On the b-pseudodifferential calculus on manifolds with corners

by

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ABSTRACT

Structure theorems for both the resolvent and the heat kernel of b-pseudodifferential operators on a compact manifold with corners (of arbitrary codimension) are presented. In both cases, the kernels are realized as classical conormal functions on appropriate manifolds with corners. To prove these results, a space of operators with complex parameter (or tempered operators) is introduced. These tempered operators are shown to be classical conormal functions on a manifold with corners called the Tempered space. The resolvent of a b-pseudodifferential operator is shown to be a tempered operator (for large values of the parameter) and so it follows that the resolvent is a classical conormal function. The Laplace transforms of holomorphic tempered operators are shown to be operators of order $-\infty$ for positive times and are also shown to be classical conormal functions on a manifold with corners called the Heat space. Since the heat kernel of a b-pseudodifferential operator is the Laplace transform of the operators resolvent, the heat kernel is of order $-\infty$ for positive times and is also a classical conormal function. The structure result for the heat kernel is used to generalize the Index formula of Atiyah, Patodi, and Singer for Dirac operators on a manifold with boundary to Fredholm b-pseudodifferential operators on arbitrary compact manifolds with corners. The formula expresses the index of an operator as a sum of two terms, the usual 'interior term' given by the integral of the Atiyah-Patodi-Singer density associated to the operator and a second contribution given by a generalization of the eta-invariant associated to the induced operators on each of the corners of the manifold.

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Contents

1 Introduction

The Atiyah-Singer index theorem [1] (AS theorem) is one of the crowning achievements of 20th century mathematics.[1] The AS theorem relates analytic invariants of Dirac operators with corresponding topological/geometric invariants of the underlying Riemannian manifold. Let $E \to X$ be a $\mathbb{Z}_2$-graded Clifford bundle over a closed compact even dimensional Riemannian manifold X and let $A^+ \in \mathrm{Diff}^1(X, E^+, E^-)$ be the positive part of a Dirac operator. Then the AS theorem states that

$$\operatorname{ind} A^+ = \int_X AS, \tag{1}$$

where AS is the Atiyah-Singer integrand. The principal aspect of (1) is that the left-hand side is an analytic object, while the right-hand side is a topological/geometric object. We note that (1) is just a special case of an 'analytic index theorem' for pseudodifferential operators. Thus, let $A \in \Psi^m(X, E^+, E^-)$ be an elliptic, positive order pseudodifferential operator. Then the analytic index theorem states that

$$\operatorname{ind} A = \zeta_{A^*A} - \zeta_{AA^*}, \tag{2}$$

where ζ_{A^*A} and ζ_{AA^*} are the constant terms in the asymptotic expansions, as $t \downarrow 0$, of $\mathrm{Tr}(e^{-tA^*A})$ and $\mathrm{Tr}(e^{-tAA^*})$ respectively. Of course, when $A = A^+$ is the positive part of a Dirac operator, then $\zeta_{A^*A} - \zeta_{AA^*} = \int_X AS$.

The Atiyah-Patoti-Singer index theorem (APS theorem) generalizes (1) to manifolds with a cylindrical end. A manifold with a cylindrical end is a Riemannian manifold (Y, g) which consists of a 'compact end' and a 'non-compact end', where the non-compact end is diffeomorphic to a cylinder $(-\infty, 1]_t \times Y_0$, where Y_0 is a closed compact manifold, and where the metric g when restricted to this cylindrical end, takes the form $g = dt^2 + g_0$, where g_0 is a metric on Y_0. Let $E \to Y$ be a $\mathbb{Z}_2$-graded Clifford bundle over an even dimensional manifold with a cylindrical end and assume that on the cylindrical end $(-\infty, 1]_t \times Y_0$, there exists an isomorphism $E \cong \pi^* E_0$, where $\pi : (-\infty, 1]_t \times Y_0 \to Y_0$ is the projection onto Y_0, and where E_0 is the bundle $E|_{\{0\}_t \times Y_0} \to Y_0$. Let A be a Dirac operator and assume that on the cylindrical end, A^+ takes the form

$$A^+ = \sigma(\partial_t + A_0), \tag{3}$$

where $\sigma : E_0^+ \to E_0^-$ is a bundle map, and A_0 is a Dirac operator associated to the Clifford bundle $E_0 \to Y_0$. Then the noted APS index theorem states that if A^+ is Fredholm, then

$$\operatorname{ind} A^+ = \int_Y AS - \frac{1}{2}\eta, \tag{4}$$

where AS is once again, the AS integrand, and where η is the eta invariant of A and can be defined as the value of the meromorphic extension of the eta function

$$\eta(z) = \sum_i \frac{\operatorname{sign} \lambda_i}{|\lambda_i|^z}$$

at $z = 0$, where $\{\lambda_i\}$ are the eigenvalues of A_0. Comparing (1) with (4), we observe that η can be thought of as the correction term in extending the formula (1) from closed manifolds to manifolds with a cylindrical end.

In [8], Melrose reproved (4) using his calculus of b-pseudodifferential operators. The fundamental feature of his b-calculus methods is that his proof of (4) is essentially the same as the proof of (1).

Melrose's b-calculus arises naturally by reinterpreting (4) as an index theorem on a compact manifold *with boundary*. Thus, consider the change of variables $x = e^t$ on the cylindrical end. Observe that as $t \to -\infty$, $x \to 0$. Hence, this change of variables compactifies the cylindrical end $(-\infty, 1]_t \times Y_0$ into the interior of the manifold with boundary $[0, e)_x \times Y_0$. Let X be the manifold with the same 'compact end' as Y and with the cylindrical end replaced by $[0, e)_x \times Y_0$. Then Y

[1] Thanks to Robert Lauter for his suggestions on this introduction.

is just the interior of X. Note that the metric $g = dt^2 + g_0$ over the cylinder becomes, under the change of variables $x = e^t$, the (b-)metric $g = (\frac{dx}{x})^2 + g_0$ over $[0, e)_x \times Y_0$ in X. Also note that E extends naturally to a vector bundle on X, and similarly, A^+ extends naturally to a differential operator on X, which takes the form

$$A^+ = \sigma(x\partial_x + A_0)$$

over $[0, e)_x \times Y_0$ in X. A^+ is an example of a first order b-differential operator on X. Thus, for any $m \in \mathbb{N}$, an mth order b-differential operator is just a usual differential operator on the the interior of X and on the neighborhood $[0, e)_x \times Y_0$, is given locally by an mth order polynomial in the vector fields $\{x\partial_x, \partial_{y_1}, \dots, \partial_{y_{n-1}}\}$, where $(y_1, \dots, y_{n-1})$ are local coordinates on Y_0. Roughly speaking, if $m \in \mathbb{R}$, an mth order b-pseudodifferential operator is a usual pseudodifferential operator on the interior of X and locally on the neighborhood $[0, e)_x \times Y_0$, is given by a formal symbol of degree m in the vector fields $\{x\partial_x, \partial_{y_1}, \dots, \partial_{y_{n-1}}\}$.

In [13], Piazza extended the analytic index formula (2) from closed manifolds to b-pseudodifferential operators on manifolds with boundary. If $A \in \Psi_b^m(X, E^+, E^-)$ is a Fredholm b-pseudodifferential operator of positive order, then Piazza showed that

$$\operatorname{ind} A = {}^b\zeta_{A^*A} - {}^b\zeta_{AA^*} - \frac{1}{2}\eta, \tag{5}$$

where ${}^b\zeta_{A^*A}$ and ${}^b\zeta_{AA^*}$ are related to the constant terms in the asymptotic expansions, as $t \downarrow 0$, of $e^{-tA^*A}|_\Delta$ and $e^{-tAA^*}|_\Delta$ respectively, and where η is the eta invariant of A, a globally defined invariant of the normal operator of A.

As stated before, a great advantage of Melrose's b-calculus methods is that his proof of the APS theorem is modeled on the proof of the AS theorem. A second advantage of his b-calculus is that such operators are naturally defined on any compact manifold with corners. Roughly speaking, a manifold with corners is a topological space which is locally homeomorphic to a product of manifolds with boundary. Thus, the definitions of b-differential and b-pseudodifferential operators naturally extend to manifolds with corners. Hence, it might be expected that the APS index theorem and Piazza's theorem can be extended to arbitrary compact manifolds with corners. This is in fact the case. Melrose [11] extended the APS theorem for Dirac operators on manifolds with boundary to Dirac operators on manifolds with corners. In this thesis, we extend Piazza's theorem to compact manifolds with corners.

Our proof of Piazza's theorem for compact manifolds with corners is modeled on the proof of (2) on closed manifolds.[2] Of course, there are some fundamental difficulties which arise due to the presence of the corners. To isolate these difficulties, we review the proof of (2). Thus, omitting vector bundles for simplicity, let $A \in \Psi^m(X)$ be a Fredholm pseudodifferential operator of positive order on a closed compact manifold X. To prove this theorem, we follow McKean and Singer and consider the function

$$h(t) = \operatorname{Tr}(e^{-tA^*A}) - \operatorname{Tr}(e^{-tAA^*}), \tag{6}$$

where $\operatorname{Tr}(e^{-tA^*A}) = \int_X e^{-tA^*A}|_\Delta$ and $\operatorname{Tr}(e^{-tAA^*}) = \int_X e^{-tAA^*}|_\Delta$. We then observe that by spectral theory,

$$\lim_{t\to\infty} h(t) = \dim\ker A - \dim\ker A^* = \operatorname{ind} A. \tag{7}$$

Hence, by the fundamental theorem of calculus, for any $t \in \mathbb{R}^+$,

$$\operatorname{ind} A = h(t) + \int_t^\infty \partial_s h(s)\, ds. \tag{8}$$

Next, by an algebraic manipulation, we find that

$$\partial_s h(s) = \operatorname{Tr}(\, [A, A^* e^{-tAA^*}]\,). \tag{9}$$

[2] We remark that Piazza's proof of (5) was not modeled on the proof of (2) in the same sense that Melrose's and the author's proofs were.

Finally, we observe that the trace vanishes on commutators and so, $\partial_s h(s) \equiv 0$. Thus, $\operatorname{ind} A \equiv h(t)$ for all $t \in \mathbb{R}^+$. Taking the constant term in the expansion, as $t \downarrow 0$, of $h(t)$ then yields (2).

Now suppose that $A \in \Psi_b^m(X)$ is a Fredholm b-pseudodifferential operator of positive order on a compact manifold with corners X and let us try to use the above argument to compute $\operatorname{ind} A$. Here, we immediately get into trouble at equation (6). Our first fundamental difficulty is that $e^{-tA^*A}|_\Delta$ and $e^{-tAA^*}|_\Delta$ are not integrable and hence $h(t)$ does not even make sense. However, Melrose [8], has defined a suitable replacement for the integral, called the b-integral, ${}^b\!\!\int$, so that $e^{-tA^*A}|_\Delta$ and $e^{-tAA^*}|_\Delta$ are b-integrable (see Section 6). Thus, in place of $h(t)$ we consider

$$ {}^b h(t) = b\text{–}\mathrm{Tr}(e^{-tA^*A}) - b\text{–}\mathrm{Tr}(e^{-tAA^*}), $$

where $b\text{–}\mathrm{Tr}(e^{-tA^*A}) = {}^b\!\!\int_X e^{-tA^*A}|_\Delta$ and $b\text{–}\mathrm{Tr}(e^{-tAA^*}) = {}^b\!\!\int_X e^{-tAA^*}|_\Delta$. Fortunately, (7) still holds for ${}^b h(t)$: $\lim_{t\to\infty} {}^b h(t) = \operatorname{ind} A$ (see Theorem 7.2) . Thus, (8) still holds with $h(t)$ replaced with ${}^b h(t)$. Since (9) was just an algebraic manipulation, it follows that

$$ \partial_s {}^b h(s) = b\text{–}\mathrm{Tr}(\,[A, A^* e^{-tAA^*}]\,). $$

The second fundamental problem arises at this point. It turns out that $b\text{–}\mathrm{Tr}(\,[A, A^* e^{-tAA^*}]\,) \neq 0$. The b-trace does not vanish on commutators. Intuitively, the b-trace is a trace on the interior of X. Hence, $b\text{–}\mathrm{Tr}(\,[A, A^* e^{-tAA^*}]\,)$ should be expressible in terms of induced 'boundary operators' of A and A^*. This is in fact the case (see Theorem 6.6). Hence, we find that

$$ \operatorname{ind} A = {}^b h(t) - \frac{1}{2} {}^b\eta(t), \tag{10} $$

where $-\frac{1}{2}{}^b\eta(t) = \int_t^\infty \partial_s {}^b h(s)\, ds$ is a 'boundary term' and ${}^b h(t)$ is an 'interior term'. Taking the constant term in the expansion of the right-hand side of (10) as $t \downarrow 0$ yields Piazza's theorem for arbitrary compact manifolds with corners.

1.1 Outline

In Section 2, we state the fundamental theorems concerning classical (that is, asymptotic) expansions of conormal functions and the fundamental theorems about the b-calculus that will be used later in the thesis. The results of this section are, to the authors knowledge, due to Melrose. More thorough treatments of the results contained in this section can be found in [8], [9], and [7].

In Section 3, we begin to lay the foundations needed to prove the APS index theorem for b-pseudodifferential operators. Here, the resolvent $(A - \lambda)^{-1}$ of such an operator is constructed. We follow the program initiated by Seeley in [14], by defining a basic space of tempered operators, which is implicitly defined in his paper. These are operators depending symbolically on a parameter $\lambda \in \mathbb{C}$. We make further refinements of these tempered spaces into resolvent tempered and resolvent one-step tempered spaces. We also define 'resolvent like' operators. These refinements, as far as we know, are original to this thesis. The resolvent of a (classical) b-pseudodifferential operator is shown in Subsection 3.7 to be a resolvent one-step tempered operator for large values of the parameter. In Subsection 3.9, it is shown that the resolvent one-step operators can be realized as classical functions on a blown-up manifold called the Tempered space. Since the resolvent is a resolvent one-step operator, it too, is a classical function on the Tempered space. The realization of these tempered operators as classical functions was discovered by the author. Using Theorem 2.1, due to Melrose, which gives a simple characterization of functions having classical expansions, we have discovered a direct method to prove that certain functions have expansions. This method of proof is used to prove the structure theorem for tempered operators (Theorem 3.34) and the structure theorem for the Laplace transforms of tempered operators (Theorem 4.11). The application subsection 3.11 is based on the work of Grubb and Seeley [5].

In Section 4, the Laplace transforms of tempered operators are analyzed. First, it is proved that the Laplace transforms of resolvent like tempered operators are b-pseudodifferential operators or order $-\infty$ for positive times and are continuous in a certain sense down to $t = 0$. The method of proof of the fundamental Lemma 4.7, which provides the essential properties of the local symbols of Laplace transforms is, to the best of the authors knowledge, due to Grubb [4]. In [8], Melrose realized the heat kernels of b-differential operators on any compact manifold with corners as classical functions on a blown-up manifold called the Heat space. Here, we present a similar result for the Laplace transforms of resolvent like, one-step operators. The classical expansions for these Laplace transforms are much more complicated than for b-differential operators, as here log terms show themselves. In particular, as the heat kernel of a b-pseudodifferential operator is the Laplace transform of the operators resolvent, the heat kernel is a classical function on the Heat space. This result, at least for closed manifolds, is probably well known; however, as pointed out by Melrose [11], this is possibly the first time the proof has been formally written down.

In Section 5, the Mellin transforms of tempered operators are analyzed. We first prove that the Mellin transform of a resolvent like tempered operator extends to be an entire family of b-pseudodifferential operators. The method of proof of the fundamental Lemma 5.8, which provides the essential properties of the local symbols of Mellin transforms is, to the best of the authors knowledge, due to Grubb [4]. We then prove that such a Mellin transform, when restricted to the diagonal, is a meromorphic function with only simple poles. In particular, as the complex power of a b-pseudodifferential operator is the Mellin transform of the operators resolvent, the complex power is an entire family of b-pseudodifferential operators, and when restricted to the diagonal, is meromorphic with only simple poles. The complex power is used to define the b-zeta and b-eta function used in an index theorem presented in Section 7.

In Section 6, we present the fundamentals of the b-trace and the b-integral. The ideas used in this section are due to Melrose [10].

Finally, in Section 7, we present the index theorem. The ideas used in this section are due to Melrose [10].

2 Preliminary Material

We assume the reader to be familar with the essentials of manifolds with corners and with b-pseudodifferential operators. This section is here to fix notation. It is based heavily on [10]. Other references are [8], [10], and [7].

2.1 Manifolds with corners and the b-category

An n dimensional *manifold with corners* is a *set* X together with a *set* of functions $C^\infty(X)$ on X (called the C^∞ structure of X), satisfying the following conditions:

1. there exists a smooth n dimensional manifold without boundary $\widetilde{X}$ and an injection $\iota : X \hookrightarrow \widetilde{X}$ such that $C^\infty(X) = \iota^*(C^\infty(\widetilde{X}))$; henceforth, we will consider $X \subseteq \widetilde{X}$;
2. there exists a finite set of smooth functions $\{\rho_i\}_{i=1}^N \subseteq C^\infty(\widetilde{X})$ having the property that
   (a) $X = \{p \in \widetilde{X} \mid \rho_i(p) \geq 0 \text{ for all } i\}$;
   (b) for each i, $\{\rho_i = 0\}$ is connected;
   (c) if $p \in \widetilde{X}$ and $\rho_{i_j}(p) = 0$ for some $1 \leq i_1 < \ldots < i_k \leq N$, then $\{(d\rho_{i_j})(p) \mid j = 1, \ldots k\}$ are a set of independent differentials.

The model example of a manifold with corners is

$$\mathbb{R}^{n,k} := [0,\infty)^k \times \mathbb{R}^{n-k}, \ \ 0 \leq k \leq n.$$

Another example is

$$\mathbb{S}^{n-1,k} := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x| = 1 \text{ and } x_i \geq 0 \text{ for } 1 \leq i \leq k\} = \mathbb{S}^{n-1} \cap \mathbb{R}^{n,k}.$$

For each $i = 1, \ldots, N$, the subset $H_i := \{\rho_i = 0\} \subseteq X$ is called a *boundary hypersurface* of X and ρ_i is called a *boundary defining function* for H_i. By condition (2) above, it follows that near H_i, there exists a local diffeomorphism

$$X \cong [0,\epsilon)_{\rho_i} \times H_i$$

for some $\epsilon > 0$. Note that by rescaling, we may assume that ϵ is always 1; henceforth, we will always assume that the ρ_i's have this property. Also note that H_i itself is a manifold with corners, and if ρ_i' is any other boundary defining function for H_i, then $\rho_i' = a\rho_i$, where $0 < a \in C^\infty(X)$. A total boundary defining function for X is a function of the form $\rho = \prod_{i=1}^N \rho_i$. Then, $X \equiv \{\rho \geq 0\}$.

If $k \in \mathbb{N}$, a *codimension k boundary face* of X is, by definition, a connected component of an intersection of k distinct boundary hypersurfaces of X. The *codimension* of X is the largest such k. The set of codimension k boundary faces of X is denoted by $M_k(X)$, and we define $M(X)$ to be the union over all such faces: $M(X) := \cup_{k\geq 1} M_k(X)$. Note that if $M \in M_k(X)$, $k \in \mathbb{N}$, is a codimension k face of X, then near M, $X \cong [0,1)^k \times M$, where the coordinate functions on $[0,1)^k$ are the boundary defining functions for the hypersurfaces which define M.

A subset $Y \subseteq X$ is said to be an l dimensional *p-submanifold* (p for 'product'), if for each $y \in Y$, there exists a coordinate patch $\mathbb{R}^{l,k} \times \mathbb{R}^{p,q}$ on X centered at y such that $Y \cong \mathbb{R}^{l,k} \times \{0\}$; Y is said to be *interior p-submanifold* if on any such coordinate patch, $q = 0$.

The space of smooth vector fields on X is denoted by $\mathcal{V}(X) := C^\infty(X, TX)$. The space of *$b$-vector fields on* X (b for 'boundary'), $\mathcal{V}_b(X)$, is the space

$$\mathcal{V}_b(X) := \{v \in \mathcal{V}(X) \mid v \text{ is tangent to each } H \in M_1(X)\}.$$

These b-vector fields form a Lie Algebra: if $v, w \in \mathcal{V}_b(X)$, then $[v,w] \in \mathcal{V}_b(X)$. The fundamental property of these b-vector fields is that they can be realized as the sections of a vector bundle ${}^bTX$ over X, called the *b-tangent bundle*. If $p \in X$, then the fibre of ${}^bTX$ at p is

$${}^bT_pX := \mathcal{V}_b(X)/\mathcal{I}_p(X) \cdot \mathcal{V}_b(X),$$

where $\mathcal{I}_p(X) \subseteq C^\infty(X)$ is the space of functions $f \in C^\infty(X)$ which vanish at p, and $\mathcal{I}_p(X) \cdot \mathcal{V}_b(X)$ consists of finite sums of products of elements of $\mathcal{I}_p(X)$ and $\mathcal{V}_b(X)$.

Given $p \in \partial X$, the map $\mathcal{V}_b(X) \ni v \to v(p) \in T_pX$ vanishes on $\mathcal{I}_p(X) \cdot \mathcal{V}_b(X)$, and hence, this map induces a map on the quotient

$$ {}^bT_pX = \frac{\mathcal{V}_b(X)}{\mathcal{I}_p(X) \cdot \mathcal{V}_b(X)} \to T_pX. \tag{11}$$

The *b-normal bundle* of X at p is defined as

$$ {}^bN_pX := \text{the kernel of the map (11).}$$

If $M \in M_k(X)$, $k \in \mathbb{N}$, then the *b-normal bundle* to M is the set

$$ {}^bNM := \overline{{}^bNX|_{\mathrm{int}\, M}} \subseteq {}^bTX|_M,$$

where the closure is in ${}^bTX|_M$. If $X \cong [0,1)^k_x \times M$, $x = (x_1, \dots, x_k)$, is a decomposition of X near M, where each x_i is a boundary defining function on X, then for each $p \in M$, ${}^bN_pM \equiv \mathrm{span}\,\{(x_i\partial_{x_i})(p)\}$. Hence, ${}^bNM$ is a trivial k dimensional bundle. Since for each i, every boundary defining function for the hypersurface $H_i := \{x_i = 0\}$ is just a multiple of x_i by a positive function, it follows that for each $p \in M$, the element $(x_i\partial_{x_i})(p) \in {}^bT_pX$ is defined independent of the boundary defining function choosen for H_i. Hence, ${}^bNM$ is a *canonically* trivial bundle. Note that for each $p \in M$,

$$ {}^bT_pX \equiv {}^bN_pM \oplus {}^bT_pM. \tag{12}$$

If $\alpha \in \mathbb{R}$, we define the α *b-density bundle*, $\Omega_b^\alpha X$, by

$$ \Omega_b^\alpha X := \bigsqcup_{p \in X} \Omega^\alpha({}^bT_pX).$$

If $\mathcal{U} = \mathbf{R}^{n,k}_{(x,y)}$ is a coordinate patch on X, then $|\frac{dx}{x}dy|^\alpha = |\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_k}{x_k} \wedge dy_1 \wedge \cdots \wedge dy_{n-k}|^\alpha$ is a local trivialization of $\Omega_b^\alpha X$. If $M \in M_k(X)$, then by (12), $\Omega_b^\alpha X|_M = \Omega^\alpha({}^bNM) \otimes \Omega^\alpha({}^bTM)$. Since ${}^bNM$ is a canonically trivial bundle, we can identify

$$ \Omega_b^\alpha X|_M \equiv \Omega_b^\alpha M. \tag{13}$$

2.2 Conormal functions

Henceforth, X will always be compact with corners. We denote by $\dot{C}^\infty(X)$ the subspace of $C^\infty(X)$ consisting of those functions which vanish to infinite order at the boundary of X. The space of *(extendible) distributions* on X, $C^{-\infty}(X)$, is the topological dual space of $\dot{C}^\infty(X, \Omega_b)$: $C^{-\infty}(X) := (\dot{C}^\infty(X, \Omega_b))^*$.

The space of mth order *b-differential operators*, $\mathrm{Diff}_b^m(X)$, is the space of operators

$$ \mathrm{Diff}_b^m(X) := \mathrm{span}_{0 \le k \le m} \mathcal{V}_b(X)^k,$$

where the span is over $C^\infty(X)$. The space of *symbols of order 0*, $S^0(X)$, is the space

$$ S^0(X) := \{u \in C^{-\infty}(X) \,|\, \mathrm{Diff}_b^*(X)u \subseteq L^\infty(X)\}.$$

If α is a multi-index on X and ρ is a total boundary defining function for X, then the space of *symbols of order* α, $S^\alpha(X)$, is given by

$$ S^\alpha(X) := \rho^{-\alpha} S^0(X) = \{\rho^{-\alpha} u \,|\, u \in S^0(X)\}.$$

Given $m \in \mathbb{N}$, we define $S^0_m(X) := \{u \in C^{-\infty}(X) \,|\, \mathrm{Diff}_b^m(X)u \subseteq L^\infty(X)\}$. Then the topology on $S^0(X)$ is the Frechet topology generated by the spaces $\{S^0_m(X)\}_{m \in \mathbb{N}_0}$.

There are two refinements of the symbol spaces that we will present; the Hölder differentiable symbols and the polyhomogeneous symbols.

Let $H \in M_1(X)$ and let η be a non-negative real number. Then a symbol $u \in S^0(X)$ is said to be *η-Hölder differentiable* at H, if for any product decomposition $X \cong [0,1)_x \times H_y$ of X near H, if we write $u = u(x,y)$ with respect to this product decomposition, the function $u(x,y)$ has the form

$$u(x,y) = u_0(y) + xu_1(y) + \cdots + x^k u_k(y) + x^\eta u_{k+1}(x,y),$$

where $0 \le \eta - k < 1$, $u_i(y) \in S^0(H)$ for each i, and $u_{k+1}(x,y) \in S^0([0,1) \times H)$ is such that for any $P \in \mathrm{Diff}_b^*(X)$, Pu_{k+1} is continuous up to H. Let $A \subseteq M_1(X)$ and η be a non-negative multi-index associated to A; that is, $\eta : A \to [0,\infty)$. Then a function $u \in S^0(X)$ is said to be an *η-Hölder differentiable function on A*, if for each $H \in A$, u is $\eta(H)$-Hölder differentiable at H. The space of such functions is denoted by $S_A^{0,\eta}(X)$. If α is a multi-index on X, we define

$$S_A^{\alpha,\eta}(X) := \rho_{tot}^{-\alpha} S_A^{0,\eta}(X).$$

A *C^∞ index set* is a discrete subset $E \subseteq \mathbb{C} \times \mathbb{N}_0$ satisfying

1. For each $M \in \mathbb{R}$, the set $E \cap (\{z \in \mathbb{C} \,|\, \mathrm{Re}\, z \le M\} \times \mathbb{N}_0)$ is finite;
2. If $(z,k) \in E$, then $(z,l) \in E$ for all $0 \le l \le k$;
3. If $(z,k) \in E$, then $(z+l,k) \in E$ for all $l \in \mathbb{N}$.

E is just called an *index set* if it satisfies only (1) and (2).

Let E be an index set and let $H \in M_1(X)$. A function $u \in S^\alpha(X)$ (for some α) is said to have an *asymptotic (or classical) expansion at H with index set E*, if for any product decomposition $X \cong [0,1)_x \times H_y$ of X near H, if we write $u = u(x,y)$ with respect to this product decomposition, then for all $(z,k) \in E$, there exists a $u_{(z,k)} \in S^{\alpha|_H}(H)$, where $\alpha|_H$ is α restricted to H, such that for each $N \in \mathbb{N}$,

$$u(x,y) - \sum_{(z,k)\in E,\, \mathrm{Re}\, z \le N} x^z (\log x)^k \, u_{(z,k)}(y) \in S^{\alpha_N}([0,1) \times H), \tag{14}$$

where α_N is an index set with $\alpha_N|_H = \alpha|_H$ and $\alpha_N(H) \to -\infty$ as $N \to \infty$. We then write

$$u \sim \sum_{(z,k)\in E} x^z (\log x)^k u_{(z,k)}.$$

We denote the space of all such functions having expansions at H with index set E, by $\mathcal{A}^E(X)$ (where E is understood to be associated to H). Note that if $E = \emptyset$, then (14) holds for all N iff u vanishes to infinite order at $x = 0$. An *index family* is a set $\mathcal{E} = \{\mathcal{E}(H) \,|\, H \in M_1(X)\}$, where each $\mathcal{E}(H)$ is an index set. We define $\mathcal{A}_{phg}^{\mathcal{E}}(X)$ to be the space of those functions u such that for each $H \in M_1(X)$, u has an expansion at H with index set $\mathcal{E}(H)$.

Let E be a C^∞ index set and let $v \in C^\infty(X, TX)$. Then, we define for each $N \in \mathbb{N}$,

$$P_N^E(v) := \prod_{(z,k)\in E,\, \mathrm{Re}\, z \le N} (v - z).$$

The following theorems are fundamental to the proofs of the asympotics of the resolvent and heat kernels proved in Sections 3.9 and 4.4.

Theorem 2.1 *Let $H \in M_1(X)$ and let x be a boundary defining function for H. Then a function $u \in S^\alpha(X)$ has an asymptotic expansion at $H \in M_1(X)$ with index set E iff for each $N \in \mathbb{N}$, we have*

$$P_N^E(x\partial_x)u \in S^{\alpha_N}(X),$$

where α_N is an index set with $\alpha_N(G) = \alpha(G)$ if $G \ne H$ and $\alpha_N(H) \to -\infty$ as $N \to \infty$.

Proposition 2.2 *Let $H \in M_1(X)$ and decompose $X \cong [0,1)_x \times H_y$ near H. Let $u \in S^\gamma(X)$ (some γ) and let β be a multi-index on H. Suppose that for each $N \in \mathbb{N}$, there exists an $m = m(N) \in \mathbb{N}$ such that $m(N) \to \infty$ as $N \to \infty$, and such that if we write $u = u(x,y)$ in this decomposition, we have*

$$u(x,y) = \sum_{(z,k)\in E,\, \mathrm{Re}\, z \le N} x^z (\log x)^k u^m_{(z,k)} + u^m_N(x,y),$$

for some $u^m_{(z,k)} \in S^\beta_m(H)$ and $u^m_N(x,y) \in S^{\alpha_N}_m([0,1)\times H)$, where $\alpha_N = (\alpha_N(H), \beta)$ with $\alpha_N(H) \to -\infty$ as $N \to \infty$. Then $u \in \mathcal{A}^E(X)$.

2.3 Blow-ups

Given any p-submanifold $Z \subseteq X$, the *inward pointing Tangent bundle of* Z, T^+Z has fibres

$$T^+_p Z := \begin{cases} T_pZ, \text{ if } p \in \operatorname{int} Z; \\ \{v \in T_pZ \,|\, d\rho_H(v) \ge 0 \text{ for all } H \in M_1(X) \text{ with } p \in H\}, \text{ if } p \in \partial Z. \end{cases}$$

Let $Y \subseteq X$ be a *closed* p-submanifold of X. Then the *inward pointing normal bundle*, N^+Y, is the vector bundle $N^+Y := T^+X/T^+Y$. We define X *blown-up along* Y, $[X;Y]$, as a *set*, by

$$[X;Y] := (N^+Y \setminus \{0\})/\mathbb{R}^+ \sqcup (X \setminus Y).$$

The set $(N^+Y \setminus \{0\})/\mathbb{R}^+$ is called the *front face* of $[X;Y]$ and is denoted by *ff*$(\,[X;Y]\,)$. The C^∞ structure on $[X;Y]$ is taken to be the set of functions on $[X;Y]$ which define smooth functions in any local coordinates in $X \setminus Y$, and which also define smooth functions in any local polar coordinates about Y. The *blow-down map* $\beta : [X;Y] \to X$ is the map defined as follows: if $x \in X \setminus Y$, then we define $\beta(x) := x$; if $[v] \in (N^+_pY \setminus \{0\})/\mathbb{R}^+$, then we define $\beta([v]) := p \in Y$. If $Z \subseteq X$ is a closed subset of X, then the *lift* of Z into $[X;Y]$, $\beta^* Z \subseteq [X;Y]$, is defined under the following conditions:

1. if $Z \subseteq Y$, we define $\beta^*Z := \beta^{-1}(Z)$;
2. if $Z = \overline{Z \setminus Y}$, we define $\beta^*Z := \overline{\beta^{-1}(Y \setminus Z)}$.

If Z satisfies either (1) or (2) above, and if in addition, β^*Z is a p-submanifold of $[X;Y]$, then $[X;Y]$ blown-up along β^*Z is defined, and we denote it by $[X;Y;Z] \equiv [\,[X;Y];\beta^*Z]$.

A family $\mathcal{Y} = \{Y_1,\dots,Y_N\}$ of p-submanifolds of X are said to intersect *normally* if given any $p \in Y_{l_1} \cap \cdots \cap Y_{l_k}$, where $1 \le l_1 < \cdots < l_k \le N$, there exists a joint product decomposition

$$X \cong \mathcal{U}_0 \times \mathcal{U}_1 \times \cdots \times \mathcal{U}_k, \tag{15}$$

where $\mathcal{U}_i = \mathbb{R}^{n_i,k_i}$ (for some n_i, k_i), such that

$$Y_{l_j} \cong \mathcal{U}_0 \times \cdots \times \mathcal{U}_{j-1} \times \{0\} \times \mathcal{U}_{j+1} \times \cdots \times \mathcal{U}_k,$$

for each $j = 1,\dots,k$. Observe that if $\{i_1,\dots,i_N\} = \{1,\dots,N\}$, then the iterated blow-up $[X;Y_{i_1};\cdots;Y_{i_N}]$ is defined and locally in the coordinate patch (15),

$$[X;Y_{i_1};\cdots;Y_{i_N}] \cong \mathcal{U}_0 \times [\mathcal{U}_1;\{0\}] \times [\mathcal{U}_2;\{0\}] \times \cdots \times [\mathcal{U}_N;\{0\}].$$

In particular, if $\{i_1,\dots,i_N\} = \{j_1,\dots,j_N\} = \{1,\dots,N\}$, then $[X;Y_{i_1};\cdots;Y_{i_N}] \equiv [X;Y_{j_1};\cdots;Y_{j_N}]$. We define $[X;\mathcal{Y}] := [X;Y_{i_1};\cdots;Y_{i_N}]$ for any $\{i_1,\dots,i_N\} = \{1,\dots,N\}$.

2.4 b-pseudodifferential operators

Let $\mathcal{B} := \{H \times H \,|\, H \in M_1(X)\}$. Then, as $\mathcal{B}$ is a normally intersecting family, $X^2_b := [X^2;\mathcal{B}]$ is defined. The b-diagonal is the p-submanifold $\Delta_b := \beta^*(\Delta)$, where Δ is the diagonal in X^2. Given $H \in M_1(X)$, we define

$$lb(H) := \beta^*(H \times X);\ rb(H) := \beta^*(X \times H);\ f\!f(H) := \beta^*(H \times H).$$

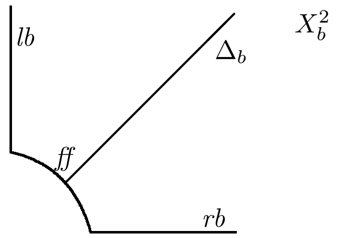


Figure 1: The b-streched product X_b^2 and the lifted diagonal $\Delta_b := \beta^*\Delta$.

Also, we define $lb := \cup_H lb(H)$, $rb := \cup_H rb(H)$, and $ff := \cup_H ff(H)$. Figure 1 gives a picture of X_b^2.

For each $m \in \mathbb{R}$, the space of *b-pseudodifferential operators of order* m is the space of kernels

$$\Psi_b^m(X, \Omega_b^{\frac{1}{2}}) := \{A \in I^m(X_b^2, \Delta_b, \Omega_b^{\frac{1}{2}}) \,|\, A \equiv 0 \text{ at } lb \cup rb\}.$$

Here, $I^m(X_b^2, \Delta_b, \Omega_b^{\frac{1}{2}})$ is the space of distributional b-half densities on X_b^2 which are conormal to Δ_b of degree m. Thus, $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$ iff

1. given any $\phi \in C_c^\infty(X_b^2 \setminus \Delta_b)$, we have $\phi\, A \in C^\infty(X_b^2, \Omega_b^{\frac{1}{2}})$ and $\phi\, A \equiv 0$ at $lb \cup rb$;
2. for any coordinate patch $\mathbb{R}_y^{n,k} \times \mathbb{R}_z^n$ on X_b^2 such that $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}$ and any compactly supported function ϕ on the coordinate patch, we have

$$\phi\, A = \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, a(y,\xi)\, d\xi \otimes \nu,$$

where $\nu \in C^\infty(X_b^2, \Omega_b^{\frac{1}{2}})$, and where $y \mapsto a(y,\xi) \in C^\infty(\mathbb{R}^{n,k}; S^m(\mathbb{R}^n)\,)$.

The (principal) symbol, ${}^b\sigma_m(A)$, of a b-pseudodifferential operator $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}}), m \in \mathbb{R}$, is an element of $S^{[m]}({}^bT^*X) = S^m({}^bT^*X)/S^{m-1}({}^bT^*X)$, where for any $p \in \mathbb{R}$, $S^p({}^bT^*X)$ are the symbols of order p on the b-cotangent bundle ${}^bT^*X$. A is called *elliptic* if ${}^b\sigma_m(A)$ is invertible.

The space $\Psi_{b,os}^m(X, \Omega_b^{\frac{1}{2}})$ consists of those elements of $\Psi_b^m(X, \Omega_b^{\frac{1}{2}})$ with classical symbols. Their symbols can be identified with elements of $C^\infty_{\hom(m)}({}^bT^*X)$, the homogeneous functions of degree m on ${}^bT^*X$.

Theorem 2.3 *Let $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}$. Then A defines a linear map*

$$A : \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}}) \,\to\, \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}}),$$

continuous in the C^∞ topology.

Assume now that each of the boundary hypersurfaces of X has a *fixed* boundary defining function associated to it.

Let $M \in M_k(X)$, $k \in \mathbb{N}$, and let $X \cong [0,1)^k \times M$ near M, where the coordinate functions on $[0,1)^k$ are the boundary defining functions for the hypersurfaces which define M. Let $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$. Then A induces an operator on $\dot{C}^\infty(M, \Omega_b^{\frac{1}{2}})$ defined as follows. Let $\phi \in \dot{C}^\infty(M, \Omega_b^{\frac{1}{2}})$ and let $\tilde{\phi} \in C^\infty(X, \Omega_b^{\frac{1}{2}})$ be such that $\tilde{\phi}|_M = \phi$ (here we use (13)). Then, $A|_M\phi := (A\tilde{\phi})|_M$. This is defined independent of the extension of ϕ and defines an operator $A|_M \in \Psi_b^m(M, \Omega_b^{\frac{1}{2}})$. The *normal operator of A at M* is the holomorphic family

$$\mathbb{C}^k \ni \tau \mapsto N_M(A)(\tau) := (x^{-i\tau} A x^{i\tau})|_M \in \Psi_b^m(M, \Omega_b^{\frac{1}{2}}),$$

where $x^{\pm i\tau} = x_1^{\pm i\tau_1} \cdots x_k^{\pm i\tau_k}$. Another way to describe $N_M(A)(\tau)$ is as follows. Observe that since $X \cong [0,1)^k \times M$ near M, we have $X_b^2 \cong ([0,1)_b^2)^k \times M_b^2$ near M_b^2. Hence by (13), we can identify

$\Omega_b^{\frac12}(X_b^2)|_{ff(M)}$ with $\Omega_b^{\frac12}(ff(([0,1)_b^2)^k)) \otimes \Omega_b^{\frac12}(M_b^2)$. Now let x and x' be the coordinate functions on the left and right factors of $[0,1)^k \times [0,1)^k$. Then, $s := x/x' := (x_1/x_1', \dots, x_k/x_k')$ define projective coordinates on $ff(([0,1)_b^2)^k)$, and $|\frac{ds}{s}|^{\frac12}$ trivializes $\Omega_b^{\frac12}(ff(([0,1)_b^2)^k))$. Hence, we can write $A|_{ff(M)} = B(s)|\frac{ds}{s}|^{\frac12}$, where $B(s) \in C^{-\infty}([0,\infty)^k \times M_b^2, \Omega_b^{\frac12}(M_b^2))$. Then, if $\tau \in \mathbb{C}^k$, $N_M(A)(\tau)$ is the Mellin transform:

$$N_M(A)(\tau) \equiv < A|_{ff(M)}, s^{-i\tau}|\frac{ds}{s}|^{\frac12} > = \int_0^\infty s^{-i\tau} B(s)\frac{ds}{s},$$

where $<,>$ denotes the distributional pairing. Define $L_b^2(X,\Omega_b^{\frac12})$ to be the completion of $\dot{C}^\infty(X,\Omega_b^{\frac12})$ with respect to the norm $\sqrt{\int_X |u|^2}$, where $u \in \dot{C}^\infty(X,\Omega_b^{\frac12})$. Then elements of $\Psi_b^0(X,\Omega_b^{\frac12})$ define bounded operators on $L_b^2(X,\Omega_b^{\frac12})$. For each $m \in \mathbb{R}$, we define the *Sobolev space of order* m by

$$H_b^m(X,\Omega_b^{\frac12}) := \{u \in C^{-\infty}(X,\Omega_b^{\frac12}) \mid \Psi_b^m(X,\Omega_b^{\frac12})u \subseteq L_b^2(X,\Omega_b^{\frac12})\}.$$

Note that $H_b^0(X,\Omega_b^{\frac12}) \equiv L_b^2(X,\Omega_b^{\frac12})$. If $A \in \Psi_b^m(X,\Omega_b^{\frac12})$, then for any $s \in \mathbb{R}$,

$$A : H_b^s(X,\Omega_b^{\frac12}) \to H_b^{s-m}(X,\Omega_b^{\frac12}).$$

2.5 Fredholm properties

Let α and β be multi-indices for lb and rb of X^2. Then we define

$$\Psi^{-\infty,\alpha,\beta}(X^2,\Omega_b^{\frac12}) := \rho_{lb}^{\alpha}\rho_{rb}^{\beta} H_b^\infty(X^2,\Omega_b^{\frac12}).$$

Let α, β, and η be multi-indices for lb, rb, and ff of X_b^2 respectively, with $\eta \geq 0$. Then we define

$$\Psi_b^{-\infty,\alpha,\beta,\eta}(X,\Omega_b^{\frac12}) := \bigcup_{\epsilon>0} \rho_{lb}^{\alpha+\epsilon}\rho_{rb}^{\beta+\epsilon} S_{ff}^{0,\eta}(X_b^2,\Omega_b^{\frac12}).$$

For any $m \in \mathbb{R}$, we define

$$\Psi_b^{m,\alpha,\beta,\eta}(X,\Omega_b^{\frac12}) := \Psi_b^m(X,\Omega_b^{\frac12}) + \Psi_b^{-\infty,\alpha,\beta,\eta}(X,\Omega_b^{\frac12}).$$

These spaces $\Psi_b^{m,*,*,*}(X,\Omega_b^{\frac12})$ form the 'calculus with bounds'. The following three fundamental results concern the Fredholm properties of b-pseudodifferential operators.

Proposition 2.4 *Let $A \in \Psi_b^m(X,\Omega_b^{\frac12})$, $m \in \mathbb{R}$, be elliptic and suppose that for some $\theta > 0$, for each $k \in \mathbb{N}$ and $M \in M_k(X)$,*

$$N_M(A)(\tau) : H_b^m(M,\Omega_b^{\frac12}) \to L_b^2(M,\Omega_b^{\frac12})$$

is invertible for all $\tau \in \mathbb{C}^k$ with $|\mathrm{Im}\,\tau_i| \leq \theta$. Then for each $\epsilon < \theta$, there exists a $B \in \Psi_b^{-m,\epsilon,\epsilon,\epsilon}(X,\Omega_b^{\frac12})$ such that

$$\begin{aligned} AB &= \mathrm{Id} - K_1; \\ BA &= \mathrm{Id} - K_2, \end{aligned}$$

where $K_1,\ K_2 \in \Psi^{-\infty,\epsilon,\epsilon}(X,\Omega_b^{\frac12})$. In particular, $A : H_b^m(X,\Omega_b^{\frac12}) \to L_b^2(X,\Omega_b^{\frac12})$ is Fredholm.

Theorem 2.5 *Let $A \in \Psi_{b,os}^m(X,\Omega_b^{\frac12})$, $m \in \mathbb{R}$. Then,*

$$A : H_b^m(X,\Omega_b^{\frac12}) \to L_b^2(X,\Omega_b^{\frac12})$$

is Fredholm iff A is elliptic and for each $k \in \mathbb{N}$ and $M \in M_k(X)$,

$$N_M(A)(\tau) : H_b^m(M,\Omega_b^{\frac12}) \to L_b^2(M,\Omega_b^{\frac12})$$

is invertible for all $\tau \in \mathbb{R}^k$.

This Theorem implies the following.

Corollary 2.6 *Let* $A \in \Psi^m_{b,os}(X, \Omega_b^{\frac{1}{2}}), m \in \mathbb{R}$. *Then*

$$A : H^m_b(X, \Omega_b^{\frac{1}{2}}) \,\rightarrow\, L^2_b(X, \Omega_b^{\frac{1}{2}})$$

is Fredholm iff A *is elliptic and for each* $H \in M_1(X)$,

$$N_H(A)(\tau) : H^m_b(H, \Omega_b^{\frac{1}{2}}) \,\rightarrow\, L^2_b(H, \Omega_b^{\frac{1}{2}})$$

is invertible for all $\tau \in \mathbb{R}$.

The following two results describe the behavior of the resolvent in the finite plane.

Proposition 2.7 *Let* $A \in \Psi^m_b(X, \Omega_b^{\frac{1}{2}}),\ m \in \mathbb{R}^+$, *be elliptic and formally self adjoint. Then given any open, relatively compact subset* $\mathcal{U} \subseteq \mathbb{C} \setminus \mathbb{R}$, *there exists an* $\epsilon > 0$ *such that*

$$\mathcal{U} \ni \lambda \,\mapsto\, (A - \lambda)^{-1} \in \mathcal{H}ol(\mathcal{U},\ \Psi_b^{-m,\epsilon,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}})).$$

If A *is also positive, then given any open, relatively compact subset* $\mathcal{U} \subseteq \mathbb{C} \setminus [0, \infty)$, *there exists an* $\epsilon > 0$ *such that*

$$\mathcal{U} \ni \lambda \,\mapsto\, (A - \lambda)^{-1} \in \mathcal{H}ol(\mathcal{U},\ \Psi_b^{-m,\epsilon,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}})).$$

Theorem 2.8 *(Analytic Fredholm Theory) Let* $A \in \Psi^m_b(X, \Omega_b^{\frac{1}{2}}),\ m \in \mathbb{R}^+$, *be elliptic and formally self adjoint and suppose that* A *is Fredholm. Then there exists an open subset* $\mathcal{U} \subseteq \mathbb{C}$ *containing zero, such that given any open, relatively compact subset* $\mathcal{U}' \subseteq \mathbb{C} \setminus \mathbb{R}$, *there exists an* $\epsilon > 0$ *such that*

$$\mathcal{U} \cup \mathcal{U}' \ni \lambda \,\mapsto\, (A - \lambda)^{-1} \in \Psi_b^{-m,\epsilon,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}})$$

is meromorphic having only simple poles, all in a discrete subset $\{\lambda_j\} \subset \mathbb{R}$, *which are (minus) the self adjoint projections onto* $\ker(A - \lambda_j)$ *at* λ_j. *If* A *is also positive, then* $\mathcal{U}'$ *can be chosen to be a subset of* $\mathbb{C} \setminus [0, \infty)$*; and the same result holds, but with* $\{\lambda_j\} \subseteq [0, \infty)$.

3 The Resolvent

3.1 Tempered symbols

Let X be compact with corners and let $A \in \Psi^m_b(X, \Omega_b^{\frac{1}{2}}),\ m \in \mathbb{R}^+$, be elliptic, formally self adjoint, and positive. Let $\Lambda \subseteq \mathbb{C} \setminus (0, \infty)$ be a *closed cone* (see Figure 2). Thus, Λ is a codimension two manifold with corners such that $\lambda \in \Lambda$ implies $r\lambda \in \Lambda$ for all $r \geq 0$. Then Proposition 2.7 gives a description of the resolvent $(A - \lambda)^{-1}$ for finite $\lambda \in \Lambda$ (away from $\lambda = 0$). In this section, we will give a precise description of the resolvent for large $\lambda \in \Lambda$. The idea is as follows. We

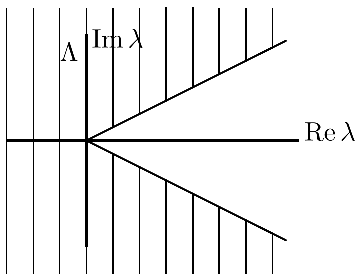


Figure 2: The closed cone $\Lambda \subseteq \mathbb{C} \setminus (0, \infty)$.

define a subspace of $C^\infty(\,\Lambda, \Psi_b^*(X, \Omega_b^{\frac{1}{2}})\,)$, of pseudodifferential operators which depend smoothly on a parameter $\lambda \in \Lambda$, where we believe the resolvent lives. Our next step is to show that this space in fact captures the resolvent. What properties should the local symbols of such operators satisfy? Well, if $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ is such that $\chi(\xi) \equiv 0$ near 0 and $\chi(\xi) \equiv 1$ outside a neighborhood of 0, then we clearly want $\chi(\xi)\, a(\lambda, \xi)$, where

$$a(\lambda, \xi) = (|\xi|^m - \lambda)^{-1} \in C^\infty(\Lambda \times (\mathbb{R}^n \setminus \{0\})\,),$$

to be such a symbol. What kind of symbol estimates does $a(\lambda, \xi)$ satisfy? Fix $\alpha \in \mathbb{N}_0^2$ and $\beta \in \mathbb{N}_0^n$. Observe that there exists a $C > 0$ such that

$$|\,(\partial_\lambda^\alpha \partial_\xi^\beta a)(\,\lambda, \xi/|\xi|\,)\,| \leq C(1 + |\lambda|)^{-1-|\alpha|}. \tag{16}$$

Now observe that $a(\lambda, \xi)$ satisfies the homogeneity property

$$a(\delta^m \lambda, \delta\xi) = \delta^{-m} a(\lambda, \xi) \text{ for all } \delta > 0.$$

Hence, $(\partial_\lambda^\alpha \partial_\xi^\beta a)(\lambda, \xi)$ satisfies the homogeneity property

$$(\partial_\lambda^\alpha \partial_\xi^\beta a)(\delta^m \lambda, \delta\xi) = \delta^{-m-m|\alpha|-|\beta|} (\partial_\lambda^\alpha \partial_\xi^\beta a)(\lambda, \xi) \text{ for all } \delta > 0.$$

Combining this property with inequality (16), we find that

$$\begin{aligned}
|\,(\partial_\lambda^\alpha \partial_\xi^\beta a)(\lambda, \xi)\,| &= |\,(\partial_\lambda^\alpha \partial_\xi^\beta a)(\,|\xi|^m\,|\xi|^{-m}\lambda, |\xi|\,\xi/|\xi|\,)\,| \\
&= |\xi|^{-m-m|\alpha|-|\beta|}\,|\,(\partial_\lambda^\alpha \partial_\xi^\beta a)(\,|\xi|^{-m}\lambda, \xi/|\xi|\,)\,| \\
&\leq C\,|\xi|^{-m-m|\alpha|-|\beta|}\,(\,1 + |\,|\xi|^{-m}\lambda\,|\,)^{-1-|\alpha|} \\
&\leq C\,(|\lambda| + |\xi|^m)^{-1-|\alpha|}\,|\xi|^{-|\beta|} \\
&\leq C'\,(|\lambda|^{1/m} + |\xi|)^{-m-m|\alpha|}\,|\xi|^{-|\beta|}.
\end{aligned}$$

Thus, for each α and β, there exists a $C > 0$ such that

$$|\,\partial_\lambda^\alpha \partial_\xi^\beta(\chi(\xi)\,a(\lambda, \xi))\,| \leq C\,(1 + |\lambda|^{1/m} + |\xi|)^{-m-m|\alpha|}\,(1 + |\xi|)^{-|\beta|}$$

for all $\lambda \in \Lambda$ and $\xi \in \mathbb{R}^n$. This computation 'motivates' the following definition.

Definition 3.1 Let m, $p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and $\Lambda \subseteq \mathbb{C}$ be a closed cone. Then the space of *tempered symbols*, $S_\Lambda^{m,p,d}(\mathbb{R}^n)$, consists of those functions $a(\lambda, \xi) \in C^\infty(\Lambda \times \mathbb{R}^n)$ satisfying the following estimates: for any α and β, there is a $C > 0$ such that

$$|\,\partial_\lambda^\alpha \partial_\xi^\beta a(\lambda, \xi)\,| \leq C\,(1 + |\lambda|^{1/d} + |\xi|)^{p-d|\alpha|}\,(1 + |\xi|)^{m-p-|\beta|}.$$

Thus, $\chi(\xi)\,(|\xi|^m - \lambda)^{-1} \in S_\Lambda^{-m,-m,m}(\mathbb{R}^n)$, where Λ is any closed cone not intersecting $(0, \infty)$.

Given closed cones Λ, $\Lambda' \subseteq \mathbb{C}$, Λ is said to be a *proper subcone* of Λ' if $\Lambda \subseteq \mathrm{int}(\Lambda') \cup \{0\}$. Let $a(\xi) \in S^m(\mathbb{R}^n)$, $m \in \mathbb{R}^+$ be elliptic and let $\Lambda \subseteq \mathbb{C}$ be a closed cone. Suppose that Λ is a proper subcone of a closed cone $\Lambda' \subseteq \mathbb{C}$ and suppose that for some constant $r > 0$, $a(\xi) \notin \Lambda'$ for all $|\xi| \geq r$. Recall that since $a(\xi)$ is elliptic, there exists a constant $\epsilon > 0$ such that

$$\epsilon|\xi|^m \;\leq\; |a(\xi)| \;\leq\; \frac{1}{\epsilon}|\xi|^m$$

for all $|\xi| \geq r$. It follows that if $N \in \mathbb{Z}$, then there is a constant $C > 0$ such that

$$|\,(\,a(\xi) - \lambda)^N\,| \leq C\,(1 + |\lambda| + |\xi|^m)^N$$

for all $\lambda \in \Lambda$ and $|\xi| \geq r$. In particular, for any $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ for $|\xi| \leq r$ and $\chi(\xi) \equiv 1$ for $|\xi| \geq r + 1$, there is a $C' > 0$ such that

$$|\,\chi(\xi)\,(\,a(\xi) - \lambda)^N\,| \leq C'\,(1 + |\lambda|^{1/m} + |\xi|)^{Nm}$$

for all $(\lambda, \xi) \in \Lambda \times \mathbb{R}^n$. In fact, we have the following.

Lemma 3.2 *Let $\Lambda \subseteq \Lambda'$ be a proper subcone of a closed cone $\Lambda' \subseteq \mathbb{C}$. Let $a(\xi) \in S^m(\mathbb{R}^n)$ be elliptic and suppose that for some $r > 0$, $a(\xi) \notin \Lambda'$ for all $|\xi| \geq r$. Then for any $N \in \mathbb{Z}$ and $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ for $|\xi| \leq r$ and $\chi(\xi) \equiv 1$ for $|\xi| \geq r+1$, we have*

$$\chi(\xi)\,(\,a(\xi) - \lambda)^N \in S_\Lambda^{Nm,Nm,m}(\mathbb{R}^n).$$

Proof: We will leave this as an exercise for the reader. ■

Lemma 3.3 *Let $\Lambda \subseteq \mathbb{C}$ be a closed cone. Then,*

1. *for each $m \in \mathbb{R}$, $S^m(\mathbb{R}^n) \subseteq S_\Lambda^{m,p,d}(\mathbb{R}^n)$ for all $p \geq 0$;*
2. *for any m, m', p, $p' \in \mathbb{R}$ and $d \in \mathbb{R}^+$, we have*

$$S_\Lambda^{m,p,d}(\mathbb{R}^n) \cdot S_\Lambda^{m',p',d}(\mathbb{R}^n) \subseteq S_\Lambda^{m+m',p+p',d}(\mathbb{R}^n);$$

3. *if $m \leq m'$ and $p \leq p'$, then for any $d \in \mathbb{R}^+$, $S_\Lambda^{m,p,d}(\mathbb{R}^n) \subseteq S_\Lambda^{m',p',d}(\mathbb{R}^n)$;*
4. *if m, $p \in \mathbb{R}$, and $d \in \mathbb{R}^+$, for any α and β,*

$$\partial_\lambda^\alpha \partial_\xi^\beta S_\Lambda^{m,p,d}(\mathbb{R}^n) \subseteq S_\Lambda^{m-d|\alpha|-|\beta|,p-d|\alpha|,d}(\mathbb{R}^n).$$

Proof: These properties follow directly from the definition of tempered symbols. The details will be left for the reader. ■

Let $\Lambda \subseteq \mathbb{C}$ be a closed cone. We define the space $S^m(\Lambda)$ as the space of functions $a(\lambda) \in S^m(\Lambda)$ satisfying the following estimates: for each α, there is a $C > 0$ such that

$$|\partial_\lambda^\alpha a(\lambda)\,| \leq C(1 + |\lambda|)^{m-|\alpha|}.$$

If $\mathcal{F}$ is any Frechet space, the space $S^m(\Lambda; \mathcal{F})$ of $\mathcal{F}$ valued symbols of order m on Λ is defined similarly.

Lemma 3.4 *Let $\Lambda \subseteq \mathbb{C}$ be a closed cone, m, $p \in \mathbb{R}$, and $d \in \mathbb{R}^+$. If $p > 0$, then*

$$S_\Lambda^{m,p,d}(\mathbb{R}^n) \subseteq S^{p/d}(\Lambda; S^m(\mathbb{R}^n)\,),$$

and if $p \leq 0$, then

$$S_\Lambda^{m,p,d}(\mathbb{R}^n) \subseteq S^{p/d}(\Lambda; S^{m-p}(\mathbb{R}^n)\,).$$

In particular, $S_\Lambda^{-\infty,p,d}(\mathbb{R}^n) \equiv S^{p/d}(\Lambda, \mathcal{S}(\mathbb{R}^n)\,)$, where $S^{p/d}(\Lambda, \mathcal{S}(\mathbb{R}^n)\,)$ are the Schwartz space valued symbols of order p/d on Λ.

Proof: Fix α. Let $p \leq 0$. Then,

$$(1 + |\lambda|^{1/d} + |\xi|)^{p-d|\alpha|} \leq (1 + |\lambda|^{1/d})^{p-d|\alpha|}.$$

It follows that $S_\Lambda^{m,p,d}(\mathbb{R}^n) \subseteq S^{p/d}(\Lambda; S^{m-p}(\mathbb{R}^n)\,)$.

Now let $p > 0$, Then, if $p - d|\alpha| \leq 0$, $(1+|\lambda|^{1/d}+|\xi|)^{p-d|\alpha|} \leq (1+|\lambda|^{1/d})^{p-d|\alpha|}$ and if $p-d|\alpha| > 0$, $(1 + |\lambda|^{1/d} + |\xi|)^{p-d|\alpha|} \leq (1 + |\lambda|^{1/d})^{p-d|\alpha|}\,(1 + |\xi|)^{p-d|\alpha|}$. In both these cases,

$$(1 + |\lambda|^{1/d} + |\xi|)^{p-d|\alpha|}(1 + |\xi|)^{m-p} \leq (1 + |\lambda|^{1/d})^{p-d|\alpha|}\,(1 + |\xi|)^m.$$

It follows that $S_\Lambda^{m,p,d}(\mathbb{R}^n) \subseteq S^{p/d}(\Lambda; S^m(\mathbb{R}^n)\,)$. ■

3.2 Resolvent tempered symbols

Fix a closed cone $\Lambda \subseteq \mathbb{C}$. In the next three subsections, we give various refinements of the tempered symbol spaces $S_{\Lambda}^{*,*,*}(\mathbb{R}^n)$. Our first refinement are the symbols that have more 'resolvent-like' behavior. Thus, let $a(\xi) \in S^m(\mathbb{R}^n)$, $m \in \mathbb{R}^+$, be elliptic. Suppose that Λ is a proper subcone of a closed cone $\Lambda' \subseteq \mathbb{C}$ and suppose that for some constant $r > 0$, $a(\xi) \notin \Lambda'$ for all $|\xi| \geq r$. Let $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ for $|\xi| \leq r$ and $\chi(\xi) \equiv 1$ for $|\xi| \geq r+1$. Then by Lemma 3.2 with $N = -1$, we know that

$$a(\lambda, \xi) := \chi(\xi)\,(\,a(\xi) - \lambda)^{-1} \in S_{\Lambda}^{-m,-m,m}(\mathbb{R}^n).$$

What other properties does $a(\lambda, \xi)$ have; especially at $\lambda = \infty$? Define $\Lambda_{cc} = \{\overline{\lambda} \,|\, \lambda \in \Lambda\}$ and set $\mu := 1/\lambda = \overline{\lambda}/|\lambda|^2 \in \Lambda_{cc}$, where $\lambda \in \Lambda \setminus \{0\}$. Then, observe that

$$a(1/\mu, \xi) = \mu\,\chi(\xi)\,(\,\mu\,a(\xi) - 1)^{-1}.$$

Thus, if we define $\tilde{a}(\mu, \xi) := \mu^{-1}\,a(1/\mu, \xi) = \chi(\xi)\,(\,\mu\,a(\xi) - 1)^{-1}$, then $\tilde{a}(\mu, \xi) \in C^\infty(\Lambda_{cc} \times \mathbb{R}^n)$. Moreover, Since $a(\xi)$ is elliptic, there exists a constant $\epsilon > 0$ such that $\epsilon|\xi|^m \leq |a(\xi)| \leq \frac{1}{\epsilon}|\xi|^m$ for all $|\xi| \geq r$. It follows that

$$|\,\tilde{a}(\mu, \xi)\,| \leq C\,(1 + |\mu|\,|\xi|^m)^{-1}$$

for some $C > 0$. In fact, one can easily check that it satisfies the estimates: for any α and β there is a $C > 0$ such that

$$|\,\partial_\mu^\alpha \partial_\xi^\beta \tilde{a}(\mu, \xi)\,| \leq C\,(1 + |\mu|\,|\xi|^m)^{-1-|\alpha|}\,(1 + |\xi|)^{m|\alpha| - |\beta|}$$

for all $(\mu, \xi) \in \Lambda_{cc} \times \mathbb{R}^n$.

Definition 3.5 Let m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. The subspace $S_{\Lambda,r}^{m,p,d}(\mathbb{R}^n) \subseteq S_{\Lambda}^{m,p,d}(\mathbb{R}^n)$ of *resolvent tempered symbols* consists of those elements $a(\lambda, \xi) \in S_{\Lambda}^{m,p,d}(\mathbb{R}^n)$ such that if we define

$$\tilde{a}(\mu, \xi) := \mu^{p/d}\,a(1/\mu, \xi) = \mu^{p/d}\,a(\overline{\mu}/|\mu|^2, \xi) \text{ for all } (\mu, \xi) \in \Lambda_{cc} \times \mathbb{R}^n,$$

then $\tilde{a}(\mu, \xi) \in C^\infty(\Lambda_{cc} \times \mathbb{R}^n)$[3] and it satisfies the estimates: for any α and β there is a $C > 0$ such that

$$|\,\partial_\mu^\alpha \partial_\xi^\beta \tilde{a}(\mu, \xi)\,| \leq C\,(1 + |\mu|\,|\xi|^d)^{p/d - |\alpha|}\,(1 + |\xi|)^{d|\alpha| + m - p - |\beta|}$$

for all $(\mu, \xi) \in \Lambda_{cc} \times \mathbb{R}^n$.

Lemma 3.6 *Let $\Lambda \subseteq \Lambda'$ be a proper subcone of a closed cone $\Lambda' \subseteq \mathbb{C}$. Let $a(\xi) \in S^m(\mathbb{R}^n)$ be elliptic and suppose that for some $r > 0$, $a(\xi) \notin \Lambda'$ for all $|\xi| \geq r$. Then for any $N \in \mathbb{Z}$ and $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ for $|\xi| \leq r$ and $\chi(\xi) \equiv 1$ for $|\xi| \geq r+1$, we have*

$$\chi(\xi)\,(\,a(\xi) - \lambda)^N \in S_{\Lambda,r}^{Nm,Nm,m}(\mathbb{R}^n).$$

Proof: We will leave this as an exercise for the reader. ■

Lemma 3.7 *Let m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. Then given any sequence $a_{m-j}(\lambda, \xi) \in S_{\Lambda,r}^{m-j,p,d}(\mathbb{R}^n)$, there exists an $a(\lambda, \xi) \in S_{\Lambda,r}^{m,p,d}(\mathbb{R}^n)$ such that for all $N \in \mathbb{N}$,*

$$a(\lambda, \xi) - \sum_{j=0}^{N-1} a_{m-j}(\lambda, \xi) \in S_{\Lambda,r}^{m-N,p,d}(\mathbb{R}^n).$$

Proof: The proof of this Lemma follows almost along the same line of reasoning used to prove the usual 'Asymptotic Summation Lemma' for the usual symbols. Thus, we will leave the details to the reader. ■

[3] Thus, $\tilde{a}(\mu, \xi)$ is smooth even at $\mu = 0$.

3.3 Polyhomogeneous tempered symbols

We will now describe a subset $S^{m,p,d}_{\Lambda,ros}(\mathbb{R}^n) \subseteq S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$ of one-step, polyhomogeneous symbols.

Lemma 3.8 *Let $a(\lambda,\xi) \in C^\infty(\,(\Lambda \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\})\,)$ be such that for some m, $e \in \mathbb{R}$,*

$$a(\delta^e \lambda, \delta\xi) = \delta^m a(\lambda,\xi) \text{ for all } \delta > 0$$

and such that $a(\lambda, \xi/|\xi|) \in C^\infty(\Lambda \times \mathbb{S}^{n-1})$. Then, $a(\lambda,\xi) \in C^\infty(\Lambda \times (\mathbb{R}^n \setminus \{0\})\,)$ and there exists an $\epsilon > 0$ such that

1. *for any β, $(\partial^\beta_\xi a)(\lambda, \xi/|\xi|) \in C^\infty(\,(\Lambda \cup B_\epsilon) \times \mathbb{S}^{n-1})$, where $B_\epsilon = \{\lambda \in \mathbb{C} \,|\, |\lambda| \le \epsilon\}$;*
2. *for any $r > 0$,*
$$a(\lambda,\xi) \in \begin{cases} C^\infty(\,(\Lambda \cup B_{\epsilon r^e}) \times \{|\xi| \ge r\}\,), & \text{if } e \ge 0; \\ C^\infty(\,(\Lambda \cup B_{\epsilon r^e}) \times \{0 < |\xi| \le r\}\,), & \text{if } e < 0. \end{cases}$$

PROOF: Since $a(\lambda,\xi/|\xi|) \in C^\infty(\Lambda \times \mathbb{S}^{n-1})$, by definition of $C^\infty(\Lambda \times \mathbb{S}^{n-1})$, there exists an $\epsilon > 0$ such that

$$a(\lambda, \xi/|\xi|) \in C^\infty(\,(\Lambda \cup B_\epsilon) \times \mathbb{S}^{n-1}). \tag{17}$$

Observe that

$$a(\lambda,\xi) = a(\,|\xi|^e|\xi|^{-e}\lambda, |\xi|\,\xi/|\xi|\,) = |\xi|^m a(\,|\xi|^{-e}\lambda, \xi/|\xi|\,). \tag{18}$$

Then (17) and (18) implies property (1). To see (2), observe that (17) and (18) implies that $a(\lambda,\xi)$ is smooth in λ and ξ if $|\xi|^{-e}|\lambda| \le \epsilon$ and $|\xi| > 0$; that is, $|\lambda| \le \epsilon|\xi|^e$ and $|\xi| > 0$. If $e \ge 0$, this holds when $|\lambda| \le \epsilon r^e$ and $|\xi| \ge r$; and if $e < 0$, this holds when $|\lambda| \le \epsilon r^e$ and $0 < |\xi| \le r$. ■

Recall that $\Lambda_{cc} = \{\overline{\lambda} \,|\, \lambda \in \Lambda\}$.

Proposition 3.9 *Let m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. Let $a(\lambda,\xi) \in C^\infty(\Lambda \times (\mathbb{R}^n \setminus \{0\})\,)$ be such that*

$$a(\delta^d\lambda, \delta\xi) = \delta^m a(\lambda,\xi) \text{ for all } \delta > 0.$$

Also, suppose that if we define

$$\tilde{a}(\mu,\xi) := \mu^{p/d}\, a(1/\mu, \xi) = \mu^{p/d}\, a(\overline{\mu}/|\mu|^2, \xi) \text{ for all } (\mu,\xi) \in \Lambda_{cc} \times \mathbb{R}^n,$$

then we have $\tilde{a}(\mu,\xi/|\xi|) \in C^\infty(\Lambda_{cc} \times \mathbb{S}^{n-1})$[4]. Then,

1. *For any α and β there is a $C > 0$ such that*
$$|\,\partial^\alpha_\lambda \partial^\beta_\xi a(\lambda,\xi)\,| \le C\,(\,|\lambda|^{1/d} + |\xi|\,)^{p-d|\alpha|}\,|\xi|^{m-p-|\beta|}.$$
2. *$\tilde{a}(\mu,\xi) \in C^\infty(\Lambda_{cc} \times \mathbb{R}^n)$ and for any α and β there is a $C > 0$ such that*
$$|\,\partial^\alpha_\mu \partial^\beta_\xi \tilde{a}(\mu,\xi)\,| \le C\,(1 + |\mu|\,|\xi|^d)^{p/d-|\alpha|}\,|\xi|^{d|\alpha|+m-p-|\beta|}.$$

In particular, for any $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ near $\xi = 0$ and $\chi(\xi) \equiv 1$ outside a neighborhood of 0, we have $\chi(\xi)\,a(\lambda,\xi) \in S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$.

PROOF: First observe that $\tilde{a}(\mu,\xi) \in C^\infty(\Lambda_{cc} \times (\mathbb{R}^n \setminus \{0\}))$ by Lemma 3.8. We will now prove (1). Since $\tilde{a}(\mu,\xi/|\xi|) \in C^\infty(\Lambda_{cc} \times \mathbb{S}^{n-1})$, by Lemma 3.8, $(\partial^\beta_\xi \tilde{a})(\mu,\xi/|\xi|) \in C^\infty(\Lambda_{cc} \times \mathbb{S}^{n-1})$ for any β. Thus, if α and β are given, there is a $C > 0$ such that for all $|\lambda| \ge 1$ (that is, $|\mu| \le 1$),

$$\begin{aligned} |\,\lambda^\alpha(\partial^\alpha_\lambda \partial^\beta_\xi a)(\lambda,\xi/|\xi|)\,| &= |\,\mu^{-\alpha}(\,(-\mu^2\partial_\mu)^\alpha \partial^\beta_\xi \mu^{-p/d}\tilde{a})(\mu,\xi/|\xi|)\,| \\ &\le C\,|\mu|^{-p/d} \\ &= C\,|\lambda|^{p/d}. \end{aligned}$$

[4]Thus, $\tilde{a}(\mu,\xi/|\xi|)$ is smooth even at $\mu = 0$

Hence, as $a(\lambda,\xi/|\xi|) \in C^\infty(\Lambda\times\mathbb{S}^{n-1})$, there is a $C>0$ such that

$$|\,(\partial_\lambda^\alpha\partial_\xi^\beta a)(\lambda,\xi/|\xi|)\,| \le C\,(1+|\lambda|)^{p/d-|\alpha|} \tag{19}$$

for all $(\lambda,\xi)\in\Lambda\times(\mathbb{R}^n\setminus\{0\})$. Since $a(\delta^d\lambda,\delta\xi)=\delta^m a(\lambda,\xi)$ for all $\delta>0$,

$$(\partial_\lambda^\alpha\partial_\xi^\beta a)(\delta^d\lambda,\delta\xi) = \delta^{m-d|\alpha|-|\beta|}(\partial_\lambda^\alpha\partial_\xi^\beta a)(\lambda,\xi) \text{ for all } \delta>0. \tag{20}$$

Combining (19) and (20), we conclude that

$$\begin{aligned}
|\,\partial_\lambda^\alpha\partial_\xi^\beta a(\lambda,\xi)\,| &= |\,(\partial_\lambda^\alpha\partial_\xi^\beta a)(\,|\xi|^d|\xi|^{-d}\lambda,|\xi|\,\xi/|\xi|\,)\,| \\
&= |\xi|^{m-d|\alpha|-|\beta|}\,|\,(\partial_\lambda^\alpha\partial_\xi^\beta a)(\,|\xi|^{-d}\lambda,\xi/|\xi|\,) \\
&\le |\xi|^{m-d|\alpha|-|\beta|}\,C\,(1+|\xi|^{-d}|\lambda|\,)^{p/d-|\alpha|} \\
&= C\,(\,|\lambda|+|\xi|^d\,)^{p/d-|\alpha|}\,|\xi|^{m-p-|\beta|} \\
&\le C\,(\,|\lambda|^{1/d}+|\xi|\,)^{p-d|\alpha|}\,|\xi|^{m-p-|\beta|}.
\end{aligned}$$

We will leave the proof of (2) to the reader. Its proof follows exactly the same line of reasoning used to prove (1). ■

Definition 3.10 Let $m,p\in\mathbb{R}$ and $d\in\mathbb{R}^+$ with $p/d\in\mathbb{Z}$. The space $C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$ consists of those functions $a(\lambda,\xi)\in C^\infty(\Lambda\times(\mathbb{R}^n\setminus\{0\}))$ such that

1. $a(\delta^d\lambda,\delta\xi)=\delta^m a(\lambda,\xi)$ for all $\delta>0$;
2. if we define $\tilde a(\mu,\xi) := \mu^{p/d}\,a(1/\mu,\xi) = \mu^{p/d}\,a(\overline{\mu}/|\mu|^2,\xi)$ for all $(\mu,\xi)\in\Lambda_{cc}\times(\mathbb{R}^n\setminus\{0\})$, then we have $\tilde a(\mu,\xi/|\xi|)\in C^\infty(\Lambda_{cc}\times\mathbb{S}^{n-1})$.

Lemma 3.11 *For any $m\in\mathbb{R}$, $p\ge 0$, and $d\in\mathbb{R}^+$ such that $p/d\in\mathbb{N}_0$, we have*

$$C^\infty_{\mathrm{hom}(m)}(\mathbb{R}^n)\subseteq C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n).$$

Proof: We will leave this as an exercise for the reader. ■

Definition 3.12 Let $m,p\in\mathbb{R}$ and $d\in\mathbb{R}^+$ with $p/d\in\mathbb{Z}$. The space of *resolvent one-step, polyhomogeneous symbols*, $S^{m,p,d}_{\Lambda,ros}(\mathbb{R}^n)\subseteq S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$ consists of those elements $a(\lambda,\xi)\in S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$ such that

$$a(\lambda,\xi)\sim\sum_{j=0}^\infty \chi(\xi)\,a_{m-j}(\lambda,\xi), \tag{21}$$

where for each j, $a_{m-j}(\lambda,\xi)\in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m-j)}(\mathbb{R}^n)$, where $\chi(\xi)\in C^\infty(\mathbb{R}^n)$ with $\chi(\xi)\equiv 0$ near $\xi=0$ and $\chi(\xi)\equiv 1$ outside a neighborhood of 0, and where the asymptotic sum (21) is in the sense of Lemma 3.7.

Lemma 3.13 *Let $m,p\in\mathbb{R}$ and $d\in\mathbb{R}^+$ with $p/d\in\mathbb{Z}$. Then for any $\chi(\xi)\in C^\infty(\mathbb{R}^n)$ with $\chi(\xi)\equiv 0$ near $\xi=0$ and $\chi(\xi)\equiv 1$ outside a neighborhood of 0, we have*

$$\chi(\xi)\,C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)\subseteq S^{m,p,d}_{\Lambda,ros}(\mathbb{R}^n).$$

Lemma 3.14 *Let $a(\xi)\in C^\infty_{\mathrm{hom}(m)}(\mathbb{R}^n)$, $m\in\mathbb{R}^+$ be elliptic and suppose that $a(\xi)$ never takes values in the cone Λ for $\xi\ne 0$. Then for all $N\in\mathbb{Z}$,*

$$(\,a(\xi)-\lambda)^N\in C^{\infty,Nm,m}_{\Lambda,\mathrm{hom}(Nm)}(\mathbb{R}^n).$$

In particular, for any $\chi(\xi)\in C^\infty(\mathbb{R}^n)$ with $\chi(\xi)\equiv 0$ near $\xi=0$ and $\chi(\xi)\equiv 1$ outside a neighborhood of 0, $\chi(\xi)\,(\,a(\xi)-\lambda)^N\in S^{Nm,Nm,m}_{\Lambda,ros}(\mathbb{R}^n)$.

PROOF: We will leave the verification that $(a(\xi)-\lambda)^N$ satisfies (1) and (2) of Definition 3.10 to the reader. ■

If $m \in \mathbb{R}$, we define $S^m_{os}(\Lambda)$ as those $a(\lambda) \in C^\infty(\Lambda)$ such that $a(\lambda) \sim \sum_{j=0}^{\infty} \chi(\lambda)\, a_{m-j}(\lambda)$, where $\chi(\lambda) \equiv 0$ near $\lambda = 0$ and outside a neighborhood of 0, and where $a_{m-j}(\lambda) \in C^\infty(\Lambda \setminus \{0\})$ is homogeneous of degree $m-j$. Here, the asymptotic sum means that $a(\lambda) - \sum_{j=0}^{N-1} \chi(\lambda)\, a_{m-j}(\lambda) \in S^{m-N}(\Lambda)$ for each $N \in \mathbb{N}$. If $\mathcal{F}$ is any Frechet space, the space $S^m_{os}(\Lambda; \mathcal{F})$ of $\mathcal{F}$ valued one-step symbols of order m on Λ is defined similarly.

Lemma 3.15 *Let $\Lambda \subseteq \mathbb{C}$ be a closed cone. Then*

$$S^{-\infty,p,d}_{\Lambda,ros}(\mathbb{R}^n) \equiv S^{p/d}_{os}(\Lambda, \mathcal{S}(\mathbb{R}^n)).$$

PROOF: We will leave this as an exercise for the reader. ■

3.4 Resolvent like symbols

Let $a(\xi) \in S^m(\mathbb{R}^n)$, $m \in \mathbb{R}^+$ be elliptic and let $\Lambda \subseteq \mathbb{C}$ be a closed cone. Suppose that Λ is a proper subcone of a closed cone $\Lambda' \subseteq \mathbb{C}$ and suppose that for some constant $r > 0$, $a(\xi) \notin \Lambda'$ for all $|\xi| \geq r$. Since $a(\xi)$ is elliptic, there exists a constant $\delta > 0$ such that

$$\delta(1+|\xi|)^m \;\leq\; |a(\xi)| \;\leq \frac{1}{\delta}(1+|\xi|)^m$$

for all $|\xi| \geq r$. Let $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ for $|\xi| \leq r$ and $\chi(\xi) \equiv 1$ for $|\xi| \geq r+1$. Then it follows that $a(\lambda,\xi) := \chi(\xi)\,(a(\xi)-\lambda)^{-1} \in S^{-m,-m,m}_{\Lambda}(\mathbb{R}^n)$ is such that given any $0 < \epsilon < \delta$, for each $\xi \in \mathbb{R}^n$, $\lambda \mapsto a(\lambda,\xi)$ extends to be a holomorphic function for all

$$\lambda \in \Lambda \cup \{\lambda \in \mathbb{C} \,|\, |\lambda| \leq \epsilon\,(1+|\xi|)^m \;\text{ or }\; \frac{1}{\epsilon}\,(1+|\xi|)^m \leq |\lambda|\}.$$

Thus, $a(\lambda,\xi)$ extends to be a smooth function, holomorphic in λ, for (λ,ξ) in

$$\{(\lambda,\xi) \in \mathbb{C}\times\mathbb{R}^n \,|\, \lambda \in \Lambda \;\text{ or }\; |\lambda| \leq \epsilon\,(1+|\xi|)^m \;\text{ or }\; \frac{1}{\epsilon}\,(1+|\xi|)^m \leq |\lambda|\}. \tag{22}$$

Moreover, observe that for any α and β, there is a $C > 0$ such that

$$|\,\partial^\alpha_\lambda \partial^\beta_\xi a(\lambda,\xi)\,| \leq C\,(1+|\lambda|^{1/m}+|\xi|\,)^{-m-d|\alpha|}\,(1+|\xi|\,)^{-|\beta|}$$

for all (λ,ξ) in the set given by (22); that is, $a(\lambda,\xi)$ continues to satisfy the same symbol estimates for (λ,ξ) in the set given by (22).

Definition 3.16 Let m, $p \in \mathbb{R}$, and let $d \in \mathbb{R}^+$. Then a symbol $a(\lambda,\xi) \in S^{m,p,d}_{\Lambda}(\mathbb{R}^n)$ is said to be *resolvent like* if there exists an $\epsilon > 0$ such that $a(\lambda,\xi)$ extends to be a smooth function, holomorphic in λ, for (λ,ξ) in

$$\{(\lambda,\xi) \in \mathbb{C}\times\mathbb{R}^n \,|\, \lambda \in \Lambda \;\text{ or }\; |\lambda| \leq \epsilon\,(1+|\xi|)^d \;\text{ or }\; \frac{1}{\epsilon}\,(1+|\xi|)^d \leq |\lambda|\}; \tag{23}$$

and moreover, $a(\lambda,\xi)$ continues to satisfy the same symbol estimates for (λ,ξ) in the set given by (23).

Definition 3.17 Let m, $p \in \mathbb{R}$, and let $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. Then a function $a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$ is said to be *resolvent like* if for any $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ near 0 and with $\chi(\xi) \equiv 1$ outside a neighborhood of 0, the symbol $\chi(\xi)\,a(\lambda,\xi) \in S^{m,p,d}_{\Lambda}(\mathbb{R}^n)$ is resolvent like.

Lemma 3.18 *Let m, $p \in \mathbb{R}$, and let $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. Let $a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$. Then the following are equivalent.*

1. *$a(\lambda,\xi)$ is resolvent like.*

2. *There exists an $\epsilon>0$ such that $a(\lambda,\xi/|\xi|\,)$ extends to be a smooth function, holomorphic in λ, for $(\lambda,\xi/|\xi|\,)\in\{(\lambda,\xi/|\xi|\,)\in\mathbb{C}\times\mathbb{S}^{n-1}\mid\lambda\in\Lambda\ \text{ or }\ |\lambda|\le\epsilon\ \text{ or }\ \frac{1}{\epsilon}\le|\lambda|\}$.*

3. *There exists an $\epsilon>0$ such that $a(\lambda,\xi)$ extends to be a smooth function, holomorphic in λ, for $(\lambda,\xi)\in\{(\lambda,\xi)\in\mathbb{C}\times(\mathbb{R}^n\setminus\{0\})\,|\,\lambda\in\Lambda\ \text{ or }\ |\lambda|\le\epsilon\,|\xi|^d\ \text{ or }\ \frac{1}{\epsilon}\,|\xi|^d\le|\lambda|\}$.*

Proof: Let $a(\lambda,\xi)\in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$. Suppose that it is resolvent like. We'll show that $a(\lambda,\xi)$ satisfies (2). Let $\chi(\xi)\in C^\infty(\mathbb{R}^n)$ with $\chi(\xi)\equiv 0$ for $|\xi|\le 1/2$ and with $\chi(\xi)\equiv 1$ for $|\xi|\ge 1$. Then $b(\lambda,\xi):=\chi(\xi)\,a(\lambda,\xi)$ is such that there exists an $\epsilon>0$ such that $b(\lambda,\xi)$ extends to be a smooth function, holomorphic in λ, for (λ,ξ) in

$$\{(\lambda,\xi)\in\mathbb{C}\times\mathbb{R}^n\,|\,\lambda\in\Lambda\ \text{ or }\ |\lambda|\le\epsilon\,|\xi|^d\ \text{ or }\ \frac{1}{\epsilon}\,|\xi|^d\le|\lambda|\}.$$

It follows that $b(\lambda,\xi/|\xi|\,)=a(\lambda,\xi/|\xi|\,)$ is such that it extends to be a smooth function, holomorphic in λ, for $(\lambda,\xi/|\xi|\,)\in\{(\lambda,\xi/|\xi|\,)\in\mathbb{C}\times\mathbb{S}^{n-1}\mid\lambda\in\Lambda\ \text{ or }\ |\lambda|\le\epsilon\ \text{ or }\ \frac{1}{\epsilon}\le|\lambda|\}$.

Now suppose that $a(\lambda,\xi)$ satisfies (2); thus, there exists an $\epsilon>0$ such that $a(\lambda,\xi/|\xi|\,)$ extends to be a smooth function, holomorphic in λ, for $(\lambda,\xi/|\xi|\,)$ in

$$\{(\lambda,\xi/|\xi|\,)\in\mathbb{C}\times\mathbb{S}^{n-1}\mid\lambda\in\Lambda\ \text{ or }\ |\lambda|\le\epsilon\ \text{ or }\ \frac{1}{\epsilon}\le|\lambda|\}.$$

We'll show that $a(\lambda,\xi)$ satisfies (3). Observe that for $(\lambda,\xi)\in\Lambda\times(\mathbb{R}^n\setminus\{0\})$, we have

$$a(\lambda,\xi)=|\xi|^m\,a(\,|\xi|^{-d}\lambda,\xi/|\xi|\,).$$

It follows that $a(\lambda,\xi)$ extends to be a smooth function, holomorphic in λ, for (λ,ξ) in

$$\{(\lambda,\xi)\in\mathbb{C}\times(\mathbb{R}^n\setminus\{0\})\,|\,\lambda\in\Lambda\ \text{ or }\ |\xi|^{-d}|\lambda|\le\epsilon\ \text{ or }\ \frac{1}{\epsilon}\le|\xi|^{-d}|\lambda|\};$$

that is, $\{(\lambda,\xi)\in\mathbb{C}\times(\mathbb{R}^n\setminus\{0\})\,|\,\lambda\in\Lambda\ \text{ or }\ |\lambda|\le\epsilon\,|\xi|^d\ \text{ or }\ \frac{1}{\epsilon}\,|\xi|^d\le|\lambda|\}$. We will leave (3) implies (1) to the reader. ■

3.5 Tempered small calculus

Let $\Lambda\subseteq\mathbb{C}$ be a closed cone and let X be compact with corners.

Definition 3.19 Let $Y\subseteq X$ be a closed p-submanifold and let $m,\,p\in\mathbb{R}$ and $d\in\mathbb{R}^+$. The space of *tempered conormal distributions to Y of degree m,p,d,* $I^{m,p,d}_\Lambda(X,Y)$, consists of those elements $u\in C^\infty(\Lambda;I^m(X,Y)\,)$ satisfying the following two conditions:

1. Given any $\phi\in C^\infty_c(X\setminus Y)$, we have $\phi\,u\in S^{p/d}(\Lambda;C^\infty(X)\,)$.

2. Given any coordinate patch $\mathbb{R}^{l,k}_y\times\mathbb{R}^q_z$ on X such that $Y\cong\mathbb{R}^{l,k}\times\{0\}$ and any compactly supported function ϕ on the coordinate patch, we have

$$\phi\,u=\frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}\,a(\lambda,y,\xi)\,d\xi,$$

where $y\mapsto a(\lambda,y,\xi)\in C^\infty(\mathbb{R}^{n,k};\,S^{m,p,d}_\Lambda(\mathbb{R}^n)\,)$.

The space of *resolvent tempered conormal distributions to Y of degree m,p,d,* $I^{m,p,d}_{\Lambda,r}(X,Y)$, consists of those elements $u\in C^\infty(\Lambda;I^m(X,Y)\,)$ satisfying (1) with $S^{p/d}$ replaced by $S^{p/d}_{os}$ and satisfying (2) with $S^{m,p,d}_\Lambda$ replaced by $S^{m,p,d}_{\Lambda,r}$. The space of *resolvent tempered, one-step, conormal distributions to Y of degree m,p,d,* $I^{m,p,d}_{\Lambda,ros}(X,Y)$, consists of those elements $u\in C^\infty(\Lambda;I^m(X,Y)\,)$ satisfying (1) with $S^{p/d}$ replaced by $S^{p/d}_{os}$ and satisfying (2) with $S^{m,p,d}_\Lambda$ replaced by $S^{m,p,d}_{\Lambda,ros}$.

Definition 3.20 Let $m,\, p \in \mathbb{R}$ and $d \in \mathbb{R}^+$. The *small calculus of tempered b-pseudodifferential operators of degree* m, p, d, $\Psi^{m,p,d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$, is the space of kernels

$$\Psi^{m,p,d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}}) := \{A \in I^{m,p,d}_{\Lambda}(X_b^2, \Delta_b, \Omega_b^{\frac{1}{2}}) \,|\, A \equiv 0 \text{ at } lb \cup rb\}.$$

If $p/d \in \mathbb{Z}$, the *small calculus of resolvent tempered b-pseudodifferential operators of degree* m, p, d, $\Psi^{m,p,d}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$, is the space of kernels

$$\Psi^{m,p,d}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}}) := \{A \in I^{m,p,d}_{\Lambda,r}(X_b^2, \Delta_b, \Omega_b^{\frac{1}{2}}) \,|\, A \equiv 0 \text{ at } lb \cup rb\}.$$

The *small calculus of resolvent tempered, one-step, b-pseudodifferential operators of degree* m, p, d, $\Psi^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$, is the space of kernels

$$\Psi^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}}) := \{A \in I^{m,p,d}_{\Lambda,ros}(X_b^2, \Delta_b, \Omega_b^{\frac{1}{2}}) \,|\, A \equiv 0 \text{ at } lb \cup rb\}.$$

Lemma 3.3 implies the following.

Lemma 3.21 *Let* $m, p \in \mathbb{R}$ *and* $d \in \mathbb{R}^+$. *Then,*

1. *for any* $m', p' \in \mathbb{R}$, $\Psi^{m,p,d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}}) \subseteq \Psi^{m',p',d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$, *provided* $m \leq m'$ *and* $p \leq p'$;
2. *for any* α, $\partial_\lambda^\alpha \Psi^{m,p,d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}}) \subseteq \Psi^{m-d|\alpha|,p-d|\alpha|,d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$;
3. *if* $p > 0$,
$$\Psi^{m,p,d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}}) \subseteq S^{p/d}(\Lambda; \Psi_b^m(X, \Omega_b^{\frac{1}{2}}))$$
and if $p \leq 0$, *then*
$$\Psi^{m,p,d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}}) \subseteq S^{p/d}(\Lambda; \Psi_b^{m-p}(X, \Omega_b^{\frac{1}{2}})).$$

Similar statements holds for the resolvent tempered and resolvent one-step spaces.

Thus, the following Theorem follows from Theorem 2.3.

Theorem 3.22 *Let* $m,\, p \in \mathbb{R}$ *and* $d \in \mathbb{R}^+$ *and let* $A \in \Psi^{m,p,d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$. *Then* A *defines a linear map*

$$A : \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}}) \;\rightarrow\; S^{p/d}(\Lambda; \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}})),$$

continuous in the C^∞ *topology.*

The following Theorem is proved just like the usual composition result in the small calculus.

Theorem 3.23 *Let* $m, m', p, p' \in \mathbb{R}$ *and* $d \in \mathbb{R}^+$. *If* $A \in \Psi^{m,p,d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$ *and* $B \in \Psi^{m',p',d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$, *then*

$$A \circ B \in \Psi^{m+m',p+p',d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}}).$$

Similar statements holds for the resolvent tempered and resolvent one-step spaces.

The proof of the following 'Asymptotic Summation' result will be left to the reader.

Lemma 3.24 *Let* $m,\, p \in \mathbb{R}$ *and* $d \in \mathbb{R}^+$ *and for each* $j \in \mathbb{N}_0$, *let* $A_{m-j} \in \Psi^{m-j,p,d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$. *Then, there exists an* $A \in \Psi^{m,p,d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$ *such that for any* $N \in \mathbb{N}$,

$$A - \sum_{j=0}^{N-1} A_{m-j} \in \Psi^{m-N,p,d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}});$$

in which case, we write $A \sim \sum_{j=0}^\infty A_{m-j}$. *Similar statements hold for the resolvent tempered and resolvent one-step spaces.*

3.6 Tempered calculus with bounds

Let α, β, and η be multi-indices for *lb*, *rb*, and *ff* of X_b^2. Let $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. We define

$$\Psi_{b,\Lambda,r}^{-\infty,p,d;\alpha,\beta,\eta}(X, \Omega_b^{\frac{1}{2}}) := \bigcup_{\epsilon>0} \rho_{lb}^{\alpha+\epsilon} \rho_{rb}^{\beta+\epsilon} S_{os}^{p/d}(\Lambda; S_{ff}^{0,\eta}(X_b^2, \Omega_b^{\frac{1}{2}})).$$

For any $m \in \mathbb{R}$, we define

$$\Psi_{b,\Lambda,r}^{m,p,d;\alpha,\beta,\eta}(X, \Omega_b^{\frac{1}{2}}) := \Psi_{b,\Lambda,r}^{m,p,d}(X, \Omega_b^{\frac{1}{2}}) + \Psi_{b,\Lambda,r}^{-\infty,p,d;\alpha,\beta,\eta}(X, \Omega_b^{\frac{1}{2}})$$

and

$$\Psi_{b,\Lambda,ros}^{m,p,d;\alpha,\beta,\eta}(X, \Omega_b^{\frac{1}{2}}) := \Psi_{b,\Lambda,ros}^{m,p,d}(X, \Omega_b^{\frac{1}{2}}) + \Psi_{b,\Lambda,r}^{-\infty,p,d;\alpha,\beta,\eta}(X, \Omega_b^{\frac{1}{2}}).$$

Lemma 3.25 *Let $R \in \Psi_{b,\Lambda,r}^{-\infty,p,d}(X, \Omega_b^{\frac{1}{2}})$, where $p < 0$ and $d > 0$ with $p/d \in \mathbb{Z}$. Then there exists a continuous increasing function $r : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $\epsilon > 0$ and $|\lambda| \geq r(\epsilon)$,*

$$(\mathrm{Id} - R(\lambda))^{-1} = \mathrm{Id} + S(\lambda),$$

where for each $\epsilon > 0$, $S(\lambda) \in \Psi_{b,\Lambda,r}^{-\infty,p,d,\epsilon,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r(\epsilon)$.

Proof: See [7]. ■

3.7 The Resolvent as a Tempered operator

Recall that given closed cones Λ, $\Lambda' \subseteq \mathbb{C}$, Λ is a proper subcone of Λ' if $\Lambda \subseteq \mathrm{int}(\Lambda') \cup \{0\}$.

Definition 3.26 Let $m \in \mathbb{R}^+$ and $\Lambda \subseteq \mathbb{C}$ be a closed cone. The set $\mathcal{E}\ell\ell_{b,\Lambda}^m(X, \Omega_b^{\frac{1}{2}})$ consists of those elliptic operators $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$ such that the principal symbol ${}^b\sigma_m(A) \in S^{[m]}({}^bT^*X)$ can be represented by an element $a(x,\xi) \in S^m({}^bT^*X)$ having the property that there exists a close cone $\Lambda' \subseteq \mathbb{C}$ with $\Lambda \subseteq \Lambda'$ a proper subcone, and a compact set $K \subseteq {}^bT^*X$ such that $a(x,\xi) \notin \Lambda'$ for all $(x,\xi) \notin K$.

We define

$$\mathcal{E}\ell\ell_{b,\Lambda,os}^m(X, \Omega_b^{\frac{1}{2}}) := \mathcal{E}\ell\ell_{b,\Lambda}^m(X, \Omega_b^{\frac{1}{2}}) \cap \Psi_{b,os}^m(X, \Omega_b^{\frac{1}{2}}).$$

This Definition makes sense because of the following Lemma.

Lemma 3.27 *Let $a(x,\xi) \in S^m({}^bT^*X)$, $m \in \mathbb{R}^+$ be elliptic and let $\Lambda \subseteq \mathbb{C}$ be a closed cone. Suppose that $a(x,\xi)$ has the property that there exists a close cone $\Lambda' \subseteq \mathbb{C}$ with $\Lambda \subseteq \Lambda'$ a proper subcone, and compact set $K \subseteq {}^bT^*X$ such that $a(x,\xi) \notin \Lambda'$ for all $(x,\xi) \notin K$. Then for any $b(x,\xi) \in S^{m-1}({}^bT^*X)$, $a(x,\xi) + b(x,\xi)$ has the same property.*

Proof: Let $\chi(x,\xi) \in C^\infty({}^bT^*X)$ with $\chi(x,\xi) \equiv 0$ for $(x,\xi) \in K$ and with $\chi(x,\xi) \equiv 1$ outside a neighborhood of K. Let $b(x,\xi) \in S^{m-1}({}^bT^*X)$. Then,

$$a(x,\xi) + b(x,\xi) = a(x,\xi) + \chi(x,\xi)\, b(x,\xi) + (1 - \chi(x,\xi))\, b(x,\xi).$$

Since $1 - \chi(x,\xi) \equiv 0$ outside of K, we just need $a(x,\xi) + \chi(x,\xi)\, b(x,\xi)$ to have the same property as $a(x,\xi)$. Since a is elliptic and non-zero outside of K, we can write

$$a(x,\xi) + \chi(x,\xi)\, b(x,\xi) = a(x,\xi)\,(1 + \frac{\chi(x,\xi)\, b(x,\xi)}{a(x,\xi)}).$$

Thus, we just have to show that for any $c(x,\xi) \in S^{-1}({}^bT^*X)$, $a(x,\xi)\,(1 + c(x,\xi))$, has the same property as $a(x,\xi)$. We will leave this as an exercise to the reader. (Hint: Since $a(x,\xi)$ is elliptic, it grows like $|\xi|^m$ as $|\xi| \to \infty$, where $|\ |$ is the norm of any metric on ${}^bT^*X$, and $c(x,\xi)$ decays like $|\xi|^{-1}$ as $|\xi| \to \infty$.) ■

Observe that if $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, is any elliptic operator, then $A^*A \in \mathcal{E}\ell\ell_{b,\Lambda}^m(X, \Omega_b^{\frac{1}{2}})$ for any closed cone $\Lambda \subseteq \mathbb{C}$ with $\Lambda \cap (0,\infty) = \emptyset$.

Lemma 3.28 *Let $A \in \Psi^m_{b,os}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, be elliptic. Then $A \in \mathcal{E}\ell\ell^m_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$ iff ${}^b\sigma_m(A) \in C^\infty_{\mathrm{hom}(m)}({}^bT^*X)$ is such that ${}^b\sigma_m(A)(x,\xi) \notin \Lambda$ for all $\xi \neq 0$.*

Proof: We will leave this as an exercise for the reader. ∎

Lemma 3.29 *Let $A \in \mathcal{E}\ell\ell^m_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$. Then for some $r > 0$, there exists a $B(\lambda) \in \Psi^{-m,-m,m}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r$, such that*

$$(A - \lambda)B(\lambda) = \mathrm{Id} - R(\lambda),$$

where $R(\lambda) \in \Psi^{-\infty,-m,m}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r$. If $A \in \mathcal{E}\ell\ell^m_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$, then we may choose $B(\lambda) \in \Psi^{-m,-m,m}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r$.

Proof: For simplicity of notation, we will only prove this Lemma for the space $\mathcal{E}\ell\ell^m_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$; the proof for the one-step space is basically the same.[5] Thus, let $A \in \mathcal{E}\ell\ell^m_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$. Let $a_m(x,\xi) \in S^m({}^bT^*X)$ be a representative of the principal symbol ${}^b\sigma_m(A) \in S^{[m]}({}^bT^*X)$ such that there exists a close cone $\Lambda' \subseteq \mathbb{C}$ with $\Lambda \subseteq \Lambda'$ a proper subcone, and a compact set $K \subseteq {}^bT^*X$ such that $a_m(x,\xi) \notin \Lambda'$ for all $(x,\xi) \notin K$. Define $r := 1 + \max_{(y,\xi)\in K} |a_m(y,\xi)|$.

Let $\mathcal{U} = \mathbb{R}^{n,k}_y \times \mathbb{R}^n_z$ be a coordinate patch on X^2_b such that $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}$. Note that the $\mathbb{R}^{n,k}$ factor of $\mathcal{U}$ can be considered as a coordinate patch on X. Let $\phi \in C^\infty_c(\mathbb{R}^{n,k})$ and let $\psi \in C^\infty_c(\mathcal{U})$ be such that $\psi(y,0) \equiv 1$ on $\operatorname{supp}\phi$. Since ψA is supported on the coordinate patch $\mathcal{U}$, we can write

$$\psi A = \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, a(y,\xi)\, d\xi \otimes \nu,$$

where $0 < \nu \in C^\infty(\mathcal{U}, \Omega_b^{\frac{1}{2}})$, and where

$$a(y,\xi) = \psi(y,0)\, a_m(y,\xi) \text{ modulo } S^{m-1}({}^bT^*X). \tag{24}$$

Then observe that $\phi(y)(\, a_m(y,\xi) - \lambda\,)^{-1}$ is smooth for $\lambda \in \Lambda$ with $|\lambda| \geq r$. Hence, by Lemma 3.6, $\phi(y)(\, a_m(y,\xi) - \lambda\,)^{-1} \in C^\infty(\mathbb{R}^{n,k}_y;\, S^{-m,-m,m}_{\Lambda,r}(\mathbb{R}^n)\,)$ for all $\lambda \in \Lambda$ with $|\lambda| \geq r$. Let $\psi' \in C^\infty_c(\mathcal{U})$ such that $\psi'(y,0) \equiv 1$ on $\operatorname{supp}\phi$ and $\psi \equiv 1$ on $\operatorname{supp}\psi'$. For $\lambda \in \Lambda$ with $|\lambda| \geq r$, define

$$B(\lambda) := \psi'(y,z)\, \phi(y) \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, (\, a_m(y,\xi) - \lambda\,)^{-1}\, d\xi \otimes \nu. \tag{25}$$

Then, $B(\lambda) \in \Psi^{-m,-m,m}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r$. Since $1 - \psi \equiv 0$ on $\operatorname{supp}\psi'$ it follows that

$$[(1-\psi)A]B(\lambda) \in \Psi^{-\infty,-m,m}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}}).$$

for $|\lambda| \geq r$. Hence, for $|\lambda| \geq r$,

$$\begin{aligned} AB(\lambda) &= [\psi A]B(\lambda) + [(1-\psi)A]B(\lambda) \\ &= [\psi A]B(\lambda) \text{ modulo } \Psi^{-\infty,-m,m}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}}). \end{aligned}$$

Also, observe that (24) implies that for $|\lambda| \geq r$, $(\psi A - \lambda)B(\lambda) = \phi$ modulo $\Psi^{-1,-m,m}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$. Hence, for $\lambda \in \Lambda$ with $|\lambda| \geq r$

$$(A - \lambda)B(\lambda) = \phi - R(\lambda), \tag{26}$$

where $R(\lambda) \in \Psi^{-1,-m,m}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r$.

[5] The only difference occurs in equation 25, where we set $a_m(y,\xi) := \chi(\xi){}^b\sigma_m(A)(y,\xi)$, where $\chi(\xi) \equiv 0$ near 0 and $\chi(\xi) \equiv 1$ outside of a neighborhood of 0.

Let $\{\mathcal{U}_i\}_{i=1}^N$ be coordinate patches of X_b^2 giving product decompositions of X_b^2 with respect to the p-submanifold Δ_b, such that $\{\mathcal{U}_i \cap \Delta_b\}$ covers Δ_b. Let $\{\phi_i\}$ be a partition of unity of Δ_b with respect to the cover $\{\mathcal{U}_i \cap \Delta_b\}$. Then for each i, by (26), there exists a $B_i(\lambda) \in \Psi_{b,\Lambda,r}^{-m,-m,m}(X, \Omega_b^{\frac{1}{2}})$ for $\lambda \in \Lambda$ with $|\lambda| \geq r$, such that $(A - \lambda)B_i(\lambda) = \phi_i - R_i(\lambda)$, where $R_i(\lambda) \in \Psi_{b,\Lambda,r}^{-1,-m,m}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r$. Setting $B(\lambda) := \sum_i^N B_i(\lambda)$, it follows that $(A-\lambda)B(\lambda) = \mathrm{Id} - R(\lambda)$, where $R(\lambda) \in \Psi_{b,\Lambda,r}^{-1,-m,m}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r$. Since $R(\lambda)^j \in \Psi_{b,\Lambda,r}^{-j,-m,m}(X, \Omega_b^{\frac{1}{2}})$ for each j, we can choose an $R'(\lambda) \in \Psi_{b,\Lambda,r}^{-1,-m,m}(X, \Omega_b^{\frac{1}{2}})$ such that $R'(\lambda) \sim \sum_{j=1}^{\infty} R(\lambda)^j$. Thus, if $B'(\lambda) := B(\lambda) \circ (\mathrm{Id} + R'(\lambda)) \in \Psi_{b,\Lambda,r}^{-m,-m,m}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r$, it follows that

$$(A - \lambda)B'(\lambda) = \mathrm{Id} - R''(\lambda), \tag{27}$$

where $R''(\lambda) \in \Psi_{b,\Lambda,r}^{-\infty,-m,m}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r$. ■

Lemma 3.30 *Let $A \in \mathcal{E}\ell\ell_{b,\Lambda}^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$. Then there exists a continuous increasing function $r : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $\epsilon > 0$, $(A-\lambda)^{-1} \in \Psi_{b,\Lambda,r}^{-m,-m,m,\epsilon,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r(\epsilon)$. Moreover, If $A \in \mathcal{E}\ell\ell_{b,\Lambda,os}^m(X, \Omega_b^{\frac{1}{2}})$, then for each $\epsilon > 0$, $(A - \lambda)^{-1} \in \Psi_{b,\Lambda,ros}^{-m,-m,m,\epsilon,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r(\epsilon)$.*

Proof: For simplicity of notation, we will only prove the Theorem for the space $\mathcal{E}\ell\ell_{b,\Lambda}^m(X, \Omega_b^{\frac{1}{2}})$; the proof for the one-step space is exactly the same. Thus, let $A \in \mathcal{E}\ell\ell_{b,\Lambda}^m(X, \Omega_b^{\frac{1}{2}})$. By the previous Lemma, for some $r > 0$, there exists a $B(\lambda) \in \Psi_{b,\Lambda,r}^{-m,-m,m}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r$, such that

$$(A - \lambda)B(\lambda) = \mathrm{Id} - R(\lambda),$$

where $R(\lambda) \in \Psi_{b,\Lambda,r}^{-\infty,-m,m}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r$. Thus, by Lemma 3.25, there exists a continuous increasing function $r : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $\epsilon > 0$ and $|\lambda| \geq r(\epsilon)$,

$$(\mathrm{Id} - R(\lambda))^{-1} = \mathrm{Id} + S(\lambda),$$

where for each $\epsilon > 0$, $S(\lambda) \in \Psi_{b,\Lambda,r}^{-\infty,p,d,\epsilon,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r(\epsilon)$. Hence, for each $\epsilon > 0$ and $|\lambda| \geq r(\epsilon)$,

$$(A - \lambda)^{-1} = B(\lambda) \circ (\mathrm{Id} + S(\lambda)) \in \Psi_{b,\Lambda,r}^{-m,-m,m,\epsilon,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}})$$

for all $\lambda \in \Lambda$ with $|\lambda| \geq r(\epsilon)$. ■

For any $\epsilon > 0$, we define $B_\epsilon := \{\lambda \in \mathbb{C} \mid |\lambda| \leq \epsilon\}$.

Definition 3.31 Let $m, p \in \mathbb{R}$ and let $d \in \mathbb{R}^+$. Then an operator $A \in \Psi_{b,\Lambda}^{m,p,d}(X, \Omega_b^{\frac{1}{2}})$ is said to be *resolvent like* if

1. $\Lambda \cup B_\epsilon \ni \lambda \mapsto A(\lambda) \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$ is holomorphic for some $\epsilon > 0$ and
2. for any coordinate patch $\mathbb{R}_y^{n,k} \times \mathbb{R}_z^n$ on X_b^2 such that $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}$ and any compactly supported function ϕ on the coordinate patch, we have

$$\phi\, A = \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, a(\lambda, y, \xi)\, d\xi \otimes \nu,$$

where $\nu \in C^\infty(X_b^2, \Omega_b^{\frac{1}{2}})$, and where for each $y \in \mathbb{R}^{n,k}$, $(\lambda, \xi) \mapsto a(\lambda, y, \xi) \in S_\Lambda^{m,p,d}(\mathbb{R}^n)$ is resolvent like.

Theorem 3.32 *Let $A \in \mathcal{E}\ell\ell_{b,\Lambda}^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$. Then there exists a resolvent like operator $B(\lambda) \in \Psi_{b,\Lambda,r}^{-m,-m,m}(X, \Omega_b^{\frac{1}{2}})$ such that for $\lambda \in \Lambda$ with $|\lambda|$ sufficiently large,*

$$(A - \lambda)^{-1} = B(\lambda) + R(\lambda),$$

where for some continuous increasing function $r : \mathbb{R}^+ \to \mathbb{R}^+$, for each $\epsilon > 0$, we have $R(\lambda) \in \Psi^{-\infty,-m,m,\epsilon,\epsilon,\epsilon}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r(\epsilon)$.

If $A \in \mathcal{E}\ell\ell^m_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$, then there exists a resolvent like operator $B(\lambda) \in \Psi^{-m,-m,m}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$ such that for $\lambda \in \Lambda$ with $|\lambda|$ sufficiently large,

$$(A - \lambda)^{-1} = B(\lambda) + R(\lambda),$$

where for some continuous increasing function $r : \mathbb{R}^+ \to \mathbb{R}^+$, for each $\epsilon > 0$, we have $R(\lambda) \in \Psi^{-\infty,-m,m,\epsilon,\epsilon,\epsilon}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r(\epsilon)$.

Proof: The proof of this Theorem follows almost exactly the same pattern as the proof of Lemma 3.29. Thus, we will leave this proof as an exercise with hints:

1. Read through the proof of Lemma 3.29 until you get to equation (25). Then,
   (a) if $A \in \mathcal{E}\ell\ell^m_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$, replace $B(\lambda)$ in equation (25) by
   $$B(\lambda) := \psi'(y,z)\,\phi(y)\frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}\,\chi(\xi)(\,a_m(y,\xi) - \lambda\,)^{-1}\,d\xi \otimes \nu \in \Psi^{-m,-m,m}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}}),$$
   where $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ near 0 and with $\chi(\xi) \equiv 1$ outside a neighborhood of 0, such that $a_m(x,\xi) \notin \Lambda'$ for all (x,ξ) with $\xi \notin \operatorname{supp}\chi$;
   (b) if $A \in \mathcal{E}\ell\ell^m_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$, replace $B(\lambda)$ in equation (25) by
   $$B(\lambda) := \psi(y,z)\,\phi(y)\frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}\,\chi(\xi)(\,a_m(y,\xi) - \lambda\,)^{-1}\,d\xi \otimes \nu \in \Psi^{-m,-m,m}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}}),$$
   where $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ is any smooth function with $\chi(\xi) \equiv 0$ near 0 and with $\chi(\xi) \equiv 1$ outside a neighborhood of 0.
2. Using a similar argument we used to prove equation (26), show that $(A - \lambda)B(\lambda) = \phi - R(\lambda)$, where
   (a) $R(\lambda) \in \Psi^{-1,0,m}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$, if $A \in \mathcal{E}\ell\ell^m_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$;
   (b) $R(\lambda) \in \Psi^{-1,0,m}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$, if $A \in \mathcal{E}\ell\ell^m_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$.
3. By taking a partition of unity of Δ_b, show that $(A - \lambda)B(\lambda) = \mathrm{Id} - R(\lambda)$, for some resolvent like operators
   (a) $B(\lambda) \in \Psi^{-m,-m,m}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$ and $R(\lambda) \in \Psi^{-1,0,m}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$, if $A \in \mathcal{E}\ell\ell^m_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$;
   (b) $B(\lambda) \in \Psi^{-m,-m,m}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$ and $R(\lambda) \in \Psi^{-1,0,m}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$, if $A \in \mathcal{E}\ell\ell^m_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$.
4. By choosing appropriate asymptotic sums, show as we did in equation (27), that
   $$(A - \lambda)B'(\lambda) = \mathrm{Id} - R''(\lambda), \tag{28}$$
   for some resolvent like operators
   (a) $B'(\lambda) \in \Psi^{-m,-m,m}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$ and $R''(\lambda) \in \Psi^{-\infty,0,m}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$, if $A \in \mathcal{E}\ell\ell^m_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$;
   (b) $B'(\lambda) \in \Psi^{-m,-m,m}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$ and $R''(\lambda) \in \Psi^{-\infty,0,m}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$, if $A \in \mathcal{E}\ell\ell^m_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$.
5. Now multiply both sides of equation (28) by $(A - \lambda)^{-1}$, using Lemma 3.30, to finish off the proof of this Theorem.

■

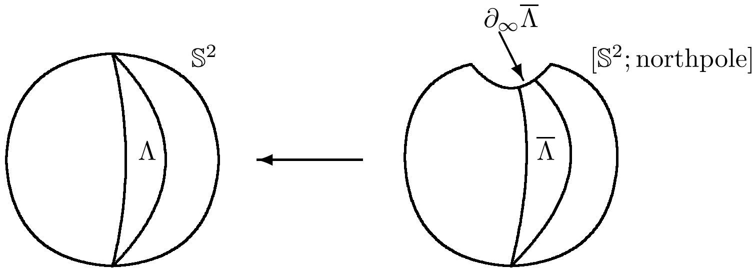


Figure 3: The manifold Λ radially compactified.

3.8 The blown-up tempered space

We will now realize the spaces $\Psi^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac{1}{2}})$ as classical conormal functions on an appropriate blown-up space.

Let $\Lambda \subseteq \mathbb{C}$ be a closed cone. Denote by $\overline{\Lambda}$, the manifold Λ radially compactified. We denote by $\partial_\infty\overline{\Lambda}$, the boundary 'at infinity'. A geometric way to view $\overline{\Lambda}$ is as follows. Consider Λ as a subset of the Riemann Sphere $\mathbb{S}^2$. Then if we blow up the north pole of $\mathbb{S}^2$, the compactification $\overline{\Lambda}$ is just the closure (of the lift) of Λ into $[\mathbb{S}^2;\text{north pole}]$ (see Figure 3). Some convenient coordinates near $\partial_\infty\overline{\Lambda}$ can be obtained as follows. Let $\Lambda_{cc} = \{\overline{\lambda}\,|\,\lambda\in\Lambda\}$ be the 'complex conjugate' cone of Λ. Observe that

$$\Lambda_{cc} \ni \mu \;\mapsto\; \lambda = 1/\mu = \overline{\mu}/|\mu|^2 \in \Lambda$$

is a biholomorphism of $\Lambda_{cc}\setminus\{0\}$ onto $\Lambda\setminus\{0\}$. Then by the definition of $\overline{\Lambda}$, $[\Lambda_{cc};\{0\}] \equiv \overline{\Lambda}\setminus\{0\}$, where $\mathit{ff}[\Lambda_{cc};\{0\}] \equiv \partial_\infty\overline{\Lambda}$.

Let X be compact with corners. We define

$$X^2_{b,\Lambda} := [X^2_b\times\overline{\Lambda};\Delta_b\times\partial_\infty\overline{\Lambda};\Delta_b\times\overline{\Lambda}].$$

Figure 4 gives a pictorial representation of $X^2_{b,\Lambda}$. We define

$$bi := \overline{\beta^{-1}(X^2_b\times\partial_\infty\overline{\Lambda}\setminus(\Delta_b\times\partial_\infty\overline{\Lambda})\,)};$$

$$\mathit{fi} := \beta^{-1}(\Delta_b\times\partial_\infty\overline{\Lambda});$$

$$df := \overline{\beta^{-1}(\Delta_b\times\overline{\Lambda}\setminus(\Delta_b\times\partial_\infty\overline{\Lambda})\,)};$$

where we call bi, the 'boundary at infinity', fi, the 'face at infinity', and df, the 'diagonal face'.

We will now fix the notation for local coordinates on $X^2_{b,\Lambda}$. We will use the identification $[\Lambda_{cc};\{0\}]\equiv\overline{\Lambda}\setminus\{0\}$, with $\mathit{ff}[\Lambda_{cc};\{0\}]\equiv\partial_\infty\overline{\Lambda}$. We define

$$r := |\mu|,\;\; \omega := \mu/|\mu|,\;\; \mu = 1/\lambda\in\Lambda_{cc}.$$

Let $\mathcal{U} = \mathbb{R}^{n,k}_y\times\mathbb{R}^n_z$ be a coordinate patch on X^2_b with $\Delta_b\cong\mathbb{R}^{n,k}\times\{0\}$. Then away from $\lambda=0$,

$$X^2_b\times\overline{\Lambda}\cong\mathbb{R}^{n,k}_y\times\mathbb{R}^n_z\times[0,\infty)_r\times(\mathbb{S}^1\cap\Lambda_{cc})_\omega,$$

with

$$\Delta_b\times\partial_\infty\overline{\Lambda}\cong\mathbb{R}^{n,k}_y\times\{0\}_z\times\{0\}_r\times(\mathbb{S}^1\cap\Lambda_{cc})_\omega$$

and

$$\Delta_b\times\overline{\Lambda}\cong\mathbb{R}^{n,k}_y\times\{0\}_z\times[0,\infty)_r\times(\mathbb{S}^1\cap\Lambda_{cc})_\omega.$$

Thus, $X^2_{b,\Lambda}\cong\mathbb{R}^{n,k}_y\times T^n_\Lambda$, where $T^n_\Lambda := [\mathbb{R}^n_z\times\overline{\Lambda};Y_1;Y_2]$ with

$$Y_1 := \{0\}_z\times\{0\}_r\times(\mathbb{S}^1\cap\Lambda_{cc})_\omega \text{ and } Y_2 := \{0\}_z\times[0,\infty)_r\times(\mathbb{S}^1\cap\Lambda_{cc})_\omega.$$

The coordinates (y,ω), together with each of the first three coordinates shown in Figure 5, give various coordinate systems on $[X^2_b\times\overline{\Lambda};\Delta_b\times\partial_\infty\overline{\Lambda}]$ and the coordinates (y,ω), together with the last

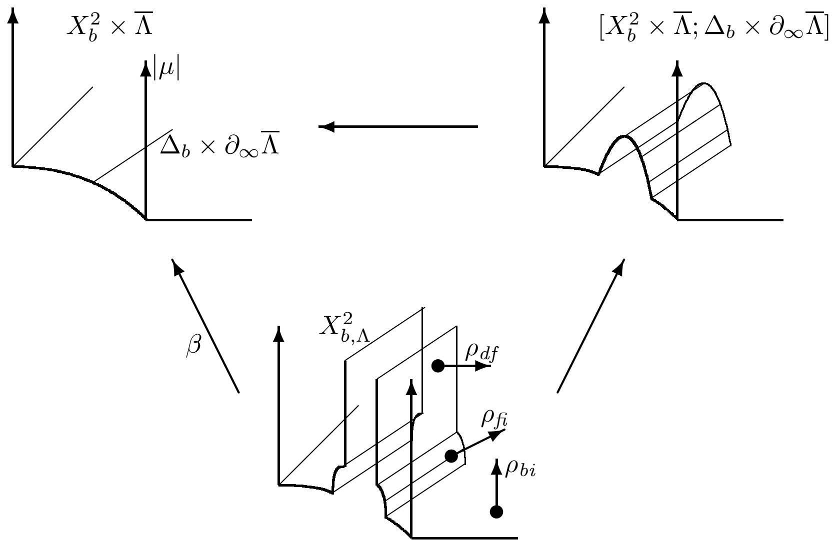


Figure 4: The manifold $X^2_{b,\Lambda}$.

set of coordinates shown in Figure 5, give coordinates on $X^2_{b,\Lambda} = [X_b^2 \times \overline{\Lambda}; \Delta_b \times \partial_\infty \overline{\Lambda}; \Delta_b \times \overline{\Lambda}]$ near df. On T^n_Λ, we define $bi := \{\omega_0 = 0\}$, $fi := \{t = 0\}$, and $df := \{u = 0\}$.

We now consider densities. We claim that $|dz\, \frac{d\lambda \wedge d\overline{\lambda}}{\lambda^2}|^{\frac{1}{2}} \in C^\infty(\mathbb{R}^n_z \times \overline{\Lambda}, \Omega_b^{\frac{1}{2}})$, when lifted to T^n_Λ, is of the form $|dz\, \frac{d\lambda \wedge d\overline{\lambda}}{\lambda^2}|^{\frac{1}{2}} = \rho_{df}^{\frac{n}{2}} \rho_{fi}^{\frac{n}{2}}\, \mu$, where $0 < \mu \in C^\infty(T^n_\Lambda, \Omega_b^{\frac{1}{2}})$. Indeed, observe that in the coordinates $r = s\gamma_0$ and $z = s\gamma'$ near bi, we have

$$|dz\, \frac{dr}{r}|^{\frac{1}{2}} = s^{\frac{n}{2}} |\frac{ds}{s} \frac{d\gamma_0}{\gamma_0}|^{\frac{1}{2}}\, \mu_1, \tag{29}$$

where $0 < \mu_1 \in C^\infty(\mathbb{S}^{n-1}_{\gamma'}, \Omega^{\frac{1}{2}})$. Also, observe that in the coordinates $r = t$ and $z = t\,u\,\gamma$ near df, we have $|dz\, \frac{dr}{r}|^{\frac{1}{2}} = t^{\frac{n}{2}} u^{\frac{n}{2}}\, |\frac{dt}{t} \frac{du}{u}|^{\frac{1}{2}}\, \mu_2$, where $\mu_2 \in C^\infty(\mathbb{S}^{n-1}_\gamma, \Omega^{\frac{1}{2}})$. This expression, together with (29) implies that $|dz\, \frac{d\lambda \wedge d\overline{\lambda}}{\lambda^2}|^{\frac{1}{2}} = \rho_{df}^{\frac{n}{2}} \rho_{fi}^{\frac{n}{2}}\, \mu$, where $0 < \mu \in C^\infty(T^n_\Lambda, \Omega_b^{\frac{1}{2}})$. More generally, we have the following Lemma.

Lemma 3.33 *Let* $0 < \nu \in C^\infty(X_b^2, \Omega_b^{\frac{1}{2}})$. *Then,*

$$\beta^*(\,\nu\, |\frac{d\lambda \wedge d\overline{\lambda}}{\lambda^2}|^{\frac{1}{2}}\,) = \rho_{df}^{\frac{n}{2}} \rho_{fi}^{\frac{n}{2}}\, \mu,$$

where $0 < \mu \in C^\infty(X^2_{b,\Lambda}, \Omega_b^{\frac{1}{2}})$.

3.9 The Structure Theorem

Let $d \in \mathbb{R}^+$. We will assume that $\Lambda \neq \mathbb{C}$. Thus, we can *fix* a branch of the logarithm function such that $\log : \Lambda \setminus \{0\} \to \mathbb{C}$ is holomorphic. In particular, for any $a \in \mathbb{C}$, $\lambda^a := e^{a \log \lambda}$, where $\lambda \in \Lambda \setminus \{0\}$, is well defined. We define $\Lambda^{1/d} := \{\lambda^{1/d} \,|\, \lambda \in \Lambda \setminus \{0\}\} \cup \{0\}$. Then $\Lambda^{1/d}$ is also a closed cone in $\mathbb{C}$ and if $\lambda \in \Lambda^{1/d}$, then $\lambda^d \in \Lambda$. Since $\Lambda^{1/d}$ is a cone, the manifold $X^2_{b,\Lambda^{1/d}}$ is defined.

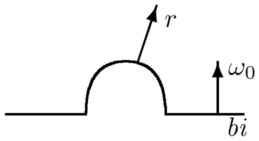
Recall that $\Psi^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}}) \subseteq C^\infty(\Lambda; \Psi^m_b(X, \Omega_b^{\frac{1}{2}}))$. If $m \in \mathbb{N}_0$, we define

$$\mathrm{Diff}^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}}) := \Psi^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}}) \cap C^\infty(\Lambda; \mathrm{Diff}^m_b(X, \Omega_b^{\frac{1}{2}})).$$

If $m \notin \mathbb{N}_0$, then we define $\mathrm{Diff}^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}}) := 0$.

Coordinates on $[X_b^2 \times \overline{\Lambda}; \Delta_b \times \partial_\infty \overline{\Lambda}]$:

$$\left.\begin{array}{l} \rho := (|z|^2 + r^2)^{1/2} \\ \omega_0 := r/(|z|^2 + r^2)^{1/2} \\ \omega' := z/(|z|^2 + r^2)^{1/2} \end{array}\right\} \leftrightarrow \left\{ \begin{array}{l} r = \rho\, \omega_0 \\ z = \rho\, \omega' \end{array}\right.$$

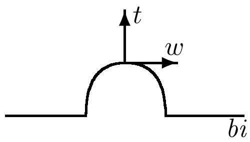


Coordinates near bi:

$$\left.\begin{array}{l} s := |z| \\ \gamma_0 := r/|z| \\ \gamma' := z/|z| \end{array}\right\} \leftrightarrow \left\{ \begin{array}{l} r = s\, \gamma_0 \\ z = s\, \gamma' \end{array}\right.$$

Coordinates near $\{\omega' \equiv 0\}$:

$$\left.\begin{array}{l} t := r \\ w := z/r \end{array}\right\} \leftrightarrow \left\{ \begin{array}{l} r = t \\ z = t\, w \end{array}\right.$$



Coordinates on $X^2_{b,\Lambda}$ near df:

$$\left.\begin{array}{l} t := t \\ u := |w| \\ \gamma := w/|w| \end{array}\right\} \leftrightarrow \left\{ \begin{array}{l} t = t \\ w = u\, \gamma \end{array}\right\} \leftrightarrow \left\{ \begin{array}{l} r = t \\ z = t\, u\, \gamma \end{array}\right.$$

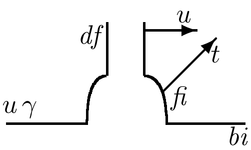


Figure 5: Various coodinates on $X^2_{b,\Lambda}$

Let m, $p \in \mathbb{R}$ with $p/d \in \mathbb{Z}$. Define the index set $\mathcal{E}_{m,p,d}$ on $X^2_{b,\Lambda^{1/d}}$ by

$$\mathcal{E}_{m,p,d}(\mathit{ff}) = \mathcal{E}_{m,p,d}(\text{edges of } \Lambda) := \mathbb{N}_0;$$

$$\mathcal{E}_{m,p,d}(lb) = \mathcal{E}_{m,p,d}(rb) := \emptyset; \ \mathcal{E}_{m,p,d}(bi) = -p + d\mathbb{N}_0;$$

$$\mathcal{E}_{m,p,d}(df) := \frac{n}{2} + \{(k - m - n + dl, 0) \,|\, k, l \in \mathbb{N}_0\} \overline{\cup} \mathbb{N}_0;$$

$$\mathcal{E}_{m,p,d}(\mathit{fi}) := \frac{n}{2} + \{(k - m - n, 0) \,|\, k \in \mathbb{N}_0\} \overline{\cup} (-p + \mathbb{N}_0 + d\mathbb{N}_0);$$

and define the index set $\mathcal{F}_{m,p,d}$ on $\overline{\Lambda^{1/d}}$, associated to $\partial_\infty \overline{\Lambda^{1/d}}$, by

$$\mathcal{F}_{m,p,d} := \{(k - m - n, 0) \,|\, k \in \mathbb{N}_0\} \overline{\cup} (-p + d\mathbb{N}_0).$$

Our Structure Theorem is the following.

Theorem 3.34 *For each m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, there exists an exact sequence of continuous linear maps*

$$0 \hookrightarrow \mathrm{Diff}^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}}) \hookrightarrow \Psi^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}}) \to \mathcal{A}^{\mathcal{E}_{m,p,d}}_{phg}(X^2_{b,\Lambda^{1/d}}, \Omega_b^{\frac{1}{2}}).$$

Moreover, if $m < -n$, there exists a continuous linear map

$$\Psi^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}}) \hookrightarrow \mathcal{A}^{\mathcal{F}_{m,p,d}}_{phg}(\overline{\Lambda^{1/d}}; C^\infty(X, \Omega_b)).$$

In the proof of this Theorem, we show that if $A(\lambda) \in \Psi^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$, then

$$A(\lambda^d)\, |\frac{d\lambda \wedge d\overline{\lambda}}{\lambda^2}|^{\frac{1}{2}} \in \mathcal{A}^{\mathcal{E}_{m,p,d}}_{phg}(X^2_{b,\Lambda^{1/d}}, \Omega_b^{\frac{1}{2}});$$

and if $m < -n$, then $A(\lambda^d)|_{\Delta_b} \in \mathcal{A}^{\mathcal{F}_{m,p,d}}_{phg}(\overline{\Lambda^{1/d}}; C^\infty(X, \Omega_b))$.

3.10 Proof of the Structure Theorem

Warning: Throughout this subsection, we will use the *same* letter λ for the variable on Λ and on $\Lambda^{1/d}$. But it will be clear during the context of an argument, whether we mean $\lambda \in \Lambda$ or $\lambda \in \Lambda^{1/d}$.
Lemma 3.35 *For each $N \in \mathbb{N}_0$, define $\mathcal{R}_N(\mathbb{S}^{n-1}_\omega \times \mathbb{R}^n_\xi)$ as those $a \in C^\infty(\mathbb{S}^{n-1} \times (\mathbb{R}^n \setminus \{0\}))$ satisfying the following estimates: for each $P \in \mathrm{Diff}^*(\mathbb{S}^{n-1})$ and β, there exists a $C > 0$ such that*

$$| P\partial_\xi^\beta a(\omega,\xi) | \le C \begin{cases} |\xi|^{-n+N+1-|\beta|}, & |\xi| \le 1; \\ |\xi|^{-n-2-|\beta|}, & |\xi| \ge 1. \end{cases} \tag{30}$$

Then, given $a \in \mathcal{R}_N(\mathbb{S}^{n-1} \times \mathbb{R}^n)$, the function

$$A(r,\omega) := \int e^{i\frac{\omega}{r}\cdot\xi}\, a(\omega,\xi)\, d\xi$$

is an element of $r^N S^0([0,\infty)_r \times \mathbb{S}^{n-1}_\omega)$.

Proof: Let $N \in \mathbb{N}$ and let $a \in \mathcal{R}_N(\mathbb{S}^{n-1} \times \mathbb{R}^n)$. Observe that the inequalities (30) imply that the integral defining $A(r,\omega)$ converges. We need to show that

$$r^{-N} A(r,\omega) = r^{-N} \int e^{i\frac{\omega}{r}\cdot\xi}\, a(\omega,\xi)\, d\xi$$

is in $S^0([0,\infty)_r \times \mathbb{S}^{n-1}_\omega)$. To see this, define for each $\omega \in \mathbb{S}^{n-1}$, $L_\omega := -\omega \cdot D_\xi = -\sum_{i=1}^n \omega_i D_{\xi_i}$. Then observe that $L_\omega e^{i\frac{\omega}{r}\cdot\xi} = -r^{-1} e^{i\frac{\omega}{r}\cdot\xi}$. Hence, using the estimates (30) to justify integrating by parts, we have

$$\begin{aligned} r^{-N} A(r,\omega) &= (-1)^N \int (L_\omega^N e^{i\frac{\omega}{r}\cdot\xi})\, a(\omega,\xi)\, d\xi \\ &= \int e^{i\frac{\omega}{r}\cdot\xi}\, L_\omega^N a(\omega,\xi)\, d\xi \\ &= \int e^{i\frac{\omega}{r}\cdot\xi}\, a_0(\omega,\xi)\, d\xi, \end{aligned}$$

where $a_0(\omega,\xi) := L_\omega^N a(\omega,\xi) \in \mathcal{R}_0(\mathbb{S}^{n-1} \times \mathbb{R}^n)$. Thus, we may assume that $a \in \mathcal{R}_0(\mathbb{S}^{n-1} \times \mathbb{R}^n)$; we'll show that $A(r,\omega) := \int e^{i\frac{\omega}{r}\cdot\xi}\, a(\omega,\xi)\, d\xi \in S^0([0,\infty)_r \times \mathbb{S}^{n-1}_\omega)$. Observe that A is bounded since a is integrable. We claim that

$$r\partial_r A(r,\omega) = \int e^{i\frac{\omega}{r}\cdot\xi}\, a_1(\omega,\xi)\, d\xi, \tag{31}$$

where $a_1 \in \mathcal{R}_0(\mathbb{S}^{n-1} \times \mathbb{R}^n)$ and that given any $v \in \mathcal{V}(\mathbb{S}^{n-1})$,

$$vA(r,\omega) = \int e^{i\frac{\omega}{r}\cdot\xi}\, a_v(\omega,\xi)\, d\xi, \tag{32}$$

where $a_v \in \mathcal{R}_0(\mathbb{S}^{n-1} \times \mathbb{R}^n)$. Note that once we prove these two statements, given any $k \in \mathbb{N}$ and $P \in \mathrm{Diff}^*(\mathbb{S}^{n-1})$, we have $(r\partial_r)^k PA(r,\omega) = \int e^{i\frac{\omega}{r}\cdot\xi}\, a_{k,P}(\omega,\xi)\, d\xi$, where $a_{k,P} \in \mathcal{R}_0(\mathbb{S}^{n-1} \times \mathbb{R}^n)$. Since $a_{k,P}$ is integrable, $(r\partial_r)^k PA(r,\omega)$ is bounded. Thus, $A(r,\omega) \in S^0([0,\infty)_r \times \mathbb{S}^{n-1}_\omega)$. Thus, it remains to prove (31) and (32). We first prove (31). Thus, using the estimates (30) to justify integrating by parts, we find that

$$\begin{aligned} r\partial_r A(r,\omega) &= -r^{-1} \int e^{i\frac{\omega}{r}\cdot\xi}\, (i\omega\cdot\xi)\, a(\omega,\xi)\, d\xi \\ &= \int (L_\omega e^{i\frac{\omega}{r}\cdot\xi})\, (i\omega\cdot\xi)\, a(\omega,\xi)\, d\xi \\ &= -\int e^{i\frac{\omega}{r}\cdot\xi}\, L_\omega[\,(i\omega\cdot\xi)\, a(\omega,\xi)\,]\, d\xi. \end{aligned} \tag{33}$$

Note that $L_\omega[\,(i\omega\cdot\xi)\,a(\omega,\xi)\,]\in\mathcal{R}_0(\mathbb{S}^{n-1}\times\mathbb{R}^n)$. Thus, (31) holds. Now let $v\in\mathrm{Diff}^1(\mathbb{S}^{n-1})$. Then,

$$vA(r,\omega)=\int e^{i\frac{\omega}{r}\cdot\xi}\,va(\omega,\xi)\,d\xi+r^{-1}\int e^{i\frac{\omega}{r}\cdot\xi}\,p_v(\omega,\xi)\,a(\omega,\xi)\,d\xi, \tag{34}$$

where, for each ω, $p_v(\omega,\xi)$ is a homogeneous polynomial in ξ of degree 1. Observe that $va\in\mathcal{R}_0(\mathbb{S}^{n-1}\times\mathbb{R}^n)$. Hence, by using a similar integration by parts argument on the second term of (34) as we used in (33) above, it follows that $vA(r,\omega)=\int e^{i\frac{\omega}{r}\cdot\xi}\,a_v(\omega,\xi)\,d\xi$, for some $a_v\in\mathcal{R}_0(\mathbb{S}^{n-1}\times\mathbb{R}^n)$. Thus, (32) is proved. ■

We use the identification $[\Lambda_{cc}^{1/d};\{0\}]\equiv\overline{\Lambda^{1/d}}\setminus\{0\}$, with $f\!f[\Lambda_{cc}^{1/d};\{0\}]\equiv\partial_\infty\overline{\Lambda^{1/d}}$. We define

$$r:=|\mu|,\ \omega:=\mu/|\mu|,\ \mu=1/\lambda\in\Lambda_{cc}^{1/d}.$$

Recall that $T^n_{\Lambda^{1/d}}=[\mathbb{R}^n_z\times\overline{\Lambda^{1/d}};Y_1;Y_2]$, with

$$Y_1=\{0\}_z\times\{0\}_r\times(\mathbb{S}^1\cap\Lambda_{cc}^{1/d})_\omega\ \text{ and }\ Y_2=\{0\}_z\times[0,\infty)_r\times(\mathbb{S}^1\cap\Lambda_{cc}^{1/d})_\omega.$$

Coordinates on $T^n_{\Lambda^{1/d}}$ are shown in Figure 5. Recall that $bi=\{\omega_0=0\}$, $f\!i=\{t=0\}$, and $df=\{u=0\}$.

We first work on the expansion at bi.

Lemma 3.36 *Then $m,\,p\in\mathbb{R}$ with $p/d\in\mathbb{Z}$ and let $\chi(\xi)\in C^\infty(\mathbb{R}^n)$ with $\chi(\xi)\equiv 0$ on a neighborhood of 0 and $\chi(\xi)\equiv 1$ outside a neighborhood of 0. Then given any*

$$A=\int e^{iz\cdot\xi}\,\chi(\xi)\,a(\lambda,\xi)\,d\xi,$$

where $a(\lambda,\xi)\in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$, we have $A(\lambda^d)\in\mathcal{A}^{\mathcal{E}_{bi}}(T^n_{\Lambda^{1/d}})$, where $\mathcal{E}_{bi}:=-p+d\mathbb{N}_0$ is the index set on $T^n_{\Lambda^{1/d}}$ associated to bi.

Proof: Coordinates on $T^n_{\Lambda^{1/d}}$ near bi are given by the second set of coordinates in Figure 5:

$$\left.\begin{array}{l} s:=|z|\\ \gamma_0:=r/|z|\\ \gamma':=z/|z|\end{array}\right\}\leftrightarrow\left\{\begin{array}{l} r=s\,\gamma_0\\ z=s\,\gamma'.\end{array}\right.$$

Let $a(\lambda,\xi)\in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$. Then $a(\lambda,\xi)\in C^\infty(\Lambda\times(\mathbb{R}^n\setminus\{0\}))$ and is such that

$$a(\delta^d\lambda,\delta\xi)=\delta^m a(\lambda,\xi)\text{ for all }\delta>0;$$

and if we define $\tilde a(\mu,\xi):=\mu^{p/d}\,a(1/\mu,\xi)=\mu^{p/d}\,a(\overline{\mu}/|\mu|^2,\xi)$ for all $(\mu,\xi)\in\Lambda_{cc}\times\mathbb{R}^n$, then $\tilde a(\mu,\xi)\in C^\infty(\Lambda_{cc}\times(\mathbb{R}^n\setminus\{0\})\,)$. Observe that since $\mu=r\omega=s\gamma_0\omega$, we have

$$\begin{aligned} A(\lambda^d)&=\int e^{is\gamma'\cdot\xi}\,\chi(\xi)\,a(\,(s\gamma_0\omega)^{-d},\xi)\,d\xi\\ &=s^{-p}\gamma_0^{-p}\omega^{-p}\int e^{is\gamma'\cdot\xi}\,\chi(\xi)\,\tilde a(s^d\gamma_0^d\omega^d,\xi)\,d\xi.\end{aligned}$$

Thus, it suffices to prove that $\int e^{is\gamma'\cdot\xi}\,\chi(\xi)\,\tilde a(s^d\gamma_0^d\omega^d,\xi)\,d\xi\in\mathcal{A}^{\mathcal{E}'_{bi}}(T^n_{\Lambda^{1/d}})$, where $\mathcal{E}'_{bi}:=d\mathbb{N}_0$. To see this, set $L_{\gamma'}:=-\gamma'\cdot D_\xi$ and choose $N\in\mathbb{N}$ with $m-N\le -n-2$. Then observe that

$$\begin{aligned}\int e^{is\gamma'\cdot\xi}\,\chi(\xi)\,\tilde a(s^d\gamma_0^d\omega^d,\xi)\,d\xi&=s^{-N}(-1)^N\int(\,L^N_{\gamma'}e^{is\gamma'\cdot\xi}\,)\,\chi(\xi)\,\tilde a(s^d\gamma_0^d\omega^d,\xi)\,d\xi\\ &=s^{-N}\int e^{is\gamma'\cdot\xi}\,\chi(\xi)\,(L^N_{\gamma'}\tilde a)(s^d\gamma_0^d\omega^d,\xi)\,d\xi\\ &\quad+s^{-N}A_N(s,s^d\gamma_0^d\omega^d,\gamma'),\end{aligned}$$

where $A_N(s,\mu,\gamma') = \int e^{is\gamma'\cdot\xi}\,[\sum_{l=1}^N \left(\begin{array}{c} N \\ l \end{array}\right) (L_{\gamma'}^l \chi(\xi)\,)\,(L_{\gamma'}^{N-l}\tilde{a})(\mu,\xi)]\,d\xi$. Since for $l \geq 1$, $L_{\gamma'}^l\chi(\xi) \equiv 0$ for ξ near 0 and for ξ outside a neighborhood of 0, it follows that $A_N(s,\mu,\gamma')$ is C^∞ in all variables (s,μ,γ'). In particular, $A_N(s,s^d\gamma_0^d\omega^d,\gamma')$ can be expanded in powers of γ_0^d. Hence, $s^{-N}A_N(s,s^d\gamma_0^d\omega^d,\gamma') \in \mathcal{A}^{\mathcal{E}'_{bi}}(T^n_{\Lambda^{1/d}})$. Thus, it remains to show that

$$\int e^{is\gamma'\cdot\xi}\,\chi(\xi)\,(L_{\gamma'}^N\tilde{a})(s^d\gamma_0^d\omega^d,\xi)\,d\xi \in \mathcal{A}^{\mathcal{E}'_{bi}}(T^n_{\Lambda^{1/d}}).$$

Since $m - N \leq n-2$, $(L_{\gamma'}^N a)(\lambda,\xi) \in C^\infty(\mathbb{S}^{n-1}_{\gamma'}; C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m')}(\mathbb{R}^n)\,)$, where $m' \leq -n-2$. Thus, it suffices to prove the following statement: given $a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$, where $m \leq -n-2$,

$$A := \int e^{is\gamma'\cdot\xi}\,\chi(\xi)\,\tilde{a}(s^d\gamma_0^d\omega^d,\xi)\,d\xi \in \mathcal{A}^{\mathcal{E}'_{bi}}(T^n_{\Lambda^{1/d}}).$$

To see this, set $v := \gamma_0^d$ and if $\omega = e^{i\theta}$, set $D_\omega := \cos(d\theta)\partial_{\mu_1} + \sin(d\theta)\partial_{\mu_2}$. Then, for any $M \in \mathbb{N}$,

$$D_v^M A = s^{dM}\int e^{is\gamma'\cdot\xi}\,\chi(\xi)\,(D_\omega^M\tilde{a})(s^d\gamma_0^d\omega^d,\xi)\,d\xi.$$

By Proposition 3.9, $\tilde{a}(\mu,\xi)$ satisfies the estimates: for each α and β, there is a C such that

$$|\,\partial_\mu^\alpha\partial_\xi^\beta\tilde{a}(\mu,\xi)\,| \leq C\,(1+|\mu|\,|\xi|^d)^{p/d-|\alpha|}\,|\xi|^{d|\alpha|+m-p-|\beta|}.$$

In particular, for any α and β and M,

$$|\,\partial_\mu^\alpha\partial_\xi^\beta D_\omega^M\tilde{a}(\mu,\xi)\,| \leq C\,(1+|\mu|\,|\xi|^d)^{p/d-M-|\alpha|}\,|\xi|^{d|\alpha|+dM+m-p-|\beta|}. \tag{35}$$

Observe that this estimate implies that if M is sufficiently large (for example, if $dM+m-p > -n$), then $(D_\omega^M\tilde{a})(w\omega^d,\xi)$ is integrable near $\xi = 0$ for all $w \in [0,\infty)$; and moreover, it remains integrable near $\xi = 0$ even after we differentiate it with respect to w or ω. Thus, if we define

$$B_M(s,w,\gamma',\omega) := \int e^{is\gamma'\cdot\xi}\,(\chi(\xi)-1)\,(D_\omega^M\tilde{a})(w\omega^d,\xi)\,d\xi,\ w \in [0,\infty),$$

then B_M is smooth in all variables. Also, since $m \leq -n-2$, observe that for any M, the estimate (35) implies that for any fixed $w > 0$, $(D_\omega^M\tilde{a})(w\omega^d,\xi)$ is integrable near $\xi = \infty$. Thus, for M sufficiently large and $s^d\gamma_0^d\omega^d > 0$, $\int e^{is\gamma'\cdot\xi}\,(D_\omega^M\tilde{a})(s^d\gamma_0^d\omega^d,\xi)\,d\xi$ converges. Thus, for M sufficiently large, we can write

$$D_v^M A = s^{dM}\int e^{is\gamma'\cdot\xi}\,(D_\omega^M\tilde{a})(s^d\gamma_0^d\omega^d,\xi)\,d\xi + s^{dM}B_M(s,s^d\gamma_0^d,\gamma',\omega). \tag{36}$$

Since $(D_\omega^M\tilde{a})(s^d\gamma_0^d\omega^d,(s\gamma_0)^{-1}\xi) = (s\gamma_0)^{-dM+p-m}(D_\omega^M\tilde{a})(\omega^d,\xi)$, making the change of variables $\xi \mapsto (s\gamma_0)^{-1}\xi$ in the integral (36), we find that

$$D_v^M A = s^{dM}(s\gamma_0)^{-dM+p-m-n}A_M + s^{dM}B_M(s,s^d\gamma_0^d,\gamma',\omega),$$

where $A_M := \int e^{i\frac{\gamma'}{\gamma_0}\cdot\xi}\,(D_\omega^M\tilde{a})(\omega^d,\xi)\,d\xi$. Since $v = \gamma_0^d$, we thus have

$$v^M D_v^M A = (s\gamma_0)^{p-m-n}A_M + s^{dM}\gamma_0^{dM}B_M(s,s^d\gamma_0^d,\gamma',\omega).$$

Let $N >> 0$. Since $m \leq -n-2$, choosing M such that $dM+m-p \geq -n-N+1$, the estimate (35) implies that given β, there is a C such that

$$\begin{aligned} |\,\partial_\xi^\beta D_\omega^M\tilde{a}(\omega^d,\xi)\,| &\leq C\,(1+|\xi|^d)^{p/d-M}\,|\xi|^{dM+m-p-|\beta|} \\ &\leq C'\left\{\begin{array}{l} |\xi|^{-n+N+1-|\beta|},\ |\xi| \leq 1; \\ |\xi|^{-n-2-|\beta|},\ |\xi| \geq 1, \end{array}\right. \end{aligned}$$

for some $C' > 0$. Thus, Lemma 3.35 implies that $A_M \in \gamma_0^N S^0([0,\infty)_{\gamma_0} \times \mathbb{S}^{n-1}_{\gamma'} \times (\mathbb{S}^1_\omega \cap \Lambda^{1/d}_{cc}))$. Now observe that $v^M \partial_v^M = (v\partial_v - M + 1)(v\partial_v - M + 2) \cdots (v\partial_v - 1)(v\partial_v)$ and that $\gamma_0 \partial_{\gamma_0} = d\, v\partial_v$. Thus,

$$\begin{aligned}&(\gamma_0\partial_{\gamma_0} - d(M+1)) \cdots (\gamma_0\partial_{\gamma_0} - d)(\gamma_0\partial_{\gamma_0})A \in \\ &\quad (s\gamma_0)^{p-m-n}\gamma_0^N S^0([0,\infty)_{\gamma_0} \times \mathbb{S}^{n-1}_{\gamma'} \times (\mathbb{S}^1_\omega \cap \Lambda^{1/d}_{cc})) \\ &\qquad + s^{dM}\gamma_0^{dM} C^\infty([0,\infty)_s \times [0,\infty)_{s^d\gamma_0^d} \times \mathbb{S}^{n-1}_{\gamma'} \times (\mathbb{S}^1_\omega \cap \Lambda^{1/d}_{cc})).\end{aligned}$$

Since N and M can be arbitrarily large, Theorem 2.1 implies that $A \in \mathcal{A}^{\mathcal{E}'_{bi}}(T^n_{\Lambda^{1/d}})$. ■

We now work on the expansion at fi.

Lemma 3.37 *Let m, $p \in \mathbb{R}$ with $p/d \in \mathbb{Z}$ and let $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ on a neighborhood of 0 and $\chi(\xi) \equiv 1$ outside a neighborhood of 0. Then given any*

$$A = \int e^{iz\cdot\xi}\,\chi(\xi)\,a(\lambda,\xi)\,d\xi,$$

where $a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$, we have

$$A(\lambda^d) \in \mathcal{A}^{\mathcal{E}_{fi}}(T^n_{\Lambda^{1/d}}),$$

where $\mathcal{E}_{fi} := \{(k-m-n,0)\,|\,k \in \mathbb{N}_0\}\overline{\cup}(-p+\mathbb{N}_0+d\mathbb{N}_0)$ is the index set on $T^n_{\Lambda^{1/d}}$ associated to fi. Moreover, if $m < -n$, then

$$A(\lambda^d)|_{z=0} \in \mathcal{A}^{\mathcal{F}}(\overline{\Lambda^{1/d}}),$$

where $\mathcal{F} := \{(k-m-n,0)\,|\,k \in \mathbb{N}_0\}\overline{\cup}(-p+d\mathbb{N}_0)$ is the index set on $\overline{\Lambda^{1/d}}$ associated to $\partial_\infty\overline{\Lambda^{1/d}}$.

Proof: Coordinates on $T^n_{\Lambda^{1/d}}$ near fi are given by the first set of coordinates in Figure 5:

$$\left.\begin{array}{l}\rho := (|z|^2+r^2)^{1/2}\\ \omega_0 := r/(|z|^2+r^2)^{1/2}\\ \omega' := z/(|z|^2+r^2)^{1/2}\end{array}\right\} \leftrightarrow \left\{\begin{array}{l} r = \rho\,\omega_0\\ z = \rho\,\omega'.\end{array}\right.$$

Let $a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$. Then, since $\mu = r\omega = \rho\omega_0\omega$ and $z = r\omega'$, we have

$$\begin{aligned}A(\lambda^d) &= \int e^{i\rho\omega'\cdot\xi}\,\chi(\xi)\,a(\,(\rho\omega_0\omega)^{-d},\xi)\,d\xi\\ &= (\rho\omega_0\omega)^{-p}\int e^{i\rho\omega'\cdot\xi}\,\chi(\xi)\,\tilde{a}(\rho^d\omega_0^d\omega^d,\xi)\,d\xi.\end{aligned}$$

Making the change of variables $\xi \mapsto \rho^{-1}\xi$ yields

$$A = \rho^{-m-n}(\omega_0\omega)^{-p}\int e^{i\rho\omega'\cdot\xi}\,\chi(\xi)\,\tilde{a}((\omega_0\omega)^d,\xi)\,d\xi.$$

Hence,

$$\begin{aligned}(\rho\partial_\rho - (-m-n))A &= -\rho^{-m-n}(\omega_0\omega)^{-p}\int e^{i\rho\omega'\cdot\xi}\,(\xi\cdot\partial_\xi\chi)(\xi/\rho)\,\tilde{a}((\omega_0\omega)^d,\xi)\,d\xi\\ &= -\rho^{-p}(\omega_0\omega)^{-p}\int e^{i\rho\omega'\cdot\xi}\,(\xi\cdot\partial_\xi\chi)(\xi)\,\tilde{a}(\rho^d(\omega_0\omega)^d,\xi)\,d\xi.\end{aligned}$$

Since $(\xi\cdot\partial_\xi\chi)(\xi) \equiv 0$ near $\xi = 0$ and outside a neighborhood of 0, it follows that

$$f(u,v) := \int e^{iu\omega'\cdot\xi}\,(\xi\cdot\partial_\xi\chi)(\xi)\,\tilde{a}(v(\omega_0\omega)^d,\xi)\,d\xi$$

is a smooth function of $(u,v)\in[0,\infty)^2$. Hence,

$$(\rho\partial_\rho-(-m-n))A\sim\sum_{i,j=0}^{\infty}\rho^{-p+i+dj}A_{i,j}(\omega_0,\omega',\omega),$$

where for each i and j, $A_{i,j}(\omega_0,\omega',\omega):=(\partial_u^i\partial_v^j f)(0,0)\in\omega_0^{-p}\omega^{-p}S^0(\mathbb{S}^{n,1}_{(\omega_0,\omega')}\times(\mathbb{S}^1_\omega\cap\Lambda^{1/d}_{cc}))$. Note that $A_{i,j}(\omega_0,\omega',\omega)\equiv 0$ at $\omega'=0$ for $i>0$. Thus, by Theorem 2.1, it follows that $A\in\mathcal{A}^{\mathcal{E}_{fi}}(T^n_{\Lambda^{1/d}})$, and if $m<-n$, then $A(\lambda^d)|_{z=0}\in\mathcal{A}^{\mathcal{F}}(\overline{\Lambda^{1/d}})$. ■

We now work on the expansion at df.

Lemma 3.38 *Let m, $p\in\mathbb{R}$ with $p/d\in\mathbb{Z}$ and let $\chi(\xi)\in C^\infty(\mathbb{R}^n)$ with $\chi(\xi)\equiv 0$ on a neighborhood of 0 and $\chi(\xi)\equiv 1$ outside a neighborhood of 0. Then given any*

$$A=\int e^{iz\cdot\xi}\,\chi(\xi)\,a(\lambda,\xi)\,d\xi,$$

where $a(\lambda,\xi)\in C^{\infty,p,d}_{\Lambda,\hom(m)}(\mathbb{R}^n)$, we have

$$A(\lambda^d)\in\mathcal{A}^{\mathcal{E}_{df}}(T^n_{\Lambda^{1/d}}),$$

where $\mathcal{E}_{df}:=\{(k-m-n+dl,0)\,|\,k,l\in\mathbb{N}_0\}\overline{\cup}\mathbb{N}_0$ is the index set on $T^n_{\Lambda^{1/d}}$ associated to bi.

Proof: Coordinates on $T^n_{\Lambda^{1/d}}$ near bi are given by the last set of coordinates in Figure 5:

$$\left.\begin{array}{l} t:=t\\ u:=|w|\\ \gamma:=w/|w|\end{array}\right\}\leftrightarrow\left\{\begin{array}{l} t=t\\ w=u\,\gamma\end{array}\right\}\leftrightarrow\left\{\begin{array}{l} r=t\\ z=t\,u\,\gamma.\end{array}\right.$$

Let $a(\lambda,\xi)\in C^{\infty,p,d}_{\Lambda,\hom(m)}(\mathbb{R}^n)$. Then, $A(\lambda^d)=\int e^{itu\gamma\cdot\xi}\,\chi(\xi)\,a(\,t^{-d}\omega^{-d},\xi)\,d\xi$. Making the change of variables $\xi\mapsto t^{-1}\xi$ and using the homogeneous properties of a, we find that

$$A=t^{-m-n}\int e^{iu\gamma\cdot\xi}\,\chi(\xi/t)\,a(\,\omega^{-d},\xi)\,d\xi.$$

Thus,

$$\begin{aligned}(t\partial_t-(-m-n))A\ &=\ -t^{-m-n}\int e^{iu\gamma\cdot\xi}\,(\xi\cdot\partial_\xi\chi)(\xi/t)\,a(\,\omega^{-d},\xi)\,d\xi\\ &=\ -\int e^{itu\gamma\cdot\xi}\,(\xi\cdot\partial_\xi\chi)(\xi)\,a(\,t^{-d}\omega^{-d},\xi)\,d\xi\\ &=:\ B(t,u,\gamma,\omega).\end{aligned}$$

Since $\xi\cdot\partial_\xi\chi(\xi)\equiv 0$ near $\xi=0$ and outside a neighborhood of 0, B is a smooth function of u at $u=0$. Since $(t\partial_t-(-m-n))A=B$, we have $t\partial_t(t^{m+n}A)=t^{m+n}B$. Integrating this equation from 1 to t yields $A=t^{-m-n}A(1,u,\gamma,\omega)+t^{-m-n}\int_1^t\tau^{m+n}B(\tau,u,\gamma,\omega)\frac{d\tau}{\tau}$. The second term in this equation is smooth at $u=0$, and so we are left to show that

$$A':=A(1,u,\gamma,\omega)=\int e^{iu\gamma\cdot\xi}\,\chi(\xi)\,a(\omega^{-d},\xi)\,d\xi\in\mathcal{A}^{\mathcal{E}_{df}}(T^n_{\Lambda^{1/d}}).$$

Note that for any α, we have $(\partial_\lambda^\alpha a)(\delta^d\lambda,\delta\xi)=\delta^{m-d|\alpha|}(\partial_\lambda^\alpha a)(\lambda,\xi)$ for all $\delta>0$. In particular, setting $\lambda=0$ yields $(\partial_\lambda^\alpha a)(0,\delta\xi)=\delta^{m-d|\alpha|}(\partial_\lambda^\alpha a)(0,\xi)$ for all $\delta>0$. Thus, expanding $[0,\infty)\ni v\mapsto a(v\omega^{-d},\xi)$ in Taylor Series at $v=0$, we find that for each $N\in\mathbb{N}$,

$$a(v\omega^{-d},\xi)=\sum_{l=0}^{N-1}v^l a_{m-dl}(\omega,\xi)+b_N(v,\omega,\xi),$$

$$\left.\begin{array}{l} t := t \\ u := |w| \\ \gamma := w/|w| \end{array}\right\} \leftrightarrow \left\{ \begin{array}{l} t = t \\ z = u\,\gamma \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} r = t \\ z = t\,u\,\gamma \end{array}\right.$$

$$\left.\begin{array}{l} s := |z| \\ \gamma_0 := r/|z| \\ \gamma' := z/|z| \end{array}\right\} \leftrightarrow \left\{ \begin{array}{l} r = s\,\gamma_0 \\ z = s\,\gamma' \end{array}\right.$$

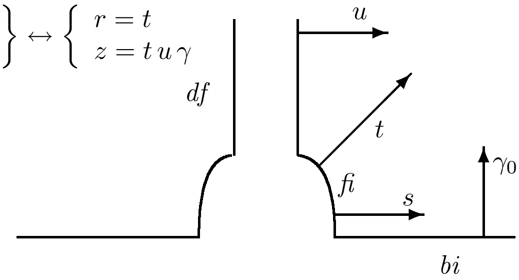


Figure 6: Coordinates near *df* and *bi*.

where for each l and ω, $a_{m-dl}(\omega,\xi) \in C^\infty_{\mathrm{hom}(m-dl)}(\mathbb{R}^n)$, and where

$$b_N(v,\omega,\xi) = \frac{v^N}{N!}\int_0^1 (1-s)^{N-1}\,(\partial_v^N a)(sv\omega^{-d},\xi)\,ds.$$

Let $N >> 0$. Then, setting $v=1$, it follows that $A' = \sum_{l=0}^{N-1} A_{m-dl} + B_N$, where

$$A_{m-dl} := \int e^{iu\gamma\cdot\xi}\,\chi(\xi)\,a_{m-dl}(\omega,\xi)\,d\xi, \text{ and } B_N := \int e^{iu\gamma\cdot\xi}\,\chi(\xi)\,b_N(1,\omega,\xi)\,d\xi.$$

By using a similar argument as we did in Lemma 3.37, one can show that for each l, $A_{m-dl} \in \mathcal{A}^{\mathcal{E}_{df}}(T^n_{\Lambda^{1/d}})$. Also, note that for each ω, $\chi(\xi)\,b_N(1,\omega,\xi) \in S^{m-dN}(\mathbb{R}^n)$. Thus,

$$B_N \in C^M([0,\infty)_u \times \mathbb{S}^{n-1}_\gamma \times (\mathbb{S}^1_\omega \cap \Lambda^{1/d}))$$

for all M with $M + m - dN < -n$. Since N is arbitrary, by Proposition 2.2, $A' \in \mathcal{A}^{\mathcal{E}_{df}}(T^n_{\Lambda^{1/d}})$. ■

Near *bi*, we can write $T^n_{\Lambda^{1/d}} \cong [0,\infty)_s \times [0,\infty)_{\gamma_0} \times F_{bi}$, where $F_{bi} := \mathbb{S}^{n-1}_{\gamma'} \times (\mathbb{S}^1_\omega \cap \Lambda^{1/d}_{cc})$; and near *df*, we can write $T^n_{\Lambda^{1/d}} \cong [0,\infty)_u \times [0,\infty)_t \times F_{df}$, where $F_{df} := \mathbb{S}^{n-1}_\gamma \times (\mathbb{S}^1_\omega \cap \Lambda^{1/d}_{cc})$. (See Figure 6.)

Lemma 3.39 *Let $N \in \mathbb{N}$, let $m, p \in \mathbb{R}$ with $p/d \in \mathbb{Z}$ and with $m + |p| \le -N - n - 1$, and let $M \in \mathbb{N}$ with $M + dM \le N$. Then given any*

$$R = \int e^{iz\cdot\xi}\,r(\lambda,\xi)\,d\xi,$$

where $r(\lambda,\xi) \in S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$, we have $R \in (s\gamma_0)^{-p} C^M(\,[0,\infty)_s \times [0,\infty)_{s^d\gamma_0^d} \times F_{bi}\,)$ near bi, and $R \in t^{-p} C^M(\,[0,\infty)_u \times [0,\infty)_t \times [0,\infty)_{t^d} \times F_{df}\,)$ near df.

Proof: We must show that if

$$R_{bi} := \int e^{is\gamma'\cdot\xi}\,r((s\gamma_0\omega)^{-d},\xi)\,d\xi \text{ and } R_{df} := \int e^{itu\gamma\cdot\xi}\,r((t\omega)^{-d},\xi)\,d\xi,$$

then

$$R_{bi} \in (s\gamma_0)^{-p} C^M(\,[0,\infty)_s \times [0,\infty)_{s^d\gamma_0^d} \times F_{bi}\,)$$

and

$$R_{df} \in t^{-p} C^M(\,[0,\infty)_u \times [0,\infty)_t \times [0,\infty)_{t^d} \times F_{df}\,).$$

Since $r \in S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$, if we define $\tilde r(\mu,\xi) := \mu^{p/d}\,r(1/\mu,\xi)$, then $\tilde r(\mu,\xi) \in C^\infty(\Lambda_{cc} \times \mathbb{R}^n)$ and it satisfies the estimates: for any α and β, there is a $C > 0$ such that

$$|\,\partial^\alpha_\mu \partial^\beta_\xi \tilde r(\mu,\xi)\,| \le C\,(1 + |\mu|\,|\xi|^d)^{p/d - |\alpha|}\,(1 + |\xi|)^{d|\alpha| + m - p - |\beta|}.$$

Observe that for $|\mu|$ bounded, for some C' (depending on the bound on $|\mu|$), we have

$$(1 + |\mu|\,|\xi|^d)^{p/d}\,(1 + |\xi|)^{-p} \le C'(1 + |\xi|)^{|p|}.$$

Thus, for $|\mu|$ bounded, for any α, there is a $C > 0$ such that $|\,\partial_\mu^\alpha \tilde{r}(\mu,\xi)\,| \le C\,(1+|\xi|)^{d|\alpha|+m+|p|}$; or, as $m+|p| \le -N-n-1$ we have

$$|\,\partial_\mu^\alpha \partial_\xi^\beta \tilde{r}(\mu,\xi)\,| \le C\,(1+|\xi|)^{d|\alpha|-N-n-1}. \tag{37}$$

Rewriting R_{bi} and R_{df} in terms of $\tilde{r}$ gives

$$R_{bi} = (s\gamma_0\omega)^{-p} \int e^{is\gamma'\cdot\xi}\,\tilde{r}((s\gamma_0)^d\omega^d,\xi)\,d\xi \text{ and } R_{df} = (t\omega)^{-p}\int e^{itu\gamma\cdot\xi}\,\tilde{r}(t^d\omega^d,\xi)\,d\xi.$$

Thus, to prove our Lemma, it suffices to show that if

$$R_{bi}(s,v,\gamma',\omega) := \int e^{is\gamma'\cdot\xi}\,\tilde{r}(v\omega^d,\xi)\,d\xi \text{ and } R_{df}(u,t,v,\gamma,\omega) := \int e^{itu\gamma\cdot\xi}\,\tilde{r}(v\omega^d,\xi)\,d\xi,$$

then for all $M \in \mathbb{N}$ with $M + dM \le N$,

$$R_{bi}(s,v,\gamma',\omega) \in C^M(\,[0,\infty)_s \times [0,\infty)_v \times F_{bi}\,)$$

and

$$R_{df}(u,t,v,\gamma,\omega) \in C^M(\,[0,\infty)_u \times [0,\infty)_t \times [0,\infty)_v \times F_{df}\,).$$

We will leave the proof of this to the reader. But we remark that the proof follows directly from the estimate (37). ■

For each m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, we define the index set $\mathcal{E}_{m,p,d}$ on $T^n_{\Lambda^{1/d}}$ by

$$\mathcal{E}_{m,p,d}(\text{edges of }\Lambda) := \mathbb{N}_0;\;\; \mathcal{E}_{m,p,d}(bi) = -p + d\mathbb{N}_0;$$

$$\mathcal{E}_{m,p,d}(df) := \frac{n}{2} + \{(k-m-n+dl,0)\,|\,k,l\in\mathbb{N}_0\}\overline{\cup}\mathbb{N}_0;$$

$$\mathcal{E}_{m,p,d}(fi) := \frac{n}{2} + \{(k-m-n,0)\,|\,k\in\mathbb{N}_0\}\overline{\cup}(-p+\mathbb{N}_0+d\mathbb{N}_0)$$

and define the index set $\mathcal{F}_{m,p,d}$ on $\overline{\Lambda^{1/d}}$, associated to $\partial_\infty\overline{\Lambda^{1/d}}$, by

$$\mathcal{F}_{m,p,d} := \{(k-m-n,0)\,|\,k\in\mathbb{N}_0\}\overline{\cup}(-p+d\mathbb{N}_0).$$

Lemma 3.40 *For each m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, there exists a continuous linear map*

$$I^{m+n/4,p,d}_{\Lambda,ros}(\mathbb{R}^n,\{0\},\Omega_b^{\frac12}) \;\rightarrow\; \mathcal{A}^{\mathcal{E}_{m,p,d}}_{phg}(\,T^n_{\Lambda^{1/d}},\Omega_b^{\frac12}\,);\;\; A \;\mapsto\; A(\lambda^d)\Big|\frac{d\lambda\wedge d\overline{\lambda}}{\lambda^2}\Big|^{\frac12},$$

such that if $m \notin \mathbb{N}_0$, then this map is injective and if $m \in \mathbb{N}_0$, then this map has kernel

$$I^{m+n/4,p,d}_{\Lambda,ros}(\mathbb{R}^n,\{0\},\Omega_b^{\frac12}) \cap C^\infty(\,\Lambda;\mathrm{span}_{\mathbb{C}}\{\delta^k\otimes|dz|^{\frac12}\,|\,0\le k\le m\}\,).$$

Moreover, if $m < -n$, then there exists a continuous linear map

$$I^{m+n/4,p,d}_{\Lambda,ros}(\mathbb{R}^n,\{0\},\Omega_b^{\frac12})|_{z=0} \hookrightarrow \mathcal{A}^{\mathcal{F}_{m,p,d}}_{phg}(\,\overline{\Lambda^{1/d}},\Omega_b);\;\; A \;\mapsto\; A(\lambda^d)|_{z=0}.$$

Proof: By Lemma 3.33, if $0 < \nu \in C^\infty(\mathbb{R}^n,\Omega_b^{\frac12})$, then when lifted to $T^n_{\Lambda^{1/d}}$, $\nu\,|\frac{d\lambda\wedge d\overline{\lambda}}{\lambda^2}|^{\frac12} = \rho_{df}^{\frac{n}{2}}\rho_{fi}^{\frac{n}{2}}\,\mu$, where $0 < \mu \in C^\infty(T^n_\Lambda,\Omega_b^{\frac12})$. Hence, if $\mathcal{E}'_{m,p,d}$ is the index set

$$\mathcal{E}'_{m,p,d}(\text{edges of }\Lambda) := \mathbb{N}_0;\;\; \mathcal{E}'_{m,p,d}(bi) = -p + d\mathbb{N}_0;$$

$$\mathcal{E}'_{m,p,d}(df) := \{(k-m-n+dl,0)\,|\,k,l\in\mathbb{N}_0\}\overline{\cup}\mathbb{N}_0;$$

$$\mathcal{E}'_{m,p,d}(fi) := \{(k-m-n,0)\,|\,k\in\mathbb{N}_0\}\overline{\cup}(-p+\mathbb{N}_0+d\mathbb{N}_0),$$

then it suffices to show that there exists a continuous linear map

$$I^{m+n/4,p,d}_{\Lambda,ros}(\mathbb{R}^n,\{0\}) \to \mathcal{A}^{\mathcal{E}'_{m,p,d}}_{phg}(\,T^n_{\Lambda^{1/d}}\,);\;\; A \mapsto A(\lambda^d),$$

such that if $m \notin \mathbb{N}_0$, then this map is injective and if $m \in \mathbb{N}_0$, then this map has kernel

$$I^{m+n/4,p,d}_{\Lambda,ros}(\mathbb{R}^n,\{0\}) \cap C^\infty(\,\Lambda;\mathrm{span}_{\mathbb{C}}\{\delta^k\,|\,0 \le k \le m\}\,);$$

and if $m < -n$, then there exists a continuous linear map

$$I^{m+n/4,p,d}_{\Lambda,ros}(\mathbb{R}^n,\{0\})|_{z=0} \hookrightarrow \mathcal{A}^{\mathcal{F}_{m,p,d}}_{phg}(\,\overline{\Lambda^{1/d}});\;\; A \mapsto A(\lambda^d)|_{z=0}.$$

Let $A \in I^{m+n/4,p,d}_{\Lambda,ros}(\mathbb{R}^n,\{0\})$. Then we can write

$$A = \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}\,a(\lambda,\xi)d\xi,$$

with $a(\lambda,\xi) \sim \sum_{j=0}^\infty \chi(\xi)\,a_{m-j}(\lambda,\xi)$, where $a_{m-j}(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\hom(m-j)}(\mathbb{R}^n)$ for each j, and where $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ is such that $\chi(\xi) \equiv 0$ on a neighborhood of 0 and $\chi(\xi) \equiv 1$ outside a neighborhood of 0. Thus, for any $N >> 0$, we can write

$$A = \sum_{j=1}^{N-1} A_{m-j} + R_N, \tag{38}$$

where $A_{m-j} = \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}\,\chi(\xi)\,a_{m-j}(\lambda,\xi)\,d\xi$, and where $R_N \in I^{m+n/4-N,p,d}_{\Lambda,ros}(\mathbb{R}^n,\{0\})$.

Now by Lemma 3.36, Lemma 3.38, and Lemma 3.37, for each $j \in \mathbb{N}_0$,

$$A_{m-j}(\lambda^d) \in \mathcal{A}^{\mathcal{E}'_{m,p,d}}_{phg}(\,T^n_{\Lambda^{1/d}}\,) \text{ and if } m < -n, \text{ then } A_{m-j}(\lambda^d)|_{z=0} \in \mathcal{A}^{\mathcal{F}_{m,p,d}}_{phg}(\,\overline{\Lambda^{1/d}}).$$

Thus, Proposition 2.2 applied to the expansion (38), using Lemma 3.39 on the remainer term R_N, shows that $A(\lambda^d) \in \mathcal{A}^{\mathcal{E}'_{m,p,d}}_{phg}(\,T^n_{\Lambda^{1/d}}\,)$, and if $m < -n$, then $A(\lambda^d)|_{z=0} \in \mathcal{A}^{\mathcal{F}_{m,p,d}}_{phg}(\,\overline{\Lambda^{1/d}})$. Note that

$$A(\lambda^d) \equiv 0 \text{ in } \mathcal{A}^{\mathcal{E}'_{m,p,d}}_{phg}(\,T^n_{\Lambda^{1/d}}\,) \text{ iff for each } \lambda \in \Lambda^{1/d},\; \mathrm{supp}\,A(\lambda^d) \subseteq \{0\}.$$

Now the only distributions having support at 0 are linear combinations of derivatives of the delta distribution at 0. Hence, $A(\lambda^d) \equiv 0$ in $\mathcal{A}^{\mathcal{E}'_{m,p,d}}_{phg}(\,T^n_{\Lambda^{1/d}}\,)$ iff

$$A(\lambda^d) \in I^{m+n/4,p,d}_{\Lambda,ros}(\mathbb{R}^n,\{0\}) \cap C^\infty(\,\Lambda;\mathrm{span}_{\mathbb{C}}\{\delta^k\,|\,0 \le k \le m\}\,).$$

■

Proof of the Structure Theorem: Let $m,\,p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. Then by definition, $\Psi^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac12}) := \{A \in I^{m,p,d}_{\Lambda,ros}(X^2_b,\Delta_b,\Omega_b^{\frac12})\,|\,A \equiv 0 \text{ at } lb\cup rb\}$. In particular, $\Psi^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac12}) \subseteq S^{p/d}_{os}(\Lambda;\,\Psi_b^{-\infty}(X,\Omega_b^{\frac12})\,)$ off of Δ_b, and in any product decomposition $X^2_b \cong \mathbb{R}^{n,k} \times \mathbb{R}^n_z$, where $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}_z$, we can identify

$$\Psi^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac12}) \equiv I^{m,p,d}_{\Lambda,ros}(\mathbb{R}^{n,k}\times\mathbb{R}^n,\mathbb{R}^{n,k}\times\{0\},\Omega_b^{\frac12}),$$

with

$$\mathrm{Diff}^{m,p,d}_{b,\Lambda,ros}(X,\Omega_b^{\frac12}) \subseteq I^{m,p,d}_{\Lambda,ros}(\mathbb{R}^{n,k}\times\mathbb{R}^n,\mathbb{R}^{n,k}\times\{0\},\Omega_b^{\frac12})$$

consisting of those elements having polynomial symbols (in ξ). Thus, Theorem 3.34 follows from Lemma 3.40.

3.11 Applications

Lemma 3.41 *Let $u(\lambda)$ be a holomorphic function on $\Lambda\setminus\{0\}$, where $\Lambda \subseteq \mathbb{C}$ is a proper subcone of $\mathbb{C}$. Suppose that in polar coordinates, (r,θ), u is of the form $u(r,\theta) = r^z(\sum_{k=0}^N u_k(\theta)(\log r)^k)$, where $z \in \mathbb{C}$, $N \in \mathbb{N}_0$, and for each k, $u_k \in C^\infty(\mathbb{S}^1 \cap \Lambda)$. Then, we can write $u(\lambda) = \lambda^z(\sum_{k=0}^N a_k (\log\lambda)^k)$, for some constants $a_k \in \mathbb{C}$.*

PROOF: By replacing $u_k(\theta)$ with $e^{-iz\theta}u_k(\theta)$, we may assume that $u(r,\theta)$ is of the form

$$u(r,\theta) = \lambda^z(\sum_{k=0}^N u_k(\theta)(\log r)^k),$$

where $\lambda = re^{i\theta}$. Observe that

$$\begin{aligned}\overline{\lambda}\,\overline{\partial}_\lambda &= (x-iy)\cdot\frac{1}{2}(\partial_x + i\partial_y) \\ &= \frac{1}{2}(x\partial_x + y\partial_y) + \frac{i}{2}(x\partial_y - y\partial_x) \\ &= \frac{1}{2}r\partial_r + \frac{i}{2}\partial_\theta.\end{aligned}$$

Also, observe that

$$r\partial_r u = zr^z e^{iz\theta}(\sum_{k=0}^N u_k(\theta)(\log r)^k) + r^z e^{iz\theta}(\sum_{k=0}^N u_k(\theta)k(\log r)^{k-1}), \tag{39}$$

and

$$i\partial_\theta u = -zr^z e^{iz\theta}(\sum_{k=0}^N u_k(\theta)(\log r)^k) + r^z e^{iz\theta}(\sum_{k=0}^N i\partial_\theta u_k(\theta)(\log r)^k). \tag{40}$$

Since u is holomorphic, $0 = 2\overline{\lambda}\,\overline{\partial}_\lambda u = r\partial_r u + i\partial_\theta u$. Thus, adding (39) and (40), we find that

$$\begin{cases} i\partial_\theta u_N(\theta) = 0; \\ ku_k(\theta) + i\partial_\theta u_{k-1}(\theta) = 0, \text{ for } k = 1,\dots,N. \end{cases} \tag{41}$$

Evidently, for each k, $u_k(\theta)$ is a polynomial of degree $N-k$. Hence, for each k, we can write

$$u_k(\theta) = \sum_{l=k}^N a_{k,l}\begin{pmatrix} l \\ k \end{pmatrix}(i\theta)^{l-k}$$

for some constants $a_{k,l} \in \mathbb{C}$. Let $1 \le k \le N$. Then,

$$\begin{aligned}\partial_\theta u_{k-1}(\theta) &= i\sum_{l=k}^N a_{k-1,l}\,(l-k+1)\begin{pmatrix} l \\ k-1 \end{pmatrix}(i\theta)^{l-k} \\ &= i\sum_{l=k}^N a_{k-1,l}\,k\begin{pmatrix} l \\ k \end{pmatrix}(i\theta)^{l-k} \\ &= ik\sum_{l=k}^N a_{k-1,l}\begin{pmatrix} l \\ k \end{pmatrix}(i\theta)^{l-k}.\end{aligned}$$

By the equations (41), $ku_k(\theta) = -i\partial_\theta u_{k-1}(\theta)$. Thus, we must have $a_{k-1,l} = a_{k,l}$ for each $l = k,\dots,N$. Hence, if we set $a_k := a_{k,k}$ for $k = 0,\dots,N$, it follows that

$$u_k(\theta) = \sum_{l=k}^N a_l\begin{pmatrix} l \\ k \end{pmatrix}(i\theta)^{l-k}, \text{ for } k = 0,\dots,N.$$

Since $\log\lambda = \log r + i\theta$, we conclude that

$$\begin{aligned}
u(\lambda) &= \lambda^z(\sum_{k=0}^N u_k(\theta)(\log r)^k\,) \\
&= \lambda^z(\sum_{k=0}^N\sum_{l=k}^N a_l \begin{pmatrix} l \\ k \end{pmatrix} (i\theta)^{l-k}(\log r)^k\,) \\
&= \lambda^z(\sum_{l=0}^N\sum_{k=0}^l a_l \begin{pmatrix} l \\ k \end{pmatrix} (i\theta)^{l-k}(\log r)^k\,) \\
&= \lambda^z(\sum_{l=0}^N a_l(\log r + i\theta)^k\,) \\
&= \lambda^z(\sum_{l=0}^N a_l(\log\lambda)^k\,).
\end{aligned}$$

■

This Lemma implies the following Proposition.

Proposition 3.42 *Let $\Lambda \subseteq \mathbb{C}$ be a closed cone and let $\mathcal{E}$ be any index set associated to $\partial\overline{\Lambda}$. Let $u \in \mathcal{A}^{\mathcal{E}}_{phg}(\overline{\Lambda})$ and suppose that u is holomorphic on Λ. Then,*

$$u(\lambda) \sim \sum_{(z,k)\in\mathcal{E}} \lambda^{-z}(\log\lambda)^k u_{(z,k)},\ u_{(z,k)} \in \mathbb{C},$$

as $|\lambda| \to \infty$ in Λ.

Thus, Theorem 3.34 and Proposition 3.42 give the following.

Theorem 3.43 *Let $m, p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$ and $m < -n$. Let $A(\lambda) \in \Psi^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$ be holomorphic. Then as $|\lambda| \to \infty$ in Λ, we have*

$$A(\lambda)|_{\Delta_b} \sim \sum_{k=0}^{\infty} \lambda^{\frac{m+n-k}{d}}\alpha_k(x) + \sum_{k,\,\frac{k-m-n+p}{d}\in\mathbb{N}_0} \lambda^{\frac{m+n-k}{d}} \log\lambda\,\alpha'_k(x) + \sum_{k=0}^{\infty}\lambda^{\frac{p}{d}-k}\alpha''_k(x),$$

where for each k, α_k, α'_k, $\alpha''_k \in C^\infty(X, \Omega_b)$.

Let $A \in \mathcal{E}\ell\ell^m_{\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$. Then by Theorem 3.32, there exists a continuous increasing function $r : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $\epsilon > 0$ and $N \in \mathbb{N}$,

$$(A-\lambda)^{-N} \in \Psi^{-Nm,-Nm,m,\epsilon,\epsilon,\epsilon}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$$

for $|\lambda| \geq r(\epsilon)$. In particular, as $\epsilon \to \infty$, $(A-\lambda)^{-N}$ becomes more and more an element of $\Psi^{-Nm,-Nm,m}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$. For each $m' \in \mathbb{R}$ and $N \in \mathbb{N}$, define the index set $\mathcal{E}_{m,m',N}$, associated to the boundary faces *bi*, *fi*, and *df* of $X^2_{b,\Lambda^{1/d}}$, by

$$\mathcal{E}_{m,m',N}(bi) = mN + m\mathbb{N}_0;$$

$$\mathcal{E}_{m,m',N}(df) := \frac{n}{2} + \{(k - m' - n + m(N+l), 0) \mid k, l \in \mathbb{N}_0\}\overline{\cup}\mathbb{N}_0;$$

$$\mathcal{E}_{m,m',N}(fi) := \frac{n}{2} + \{(k - m' - n + mN, 0) \mid k \in \mathbb{N}_0\}\overline{\cup}(mN + \mathbb{N}_0 + m\mathbb{N}_0).$$

Then the following Theorem follows from Theorem 3.34 and Theorem 3.43.

Theorem 3.44 *Let* $A \in \mathcal{E}\ell\ell^m_{\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, *and let* $B \in \Psi_b^{m'}(X, \Omega_b^{\frac{1}{2}})$. *Then, for any* $N \in \mathbb{N}$,

$$B(A - \lambda^m)^{-N} \in \mathcal{A}^{\mathcal{E}_{m,m',N}}(\, X^2_{b,\Lambda^{1/d}}, \Omega_b^{\frac{1}{2}}\,).$$

Moreover, if $Nm - m' > n$, *then as* $|\lambda| \to \infty$ *in* Λ, *we have*

$$\begin{aligned} B(A-\lambda)^{-N}|_{\Delta_b} \quad \sim \quad & \sum_{k=0}^{\infty} \lambda^{\frac{n+m'-k}{m} - N}\, \alpha_{N,k}(x) + \sum_{k,\, \frac{k-m'-n}{m} \in \mathbb{N}_0} \lambda^{\frac{n+m'-k}{m} - N} \, \log \lambda \, \alpha'_{N,k}(x) \\ & + \sum_{k=0}^{\infty} \lambda^{-k-N} \alpha''_{N,k}(x), \end{aligned}$$

where for each k, $\alpha_{N,k}$, $\alpha'_{N,k}$, $\alpha''_{N,k} \in C^\infty(X, \Omega_b)$.

4 Laplace transforms

4.1 Laplace transforms and the heat kernel

If $\Lambda \subseteq \mathbb{C}$ is a proper subcone of $\mathbb{C}$, then Λ is said to be *positive* if Λ contains a cone of the form $\{\lambda \in \mathbb{C} \,|\, \epsilon \leq \arg(\lambda) \leq 2\pi - \epsilon\}$ for some $0 \leq \epsilon < \pi/2$. Figure 7 gives an example of a positive cone.

Any Tempered operator $A \in \Psi^{*,*,*}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$, where Λ is positive, will be called a *positive operator*.

Let $m,\, p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and let $A(\lambda) \in \Psi^{m,p,d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$ be holomorphic and positive. Let Γ be any contour in Λ of the form given in Figure 8. Let $\phi \in \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}})$. Then, by Theorem 3.22,

$$A(\lambda)\phi \in S^{p/d}(\Lambda; \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}})\,).$$

Thus, for each $t > 0$, $e^{-t\lambda}\, A(\lambda)\phi$ is exponentially decreasing in $\dot{C}^\infty(X, \Omega_b^{\frac{1}{2}})$ as $|\lambda| \to \infty$ for $\lambda \in \Gamma$. Hence, for each $t > 0$, the integral

$$\mathcal{L}(A)(t)\phi := \int_\Gamma e^{-t\lambda}\, A(\lambda)\phi\, d\lambda \tag{42}$$

converges in $\dot{C}^\infty(X, \Omega_b^{\frac{1}{2}})$. Observe that since $A(\lambda)$ is holomorphic, the integral (42) is defined *independent* of the contour Γ choosen (where Γ is of the form given in Figure 8). Since for any $k \in \mathbb{N}$, $e^{-t\lambda}\, \lambda^k A(\lambda)\phi$ is still exponentially decreasing in $\dot{C}^\infty(X, \Omega_b^{\frac{1}{2}})$ as $|\lambda| \to \infty$ for $\lambda \in \Gamma$, it follows that $\mathcal{L}(A)(t)\phi \in C^\infty(\,(0,\infty)_t;\, \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}})\,)$.

Definition 4.1 Let $m,\, p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and let $A(\lambda) \in \Psi^{m,p,d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$ be holomorphic and positive. Then the *Laplace Transform* of A is the map

$$\mathcal{L}(A)(t) : \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}}) \;\to\; C^\infty(\,(0,\infty)_t;\, \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}})\,)$$

defined by equation (42) for any contour Γ of the form given in Figure 8.

Theorem 4.2 *Let $m,\, p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and let $A(\lambda) \in \Psi^{m,p,d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$ be holomorphic and positive. Then, $\mathcal{L}(A)(t) \in C^\infty(\,(0,\infty)_t;\, \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})\,)$. In fact, if $k \in \mathbb{N}_0$ and $M \in \mathbb{N}_0$ is such that $p/d - M < -k-1$, then for any $N \in \mathbb{N}_0$,*

$$t^{M+N}\mathcal{L}(A)(t) \in C^\infty(\,(0,\infty)_t;\, \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})\,) \cap C^k(\,[0,\infty)_t;\, \Psi_b^{m-p-dN}(X, \Omega_b^{\frac{1}{2}})\,). \tag{43}$$

Moreover, if $A(\lambda)$ extends to be holomorphic on a neighborhood of $\lambda = 0$, then

$$\mathcal{L}(A)(t) \to 0 \text{ exponentially in } \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}}) \text{ as } t \to \infty. \tag{44}$$

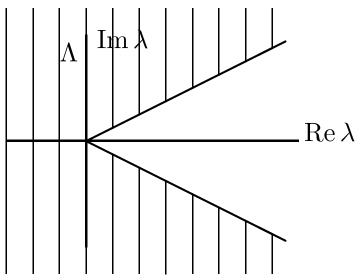


Figure 7: A positive cone Λ.

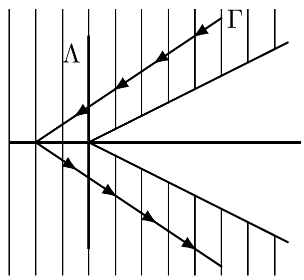


Figure 8: The contour Γ.

Proof: Let $k \in \mathbb{N}_0$ and let $M \in \mathbb{N}_0$ be such that $p/d - M < -k-1$. Then to prove (43), it suffices to show that for any $N \in \mathbb{N}_0$,

$$t^{M+N}\mathcal{L}(A)(t) \in C^\infty(\,(0,\infty)_t;\, \Psi_b^{m-p-dN}(X,\Omega_b^{\frac{1}{2}})\,) \cap C^k(\,[0,\infty)_t;\, \Psi_b^{m-p-dN}(X,\Omega_b^{\frac{1}{2}})\,). \tag{45}$$

To see this, observe that

$$\begin{aligned} t^{M+N}\mathcal{L}(A)(t) &= (-1)^{M+N}\int_\Gamma (\partial_\lambda^{M+N} e^{-t\lambda})\, A(\lambda)\, d\lambda \\ &= \int_\Gamma e^{-t\lambda}(\partial_\lambda^{M+N} A(\lambda))\, d\lambda, \end{aligned} \tag{46}$$

By Lemma 3.21, and the fact that $p/d - M < 0$, we have

$$\begin{aligned} \partial_\lambda^{M+N} A(\lambda) &\in \Psi_{b,\Lambda}^{m-dM-dN,p-dM-dN,d}(X,\Omega_b^{\frac{1}{2}}) \\ &\subseteq \Psi_{b,\Lambda}^{m-dM-dN,p-dM,d}(X,\Omega_b^{\frac{1}{2}}) \\ &\subseteq S^{(p-dM)/d}(\Lambda; \Psi_b^{m-dM-dN-(p-dM)}(X,\Omega_b^{\frac{1}{2}})\,) \\ &= S^{-k-1-\epsilon}(\Lambda; \Psi_b^{m-p-dN}(X,\Omega_b^{\frac{1}{2}})\,), \end{aligned}$$

where $\epsilon = -k-1-p/d+M > 0$. Thus, (45) follows from (46).

Now suppose that $A(\lambda)$ extends to be holomorphic on $\Lambda \cup B_\epsilon$ for some $\epsilon > 0$. Then it follows that

$$\mathcal{L}(A)(t) = \int_{\Gamma'} e^{-t\lambda}\, A(\lambda)\, d\lambda, \tag{47}$$

where Γ' is the contour shown in Figure 9, where the radius of the arc in Γ' is $\epsilon/2$. Using equation (47), we will leave the reader to verify that $\mathcal{L}(A)(t) \to 0$ exponentially in $\Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}})$ as $t \to \infty$. ■

In the next subsection, we will show that under more assumptions on A, $\mathcal{L}(A)(t)$ is continuous at $t = 0$ in an appropriate sense.

The 'most important' example of a Laplace transform is the Heat kernel of a b-pseudodifferential operator. Thus, let $A \in \mathcal{E}\ell\ell_{b,\Lambda}^m(X,\Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, where Λ is a positive cone. Then by Theorem 3.32, there exists a continuous increasing function $r : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $\epsilon > 0$,

$$(A-\lambda)^{-1} \in \Psi_{b,\Lambda,r}^{-m,-m,m,\epsilon,\epsilon,\epsilon}(X,\Omega_b^{\frac{1}{2}}) \tag{48}$$

for $|\lambda| \geq r(\epsilon)$. Let Γ be any contour in Λ of the form given in Figure 8, where $(A-\lambda)^{-1}$ defined on Γ (for example, choose Γ such that $|\lambda| > r(1)$ for all $\lambda \in \Gamma$). Then the *Heat kernel*, e^{-tA}, of A is the operator

$$e^{-tA} := \frac{i}{2\pi}\int_\Gamma e^{-t\lambda}\,(A-\lambda)^{-1}\, d\lambda = \frac{i}{2\pi}\mathcal{L}(\,(A-\lambda)^{-1})(t).$$

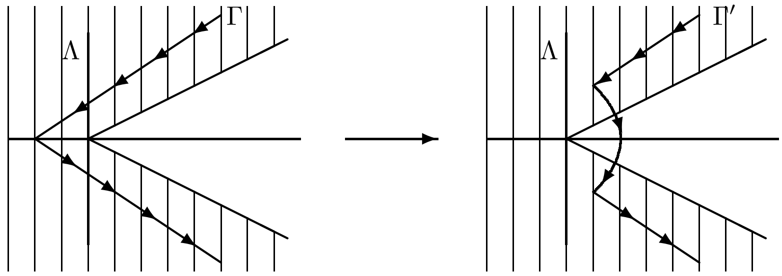


Figure 9: Deformation of the contour Γ into the contour Γ'.

Note that since $(A-\lambda)^{-1}$ is holomorphic, the integral defining e^{-tA} is well-defined, independent of the contour Γ choosen. By (48) and Theorem 4.2, it follows that if $\epsilon > 0$ is given, then by choosing Γ such that $|\lambda| \geq r(\epsilon)$ for all $\lambda \in \Gamma$, we have $e^{-tA} \in C^\infty(\,(0,\infty)_t;\ \Psi_b^{-\infty,\epsilon,\epsilon,\epsilon}(X,\Omega_b^{\frac{1}{2}})\,)$. Since $\epsilon > 0$ is arbitrary, we have $e^{-tA} \in C^\infty(\,(0,\infty)_t;\ \Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}})\,)$.

Theorem 4.3 *Let $A \in \mathcal{E}\ell\ell^m_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, where Λ is a positive cone. Then e^{-tA} satisfies the heat equation: $(\partial_t + A)e^{-tA} = 0,\ t > 0$; and $e^{-tA}|_{t=0} = \mathrm{Id}$. That is, for any $\phi \in \dot{C}^\infty(X,\Omega_b^{\frac{1}{2}})$,*

$$e^{-tA}\phi \in C^\infty(\,(0,\infty)_t;\ C^\infty(X,\Omega_b^{\frac{1}{2}})\,) \cap C^0(\,[0,\infty)_t;\ C^\infty(X,\Omega_b^{\frac{1}{2}})\,) \tag{49}$$

and $(\partial_t + A)(e^{-tA}\phi) = 0$, $t > 0$; and $(e^{-tA}\phi)|_{t=0} = \phi$. Moreover, if $\psi(t)$ is in the same space (49) with $(\partial_t + A)\psi(t) = 0$, $t > 0$; and $\psi(0) = \phi$, then $\psi(t) \equiv e^{-tA}\phi$.

Proof: We leave the reader to verify that $(\partial_t + A)(e^{-tA}\phi) = 0$ for $t > 0$. To see (49), observe that

$$\begin{aligned}
(A-\lambda)^{-1} &= -\lambda^{-1}(-\lambda)(A-\lambda)^{-1} \\
&= -\lambda^{-1}(A-\lambda-A)(A-\lambda)^{-1} \\
&= -\lambda^{-1} + \lambda^{-1}A(A-\lambda)^{-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
e^{-tA}\phi &= \frac{i}{2\pi}\int_\Gamma e^{-t\lambda}\,(A-\lambda)^{-1}\phi\,d\lambda \\
&= -\frac{i}{2\pi}\int_\Gamma e^{-t\lambda}\,\phi\,d\lambda + \frac{i}{2\pi}\int_\Gamma e^{-t\lambda}\lambda^{-1}A(A-\lambda)^{-1}\phi\,d\lambda \\
&= \phi + \frac{i}{2\pi}\int_\Gamma e^{-t\lambda}\lambda^{-1}A(A-\lambda)^{-1}\phi\,d\lambda.
\end{aligned}$$

Since $\lambda^{-1}A(A-\lambda)^{-1}$ decreases like $|\lambda|^{-2}$ as $|\lambda| \to \infty$ (see Theorem 3.22), we leave the reader to verify that $\frac{i}{2\pi}\int_\Gamma e^{-t\lambda}\lambda^{-1}A(A-\lambda)^{-1}\phi\,d\lambda$ is continuous at $t = 0$ with value 0. Thus, $(e^{-tA}\phi)|_{t=0} = \phi$. For the uniqness statement, see [8, p. 271]. ■

Lemma 4.4 *Let m, $p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and let $A(\lambda) \in \Psi^{m,p,d}_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}})$ be holomorphic and positive. Let $M \in M_k(X)$ and $t > 0$. Then,*

$$N_M(\,\mathcal{L}(A)(t)\,)(\tau) = \mathcal{L}(\,N_M(A(\lambda)\,)(\tau)\,)(t) \tag{50}$$

for all $\tau \in \mathbb{C}^k$. In particular, if $A \in \mathcal{E}\ell\ell^m_{b,\Lambda}(X,\Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, then $N_M(e^{-tA})(\tau) = e^{-tN_M(A)(\tau)}$.

PROOF: First of all, note that if $p \leq 0$ (for example), then as $A(\lambda) \in S^{p/d}(\Lambda; \Psi_b^{m-p}(X, \Omega_b^{\frac{1}{2}}))$, we have $N_M(A(\lambda))(\tau) \in S^{p/d}(\Lambda; \Psi_{b,\mathbb{C}^k}^{m-p}(X, \Omega_b^{\frac{1}{2}}))$; thus, the Laplace transform $\mathcal{L}(N_M(A(\lambda))(\tau))(t)$ is well defined. To see the identity (50), let $\phi \in \dot{C}^\infty(M, \Omega_b^{\frac{1}{2}})$ and let $\tilde{\phi} \in C^\infty(X, \Omega_b^{\frac{1}{2}})$ be an extension of ϕ to X. Then, if $x_1, \ldots, x_k$ are the boundary defining functions of X which define M, for any $\tau \in \mathbb{C}^k$,

$$\begin{aligned}
N_M(\mathcal{L}(A)(t))(\tau)\phi &= (x^{-i\tau}\mathcal{L}(A)(t)x^{i\tau}\tilde{\phi})|_{x=0} \\
&= (x^{-i\tau}\int_\Gamma e^{-t\lambda}\, A(\lambda)(x^{i\tau}\tilde{\phi})\, d\lambda)|_{x=0} \\
&= (\int_\Gamma e^{-t\lambda}\, x^{-i\tau} A(\lambda)(x^{i\tau}\tilde{\phi})\, d\lambda)|_{x=0} \\
&= \int_\Gamma e^{-t\lambda}\, N_M(A(\lambda))(\tau)\phi\, d\lambda \\
&= \mathcal{L}(N_M(A(\lambda))(\tau))(t)\phi.
\end{aligned}$$

■

4.2 Conormal nature of Laplace Transforms

Throughout this section, fix a positive cone $\Lambda \subseteq \mathbb{C}$. Let $m, p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. Recall that a symbol $a(\lambda, \xi) \in S_\Lambda^{m,p,d}(\mathbb{R}^n)$ is resolvent like if there exists an $\epsilon > 0$ such that $a(\lambda, \xi)$ extends to be a smooth function, holomorphic in λ, for (λ, ξ) in

$$\{(\lambda, \xi) \in \mathbb{C} \times \mathbb{R}^n \mid \lambda \in \Lambda \ \text{ or } \ |\lambda| \leq \epsilon\,(1 + |\xi|)^d \ \text{ or } \ \frac{1}{\epsilon}\,(1 + |\xi|)^d \leq |\lambda|\}; \tag{51}$$

and moreover, $a(\lambda, \xi)$ continues to satisfy similar symbol estimates for (λ, ξ) in the set given by (51) as for (λ, ξ) in $\Lambda \times \mathbb{R}^n$.

Recall that an operator $A \in \Psi_{b,\Lambda,r}^{m,p,d}(X, \Omega_b^{\frac{1}{2}})$ is resolvent like if

1. $\Lambda \cup B_\epsilon \ni \lambda \mapsto A(\lambda) \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$ is holomorphic for some $\epsilon > 0$ and
2. for any coordinate patch $\mathbb{R}_y^{n,k} \times \mathbb{R}_z^n$ on X_b^2 such that $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}$ and any compactly supported function ϕ on the coordinate patch, we have

$$\phi\, A = \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, a(\lambda, y, \xi)\, d\xi \otimes \nu,$$

   where $\nu \in C^\infty(X_b^2, \Omega_b^{\frac{1}{2}})$ and where for each $y \in \mathbb{R}^{n,k}$, $(\lambda, \xi) \mapsto a(\lambda, y, \xi) \in S_\Lambda^{m,p,d}(\mathbb{R}^n)$ is resolvent like.

We will prove the following Theorem.

Theorem 4.5 *Let $A(\lambda) \in \Psi_{b,\Lambda,r}^{m,p,d}(X, \Omega_b^{\frac{1}{2}})$, where $m, p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, be a resolvent like operator. Then for any $k \in \mathbb{N}_0$,*

$$\mathcal{L}(A)(t) \in C^\infty(\,(0, \infty)_t;\ \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})\,) \cap C^k(\,[0, \infty)_t;\ \Psi_b^{dk+m+d}(X, \Omega_b^{\frac{1}{2}})\,).$$

Moreover, as $t \to \infty$, $\mathcal{L}(A)(t) \to 0$ exponentially in $\Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})$.

Let $A \in \mathcal{E}\ell\ell_{b,\Lambda}^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$. Then by Theorem 3.32, there exists a resolvent like operator $B(\lambda) \in \Psi_{b,\Lambda,r}^{-m,-m,m}(X, \Omega_b^{\frac{1}{2}})$ such that for $\lambda \in \Lambda$ with $|\lambda|$ sufficiently large, $(A - \lambda)^{-1} = B(\lambda) + R(\lambda)$, where for some continuous increasing function $r : \mathbb{R}^+ \to \mathbb{R}^+$, for each $\epsilon > 0$, we have $R(\lambda) \in \Psi^{-\infty,-m,m,\epsilon,\epsilon,\epsilon}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$ for $|\lambda| \geq r(\epsilon)$. Hence,

$$e^{-tA} := \frac{i}{2\pi}\mathcal{L}(\,(A-\lambda)^{-1}\,)(t) = \frac{i}{2\pi}\mathcal{L}(B)(t) + \frac{i}{2\pi}\mathcal{L}(R)(t). \tag{52}$$

By Theorem 4.5, for any $k \in \mathbb{N}_0$,

$$\mathcal{L}(B)(t) \in C^\infty(\,(0,\infty)_t;\, \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})\,) \cap C^k(\,[0,\infty)_t;\, \Psi_b^{mk}(X, \Omega_b^{\frac{1}{2}})\,).$$

If $\epsilon > 0$ is given, we can choose the contour Γ as in Figure 8 such that $|\lambda| \geq r(\epsilon)$ for all $\lambda \in \Gamma$; in which case, by the remark after Lemma 4.9 below, we have $\mathcal{L}(R)(t) \in C^\infty(\,[0,\infty)_t;\, \Psi_b^{-\infty,\epsilon,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}})\,)$. Since $\epsilon > 0$ is arbitrary, it follows that

$$\mathcal{L}(R)(t) \in C^\infty(\,[0,\infty)_t;\, \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})\,). \tag{53}$$

Thus, for any $k \in \mathbb{N}_0$, $e^{-tA} \in C^\infty(\,(0,\infty)_t;\, \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})\,) \cap C^k(\,[0,\infty)_t;\, \Psi_b^{mk}(X, \Omega_b^{\frac{1}{2}})\,)$.

Corollary 4.6 *Let $A \in \mathcal{E}\ell\ell^m_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, where Λ is a positive cone. Then for any $k \in \mathbb{N}_0$,*

$$e^{-tA} \in C^\infty(\,(0,\infty)_t;\, \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})\,) \cap C^k(\,[0,\infty)_t;\, \Psi_b^{mk}(X, \Omega_b^{\frac{1}{2}})\,).$$

To prove Theorem 4.5, we start with the following Lemma.

Lemma 4.7 *Let m, $p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and let $a(\lambda,\xi) \in S^{m,p,d}_\Lambda(\mathbb{R}^n)$ be a resolvent like symbol. Let Γ be any contour in Λ of the form given in Figure 8 and define*

$$\mathcal{L}(a)(t,\xi) := \int_\Gamma e^{-t\lambda}\, a(\lambda,\xi)\, d\lambda.$$

Then, $\mathcal{L}(a)(t,\xi) \in C^\infty(\,[0,\infty) \times \mathbb{R}^n)$ and moreover,

1. *for any $k \in \mathbb{N}_0$, $\mathcal{L}(a)(t) \in C^\infty(\,(0,\infty)_t;\, S^{-\infty}(\mathbb{R}^n)\,) \cap C^k(\,[0,\infty)_t;\, S^{dk+m+d}(\mathbb{R}^n)\,)$;*
2. *there exists a constant $C' > 0$ such that $\mathcal{L}(a)(t,\xi)$ satisfies the estimates: for any k and β, there exists a C such that*

$$|\,\partial_t^k \partial_\xi^\beta \mathcal{L}(a)(t,\xi)\,| \leq C\,(1+|\xi|)^{dk+m+d-|\beta|}\, e^{-tC'|\xi|^d}. \tag{54}$$

Proof:[6] Since $a(\lambda,\xi)$ is resolvent like, there exists an $\epsilon > 0$ such that $a(\lambda,\xi)$ extends to be a smooth function, holomorphic in λ, for (λ,ξ) in

$$\{(\lambda,\xi) \in \mathbb{C} \times \mathbb{R}^n \mid \lambda \in \Lambda \;\text{ or }\; |\lambda| \leq \epsilon\,(1+|\xi|)^d \;\text{ or }\; \frac{1}{\epsilon}\,(1+|\xi|)^d \leq |\lambda|\}; \tag{55}$$

and moreover, $a(\lambda,\xi)$ continues to satisfy similar symbol estimates for (λ,ξ) in the set given by (55) as for (λ,ξ) in $\Lambda \times \mathbb{R}^n$. Let $\xi \in \mathbb{R}^n$. Then, as

$$\Lambda \cup \{\lambda \in \mathbb{C} \mid |\lambda| \leq \epsilon\,(1+|\xi|)^d \;\text{ or }\; \frac{1}{\epsilon}\,(1+|\xi|)^d \leq |\lambda|\} \ni \lambda \mapsto a(\lambda,\xi)$$

is holomorphic, it follows that $\mathcal{L}(a)(t,\xi) = \int_{\Gamma_\xi} e^{-t\lambda}\, a(\lambda,\xi)\, d\lambda$, where Γ_ξ is the contour shown in Figure 10, where the radius of the inner arc of Γ_ξ is $\frac{\epsilon}{2}(1+|\xi|)^d$ and the radius of the outer arc of Γ_ξ is $\frac{2}{\epsilon}(1+|\xi|)^d$. It follows that $\mathcal{L}(a)(t,\xi) \in C^\infty([0,\infty) \times \mathbb{R}^n)$. We will prove that $\mathcal{L}(a)(t,\xi)$ satisfies the estimates (54). These estimates automatically prove that for any $k \in \mathbb{N}_0$, $\mathcal{L}(a)(t) \in C^\infty(\,(0,\infty)_t;\, S^{-\infty}(\mathbb{R}^n)\,) \cap C^k(\,[0,\infty)_t;\, S^{dk+m+d}(\mathbb{R}^n)\,)$. Let $k \in \mathbb{N}_0$ and $\beta \in \mathbf{N}_0^n$. Then, $\partial_t^k \partial_\xi^\beta \mathcal{L}(a)(t,\xi) = \int_{\Gamma_\xi} e^{-t\lambda}(-\lambda)^k\, \partial_\xi^\beta a(\lambda,\xi)\, d\lambda$. Observe that

[6] This proof is based on the proof given in Grubb [4, Lemma 4.2.3].

1. the length of $\Gamma_\xi \leq C_1\,(1+|\xi|)^d$ for some $C_1 > 0$ (independent of ξ);
2. $|e^{-t\lambda}| \leq e^{-tC'|\xi|^d}$ for all $\lambda \in \Gamma_\xi$, for some $C' > 0$ (independent of ξ);
3. $|\lambda|^k \leq \frac{2^k}{\epsilon^k}(1+|\xi|)^{dk}$ for all $\lambda \in \Gamma_\xi$;
4. there exists constants C_2 and C_2' such that

$$\begin{aligned} |\,\partial_\xi^\beta a(\lambda,\xi)\,| &\leq C_2\,(1+|\lambda|^{1/d}+|\xi|)^p\,(1+|\xi|)^{m-p-|\beta|} \\ &\leq C_2'(1+|\xi|)^{m-|\beta|} \text{ for all } \lambda \in \Gamma_\xi. \end{aligned}$$

Hence,

$$\begin{aligned} |\,\partial_t^k \partial_\xi^\beta \mathcal{L}(a)(t,\xi)\,| &\leq \int_{\Gamma_\xi} |e^{-t\lambda}|\,|\lambda|^k\,|\,\partial_\xi^\beta a(\lambda,\xi)\,|\,d\lambda \\ &\leq C_1\,(1+|\xi|)^d \cdot e^{-tC'|\xi|^d} \cdot \frac{2^k}{\epsilon^k}(1+|\xi|)^{dk} \cdot C_2'(1+|\xi|)^{m-|\beta|} \\ &\leq C\,(1+|\xi|)^{dk+m+d-|\beta|}\,e^{-tC'|\xi|^d}. \end{aligned}$$

■

Lemma 4.8 *Let $a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$ be resolvent like, where $m,\,p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. Let Γ be any contour in Λ of the form given in Figure 8 and define*

$$\mathcal{L}(a)(t,\xi) := \int_\Gamma e^{-t\lambda}\,a(\lambda,\xi)\,d\lambda.$$

Then, $\mathcal{L}(a)(t,\xi) \in C^\infty(\,[0,\infty) \times (\mathbb{R}^n \setminus \{0\})\,)$ and it satisfies

1. *$\mathcal{L}(a)(\delta^{-d}t,\delta\xi) = \delta^{m+d}\mathcal{L}(a)(t,\xi)$ for all $\delta > 0$;*
2. *there exists a constant $C' > 0$ such that $\mathcal{L}(a)(t,\xi)$ satisfies the estimates: for any k and β, there exists a C such that*

$$|\,\partial_t^k \partial_\xi^\beta \mathcal{L}(a)(t,\xi)\,| \leq C\,|\xi|^{dk+m+d-|\beta|}\,e^{-tC'|\xi|^d}.$$

Proof: Since the proof that $\mathcal{L}(a)(t,\xi) \in C^\infty(\,[0,\infty)\times(\mathbb{R}^n\setminus\{0\})\,)$ and that it satisfies the estimates in (2) are proved much like the results of the previous Lemma, we will leave these proofs as an exercise

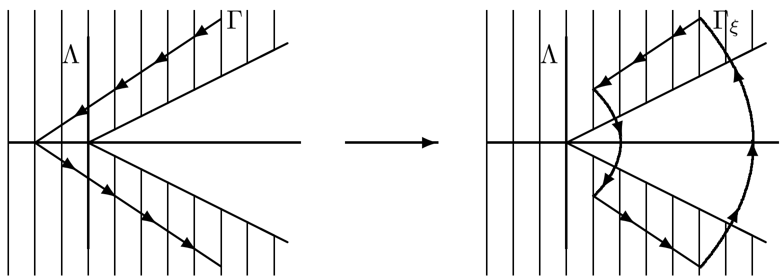


Figure 10: Deformation of the contour Γ into the contour Γ_ξ.

for the reader. To see (1), we make the change of variables $\lambda \mapsto \delta^d \lambda$ and use the homogeneous properties of $a(\lambda,\xi)$ to find that

$$\begin{aligned}
\mathcal{L}(a)(\delta^{-d}t,\delta\xi) &= \int_\Gamma e^{-t\delta^{-d}\lambda}\, a(\lambda,\delta\xi)\, d\lambda \\
&= \delta^d \int_{\delta^{-d}\Gamma} e^{-t\lambda}\, a(\delta^d\lambda,\delta\xi)\, d\lambda \\
&= \delta^{m+d} \int_\Gamma e^{-t\lambda}\, a(\lambda,\xi)\, d\lambda \\
&= \delta^{m+d}\mathcal{L}(a)(t,\xi).
\end{aligned}$$

■

Lemma 4.9 *Let $R \in \Psi^{-\infty,p,d}_{b,\Lambda,r}(X,\Omega_b^{\frac12}) \equiv S^{p/d}_{os}(\Lambda;\, \Psi_b^{-\infty}(X,\Omega_b^{\frac12}))$, where $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, be holomorphic. Then,*

$$\mathcal{L}(R)(t) \in C^\infty(\,[0,\infty)_t;\, \Psi_b^{-\infty}(X,\Omega_b^{\frac12})\,). \tag{56}$$

Proof: Fix any $N >> 0$. Then by Proposition 3.42, we can write, away from $\lambda = 0$,

$$R = \sum_{j=0}^{N-1} \lambda^{p/d-j} R_j + \lambda^{p/d-N} R_N(\lambda),$$

where $R_j \in \Psi_b^{-\infty}(X,\Omega_b^{\frac12})$ and, away from $\lambda = 0$, $R_N(\lambda) \in S^0(\Lambda;\, \Psi_b^{-\infty}(X,\Omega_b^{\frac12}))$. Hence,

$$\mathcal{L}(R)(t) = \sum_{j=0}^{N-1} \int_\Gamma e^{-t\lambda}\lambda^{p/d-j} d\lambda \cdot R_j + \int_\Gamma e^{-t\lambda}\lambda^{p/d-j} R_N(\lambda) d\lambda,$$

where Γ is any contour in Λ of the form given in Figure 8. Observe that for any $j \in \mathbb{N}_0$,

$$\int_\Gamma e^{-t\lambda}\lambda^{p/d-j} d\lambda = t^{j-p/d-1} \int_\Gamma e^{-\lambda}\lambda^{p/d-j} d\lambda = \begin{cases} 0, & j - p/d \notin \mathbb{N}; \\ (-t)^{j-p/d-1}, & j - p/d \in \mathbb{N}. \end{cases} \tag{57}$$

Also, observe that Cauchy's Theorem implies $\int_\Gamma \lambda^{-k} R_N(\lambda)\, d\lambda \equiv 0$ for all $k \geq 2$. Hence, for any $M \in \mathbb{N}$ such that $p/d - N + M \leq -2$,

$$R_N(t) := \int_\Gamma e^{-t\lambda}\lambda^{p/d-N} R_N(\lambda)\, d\lambda \in C^M(\,[0,\infty)_t;\, \Psi_b^{-\infty}(X,\Omega_b^{\frac12})\,), \tag{58}$$

with $\partial_t^j R_N(t) \equiv 0$ for $j = 0,\ldots,M$. Now by (57) and (58), we can write

$$\mathcal{L}(R)(t) = \sum_{0\leq j\leq N-1,\, j-p/d-1\geq 0} t^{j-p/d-1} R_j + R_N(t);$$

and thus, $\mathcal{L}(R)(t) \in C^\infty(\,[0,\infty)_t;\, \Psi_b^{-\infty}(X,\Omega_b^{\frac12}))$ since N is arbitrary. ■

Remark: Observe that the result (56) still holds if we only assume that $R(\lambda)$ is holomorphic for all $|\lambda| \geq r$ for some $r > 0$ (in which case, to define $\mathcal{L}(R)$, we take the contour Γ of the form given in Figure 8 such that $|\lambda| \geq r$ for all $\lambda \in \Gamma$).

Proof of Theorem 4.5: By Theorem 4.2, $\mathcal{L}(A)(t) \to 0$ exponentially in $\Psi_b^{-\infty}(X,\Omega_b^{\frac12})$ as $t \to \infty$. Note that it suffices to prove Theorem 4.5 for $\phi\, A(\lambda)$, when $\phi \in C_c^\infty(X_b^2 \setminus \Delta_b)$, and when ϕ is a compactly supported function on a coordinate patch $\mathbb{R}^{n,k} \times \mathbb{R}^n$ on X_b^2 such that $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}$.

Let $\phi \in C_c^\infty(X_b^2 \setminus \Delta_b)$. Then, $\phi\, A \in \Psi^{-\infty,p,d}_{b,\Lambda,r}(X,\Omega_b^{\frac12}) \equiv S^{p/d}_{os}(\Lambda;\, \Psi_b^{-\infty}(X,\Omega_b^{\frac12}))$. Thus, by Lemma 4.9, $\phi\, \mathcal{L}(A)(t) \in C^\infty(\,[0,\infty)_t;\, \Psi_b^{-\infty}(X,\Omega_b^{\frac12}))$. Now let $\mathbb{R}^{n,k} \times \mathbb{R}^n$ be a coordinate patch on X_b^2 such

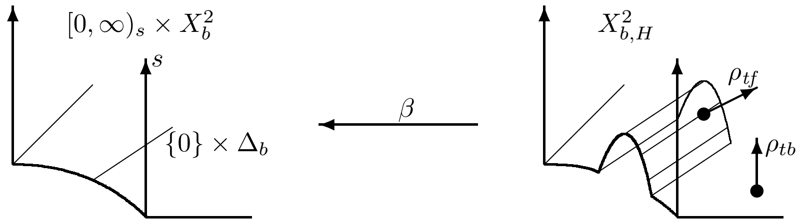


Figure 11: The manifold $X_{b,H}^2$.

that $\Delta_b \cong \mathbb{R}^{n,k}\times\{0\}$ and let ϕ be any compactly supported function on the coordinate patch. Then, we can write

$$\phi\, A(\lambda) = \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}\, a(\lambda, y, \xi)\, d\xi \otimes \nu,$$

where $\nu \in C^\infty(X_b^2, \Omega_b^{\frac{1}{2}})$, and where for each $y \in \mathbb{R}^{n,k}$, $(\lambda,\xi)\mapsto a(\lambda,y,\xi) \in S_\Lambda^{m,p,d}(\mathbb{R}^n)$ is resolvent like. Hence,

$$\phi\, \mathcal{L}(A)(t) = \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}\, \mathcal{L}(a)(t, y, \xi)\, d\xi \otimes \nu.$$

By Lemma 4.7, for any $k\in\mathbb{N}_0$,

$$\mathcal{L}(a)(t,y,\xi) \in C^\infty(\,(0,\infty)_t\times\mathbb{R}_y^{n,k};\, S^{-\infty}(\mathbb{R}_\xi^n)\,)\cap C^k(\,[0,\infty)_t\times\mathbb{R}_y^{n,k};\, S^{dk+m+d}(\mathbb{R}_\xi^n)\,).$$

Thus, for any $k\in\mathbb{N}_0$, $\phi\,\mathcal{L}(A)(t)\in C^\infty(\,(0,\infty)_t;\,\Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}})\,)\cap C^k(\,[0,\infty)_t;\,\Psi_b^{dk+m+d}(X,\Omega_b^{\frac{1}{2}})\,)$. ■

4.3 The blown-up heat space

Recall that in Section 3.9, we realized the space of one-step tempered operators as classical conormal functions on an appropriate blown-up space. In this section, we will realize their Laplace Transforms as classical conormal functions on an appropriate blown-up space.

Let X be compact with corners and let $[0,\infty)_s$ be the half-line with variable s. We define

$$X_{b,H}^2 := [\,[0,\infty)_s\times X_b^2; \{0\}\times\Delta_b\,].$$

Figure 11 gives a pictorial representation of $X_{b,H}^2$.

We define $tf := \beta^{-1}(\{0\}\times\Delta_b)$ and $tb := \overline{\beta^{-1}(\{0\}\times X_b^2\setminus\{0\}\times\Delta_b)}$, and we call tf the 'temporal face' and tb the 'temporal boundary' respectively.

We now fix the notation for local coordinates on $X_{b,H}^2$. Let $\mathcal{U}=\mathbb{R}_y^{n,k}\times\mathbb{R}_z^n$ be a coordinate patch on X_b^2 with $\Delta_b\cong\mathbb{R}^{n,k}\times\{0\}$. Then,

$$[0,\infty)_s\times X_b^2 \cong [0,\infty)_s\times\mathbb{R}_y^{n,k}\times\mathbb{R}_z^n$$

with $\{0\}\times\Delta_b\cong\{0\}\times\mathbb{R}_y^{n,k}\times\{0\}$. Hence,

$$X_{b,H}^2\cong H^n\times\mathbb{R}_y^{n,k},\ \text{ where }\ H^n := [\,[0,\infty)_s\times\mathbb{R}_z^n;\{0\}\times\{0\}].$$

Now by definition of blow-up, $H^n\equiv[0,\infty)_\rho\times\mathbb{S}^{n,1}_{(\omega_0,\omega')}$, where

$$\left.\begin{array}{l}\rho := (|z|^2+s^2)^{1/2}\\ \omega_0 := s/(|z|^2+s^2)^{1/2}\\ \omega' := z/(|z|^2+s^2)^{1/2}\end{array}\right\}\leftrightarrow\left\{\begin{array}{l} s=\rho\,\omega_0\\ z=\rho\,\omega'.\end{array}\right.$$

$$\left.\begin{array}{l} r := |z| \\ \gamma_0 := s/|z| \\ \gamma' := z/|z| \end{array}\right\} \leftrightarrow \left\{ \begin{array}{l} s = r\,\gamma_0 \\ z = r\,\gamma' \end{array}\right.$$

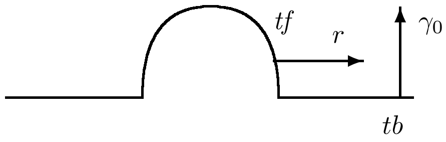


Figure 12: Coordinates on H^n near tb

In H^n, we define $tf := \{\rho = 0\}$ and $tb := \{\omega_0 = 0\}$. Then useful coordinates near tb in H^n are given in Figure 12. These coordinates give a product decompostion

$$H^n \setminus \{\omega_0 = 1\} \cong [0,\infty)_r \times [0,\infty)_{\gamma_0} \times \mathbb{S}^{n-1}_{\gamma'}.$$

We now touch on the subject of densities. We claim that if $t = s^d$, then

$$|dz\,\frac{dt}{t}|^{\frac{1}{2}} \in C^\infty(\mathbb{R}^n_z \times [0,\infty)_t, \Omega_b^{\frac{1}{2}}),$$

when lifted to H^n, is of the form $|dz\,\frac{dt}{t}|^{\frac{1}{2}} = \rho^{\frac{n}{2}}\,\mu$, where $0 < \mu \in C^\infty(H^n, \Omega_b^{\frac{1}{2}})$. Indeed, observe that since $t = \rho^d \omega_0^d$ and $z = \rho\omega'$,

$$\frac{dt}{t} = d \cdot \frac{d\rho}{\rho} + d \cdot \frac{d\omega_0}{\omega_0}$$

and for each $i = 1, \ldots, n$, $dz_i = \omega'_i\,d\rho + \rho\,d\omega'_i$. Thus, $|dz\,\frac{dt}{t}|^{\frac{1}{2}} = \rho^n \frac{d\rho}{\rho}\nu$, where $\nu \in C^\infty(\mathbb{S}^{n,1}_{(\omega_0,\omega')}, \Omega_b^{\frac{1}{2}})$. Since $|dz\,\frac{dt}{t}|^{\frac{1}{2}} > 0$, we must have $\nu > 0$. Thus, $|dz\,\frac{dt}{t}|^{\frac{1}{2}} = \rho^{\frac{n}{2}}\,\mu$, where $0 < \mu \in C^\infty(H^n, \Omega_b^{\frac{1}{2}})$. More generally, we have the following Lemma.

Lemma 4.10 *Let $0 < \nu \in C^\infty(X_b^2, \Omega_b^{\frac{1}{2}})$. Then,*

$$\beta^*(\,\nu\,|\frac{dt}{t}|^{\frac{1}{2}}\,) = \rho_{tf}^{\frac{n}{2}}\,\mu,$$

where $0 < \mu \in C^\infty(X^2_{b,H}, \Omega_b^{\frac{1}{2}})$.

4.4 The Structure Theorem

Let $m \in \mathbb{R}$ with $p/d \in \mathbb{Z}$. Define the index set $\mathcal{E}_{m,d}$ on $X^2_{b,H}$ by

$$\mathcal{E}_{m,d}(ff) := \mathbb{N}_0;\; \mathcal{E}_{m,d}(lb) = \mathcal{E}_{m,d}(rb) := \emptyset;\; \mathcal{E}_{m,d}(tb) := d\mathbb{N}_0;$$

$$\mathcal{E}_{m,d}(tf) := \frac{n}{2} + \{(k - m - n - d, 0)\,|\,k \in \mathbb{N}_0\}\overline{\cup}(\mathbb{N}_0 + d\mathbb{N}_0).$$

The Structure Theorem is the following.

Theorem 4.11 *Let $A(\lambda) \in \Psi^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$, where m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, be resolvent like. Then if $t = s^d$, $\mathcal{L}(A)(t)\,|\frac{dt}{t}|^{\frac{1}{2}} \in \mathcal{A}^{\mathcal{E}_{m,d}}_{phg}(X^2_{b,H}, \Omega_b^{\frac{1}{2}})$. Moreover, as $t \downarrow 0$,*

$$\mathcal{L}(A)(t)|_{\Delta_b} \sim \sum_{k\in\mathbb{N}_0} t^{\frac{k-m-n}{d}-1}\,\gamma_k(x) + \sum_{k,\,\frac{k-m-n}{d}-1\in\mathbb{N}_0} t^{\frac{k-m-n}{d}-1}\,\log t\,\gamma'_k(x) + \sum_{k\in\mathbb{N}_0} t^k \gamma''_k(x),$$

where for each k, γ_k, γ'_k, $\gamma''_k \in C^\infty(X, \Omega_b)$.

Corollary 4.12 *Let $A \in \mathcal{E}\ell\ell^m_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$. Then if $t = s^d$,*

$$e^{-tA}\,|\frac{dt}{t}|^{\frac{1}{2}} \in \mathcal{A}^{\mathcal{E}_m}_{phg}(X^2_{b,H}, \Omega_b^{\frac{1}{2}}).$$

Moreover, given any $B \in \Psi^{m'}_{b,os}(X, \Omega_b^{\frac{1}{2}})$, $m' \in \mathbb{R}$, *as* $t \downarrow 0$,

$$Be^{-tA}|_{\Delta_b} \sim \sum_{k \in \mathbb{N}_0} t^{\frac{k-m'-n}{m}} \gamma_k(x) + \sum_{k,\, \frac{k-m'-n}{m} \in \mathbb{N}_0} t^{\frac{k-m'-n}{m}} \log t\, \gamma'_k(x) + \sum_{k \in \mathbb{N}_0} t^k \gamma''_k(x), \tag{59}$$

where for each k, γ_k, γ'_k, $\gamma''_k \in C^\infty(X, \Omega_b)$.

Proof: Just as we showed (52) and (53), one can show that

$$e^{-tA} = \frac{i}{2\pi} \mathcal{L}(B)(t) + \frac{i}{2\pi} \mathcal{L}(R)(t),$$

where $B \in \Psi^{-m,-m,m}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$ is resolvent like, and $\mathcal{L}(R)(t) \in C^\infty([0,\infty)_t;\, \Psi^{-\infty}(X, \Omega_b^{\frac{1}{2}}))$. By Theorem 4.11, $\mathcal{L}(B)(t)\,|\frac{dt}{t}|^{\frac{1}{2}} \in \mathcal{A}^{\mathcal{E}'_m}_{phg}(X^2_{b,H}, \Omega_b^{\frac{1}{2}})$, where $\mathcal{E}'_m(G) = \mathcal{E}_m(G)$ if $G \neq tb$ and $\mathcal{E}'_m(tb) = m\mathbb{N}_0$. Also, observe that $\mathcal{L}(R)(t)\,|\frac{dt}{t}|^{\frac{1}{2}} \in \mathcal{A}^{\mathcal{E}'_m}_{phg}(X^2_{b,H}, \Omega_b^{\frac{1}{2}})$. Thus, $e^{-tA}\,|\frac{dt}{t}|^{\frac{1}{2}} \in \mathcal{A}^{\mathcal{E}'_m}_{phg}(X^2_{b,H}, \Omega_b^{\frac{1}{2}})$. But note that as operators, $e^{-tA}|_{t=0} = \mathrm{Id}$, and the identity is zero off of Δ_b. It follows that $e^{-tA}\,|\frac{dt}{t}|^{\frac{1}{2}}$ must vanish at tb. Hence, $e^{-tA}\,|\frac{dt}{t}|^{\frac{1}{2}} \in \mathcal{A}^{\mathcal{E}_m}_{phg}(X^2_{b,H}, \Omega_b^{\frac{1}{2}})$.

Observe that if $C \in \Psi^{m'}_{b,os}(X, \Omega_b^{\frac{1}{2}})$, $m' \in \mathbb{R}$, then $Ce^{-tA} = \frac{i}{2\pi}\mathcal{L}(CB)(t) + \frac{i}{2\pi} C\mathcal{L}(R)(t)$, where $C \circ B(\lambda) \in \Psi^{-m+m',-m,m}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$ is resolvent like, and $C\mathcal{L}(R)(t) \in C^\infty([0,\infty)_t;\, \Psi^{-\infty}(X, \Omega_b^{\frac{1}{2}}))$. Thus, the expansion for $Ce^{-tA}|_{\Delta_b}$ follows from Theorem 4.11. ■

4.5 Proof of the Structure Theorem

Lemma 4.13 *Let* $a(\lambda, \xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$, *where* m, $p \in \mathbb{R}$ *and* $d \in \mathbb{R}^+$ *with* $p/d \in \mathbb{Z}$, *be resolvent like and let* $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ *with* $\chi(\xi) \equiv 0$ *on a neighborhood of* 0 *and* $\chi(\xi) \equiv 1$ *outside a neighborhood of* 0. *Define*

$$A(t) := \int e^{iz\cdot\xi}\, \chi(\xi)\, \mathcal{L}(a)(t,\xi)\, d\xi.$$

Then if $t = s^d$,

$$A(t) \in \mathcal{A}^{\mathcal{E}_{tf}}(H^n),$$

where $\mathcal{E}_{tf} := \{(k-m-n-d, 0) \mid k \in \mathbb{N}_0\} \overline{\cup} (\mathbb{N}_0 + d\mathbb{N}_0)$ *is the index set on* H^n *associated to* tf. *Also,*

$$A(t)|_{z=0} \in \mathcal{A}^{\mathcal{F}_{m,d}}_{phg}([0,\infty)_s),$$

where $\mathcal{F}_{m,d}$ *is the index set* $\mathcal{F}_{m,d} := \{(k-m-n-d, 0) \mid k \in \mathbb{N}_0\} \overline{\cup} d\mathbb{N}_0$.

Proof: In the coordinates

$$\left.\begin{array}{l} \rho := (|z|^2 + s^2)^{1/2} \\ \omega_0 := s/(|z|^2+s^2)^{1/2} \\ \omega' := z/(|z|^2+s^2)^{1/2} \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} s = \rho\,\omega_0 \\ z = \rho\,\omega' \end{array}\right.$$

on H^n, we can write $A(s^d) := \int e^{i\rho\omega'\cdot\xi}\, \chi(\xi)\, \mathcal{L}(a)(\rho^d \omega_0^d, \xi)\, d\xi$. Thus, making the change of variables $\xi \mapsto \rho^{-1}\xi$ and using the homogeneity property of $\mathcal{L}(a)(t,\xi)$ found in Lemma 4.8, yields

$$A(s^d) = \rho^{-m-n-d} \int e^{i\omega'\cdot\xi}\, \chi(\xi/\rho)\, \mathcal{L}(a)(\omega_0^d, \xi)\, d\xi.$$

Hence,

$$\begin{aligned} (\rho\partial_\rho - (-m-n-d))A(s^d) &= -\rho^{-m-n-d} \int e^{i\omega'\cdot\xi}\, (\xi \cdot D_\xi \chi)(\xi/\rho)\, \mathcal{L}(a)(\omega_0^d, \xi)\, d\xi \\ &= -\int e^{i\rho\omega'\cdot\xi}\, (\xi \cdot D_\xi\chi)(\xi)\, \mathcal{L}(a)(\rho^d\omega_0^d, \xi)\, d\xi. \end{aligned} \tag{60}$$

Since $(\xi \cdot D_\xi \chi)(\xi) \equiv 0$ near $\xi = 0$ and outside a neighborhood of 0, the function

$$f(u,v) := \int e^{iu\omega'\cdot\xi}\,(\xi \cdot D_\xi \chi)(\xi)\,\mathcal{L}(a)(v\omega_0^d,\xi)\,d\xi$$

is a smooth function of $(u,v) \in [0,\infty)^2$. Thus, we can expand the right hand side of (60) in powers of ρ and ρ^d:

$$(\rho\partial_\rho - (-m-n-d)\,)A(s^d) \sim \sum_{i,j=0}^{\infty} \rho^{i+dj} A_{i,j}(\omega_0,\omega'),$$

where for each i and j, $A_{i,j}(\omega_0,\omega') = (\partial_u^i \partial_v^j f)(0,0) \in S^0(\mathbb{S}^{n,1}_{(\omega_0,\omega')})$. Observe that $A_{i,j}(\omega_0,0) \equiv 0$ for $i > 0$. Thus, by Theorem 2.1, it follows that $A(s^d) \in \mathcal{A}^{\mathcal{E}_{tf}}(H^n)$ and $A(s^d)|_{z=0} \in \mathcal{A}^{\mathcal{F}_{m,d}}_{phg}([0,\infty)_s)$. ■

We now work on the expansion at tb.

Lemma 4.14 *Let $a(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m)}(\mathbb{R}^n)$, where m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, be resolvent like and let $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) \equiv 0$ on a neighborhood of 0 and $\chi(\xi) \equiv 1$ outside a neighborhood of 0. Define*

$$A(t) := \int e^{iz\cdot\xi}\,\chi(\xi)\,\mathcal{L}(a)(t,\xi)\,d\xi.$$

Then if $t = s^d$, $A(t) \in \mathcal{A}^{\mathcal{E}_{tb}}(H^n)$, where $\mathcal{E}_{tb} := d\mathbb{N}_0$ is the index set on H^n associated to tb.

Proof: We will use the coordinates

$$\left.\begin{array}{l} r := |z| \\ \gamma_0 := s/|z| \\ \gamma' := z/|z| \end{array}\right\} \leftrightarrow \left\{ \begin{array}{l} s = r\,\gamma_0 \\ z = r\,\gamma' \end{array}\right. \tag{61}$$

on H^n found in Figure 12. Fix any $N >> 0$ and choose $M >> 0$ such that $dM+m+d \geq -n+N+1$. Consider the function

$$g(u,v,\gamma_0,\gamma') := u^M \int e^{iv\gamma'\cdot\xi}\,(\chi(\xi)-1)\,(\partial_t^M \mathcal{L}(a))(u\gamma_0,\xi)\,d\xi,$$

where $(u,v) \in [0,\infty)^2$. By Lemma 4.8, for any k and $|\xi| \leq 1$, there is a $C > 0$ such that

$$\begin{aligned} |\,\partial_t^{M+k}\mathcal{L}(a)(t,\xi)\,| &\leq C\,|\xi|^{d(M+k)+m+d} \\ &\leq C\,|\xi|^{-n+1}, \end{aligned}$$

where we used that $dM+m+d \geq -n+N+1 \geq -n+1$. It follows that we can differentiate g as many times as we want, with respect to any of its variables, and its integrand still remains integrable. Thus, $g \in C^\infty(\,[0,\infty)^2_{(u,v)} \times [0,\infty)_{\gamma_0} \times \mathbb{S}^{n-1}_{\gamma'})$. Now in the coordinates (61), we can write $A(s^d) := \int e^{ir\gamma'\cdot\xi}\,\chi(\xi)\,\mathcal{L}(a)(r^d\gamma_0^d,\xi)\,d\xi$. Set $w := \gamma_0^d$. Then observe that

$$\begin{aligned} \partial_w^M A(s^d) &= r^{dM} \int e^{ir\gamma'\cdot\xi}\,\chi(\xi)\,(\partial_t^M\mathcal{L}(a))(r^d\gamma_0^d,\xi)\,d\xi \\ &= r^{dM} \int e^{ir\gamma'\cdot\xi}\,(\partial_t^M\mathcal{L}(a))(r^d\gamma_0^d,\xi)\,d\xi + g(r^d,r,\gamma_0^d,\gamma'). \end{aligned}$$

Making the change of coordinates $\xi \mapsto r^{-1}\gamma_0^{-1}\xi$ in the first integral and using the homogeneity property of $\mathcal{L}(a)(t,\xi)$ found in Lemma 4.8 yields

$$\partial_w^M A(s^d) = r^{-m-n-d}\gamma_0^{-dM-m-n-d} \int e^{i\frac{\gamma'}{\gamma_0}\cdot\xi}\,(\partial_t^M\mathcal{L}(a))(1,\xi)\,d\xi + g(r^d,r,\gamma_0^d,\gamma').$$

Hence,

$$w^M\partial_w^M A(s^d) = r^{-m-n-d}\gamma_0^{-m-n-d} f + \gamma_0^{dM} g(r^d,r,\gamma_0^d,\gamma'), \tag{62}$$

where $f := \int e^{i\frac{\gamma'}{\gamma_0}\cdot\xi}\,(\partial_t^M\mathcal{L}(a))(1,\xi)\,d\xi$. We claim that $f \in \gamma_0^N S^0([0,\infty)_{\gamma_0}\times\mathbb{S}^{n-1}_{\gamma'})$. Indeed, by Lemma 4.8, there is a $C'>0$ such that for any β, there is a $C>0$ such that

$$|\,\partial_\xi^\beta(\partial_t^M\mathcal{L}(a))(1,\xi)\,| \le C\,|\xi|^{dM+m+d-|\beta|}\,e^{-C'|\xi|^d}.$$

Observe that since $dM+m+d \ge -n+N+1$, for some C'' we have

$$C|\xi|^{dM+m+d-|\beta|}\,e^{-C'|\xi|^d} \le C''\begin{cases} |\xi|^{-n+N+1-|\beta|}, & |\xi|\le 1;\\ |\xi|^{-n-2-|\beta|}, & |\xi|\ge 1.\end{cases}$$

Thus, for any β,

$$|\,\partial_\xi^\beta(\partial_t^M\mathcal{L}(a))(1,\xi)\,| \le C''\begin{cases} |\xi|^{-n+N+1-|\beta|}, & |\xi|\le 1;\\ |\xi|^{-n-2-|\beta|}, & |\xi|\ge 1.\end{cases}$$

Hence, by Lemma 3.35, $f\in\gamma_0^N S^0([0,\infty)_{\gamma_0}\times\mathbb{S}^{n-1}_{\gamma'})$. Since

$$w^M\partial_w^M = (w\partial_w - M+1)(w\partial_w - M+2)\cdots(w\partial_w - 1)(w\partial_w)$$

and $\gamma_0\partial_{\gamma_0} = d\,w\partial_w$, (62) implies that

$$\begin{aligned}(\gamma_0\partial_{\gamma_0} - d(M+1))(\gamma_0\partial_{\gamma_0} - (M+2))\cdots(\gamma_0\partial_{\gamma_0} - d)(\gamma_0\partial_{\gamma_0})A(s^d) &\in \\ \gamma_0^{N-m-n-d}S^0([0,\infty)_{\gamma_0}\times\mathbb{S}^{n-1}_{\gamma'}) + \gamma_0^{dM}S^0_{loc}(H^n).&\end{aligned}$$

Since N and M can be made arbitrarily large, by Theorem 2.1, $A(s^d)\in\mathcal{A}^{\mathcal{E}_{tb}}(H^n)$. ■

Lemma 4.15 *Let $N\in\mathbb{N}$, let $m,\,p\in\mathbb{R}$ and $d\in\mathbb{R}^+$ with $p/d\in\mathbb{Z}$ and with $m+|p|\le -2N$, and let $M\in\mathbb{N}$ with $0<p+dM\le N$. Let $r(\lambda,\xi)\in S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$ and suppose that $\Lambda\ni\lambda\mapsto r(\lambda,\xi)\in S^m(\mathbb{R}^n)$ is holomorphic. Define*

$$R(t) := \int e^{iz\cdot\xi}\,\mathcal{L}(r)(t,\xi)\,d\xi.$$

Then, we can write

$$R(t) = \sum_{l=0}^{M-2} t^l R_l + R_M(t),$$

where $R_l\in I^{-N+n/4}(\mathbb{R}^n;\{0\})$ and

$$R_M(t)\in C^\infty(\,(0,\infty)_t; I^{-N+n/4}(\mathbb{R}^n;\{0\})\,)\cap C^{M-2}(\,[0,\infty)_t; I^{-N+n/4}(\mathbb{R}^n;\{0\})\,)$$

with $\partial_t^j R_M(t)|_{t=0}\equiv 0$ for $j=0,\ldots,M-2$.

Proof: Since $r\in S^{m,p,d}_{\Lambda,r}(\mathbb{R}^n)$, if we define $\tilde r(\mu,\xi) := \mu^{p/d}\,r(1/\mu,\xi)$, then $\tilde r(\mu,\xi)\in C^\infty(\Lambda_{cc}\times\mathbb{R}^n)$ and it satisfies the estimates: for any α and β, there is a $C>0$ such that

$$|\,\partial_\mu^\alpha\partial_\xi^\beta\tilde r(\mu,\xi)\,| \le C\,(1+|\mu|\,|\xi|^d)^{p/d-|\alpha|}\,(1+|\xi|)^{d|\alpha|+m-p-|\beta|}.$$

Observe that for $|\mu|$ bounded, for some C' (depending on the bound on $|\mu|$), we have

$$(1+|\mu|\,|\xi|^d)^{p/d}\,(1+|\xi|)^{-p} \le C'(1+|\xi|)^{|p|}.$$

Thus, for $|\mu|$ bounded, for any l and β, there is a $C>0$ such that

$$|\,\partial_\mu^l\partial_\xi^\beta\tilde r(\mu,\xi)\,| \le C\,(1+|\xi|)^{dl+m+|p|-|\beta|};$$

or, as $m+|p|\le -2N$, we have

$$|\,\partial_\mu^l\partial_\xi^\beta\tilde r(\mu,\xi)\,| \le C\,(1+|\xi|)^{dl-2N-|\beta|}. \tag{63}$$

Thus, for any l and β, there is a $C > 0$ such that

$$| \, \partial_\mu^l \partial_\xi^\beta \tilde{r}(0,\xi) \, | \leq C \, (1 + |\xi|)^{dl - 2N - |\beta|}.$$

In particular, since $0 < p + dM \leq N$, for any $0 \leq l \leq M + p/d - 1$, we have

$$\tilde{r}_l(\xi) := \partial_\mu^l \tilde{r}(0,\xi) \in S^{-N}(\mathbb{R}^n).$$

Let $\mu = \rho\omega$, where $\rho = |\mu| \in [0,\infty)$ and $\omega = \mu/|\mu| \in \mathbb{S}^1 \cap \Lambda_{cc}$ be polar coordinates for μ. Then, as $\rho^l \partial_\rho^l \tilde{r}(\rho\omega,\xi) = \mu^l (\partial_\mu^l \tilde{r})(\rho\omega,\xi)$, Taylor's Theorem applied to the function $\rho \mapsto \tilde{r}(\rho\omega,\xi)$, at $\rho = 0$, yields

$$\tilde{r}(\mu,\xi) = \sum_{l=0}^{M+p/d-1} \mu^l \, \tilde{r}_l(\xi) + \mu^{M+p/d} \, \tilde{r}_M(\mu,\xi),$$

where

$$\tilde{r}_M(\mu,\xi) := \frac{1}{(M+p/d)!} \int_0^1 (1-s)^{M+p/d-1} (\partial_\mu^{M+p/d} \tilde{r})(s\,\mu,\xi) \, ds.$$

Note that $\tilde{r}_M(\mu,\xi) \in C^0(\Lambda_{cc}; S^{-N}(\mathbb{R}^n)\,)$ by (63). Since $r(\lambda,\xi) = \lambda^{p/d} \tilde{r}(1/\lambda,\xi)$, we have, away from $\lambda = 0$,

$$r(\lambda,\xi) = \sum_{l=0}^{M+p/d-1} \lambda^{p/d-l} \, \tilde{r}_l(\xi) + \lambda^{-M} \, \tilde{r}_M(\lambda^{-1},\xi).$$

Thus,

$$\mathcal{L}(r)(t,\xi) = \sum_{l=0}^{M+p/d-1} \int_\Gamma e^{-t\lambda} \lambda^{p/d-l} d\lambda \cdot \tilde{r}_l(\xi) + \int_\Gamma e^{-t\lambda} \lambda^{-M} \, \tilde{r}_M(\lambda^{-1},\xi) \, d\lambda,$$

where Γ is any contour in Λ of the form given in Figure 8. Observe that

$$\int_\Gamma e^{-t\lambda} \lambda^{p/d-l} d\lambda = t^{l-p/d-1} \int_\Gamma e^{-\lambda} \lambda^{p/d-l} d\lambda = \begin{cases} 0, & l - p/d \notin \mathbb{N}; \\ (-t)^{l-p/d-1}, & l - p/d \in \mathbb{N}. \end{cases} \tag{64}$$

Thus, we can write

$$\sum_{l=0}^{M+p/d-1} \int_\Gamma e^{-t\lambda} \lambda^{p/d-l} d\lambda \cdot \tilde{r}_l(\xi) = \sum_{l=0}^{M-2} t^l \, r_l(\xi), \quad r_l(\xi) \in S^{-N}(\mathbb{R}^n).$$

Also, observe that Cauchy's Theorem implies $\int_\Gamma \lambda^{-k} \tilde{r}_M(\lambda^{-1},\xi) \, d\lambda \equiv 0$ for all $k \geq 2$. Hence,

$$r_M(t,\xi) := \int_\Gamma e^{-t\lambda} \lambda^{-M} \tilde{r}_M(\lambda^{-1},\xi) \, d\lambda$$

is such that $r_M(t,\xi) \in C^\infty(\,(0,\infty)_t; S^{-N}(\mathbb{R}^n)\,) \cap C^{M-2}(\,[0,\infty)_t; S^{-N}(\mathbb{R}^n)\,)$ with $\partial_t^j r_M(t,\xi)|_{t=0} \equiv 0$ for $j = 0, \ldots, M-2$. ■

Remark: Observe that this Lemma still holds if we only assume that

$$\Lambda \ni \lambda \mapsto r(\lambda,\xi) \in S^m(\mathbb{R}^n)$$

is holomorphic for all $|\lambda| \geq R$ for some $R > 0$ (in which case, to define $\mathcal{L}(r)$, we take the contour Γ of the form given in Figure 8 such that $|\lambda| \geq R$ for all $\lambda \in \Gamma$).

For each $m \in \mathbb{R}$ and $d \in \mathbb{R}^+$, define the index set $\mathcal{E}_{m,d}$ on H^n by

$$\mathcal{E}_{m,d}(tb) := d\mathbb{N}_0; \;\; \mathcal{E}_{m,d}(tf) := \frac{n}{2} + \{(k - m - n - d, 0) \,|\, k \in \mathbb{N}_0\} \overline{\cup} (\mathbb{N}_0 + d\mathbb{N}_0).$$

Lemma 4.16 *Let $a(\lambda,\xi) \in S^{m,p,d}_{\Lambda,ros}(\mathbb{R}^n)$, where m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, be resolvent like and define*

$$A(t) := \int e^{iz\cdot\xi}\, \mathcal{L}(a)(t,\xi)\, d\xi \otimes |dz|^{\frac{1}{2}}.$$

Then if $t = s^d$, $A(t)\, |\frac{dt}{t}|^{\frac{1}{2}} \in \mathcal{A}^{\mathcal{E}_{m,d}}_{phg}(H^n, \Omega_b^{\frac{1}{2}})$. Moreover, as $t \downarrow 0$,

$$A(t)|_{z=0} \sim \sum_{k\in\mathbb{N}_0} t^{\frac{k-m-n}{d}-1}\, \gamma_k + \sum_{k,\, \frac{k-m-n}{d}-1\in\mathbb{N}_0} t^{\frac{k-m-n}{d}-1}\, \log t\, \gamma'_k + \sum_{k\in\mathbb{N}_0} t^k \gamma''_k,$$

for some constants γ_k, γ'_k, $\gamma''_k \in \mathbb{C}$.

Proof: By Lemma 4.10, when lifted to H^n, $|dz\frac{dt}{t}|^{\frac{1}{2}} = \rho^{\frac{n}{2}}\mu$, where $0 < \mu \in C^\infty(H^n, \Omega_b^{\frac{1}{2}})$. Hence, if $\mathcal{E}'_{m,d}$ is the index set

$$\mathcal{E}'_{m,d}(tb) := d\mathbb{N}_0;\;\; \mathcal{E}'_{m,d}(tf) := \{(k-m-n-d, 0)\,|\, k \in \mathbb{N}_0\}\overline{\cup}(\mathbb{N}_0 + d\mathbb{N}_0),$$

then it suffices to show that if

$$A(t) := \int e^{iz\cdot\xi}\, \mathcal{L}(a)(t,\xi)\, d\xi,$$

then $A(s^d) \in \mathcal{A}^{\mathcal{E}'_{m,d}}_{phg}(H^n)$ and $A(s^d)|_{z=0} \in \mathcal{A}^{\mathcal{F}_{m,d}}_{phg}([0,\infty)_s)$, where $\mathcal{F}_{m,d}$ is the index set

$$\mathcal{F}_{m,d} := \{(k-m-n-d, 0)\,|\, k \in \mathbb{N}_0\}\overline{\cup} d\mathbb{N}_0.$$

Since $a(\lambda,\xi) \in S^{m,p,d}_{\Lambda,ros}(\mathbb{R}^n)$, $a(\lambda,\xi) \sim \sum_{j=0}^\infty \chi(\xi)\, a_{m-j}(\lambda,\xi)$, where $a_{m-j}(\lambda,\xi) \in C^{\infty,p,d}_{\Lambda,\mathrm{hom}(m-j)}(\mathbb{R}^n)$ for each j, and where $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ is such that $\chi(\xi) \equiv 0$ on a neighborhood of 0 and $\chi(\xi) \equiv 1$ outside a neighborhood of 0.

Thus, for any $N >> 0$, we can write

$$A(s^d) = \sum_{j=1}^{N-1} A_{m-j}(s^d) + R_N(s^d), \tag{65}$$

where

$$A_{m-j}(s^d) = \int e^{iz\cdot\xi}\, \chi(\xi)\, \mathcal{L}(a_{m-j})(s^d,\xi) d\xi \;\text{ and }\; R_N(s^d) = \int e^{iz\cdot\xi}\, \mathcal{L}(r_N)(s^d,\xi) d\xi,$$

where $r_N(\lambda,\xi) \in S^{m-N,p,d}_{\Lambda,r}(\mathbb{R}^n)$. By Lemmas 4.13 and 4.14, for each j,

$$A_{m-j}(s^d) \in \mathcal{A}^{\mathcal{E}'_{m,d}}_{phg}(H^n) \;\text{ and }\; A_{m-j}(s^d)|_{z=0} \in \mathcal{A}^{\mathcal{F}_{m,d}}_{phg}([0,\infty)_s).$$

Also, by Lemma 4.15, for any $N >> 0$, there is an $M >> 0$ such that

$$R_N(s^d) = \sum_{l=0}^{M-2} s^{dl} R_l + R_M(s^d),$$

where $R_l(z) = \int e^{iz\cdot\xi}\, r_l(\xi) d\xi$ with $r_l(\xi) \in S^{-N-n-1}(\mathbb{R}^n)$, and $R_M(s^d, z) = \int e^{iz\cdot\xi}\, r_M(s^d,\xi) d\xi$ with $r_M(t,\xi) \in C^\infty(\,(0,\infty)_t; S^{-N-n-1}(\mathbb{R}^n)\,) \cap C^{M-2}(\,[0,\infty)_t; S^{-N-n-1}(\mathbb{R}^n)\,)$ such that $\partial_t^j r_M(t,\xi) \equiv 0$ for $j = 0, \ldots, M-2$. Hence, for some $M >> 0$,

$$R_N(s^d) = \sum_{l=0}^{M-2} \rho^{dl}\omega_0^{dl} R_l(\rho\omega') + R_M(\rho^d\omega_0^d, \rho\omega').$$

Since $r_l(\xi) \in S^{-N-n-1}(\mathbb{R}^n)$, one can check that $R_l(\rho\omega') \in C^N(H^n)$, and since

$$r_M(t,\xi) \in C^\infty(\,(0,\infty)_t; S^{-N-n-1}(\mathbb{R}^n)\,) \cap C^{M-2}(\,[0,\infty)_t; S^{-N-n-1}(\mathbb{R}^n)\,)$$

with $\partial_t^j r_M(t,\xi) \equiv 0$ for $j = 0, \ldots, M-2$, one can check that $R_M(\rho^d \omega_0^d, \rho\omega') \in \rho^{dk}\omega_0^{dk} S^0_{k,loc}(H^n)$ for some $k >> 0$ (depending on M). Thus, Proposition 2.2 applied to the expansion (65) shows that

$$A(s^d) \in \mathcal{A}^{\mathcal{E}'_{m,d}}_{phg}(H^n, \Omega_b^{\frac{1}{2}}) \text{ and } A(s^d)|_{z=0} \in \mathcal{A}^{\mathcal{F}_{m,d}}_{phg}([0,\infty)_s).$$

■

Proof of Structure Theorem: By definition,

$$\Psi^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}}) := \{A \in I^{m,p,d}_{\Lambda,ros}(X_b^2, \Delta_b, \Omega_b^{\frac{1}{2}}) \,|\, A \equiv 0 \text{ at } lb \cup rb\}.$$

Thus, off of Δ_b,

$$\Psi^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}}) \subseteq S^{p/d}_{os}(\Lambda;\, \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})),$$

and in any product decomposition $X_b^2 \cong \mathbb{R}^{n,k} \times \mathbb{R}^n_z$, where $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}_z$, we can identify

$$\Psi^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}}) \equiv I^{m,p,d}_{\Lambda,ros}(\mathbb{R}^{n,k} \times \mathbb{R}^n, \mathbb{R}^{n,k} \times \{0\}, \Omega_b^{\frac{1}{2}}).$$

Thus, Theorem 4.11 follow from Lemmas 4.16 and 4.15. ■

Note that when $B = \mathrm{Id}$, there is a $\log t$ term in the second sum in the expansion (59) when $k = n$. The following Corollary shows that there is actually *no* $\log t$ term.

Corollary 4.17 *Let $A \in \mathcal{E}\ell\ell^m_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$. Then as $t \downarrow 0$,*

$$e^{-tA}|_{\Delta_b} \sim \sum_{k\in\mathbb{N}_0} t^{\frac{k-n}{m}}\, \gamma_k(x) + \sum_{k,\,\frac{k-n}{m}\in\mathbb{N}} t^{\frac{k-n}{m}} \,\log t\, \gamma'_k(x) + \sum_{k\in\mathbb{N}_0} t^k \gamma''_k(x),$$

where for each k, γ_k, γ'_k, $\gamma''_k \in C^\infty(X, \Omega_b)$.

Proof: We will leave this as an exercise with hints.

1. Let $a(\lambda,\xi) \in S^{-m,-m,m}_{\Lambda,ros}(\mathbb{R}^n)$ be resolvent like and let

$$a(\lambda,\xi) \sim \sum_{j=0}^{\infty} \chi(\xi)\, a_{m-j}(\lambda,\xi),$$

where $a_{m-j}(\lambda,\xi) \in C^{\infty,-m,m}_{\Lambda,\mathrm{hom}(-m-j)}(\mathbb{R}^n)$ for each j, and where $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ is such that $\chi(\xi) \equiv 0$ on a neighborhood of 0 and $\chi(\xi) \equiv 1$ outside a neighborhood of 0. Define $A(t) := \int \mathcal{L}(a)(t,\xi)\, d\xi$. Show that in Lemma 4.16, the coefficient of the $\log t$ term in the expansion of this $A(t)$ is given by

$$-\int (\xi \cdot \partial_\xi \chi)(\xi)\, \mathcal{L}(a_{-m-n})(0,\xi)\, d\xi = \int_{\mathbb{S}^{n-1}} \mathcal{L}(a_{-m-n})(0,\omega)\, d\omega.$$

Subhint: See the proof of Lemma 4.13.

2. With the same set-up as in (1), suppose that $\mathcal{L}(a)(0,\xi) \equiv C$, where C is a constant. Show that $\mathcal{L}(a_{-m-j})(0,\xi) \equiv 0$ for all $j \geq 1$. In particular, if $\mathcal{L}(a)(0,\xi)$ is constant, then the expansion of $A(t)$ as $t \downarrow 0$ has no $\log t$ term.

3. Locally decompose $X_b^2 \cong \mathbb{R}^{n,k}_y \times \mathbb{R}^n_z$, where $\Delta_b \cong \mathbb{R}^{n,k}_y \times \{0\}_z$. Then in this decomposition, we can identify

$$\Psi^{-m,-m,m}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}}) \equiv I^{-m,-m,m}_{\Lambda,ros}(\mathbb{R}^{n,k} \times \mathbb{R}^n, \mathbb{R}^{n,k} \times \{0\}, \Omega_b^{\frac{1}{2}}).$$

Let $\phi \in C^\infty(X_b^2)$ be supported in this product decomposition. Then observe that $\phi\, e^{-tA}|_{t=0} = \phi \circ \mathrm{Id} \equiv \phi(y,0)$. Hence, using (2) above, show that the expansion of $(\phi\, e^{-tA})|_{\Delta_b}$ as $t \downarrow 0$ has no $\log t$ term.

■

5 Mellin Transforms

5.1 Mellin Transforms and Complex Powers

Let m, $p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and let $A(\lambda) \in \Psi^{m,p,d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$ be holomorphic and positive and suppose that $\Lambda \cup B_\epsilon \ni \lambda \mapsto A(\lambda) \in \Psi^m_b(X, \Omega_b^{\frac{1}{2}})$ is holomorphic for some $\epsilon > 0$. Let $\phi \in \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}})$. Then, by Theorem 3.22, $A(\lambda)\phi \in S^{p/d}(\Lambda; \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}}))$. Let Γ be any contour in $\Lambda \cup B_\epsilon$ of the form given in Figure 13. Then for each $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau < -p/d - 1$, the integral

$$\mathcal{M}(A)(\tau)\phi := \int_\Gamma \lambda^{-i\tau} A(\lambda)\phi \, d\lambda \tag{66}$$

converges in $\dot{C}^\infty(X, \Omega_b^{\frac{1}{2}})$, where $\lambda^{-i\tau}$ is defined by using the *principal branch* of the logarithm. Observe that since $A(\lambda)$ is holomorphic on $\Lambda \cup B_\epsilon$, the integral (66) is defined independent of the contour Γ choosen (where Γ is of the form given in Figure 13). Since for any $k \in \mathbb{N}$ and $\operatorname{Im}\tau < -p/d - 1$, $\lambda^{-i\tau} (\log\lambda)^k A(\lambda)\phi$ is still integrable in $\dot{C}^\infty(X, \Omega_b^{\frac{1}{2}})$, it follows that

$$\mathcal{M}(A)(\tau)\phi \in \mathcal{H}ol(\{\operatorname{Im}\tau < -p/d - 1\}; \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}})).$$

Definition 5.1 Let m, $p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and let $A(\lambda) \in \Psi^{m,p,d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$ be holomorphic and positive and suppose that $A(\lambda)$ extends to be holomorphic on a neighborhood of $\lambda = 0$. Then the *Mellin Transform* of A is the map

$$\mathcal{M}(A)(\tau) : \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}}) \to \mathcal{H}ol(\{\operatorname{Im}\tau < -p/d - 1\}; \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}}))$$

defined by equation (66) for any contour Γ of the form given in Figure 13.

Remark: Since Resolvent like operators are, by definition, holomorphic on a neighborhood of 0, their Mellin transforms are always defined.

The 'most important' examples of Mellin transforms are Complex powers. Thus, let $A \in \mathcal{E}\ell\ell^m_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, where Λ is a positive cone and suppose that the resolvent $(A-\lambda)^{-1}$ extends to be holomorphic on $\Lambda \cup \mathcal{U}$ for some connected neighborhood $\mathcal{U}$ of $\lambda = 0$. Then by Theorem 3.32 and Theorem 2.8, there exists an $\epsilon > 0$ such that $(A - \lambda)^{-1} \in \Psi^{-m,-m,m,\epsilon,\epsilon,\epsilon}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$ is resolvent like. Let Γ be any contour in Λ of the form given in Figure 13, where $\Gamma \subseteq \Lambda \cup \mathcal{U}$. For $\operatorname{Re} z < 0$, we define A^z by

$$A^z := \frac{i}{2\pi} \int_\Gamma \lambda^z (A - \lambda)^{-1} \, d\lambda = \frac{i}{2\pi} \mathcal{M}((A - \lambda)^{-1})(iz).$$

Note that since $(A-\lambda)^{-1}$ is holomorphic on $\Lambda \cup \mathcal{U}$, the integral defining A^z is well-defined, independent of the contour Γ choosen. Observe that $A^z : \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}}) \to \mathcal{H}ol(\{\operatorname{Re} z < 0\}; \rho^\epsilon H^\infty_b(X, \Omega_b^{\frac{1}{2}}))$, and it extends to an operator

$$A^z : \rho^\epsilon H^\infty_b(X, \Omega_b^{\frac{1}{2}}) \to \mathcal{H}ol(\{\operatorname{Re} z < 0\}; \rho^\epsilon H^\infty_b(X, \Omega_b^{\frac{1}{2}})).$$

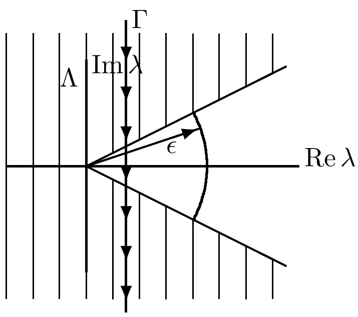


Figure 13: The contour Γ.

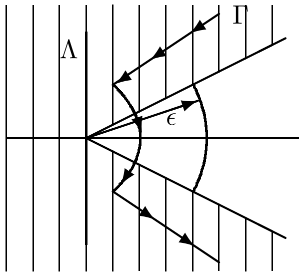


Figure 14: The contour Γ.

Now let $A \in \mathcal{E}\ell\ell^m_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, where Λ is a positive cone, be formally self adjoint, positive, and Fredholm. Then by Theorem 3.32 and Corollary 2.8, there exists a connected neighborhood $\mathcal{V} \subseteq \mathbb{C}$ containing 0 and an $\epsilon > 0$ such that on $\Lambda \cup \mathcal{V}$, $(A - \lambda)^{-1} \in \Psi^{-m,-m,m,\epsilon,\epsilon,\epsilon}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$ is meromorphic with only a simple pole at $\lambda = 0$ given by $-\pi$, where $\pi \in \Psi^{-\infty,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}})$ is the orthogonal projection onto $\ker A$.

Lemma 5.2 *Let $A \in \mathcal{E}\ell\ell^m_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, where Λ is a positive cone, be formally self adjoint, positive, and Fredholm. Let π be the orthogonal projection onto the kernel of A and define $A_0 := A + \pi$. Then,*

$$(A_0 - \lambda)^{-1} = (A - \lambda)^{-1} + \lambda^{-1}(1 - \lambda)^{-1}\pi.$$

It follows that $(A_0 - \lambda)^{-1} \in \Psi^{-m,-m,m,\epsilon,\epsilon,\epsilon}_{b,\Lambda,r}(X, \Omega_b^{\frac{1}{2}})$ is resolvent like. In particular, the complex power, A_0^z is well defined, and the heat kernel e^{-tA_0} vanishes exponentially as $t \to \infty$. Moreover, $e^{-tA_0} = (e^{-t} - 1)\pi + e^{-tA}$.

Proof: We will leave the resolvent identity to the reader as an exercise. Then the heat kernel identity follows by taking the Laplace transform of the resolvent identity. ■

5.2 Mellin and Laplace Transforms

Recall that the *gamma function*, $\Gamma(z)$, is the function

$$\Gamma(z) := \int_0^\infty t^{z-1}\, e^{-t}\, dt \tag{67}$$

defined for all $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. Recall that the gamma function extends to be a non-vanishing meromorphic function on all of $\mathbb{C}$ having only simple poles on $-\mathbb{N}_0$, with residue $(-1)^k/k!$ at $-k \in -\mathbb{N}_0$. In particular, $1/\Gamma(z)$ is an *entire* function, vanishing on $-\mathbb{N}_0$. Also recall that the Gamma function has the 'factorial property' $\Gamma(z+1) = z\,\Gamma(z)$. Let $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$. Then making the change of variables $t \mapsto \lambda t$ in the intergral (67), we find that for any $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, $\Gamma(z) = \lambda^z \int_0^\infty t^{z-1}\, e^{-t\lambda}\, dt$; or $\lambda^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1}\, e^{-t\lambda}\, dt$.

Lemma 5.3 *For all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$ and $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, we have*

$$\lambda^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1}\, e^{-t\lambda}\, dt.$$

Let m, $p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and let $A(\lambda) \in \Psi^{m,p,d}_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$ be holomorphic and positive and suppose that $A(\lambda)$ extends to be holomorphic on $\Lambda \cup B_\epsilon$ for some $\epsilon > 0$. Then the Mellin Transform

$$\mathcal{M}(A)(\tau) : \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}}) \to \mathcal{H}ol(\{\operatorname{Im} \tau < -p/d - 1\}; \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}}))$$

is defined. We defined the Mellin transform by using any contour as in Figure 13, but of course, we can deform the contour so that it looks like the contour Γ in Figure 14. For the rest of this subsection, we will use the contour in Figure 14 to define the Mellin transform. Using this new contour Γ, we can also define the Laplace transform $\mathcal{L}(A)(t) := \int_\Gamma e^{-t\lambda}\, A(\lambda)\, d\lambda$. By Theorem 4.2, $\mathcal{L}(A)(t) \in C^\infty(\,(0,\infty)_t;\, \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})\,)$ and $\mathcal{L}(A)(t) \to 0$ exponentially in $\Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})$ as $t \to \infty$. Also, given $k \in \mathbb{N}_0$ and $M \in \mathbb{N}_0$ such that $p/d - M < -k - 1$, for any $N \in \mathbb{N}_0$, we have

$$t^{M+N}\mathcal{L}(A)(t) \in C^\infty(\,(0,\infty)_t;\, \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})\,) \cap C^k(\,[0,\infty)_t;\, \Psi_b^{m-p-dN}(X, \Omega_b^{\frac{1}{2}})\,).$$

Thus, the following interchanging of integrals is allowed for $\operatorname{Im}\tau << 0$:

$$\begin{aligned}
\mathcal{M}(A)(\tau) &= \int_\Gamma \lambda^{-i\tau}\, A(\lambda)\, d\lambda \\
&= \frac{1}{\Gamma(i\tau)} \int_\Gamma \int_0^\infty t^{i\tau-1}\, e^{-t\lambda}\, A(\lambda)\, dt\, d\lambda \\
&= \frac{1}{\Gamma(i\tau)} \int_0^\infty t^{i\tau-1} \int_\Gamma e^{-t\lambda}\, A(\lambda)\, d\lambda\, dt \\
&= \frac{1}{\Gamma(i\tau)} \int_0^\infty t^{i\tau-1} \mathcal{L}(A)(t)\, dt.
\end{aligned}$$

Lemma 5.4 *Let m, $p \in \mathbb{R}$, $d \in \mathbb{R}^+$, and let $A(\lambda) \in \Psi_{b,\Lambda}^{m,p,d}(X, \Omega_b^{\frac{1}{2}})$ be holomorphic and positive and suppose that $A(\lambda)$ extends to be holomorphic on a neighborhood of $\lambda = 0$. Then, for $\operatorname{Im}\tau << 0$, we have*

$$\mathcal{M}(A)(\tau) = \frac{1}{\Gamma(i\tau)} \int_0^\infty t^{i\tau-1} \mathcal{L}(A)(t)\, dt.$$

5.3 Conormal nature of Mellin Transforms

Throughout this section, fix a positive cone $\Lambda \subseteq \mathbb{C}$. Let m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$. Recall that a symbol $a(\lambda, \xi) \in S_\Lambda^{m,p,d}(\mathbb{R}^n)$ is resolvent like if there exists an $\epsilon > 0$ such that $a(\lambda, \xi)$ extends to be a smooth function, holomorphic in λ, for (λ, ξ) in

$$\{(\lambda, \xi) \in \mathbb{C} \times \mathbb{R}^n \mid \lambda \in \Lambda \text{ or } |\lambda| \le \epsilon\,(1 + |\xi|)^d \text{ or } \frac{1}{\epsilon}\,(1 + |\xi|)^d \le |\lambda|\}; \qquad (68)$$

and moreover, $a(\lambda, \xi)$ continues to satisfy similar symbol estimates for (λ, ξ) in the set given by (68) as for (λ, ξ) in $\Lambda \times \mathbb{R}^n$.

Recall that an operator $A \in \Psi_{b,\Lambda,r}^{m,p,d}(X, \Omega_b^{\frac{1}{2}})$ is resolvent like if

1. $\Lambda \cup B_\epsilon \ni \lambda \mapsto A(\lambda) \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$ is holomorphic for some $\epsilon > 0$ and
2. for any coordinate patch $\mathbb{R}_y^{n,k} \times \mathbb{R}_z^n$ on X_b^2 such that $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}$ and any compactly supported function ϕ on the coordinate patch, we have

$$\phi\, A = \frac{1}{(2\pi)^n} \int e^{iz\cdot\xi}\, a(\lambda, y, \xi)\, d\xi \otimes \nu,$$

   where $\nu \in C^\infty(X_b^2, \Omega_b^{\frac{1}{2}})$ and where for each $y \in \mathbb{R}^{n,k}$, $(\lambda, \xi) \mapsto a(\lambda, y, \xi) \in S_\Lambda^{m,p,d}(\mathbb{R}^n)$ is resolvent like.

We will prove the following Theorem.

Theorem 5.5 *Let $A(\lambda) \in \Psi_{b,\Lambda,r}^{m,p,d}(X, \Omega_b^{\frac{1}{2}})$, where m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, be a resolvent like operator. Then, the Mellin transform*

$$\mathcal{M}(A)(\tau) : \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}}) \to \mathcal{H}ol(\,\{\operatorname{Im}\tau < -p/d - 1\};\, \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}})\,)$$

extends to be an entire map

$$\mathcal{M}(A)(\tau) : \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}}) \;\to\; \mathcal{H}ol(\mathbb{C}; \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}})).$$

Moreover, for each $\tau \in \mathbb{C}$, $\mathcal{M}(A)(\tau) \in \Psi_b^{d\operatorname{Im}\tau+m+d}(X, \Omega_b^{\frac{1}{2}})$ *and is an entire family of b-pseudodifferential operators in the sense that for each* $R \in \mathbb{R}$, *the map*

$$\{\tau \in \mathbb{C} : \operatorname{Im}\tau < R\} \ni \tau \;\mapsto\; \mathcal{M}(A)(\tau) \in \Psi_b^{d\operatorname{Im}\tau+m+d}(X, \Omega_b^{\frac{1}{2}})$$

is an element of $\mathcal{H}ol(\operatorname{Im}\tau < R,\, \Psi_b^{dR+m+d}(X, \Omega_b^{\frac{1}{2}}))$.

Corollary 5.6 *Let* $A \in \mathcal{E}\ell\ell^m_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, *where* Λ *is a positive cone and suppose that the resolvent* $(A-\lambda)^{-1}$ *extends to be holomorphic on* $\Lambda \cup \mathcal{U}$, *where* $\mathcal{U}$ *is a neighborhood of* $\lambda = 0$. *Then for each* $z \in \mathbb{C}$, $A^z \in \Psi_b^{m\operatorname{Re} z,\epsilon,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}})$ *for some* $\epsilon > 0$, *and is an entire family of b-pseudodifferential operators in the sense that for each* $R \in \mathbb{R}$, *the map*

$$\{z \in \mathbb{C} : \operatorname{Re} z < R\} \ni z \;\mapsto\; A^z \in \Psi_b^{m\operatorname{Re} z,\epsilon,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}})$$

is an element of $\mathcal{H}ol(\operatorname{Re} z < R,\, \Psi_b^{mR,\epsilon,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}}))$.

Corollary 5.7 *Let* $A \in \mathcal{E}\ell\ell^m_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, *where* Λ *is a positive cone, be formally self adjoint, positive, and Fredholm. Then for each* $z \in \mathbb{C}$, $A_0^z \in \Psi_b^{m\operatorname{Re} z,\epsilon,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}})$ *for some* $\epsilon > 0$, *and is an entire family of b-pseudodifferential operators in the sense that for each* $R \in \mathbb{R}$, *the map*

$$\{z \in \mathbb{C} : \operatorname{Re} z < R\} \ni z \;\mapsto\; A_0^z \in \Psi_b^{m\operatorname{Re} z,\epsilon,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}})$$

is an element of $\mathcal{H}ol(\operatorname{Re} z < R,\, \Psi_b^{mR,\epsilon,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}}))$.

To prove Theorem 5.5, we start with the following Lemma.

Lemma 5.8 *Let* $m, p \in \mathbb{R}$, $d \in \mathbb{R}^+$, *and let* $a(\lambda, \xi) \in S_\Lambda^{m,p,d}(\mathbb{R}^n)$ *be a resolvent like symbol. Let* Γ *be any contour in* Λ *of the form given in Figure 14 and define*

$$\mathcal{M}(a)(\tau, \xi) := \int_\Gamma \lambda^{-i\tau}\, a(\lambda, \xi)\, d\lambda$$

for $\operatorname{Im}\tau << 0$. *Then,* $\mathcal{M}(a)(\tau, \xi)$ *extends to be entire in* τ. *Moreover, for each* $\tau \in \mathbb{C}$, $\mathcal{M}(a)(\tau, \xi) \in S^{d\operatorname{Im}\tau+m+d}(\mathbb{R}^n)$ *and is an entire family of symbols in the sense that for each* $R \in \mathbb{R}$, *the map*

$$\{\tau \in \mathbb{C} : \operatorname{Im}\tau < R\} \ni \tau \;\mapsto\; \mathcal{M}(a)(\tau, \xi) \in S^{d\operatorname{Im}\tau+m+d}(\mathbb{R}^n)$$

is an element of $\mathcal{H}ol(\operatorname{Im}\tau < R,\, S^{dR+m+d}(\mathbb{R}^n))$.

Proof:[7] Since $a(\lambda, \xi)$ is resolvent like, there exists an $\epsilon > 0$ such that $a(\lambda, \xi)$ extends to be a smooth function, holomorphic in λ, for (λ, ξ) in

$$\{(\lambda, \xi) \in \mathbb{C} \times \mathbb{R}^n \mid \lambda \in \Lambda \;\text{ or }\; |\lambda| \le \epsilon\,(1 + |\xi|)^d \;\text{ or }\; \frac{1}{\epsilon}\,(1 + |\xi|)^d \le |\lambda|\}; \tag{69}$$

and moreover, $a(\lambda, \xi)$ continues to satisfy similar symbol estimates for (λ, ξ) in the set given by (69) as for (λ, ξ) in $\Lambda \times \mathbb{R}^n$. Let $\xi \in \mathbb{R}^n$. Then, as

$$\Lambda \cup \{\lambda \in \mathbb{C} \mid \; |\lambda| \le \epsilon\,(1 + |\xi|)^d \;\text{ or }\; \frac{1}{\epsilon}\,(1 + |\xi|)^d \le |\lambda|\} \ni \lambda \mapsto a(\lambda, \xi)$$

is holomorphic, it follows that $\mathcal{M}(a)(t,\xi) = \int_{\Gamma_\xi} \lambda^{-i\tau}\, a(\lambda,\xi)\, d\lambda$, where Γ_ξ is the contour shown in Figure 15, where the radius of the inner arc of Γ_ξ is $\frac{\epsilon}{2}(1+|\xi|)^d$ and the radius of the outer arc of Γ_ξ is $\frac{2}{\epsilon}(1+|\xi|)^d$. It follows that $\mathcal{M}(a)(\tau,\xi)$ extends to be entire in τ. We now prove that for each $\tau \in \mathbb{C}$, $\mathcal{M}(a)(\tau,\xi) \in S^{d\operatorname{Im}\tau+m+d}(\mathbb{R}^n)$ and is an entire family of symbols in the sense that for each $R \in \mathbb{R}$, the map

$$\{\tau \in \mathbb{C} : \operatorname{Im}\tau < R\} \ni \tau \mapsto \mathcal{M}(a)(\tau,\xi) \in S^{d\operatorname{Im}\tau+m+d}(\mathbb{R}^n)$$

is an element of $\mathcal{H}ol(\operatorname{Im}\tau < R,\, S^{dR+m+d}(\mathbb{R}^n))$. Let $k \in \mathbb{N}_0$ and $\beta \in \mathbf{N}_0^n$. Then,

$$\partial_\tau^k \partial_\xi^\beta \mathcal{M}(a)(\tau,\xi) = \int_{\Gamma_\xi} \lambda^{-i\tau}(-i\log\lambda)^k\, \partial_\xi^\beta a(\lambda,\xi)\, d\lambda.$$

Observe that

1. the length of $\Gamma_\xi \leq C_1\,(1+|\xi|)^d$ for some $C_1 > 0$ (independent of ξ);
2. $|\lambda^{-i\tau}| \leq C'(1+|\xi|)^{d\operatorname{Im}\tau}$ for all $\lambda \in \Gamma_\xi$, for some $C' > 0$ (independent of ξ);
3. $|\log\lambda|^k \leq C''|\log((1+|\xi|)^d)|^k$ for all $\lambda \in \Gamma_\xi$, for some $C'' > 0$ (independent of ξ);
4. there exists constants C_2 and C_2' such that for all $\lambda \in \Gamma_\xi$,

$$\begin{aligned} |\,\partial_\xi^\beta a(\lambda,\xi)\,| &\leq C_2\,(1+|\lambda|^{1/d}+|\xi|)^p\,(1+|\xi|)^{m-p-|\beta|} \\ &\leq C_2'(1+|\xi|)^{m-|\beta|}. \end{aligned}$$

Hence,

$$\begin{aligned} |\,\partial_\tau^k \partial_\xi^\beta \mathcal{M}(a)(\tau,\xi)\,| &\leq \int_{\Gamma_\xi} |\lambda^{-i\tau}|\,|\log\lambda|^k\,|\,\partial_\xi^\beta a(\lambda,\xi)\,|\,d\lambda \\ &\leq C_1\,(1+|\xi|)^d \cdot C'(1+|\xi|)^{d\operatorname{Im}\tau} \cdot C''|\log((1+|\xi|)^d)|^k \cdot C_2'(1+|\xi|)^{m-|\beta|} \\ &\leq C\,(1+|\xi|)^{d\operatorname{Im}\tau+m+d-|\beta|}\,|\log(1+|\xi|)|^k. \end{aligned}$$

Thus, for any $k \in \mathbb{N}_0$ and $\beta \in \mathbf{N}_0^n$, there is a $C > 0$ such that

$$|\,\partial_\tau^k \partial_\xi^\beta \mathcal{M}(a)(\tau,\xi)\,| \leq C\,(1+|\xi|)^{d\operatorname{Im}\tau+m+d-|\beta|}\,|\log(1+|\xi|)|^k.$$

This estimate implies that for fixed τ, $\mathcal{M}(a)(\tau,\xi) \in S^{d\operatorname{Im}\tau+m+d}(\mathbb{R}^n_\xi)$ and that for each $R \in \mathbb{R}$, the map

$$\{\tau \in \mathbb{C} : \operatorname{Im}\tau < R\} \ni \tau \mapsto \mathcal{M}(a)(\tau,\xi) \in S^{d\operatorname{Im}\tau+m+d}(\mathbb{R}^n)$$

is an element of $\mathcal{H}ol(\operatorname{Im}\tau < R,\, S^{dR+m+d}(\mathbb{R}^n))$. ■

[7] This proof is based on the proof given in Grubb [4, Lemma 4.2.3].

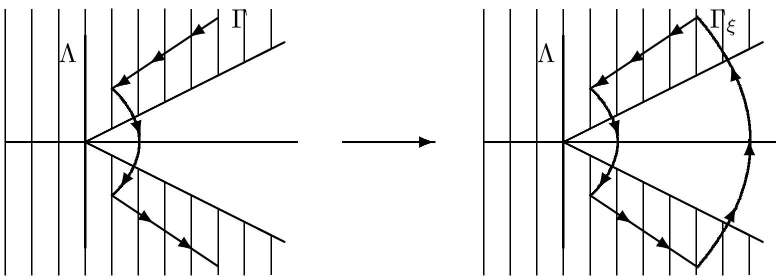


Figure 15: Deformation of the contour Γ into the contour Γ_ξ.

Lemma 5.9 *Let* $R \in \Psi_{b,\Lambda,r}^{-\infty,p,d}(X,\Omega_b^{\frac{1}{2}}) \equiv S_{os}^{p/d}(\Lambda;\, \Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}}))$*, where* $p \in \mathbb{R}$ *and* $d \in \mathbb{R}^+$ *with* $p/d \in \mathbb{Z}$*, be holomorphic and suppose that* $R(\lambda)$ *extends to be holomorphic on a neighborhood of* $\lambda = 0$*. Then,*

$$\mathcal{M}(R)(\tau) \in \mathcal{H}ol(\mathbb{C};\, \Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}})).$$

Proof: By Lemma 5.4, for $\operatorname{Im}\tau << 0$, we have $\mathcal{M}(R)(\tau) = \frac{1}{\Gamma(i\tau)}\int_0^\infty t^{i\tau-1}\mathcal{L}(R)(t)\,dt$. By Lemma 4.9, $\mathcal{L}(R)(t) \in C^\infty([0,\infty)_t;\, \Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}}))$ and $\mathcal{L}(R)(t) \to 0$ exponentially in $\Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}})$ as $t \to \infty$. It follows that $\frac{1}{\Gamma(i\tau)}\int_1^\infty t^{i\tau-1}\mathcal{L}(R)(t)\,dt$ is an *entire* function of $\tau \in \mathbb{C}$. Thus, we just have to show that $\frac{1}{\Gamma(i\tau)}\int_0^1 t^{i\tau-1}\mathcal{L}(R)(t)\,dt$ extends to be entire. Since, $\mathcal{L}(R)(t) \in C^\infty([0,\infty)_t;\, \Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}}))$, for any $M \in \mathbb{N}$, we can write

$$\mathcal{L}(R)(t) = \sum_{l=0}^{M-1} t^l R_l + t^M R_M(t),$$

where $R_l \in \Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}})$, and $R_M(t) \in C^\infty([0,\infty)_t;\, \Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}}))$. Hence, for any $M \in \mathbb{N}$, for $\operatorname{Im}\tau << 0$,

$$\begin{aligned}\frac{1}{\Gamma(i\tau)}\int_0^1 t^{i\tau-1}\mathcal{L}(R)(t)\,dt &= \sum_{l=0}^{M-1}\frac{1}{\Gamma(i\tau)}\int_0^1 t^{i\tau+l-1}dt\cdot R_l + \frac{1}{\Gamma(i\tau)}\int_0^1 t^{i\tau+M-1}R_M(t)\,dt\\ &= \sum_{l=0}^{M-1}\frac{1}{\Gamma(i\tau)}\,\frac{1}{i\tau+l}\,R_l + \frac{1}{\Gamma(i\tau)}\int_0^1 t^{i\tau+M-1}R_M(t)\,dt.\end{aligned}$$

Observe that since $1/\Gamma(i\tau)$ is entire and it vanishes when $i\tau \in -\mathbb{N}_0$, for any $l \in \mathbb{N}_0$, $\frac{1}{\Gamma(i\tau)}\,\frac{1}{i\tau+l}$ is entire. Also, observe that $\int_0^1 t^{i\tau+M-1}R_M(t)\,dt$ is holomorphic for $\operatorname{Im}\tau < M$. Thus, as $M \in \mathbb{N}$ is arbitrary, $\frac{1}{\Gamma(i\tau)}\int_0^1 t^{i\tau-1}\mathcal{L}(R)(t)\,dt$ extends to be an entire function of τ. ■

Proof of Theorem 5.5: Note that it suffices to prove Theorem 5.5 for $\phi\, A(\lambda)$, when $\phi \in C_c^\infty(X_b^2 \setminus \Delta_b)$, and when ϕ is a compactly supported function on a coordinate patch $\mathbb{R}^{n,k} \times \mathbb{R}^n$ on X_b^2 such that $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}$.

Let $\phi \in C_c^\infty(X_b^2 \setminus \Delta_b)$. Then,

$$\phi\, A \in \Psi_{b,\Lambda,r}^{-\infty,p,d}(X,\Omega_b^{\frac{1}{2}}) \equiv S_{os}^{p/d}(\Lambda;\, \Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}}))$$

and $\phi\, A(\lambda)$ is holomorphic on a neighborhood of $\lambda = 0$. Thus, by Lemma 5.9, $\mathcal{M}(\phi\, A)(\tau) \in \mathcal{H}ol(\mathbb{C};\, \Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}}))$.

Now let $\mathbb{R}^{n,k} \times \mathbb{R}^n$ be a coordinate patch on X_b^2 such that $\Delta_b \cong \mathbb{R}^{n,k} \times \{0\}$ and let ϕ be any compactly supported function on the coordinate patch. Then, we can write

$$\phi\, A(\lambda) = \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}\, a(\lambda,y,\xi)\,d\xi \otimes \nu,$$

where $\nu \in C^\infty(X_b^2,\Omega_b^{\frac{1}{2}})$, and where for each $y \in \mathbb{R}^{n,k}$, $(\lambda,\xi) \mapsto a(\lambda,y,\xi) \in S_\Lambda^{m,p,d}(\mathbb{R}^n)$ is resolvent like. Hence,

$$\mathcal{M}(\phi\, A)(\tau) = \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}\,\mathcal{M}(a)(\tau,y,\xi)\,d\xi \otimes \nu.$$

By Lemma 5.8, $\mathcal{M}(a)(\tau,y,\xi)$ extends to be entire in τ and for each $\tau \in \mathbb{C}$, $\mathcal{M}(a)(\tau,y,\xi) \in C^\infty(\mathbb{R}^{n,k}; S^{d\operatorname{Im}\tau+m+d}(\mathbb{R}^n))$ and is an entire family of symbols in the sense that for each $R \in \mathbb{R}$, the map

$$\{\tau \in \mathbb{C} : \operatorname{Im}\tau < R\} \ni \tau \;\mapsto\; \mathcal{M}(a)(\tau,y,\xi) \in C^\infty(\mathbb{R}^{n,k}; S^{d\operatorname{Im}\tau+m+d}(\mathbb{R}^n))$$

is an element of $\mathcal{H}ol(\operatorname{Im}\tau < R; C^\infty(\mathbb{R}^{n,k}; S^{dR+m+d}(\mathbb{R}^n)))$. It follows that $\mathcal{M}(\phi\, A)(\tau)$ extends to be an entire function of τ such that for each $\tau \in \mathbb{C}$, $\mathcal{M}(\phi\, A)(\tau) \in \Psi_b^{d\operatorname{Im}\tau+m+d}(X,\Omega_b^{\frac{1}{2}})$ and is an entire family of b-pseudodifferential operators in the sense that for each $R \in \mathbb{R}$, the map

$$\{\tau \in \mathbb{C} : \operatorname{Im}\tau < R\} \ni \tau \;\mapsto\; \mathcal{M}(\phi\, A)(\tau) \in \Psi_b^{d\operatorname{Im}\tau+m+d}(X,\Omega_b^{\frac{1}{2}})$$

is an element of $\mathcal{H}ol(\operatorname{Im}\tau < R,\, \Psi_b^{dR+m+d}(X, \Omega_b^{\frac{1}{2}}))$. ■

5.4 Restriction to the Diagonal

Lemma 5.10 *Let $\mathcal{E}$ be an index set (not necessarily C^∞) such that $\mathcal{E} \subseteq \mathbb{C} \times \{0,1\}$ and such that $\mathcal{E} \cap (\mathbb{C} \times \{1\}) \subseteq \mathbb{N}_0 \times \{1\}$. Let $u(t) \in \mathcal{A}^{\mathcal{E}}_{phg}([0,\infty)_t)$ and suppose that $u(t)$ is Schwartz at ∞; thus, for any $k \in \mathbb{N}$, $t^k u(t)$ is bounded outside a neighborhood of 0. Denote the coefficients of the expansion of $u(t)$ by $u_{(z,k)}$, where $(z,k) \in \mathcal{E}$. Define for $\operatorname{Im}\tau << 0$,*

$$v(\tau) := \frac{1}{\Gamma(i\tau)} \int_0^\infty t^{i\tau - 1} u(t)\, dt.$$

Then, $v(\tau)$ extends to be a meromorphic function on all of $\mathbb{C}$ having only simple poles at the points

$$i \cdot \{z \,|\, (z,0) \in \mathcal{E} \text{ and } z \notin \mathbb{N}_0, \text{ or } (z,1) \in \mathcal{E}\}.$$

The residue of $v(\tau)$ at $\tau = iz$, when $z \notin \mathbb{N}_0$, is $-i\frac{u_{(z,0)}}{\Gamma(-z)}$, and when $z \in \mathbb{N}_0$, the residue is $i(-1)^{z+1}\Gamma(z+1)\, u_{(z,1)}$. Moreover, the regular value of $v(\tau)$ at $\tau = 0$ is $u_{(0,0)} - \gamma \cdot u_{(0,1)}$, where $\gamma := \lim_{k\to\infty}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} - \log k)$ is Euler's constant.

Proof: For $\operatorname{Im}\tau << 0$, we can write

$$\begin{aligned} v(\tau) &= \frac{1}{\Gamma(i\tau)} \int_0^\infty t^{i\tau-1} u(t)\, dt \\ &= \frac{1}{\Gamma(i\tau)} \int_0^1 t^{i\tau-1} u(t)\, dt + \frac{1}{\Gamma(i\tau)} \int_1^\infty t^{i\tau-1} u(t)\, dt. \end{aligned}$$

Since $u(t)$ is Schwartz at $t = \infty$ and since $1/\Gamma(i\tau)$ is an entire function vanishing at $\tau = 0$, the term $\frac{1}{\Gamma(i\tau)} \int_1^\infty t^{i\tau-1} u(t)\, dt$ is an entire function vanishing at $\tau = 0$. Thus, we need only concern ourselves with the term $\frac{1}{\Gamma(i\tau)} \int_0^1 t^{i\tau-1} u(t)\, dt$. Fix $N >> 0$. Then there is an $M >> 0$ such that

$$u(t) = \sum_{k \notin \mathbb{N}_0, z \le N} t^z u_{(z,0)} + \sum_{z \in \mathbb{N}_0, z \le N} t^z \log t\, u_{(z,1)} + \sum_{k \in \mathbb{N}_0, k \le N} t^k u_{(k,0)} + t^M u_M(t),$$

where $u_M(t) \in S^0_{loc}([0,\infty)_t)$. Observe that for any $z \in \mathbb{C}$ with $\operatorname{Re} z >> 0$, we have

$$\int_0^1 t^{z-1}\, dt = \frac{1}{z} \quad \text{and} \quad \int_0^1 t^{z-1} \log t\, dt = \partial_z \int_0^1 t^{z-1}\, dt = -\frac{1}{z^2}.$$

Thus, for $\operatorname{Im}\tau << 0$,

$$\begin{aligned} \frac{1}{\Gamma(i\tau)} \int_0^1 t^{i\tau-1} u(t)\, dt &= \sum_{k \notin \mathbb{N}_0, z \le N} \frac{1}{\Gamma(i\tau)(i\tau + z)} u_{(z,0)} - \sum_{z \in \mathbb{N}_0, z \le N} \frac{1}{\Gamma(i\tau)(i\tau+z)^2} u_{(z,1)} \\ &\quad + \sum_{k \in \mathbb{N}_0, k \le N} \frac{1}{\Gamma(i\tau)(i\tau + k)} u_{(k,0)} + \frac{1}{\Gamma(i\tau)} \int_0^1 t^{i\tau + M - 1} u_M(t)\, dt. \end{aligned} \tag{70}$$

Now observe that since $\Gamma(z)$ has simple poles only on the set $-\mathbb{N}_0$, with residue $(-1)^j/j! = (-1)^j/\Gamma(j+1)$ at $-j \in -\mathbb{N}_0$,

1. if $z \notin \mathbb{N}_0$, $1/[\Gamma(i\tau)(i\tau + z)]$ is meromorphic having only a simple pole at $i\tau = -z$, with residue $-i/\Gamma(-z)$, and it vanishes at $\tau = 0$;
2. if $z \in \mathbb{N}_0$, $1/[\Gamma(i\tau)(i\tau + z)^2]$ is meromorphic having only a simple pole at $i\tau = -z$, with residue $i(-1)^{z+1}\Gamma(z+1)$;

3. if $z \in \mathbb{N}_0$, $1/[\Gamma(i\tau)(i\tau + z)]$ is entire and vanishes at $\tau = 0$, unless $z = 0$, in which case it has the value 1 at $\tau = 0$.

Also, observe that $\frac{1}{\Gamma(i\tau)} \int_0^1 t^{i\tau+M-1} u_M(t)\, dt$ is holomorphic on $\{\mathrm{Im}\,\tau < M - 1\}$ and it vanishes at $\tau = 0$ for any $M > 1$.

It follows that $v(\tau)$ extends to be meromorphic on all of $\mathbb{C}$, having only simple poles at the points

$$i \cdot \{z \,|\, (z, 0) \in \mathcal{E} \text{ and } z \notin \mathbb{N}_0, \text{ or } (z, 1) \in \mathcal{E}\}$$

where the residue of $v(\tau)$ at $\tau = iz$, when $z \notin \mathbb{N}_0$, is $-i \frac{u_{(z,0)}}{\Gamma(-z)}$, and when $z \in \mathbb{N}_0$, the residue is $i(-1)^{z+1}\Gamma(z+1)\, u_{(z,1)}$. It also follows from (1)–(3) that the regular value of (70) at $\tau = 0$ is

$$u_{(0,0)} - [\text{the regular value of } \frac{1}{\Gamma(i\tau)(i\tau)^2} \text{ at } \tau = 0] \cdot u_{(0,1)}.$$

Since $\Gamma(z)z^2 = z\Gamma(z+1)$, the regular value of $1/[\Gamma(z)z^2]$ at $z = 0$ is $\partial_z(z \cdot \frac{1}{z\Gamma(z+1)})\Big|_{z=0} = -\frac{\Gamma'(z+1)}{\Gamma(z+1)}\Big|_{z=0}$. Since (see [3, page 179]), $\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma - \sum_{k=1}^{\infty}(\frac{1}{z+k} - \frac{1}{k})$, it follows that $\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma$; and hence, the regular value of $1/[\Gamma(z)z^2]$ at $z = 0$ is γ. ■

Let $A(\lambda) \in \Psi^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$, where m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, be resolvent like. Then by Theorem 4.11, as $t \downarrow 0$,

$$\mathcal{L}(A)(t)|_{\Delta_b} \sim \sum_{k \in \mathbb{N}_0} t^{\frac{k-m-n}{d}-1} \gamma_k(x) + \sum_{k,\, \frac{k-m-n}{d}-1 \in \mathbb{N}_0} t^{\frac{k-m-n}{d}-1} \log t\, \gamma'_k(x) + \sum_{k \in \mathbb{N}_0} t^k \gamma''_k(x), \tag{71}$$

where for each k, γ_k, γ'_k, $\gamma''_k \in C^\infty(X, \Omega_b)$. Also, by Theorem 5.5, for each $\tau \in \mathbb{C}$, $\mathcal{M}(A)(\tau) \in \Psi_b^{d\mathrm{Im}\,\tau+m+d}(X, \Omega_b^{\frac{1}{2}})$ and is an entire family of b-pseudodifferential operators in the sense that for each $R \in \mathbb{R}$, the map

$$\{\tau \in \mathbb{C} : \mathrm{Im}\,\tau < R\} \ni \tau \mapsto \mathcal{M}(A)(\tau) \in \Psi_b^{d\mathrm{Im}\,\tau+m+d}(X, \Omega_b^{\frac{1}{2}})$$

is an element of $\mathcal{H}ol(\,\mathrm{Im}\,\tau < R,\, \Psi_b^{dR+m+d}(X, \Omega_b^{\frac{1}{2}})\,)$. Thus,

$$\{\tau \in \mathbb{C} : \mathrm{Im}\,\tau < \frac{-m-n}{d} - 1\} \ni \tau \mapsto \mathcal{M}(A)(\tau)|_{\Delta_b} \in C^\infty(X, \Omega_b)$$

is holomorphic. By Lemma 5.4, for $\mathrm{Im}\,\tau << 0$, we have $\mathcal{M}(A)(\tau) = \frac{1}{\Gamma(i\tau)} \int_0^\infty t^{i\tau-1} \mathcal{L}(A)(t)\, dt$. Also, by properties (43) and (44) of Theorem 4.2, it follows that for $\mathrm{Im}\,\tau << 0$,

$$\mathcal{M}(A)(\tau)|_{\Delta_b} = \frac{1}{\Gamma(i\tau)} \int_0^\infty t^{i\tau-1} (\mathcal{L}(A)(t)|_{\Delta_b})\, dt.$$

Thus, Lemma 5.10 gives the following.

Theorem 5.11 *Let $A(\lambda) \in \Psi^{m,p,d}_{b,\Lambda,ros}(X, \Omega_b^{\frac{1}{2}})$, where m, $p \in \mathbb{R}$ and $d \in \mathbb{R}^+$ with $p/d \in \mathbb{Z}$, be resolvent like. Then the Schwartz kernel of the Mellin transform $\mathcal{M}(A)(\tau)$, when restricted to Δ_b, extends from $\mathrm{Im}\,\tau < \frac{-m-n}{d} - 1$ to be a meromorphic function on all of $\mathbb{C}$, with values in $C^\infty(X, \Omega_b)$, having only simple poles at the points*

$$i \cdot \{z_k = \frac{k-m-n}{d} - 1 \,:\, k \in \mathbb{N}_0\}.$$

The residue of $\mathcal{M}(A)(\tau)|_{\Delta_b}$ at $\tau = iz_k$, when $z_k \notin \mathbb{N}_0$, is $-i \frac{\gamma_k}{\Gamma(-z_k)}$, and when $z_k \in \mathbb{N}_0$, the residue is $i(-1)^{z_k+1}\Gamma(z_k+1)\, \gamma'_k$, where γ_k and γ'_k are given in the expansion (71) above. Moreover, the regular value of $\mathcal{M}(A)(\tau)|_{\Delta_b}$ at $\tau = 0$ is

the constant term in the expansion of $\mathcal{L}(A)(t)|_{\Delta_b}$ as $t \downarrow 0$
$- \gamma \cdot$ (the coefficient of $\log t$ in the expansion of $\mathcal{L}(A)(t)|_{\Delta_b}$ as $t \downarrow 0$),

where γ is Euler's constant. In particular, if the expansion of $\mathcal{L}(A)(t)$ as $t \downarrow 0$ has no $\log t$ term, then $\mathcal{M}(A)(\tau)$ is holomorphic at $\tau = 0$ and

$$\begin{aligned}&\text{the value of } \mathcal{M}(A)(\tau)|_{\Delta_b} \text{at } \tau = 0\\ &\quad = \text{ the constant term in the expansion of } \mathcal{L}(A)(t)|_{\Delta_b} \text{ as } t \downarrow 0.\end{aligned}$$

Let $A \in \Psi^m_{b,os}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be elliptic, formally self adjoint, and positive. Then it follows that A has a positive principal symbol (see, for example, [7]). Thus, $A \in \mathcal{E}\ell\ell^m_{b,\Lambda,os}(X, \Omega_b^{\frac{1}{2}})$ for any positive cone Λ. Hence, by Corollary 4.17,

$$e^{-tA}|_{\Delta_b} \sim \sum_{k\in\mathbb{N}_0} t^{\frac{k-n}{m}}\, \gamma_k(x) + \sum_{k,\, \frac{k-n}{m}\in\mathbb{N}} t^{\frac{k-n}{m}}\, \log t\, \gamma'_k(x) + \sum_{k\in\mathbb{N}_0} t^k \gamma''_k(x), \tag{72}$$

where for each k, γ_k, γ'_k, $\gamma''_k \in C^\infty(X, \Omega_b)$. Now suppose that A is Fredholm. Then by Lemma 5.2, $e^{-tA_0} = (e^{-t} - 1)\pi + e^{-tA}$, where $A_0 := A + \pi$, with π being the orthogonal projection onto the kernel of A. . In particular, $e^{-tA_0}|_{\Delta_b}$ has an expansion of the same form as in (72). By Corollary 5.7, for each $z \in \mathbb{C}$, $A_0^z \in \Psi_b^{m\mathrm{Re}\, z,\epsilon,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}})$ for some $\epsilon > 0$. In particular, $A_0^z|_{\Delta_b} \in S^{0,\epsilon}(X, \Omega_b)$ for $\mathrm{Re}\, z > n/m$.

Corollary 5.12 *Let $A \in \Psi^m_{b,os}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be elliptic, formally self adjoint, positive, and Fredholm. Then the Schwartz kernel of the Complex power A_0^z, when restricted to Δ_b, extends from $\mathrm{Re}\, z > n/m$ to be a meromorphic function on all of $\mathbb{C}$, with values in $S^{0,\epsilon}(X, \Omega_b)$ for some $\epsilon > 0$, having only simple poles at the points*

$$\{z_k = \frac{n-k}{m} \;:\; k \in \mathbb{N}_0,\, z_k \neq 0\,\}.$$

The residue of $A_0^z|_{\Delta_b}$ at $z = z_k$, when $z_k \notin \mathbb{N}$, is $\frac{\gamma_k}{\Gamma(-z_k)}$, and when $z_k \in \mathbb{N}$, the residue is $(-1)^{z_k+1}\Gamma(z_k + 1)\, \gamma'_k$, where γ_k and γ'_k are given in the expansion (72) above. Moreover,

$$\begin{aligned}&\text{the value of } A_0^z|_{\Delta_b} \text{at } z = 0\\ &\quad = \text{ the constant term in the expansion of } e^{-tA}|_{\Delta_b} \text{ as } t \downarrow 0.\end{aligned}$$

6 The b-Trace

6.1 b-Trace

Lemma 6.1 *Let $f \in S^{0,\eta}([0,1)^k)$, where $0 < \eta \leq 1$. Then we can write*

$$f(x_1, \ldots, x_k) = f(0) + \sum_I x_{i_1}^{\eta_{i_1}} \cdots x_{i_l}^{\eta_{i_l}} f_I(x_I),$$

where the sum is over all $I = (i_1, \ldots, i_l), 1 \leq i_1 < \cdots < i_l \leq k$, and where for each $I = (i_1, \ldots, i_l)$, $x_I = (x_{i_1}, \ldots, x_{i_l})$ and $f_I \in S^{0,0}([0,1)^l)$.

Moreover, if $\eta \equiv 1$, then for each I,

$$f_I(x_I) = \int_0^1 \cdots \int_0^1 (\partial_{x_{i_1}} \cdots \partial_{x_{i_l}} f \circ g_I)(t_1 x_{i_1}, \ldots, t_l x_{i_l}) dt_1 \cdots dt_l,$$

where $g_I(x_I) = (y_1, \ldots, y_k)$, with

$$y_i = \begin{cases} 0, & \text{if } i \notin I; \\ x_{i_j}, & \text{if } i = i_j. \end{cases}$$

Proof: The proof is by induction on k. Thus, assume $k = 1$. Then by definition of $S^{0,\eta}([0,1))$,

$$f(x) = f(0) + x^\eta f_1(x), \tag{73}$$

where $f_1(x) \in S^{0,0}([0,1))$, and where by the fundamental theorem of calculus, if $\eta = 1$, $f_1(x) = \int_0^1 (\partial_x f)(tx)dt$. Thus, our Lemma is true if $k=1$. Assume our Lemma is true for k; we'll prove it is true for $k+1$. Applying our Lemma to the first k variables of $f(x_1, \ldots, x_{k+1})$ yields

$$f(x_1, \ldots, x_{k+1}) = f(0, x_{k+1}) + \sum_I x_{i_1}^{\eta_{i_1}} \cdots x_{i_l}^{\eta_{i_l}} f_I(x_I, x_{k+1}),$$

where if $\eta \equiv 1$, then $f_I(x_I, x_{k+1}) = \int_0^1 \cdots \int_0^1 (\partial_{x_{i_1}} \cdots \partial_{x_{i_l}} f \circ g_I)(t_1 x_{i_1}, \ldots, t_l x_{i_l}, x_{k+1}) dt_1 \cdots dt_l$, where $g_I(x_I, x_{k+1}) = (y_1, \ldots, y_k, x_{k+1})$, with

$$y_i = \begin{cases} 0, & \text{if } i \notin I; \\ x_{i_j}, & \text{if } i = i_j. \end{cases}$$

Applying (73) to each of the terms $f(0, x_{k+1})$ and $f_I(x_I, x_{k+1})$ with respect to the variable x_{k+1} yields our Lemma for $k+1$. ■

Let X be compact with corners. Fix an ordering of the boundary hypersurfaces of X: $M_1(X) = \{H_1, \ldots, H_N\}$, and fix boundary defining functions $\{\rho_i\}$, where ρ_i is a boundary defining function for H_i. Define

$$\mathbb{C}_+^N := \{z = (z_1, \ldots, z_N) \in \mathbb{C}^N \mid \operatorname{Re} z_i > 0 \text{ for all } i\}.$$

Observe that if $A \in \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})$, then $\rho^z A$, where $\rho^z = \rho_1^{z_1} \cdots \rho_N^{z_N}$, is trace class on $L_b^2(X, \Omega_b^{\frac{1}{2}})$ for all $z \in \mathbb{C}_+^N$ with trace given by $\operatorname{Tr}(\rho^z A) = \int_X (\rho^z A)|_{\Delta_b}$. Thus, $\mathbb{C}_+^N \ni z \mapsto \operatorname{Tr}(\rho^z A)$ is holomorphic.

Theorem 6.2 *Let $A \in \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})$. Then we can write*

$$\operatorname{Tr}(\rho^z A) = \sum_{I,J} \frac{f_I(z_I)}{z_J}, \tag{74}$$

where the sum is over all $I \cup J = \{1, \ldots, N\}$ with $I \cap J = \emptyset$, and where for each I, $f_I : \mathbb{C}^{|I|} \to \mathbb{C}$ is meromorphic, with only simple poles, all on the set

$$\{z_I \in \mathbb{C}^{|I|} \mid z_i \in -\mathbb{N} \text{ for some } i \in I\}.$$

In particular, for each I, f_I is holomorphic near $z_I = 0$.

PROOF: Let $\{\mathcal{U}_i\}$, where $\mathcal{U}_i = [0,1)_x^k \times \mathbb{R}_y^{n-k} \times \mathbb{R}_z^n$, $(k \geq 0)$, be a coordinate cover of Δ_b with $\Delta_b \cong [0,1)^k \times \mathbb{R}^{n-k} \times \{0\}$, and with the appropriate ρ_j's defining the $[0,1)$ factors, and let $\{\phi_i\} \subseteq C^\infty(X_b^2)$ be such that $\phi_i|_{\Delta_b}$ is a partition of unity of Δ_b subordinate to the cover $\{\mathcal{U}_i \cap \Delta_b\}$ of Δ_b. Then,

$$A|_{\Delta_b} = \sum_i (\phi_i A)|_{\Delta_b},$$

and so $\operatorname{Tr}(\rho^z A) = \sum_i \operatorname{Tr}(\rho^z \phi_i A)$. Thus, we may assume that A is a supported in some $\mathcal{U}_i$, which we now fix. For simplicity, assume that $x_j = \rho_j$ for $j = 1, \ldots, k$. Then, $A = A(x,y,z)|\frac{dx}{x} dy dz|^{1/2}$, where $A(x,y,z) \in C_c^\infty(\mathcal{U}_i)$. We can write $\rho^z = x^w \cdot r^{w'}$, where $r = \rho_{k+1} \cdots \rho_N$, $w = (z_1, \ldots, z_k)$, $w' = (z_{k+1}, \ldots, z_N)$, $x^w = x_1^{z_1} \cdots x_k^{z_k}$, and $r^{w'} = \rho_{k+1}^{z_{k+1}} \cdots \rho_N^{z_N}$. Hence,

$$\begin{aligned} \operatorname{Tr}(\rho^z A) &= \int \rho^z A(x,y,0) \frac{dx}{x} dy \\ &= \int x^w B(w', x, y) \frac{dx}{x} dy, \end{aligned}$$

where $B(w', x, y) = r(x,y)^{w'} A(x,y,0)$. Observe that since $r(x,y) > 0$ for any $(x,y) \in [0,1)^k \times \mathbb{R}^{n-k}$, $B(w', x, y)$ is holomorphic for $w' \in \mathbb{C}^{N-k}$.

By Lemma 6.1, we can write

$$B(w',x,y) = B(w',0,y) + \sum_I x_{i_1}\cdots x_{i_l} B_I(w',x_I,y)$$

for some smooth functions $B_I(w',x_I,y)$. Since for any $a \in \mathbb{C}_+$, $\int_0^1 s^a \frac{ds}{s} = \frac{1}{a}$, we have

$$\operatorname{Tr}(\rho^z A) = \frac{1}{z_1\cdots z_k}\int B(w',0,y)dy + \sum_{I,J} \frac{1}{z_{j_1}\cdots z_{j_{k-l}}} \int x_{i_1}^{z_{i_1}}\cdots x_{i_l}^{z_{i_l}} B_I(w',x_I,y)dx_I dy, \tag{75}$$

where $I \cup J = (1,\dots,k)$.

By Taylors theorem, $B_I(w',x_I,y) \sim \sum_\alpha x_I^\alpha B_{I,\alpha}(w',y)$, and so

$$\int x_{i_1}^{z_{i_1}}\cdots x_{i_l}^{z_{i_l}} B_I(w',x_I,y)dx_I dy \sim \sum_\alpha \frac{1}{z_{i_1}+\alpha_1+1}\cdots\frac{1}{z_{i_l}+\alpha_l+1}\int B_{I,\alpha}(w',y)dy.$$

This formula, together with (75), proves our Theorem. ■

Observe that we can write (74) as $\operatorname{Tr}(\rho^z A) = \sum_{I,J,|I|<N} \frac{f_I(z_I)}{z_J} + f(z)$, where $f(z)$ is *holomorphic* at 0. The number $f(0)$ is called the *regular value at* $z=0$ of the meromorphic function $\operatorname{Tr}(\rho^z A)$.

Definition 6.3 Let $A \in \Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}})$. Then the regular value of meromorphic function $\operatorname{Tr}(\rho^z A)$ is called the *b-Trace* of A and is denoted by $b\text{–}\operatorname{Tr}(A)$. Thus,

$$b\text{–}\operatorname{Tr}(A) := \text{ regular value of } \operatorname{Tr}(\rho^z A) \text{ at } z=0.$$

Remark: If $A \in \Psi_b^{-\infty,\alpha,\beta,\eta}(X,\Omega_b^{\frac{1}{2}})$ is in the calculus with bounds with $\alpha,\beta \ge 0$ and $0<\eta\le 1$, then a proof similar to that of Theorem 6.2 shows that we can write $\operatorname{Tr}(\rho^z A) = \sum_{I,J}\frac{f_I(z_I)}{z_J}$, where the sum is over all $I\cup J = \{1,\dots,N\}$ with $I\cap J=\emptyset$, and where for each I,

$$f_I : \{z_I \in \mathbb{C}^{|I|} \,|\, z_i > -\eta_i \text{ for all } i\in I\} \to \mathbb{C}$$

is holomorphic. Then, $b\text{–}\operatorname{Tr}(A)$ is defined to be the regular value of $\operatorname{Tr}(\rho^z A)$ at $z=0$ just as in the case $A\in \Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}})$.

Proposition 6.4 *If $A \in \rho_{ff}^\gamma \Psi_b^{-\infty,\alpha,\beta}(X,\Omega_b^{\frac{1}{2}})$ with $\gamma>0$, then $b\text{–}\operatorname{Tr}(A)\equiv \operatorname{Tr}(A)$.*

Proof: Note that $\operatorname{Tr}(\rho^z A)$ is holomorphic at $z=0$, and so its regular value at $z=0$ is just its value at $z=0$. ■

Thus, the b-Trace is a generalization of the usual trace. However, unlike the usual trace, the b-Trace does not vanish on commutators, as Theorem 6.6 below shows.

Let $H_{i_1},\dots,H_{i_k} \in M_1(X)$. Then observe that $M := H_{i_1}\cap\cdots\cap H_{i_k}$ is a disjoint union of codimension k boundary faces of X. Given $A\in\Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}})$, we define

$$\int_{\mathbb{R}^k} b\text{–}\operatorname{Tr}(\,N_M(A)(\tau)\,)d\tau := \sum_{F=\text{component of } M}\int_{\mathbb{R}^k} b\text{–}\operatorname{Tr}(\,N_F(A)(\tau)\,)d\tau.$$

Lemma 6.5 *If $A\in\Psi_b^{-\infty}(X,\Omega_b^{\frac{1}{2}})$, the regular value of $z_1\cdots z_k\operatorname{Tr}(\rho^z A)$ at $z=0$ is*

$$\frac{1}{(2\pi)^k}\int_{\mathbb{R}^k} b\text{–}\operatorname{Tr}(\,N_M(A)(\tau)\,)d\tau,$$

where $M = H_1\cap\cdots\cap H_k$ and where the b-Trace appearing in the integral is the b-Trace on $\Psi_b^{-\infty}(M,\Omega_b^{\frac{1}{2}})$.

Proof: Let $\chi \in C_c^\infty([0,1))$ with $\chi \equiv 0$ near $x = 0$. Set $\chi_i := \chi(x_i) \in C_c^\infty(X)$. Then, we can write

$$A = \chi_1 \cdots \chi_k A - (\chi_1 \cdots \chi_k - 1)A.$$

Observe that

$$\begin{aligned}
\chi_1 \cdots \chi_k - 1 &= (\chi_1 - 1)\chi_2 \cdots \chi_k + \chi_2 \cdots \chi_k - 1 \\
&= (\chi_1 - 1)\chi_2 \cdots \chi_k + (\chi_2 - 1)\chi_3 \cdots \chi_k + \chi_3 \cdots \chi_k - 1 \\
&\vdots \\
&= \sum_{i=1}^{k} (\chi_i - 1)\chi_{i+1} \cdots \chi_k.
\end{aligned}$$

Also, observe that if $1 \le i \le k$, then since $\chi_i - 1 \equiv 0$ near $x_i = 0$,

$$\mathrm{Tr}(\rho^z (\chi_i - 1)\chi_{i+1} \cdots \chi_k A)$$

is holomorphic in z_i down to $z_i = 0$. Thus, the regular value of $z_1 \cdots z_k \mathrm{Tr}(\rho^z A)$ at $z = 0$ is the regular value of $z_1 \cdots z_k \mathrm{Tr}(\rho^z \chi_1 \cdots \chi_k A)$ at $z = 0$. Thus, we may assume that A is a supported on a coordinate patch $\mathcal{U}_b^2$, where $\mathcal{U} = [0,1)_x^k \times M$ with $x_j = \rho_j$ for $j = 1, \ldots, k$, is a decomposition of X near M. Let $s_j = x_j / x_j'$. Then (x,s) are coordinates on the $[0,1)_b^{2k}$ factor of $\mathcal{U}_b^2$ near Δ_b, and so we can write $A = A(x,s)|\frac{dx}{x}\frac{ds}{s}|^{1/2}$, where $A(x,s) \in C_c^\infty(\mathcal{U}_b^2, \Omega_b^{\frac{1}{2}}(M_b^2))$. Let $z = (w, w')$, where $w = (z_1, \ldots, z_k)$ and $w' = (z_{k+1}, \ldots, z_N)$. Then, $\rho^z = x^w \cdot r^{w'}$, where $r = \rho_{k+1} \cdots \rho_N$, and so

$$\mathrm{Tr}(\rho^z A) = \int x^w B(w', x) \frac{dx}{x},$$

where $B(w', x) = \int_M r^{w'} A(x,1)|_{\Delta_b(M)}$. We can now proceed exactly as we did in the proof of Theorem 6.2 to see that the regular value of $z_1 \cdots z_k \mathrm{Tr}(\rho^z A)$ at $z = 0$ is equal to the regular value of $\int_M r^{w'} A(0,1)|_{\Delta_b(M)}$ at $z = 0$. By the Mellin inversion formula,

$$\begin{aligned}
A(0,1) &= \Big(\frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} s^{i\tau} N_M(A)(\tau) d\tau\Big)|_{s=1} \\
&= \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} N_M(A)(\tau) d\tau.
\end{aligned}$$

Thus, by the definition of the b-Trace on $\Psi_b^{-\infty}(M, \Omega_b^{\frac{1}{2}})$, the regular value of $\int_M r^{w'} A(0,1)|_{\Delta_b(M)}$ at $z = 0$ is $\frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} b\text{–}\mathrm{Tr}(\, N_M(A)(\tau)\,) d\tau$. ■

The formula in the following theorem is called the *trace defect formula* and it measure how non-commutative the b-Trace is.

Theorem 6.6 *Let $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$ and $B \in \Psi_b^{m'}(X, \Omega_b^{\frac{1}{2}})$ with either m or m' equal to $-\infty$. Then*

$$b\text{–}\mathrm{Tr}[A,B] = -\sum_{M \in M_k(X), k \ge 1} \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} b\text{–}\mathrm{Tr}(\, \mathbf{D}_\tau N_M(A)(\tau)\, N_M(B)(\tau)\,)\, d\tau, \tag{76}$$

where the sum is over all boundary faces of X, and where in each term of the sum (76), if M is a component of $H_{i_1} \cap \cdots \cap H_{i_k}$, then $\mathbf{D}_\tau = D_{\tau_{i_1}} \cdots D_{\tau_{i_k}}$.

Proof: Observe that

$$\begin{aligned}
\rho^z [A,B] &= [\rho^z, A]B + A\rho^z B - \rho^z BA \\
&= [\rho^z, A]B + [A, \rho^z B].
\end{aligned}$$

Since the Trace vanishes on commutators, $\mathrm{Tr}([A, \rho^z B]) \equiv 0$ for $z \in \mathbb{C}_+^N$. Thus,

$$\begin{aligned}
b\text{–}\mathrm{Tr}[A,B] &= \text{regular value of } \mathrm{Tr}([\rho^z, A]B) \text{ at } z = 0 \\
&= \text{regular value of } \mathrm{Tr}(\rho^z C(z)) \text{ at } z = 0,
\end{aligned}$$

where $C(z) := \rho^{-z}[\rho^z, A]B = AB - \rho^{-z} A \rho^z B$. Note that $C(0) = 0$ and so by Lemma 6.1, we can write $C(z) = \sum_I z_{i_1} \cdots z_{i_k} C_I(z_I)$, where

$$C_I(x_I) = \int_0^1 \cdots \int_0^1 (\partial_{z_{i_1}} \cdots \partial_{z_{i_k}} C \circ g_I)(t_1 z_{i_1}, \ldots, t_l z_{i_k}) dt_1 \cdots dt_k, \tag{77}$$

where $g_I(z_I) = (y_1, \ldots, y_N)$, with

$$y_i = \begin{cases} 0, & \text{if } i \notin I; \\ z_{i_j}, & \text{if } i = i_j. \end{cases}$$

Thus, $\mathrm{Tr}(\rho^z C(z)) = \sum_I z_{i_1} \cdots z_{i_k} \mathrm{Tr}(C_I(z_I))$, and so, by Lemma 6.5, the regular value of $\mathrm{Tr}(\rho^z C(z))$ at $z = 0$ is

$$\sum_{k=1}^N \sum_{|I|=k} \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} b\text{–}\mathrm{Tr}(N_{M_I}(C_I(0))(\tau)) d\tau,$$

where $M_I := H_{i_1} \cap \cdots \cap H_{i_k}$. We will show that

$$\int_{\mathbb{R}^k} b\text{–}\mathrm{Tr}(N_{M_I}(C_I(0))(\tau)) d\tau = -\int_{\mathbb{R}^k} b\text{–}\mathrm{Tr}(\mathbf{D}_\tau N_{M_I}(A)(\tau) N_{M_I}(B)(\tau)) d\tau,$$

where $\mathbf{D}_\tau = D_{\tau_{i_1}} \cdots D_{\tau_{i_k}}$, which proves our Theorem. To see this, let $s_i = x_i/x_i'$. Then (77) implies that

$$\begin{aligned} C_I(0) &= (\partial_{z_{i_1}} \cdots \partial_{z_{i_k}} C)(0) \\ &= -\partial_{z_{i_1}} \cdots \partial_{z_{i_k}} ([\rho^{-z} A \rho^z] B)|_{z=0} \\ &= -\partial_{z_{i_1}} \cdots \partial_{z_{i_k}} ([s^{-z} A] B)|_{z=0} \\ &= -(-1)^k [(\log s_{i_1}) \cdots (\log s_{i_k}) A] B. \end{aligned}$$

Thus,

$$\begin{aligned} N_{M_I}(C_I(0))(\tau) &= -N_{M_I}(\,(-1)^k [(\log s_{i_1}) \cdots (\log s_{i_k}) A] B\,)(\tau) \\ &= -N_{M_I}(\,(-1)^k (\log s_{i_1}) \cdots (\log s_{i_k}) A\,)(\tau)\, N_{M_I}(B)(\tau) \\ &= -D_{\tau_{i_1}} \cdots D_{\tau_{i_k}} N_{M_I}(A)(\tau)\, N_{M_I}(B)(\tau). \end{aligned}$$

■

6.2 b-integral

We will now give another description of the b-Trace. Let $A \in \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})$. Then, $A|_{\Delta_b} \in C^\infty(X, \Omega_b)$. Thus, one might think that $b\text{–}\mathrm{Tr}(A) = \int_X A|_{\Delta_b}$. This is in fact true — if we interpret $\int_X$ appropriately. Let $\nu \in C^\infty(X, \Omega_b)$. Then for $z \in \mathbb{C}_+^N$, $\rho^z \nu \in L^1(X, \Omega_b)$ and so $\mathbb{C}_+^N \ni z \mapsto \int_X \rho^z \nu \in \mathbb{C}$ is a holomorphic map of $\mathbb{C}_+^N$ into $\mathbb{C}$. The proof of the following Theorem is similar to the proof of Theorem 6.2.

Theorem 6.7 *Let $\nu \in C^\infty(X, \Omega_b)$. Then we can write*

$$\int_X \rho^z \nu = \sum_{I,J} \frac{f_I(z_I)}{z_J}, \tag{78}$$

where the sum is over all $I \cup J = \{1, \ldots, N\}$ with $I \cap J = \emptyset$, and where for each I, $f_I : \mathbb{C}^{|I|} \to \mathbb{C}$ is meromorphic, with only simple poles, all on the set

$$\{z_I \in \mathbb{C}^{|I|} \mid z_i \in -\mathbb{N} \text{ for some } i \in I\}.$$

In particular, for each I, f_I is holomorphic near $z_I = 0$.

Definition 6.8 Let $\nu \in C^\infty(X, \Omega_b)$. Then the regular value of the meromorphic function $\int_X \rho^z \nu$ is called the *b-integral* of ν and is denoted by ${}^b\!\int_X \nu$. Thus,

$$ {}^b\!\int_X \nu := \text{ the regular value of } \int_X \rho^z \nu \text{ at } z = 0. $$

Proposition 6.9 *Let $A \in \Psi_b^{-\infty}(X, \Omega_b^{\frac{1}{2}})$. Then, b–$\mathrm{Tr}(A) = {}^b\!\int_X A|_{\Delta_b}$.*

PROOF: We will leave this to the reader. ■

7 The Index Theorem via the Heat Kernel

7.1 Trace of the Heat Kernel

Theorem 7.1 *Let $A \in \mathcal{E}\ell\ell^m_{b,\Lambda}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, where Λ is a positive cone, be formally self adjoint, positive, and Fredholm. Then,*

$$ \lim_{t\to\infty} b\text{–}\mathrm{Tr}(e^{-tA}) = \dim\ker A. $$

PROOF: Let π be the orthogonal projection onto the kernel of A and set $A_0 := A + \pi$. Then by Lemma 5.2, $e^{-tA_0} = (e^{-t} - 1)\pi + e^{-tA}$, where $e^{-tA_0} \to 0$ exponentially in $\Psi_b^{-\infty,\epsilon,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}})$ for some $\epsilon > 0$. Hence,

$$ \begin{aligned} \lim_{t\to\infty} b\text{–}\mathrm{Tr}(e^{-tA}) &= \lim_{t\to\infty} [\,(1 - e^{-t})\,\dim\ker A + b\text{–}\mathrm{Tr}(e^{-tA_0})\,] \\ &= (1-0)\,\dim\ker A + 0 \\ &= \dim\ker A. \end{aligned} $$

■

Corollary 7.2 *Let $A \in \Psi_b^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be Fredholm. Then,*

$$ \mathrm{ind}\, A = \lim_{t\to\infty} [\, b\text{–}\mathrm{Tr}(e^{-tA^*A}) - b\text{–}\mathrm{Tr}(e^{-tAA^*})\,]. $$

PROOF: This follows from the previous Theorem, using the fact that $\ker A^*A = \ker A$ and $\ker AA^* = \ker A^*$. ■

7.2 The Index formula

Let $A \in \Psi^m_{b,os}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$, be Fredholm. We will find a formula for the index of A. To do so, we consider the function

$$ h(t) := b\text{–}\mathrm{Tr}(e^{-tA^*A}) - b\text{–}\mathrm{Tr}(e^{-tAA^*}). $$

Then by Corollary 7.2, $\mathrm{ind}\, A = \lim_{t\to\infty} h(t)$. Hence, by the fundamental theorem of calculus, for all $t > 0$,

$$ \mathrm{ind}\, A = h(t) + \int_t^\infty \partial_s h(s)\, ds. \tag{79} $$

We now compute $\partial_s h(s)$. We first observe that $A^*Ae^{-tA^*A} \equiv A^*e^{-tAA^*}A$. Indeed, let $\phi \in \dot{C}^\infty(X, \Omega_b^{\frac{1}{2}})$. Then, $u(t) := A^*Ae^{-tA^*A}\phi$ and $v(t) := A^*e^{-tAA^*}A\phi$ agree at $t = 0$ and they both satisfy the equation $(\partial_t + A^*A)\psi(t) = 0,\ t > 0$. Thus, by uniqueness of solutions to the heat equation, $u(t) \equiv v(t)$ (see Theorem 4.3). Hence, $A^*Ae^{-tA^*A} \equiv A^*e^{-tAA^*}A$. Thus,

$$ \begin{aligned} \partial_s h(s) &= b\text{–}\mathrm{Tr}(\,-A^*Ae^{-sA^*A} + AA^*e^{-sAA^*}\,) \\ &= b\text{–}\mathrm{Tr}(\,AA^*e^{-sAA^*} - A^*Ae^{-sA^*A}\,) \\ &= b\text{–}\mathrm{Tr}(\,AA^*e^{-sAA^*} - A^*e^{-sAA^*}A\,) \\ &= b\text{–}\mathrm{Tr}(\,[A, A^*e^{-sAA^*}]\,). \end{aligned} $$

By the Trace defect formula in Theorem 6.6, we have

$$b\text{–Tr}(\,[A, A^* e^{-sAA^*}]\,) = -\sum_{M\in M_k(X),k\geq 1} \frac{1}{(2\pi)^k}\int_{\mathbb{R}^k} b\text{–Tr}(\,\mathbf{D}_\tau N_M(A)(\tau)\, N_M(A^*)(\tau)\, N_M(e^{-sAA^*})(\tau)\,)\, d\tau.$$

Hence,

$$\partial_s h(s) = -\sum_{M\in M_k(X),k\geq 1} \frac{1}{(2\pi)^k}\int_{\mathbb{R}^k} b\text{–Tr}(\,\mathbf{D}_\tau N_M(A)(\tau)\, N_M(A^*)(\tau)\, N_M(e^{-sAA^*})(\tau)\,)\, d\tau. \tag{80}$$

By Lemma 5.2, if $(AA^*)_0 = AA^* + \pi$, where π is the orthogonal projection onto the kernel of A^*, then $e^{-t(AA^*)_0} = (e^{-t}-1)\pi + e^{-tAA^*}$, where $e^{-t(AA^*)_0} \to 0$ exponentially in $\Psi_b^{-\infty,\epsilon,\epsilon,\epsilon}(X, \Omega_b^{\frac{1}{2}})$ for some $\epsilon > 0$. Hence, for any $M \in M_k(X)$, $k \in \mathbb{N}$, $N_M(e^{-tAA^*})(\tau) = N_M(e^{-t(AA^*)_0})(\tau)$ is rapidly decreasing in $\Psi_b^{-\infty,\epsilon,\epsilon,\epsilon}(M, \Omega_b^{\frac{1}{2}})$ as $t \to \infty$ and as $\tau \to \infty$. Hence, we can interchange integrals in the following computation: if $M \in M_k(X)$, $k \in \mathbb{N}$,

$$\begin{aligned}
&\int_t^\infty \int_{\mathbb{R}^k} b\text{–Tr}(\,\mathbf{D}_\tau N_M(A)(\tau)\, N_M(A^*)(\tau)\, N_M(e^{-sAA^*})(\tau)\,)\, d\tau ds \\
&= \int_{\mathbb{R}^k}\int_t^\infty b\text{–Tr}(\,\mathbf{D}_\tau N_M(A)(\tau)\, N_M(A^*)(\tau)\, N_M(e^{-sAA^*})(\tau)\,)\, ds d\tau \\
&= -\int_{\mathbb{R}^k}\int_t^\infty b\text{–Tr}(\,\mathbf{D}_\tau N_M(A)(\tau)\, N_M(A)(\tau)^{-1}\, \partial_s N_M(e^{-sAA^*})(\tau)\,)\, ds d\tau \\
&= -\int_{\mathbb{R}^k} b\text{–Tr}(\,\mathbf{D}_\tau N_M(A)(\tau)\, N_M(A)(\tau)^{-1}\, [\, N_M(e^{-sAA^*})(\tau)\,]_{s=t}^{s=\infty}\,)\, d\tau \\
&= \int_{\mathbb{R}^k} b\text{–Tr}(\,\mathbf{D}_\tau N_M(A)(\tau)\, N_M(A)(\tau)^{-1}\, N_M(e^{-tAA^*})(\tau)\,)\, d\tau.
\end{aligned}$$

Thus, equations (79) and (80) give the following Lemma.

Lemma 7.3 *Let $A \in \Psi_{b,os}^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be Fredholm. Then for all $t > 0$,*

$$\operatorname{ind} A = h(t) - \frac{1}{2}{}^b\eta_A(t), \tag{81}$$

where $h(t) = b\text{–Tr}(e^{-tA^*A}) - b\text{–Tr}(e^{-tAA^*})$ and ${}^b\eta_A(t) = \sum_{M\in M_k(X),k\geq 1} {}^b\eta_M(t)$, where

$$\begin{aligned}
{}^b\eta_M(t) &= \frac{2}{(2\pi)^k}\int_t^\infty\int_{\mathbb{R}^k} b\text{–Tr}(\,\mathbf{D}_\tau N_M(A)(\tau)\, N_M(A^*)(\tau)\, N_M(e^{-sAA^*})(\tau)\,)\, d\tau ds &\quad (82)\\
&= -\frac{2}{(2\pi)^k}\int_{\mathbb{R}^k} b\text{–Tr}(\,\mathbf{D}_\tau N_M(A)(\tau)\, N_M(A)(\tau)^{-1}\, N_M(e^{-tAA^*})(\tau)\,)\, d\tau. &\quad (83)
\end{aligned}$$

Observe that since $e^{-tA^*A}|_{\Delta_b}$ and $e^{-tAA^*}|_{\Delta_b}$ both have asymptotic expansions in t as $t \downarrow 0$, the function $h(t)$ also has an asymptotic expansion as $t \downarrow 0$. In the following Lemma, we show that each ${}^b\eta_M(t)$ has an asymptotic expansion as $t \downarrow 0$.

Lemma 7.4 *Let $A \in \Psi_{b,os}^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be Fredholm. Let $M \in M_k(X)$, $k \in \mathbb{N}$, and let ${}^b\eta_M(t)$ be defined as in (82) above. Then as $t \downarrow 0$,*

$${}^b\eta_M(t) \sim \sum_{j=0}^\infty t^{\frac{j-k-n}{2m}}\,\eta_j + \sum_{j,\frac{j-k-n}{2m}\in\mathbb{N}_0} t^{\frac{j-k-n}{2m}}\,\log t\,\eta'_j + \sum_{j\in\mathbb{N}_0} t^j \eta''_j,$$

for some constants η_j, η'_j, $\eta''_j \in C^\infty(X, \Omega_b)$.

PROOF: Let $B \in \Psi^{-m,\epsilon,\epsilon,\epsilon}_{b,os}(X,\Omega_b^{\frac{1}{2}})$, where $\epsilon > 0$, be a parametrix for A as in Proposition 2.4; thus, $AB - \mathrm{Id}$, $BA - \mathrm{Id} \in \Psi^{-\infty,\epsilon,\epsilon}(X,\Omega_b^{\frac{1}{2}})$. Then it follows that $N_M(B)(\tau) = N_M(A)(\tau)^{-1}$. Let $M_1(X) = \{H_1,\ldots,H_N\}$ be an ordering of the boundary hypersurfaces of X and denote by ρ_i, the fixed boundary defining function for H_i. For simplicity, we will assume that M is a component of $H_1 \cap \cdots \cap H_k$. Let $t_i := \rho_i'/\rho_i$. Then note that $\mathbf{D}_\tau N_M(A)(\tau) = N_M(\,(\log t_1)\cdots(\log t_k)A\,)(\tau)$. Hence,

$$\mathbf{D}_\tau N_M(A)(\tau)\, N_M(A)(\tau)^{-1}\, N_M(e^{-tAA^*})(\tau) = N_M(\,[(\log t_1)\cdots(\log t_k)A]\, B\, e^{-tAA^*}\,)(\tau).$$

Thus, by Lemma 6.5,

$$\begin{aligned}
-\frac{1}{2}{}^b\eta_M(t) &= \frac{1}{(2\pi)^k}\int_{\mathbb{R}^k} b\text{–Tr}(\,\mathbf{D}_\tau N_M(A)(\tau)\, N_M(A)(\tau)^{-1}\, N_M(e^{-tAA^*})(\tau)\,)\, d\tau \\
&= \frac{1}{(2\pi)^k}\int_{\mathbb{R}^k} b\text{–Tr}(\, N_M(\,[(\log t_1)\cdots(\log t_k)A]\, B\, e^{-tAA^*}\,)(\tau)\,)\, d\tau \\
&= \text{the regular value of } z_1\cdots z_k \mathrm{Tr}(\,\rho^z\,[(\log t_1)\cdots(\log t_k)A]\, B\, e^{-tAA^*}\,) \\
&\quad\ \text{at } z = 0 \\
&= \text{the regular value of } z_1\cdots z_k \int_X (\,\rho^z\,[(\log t_1)\cdots(\log t_k)A]\, B\, e^{-tAA^*}\,)|_{\Delta_b} \\
&\quad\ \text{at } z = 0.
\end{aligned}$$

Since $\log t_i$ vanishes on Δ_b,

$$[(\log t_1)\cdots(\log t_k)A]\, B \in \Psi^{m-k+(-m),\epsilon,\epsilon,\epsilon}_{b,os}(X,\Omega_b^{\frac{1}{2}}) = \Psi^{-k,\epsilon,\epsilon,\epsilon}_{b,os}(X,\Omega_b^{\frac{1}{2}}).$$

Thus, by Corollary 4.12,

$$\begin{aligned}
&(\,[(\log t_1)\cdots(\log t_k)A]\, B\, e^{-tAA^*}\,)|_{\Delta_b} \sim \\
&\quad \sum_{j=0}^{\infty} t^{\frac{j-k-n}{2m}}\,\eta_j(x) + \sum_{j,\,\frac{j-k-n}{2m}\in\mathbb{N}_0} t^{\frac{j-k-n}{2m}}\,\log t\,\eta_j'(x) + \sum_{j\in\mathbb{N}_0} t^j\eta_j''(x),
\end{aligned}$$

for some $\eta_j(x)$, $\eta_j'(x)$, $\eta_j''(x)$. Our Lemma follows. ■

Definition 7.5 Let $A \in \Psi^m_{b,os}(X,\Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be Fredholm. Then the *b-eta invariant of* A, ${}^b\eta_A$, is the constant term in the expansion of ${}^b\eta_A(t)$ as $t \downarrow 0$.

Taking the constant term in expansion of the right hand side of equation (81) as $t \downarrow 0$ gives the following Theorem.

Theorem 7.6 *Let $A \in \Psi^m_{b,os}(X,\Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be Fredholm. Then,*

$$\operatorname{ind} A = {}^b\zeta_{A^*A} - {}^b\zeta_{AA^*} - \frac{1}{2}{}^b\eta_A,$$

where ${}^b\zeta_{A^*A}$ and ${}^b\zeta_{AA^*}$ are the constant terms in the expansions, as $t \downarrow 0$, of b–$\mathrm{Tr}(e^{-tA^*A})$ and b–$\mathrm{Tr}(e^{-tAA^*})$ respectively, and where ${}^b\eta_A$ is the b-eta invariant of A.

7.3 Exact operators

Although Theorem 7.6 was stated only for the bundle of b-half densities, it of course holds for any vector bundles. Thus, let $0 < \nu \in C^\infty(X,\Omega_b)$ be a fixed positive b-density, and let E and F be hermitian vector bundles over X. Then, Theorem 7.6 takes the following form: If $A \in \Psi^m_{b,os}(X,E,F)$, $m \in \mathbb{R}^+$, is Fredholm, then

$$\operatorname{ind} A = {}^b\zeta_{A^*A} - {}^b\zeta_{AA^*} - \frac{1}{2}{}^b\eta_A,$$

where ${}^b\zeta_{A^*A}$ and ${}^b\zeta_{AA^*}$ are the constant terms in the expansions, as $t \downarrow 0$, of $b\text{–}\mathrm{Tr}(e^{-tA^*A})$ and $b\text{–}\mathrm{Tr}(e^{-tAA^*})$ respectively, and where ${}^b\eta_A$ is the b-eta invariant of A.

In this section, we'll find an explicit formula for the b-eta invariant, ${}^b\eta_A$, where $A \in \mathrm{Diff}^1_b(X,E,F)$ is a special operator called an exact operator.

Let $A \in \mathrm{Diff}^1_b(X,E,F)$ be elliptic. Let $H \in M_1(X)$ be a boundary hypersurface of X. Denote the fixed boundary defining function for H by x and identify E and F over the product decomposition $X \cong [0,1)_x \times H$ with $E_H := E|_H$ and $F_H := F|_H$ respectively. Then on the product decomposition $[0,1)_x \times H$, we can write $A = a(x,y)\, x\partial_x + B(x)$, where $a(x,y) \in C^\infty(\,[0,1)_x \times H, \hom(E_H,F_H)\,)$ and $B(x) \in C^\infty(\,[0,1)_x; \mathrm{Diff}^1_b(H,E_H,F_H)\,)$. Note that by the definition of the symbol, $a(x,y) = \frac{1}{i}\,{}^b\sigma_1(A)(\frac{dx}{x})$. Since A is elliptic, ${}^b\sigma_1(A)(\frac{dx}{x})$ is invertible. Thus, we can write

$$A = \frac{1}{i}\,{}^b\sigma_1(A)(\frac{dx}{x})[\,x\partial_x + A(x)\,],$$

where $A(x) := i\,{}^b\sigma_1(A)(\frac{dx}{x})^{-1}B(x) \in C^\infty([0,1)_x; \mathrm{Diff}^1_b(H,E_H)\,)$. In particular,

$$N_H(A)(\tau) = \frac{1}{i}\,\sigma_H[\,i\tau + A_H\,], \tag{84}$$

where $\sigma_H := {}^b\sigma_1(A)(\frac{dx}{x})|_H$ and $A_H := A(0) \in \mathrm{Diff}^1_b(H,E_H)$.

Definition 7.7 An operator $A \in \mathrm{Diff}^1_b(X,E,F)$ is said to be *exact* if it is elliptic and if for each $H \in M_1(X)$,

1. $\sigma_H : E_H \to F_H$ is an isometry (that is, $\sigma_H^* = \sigma_H^{-1}$);
2. $A_H \in \mathrm{Diff}^1_b(H,E_H)$ is formally self adjoint,

where σ_H and A_H are defined in formula (84) above.

Lemma 7.8 *Let $A \in \mathrm{Diff}^1_b(X,E,F)$ be exact. Then A is Fredholm iff for each $H \in M_1(X)$,*

$$A_H : H^1_b(H,E_H) \to L^2_b(H,E_H)$$

is invertible.

PROOF: By Corollary 2.6, A is Fredholm iff for each $H \in M_1(X)$,

$$N_H(A)(\tau) : H^1_b(H,E_H) \to L^2_b(H,F_H)$$

is invertible for all $\tau \in \mathbb{R}$. Since $N_H(A)(\tau) = \frac{1}{i}\,\sigma_H[\,i\tau + A_H\,]$ and σ_H is invertible, $N_H(A)(\tau)$ is invertible for all $\tau \in \mathbb{R}$ iff $(i\tau + A_H)$ is invertible for all $\tau \in \mathbb{R}$. Since A_H is self adjoint, $(i\tau + A_H)$ is invertible for all $\tau \in \mathbb{R}\setminus\{0\}$. Thus $(i\tau + A_H)$ is invertible for all $\tau \in \mathbb{R}$ iff A_H is invertible. ■

Lemma 7.9 *Let $A \in \mathrm{Diff}^1_b(X,E,F)$ be exact and Fredholm. Then, ${}^b\eta_A(t) = \sum_{H\in M_1(X)} {}^b\eta_H(t)$, where for each $H \in M_1(X)$,*

$$ {}^b\eta_H(t) = \frac{1}{\sqrt{\pi}}\int_t^\infty s^{-1/2}\, b\text{–}\mathrm{Tr}(\,A_H e^{-sA_H^2}\,)\, ds.$$

PROOF: Observe that if $M \in M_k(X)$ with $k \geq 2$, then $\mathbf{D}_\tau N_M(A)(\tau) \equiv 0$ Indeed, since A is a first order b-differential operator, $N_M(A)(\tau)$ is a polynomial of degree *one* in τ. Hence, $\mathbf{D}_\tau N_M(A)(\tau) \equiv 0$ for $k \geq 2$. Thus, ${}^b\eta_M(t) \equiv 0$ if $\operatorname{codim} M \geq 2$, and so ${}^b\eta_A(t) = \sum_{M\in M_k(X),k\geq 1} {}^b\eta_M(t) = \sum_{H\in M_1(X)} {}^b\eta_H(t)$. Let $H \in M_1(X)$. We need to show that

$$ {}^b\eta_H(t) = \frac{1}{\sqrt{\pi}}\int_t^\infty s^{-1/2}\, b\text{–}\mathrm{Tr}(\,A_H e^{-sA_H^2}\,)\, ds.$$

To see this, observe that for $\tau \in \mathbb{R}$,

$$D_\tau N_M(A)(\tau) = \frac{1}{i}\,\sigma_H \quad\text{and}\quad N_H(A^*)(\tau) = -\frac{1}{i}\,(-i\tau + A_H)\sigma_H^{-1};$$

and, since $N_H(AA^*)(\tau) = N_H(A)(\tau) \circ N_H(A^*)(\tau) = \sigma_H(\tau^2 + A_H^2)\sigma_H^{-1}$, by Lemma 4.4,

$$N_H(e^{-sAA^*})(\tau) = e^{-s\sigma_H(\tau^2+A_H^2)\sigma_H^{-1}}.$$

We claim that $e^{-s\sigma_H(\tau^2+A_H^2)\sigma_H^{-1}} = \sigma_H e^{-s\tau^2} e^{-sA_H^2}\sigma_H^{-1}$. Indeed, both the left and right hand sides of this equation agree at $s = 0$ and they both satisfy the heat equation $(\partial_s + \sigma_H(\tau^2 + A_H^2)\sigma_H^{-1})\psi(s) = 0,\ s > 0$; thus, by uniqueness of solutions to the heat equation, they must be equal. It follows that

$$D_\tau N_H(A)(\tau)\, N_H(A^*)(\tau)\, N_H(e^{-sAA^*})(\tau) = \sigma_H(-i\tau + A_H)e^{-s\tau^2}e^{-sA_H^2}\sigma_H^{-1},$$

and so,

$$b\text{–Tr}(\, D_\tau N_H(A)(\tau)\, N_H(A^*)(\tau)\, N_H(e^{-sAA^*})(\tau)\,) = b\text{–Tr}(\, (-i\tau + A_H)e^{-s\tau^2}e^{-sA_H^2}\,).$$

Note that $\int_{\mathbb{R}} \tau e^{-s\tau^2}\, d\tau = 0$ and $\int_{\mathbb{R}} e^{-s\tau^2}\, d\tau = s^{-1/2}\int_{\mathbb{R}} e^{-\tau^2}\, d\tau = s^{-1/2}\sqrt{\pi}$. Thus,

$$\begin{aligned} {}^b\eta_H(t) &= \frac{1}{\pi}\int_t^\infty \int_{\mathbb{R}} b\text{–Tr}(\, D_\tau N_H(A)(\tau)\, N_H(A^*)(\tau)\, N_H(e^{-sAA^*})(\tau)\,)\, d\tau ds \\ &= \frac{1}{\sqrt{\pi}}\int_t^\infty s^{-1/2}\, b\text{–Tr}(\, A_H e^{-sA_H^2}\,)\, ds. \end{aligned}$$

■

Theorem 7.10 *Let $A \in \mathrm{Diff}_b^1(X, E, F)$ be exact and Fredholm. Then,*

$$\mathrm{ind}\, A = {}^b\zeta_{A^*A} - {}^b\zeta_{AA^*} - \frac{1}{2}{}^b\eta_A,$$

where ${}^b\zeta_{A^*A}$ and ${}^b\zeta_{AA^*}$ are the constant terms in the expansions, as $t \downarrow 0$, of b–$\mathrm{Tr}(e^{-tA^*A})$ and b–$\mathrm{Tr}(e^{-tAA^*})$ respectively, and where ${}^b\eta_A = \sum_{H\in M_1(X)} {}^b\eta_H$, where ${}^b\eta_H$ is the constant term in the expansion, as $t \downarrow 0$, of

$$\frac{1}{\sqrt{\pi}}\int_t^\infty s^{-1/2}\, b\text{–Tr}(\, A_H e^{-sA_H^2}\,)\, ds.$$

8 The Index Theorem via the Complex Power

8.1 The b-zeta function

Let $A \in \Psi_{b,os}^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be elliptic, formally self adjoint, positive, and Fredholm. Then by Corollary 5.12, $A_0^{-z}|_{\Delta_b}$ is holomorphic for $\mathrm{Re}\, z > n/m$. In particular, for $\mathrm{Re}\, z > n/m$, b–$\mathrm{Tr}(A_0^{-z}) = {}^b\!\int A_0^{-z}|_{\Delta_b}$ is well defined.

Definition 8.1 Let $A \in \Psi_{b,os}^m(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be elliptic, formally self adjoint, positive, and Fredholm. Then the *b-zeta function*, ${}^b\zeta_A(z)$, defined for $z \in \mathbb{C}$ with $\mathrm{Re}\, z > n/m$, is the function

$${}^b\zeta_A(z) := b\text{–Tr}(A_0^{-z}).$$

By Corollary 4.17, we have

$$e^{-tA}|_{\Delta_b} \sim \sum_{k=0}^\infty t^{\frac{k-n}{m}}\, \gamma_k(x) + \sum_{k,\, \frac{k-n}{m}\in\mathbb{N}} t^{\frac{k-n}{m}}\, \log t\, \gamma_k'(x) + \sum_{k\in\mathbb{N}_0} t^k \gamma_k''(x), \tag{85}$$

where for each k, γ_k, γ_k', $\gamma_k'' \in C^\infty(X, \Omega_b)$. Hence, Corollary 5.12 implies the following.

Theorem 8.2 *Let $A \in \Psi^m_{b,os}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be elliptic, formally self adjoint, positive, and Fredholm. Then the b-zeta function, ${}^b\zeta_A(z)$, extends from $\operatorname{Re} z > n/m$ to be a meromorphic function on all of $\mathbb{C}$, having only simple poles at the points*

$$\{ z_k = \frac{n-k}{m} \, : \, k \in \mathbb{N}_0, \, z_k \neq 0 \}.$$

The residue of ${}^b\zeta_A(z)$ at $z = z_k$, when $z_k \notin \mathbb{N}$, is $\frac{{}^b\!\int \gamma_k}{\Gamma(-z_k)}$, and when $z_k \in \mathbb{N}$, the residue is $(-1)^{z_k+1}\Gamma(z_k+1)\, {}^b\!\int \gamma'_k$, where γ_k and γ'_k are given in the expansion (85) above. Moreover,

$$\begin{aligned} &\textit{the value of } {}^b\zeta_A(z) \textit{ at } z = 0 \\ &\quad = \textit{ the constant term in expansion of } b\text{–}\mathrm{Tr}(e^{-tA}) \textit{ as } t \downarrow 0. \end{aligned}$$

8.2 The Index formula

Let $A \in \Psi^m_{b,os}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be Fredholm. We will find a formula for the index of A. Define $(A^*A)_0 := A^*A + \pi_A$, where π_A is the orthogonal projection onto the kernel of A and define $(AA^*)_0 := AA^* + \pi_{A^*}$, where π_{A^*} is the orthogonal projection onto the kernel of A^* . Then by Lemma 5.2,

$$e^{-t(A^*A)_0} = (e^{-t} - 1)\pi_A + e^{-tA^*A} \text{ and } e^{-t(AA^*)_0} = (e^{-t} - 1)\pi_{A^*} + e^{-tAA^*}. \tag{86}$$

For each $t > 0$, define $h_0(t) = b\text{–}\mathrm{Tr}(e^{-t(A^*A)_0}) - b\text{–}\mathrm{Tr}(e^{-t(AA^*)_0})$. Then, if $h(t) = b\text{–}\mathrm{Tr}(e^{-tA^*A}) - b\text{–}\mathrm{Tr}(e^{-tAA^*})$, the equations in (86) imply that

$$h_0(t) = (e^{-t} - 1) \operatorname{ind} A + h(t). \tag{87}$$

By Lemma 7.3, $\operatorname{ind} A = h(t) - \frac{1}{2}{}^b\eta_A(t)$, and so, by equation (87), $e^{-t} \operatorname{ind} A = h_0(t) - \frac{1}{2}{}^b\eta_A(t)$. Thus, we have proved the following Lemma.

Lemma 8.3 *Let $A \in \Psi^m_{b,os}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be Fredholm. Then for all $t > 0$,*

$$e^{-t} \operatorname{ind} A = h_0(t) - \frac{1}{2}{}^b\eta_A(t), \tag{88}$$

where $h_0(t) = b\text{–}\mathrm{Tr}(e^{-t(A^*A)_0}) - b\text{–}\mathrm{Tr}(e^{-t(AA^*)_0})$ and ${}^b\eta_A(t) = \sum_{M \in M_k(X), k \geq 1} {}^b\eta_M(t)$, where

$${}^b\eta_M(t) = -\frac{2}{(2\pi)^k} \int_{\mathbb{R}^k} b\text{–}\mathrm{Tr}(\, \mathbf{D}_\tau N_M(A)(\tau)\, N_M(A)(\tau)^{-1}\, N_M(e^{-tAA^*})(\tau)\,)\, d\tau.$$

Let $\operatorname{Re} z >> 0$. Then by properties (43) and (44) of Theorem 4.2, it follows that for $\operatorname{Re} z >> 0$,

$$\begin{aligned} \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} b\text{–}\mathrm{Tr}(\, e^{-t(A^*A)_0}\,)\, dt &= \frac{1}{\Gamma(z)} \int_0^\infty {}^b\!\!\int t^{z-1} e^{-t(A^*A)_0}|_{\Delta_b}\, dt \\ &= {}^b\!\!\int \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-t(A^*A)_0}|_{\Delta_b}\, dt \\ &= {}^b\!\!\int (A^*A)_0^{-z}|_{\Delta_b} \\ &= b\text{–}\mathrm{Tr}(\, (A^*A)_0^{-z}\,) \\ &= {}^b\zeta_{A^*A}(z); \end{aligned}$$

similarly, we have $\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}b\text{–Tr}(\,e^{-t(AA^*)_0}\,)\,dt = {}^b\zeta_{AA^*}(z)$; and, for any $M \in M_k(X)$, $k \in \mathbb{N}$,

$$\begin{aligned}
&\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\int_{\mathbb{R}^k} b\text{–Tr}(\,\mathbf{D}_\tau N_M(A)(\tau)\,N_M(A)(\tau)^{-1}\,e^{-tN_M(AA^*)(\tau)}\,)\,d\tau\,dt \\
&= \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\int_{\mathbb{R}^k} {}^b\!\!\int_M [\mathbf{D}_\tau N_M(A)(\tau)\,N_M(A)(\tau)^{-1}e^{-tN_M(AA^*)(\tau)}]|_{\Delta_b}d\tau\,dt \\
&= \int_{\mathbb{R}^k} {}^b\!\!\int_M [\mathbf{D}_\tau N_M(A)(\tau)\,N_M(A)(\tau)^{-1}(\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}e^{-tN_M(AA^*)(\tau)}dt)]|_{\Delta_b}d\tau \\
&= \int_{\mathbb{R}^k} {}^b\!\!\int_M [\,\mathbf{D}_\tau N_M(A)(\tau)\,N_M(A)(\tau)^{-1}\,N_M(AA^*)(\tau)^{-z}\,]|_{\Delta_b}\,d\tau \\
&= \int_{\mathbb{R}^k} b\text{–Tr}(\,\mathbf{D}_\tau N_M(A)(\tau)\,N_M(A)(\tau)^{-1}\,N_M(AA^*)(\tau)^{-z}\,)\,d\tau \\
&= \int_{\mathbb{R}^k} b\text{–Tr}(\,\mathbf{D}_\tau N_M(A)(\tau)\,N_M(A^*)(\tau)\,N_M(AA^*)(\tau)^{-z-1}\,)\,d\tau.
\end{aligned}$$

Thus, multiplying both sides of equation (88) by $1/\Gamma(z)\cdot t^{z-1}$, and integrating the resulting equation from $t=0$ to $t=\infty$ yields

$$\operatorname{ind} A = {}^b\zeta_{A^*A}(z) - {}^b\zeta_{AA^*}(z) - \frac{1}{2}{}^b\eta_A(z), \tag{89}$$

where ${}^b\eta_A(z) = \sum_{M\in M_k(X),k\geq 1} {}^b\eta_M(z)$, where

$$\begin{aligned}
{}^b\eta_M(z) &= -\frac{2}{(2\pi)^k}\int_{\mathbb{R}^k} b\text{–Tr}(\mathbf{D}_\tau N_M(A)(\tau)\,N_M(A)(\tau)^{-1}\,N_M(AA^*)(\tau)^{-z})\,d\tau \\
&= -\frac{2}{(2\pi)^k}\int_{\mathbb{R}^k} b\text{–Tr}(\mathbf{D}_\tau N_M(A)(\tau)\,N_M(A^*)(\tau)\,N_M(AA^*)(\tau)^{-z-1})\,d\tau.
\end{aligned}$$

Note that Corollary 5.12 implies that the values of ${}^b\zeta_{A^*A}(z)$ and ${}^b\zeta_{AA^*}(z)$ at $z=0$ are ${}^b\zeta_{A^*A}$ and ${}^b\zeta_{AA^*}$ respectively, where ${}^b\zeta_{A^*A}$ and ${}^b\zeta_{AA^*}$ are the constant terms in the expansions, as $t\downarrow 0$, of $b\text{–Tr}(e^{-tA^*A})$ and $b\text{–Tr}(e^{-tAA^*})$ respectively.

Lemma 8.4 *Let $A \in \Psi^m_{b,os}(X,\Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be Fredholm. Let $M \in M_k(X)$, $k \in \mathbb{N}$. Then, ${}^b\eta_M(z)$ extends to be a meromorphic function on all of $\mathbb{C}$, having only simple poles at the points*

$$\{z_j = \frac{n-k-j}{2m}\ :\ j \in \mathbb{N}_0\}.$$

In particular, ${}^b\eta_A(z) = \sum_{M\in M_k(X),k\geq 1} {}^b\eta_M(z)$ extends to be a meromorphic function on all of $\mathbb{C}$, having only simple poles at the points

$$\{z_j = \frac{n-j}{2m}\ :\ j \in \mathbb{N}_0\}.$$

Moreover, ${}^b\eta_A(z)$ is holomorphic at $z=0$, with value ${}^b\eta_A$, where ${}^b\eta_A$ is the b-eta invariant of A.

PROOF: By Lemma 5.10 and Lemma 7.4, ${}^b\eta_M(z)$ extends to be a meromorphic function on all of $\mathbb{C}$, having only simple poles at the points

$$\{z_j = \frac{n-k-j}{2m}\ :\ j \in \mathbb{N}_0\}.$$

Since $\frac{1}{2}{}^b\eta_A(t) = \operatorname{ind} A - h(t)$, and the right hand side of this equation has *no* $\log t$ term in its expansion as $t\downarrow 0$, and with constant term equal to $\frac{1}{2}\,{}^b\eta_A$, by Lemma 5.10, ${}^b\eta_A(z) = \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\,{}^b\eta_A(t)\,dt$ is holomorphic at $z=0$ with value ${}^b\eta_A$. ■

Definition 8.5 Let $A \in \Psi^m_{b,os}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ be Fredholm. Then the *b-eta function of* A is the meromorphic function ${}^b\eta_A(z)$. It extends to be a meromorphic function on all of $\mathbb{C}$, having only simple poles at the points

$$\{z_j = \frac{n-j}{2m} \ : \ j \in \mathbb{N}_0\};$$

and it is holomorphic at $z = 0$, with value ${}^b\eta_A$, the b-eta invariant of A.

Formula (89) gives the following Theorem.

Theorem 8.6 *Let* $A \in \Psi^m_{b,os}(X, \Omega_b^{\frac{1}{2}})$, $m \in \mathbb{R}^+$ *be Fredholm. Then,*

$$\operatorname{ind} A = {}^b\zeta_{A^*A}(z) - {}^b\zeta_{AA^*}(z) - \frac{1}{2}{}^b\eta_A(z),$$

where ${}^b\zeta_{A^*A}(z)$ *and* ${}^b\zeta_{AA^*}(z)$ *are the b-zeta functions of* A^*A *and* AA^* *respectively, and where* ${}^b\eta_A(z) = \sum_{M \in M_k(X), k \geq 1} {}^b\eta_M(z)$ *is the b-eta function of* A*, where*

$$\begin{aligned} {}^b\eta_M(z) &= -\frac{2}{(2\pi)^k}\int_{\mathbb{R}^k} b\text{–}\mathrm{Tr}(\mathbf{D}_\tau N_M(A)(\tau)\, N_M(A)(\tau)^{-1}\, N_M(AA^*)(\tau)^{-z})\, d\tau \\ &= -\frac{2}{(2\pi)^k}\int_{\mathbb{R}^k} b\text{–}\mathrm{Tr}(\mathbf{D}_\tau N_M(A)(\tau)\, N_M(A^*)(\tau)\, N_M(AA^*)(\tau)^{-z-1})\, d\tau. \end{aligned}$$

In particular, at $z = 0$*, we recover the index formula of Theorem 7.6:*

$$\operatorname{ind} A = {}^b\zeta_{A^*A} - {}^b\zeta_{AA^*} - \frac{1}{2}{}^b\eta_A.$$

8.3 Exact operators

Let $A \in \mathrm{Diff}^1_b(X, E, F)$ be exact and Fredholm, where E and F are hermitian vector bundles over X. Then by Lemmas 7.9 and 8.3, for all $t > 0$,

$$e^{-t} \operatorname{ind} A = h_0(t) - \frac{1}{2}{}^b\eta_A(t), \tag{90}$$

where $h_0(t) = b\text{–}\mathrm{Tr}(e^{-t(A^*A)_0}) - b\text{–}\mathrm{Tr}(e^{-t(AA^*)_0})$ and ${}^b\eta_A(t) = \sum_{H \in M_1(X)} {}^b\eta_H(t)$, where for each $H \in M_1(X)$, ${}^b\eta_H(t) = \frac{1}{\sqrt{\pi}} \int_t^\infty s^{-1/2}\, b\text{–}\mathrm{Tr}(\, A_H e^{-sA_H^2}\,)\, ds$. Observe that for $\operatorname{Re} z >> 0$ and $H \in M_1(X)$,

$$\begin{aligned} \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} {}^b\eta_H(t)\, dt &= \frac{1}{\Gamma(z)} \int_0^\infty \int_t^\infty t^{z-1} s^{-1/2}\, b\text{–}\mathrm{Tr}(\, A_H e^{-sA_H^2}\,)\, ds\, dt \\ &= \frac{1}{\Gamma(z)} \int_0^\infty \int_0^s t^{z-1} s^{-1/2}\, b\text{–}\mathrm{Tr}(\, A_H e^{-sA_H^2}\,)\, dt\, ds \\ &= \frac{1}{\Gamma(z)} \int_0^\infty (\int_0^s t^{z-1}\, dt)\, s^{-1/2}\, b\text{–}\mathrm{Tr}(\, A_H e^{-sA_H^2}\,)\, ds \\ &= \frac{1}{z\Gamma(z)} \int_0^\infty s^{z-1/2}\, b\text{–}\mathrm{Tr}(\, A_H e^{-sA_H^2}\,)\, ds \\ &= \frac{\Gamma(z+1/2)}{\Gamma(z+1)} \cdot \frac{1}{\Gamma(z+1/2)} \int_0^\infty s^{(z+1/2)-1}\, b\text{–}\mathrm{Tr}(\, A_H e^{-sA_H^2}\,)\, ds \\ &= \frac{\Gamma(z+1/2)}{\Gamma(z+1)}\, b\text{–}\mathrm{Tr}(\, A_H\, (A_H^2)^{-z-1/2}\,) \\ &= \frac{\Gamma(z+1/2)}{\Gamma(z+1)}\, b\text{–}\mathrm{Tr}(\, \operatorname{sign} A_H \cdot |A_H|^{-2z}\,), \end{aligned}$$

where $|A_H| := (A_H^2)^{1/2}$ and $\operatorname{sign} A_H := A_H \cdot |A_H|^{-1}$. Thus, multiplying both sides of (90) by $1/\Gamma(z) \cdot t^{z-1}$, where $\operatorname{Re} z >> 0$, and integrating the resulting equation from $t = 0$ to $t = \infty$ gives the following Theorem.

Theorem 8.7 *Let $A \in \mathrm{Diff}^1_b(X,E,F)$ be exact and Fredholm. Then,*

$$\operatorname{ind} A = {}^b\zeta_{A^*A}(z) - {}^b\zeta_{AA^*}(z) - \frac{1}{2}{}^b\eta_A(z),$$

where ${}^b\zeta_{A^*A}(z)$ and ${}^b\zeta_{AA^*}(z)$ are the b-zeta functions of A^*A and AA^* respectively, and ${}^b\eta_A(z) = \sum_{H \in M_1(X)} {}^b\eta_H(z)$ is the b-eta function of A, where for each $H \in M_1(X)$,

$${}^b\eta_H(z) = \frac{\Gamma(z+1/2)}{\sqrt{\pi}\,\Gamma(z+1)}\, b\text{–}\mathrm{Tr}(\,\operatorname{sign} A_H \cdot |A_H|^{-2z}\,).$$

The following Corollary is (basically) the original Atiyah-Patoti-Singer Index theorem found in [2].

Corollary 8.8 *Let $A \in \mathrm{Diff}^1_b(X,E,F)$ be exact and Fredholm. Suppose that* $\operatorname{codim} X = 1$. *Then,*

$$\operatorname{ind} A = {}^b\zeta_{A^*A}(z) - {}^b\zeta_{AA^*}(z) - \frac{1}{2}{}^b\eta_A(z),$$

where ${}^b\zeta_{A^*A}(z)$ and ${}^b\zeta_{AA^*}(z)$ are the b-zeta functions of A^*A and AA^* respectively, and ${}^b\eta_A(z) = \sum_{H \in M_1(X)} {}^b\eta_H(z)$ is the b-eta function of A, where for each $H \in M_1(X)$,

$${}^b\eta_H(z) = \frac{\Gamma(z+1/2)}{\sqrt{\pi}\,\Gamma(z+1)} \sum_j (\operatorname{sign}\lambda_j)\,|\lambda_j|^{-2z},$$

where $\{\lambda_j\}$ are the eigenvalues of A_H.

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