A}^E(X)$.

Let $N \in \mathbb{N}$ and choose $a \in \mathbb{R}$ with $a > N$ such that there does not exist any $(z,k) \in E$ with $N < \operatorname{Re} z \le a$. Now by assumption, there exists an $M \in \mathbb{N}$ with $M > N$ such that $P^E_M u \in x^a S^{(0,\alpha_Y)}(X)$. Let $F$ be the index set $F = \{(z,k) \in E \,|\, a < \operatorname{Re} z \le M\}$. Then, $P^E_M u = P^F_M(P^E_N u) \in x^a S^{(0,\alpha_Y)}(X)$. Since $F - a > 0$, Lemma 58 implies that $P^E_N u \in x^a S^{(0,\alpha_Y)}(X)$. Then, since $N - a < 0$, Lemma 59 implies that there exists a $v \in x^a S^{(0,\alpha_Y)}(X)$ such that $P^E_N v = P^E_N u$. It follows that $P^E_N(u - v) \equiv 0$. Hence, by Lemma 56,

$$u = \sum_{(z,k)\in E,\, \operatorname{Re} z \le N} x^z (\log x)^k u_{(z,k)} + v,$$

where $u_{(z,k)} \in S^{\alpha_Y}(Y)$. Our theorem is proved. $\square$

PROOF OF PROPOSITION 2:

*Proof.* Fix $N \in \mathbb{N}$. Then we claim that there exists $u_{(z,k)} \in S^{\beta_Y}(Y)$ such that

$$u - \sum_{(z,k)\in E,\, \operatorname{Re} z \le N} x^z (\log x)^k u_{(z,k)} \in x^N S^{(0,\beta_Y)}(X). \tag{82}$$

Since $N$ is arbitrary, this will prove that $u$ has an asymptotic expansion at $Y$ with index set $E$. To prove (82), let $N_0 > N$. Then by (6), we can choose an $M \in \mathbb{N}$ such that $M > N$, and such that

$$u - \sum_{(z,k)\in E,\, \operatorname{Re} z \le M} x^z (\log x)^k u_{(z,k,N_0)} \in x^{N_0} S^{(0,\beta_Y)}_{N_0}(X), \tag{83}$$

where $u_{(z,k,N_0)} \in S^{\beta_Y}_{N_0}(Y)$. Splitting the sum in (83) into a sum with $\operatorname{Re} z \le N$ and another sum with $N < \operatorname{Re} z \le M$, we find that

$$u - \sum_{(z,k)\in E,\, \operatorname{Re} z \le N} x^z (\log x)^k u_{(z,k,N_0)} \in x^{N} S^{(0,\beta_Y)}_{N_0}(X).$$

Since $N_0$ was arbitrary, (82) follows. $\square$

We end this appendix by proving the coordinate independence of asymptotic expansions.

**Proposition 60.** *Let $F : X \to X$ be a diffeomorphism preserving $Y$ and let $u \in \mathcal{A}^E(X)$. Then $F^* u \in \mathcal{A}^E(X)$.*

*Proof.* Let $G : X \to X$ be the map $(x, y) \mapsto (x, F|_Y(y))$. Then $G$ is a diffeomorphism of $X$. Now we can write $F = H \circ G$, where $H := F \circ G^{-1}$. Hence, $F^*u = G^*(H^*u)$. We first show that $H^*u \in \mathcal{A}^E(X)$. Indeed, for each $x \in [0, 1)$, define $H_x : Y \to Y$ by $H_x(y) := \pi H(x, y)$ for all $y \in Y$, where $\pi : X \to Y$ is the projection onto $Y$. Then, since $H|_Y = \mathrm{Id}_Y$, for $x$ near 0, $H_x$ is a 1-parameter family of diffeomorphisms of $Y$ with $H_0 = \mathrm{Id}_Y$. Thus, for $x$ near 0, there exists a 1-parameter family of smooth vector fields $x \mapsto v_x$ such that

$$\partial_x H_x(y) = v_x(H_x(y)) \text{ for all } y \in Y. \tag{84}$$

Observe that (84) implies that

$$v_x\phi = \partial_x(H_x^*\phi) \text{ for all } \phi \in C^\infty(Y). \tag{85}$$

We claim that $v_x$ is a 1-parameter family of $b$-vector fields. Indeed, if $\rho$ is a boundary defining function for some boundary hypersurface of $Y$, then as $H|_Y = \mathrm{Id}_Y$ and $H_x$ is a 1-parameter family of diffeomorphisms, for $x$ near 0, we must have $H_x^*\rho = \rho\, b(x, y)$ for some $b(x, y) > 0$. Hence, $v_x\rho = \partial_x(H_x^*\rho) = \rho\, \partial_x b$. Thus, $v_x \in \mathcal{V}_b(Y)$.

Let $N \in \mathbb{N}$ and assume that $u \in S^\alpha(X)$. Then we can write

$$u(x, y) = \sum_{(z,k)\in E,\, \mathrm{Re}\, z \le N} x^z (\log x)^k u_{(z,k)} + u_N(x, y),$$

for some $u_{(z,k)} \in S^{\alpha_Y}(Y)$ and $u_N(x, y) \in x^{a_N} S^{(0,\alpha_Y)}(X)$, where $a_N$ can be made as large as we wish by choosing $N$ larger. Now, since $H^*x$ is a multiple of $x$ by a smooth positive function, it follows that

$$H^*u = \sum_{(z,k)\in E,\, \mathrm{Re}\, z \le N} \Big(\sum_{l=0}^{k} x^z (\log x)^l a^{(l)}_{(z,k)}(x, y)\Big) H_x^* u_{(z,k)} + H^*u_N, \tag{86}$$

for some $a^{(l)}_{(z,k)}(x, y) \in C^\infty(X)$. Observe that (85) implies that for each $j \in \mathbb{N}_0$, $\partial_x^j(H_x^* u_{(z,k)}) = H_x^*(v_x^j u_{(z,k)})$. Hence, as $H_0 = \mathrm{Id}_Y$, Taylor's Theorem implies that for $x$ near 0, for any $M \in \mathbb{N}$,

$$H_x^* u_{(z,k)} = \sum_{j=0}^{M-1} x^j (v_0^j u_{(z,k)}) + x^M u^M_{(z,k)}(x, y), \tag{87}$$

where $u^M_{(z,k)}(x, y) = \frac{1}{(M-1)!}\int_0^1 (1-s)^{M-1} H_{xs}^*(v_{xs}^M u_{(z,k)})ds$. Note that since $v_x$ is a 1-parameter family of $b$-vector fields for $x$ near 0, $v_0^j u_{(z,k)} \in S^{\alpha_Y}(Y)$ for each $j$, and $u^M_{(z,k)} \in S^{(0,\alpha_Y)}(X)$ near $x = 0$. Also, taking the Taylor series of $a^{(l)}_{(z,k)}(x, y)$ at $x = 0$ yields

$$a^{(l)}_{(z,k)}(x, y) = \sum_{j=0}^{M-1} x^j a^{(l)}_{(z,k,j)}(y) + x^M a^{(l)}_{(z,k,M)}(x, y), \tag{88}$$

where $a^{(l)}_{(z,k,j)}(y)$ and $a^{(l)}_{(z,k,M)}(x, y)$ are smooth. Thus, plugging (87) and (88) into (86) and choosing $M$ sufficiently large, it follows that we can write

$$H^*u = \sum_{(z,k)\in E,\, \mathrm{Re}\, z \le N} x^z (\log x)^k \tilde{u}_{(z,k)} + \tilde{u}_N,$$

where $\tilde{u}_{(z,k)} \in S^{\alpha_Y}(Y)$ and $\tilde{u}_N \in x^{a_N} S^{(0,\alpha_Y)}(X)$. Thus, as $F^*u = G^*H^*u$ and $G(x, y) = (x, F|_Y(y))$, we have

$$F^*u = G^*H^*u = \sum_{(z,k)\in E,\, \mathrm{Re}\, z \le N} x^z (\log x)^l F|_Y^* \tilde{u}_{(z,k)} + G^*\tilde{u}_N. \tag{89}$$

Hence, $F^*u \in \mathcal{A}^E(X)$. □

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MIT, Department of Mathematics, Cambridge, MA 02139, USA
*E-mail address*: `ploya@alum.mit.edu`