Residue Traces and Kernel Expansions of Pseudodifferential Operators

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Abstract

We survey and examine the methodologies of Connes–Moscovici and Kontsevich–Vishik in defining the Wodzicki residue density and canonical density for pseudodifferential operators on closed compact manifolds.
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1. Introduction

The Wodzicki residue and Kontsevich–Vishik densities arise when defining traces on pseudodifferential operators. Here, by a "trace" we mean a functional which vanishes on commutators. Consider a classical pseudodifferential operator A of real order m on a closed compact manifold X. If $m < -n$, where $n = \dim X$, then A is trace class on L^2, and $\mathrm{Tr}(A) = \int_X K_A|_{\mathrm{Diag}}$, where Diag is the diagonal of X^2 and $K_A|_{\mathrm{Diag}}$ is the Schwartz kernel of A restricted to the diagonal. To define a trace for $m \geq -n$, we can follow two roads. One road is to regularize (that is, somehow give a meaning to) the integral $\int_X K_A|_{\mathrm{Diag}}$. This was the original approach of Wodzicki [1, 2] in defining his noncommutative residue; he used the theory of complex powers, due to Seeley [3]. We give a synopsis of his residue trace in Section 4. The other approach is to regularize the density $K_A|_{\mathrm{Diag}}$ itself. This is the approach taken by Connes–Moscovici [4] when $m \in \mathbb{Z}$, and Kontsevich–Vishik [5, 6] in the case that $m \notin \mathbb{Z}$. This is the approach that we will focus on in this paper.

Let $\mathcal{U} \cong \mathbb{R}^n_x$ be a coordinate patch. Then on $\mathcal{U} \times \mathcal{U} \cong \mathbb{R}^n_x \times \mathbb{R}^n_y$, the Schwartz kernel of A can be written as an inverse Fourier transform:

$$K_A(x,z) = \int_{\mathbb{R}^n} e^{iz\cdot\xi}\, a(x,\xi)\, đ\xi, \qquad z = x - y. \tag{1.1}$$

Here, $a(x,\xi)$ is a classical symbol of order m. Since the singular support of K_A is at the diagonal $z = 0$ (see [7]), $K_A(x,z)$ is smooth for $z \neq 0$, and is singular at $z = 0$. To define "$K_A|_{\mathrm{Diag}}$", we determine the asymptotics of $K_A(x,z)$ as $|z| \to 0$.

Let $z = r\gamma$ where $r = |z|$ and $\gamma = z/|z|$ be polar coordinates about the diagonal $z = 0$. We prove that $K_A(x, r\gamma)$ has the following asymptotic expansion (see Theorem 3.1):

$$K_A(x,r\gamma) \sim_{r\to 0} \sum_{k\in\mathbb{N}_0,\, k-m-n\notin\mathbb{N}_0} \alpha_k(x,\gamma)\, r^{k-m-n} + \sum_{k=0}^{\infty} \beta_k(x,\gamma)\, r^k \log r + \sum_{k=0}^{\infty} \gamma_k(x,\gamma)\, r^k, \tag{1.2}$$

where $\alpha_k, \beta_k, \gamma_k \in C^\infty(\mathcal{U} \times \mathbb{S}^{n-1})$. Moreover, when considered as densities on X, among the coefficients α_k, β_k, and γ_k, there is *exactly one* that is invariant under coordinate changes: For $m \in \mathbb{Z}$, this coefficient is β_0, which happens to be minus the Wodzicki residue density, and for $m \notin \mathbb{Z}$, it is γ_0, which is the Kontsevich–Vishik density. In this sense, the Wodzicki and Kontsevich–Vishik densities are the only "natural" densities that can be associated to an operator in order to define "$K|_{\mathrm{Diag}}$". We will discuss these densities shortly.

The fact that K_A has the expansion (1.2), and the facts that the log coefficient (minus the Wodzicki density) when $m \in \mathbb{Z}$ and the constant coefficient (the Kontsevich–Vishik density) when $m \notin \mathbb{Z}$ are coordinate invariant, are not new. Indeed, these results are already in the works of Connes–Moscovici [4] and Kontsevich–Vishik [5, 6]. We especially point out the work of Beals–Greiner [8], where they prove the expansion (1.2) for kernels of $\mathcal{V}$-pseudodifferential operators which are more general than the operators we consider. The only novelty that we profess here is to unify the presentations of the Wodzicki and Kontsevich–Vishik densities and to go one step further in saying that these densities are precisely the coordinate invariant ones in the expansion of K_A. We also mention that we achieve the same expansion (1.2) for a general class of "h-transforms" of classical symbols; the Fourier transform which characterizes the kernels of pseudodifferential operators being a special case. Our proof of (1.2) (for general h-transforms) is of a different flavor than those of [4, 5, 6, 8] in that we use Mellin transform methods. Similar techniques can be found in the works of Brüning–Seeley [9], Grubb–Seeley [10], and Lesch [11], amongst others. Using these methods, we attain explicit formulas for all the α_k and β_k coefficients of (1.2) in terms of the homogeneous components in the symbol expansion of $a(x,\xi)$, and an explicit formula for γ_0. In particular, β_0 is always minus the Wodzicki density and γ_0 is always the Kontsevich-Vishik density for general h-transforms. The explicit formula for γ_0 is especially useful in deriving properties of the Kontsevich–Vishik density; principally, its relation to the regularized integral of Lesch and to the Wodzicki residue density, cf. Lemma 3.2.

Consider the expansion (1.2) for $m \notin \mathbb{Z}$. Then it turns out that the log terms drop out and the constant term γ_0 is the only invariantly defined coefficient. Moreover, this coefficient is computed to be $⨍_{\mathbb{R}^n} a(x,\xi)đ\xi$, the regularized integral of $a(x,\xi)$ as defined by Lesch [11, Sec. 5]; namely, the regularized value of $\int_{|\xi|\le R} a(x,\xi)đ\xi$ as $R\to\infty$. In [5, 6], Kontsevich and Vishik call the constant term γ_0 the canonical trace density, which we denote by $\omega_{KV}(A)$. In [11], Lesch defined the regularized integral $⨍_{\mathbb{R}^n} a(x,\xi)đ\xi$ to be $\omega_{KV}(A)$. Using an analytic continuation argument, see [11, Thm. 5.6], it was shown that these two definitions for the canonical density are the same. In our proof of (1.2), we obtain an explicit formula for the constant term γ_0 proving this equality directly. The integral $\mathrm{TR}(A) := \int_X \omega_{KV}(A)$ is called the Kontesevich–Vishik trace. It is a trace in that $\mathrm{TR}[A,B]=0$ for operators A and B such that $\mathrm{ord}(A)+\mathrm{ord}(B)\notin\mathbb{Z}$. Properties and applications of this trace can be found in [5, 6].

Assume now that $m\in\mathbb{Z}$. Then of all the coefficients of (1.2), the $\log r$ coefficient β_0 is the only invariantly defined coefficient. Moreover, this coefficient is directly computed to be $\beta_0 = -\int_{|\xi|=1} a_{-n}(x,\xi)đ\xi$, which is minus the Wodzicki residue density [1, 2]. Here, $a_{-n}(x,\xi)$ is the homogeneous term of degree $-n$ in the expansion of $a(x,\xi)$. The integral $\mathrm{res}(A) := \int_X \omega_W(A)$, where $\omega_W(A)$ is the residue density, is called the Wodzicki residue trace or noncommutative residue; it is a (in fact, the unique) trace on the algebra of integer order pseudodifferential operators. Connes–Moscovici [4] (see also Beals–Greiner [8]) were the first to observe the relation between β_0 and $\omega_W(A)$. Applications of the Wodzicki residue can be found in Wodzicki's paper [2], and in the books by Connes [12] and Landi [13].

In Section 2 we prove a fundamental lemma that will allow us to find explicit formulas for the coefficients in an asymptotic expansion. In Section 3, we apply the results of Section 2 to define the Wodzicki residue and Kontesevich–Vishik densities and we also prove some properties of these densities. To end this paper, in Section 4 we give a brief survey of the various ways in which the Wodzicki residue has been characterized.

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2. Expansions of Integrals on $\mathbb{R}^n$

Given a function $f(x)\in C^\infty((0,\infty))$ that vanishes to infinite order at $x=\infty$ and that is $O(x^b)$ at $x=0$, we denote by $\mathcal{M}(f)(\tau)$ the Mellin transform of f:

$$\mathcal{M}(f)(\tau) := \int_0^\infty x^{-i\tau} f(x)\frac{dx}{x} \quad\longleftrightarrow\quad f(x) = \frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic} x^{i\tau}\mathcal{M}(f)(\tau)d\tau. \tag{2.1}$$

The left-hand integral converges for $\mathrm{Im}\,\tau > b$, and the right-hand integral, for any $c>b$. See the book by Bleistein–Handelsman [14] for a detailed treatment of Mellin transforms.

The following simple lemma is key to proving the existence of expansions.

Lemma 2.1 (The Fundamental Lemma) *Let $f(x)\in C^\infty((0,\infty))$ vanish to infinite order as $x\to\infty$, and suppose that for some $z\in\mathbb{C}$, we have*

$$(x\partial_x - z)f(x) = g(x), \tag{2.2}$$

where $g(x)\in C^\infty([0,\infty))$. Then we can write

$$f(x) = \alpha\, x^z + \beta\, x^z\log x + \gamma(x),$$

where $\alpha,\beta\in\mathbb{C}$ and $\gamma(x)\in C^\infty([0,\infty))$. Moreover, if $z\notin\mathbb{N}_0$, then $\beta=0$ and $\alpha=-\mathcal{M}(g)(-iz)$, and if $z\in\mathbb{N}_0$, then $\alpha=0$ and $\beta=\frac{1}{z!}\partial_x^z g(0)$. Furthermore, if $z\neq0$, then $\gamma(0)=-\frac{1}{z}g(0)$.

Proof: To begin, we note that since g is smooth at $x=0$ and $\int_0^1 x^{-i\tau+k}\frac{dx}{x} = 1/(-i\tau+k)$, it follows that $\mathcal{M}(g)(\tau)$ has poles for $\tau \in -i\mathbb{N}_0$ with residue at $\tau = -ik$ given by $i\frac{1}{k!}\partial_x^k g(0)$. Now, taking the Mellin transform of (2.2) gives $(i\tau - z)\mathcal{M}(f)(\tau) = \mathcal{M}(g)(\tau)$. Thus, $\mathcal{M}(f)(\tau)$ has poles only at $\tau = -iz$ and $\tau \in -i\mathbb{N}_0$. Hence, for any $c > \max\{0, -\mathrm{Im}\, z\}$, we can write $f(x)$ as the inverse transform given in (2.1). Choose any $c' \in \mathbb{R}\setminus\mathbb{Z}$ with $c' < -\mathrm{Im}\, z$. Assume that $z \notin \mathbb{N}_0$. Then by Cauchy's theorem and the fact that $\mathcal{M}(g)(\tau)$ has poles at $\tau = -ik$, where $k \in \mathbb{N}_0$, given by $i\frac{1}{k!}\partial_x^k g(0)$, shifting to the contour $\mathrm{Im}\,\tau = c'$ yields

$$f(x) = -x^z \mathcal{M}(g)(-iz) - \sum_{0\le k<c'} \frac{x^k}{-ik+iz}\, i\frac{1}{k!}\partial_x^k g(0) + \frac{1}{2\pi i}\int_{\mathrm{Im}\,\tau=c'} x^{i\tau}\frac{\mathcal{M}(g)(\tau)}{\tau+iz}d\tau. \tag{2.3}$$

The last two terms combine to make a function $\gamma(x)$. Note that continuing to move the contour $\mathrm{Im}\,\tau = c'$ down to $\mathrm{Im}\,\tau = -\infty$ implies that $\gamma(x) \sim -\sum_{k=0}^{\infty} x^k\,\partial_x^k g(0)/(k!(-k+z))$ as $x \to 0$. Thus, $\gamma(x)$ is smooth and $\gamma(0) = -\frac{1}{z}g(0)$. Thus, our lemma is proved for $z \notin \mathbb{N}_0$.

Assume now that $z \in \mathbb{N}_0$. Then the only change in the above argument is that $\tau = -iz$ is now a (possible) double pole. This double pole accounts for an added log term in (2.3):

$$f(x) = -x^z g_0(-iz) - i\, x^z \log x\, g_1 + \sum_{0\le k<c',k\neq z} \frac{x^k\, i\partial_x^k g(0)}{k!(-ik+iz)} + \frac{1}{2\pi i}\int_{\mathrm{Im}\,\tau=c'} x^{i\tau}\frac{\mathcal{M}(g)(\tau)}{\tau+iz}d\tau,$$

where $\mathcal{M}(g)(\tau) = g_0(\tau) + g_1/(\tau+iz)$, where $g_0(\tau)$ is regular at $\tau = -iz$ and where $g_1 = i\frac{1}{z!}\partial_x^z g(0)$. Let $\gamma(x)$ be the sum of all the terms on the right-hand side of $f(x)$ except for the log term. Then by the same argument as we used in the case that $z \notin \mathbb{N}_0$, it follows that $\gamma(x) \in C^\infty([0,\infty))$ and that $\gamma(0) = -\frac{1}{z}g(0)$ if $z \neq 0$. Since $g_1 = i\frac{1}{z!}\partial_x^z g(0)$, our lemma is proved for $z \in \mathbb{N}_0$. □

For the rest of this paper, fix a function $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ such that $\chi(\xi) \equiv 0$ near 0 and $\chi(\xi) \equiv 1$ outside a neighborhood of 0. Let $a(\xi) \in C^\infty_{\mathrm{hom}(-n-m)}(\mathbb{R}^n)$ where $m \in \mathbb{R}$; thus, $a(\xi)$ is smooth away from $\xi = 0$ and $a(\delta\xi) = \delta^{-n-m}a(\xi)$ for all $\delta > 0$. Let $h(\xi) \in C^\infty(\mathbb{R}^n)$ be a smooth bounded function and set

$$I(r) = \int_{\mathbb{R}^n} h(r\xi)\,\chi(\xi)a(\xi)\,đ\xi, \qquad đ\xi = \frac{1}{(2\pi)^n}d\xi. \tag{2.4}$$

In general, we will assume that $h(0) = 1$ and that $h(\xi) \to 0$ to infinite order as $|\xi| \to \infty$, with the exception of *Example 2* below. For instance, $h(\xi) = e^{-|\xi|^2}$ gives a Laplace type transform. The functions $h(\xi)$ of most relevance to this paper are the following.

Example 1: $h(\xi)$ is such that $h(\xi) \equiv 1$ near 0 and $h(\xi) \equiv 0$ outside a neighborhood of 0. In this case, the integral (2.4) is essentially an integral over the region $\frac{1}{r}\cdot \operatorname{supp} h$. Then the constant term in the expansion of $I(r)$ as $r \to 0$ represents the regularized integral of $\chi(\xi)a(\xi)$ over $\mathbb{R}^n$ as discussed by Lesch [11, Sec. 5].

Example 2: $h(\xi) = e^{i\gamma\cdot\xi}$ (where $\gamma \in \mathbb{S}^{n-1}$ is fixed). Then the integral (2.4) must be interpreted as an oscillatory integral, see [7, Sec. 7.8]. The asymptotics of $I(r)$ as $r \to 0$ now represent the asymptotics of the Schwartz kernel of a pseudodifferential operator at the diagonal as discussed in the introduction around Equation (1.1).

Lemma 2.2 *For any given $h(\xi)$ as described above, we can write*

$$I(r) = \int h(r\xi)\,\chi(\xi)\,a(\xi)đ\xi = \alpha\, r^m + \beta\, r^m \log r + \gamma(r), \tag{2.5}$$

where $\alpha, \beta \in \mathbb{C}$, and $\gamma(r) \in C^\infty([0,\infty))$. Moreover, if $m \notin \mathbb{N}_0$, then $\beta = 0$ and

$$\alpha = \begin{cases} \int_{\mathbb{R}^n} h(\xi)\,a(\xi)đ\xi & \text{if } m<0; \\ \int_{\mathbb{R}^n}\{h(\xi) - \sum_{j=0}^{\lfloor m\rfloor}\frac{1}{j!}\partial_r^j h(r\xi)|_{r=0}\}\,a(\xi)đ\xi & \text{if } m>0, \end{cases} \tag{2.6}$$

where $\lfloor\cdot\rfloor$ is the greatest integer function; and if $m \in \mathbb{N}_0$, then $\alpha = 0$ and

$$\beta = -\frac{1}{m!}\int_{|\xi|=1}\{\partial_r^m h(r\xi)|_{r=0}\}\,a(\xi)đ\xi. \tag{2.7}$$

Finally, if $m \neq 0$, then

$$\gamma(0) = \frac{1}{m}\int_{\mathbb{R}^n} (\xi \cdot \partial_\xi \chi)(\xi)\, a(\xi) đ\xi, \quad \xi \cdot \partial_\xi = \sum_{i=1}^{n} \xi_i\, \partial_{\xi_i}. \tag{2.8}$$

Remarks: It is interesting to note that there are only two coefficients in the expansion (2.5) of $I(r)$ which are completely *independent* of the choice of h; namely for $m = 0$, the $r^0 \log r$ coefficient β, and for $m \neq 0$, the constant term $\gamma(0)$. Indeed, by studying the proofs of Lemmas 2.1 and 2.2, one can see that for $m \neq 0$, all derivatives $\partial_r^j \gamma(0)$, $j > 0$, depend on $h(\xi)$; and if $m = 0$, then all derivatives $\partial_r^j \gamma(0)$, $j \geq 0$, depend on $h(\xi)$. The coefficients which are independent of h are the coefficients that have natural interpretations on manifolds. If $h(\xi) = e^{i\gamma\cdot\xi}$, then the integral defining α for $m < 0$ in (2.6) must be interpreted as an oscillatory integral. Note that since $a(\xi)$ is of the order $|\xi|^{-n-m}$, for $m < 0$, $h(\xi)a(\xi)$ is integrable near $\xi = 0$; it is integrable near $\xi = \infty$ as an oscillatory integral. The integral defining α for $m > 0$ is well-defined regardless of the choice of $h(\xi)$.

Proof of Lemma 2.2: The change of variables $\xi \mapsto r^{-1}\xi$ and the homogeneity property of $a(\xi)$ imply that $I(r) = r^m \int h(\xi)\, \chi(\xi/r)\, a(\xi) đ\xi$. Since $(r\partial_r - m)r^m = 0$ and $r\partial_r \chi(\xi/r) = -(\xi \cdot \partial_\xi \chi)(\xi/r)$, we have

$$(r\partial_r - m)I(r) = -r^m \int h(\xi)\, (\xi \cdot \partial_\xi \chi)(\xi/r)\, a(\xi) đ\xi.$$

Changing back again, $\xi \mapsto r\xi$, we find that $(r\partial_r - m)I(r) = g(r)$, where

$$g(r) = -\int h(r\xi)\, (\xi \cdot \partial_\xi \chi)(\xi)\, a(\xi) đ\xi.$$

Since $(\xi \cdot \partial_\xi \chi)(\xi) \equiv 0$ near $\xi = 0$ and outside a neighborhood of 0, the integral defining $g(r)$ is over a compact subset of $\mathbb{R}^n \setminus \{0\}$. Hence, $g(r) \in C^\infty([0,\infty))$. Thus, as $g(0) = -\int (\xi \cdot \partial_\xi \chi)(\xi)\, a(\xi) đ\xi$, the expansion (2.5) and the formula for $\gamma(0)$ follow from Lemma 2.1. It remains to compute formulas (2.6) and (2.7) for α and β.

First, assume that $m \in \mathbb{N}_0$. Then, by Lemma 2.1, $\alpha = 0$ and $\beta = \frac{1}{m!}\partial_r^m g(r)|_{r=0}$. We can compute β using polar coordinates $\xi = \rho\gamma$ ($\rho = |\xi|$, $\gamma = \xi/|\xi|$), the homogeneity property of $a(\xi)$, $a(\rho\gamma) = \rho^{-n-m} a(\gamma)$, and the fact that $\xi \cdot \partial_\xi = \rho\partial_\rho$:

$$\begin{aligned}
\frac{1}{m!}\partial_r^m g(r)|_{r=0} &= -\frac{1}{m!}\int \{\partial_r^m h(r\xi)|_{r=0}\}\, (\xi \cdot \partial_\xi \chi)(\xi)\, a(\xi) đ\xi \\
&= -\frac{1}{m!}\int_{\mathbb{S}^{n-1}} \{\partial_r^m h(r\gamma)|_{r=0}\} \int_0^\infty \partial_\rho \chi(\rho\gamma) d\rho\, a(\gamma) đ\gamma \\
&= -\frac{1}{m!}\int_{\mathbb{S}^{n-1}} \{\partial_r^m h(r\gamma)|_{r=0}\}\, a(\gamma) đ\gamma,
\end{aligned} \tag{2.9}$$

where we used that $\int_0^\infty \partial_\rho \chi(\rho\gamma) d\rho = \chi(\infty) - \chi(0) = 1$.

Assume now that $m \notin \mathbb{N}_0$. Then by Lemma 2.1, $\alpha = -\mathcal{M}(g)(-im)$. We will compute $\mathcal{M}(g)(-im)$ under the cases that $m < 0$ or $m > 0$. Assume first that $m < 0$ and let $\varepsilon > 0$. Then using polar coordinates $\xi = \rho\gamma$ and that $a(\rho\gamma) = \rho^{-n-m}a(\gamma)$, yield

$$\begin{aligned}
\int_0^{\frac{1}{\varepsilon}} r^{-m-1} g(r)\, dr &= -\int_0^{\frac{1}{\varepsilon}} \int_0^\infty \int_{\mathbb{S}^{n-1}} h(r\rho\gamma)\rho\partial_\rho \chi(\rho\gamma)\, (\rho r)^{-m-1}\, a(\gamma) đ\gamma\, d\rho\, dr \\
&= -\int_{\mathbb{S}^{n-1}} \int_0^\infty \int_0^{\frac{\rho}{\varepsilon}} h(r\gamma)\partial_\rho \chi(\rho\gamma)\, r^{-m-1}\, dr\, d\rho\, a(\gamma) đ\gamma \ (r \mapsto r/\rho).
\end{aligned}$$

Interchanging the r and ρ integrals and using the homogeneity property of $a(\xi)$ give

$$\begin{aligned}
\int_0^{\frac{1}{\varepsilon}} r^{-m-1} g(r)\, dr &= -\int_{\mathbb{S}^{n-1}} \int_0^\infty \int_{\varepsilon r}^\infty h(r\gamma)\partial_\rho \chi(\rho\gamma)\, r^{-m-1}\, d\rho\, dr\, a(\gamma) đ\gamma \\
&= -\int_{\mathbb{S}^{n-1}} \int_0^\infty h(r\gamma)(1 - \chi(\varepsilon r\gamma))\, a(r\gamma) r^{n-1} dr\, đ\gamma \\
&= -\int h(\xi)\, (1 - \chi(\varepsilon\xi))\, a(\xi) đ\xi.
\end{aligned}$$

Letting $\varepsilon \to 0$ proves that $\mathcal{M}(g)(-im) = -\int h(\xi)a(\xi)đ\xi$ for $m < 0$.

Finally, assume that $m > 0$ but $m \notin \mathbb{N}$. Let $\varepsilon > 0$. Then the same argument used in the previous paragraph implies that

$$\int_\varepsilon^\infty r^{-i\tau-1} g(r)dr|_{\tau=-im} = \int h(\xi)\chi(\xi/\varepsilon)\, a(\xi)đ\xi. \tag{2.10}$$

Expressing $g(r) = \sum_{j=0}^{\lfloor m\rfloor} r^j \partial_r^j g(0)/j! + r^{\lfloor m\rfloor+1}R(r)$ in Taylor expansion and using the fact that $\int_0^\varepsilon r^{-i\tau+j-1}dr = \varepsilon^{-i\tau+j}/(-i\tau+j)$ give

$$\int_0^\varepsilon r^{-i\tau-1} g(r)dr|_{\tau=-im} = \sum_{j=0}^{\lfloor m\rfloor} \frac{\varepsilon^{-m+j}}{-m+j}\frac{1}{j!}\partial_r^j g(0) + \int_0^\varepsilon r^{-m+\lfloor m\rfloor} R(r)dr. \tag{2.11}$$

Switching to polar coordinates $\xi = \rho\gamma$, and using the homogeneity property of $a(\xi)$, integration by parts, and the fact that $-m+j<0$ for $j < \lfloor m\rfloor$, yield

$$\begin{aligned}
\partial_r^j g(0) \;&=\; -\int_0^\infty \int_{\mathbb{S}^{n-1}} \{\partial_r^j h(r\rho\gamma)|_{r=0}\}\rho\partial_\rho\chi(\rho\gamma)\,\rho^{-n-m}\,a(\gamma)đ\gamma\,\rho^{n-1}d\rho \\
&= -\int_{\mathbb{S}^{n-1}} \{\partial_r^j h(r\gamma)|_{r=0}\}\int_0^\infty \partial_\rho\chi(\rho\gamma)\,\rho^{-m+j}d\rho\, a(\gamma)đ\gamma \\
&= (-m+j)\int_{\mathbb{S}^{n-1}} \{\partial_r^j h(r\gamma)|_{r=0}\}\int_0^\infty \chi(\rho\gamma)\,\rho^{-m+j-1}d\rho\, a(\gamma)đ\gamma \\
&= \frac{-m+j}{\varepsilon^{-m+j}}\int_{\mathbb{S}^{n-1}} \{\partial_r^j h(r\gamma)|_{r=0}\}\int_0^\infty \chi(\rho\gamma/\varepsilon)\,\rho^{-m+j-1}d\rho\, a(\gamma)đ\gamma \quad (\rho\mapsto\rho/\varepsilon) \\
&= \frac{-m+j}{\varepsilon^{-m+j}}\int_{\mathbb{S}^{n-1}}\int_0^\infty \{\partial_r^j h(r\rho\gamma)|_{r=0}\}\chi(\rho\gamma/\varepsilon)\; a(\rho\gamma)\rho^{n-1}d\rho đ\gamma \\
&= \frac{-m+j}{\varepsilon^{-m+j}}\int \{\partial_r^j h(r\xi)|_{r=0}\}\chi(\xi/\varepsilon)a(\xi)đ\xi.
\end{aligned}$$

Hence, by (2.10) and (2.11), we have

$$\begin{aligned}
\mathcal{M}(g)(-im) &= \left(\int_0^\varepsilon r^{-i\tau-1}g(r)dr + \int_\varepsilon^\infty r^{-i\tau-1}g(r)dr\right)\Big|_{\tau=-im} \\
&= -\int_{\mathbb{R}^n}\{h(\xi) - \sum_{j=0}^{\lfloor m\rfloor}\frac{1}{j!}\partial_r^j h(r\xi)|_{r=0}\}\chi(\xi/\varepsilon)a(\xi)đ\xi + \int_0^\varepsilon r^{-m+\lfloor m\rfloor}R(r)dr.
\end{aligned}$$

Letting $\varepsilon \to 0$ gives $\mathcal{M}(g)(-im) = -\int_{\mathbb{R}^n}\{h(\xi) - \sum_{j=0}^{\lfloor m\rfloor}\frac{1}{j!}\partial_r^j h(r\xi)|_{r=0}\}a(\xi)đ\xi$. □

Now let $a(\xi) \in S^m(\mathbb{R}^n)$ be a classical symbol. Consider the integral

$$I(r) = \int_{\mathbb{R}^n} h(r\xi)\,a(\xi)đ\xi,$$

where the properties of $h(\xi)$ are described before Lemma 2.2. Expanding $a(\xi)$ in its homogeneous components, $a(\xi) = \sum_{k=0}^{N-1}\chi(\xi)a_{m-k}(\xi) + r_N(\xi)$, where $N >> 0$, $a_{m-k}(\xi) \in C^\infty_{\mathrm{hom}(m-k)}(\mathbb{R}^n)$, and where $r_N(\xi) \in S^{m-N}(\mathbb{R}^n)$, then applying Lemma 2.2 to each homogeneous term implies the following.

Lemma 2.3 *As $r \to 0$, we have*

$$I(r) \sim \sum_{k\in\mathbb{N}_0,\, k-m-n\notin\mathbb{N}_0} \alpha_k\, r^{k-m-n} + \sum_{k=0}^\infty \beta_k\, r^k\log r + \gamma(r), \tag{2.12}$$

where $\alpha_k, \beta_k \in \mathbb{C}$ and $\gamma(r) \in C^\infty([0,\infty))$. Moreover,

$$\alpha_k = \begin{cases} \int h(\xi)\,a_{m-k}(\xi)đ\xi & \text{if } k-m-n<0; \\ \int\{h(\xi) - \sum_{j=0}^{\lfloor k-m-n\rfloor}\frac{1}{j!}\partial_r^j h(r\xi)|_{r=0}\}\,a_{m-k}(\xi)đ\xi & \text{if } k-m-n>0. \end{cases}$$

If $m \notin \mathbb{Z}$, then $\beta_k = 0$ for all k, and if $m \in \mathbb{Z}$, then

$$\beta_k = -\frac{1}{k!}\int_{|\xi|=1} \{\partial_r^k h(r\xi)|_{r=0}\}\, a_{-n-k}(\xi) đ\xi, \quad k \geq 0.$$

Moreover, if $m \notin \mathbb{Z}$, and we write $a(\xi) = \sum_{k=0}^{N-1} \chi(\xi) a_{m-k}(\xi) + r_N(\xi)$ where N is chosen so that $r_N(\xi) \in S^{m-N}(\mathbb{R}^n)$ is integrable on $\mathbb{R}^n$ (thus, so that $m - N < -n$), then we have

$$\gamma(0) = \sum_{k=0}^{N-1} \frac{1}{-m-n+k}\int_{\mathbb{R}^n} (\xi \cdot \partial_\xi \chi)(\xi)\, a_{m-k}(\xi) đ\xi + \int_{\mathbb{R}^n} r_N(\xi) đ\xi. \tag{2.13}$$

As before, there are two distinguished terms in the expansion of $I(r)$ that are independent of $h(\xi)$. If $m \notin \mathbb{Z}$, then $\gamma(0)$ is independent of $h(\xi)$; it is called the *regularized integral* of $a(\xi)$, denoted by

$$⨍_{\mathbb{R}^n} a(\xi) đ\xi \;:=\; \text{constant term in } I(r) \text{ as } r \to 0 \;=\; (2.13).$$

Note that if $m < -n$, then we may choose $N = 0$, in which case $⨍_{\mathbb{R}^n} a(\xi) đ\xi = \int_{\mathbb{R}^n} a(\xi) đ\xi$ is just the usual integral. If $m \in \mathbb{Z}$, then the value β_0 is independent of $h(\xi)$:

$$\beta_0 := \text{ coefficient of } \log r = -\int_{|\xi|=1} a_{-n}(\xi) đ\xi.$$

3. Expansions of Kernels on Manifolds

Let X be an n-dimensional closed compact manifold. Let $A \in \Psi^m_{c\ell}(X)$ be a classical pseudodifferential operator of order $m \in \mathbb{R}$ and let $\mathcal{U} \cong \mathbb{R}^n_x$ be a coordinate patch on X. Then on $\mathcal{U} \times \mathcal{U}$, the Schwartz kernel of A can be written as an inverse Fourier transform:

$$K_A(x, z) = \int_{\mathbb{R}^n} e^{iz\cdot\xi}\, a(x, \xi) đ\xi, \qquad z = x - y, \tag{3.1}$$

where $a(x, \xi)$ is a classical symbol of order m.

Theorem 3.1 *Let $z = r\gamma$ where $r = |z|$ and $\gamma = z/|z|$. Then as $r \to 0$, we have*

$$K_A(x, r\gamma) \sim \sum_{k\in\mathbb{N}_0,\, k-m-n\notin\mathbb{N}_0} \alpha_k(x,\gamma)\, r^{k-m-n} + \sum_{k=0}^{\infty} \beta_k(x,\gamma)\, r^k \log r + \sum_{k=0}^{\infty} \gamma_k(x,\gamma)\, r^k, \tag{3.2}$$

where $\alpha_k, \beta_k, \gamma_k \in C^\infty(\mathcal{U} \times \mathbb{S}^{n-1})$. If $m \notin \mathbb{Z}$, then $\gamma_0 = ⨍_{\mathbb{R}^n} a(x, \xi) đ\xi$ (see Formula (2.13) for a more explicit formula). Moreover,

$$\alpha_k = \begin{cases} \int_{\mathbb{R}^n} e^{i\gamma\cdot\xi} a_{m-k}(x,\xi) đ\xi & \text{if } k - m - n < 0; \\ \int_{\mathbb{R}^n} \{e^{i\gamma\cdot\xi} - \sum_{j=0}^{\lfloor k-m-n \rfloor} \frac{1}{j!}(i\gamma\cdot\xi)^j\} a_{m-k}(x,\xi) đ\xi & \text{if } k - m - n > 0; \end{cases} \tag{3.3}$$

and, if $m \notin \mathbb{Z}$, then $\beta_k = 0$ for all k, and if $m \in \mathbb{Z}$, then

$$\beta_k = -\frac{1}{k!}\int_{|\xi|=1} (i\gamma\cdot\xi)^k\, a_{-n-k}(x,\xi) đ\xi, \quad k \geq 0. \tag{3.4}$$

Finally, when considered as densities on X, among the coefficients α_k, β_k, and γ_k, the coefficient β_0 is the only coordinate invariant one if $m \in \mathbb{Z}$, and γ_0 is the only coordinate invariant one if $m \notin \mathbb{Z}$.

Proof: Setting $h(\xi) := e^{i\gamma\cdot\xi}$ in Lemma 2.3 proves everything except the very last statement. Before proving this, we first recall that K_A is really a distributional *density*: There is a hidden factor of dy in (3.1). The factor dy is usually understood and omitted. With this said, we now prove the last statement. Let x' be other coordinates on X defined on $\mathcal{U}$, and let $r' = |x' - y'|$ and $\gamma' = (x' - y')/r'$ be the corresponding polar coordinates. Then, $r = br'$ where $b = b(x', r', \gamma') > 0$ is smooth, and $dy = J(y')dy' = J(x' - r'\gamma')dy'$, where J is the Jacobian determinant of the change of variables $x' \mapsto x$. Suppose now that $m \notin \mathbb{Z}$ — the proof when $m \in \mathbb{Z}$ follows the same general outline. Now, substituting the primed coordinates into (3.2) and taking into account the density factor, imply that

$$K_A \sim \sum_{k=0}^{\infty} a_k(x', r', \gamma')\,(r')^{k-m-n} + \sum_{k=0}^{\infty} b_k(x', r', \gamma')\,(r')^{k},$$

where $a_k = \alpha_k(x', r', \gamma')\, b(x', r', \gamma')^{k-m-n}\, J(x' - r'\gamma')$ and $b_k = \gamma_k(x', r', \gamma')\, b(x', r', \gamma')^{k}\, J(x' - r'\gamma')$. Expanding a_k and b_k in Taylor series in r' at $r' = 0$ give

$$K_A \sim \sum_{k=0}^{\infty} \alpha'_k(x', \gamma')\,(r')^{k-m-n} + \sum_{k=0}^{\infty} \gamma'_k(x', \gamma')\,(r')^{k},$$

where

$$\alpha'_k(x', \gamma') = \sum_{j+l=k} \frac{1}{j!}\partial^j_{r'} a_l(x', r', \gamma')|_{r'=0}, \quad \gamma'_k(x', \gamma') = \sum_{j+l=k} \frac{1}{j!}\partial^j_{r'} b_l(x', r', \gamma')|_{r'=0}.$$

In particular, $\gamma'_0(x', \gamma') = b_0(x', 0, \gamma') = \gamma_0(x', 0, \gamma')\, J(x')$, and hence, is coordinate invariant as a density. To see that the other coefficients are not coordinate invariant, let $x' = b^{-1}x$, where $b \in (0, \infty)$. Then, $\gamma = \gamma'$, $r = br'$, and J is a constant (equal to b^n). Moreover, $\alpha'_k(x', \gamma') = \alpha_k(x', \gamma')\, b^{k-m-n}\, J(x')$ and $\gamma'_k(x', \gamma') = \gamma_k(x', \gamma')\, b^k\, J(x')$. Because of the powers of b, it follows that the coefficients of non-zero powers are not coordinate invariant. $\square$

If $m \in \mathbb{Z}$, then the local β_0 terms construct a global density $\omega_W(A) \in C^\infty(X, \Omega)$, where Ω is the density bundle, called the *Wodzicki residue density*; defined by locally by

$$\omega_W(A)(x) = \int_{|\xi|=1} a_{-n}(x, \xi) đ\xi \cdot dx.$$

If $m \notin \mathbb{Z}$, then the local γ_0 terms define a global density $\omega_{KV}(A) \in C^\infty(X, \Omega)$ called the *Kontesevich–Vishik density.* It is given locally by

$$\omega_{KV}(A)(x) = ⨍_{\mathbb{R}^n} a(x, \xi) đ\xi \cdot dx. \tag{3.5}$$

Note that by definition, $\omega_{KV}(A) = K_A|_{\mathrm{Diag}}$ if $\mathrm{ord}(A) < -n$.

As we mentioned in the introduction, Kontsevich and Vishik defined the density $\omega_{KV}(A)$ to be the constant term in the expansion (3.2). In [11], Lesch defined the density $\omega_{KV}(A)$ by the regularized integral on the right-hand side of (3.5). Here, we have shown directly that these two definitions are equivalent. For simplicity in notation, we defined these densities only for the trivial bundle. The same definitions are also valid for $A \in \Psi^m_{c\ell}(X, E)$, where E is a vector bundle over X. In this case, $\omega_W(A), \omega_{KV}(A) \in C^\infty(X, \mathrm{hom}(E) \otimes \Omega)$. Henceforth, we will fix a vector bundle E over X and will consider these densities for $\Psi^m_{c\ell}(X, E)$.

So far we have only been considering operators of real order. However, Theorem 3.1 holds for $m \in \mathbb{C}$. The only difference is that in (3.3), we must replace m with $\mathrm{Re}\, m$. The other formulas, including (2.13), remain the same. The following lemma is proved in [5, Prop. 3.4]. Our proof is quite straightforward given the explicit formula (2.13).

Lemma 3.2 (Kontsevich–Vishik) *Let $A(\tau) \in \Psi^{m+\tau}_{c\ell}(X, E)$, $m \in \mathbb{C}$, be an entire family of operators. Then $\omega_{KV}(A(\tau))$ is meromorphic on $\mathbb{C}$ with simple poles on the discrete set $-m - n + \mathbb{N}_0$. The residue at a pole $\tau = \nu \in -m - n + \mathbb{N}_0$ is given by*

$$\mathrm{Res}_{\tau=\nu}\, \omega_{KV}(A(\tau)) = -\omega_W(A(\nu)).$$

Proof: By (2.13), we can write $\omega_{KV}(A(\tau))$ in terms of the homogeneous components of the local symbol of $A(\tau)$, together with a remainder term:

$$\omega_{KV}(A(\tau)) = \sum_{k=0}^{N-1} \frac{1}{-m-\tau-n+k} \int_{\mathbb{R}^n} (\xi \cdot \partial_\xi \chi)(\xi)\, a_{m+\tau-k}(\tau,x,\xi) đ\xi + \int_{\mathbb{R}^n} r_N(\tau,x,\xi) đ\xi.$$

Thus, $\omega_{KV}(A(\tau))$ has a simple pole at $\tau = -m-n+k$; the residue at such a pole is $-\int_{\mathbb{R}^n}(\xi \cdot \partial_\xi \chi)(\xi)\, a_{-n}(\nu,x,\xi) đ\xi$ with $\nu = -m-n+k$. This integral can be evaluated by introducing polar coordinates in ξ (cf. the computation (2.9)) resulting in $-\omega_W(A(\nu))$. □

The *Wodzicki residue trace* is the functional

$$\mathrm{res}(A) := \int_X \mathrm{tr}\, \omega_W(A), \quad A \in \Psi^m_{c\ell}(X,E),\ m \in \mathbb{Z}.$$

and the *Kontesevich–Vishik trace* is the functional

$$\mathrm{TR}(A) := \int_X \mathrm{tr}\, \omega_{KV}(A), \quad A \in \Psi^m_{c\ell}(X,E),\ m \notin \mathbb{Z}.$$

Note that if $\mathrm{ord}(A) < -n$, then $\mathrm{TR}(A) = \int_X \mathrm{tr}\, K_A|_{\mathrm{Diag}} = \mathrm{Tr}_{L^2}(A)$.

The following proposition shows that the Kontesevich–Vishik trace is a trace.

Proposition 3.3 (Kontesevich–Vishik) *The Kontesevich–Vishik trace vanishes on commutators of the form* $[A,B]$ *where* $\mathrm{ord}(A) + \mathrm{ord}(B) \notin \mathbb{Z}$.

Proof: Let $Q(\tau) \in \Psi^\tau_{c\ell}(X,E)$ be any entire family of operators such that $Q(0) = \mathrm{Id}$ (e.g. take a partition of unity of X with respect to a coordinate cover such that in a coordinate patch, the local symbol of Q is $|\xi|^\tau$). Then we can write

$$\begin{aligned} Q(\tau)[A,B] &= Q(\tau)AB - AQ(\tau)B + AQ(\tau)B - Q(\tau)BA \\ &= C(\tau) + [A, Q(\tau)B], \end{aligned}$$

where $C(\tau) = Q(\tau)AB - AQ(\tau)B \in \Psi^{\mathrm{ord}(A)+\mathrm{ord}(B)+\tau}_{c\ell}(X,E)$ is a holomorphic family. Thus, by Lemma 3.2, for $\tau \notin -\mathrm{ord}(A) - \mathrm{ord}(B) - n + \mathbb{N}_0$, we have

$$\mathrm{TR}(Q(\tau)[A,B]) = \mathrm{TR}(C(\tau)) + \mathrm{TR}([A,Q(\tau)B]).$$

As the L^2 trace vanishes on commutators, for $\mathrm{Re}\,\tau << 0$, the trace of $[A,Q(\tau)B]$ on $L^2(X,E)$ zero. Thus, for such τ, $\mathrm{TR}([A,Q(\tau)B]) = 0$. By analytic continuation, $\mathrm{TR}([A,Q(\tau)B]) = 0$ for all $\tau \in \mathbb{C}$. It follows that $\mathrm{TR}(Q(\tau)[A,B]) = \mathrm{TR}(C(\tau))$. By Lemma 3.2, both sides of this equality are meromorphic functions of τ and regular at $\tau = 0$. Thus, as $Q(0) = \mathrm{Id}$ and $C(0) = 0$, our proposition follows. □

This proposition is proved in [5, Prop. 3.2]. Kontsevich–Vishik also use holomorphic families of operators in their proof, but in a different way.

4. Other approaches to the Wodzicki Residue

For contrast, we survey other ways that have been used to define the Wodzicki residue. As mentioned in the introduction, Wodzicki originally defined $\mathrm{res}(A)$ by using the theory of complex powers: One can choose an operator P such that the complex powers $(P+tA)^{-z}$ exist for all small $t \in \mathbb{R}$. Then the trace $\mathrm{Tr}(P+tA)^{-z}$ extends from $\mathrm{Re}\, z >> 0$ to be a meromorphic function, and $\mathrm{res}(A) := \mathrm{ord}(P)\, d/dt|_{t=0} \mathrm{Res}_{z=1} \mathrm{Tr}(P+tA)^{-z}$. Wodzicki then shows that $\mathrm{res}(A)$ is given as the integral of $\omega_W(A)$. A related formula for the residue trace is $\mathrm{res}(A) = \mathrm{ord}(P)\, \mathrm{Res}_{z=0} \mathrm{Tr}(AP^{-z})$. An advantage of defining $\mathrm{res}(A)$ via complex powers is that the trace property (i.e., res vanishes on commutators) is straightforward to prove; however, a disadvantage is that the theory of complex powers is not at all trivial.

The residue trace can also be defined in terms of heat expansions. Thus, let e^{-tP} be the heat operator of P. Then, $\mathrm{res}(A)$ can be defined as $-\mathrm{ord}(P)\times$ coefficient of $\log t$ in the expansion of $\mathrm{Tr}(Ae^{-tP})$ as $t \to 0$. There is also a relation to resolvent expansions. We refer the reader to [11] for a treatment of these relationships. Once the theory of heat expansions or complex powers is sufficiently understood for a particular setting, these theories can be used to define residue traces; as was done for the cusp [15], the Boutet de Monvel [16], and for the cone algebras [17].

The density $\omega_W(A)$ can also be defined directly via the local formula $\int_{|\xi|=1} a_{-n}(x,\xi) đ\xi$. It is then left to prove coordinate independence and that its integral over X defines a trace. This is the approach of Fedosov–Golse–Leichtnam–Schrohe [18] for the closed manifold case and for the Boutet de Monvel algebra, and Schrohe [19] for an algebra on conic manifolds. This approach generalizes the original approach taken by Alder and Manin [22], [23] for pseudodifferential operators in the one dimensional setting. Kassel [20] provides an overview of the various means to define the Wodzicki residue trace, including the connection with the symplectic residue of Guillemin [21] (see also [11, Sec. 2.2]).

Finally, Lemma 3.2 gives a new way to characterize the Wodzicki residue. Namely, let $A(\tau) \in \Psi^{m+\tau}_{c\ell}(X,E)$ be any entire family of classical pseudodifferential operators on X such that $A(0) = A$. Then, one can define $\omega_W(A) := -\mathrm{Res}_{\tau=0}\, \omega_{KV}(A(\tau))$. As a simple application of this characterization, we prove that the Wodzicki residue trace is a trace.

Proposition 4.1 (Wodzicki) *The Wodzicki residue trace vanishes on commutators of the form* $[A,B]$ *where* $\mathrm{ord}(A) + \mathrm{ord}(B) \in \mathbb{Z}$.

Proof: The proof of this proposition follows verbatim the proof of Proposition 3.3, except we substitute the following statements for the last two sentences of Proposition 3.3: By Lemma 3.2, both sides of $\mathrm{TR}(Q(\tau)[A,B]) = \mathrm{TR}(C(\tau))$ are meromorphic functions of τ, with residue at $\tau = 0$ given by

$$-\mathrm{res}[A,B] = \mathrm{Res}_{\tau=0}\,\mathrm{TR}(Q(\tau)[A,B]) = \mathrm{Res}_{\tau=0}\,\mathrm{TR}(C(\tau)).$$

Since $C(0) = 0$, $\mathrm{TR}(C(\tau))$ is regular at $\tau = 0$. Thus, $\mathrm{res}[A,B] = 0$. □

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