Spectra and stable homotopy theory (draft version, first 10 chapters)

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This book provides an elementary, illustrated introduction to stable homotopy theory, or "the study of spectra." It also gives an introduction to all of the modern machinery of homotopy theory, with a focus on how this machinery is used to understand and work with spectra.

The first third of this book (Chapters 1–4) develops the core theory of spectra using only elementary point-set topology and algebraic topology, and the final chapter gives an elementary introduction to spectral sequences. In these chapters, we cover much of the same material from Adams' classical reference [Ada74, III] but from a more modern, and hopefully simpler, point of view.

In the remaining chapters, we describe many of the developments in the subject since Adams' time, including the "higher algebra" of ring spectra and module spectra. In order to do this properly, we give an introduction to all of the modern tools of homotopy theory, including model categories (Chapter 5), simplicial sets (Chapter 7), homotopy colimits (Chapter 8), and ∞-categories (Chapter 10). This book could therefore be used as an introduction to homotopy theory as a whole, even though it is focused on spectra in particular.

Our goal is to make the tools and machinery of this subject accessible to mathematicians and graduate students in a wide variety of fields, not just those with a strong background in algebraic topology and category theory. The book is designed to be read by someone who has just completed a semester- or year-long course on algebraic topology out of [Hat02], or any other book at a similar level. We only assume some background with the fundamental group and covering spaces, homology and cohomology, homotopy groups, and a basic familiarity with categories and functors. We do *not* assume that the reader is already familiar with model categories, ∞-categories, or even simplicial sets.

Our approach may be described as a user's guide to the subject, as it is focused around "how-to:" how to define a map in the homotopy category of spectra, how to check that something is a model category, how to take a homotopy colimit of spectra, and so on. We believe that this is especially important since homotopy theory has accomplished so much in the last few decades, but it is an intimidating subject to get into, primarily because of the depth and complexity of our machinery.

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0.1 Introduction

Algebraic topology is the study of topological spaces by algebraic invariants. What is an algebraic invariant? You've seen them before – an algebraic invariant is a rule that assigns each space to a group, or to some other algebraic object, so that homeomorphic spaces go to isomorphic groups. If you are reading this book, then you have probably already seen:

- the fundamental group $\pi_1(X)$,
- the homotopy groups $\pi_n(X)$,
- the homology groups $H_n(X;G)$, and
- the cohomology groups $H^n(X;G)$.

This list is common and familiar, but it is far from complete. There are many more algebraic invariants of spaces out there. For example:

Example 0.1.1 (Bordism)**.** For a topological space X, define the abelian group $\mathfrak{N}_k(X)$ by taking equivalence classes of k-dimensional closed smooth manifolds M, with continuous "reference maps" $f : M \to X$. Two such manifolds

$$f_0 \colon M_0 \to X, \quad f_1 \colon M_1 \to X$$

are equivalent if there is a cobordism between them. That is, if there is a compact $(k+1)$-dimensional smooth manifold W whose boundary is the disjoint union $M_0 \amalg M_1$, and a map $f : W \to X$ that restricts to $f_0 \amalg f_1$ on the boundary.

The groups $\mathfrak{N}_*(X)$ are an example of an **extraordinary homology theory** – they satisfy all of the Eilenberg-Steenrod axioms for homology except the dimension axiom, which says that the homology of a point is concentrated in degree zero. In fact, $\mathfrak{N}_2(*) = \mathbb{Z}/2$, generated by the closed manifold $\mathbb{RP}^2$, so $\mathfrak{N}_*(*)$ is not concentrated in degree zero.

Example 0.1.2 (Topological K-theory)**.** For a compact topological space X, define the abelian group $KU^0(X)$ by taking isomorphism classes of complex vector bundles over X, and formally completing the monoid of such into an abelian group. Equivalently, we define

$$KU^0(X) = [X, \mathbb{Z} \times BU]$$

where $BU = \operatorname*{colim}_{n\to\infty} BU(n)$ is the classifying space for stable complex vector bundles, and $[-,-]$ denotes homotopy classes of maps. We similarly define

$$KU^{2n}(X) = [X, \mathbb{Z} \times BU], \qquad KU^{2n+1}(X) = [X, U].$$

The groups $KU^*(X)$ are an example of an **extraordinary cohomology theory**, a contravariant theory that satisfies the Eilenberg-Steenrod axioms except the dimension axiom. In fact, $KU^2(*) = \mathbb{Z}$, so $KU^*(*)$ is not concentrated in degree zero.

Now, for ordinary homology and cohomology, we prove the basic properties by working in the category of chain complexes. For instance, to show that homotopic maps $f, g : X \rightrightarrows Y$ induce the same map on homology, we first show that the homotopy h induces a chain homotopy between the maps $f_\#, g_\# \colon C_*(X) \rightrightarrows C_*(Y)$.

The study of chain complexes is called **homological algebra**. This subject has all the tools you need to prove anything you would like to know about ordinary homology.

However, it turns out that homological algebra is not sufficient for extraordinary homology and cohomology. In general, we cannot give a chain complex whose homology is $E_*(X)$, for a given extraordinary homology theory E.

Instead, we have to use a new kind of homological algebra, where the objects are **spectra** instead of chain complexes. This new kind of homological algebra is called **stable homotopy theory**. It's also called higher algebra, spectral algebra, or brave new algebra.

This is a richer theory than classical homological algebra, and it has a far greater reach. The simplest way to illustrate this is to consider how many homomorphisms there are from $\mathbb{Z}/2$ to $\mathbb{Z}/2$. In ordinary algebra, there are two of them, the identity and the zero map. In homological algebra, there are more, because the group

$$\mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$$

gives an extra, "degree-shifting homomorphism" from $\mathbb{Z}/2$ to $\mathbb{Z}/2$.

In stable homotopy, there are many, many more. The homomorphisms from $\mathbb{Z}/2$ to $\mathbb{Z}/2$ in this setting can be organized into the Steenrod algebra $\mathscr{A}$, an infinite-dimensional noncommutative graded $\mathbb{Z}/2$-algebra. Each element of this algebra corresponds to a natural transformation on cohomology $H^*(X; \mathbb{Z}/2)$ with $\mathbb{Z}/2$ coefficients.

So stable homotopy is a much richer theory. The tradeoff for this richness, unfortunately, is that stable homotopy theory is more difficult to learn. Still, spectra and the extraordinary homology theories they represent have become increasingly important in many fields of mathematics, including:

- geometric topology (surgery theory),
- symplectic geometry (Floer homotopy theory),
- algebraic geometry (motivic homotopy theory),
- number theory (algebraic K-theory and higher algebra), and of course
- algebraic topology and homotopy theory.

0.2 What is a spectrum?

If you had to use a single phrase to say what a spectrum is, you might say one of the following. Spectra are:

- topological spaces in which suspension Σ has been inverted,
- CW complexes with negative-dimensional cells,
- abelian groups up to homotopy, or perhaps
- chain complexes with coefficients in the space of maps $S^n \to S^n$.

There are actually two categories of spectra:[1]

Sp	Ho **Sp**
The point-set category of spectra, aka the model category of spectra, and	The homotopy category of spectra, aka the stable homotopy category.

We pass from **Sp** to Ho **Sp** by localizing, taking a certain class of maps called the "stable equivalences" and turning them into isomorphisms. This has the effect of passing to homotopy classes of maps, just like when we form the classical homotopy category of spaces (CW complexes and homotopy classes of maps between them) or the derived category of a ring R (projective chain complexes of R-modules and chain homotopy classes of maps between them).

There are actually different versions of the point-set category **Sp**, that all give the same homotopy category Ho **Sp**. We focus on three of them in this book: sequential spectra (also called prespectra), symmetric spectra, and orthogonal spectra. These objects are not the same up to isomorphism! They are only the same up to equivalence, and therefore they have the same homotopy category.

The category of spectra has a smash product operation $X \wedge Y$. In the analogy between spectra and abelian groups, this plays the role of the tensor product $X \otimes Y$. Using this new tensor product, we can define rings and modules in the world of spectra. Higher algebra is the study of these ring spectra and their modules.

The most fundamental ring spectrum is the sphere spectrum $\mathbb{S}$. Every spectrum is a module over $\mathbb{S}$, in the same way that every abelian group is a module over $\mathbb{Z}$. There is also a ring spectrum $H\mathbb{Z}$, with the property that module spectra over $H\mathbb{Z}$ are essentially the same thing as unbounded chain complexes of abelian groups.

Along the ring homomorphism $\mathbb{S} \to H\mathbb{Z}$, every chain complex can be turned into a spectrum by "restriction of scalars," and every spectrum can be turned into a chain complex by "extension of scalars," or "taking chains with $\mathbb{Z}$ coefficients." This explains the last of our four bullet points above: spectra are what you get when you start with chain complexes and pass from $\mathbb{Z}$ coefficients to the "deeper base ring" $\mathbb{S}$, whose elements are continuous maps between spheres.

[1] Actually, there are three: the ∞-*category of spectra* sits in between these two. The ∞-category is like the homotopy category, except that it also includes "higher" information about homotopies between maps, homotopies between the homotopies, and so on.

0.3 Chapter summaries

Now we are prepared to tell the reader what is in the book, and where to find it.

Chapter 1 recalls basic facts from algebraic topology that we need to know before starting into spectra. Most of this should be review, but the results on basic homotopy colimits are worth looking over.

Chapter 2 develops the category of sequential spectra **Sp**. We prove the basic properties directly, and show how spectra represent extraordinary homology and cohomology theories.

Chapter 3 introduces homotopy categories and derived functors, allowing us to pass the results in Chapter 2 from the category of spectra **Sp** to the stable homotopy category Ho **Sp**.

Chapter 4 introduces the smash product $\wedge$ as a black box, and explains how this makes both spectra **Sp** and the homotopy category Ho **Sp** into symmetric monoidal categories. There is also some discussion of rings and modules in a general symmetric monoidal category, along with duality and traces.

Chapter 5 introduces model categories, and proves that **Sp** is a model category. This is an important technical framework that is needed for the more sophisticated mathematics that occurs in the second half of this book.

Chapter 6 defines symmetric spectra $\mathbf{Sp}^\Sigma$, orthogonal spectra $\mathbf{Sp}^O$, and their smash product. The black-boxed properties from Chapter 4 are proven explicitly here, using the technology of model categories from Chapter 5.

Chapter 7 introduces simplicial sets, simplicial spaces, and more generally simplicial objects in any category. We then define the bar construction $B(X, \mathbf{I}, Y)$, which plays a crucial role in the next couple of chapters.

Chapter 8 uses the bar construction to define homotopy colimits and limits of spectra, and proves that they are the left- and right-derived functors of the ordinary colimit and limit, respectively. We do this more generally in the context of any simplicial model category.

Chapter 9 introduces operads and grouplike E_∞ spaces, which we think of as "abelian groups up to coherent homotopy." Such spaces are equivalent to connective spectra, using the bar construction. This makes the idea precise that spectra are "essentially the same thing" as abelian groups. In particular, this is how every bounded-below chain complex creates a spectrum.

Chapter 10 introduces ∞-categories as another framework for understanding spectra and their homotopy colimits. To prevent this chapter from being 800 pages long, all of the proofs are skipped, but we still give precise definitions and theorem statements.

Chapter 11 develops the higher algebra of ring spectra and module spectra, phrased in both model-category and ∞-category language. A fundamental result is the Morita theorem that says every stable homotopy theory is equivalent to modules over some ring spectrum. In particular, unbounded chain complexes are equivalent to modules over the Eilenberg–Mac Lane spectrum $H\mathbb{Z}$.

Chapter 12 discusses how to construct spectral sequences, with a particular focus on the Serre, homotopy orbits, and Adams spectral sequences. We don't go especially far into computational techniques in this book, especially those involving the Adams spectral sequence and its generalizations – see e.g. [Rav86] for more in that direction.

0.4 Why study spectra?

What are spectra good for?

We've already seen that spectra give us a framework for studying algebraic invariants of spaces, like bordism and K-theory. More generally, spectra allow us to work with extraordinary homology theories. These theories are an important tool in homotopy theory, because in many situations ordinary homology $H_*(X)$ is just not powerful enough.

You see, the problem with ordinary homology $H_*(X)$ is that it has too few groups. It only has a single group H_0 when X is a point. This is great when we're trying to distinguish spaces up to homotopy equivalence, but it's terrible when we're trying to count homotopy classes of based maps, $[X,Y]_*$.

For instance, the set of homotopy classes $[S^3,S^2]_*$ is nonzero. But ordinary homology can't see that, because every map $S^3 \to S^2$ must be zero on homology. Unpacking the reason why, it is precisely because the homology of a sphere is concentrated in a single degree. If we pass to *extraordinary* homology, then a map $S^3 \to S^2$ induces a map

$$E_3(S^3) \longrightarrow E_3(S^2)$$

which could be (and in many cases is) nonzero.

Now, there are also limits to what extraordinary homology can see. Since suspension is an isomorphism on extraordinary homology, we can't see the difference between maps $[X,Y]_*$ and the direct limit of these maps under the suspension operation:

$$\{X,Y\} = \operatorname{colim}\Big([X,Y]_* \longrightarrow [\Sigma X, \Sigma Y]_* \longrightarrow [\Sigma^2 X, \Sigma^2 Y]_* \longrightarrow \cdots \Big).$$

These are the **stable maps** from X to Y. In the stable homotopy category $\mathrm{Ho}\,\mathbf{Sp}$, these are exactly the maps that appear from X to Y. All of the "unstable" information from the finite stages of this system has been thrown out.

So extraordinary homology can only see stable maps $\{X, Y\}$, not the unstable maps $[X, Y]_*$. Still, the stable maps contain a lot of information. And they are much easier to compute, in part because we have long exact sequences for ths stable maps, just like the long exact sequence for homology. So for instance, the **stable homotopy groups of spheres**

$$\pi_k(\mathbb{S}) = \{S^k, S^0\}$$

are far easier to compute, and far better understood, than the unstable groups $\pi_{k+n}(S^n)$.[2]

Philosophically, the stable maps are a "linear approximation" of the unstable ones. Or, to put it differently:

Spectra are to spaces what linear algebra is to algebra.

And in practice, this is the primary reason why homotopy theorists use spectra – they are *easier than spaces*. That might sound strange to you if you're just starting out. We need so much technology to define and work with spectra – how could they possibly be easier? But this is part of a familiar pattern in mathematics. Sometimes the more complicated definition gives the object that's easier to understand. For instance, fields are much easier to classify than groups, even though (or perhaps because) they have a longer definition. Spectra are the same way. It takes much more time to define them, but much less time to compute with them.

So spectra are easier, great. Don't spaces still matter? Aren't we giving up by leaving spaces behind and settling for spectra? It turns out that we aren't, because it's possible to go back and get information about the unstable maps, starting with the stable maps. The stable maps are the linear approximation, but then there is also a "quadratic" approximation, a "cubic" approximation, and so forth. And the amazing thing is that the difference between each of these approximations is *also governed by spectra*. This is like how in algebra, a quadratic or cubic function of several variables can still be described using linear and multilinear objects (matrices and tensors), even though the function itself is nonlinear. We still use the language of linear algebra even when we're describing nonlinear things. So we use the language of spectra even when we're describing unstable homotopy theory.

Using spectra to count homotopy classes of maps is a big application, perhaps the main application, but there are many others. One of the largest and oldest of these applications comes from differential topology. The **Pontyagin-Thom isomorphism** tells us that the homotopy groups of certain spectra classify manifolds up to cobordism. As a special case, the stable maps $\pi_d(\mathbb{S}) = \{S^d, S^0\}$ are isomorphic to the set of closed framed d-dimensional manifolds, up to cobordisms that respect the framing. (A framing is a trivialization of the tangent bundle, or the normal bundle in $\mathbb{R}^n$, up to adding trivial line bundles.)

This means that any calculation of stable homotopy groups of spheres gives as an immediate corollary a statement about framed manifolds. Our algebraic results have geometric

[2] But even computing the *stable* homotopy groups is still an open problem!

consequences. More generally, variants of the Pontryagin-Thom isomorphism result play a central role in surgery theory, and therefore in the classification of high-dimensional manifolds.

It's also important to remember that even if we set all of the applications to the side, spectra are still fascinating on their own. They are a world where topology and algebra merge, having some of the best features of chain complexes and of spaces. The algebra of spectra is richer than classical algebra. Taking homology with "exotic" coefficients in a spectrum is more powerful than being limited to "ordinary" coefficients in an abelian group. This richness has striking and dramatic consequences in many fields of mathematics, far beyond the most obvious applications to topological spaces and continuous maps between them.

0.5 History

The history of spectra is somewhat backwards – the homotopy category Ho **Sp** was invented before the point-set category **Sp**.

The homotopy category was first defined by Lima in [Lim58]; see also [SW53, SW55] for early works anticipating the concept, and [Ada74] for the state of the art in the 1970s. The original motivations for the homotopy category of spectra included

- using stable maps $\{X, Y\} = \operatorname{colim}_{n\to\infty} [\Sigma^n X, \Sigma^n Y]_*$ as a "linear approximation" to homotopy classes of maps, in order to compute new homotopy groups of spheres,
- being able to "dualize" a topological space X in the way one dualizes a vector space $V^* = \mathrm{Hom}_k(V, k)$,
- classifying smooth manifolds up to cobordism, and
- working with extraordinary homology and cohomology theories.

On the other hand, the point-set category **Sp** took longer to develop. One might say the basic foundations only came into place in the 1990s with the papers [HSS00, EKMM97, MMSS01, MM02]. Before then, it was common to work with the stable homotopy category, and the models for it were complicated and clunky. Peter May once remarked that the situation was like knowing what the derived category of a ring is, without knowing what an R-module is.

The problem wasn't how to define spectra, it was how to define the *smash product*. Like the tensor product of vector spaces, the smash product is an operation that combines two or more spectra together into a single spectrum, and it is supposed to be associative and commutative up to isomorphism. In other words, it makes spectra into a symmetric

monoidal category. And the problem is that no one knew how to do this on the point-set level (it was even believed to be impossible) until the groundbreaking works of Hovey, Shipley, and Smith [HSS00], and Elmendorf, Kriz, Mandell, and May [EKMM97].

In the next few years these foundations were fleshed out by the work of many people, see e.g. [HPS97, SS00, MMSS01, SS03b], giving a fully-developed theory that could be gleaned from research papers. In the past two decades, this theory has been worked over and distilled into more expository works, including the present one.

The theory of spectra has seen many new developments and applications since then. It would be impossible to give a complete survey of these developments, but to name a few:

- The smash product of spectra allowed for an era of "trace methods" in algebraic K-theory, greatly increasing our understanding and allowing for new calculations that connect to both differential topology and to number theory.
- Goodwillie and others further developed the theory of functor calculus, explaining how spectra emerge in the process of forming "linear approximations," "quadratic approximations," and so on, to functors on spaces.
- The theory of ∞-categories emerged as a new framework for understanding homotopy theoretic constructions and extending their reach to new settings. Lurie characterized the ∞-category of spectra and its smash product by a universal property.
- Chromatic homotopy theory saw spectacular further development, including the creation of the ring spectrum of topological modular forms, allowing for further computations and insight into the homotopy groups of spheres, arguably the most fundamental problem in algebraic topology.
- Motivic homotopy theory and stable motivic homotopy theory were created, building new connections between the theory of spectra and algebraic geometry, important even to those who are only concerned with one of the two subjects.
- Floer homotopy theory created an increasingly important connection between stable homotopy theory and symplectic geometry.
- The Kervaire invariant one problem was solved, and along with it came a renaissance in equivariant stable homotopy theory (G-spectra), built on the theory of orthogonal spectra discussed in this book. (See [HHR21].)
- Higher algebra now plays an increasingly important role in classical algebra and number theory, building on the fundamental idea that spectra are the next step of the progression after chain complexes.

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The current work is not the first book on spectra to appear, and the existing books and expository works have had a significant influence on this one. Frank Adams' classic work [Ada74] has been the standard reference for decades. On a technical level it is now out of date, but Adams' writing never goes out of style. Though it is not focused on spectra, one of the chapters in Switzer's book [Swi75] also had a large influence on the author's perspective. Stefan Schwede's project [Sch07] was the book the author used in graduate school, and it continues to be the only place you can go to find a careful treatment of the finer points of symmetric spectra. Urs Schreiber's online book [Sch16] has been a valuable reference to the author during the writing of the present book. David Barnes was kind enough to share drafts of [BR20] while it was being written. The author has also read expository works by Agnes Beaudry and Jonathan Campbell [BC18], Kirsten Wickelgren [Wic15], Bruno Stonek [Sto22], and Bastiaan Cnossen [Cno25], along with expository tidbits by many other mathematicians, all of which have informed the perspective of this book.

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Cary Malkiewich

Chapter 1

Preliminaries

This chapter is a fast review and a reference for concepts from algebraic topology and homotopy theory. We give only definitions and theorem statements, no proofs. More details can be found in standard algebraic topology texts such as [Hat02, May99, tD08, Swi75, Spa66]. We expect that the reader has seen most of this already, but this chapter might be helpful in filling in some gaps.

1.1 Basic operations on topological spaces

1.1.1 Elementary topology, CGWH spaces

A **topological space** X is a set equipped with a choice of which subsets are "open," subject to the axioms that open sets are preserved by arbitrary unions (including empty unions, hence $\varnothing$ is open) and finite intersections (including empty intersections, hence X itself is open). A set is closed precisely when its complement is open.

A map $f : X \to Y$ is **continuous** if the preimage of every open set $U \subseteq Y$ is open in X. f is a **homeomorphism** if it is a bijection and its inverse is also continuous. f is a **topological inclusion** if it is a homeomorphism onto its image; when the image is open or closed we call f a **open inclusion**, respectively a **closed inclusion**. A **retraction** of $i : A \to X$ is a map $r : X \to A$ such that $r \circ i = \mathrm{id}_A$. In this case we call A a **retract** of X.

There exist continuous bijections that are not homeomorphisms – for instance, the map from $\mathbb{R}$ with the discrete topology to $\mathbb{R}$ with the usual topology, or the map that winds the half-open interval $[0, 1)$ bijectively around the circle S^1. However, if X is compact and Y is Hausdorff then every continuous bijection $X \to Y$ is a homeomorphism, and therefore also every continuous injection is a topological inclusion.

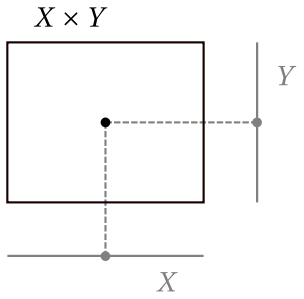
A topological space X is **compact** if every open cover has a finite subcover, **connected**

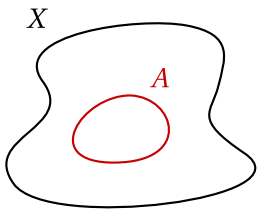
if there does not exist a separation $X = A \amalg B$ into nonempty disjoint open sets, and **Hausdorff** if every pair of points can be separated by a pair of disjoint open sets.

A topological space X is **compactly generated** if for each subset $C \subseteq X$, C is closed iff for every compact Hausdorff space K and continuous map $f : K \to X$, $f^{-1}(C)$ is closed in K. Furthermore X is **weak Hausdorff** if for each such f, the image $f(K)$ is closed. Open and closed subspaces of $\mathbb{R}^n$, along with just about every other space you usually encounter, are compactly generated weak Hausdorff (CGWH). For those that aren't, there are standard operations that make them CGWH. See [Lew78, App A] and [Str09] for a detailed treatment of the properties of these spaces. We will recall any relevant properties as they arise.

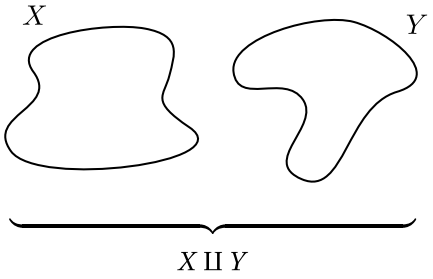
Convention: All spaces in this book are compactly generated weak Hausdorff (CGWH).

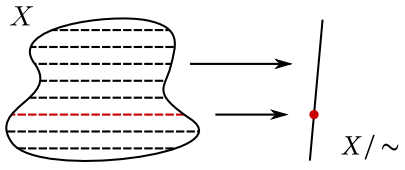
A **product** $X \times Y$ is the Cartesian product with a topology that ensures that a map $Z \to X \times Y$ is continuous iff the two factors $Z \to X$ and $Z \to Y$ are continuous. A **subspace** $A \subseteq X$ is a subset topologized so that a map $Z \to A$ is continuous iff the resulting map $Z \to X$ is continuous.





Dually, a **coproduct** is the disjoint union $X \amalg Y$ topologized so that $X \amalg Y \to Z$ is continuous iff each of the two component maps $X \to Z$ and $Y \to Z$ is continuous. A **quotient** $X \to Y$ is a surjective map (or what is the same thing, an equivalence relation on X), with Y topologized so that $Y \to Z$ is continuous iff the resulting map $X \to Z$ is continuous. The universal properties of these constructions are shown in Figure 1.1.1.





The product $X \times Y$ and subspace $A \subseteq X$ have the same underlying set whether we take them in CGWH spaces or in plain-vanilla topological spaces, but the topologies are a little different: the version in CGWH spaces sometimes has more open sets. The coproduct

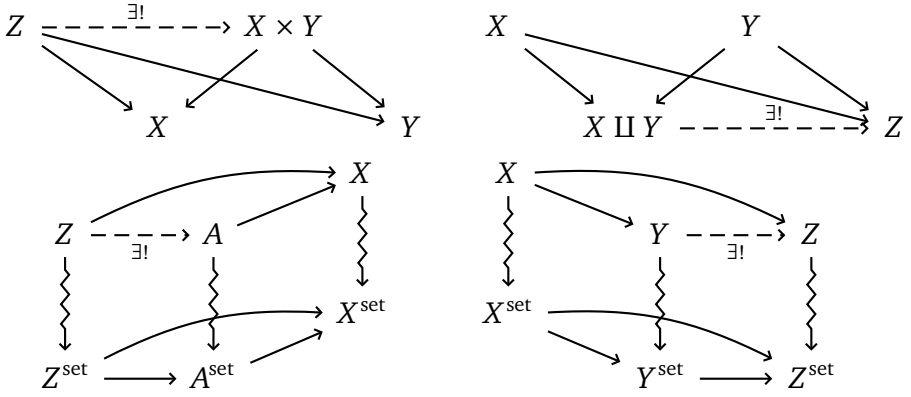


Figure 1.1.1: The defining properties of products, coproducts, subspaces, and quotients.

$X \sqcup Y$ has the same topology in both settings. The quotient is different, even on the underlying set, because if the equivalence relation on X is not closed as a subspace of $X \times X$ (with the CGWH product), then we have to take its closure first before modding out by the relation. This ensures the quotient is CGWH.

Example 1.1.2. The quotient of $\mathbb{R}$ that identifies all of the rational numbers $\mathbb{Q}$ to a point, when carried out in plain topological spaces, gives $\{*\} \sqcup (\mathbb{R} \setminus \mathbb{Q})$ with a strange topology. In CGWH spaces, the quotient is just the one-point space $*$.

We can similarly take a product of three spaces, or countably many spaces, or uncountably many spaces. The Cartesian product of arbitrarily many spaces $\prod_{s \in S} X_s$ is topologized so that a map into this product is the same thing as a continuous map into each of the spaces X_s. This allows us to see that the two-fold product is associative up to isomorphism,

$$(X \times Y) \times Z \cong X \times (Y \times Z), \tag{1.1.3}$$

because both sides have the same universal property as the three-fold product $X \times Y \times Z$. The same statements hold for the coproduct $\coprod_{s \in S} X_s$ as well.

1.1.2 Colimits and limits

Topological spaces (that are CGWH) form a **category**, that we refer to as **Top**. This means there is a collection of objects (the spaces), for each pair of objects X, Y there is a set of "morphisms" or "maps" $f : X \to Y$ (the continuous maps), and there is a rule for how to compose morphisms $f : X \to Y$ with morphisms $g : Y \to Z$, giving $g \circ f : X \to Z$. This must be associative and have identity morphisms:

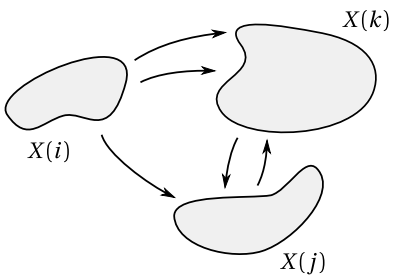
$$(f \circ g) \circ h = f \circ (g \circ h), \qquad \mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X.$$

In a category **C**, the collection of morphisms $f : X \to Y$ between any two objects X and Y must be a set, not a proper class, but the collection of objects $X \in \operatorname{ob}\mathbf{C}$ is allowed to be a proper class. We say that **C** is a **small category** if the collection of objects is a set as well. (It is usually harmless to ignore this distinction.)

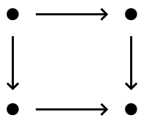
Let **Set** be the category in which the objects are sets, and the morphisms are functions $f : X \to Y$. The rule **Top** $\to$ **Set** that assigns each space X to its underlying set X^{set} is an example of a **functor**. That means that it assigns each object to an object (each space X goes to a set X^{set}), between two objects it assigns each morphism $f : X \to Y$ to a morphism $X^{\text{set}} \to Y^{\text{set}}$ (forget that f is continuous), and this respects composition and identity maps. A functor is very much like a homomorphism of groups, thinking of the maps in the category as the group elements. See [Mac71, Rie17] for more discussion of categories in general.

Those readers who are already category-lovers might recognize the definitions in Figure 1.1.1 as universal properties for products, cartesian morphisms (over the forgetful functor to sets), coproducts, and cocartesian morphisms.

Using these elementary operations, we can define all limits and colimits of diagrams of spaces. A **diagram** of spaces is informally a collection of topological spaces and continuous maps between them. Formally, it is a functor $X : \mathbf{I} \to \mathbf{Top}$ where **I** is any small category. We call **I** an **indexing category**. So we get a space $X(i)$ for each object $i \in \operatorname{ob}\mathbf{I}$, and a map $X(f) : X(i) \to X(j)$ for each morphism $f : i \to j$, respecting composition and sending each identity map $i \to i$ to the identity map of $X(i)$.

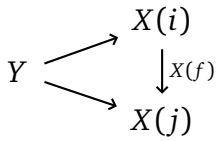


For instance, if **I** has four objects and just enough morphisms to form a commuting square



then a diagram on **I** is the same thing as a commuting square of spaces. We illustrate some common choices of indexing category **I** in Figure 1.1.4, see also Exercise 1.7.10. In some of these cases the colimit or limit is just one of the spaces in the diagram (Exercise 1.7.2).

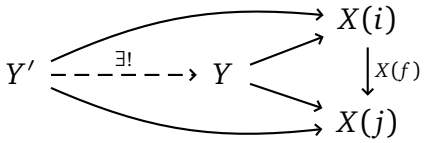
The **limit** of a diagram $X : \mathbf{I} \to \mathbf{Top}$ is the universal space Y that is equipped with maps $Y \to X(i)$ for all $i \in \operatorname{ob}\mathbf{I}$, commuting with each of the maps of the diagram:



Indexing category	Colimit is called...	Limit is called...
$\varnothing$	initial object	terminal object
• •	coproduct	product
• ⇉ •	coequalizer	equalizer
• → • ↓ •	pushout	(trivial)
• ↓ • → •	(trivial)	pullback
• → • → • → ⋯	sequential colimit	(trivial)
⋯ → • → • → •	(trivial)	sequential limit

Figure 1.1.4: Some common indexing categories and the names for their (co)limits.

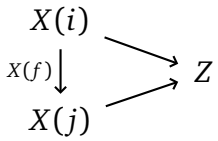
In other words, if Y' is another such space, then there is a unique map $Y' \to Y$ commuting with the map to $X(i)$ for each i:



The limit of a diagram in **Top** can be constructed as the product $\prod_{i\in \mathrm{ob}\,\mathbf{I}} X(i)$, restricted to the subspace of those points that agree along all of the maps in the diagram:

$$\lim_{i\in\mathbf{I}} X(i) = \left\{ x_i \in X(i) \quad \forall i \in \mathrm{ob}\,\mathbf{I} \;:\; \forall f : i \to j, \quad X(f)(x_i) = x_j \right\}. \tag{1.1.5}$$

The **colimit** of a diagram $X : \mathbf{I} \to \mathbf{Top}$ is the universal space Z that is equipped with maps $X(i) \to Z$, commuting with all the maps of the diagram:



The colimit of a diagram in **Top** be constructed as the coproduct (disjoint union) of the $X(i)$, modulo an identification along each morphism $X(f)$:

$$\operatorname*{colim}_{i\in\mathbf{I}} X(i) = \left(\coprod_{i\in \mathrm{ob}\,\mathbf{I}} X(i)\right) \Big/ \big(x \sim X(f)(x) \quad \forall f : i \to j,\ x \in X(i)\big). \tag{1.1.6}$$

In particular, a point in a colimit is given by a point in *one* space $X(i)$, while a point in a limit requires a point in *every* X_i. For this reason, colimits are often simpler and easier to understand than limits.

Remark 1.1.7. Of course, we can define the limit or colimit of a diagram in any category **C** in the same way. Limits and colimits in **Set** are defined by the same formulas as for **Top**, while in the category of abelian groups **Ab** the colimit is different – see Exercise 1.7.6.

Recall from Figure 1.1.4 that a pushout is a colimit along a diagram of the form $X \leftarrow A \rightarrow Y$, as shown below on the left. We call the pushout $X \cup_A Y$. Similarly a pullback is a limit of a diagram of the form shown below on the right, and we call the pullback $X \times_B Y$.

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \\ Y & & \end{array} \qquad\qquad \begin{array}{ccc} & & X \\ & & \downarrow \\ Y & \longrightarrow & B \end{array}$$

The further special case $X \cup_A \{*\}$ is called the quotient or **cofiber** X/A. Note that $X/\varnothing = X_+$ is X with a disjoint point added. Dually, a pullback of the form $X \times_B \{*\}$ is called the **fiber** X_b, because it is the set of all points of X that map to a given point $b \in B$.

Remark 1.1.8. In CGWH spaces, limits commute with the forgetful functor to sets. In other words, you can define the limit of a diagram of spaces, by taking the limit of the underlying sets first, then giving that limit a topology. For colimits, this is not always true, because when we take the quotient in the definition of the colimit, the quotient in CGWH spaces might use a larger equivalence relation than the quotient in ordinary topological spaces. However, the underlying sets do agree for the following kinds of colimits (as we defined in Figure 1.1.4):

- coproducts,
- pushouts in which one map is a closed inclusion, and
- sequential colimits in which every map is a closed inclusion.

So in these cases, you can argue in terms of the underlying set. Outside of these cases, you should use the universal property instead. (It's usually easier to use the universal property anyway.)

Example 1.1.9. Let D^n be the closed unit disc in $\mathbb{R}^n$ and let S^{n-1} the closed unit sphere. Then if we take the pushout of two copies of D^n along the closed inclusion $S^{n-1} \subset D^n$, the result is homeomorphic to S^n:

$$D^n \cup_{S^{n-1}} D^n \xrightarrow{\cong} S^n.$$

To prove this, you define the above map using the universal property of the pushout, and show that it is a continuous bijection. The pushout is compact, since it is a finite union of images of compact spaces, and S^n is Hausdorff, so the map must be a homeomorphism.

Definition 1.1.10. Given a commuting square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow[g]{} & D, \end{array}$$

we say it is a **pushout square** if the induced map $B \cup_A C \to D$ is a homeomorphism. We also say that g is the pushout of the map f, and sometimes place the symbol $\ulcorner$ inside the square.

Similarly, this is a **pullback square** if the induced map $A \to B \times_D C$ is a homeomorphism. In this case we say that f is the pullback of the map g, and sometimes place the symbol $\lrcorner$ inside the square.

1.1.3 Cell complexes and CW complexes

Definition 1.1.11. A **cell complex** is a space X that is the sequential colimit of a sequence of spaces

$$\varnothing = X^{(-1)} \longrightarrow X^{(0)} \longrightarrow X^{(1)} \longrightarrow X^{(2)} \longrightarrow \cdots \longrightarrow X^{(n-1)} \longrightarrow X^{(n)} \longrightarrow \cdots.$$

We usually use "$n \to \infty$" notation for a sequential colimit:

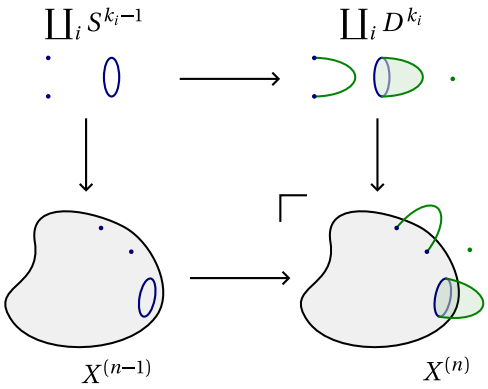
$$X = \operatorname*{colim}_{n\to\infty} X^{(n)}.$$

We also call $X^{(-1)} \to X$ the **countable composition** of the smaller maps $X^{(n-1)} \to X^{(n)}$ for $n \geq 0$. Furthermore, we ask that each of the maps $X^{(n-1)} \longrightarrow X^{(n)}$ is a pushout of a coproduct of "cells" $S^{k-1} \to D^k$, with varying $k \geq 0$:

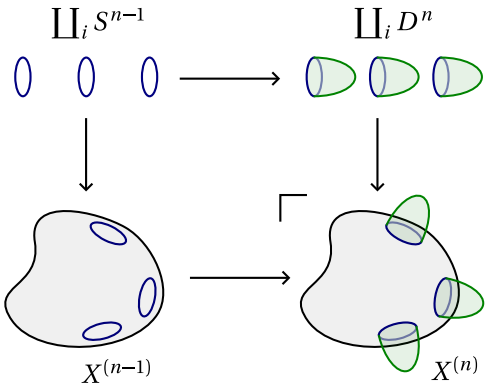
$$\begin{array}{ccc} \coprod_i S^{k_i-1} & \longrightarrow & \coprod_i D^{k_i} \\ \downarrow & & \downarrow \\ X^{(n-1)} & \longrightarrow & X^{(n)}. \end{array}$$

In the case that $k = 0$, the sphere S^{0-1} is defined to be the empty set, and D^0 is a point. So, attaching a 0-cell has the effect of taking disjoint union with a point.

Pictorially, we can imagine this process as follows.



We say that X is a **CW complex** if $k_i = n$ for all i. In other words, we only attach cells of dimension n at stage n:



A **relative cell complex** $A \to X$ is defined in the same way, except that we take $X^{(-1)} = A$. We sometimes abbreviate this by saying that "$A \to X$ is a cell complex."

A cell complex always has the **cellular topology**: a set in X is open precisely when its preimage in each closed cell $D^n \to X$ and its preimage in A are both open. Equivalently, a map $X \to Y$ is continuous iff the resulting maps $D^n \to Y$ and $A \to Y$ are all continuous. This is just a rearrangement of the universal properties for the above colimit, pushouts, and coproducts.

Proposition 1.1.12. *A product $X \times Y$ of CW (or cell) complexes is a CW (or cell) complex with an $(m+n)$-cell for each pair of an m-cell in X and an n-cell in Y.*

Remark 1.1.13. This proposition is actually not true in ordinary spaces, only in CGWH spaces. The underlying set is always correct, but the CGWH product topology is the one that agrees with the cellular topology. This is one reason why the product in CGWH spaces

is better than the ordinary product topology. (Although, if at least one of the complexes has only finitely many cells, then both topologies are correct.)

1.1.4 Based spaces

A **based space** is a space X with a chosen basepoint $x_0 \in X$. A based map $f : X \to Y$ is a (continuous) map sending x_0 to y_0. We often use the symbol $*$ to refer to the basepoint of every based space, and also to refer to a generic one-point space. So $f : X \to Y$ is based if it sends $* \in X$ to $* \in Y$.

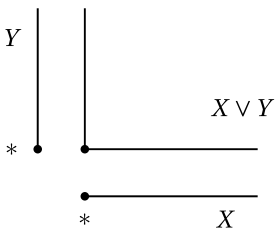
A based cell complex is just a cell complex relative to the one-point space $*$. This is the same thing as a cell complex with a chosen 0-cell as the basepoint.

Definition 1.1.14. If X is any unbased space, we let X_+ denote the same space with a **disjoint basepoint**. That is, it is a disjoint union of X and an additional point $*$, and we select $*$ as the new basepoint.

Based spaces and based maps form a category $\mathbf{Top}_*$. So we can define a limit or a colimit of a diagram of based spaces and based maps, using the same universal property as before. But beware! The space we get is formed using the universal property in $\mathbf{Top}_*$, not in $\mathbf{Top}$, so it doesn't have to have the same formula as before!

For the limit, it turns out that we do get the same space. So the limit of a diagram of based spaces is still given by the formula in (1.1.5). This space has a basepoint, given by choosing the basepoint $* \in X(i)$ for all i.

For colimits, we don't always get the same space. The colimit given in (1.1.6) may have multiple basepoints, and they have to be quotiented together in order to get the colimit of the $X(i)$ in the category of based spaces $\mathbf{Top}_*$.



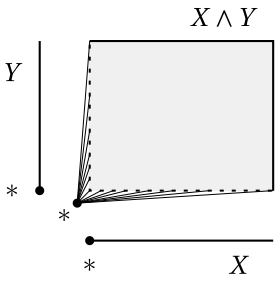
For example, the coproduct in based spaces is not the disjoint union, because that has two basepoints when it really should have one. Instead, the coproduct is the **wedge sum**

$$X \vee Y = (X \amalg Y)/(x_0 \sim y_0).$$

The general formula for a colimit in based spaces is the same as (1.1.6) but with the disjoint unions replaced by wedge sums:

$$\operatorname*{colim}^{(b)}_{i \in \mathbf{I}} X(i) = \left(\bigvee_{i \in \mathrm{ob}\,\mathbf{I}} X(i) \right) \Big/ \big(x \sim f(x) \quad \forall f : i \to j,\ x \in X(i) \big). \tag{1.1.15}$$

Usually this is just written as colim, without the (b) decoration. But remember that the based and unbased versions of the colimit are not always homeomorphic. See Exercise 1.7.4.



Definition 1.1.16. The wedge sum $X \vee Y$ is homeomorphic to a subspace of the product $X \times Y$, consisting of the union of the two subspaces $X \times \{y_0\}$ and $\{x_0\} \times Y$. The quotient $(X \times Y)/(X \vee Y)$ is called the **smash product** $X \wedge Y$. You should think of this as the "Cartesian product of the complements of the basepoints." It is illustrated on the left.

1.1.5 Mapping spaces

For any two (CGWH) topological spaces X and Y, the **mapping space** $\mathrm{Map}(X,Y) = Y^X$ is the set of continuous maps $X \to Y$, topologized so that for every space K, the "currying" rule

$$F(-,-) \mapsto (k \mapsto F(k,-))$$

gives a bijection between the set of continuous maps $K \times X \to Y$ and the set of continuous maps $K \to \mathrm{Map}(X,Y)$. (It isn't necessary to say what the topology is! This property determines the topology uniquely.)

In fact, this correspondence is continuous, giving a homeomorphism

$$\mathrm{Map}(K \times X, Y) \cong \mathrm{Map}(K, \mathrm{Map}(X,Y)). \tag{1.1.17}$$

If X and Y have basepoints then the based mapping space $\mathrm{Map}_*(X,Y) = (Y,*)^{(X,*)}$ is the closed subspace of $\mathrm{Map}(X,Y)$ of all maps preserving the basepoint. The same rule gives a bijection betweeen based (continuous) maps $K \wedge X \to Y$ and based (continuous) maps $K \to \mathrm{Map}_*(X,Y)$. Again, this passes to a homeomorphism

$$\mathrm{Map}_*(K \wedge X, Y) \cong \mathrm{Map}_*(K, \mathrm{Map}_*(X,Y)). \tag{1.1.18}$$

There are a few useful identities involving these. For instance, maps out of a coproduct becomes a product of mapping spaces,

$$\mathrm{Map}\left(\coprod_\alpha X_\alpha, Y\right) \cong \prod_\alpha \mathrm{Map}(X_\alpha, Y).$$

See Exercises 1.7.27 and 1.7.28.

1.2 Homotopy, cofibrations, and fibrations

Now that we have covered the elementary constructions, we can begin to do homotopy theory.

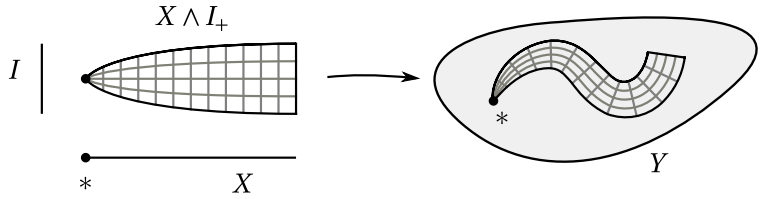
1.2.1 Homotopy

Given two continuous maps $f, g : X \rightrightarrows Y$, a **homotopy** from f to g is a (continuous) map

$$h : X \times I \to Y$$

such that $h(-,0) = f$ and $h(-,1) = g$. This is equivalent to giving a map $\tilde{h} : X \to Y^I$, in other words a map from X to the space of paths in Y. If X and Y are based, a **based homotopy** is one in which $h(*,t) = *$ at all times t.

Equivalently, a based homotopy is a based map $h : X \wedge I_+ \to Y$, where I_+ is the interval with a disjoint basepoint added. It is an exercise to check that $X \wedge I_+$ is homeomorphic to the quotient $(X \times I)/(\{*\} \times I)$, so that a based homotopy looks like the illustration below.



"Homotopy" is an equivalence relation on the set of maps from X to Y. The quotient set is called $[X, Y]$, the set of **homotopy classes** of maps from X to Y. If X and Y are based, we let $[X, Y]_*$ denote the equivalence classes of based maps up to based homotopy. There is a forgetful map of sets $[X, Y]_* \to [X, Y]$.

Example 1.2.1. The forgetful map $[S^1, S^1]_* \to [S^1, S^1]$ is a bijection. Both of these sets are in bijection with $\mathbb{Z}$, by taking the winding number of the map $S^1 \to S^1$.

The forgetful map $[S^1, S^1 \vee S^1]_* \to [S^1, S^1 \vee S^1]$ is not injective. It can be identified with the map from the free group $\mathbb{Z} * \mathbb{Z}$ to the set of conjugacy classes in $\mathbb{Z} * \mathbb{Z}$.

The forgetful map $[S^1, S^1 \amalg S^1]_* \to [S^1, S^1 \amalg S^1]$ is not surjective, if we pick a basepoint on one of the two copies of S^1. It can be identified with the inclusion of $\mathbb{Z}$ into the disjoint union $\mathbb{Z} \amalg \mathbb{Z}$.

Homotopy respects composition: if $f \sim f'$ and $g \sim g'$ then $gf \sim g'f'$. Therefore the sets $[X, Y]$ form the morphisms of a category, called the classical homotopy category $h\mathbf{Top}$. Isomorphisms in $h\mathbf{Top}$ are the same thing as **homotopy equivalences**, i.e. pairs of maps $f : X \to Y$, $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are each homotopic to an identity map.[1]

Similarly the sets $[X, Y]_*$ are the morphisms of a category, the classical based homotopy category $h\mathbf{Top}_*$. An isomorphism in $h\mathbf{Top}_*$ is a **based homotopy equivalence**, which is like a homotopy equivalence but everything has to be basepoint-preserving.

[1] In general, an isomorphism in a category is a morphism $f : X \to Y$ such that there is another morphism $g : Y \to X$ with $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$. In $h\mathbf{Top}$, the notion of "equals" becomes "homotopic," so an isomorphism in $h\mathbf{Top}$ is a homotopy equivalence.

We typically only consider these categories when we restrict to the objects that are CW complexes. This is an example of a **full subcategory**: we take some subset of the objects, and we take all of the morphisms between them. Let $h\mathbf{CW} \subseteq h\mathbf{Top}$ and $h\mathbf{CW}_* \subseteq h\mathbf{Top}_*$ refer to the full subcategories in which we only take the CW complexes.

If A is a subspace of X, we say that X **deformation retracts** onto A if there is a retraction $r : X \to A$ and a homotopy $r \sim \mathrm{id}_X$ as maps $X \to X$, that is the constant homotopy on A. In other words, the identity map $X \to X$ can be deformed to r while doing nothing along the subspace A. A deformation retraction is always a homotopy equivalence. A retraction does not have to be a homotopy equivalence: for instance, there is a retraction $S^1 \to *$.

1.2.2 Cofibrations

Definition 1.2.2. A **cofibration** is a map of topological spaces $i : A \to X$ with the homotopy extension property: for any map $f : X \to Y$, and any homotopy of the map $f \circ i$ on A, the homotopy can be extended to a homotopy of f on X. In other words, for each diagram as below, a dashed lift exists.

$$\begin{array}{ccc} (A \times I) \cup_{A \times \{0\}} (X \times \{0\}) & \longrightarrow & Y \\ \downarrow & \nearrow & \\ X \times I & & \end{array}$$

It turns out that for any cofibration $i : A \to X$, the image $i(A) \subseteq X$ has a closed neighborhood $C \subseteq X$ and a retraction $r : C \to A$, see Exercise 1.7.14.

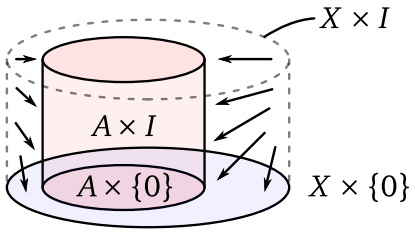
Lemma 1.2.3. *[Str09, Cor 2.29] In CGWH spaces, every map $A \to Y$ that has a retraction $Y \to A$ is a closed inclusion. Therefore every cofibration is a closed inclusion.*[2]

So when we talk about cofibrations, there is no loss of generality if we just assume that i is the inclusion of a closed subspace $A \subseteq X$.

If we assume that A is a subspace of X, we can also rephrase the condition of being a cofibration. Notice that to prove the homotopy extension property for all Y, it suffices to prove it when $Y = (A \times I) \cup_{A \times \{0\}} (X \times \{0\})$.

Lemma 1.2.4. *The inclusion $A \subseteq X$ is a cofibration iff there exists a retraction of $X \times I$ onto the subspace $(A \times I) \cup_{A \times \{0\}} (X \times \{0\})$.*

[2] This also appears in [MS06, Thm 4.4.4(ii)].



Cofibrations are sometimes called "Hurewicz cofibrations" in order to distinguish them from a different notion that appears later (Definition 5.1.19).

Lemma 1.2.5. *Every relative cell complex is a cofibration. Any coproduct, pushout, countable composition, or retract of a cofibration is again a cofibration.*

We've already seen how to define coproducts, pushouts, and compositions of maps. A retract of a map $g : B \to Y$ is any map $f : A \to X$ that fits into a commuting diagram of this form:

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X & \longrightarrow & Y & \longrightarrow & X. \end{array}$$

Every map $f : A \to X$ can be replaced by a homotopy-equivalent map that is a cofibration. Simply replace X by the mapping cylinder

$$(A \times I) \cup_{A \times \{1\}} (X \times \{1\}).$$

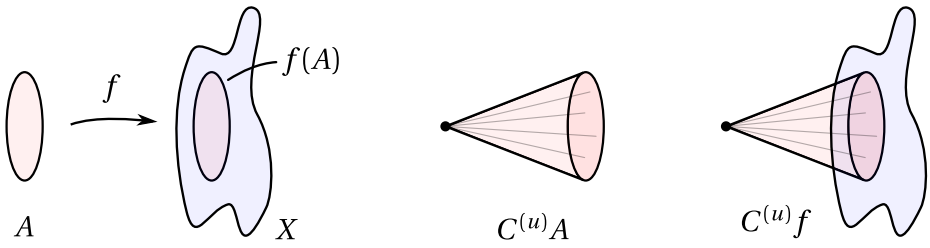
This is homotopy equivalent to X, but now the inclusion of A into the front end $A \times \{0\}$ is a cofibration. We summarize this with the following commuting diagram, in which the vertical maps are homotopy equivalences.

$$\begin{array}{ccc} A & \xrightarrow{\text{cofibration}} & (A \times I) \cup_{A \times \{1\}} (X \times \{1\}) \\ \| & & \downarrow{\scriptstyle \sim} \\ A & \xrightarrow{\quad f \quad} & X. \end{array}$$

The **cofiber** of $A \to X$ is just the quotient X/A. The **unbased homotopy cofiber** $C^{(u)}f$ is the cofiber of the new map that we just constructed:

$$\begin{aligned} C^{(u)}f &= \frac{(A \times I) \cup_{A \times \{1\}} (X \times \{1\})}{(A \times \{0\})} \\ &\cong * \cup_{A \times \{0\}} (A \times I) \cup_{A \times \{1\}} X \\ &\cong C^{(u)}A \cup_A X. \end{aligned}$$

Here $C^{(u)}A = (A \times I)/(A \times \{0\})$ is the unbased cone on A, as shown below.



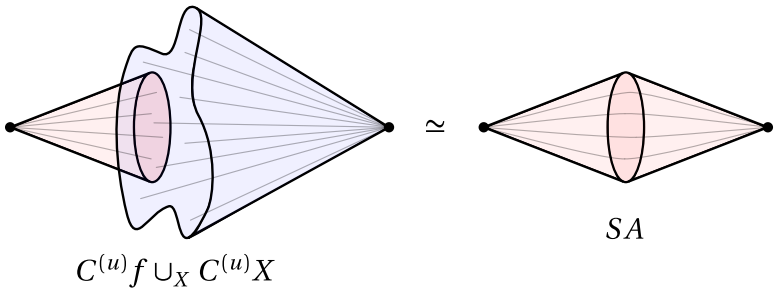
Lemma 1.2.6. *If $f : A \to X$ is a cofibration then the map $C^{(u)}f \to X/A$ that collapses the cone to a point, is a homotopy equivalence.*

So when f is a cofibration, both $C^{(u)}f$ and X/A are models for the quotient of X by A. When f is not a cofibration, we think of $C^{(u)}f$ as the "correct" way to quotient X by A; the actual quotient X/A is "incorrect" because it isn't well-behaved in that case.[3]

What is the correct quotient of $C^{(u)}f$ by X? We replace $X \to C^{(u)}f$ by a cofibration and take its cofiber! The result is homotopy equivalent to the suspension

$$SA = C^{(u)}A \cup_A C^{(u)}A$$

$$\cong (A \times I)/\big(\ \forall a, b \in A,\ (a,0) \sim (b,0) \text{ and } (a,1) \sim (b,1)\ \big).$$



Repeating this process indefinitely produces the unbased cofiber Puppe sequence

$$A \xrightarrow{f} X \longrightarrow C^{(u)}f \longrightarrow SA \xrightarrow{-Sf} SX \longrightarrow SC^{(u)}f \longrightarrow S^2A \xrightarrow{S^2f} S^2X \longrightarrow \cdots$$

Here $-Sf$ refers to the map that applies f but then flips the two cones around.

The following lemma about products of cofibrations is especially important. To state it, we recall that cofibrations $A \to X$ are always closed inclusions, so without loss of generality we think of them as subspaces $A \subseteq X$.

[3] This is a glimpse into how homotopy theorists think. We always want to replace badly-behaved constructions like X/A by well-behaved, homotopy-invariant constructions like $C^{(u)}f$.

Lemma 1.2.7. *[May99, §6.4] If $A \subseteq X$ and $B \subseteq Y$ are cofibrations, then*

$$(A \times Y) \cup_{(A \times B)} (X \times B) \subseteq X \times Y$$

is also a cofibration.

This larger inclusion is called the **pushout-product** of $A \subseteq X$ and $B \subseteq Y$ (Definition 5.5.6).

Definition 1.2.8. Suppose X has a basepoint. We say X is **well-based** or nondegenerately based if the inclusion of the basepoint $* \to X$ is a cofibration.

Remark 1.2.9. By Lemma 1.2.7, if X and Y are well-based then $(X \vee Y) \subseteq (X \times Y)$ is a cofibration. Therefore the smash product $X \wedge Y$ is equivalent to the homotopy cofiber of the inclusion $(X \vee Y) \subseteq (X \times Y)$.

There is a based version of the homotopy extension property, where all spaces are based and the homotopies preserve the basepoint. So we ask for a retraction of $X \wedge I_+$ onto the subspace $(A \wedge I_+) \cup_{A \wedge \{0\}_+} (X \wedge \{0\}_+)$.

Lemma 1.2.10. *[May99, §8.3] When A and X are based spaces, every unbased cofibration $A \to X$ is also a based cofibration. If A and X are well-based spaces, then every based cofibration is also an unbased cofibration, so the two notions coincide.*

Let CA be the based cone on A, the quotient of the unbased cone $C^{(u)}A$ by the interval $\{*\} \times I$ consisting of the basepoint of A times I:

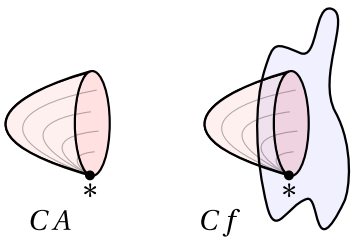
$$CA = \big(* \cup_{A \times \{0\}} (A \times I)\big) \big/ (\{*\} \times I)$$

Alternatively, it is the reduced cylinder $A \wedge I_+$ modulo one of its ends, or just the smash product $A \wedge I$ if we give I the basepoint 0:

$$CA \cong * \cup_{A \wedge 0_+} (A \wedge I_+) \cong A \wedge I.$$

Let Cf denote the based version of the homotopy cofiber,

$$Cf = CA \cup_{A \wedge \{1\}_+} X = (C^{(u)} f)/(\{*\} \times I).$$



Lemma 1.2.11. *If $f : A \to X$ is a cofibration (in either sense) and both A and X are well-based, then the collapse map $Cf \to X/A$ is a homotopy equivalence.*

In fact, we only need f to be a cofibration in the based sense for this to be true, but if A and X are not well-based, then these two spaces Cf and X/A don't have the homotopy type that we want, so it doesn't matter whether they are equivalent to each other.

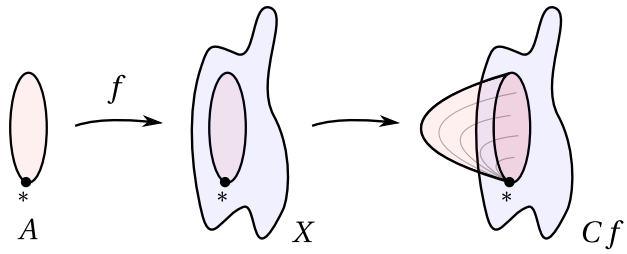
Definition 1.2.12. A **cofiber sequence** is any diagram of well-based spaces of the form

$$A \xrightarrow{f} X \longrightarrow Cf,$$

or anything weakly equivalent to such a diagram.[4] So if $A \to X$ is a cofibration (in either sense) and both A and X are well-based, then

$$A \longrightarrow X \longrightarrow X/A$$

is a cofiber sequence. If we only know that A and X are well-based and we don't have a cofibration, we should take Cf instead of X/A.



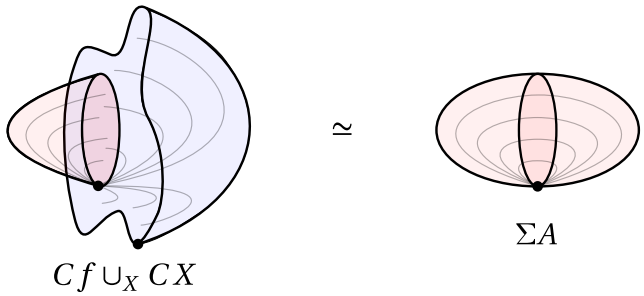
The based version of the cofiber Puppe sequence is the same, except that the cones and suspensions are reduced, i.e. the segment $* \times I$ is collapsed to the basepoint. This produces

$$A \xrightarrow{f} X \longrightarrow Cf \longrightarrow \Sigma A \xrightarrow{-\Sigma f} \Sigma X \longrightarrow \Sigma Cf \longrightarrow \Sigma^2 A \xrightarrow{\Sigma^2 f} \Sigma^2 X \longrightarrow \cdots$$

where ΣA is the reduced suspension

$$\Sigma A = S^1 \wedge A \cong SA/S(*).$$

We only consider this sequence when the spaces A and X are well-based.



[4] If you want more precision, we mean that there is a zig-zag of pointwise weak homotopy equivalences of such diagrams, in the sense of Definition 3.5.1. We will recall what a weak homotopy equivalence is in Section 1.4.

Proposition 1.2.13. *For any based space Z, taking based homotopy classes of maps from the Puppe sequence into Z produces a long exact sequence of pointed sets*

$$[A,Z]_* \longleftarrow [X,Z]_* \longleftarrow [Cf,Z]_* \longleftarrow [\Sigma A,Z]_* \longleftarrow [\Sigma X,Z]_* \longleftarrow [\Sigma Cf,Z]_* \longleftarrow \cdots$$

By "long exact sequence of pointed sets," we mean that every set has a basepoint $*$, and that the "kernel" of each map (those elements mapping to $*$) coincides with the image of the previous map.

Remark 1.2.14. Starting at the term $[\Sigma A,Z]_*$, the sets are all groups and the maps are group homomorphisms. The group operation pinches the suspension in the middle and adds two maps together:

$$\Sigma A \xrightarrow{\text{pinch}} \Sigma A \vee \Sigma A \xrightarrow{(f,g)} Z.$$

Starting at $[\Sigma^2 A,Z]_*$, these group structures are all abelian.

1.2.3 Fibrations

Definition 1.2.15. A **Hurewicz fibration** is a map of topological spaces $p\colon E \to B$ with the homotopy lifting property: for any map $f : X \to E$ and any homotopy of the map $p \circ f$, the homotopy can be lifted to a homotopy of f. In other words, for each diagram as below, a dashed lift exists.

$$\begin{array}{ccc} & & E^I \\ & \nearrow & \downarrow{\scriptstyle \alpha \mapsto (\alpha(0),p\circ\alpha)} \\ X & \longrightarrow & E \times_B B^I \end{array}$$

Writing out the definition of the pullback $E \times_B B^I$, it is the space of points $e \in E$ and paths $\gamma\colon I \to B$ such that $p(e) = \gamma(0)$. The map from E^I to this pullback takes each path $\alpha\colon I \to E$ to the data of $\alpha(0) \in E$ and $p \circ \alpha\colon I \to B$.

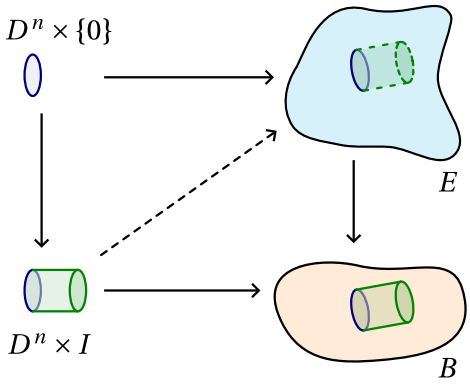
As before, it suffices to check the lifting condition when $X = E \times_B B^I$. So being a fibration is equivalent to having a map $E \times_B B^I \to E^I$ that lifts the identity of $E \times_B B^I$. In other words, we can continuously lift all of the paths in B with chosen starting points in E, to paths in E.

Using the bijections from Section 1.1.5, we can rearrange the lifting property as follows: for any commuting square as shown, a dashed lift exists.

$$\begin{array}{ccc} X \times \{0\} & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ X \times I & \longrightarrow & B. \end{array}$$

Taking $X = *$, this tells us that each path in B with a lift of the starting point to E, can be lifted to E. More generally, each "Xs worth" of paths in B, and lifts of the starting points to E, can be lifted to E.

A **Serre fibration** is a map $E \to B$ that has the homotopy lifting property for $X = D^n$ when $n \geq 0$, but not necessarily for all spaces X. So every square of the following form has a lift.



Recall that $E \to B$ is a fiber bundle if it is locally a product projection $U \times F \to U$. In other words, B can be covered by open sets U_α such that there are homeomorphisms $p^{-1}(U_\alpha) \cong U_\alpha \times F$, commuting with the projection to U_α. A covering space is a fiber bundle in which the fiber F is discrete.

Lemma 1.2.16. *Every fiber bundle, and in particular every covering space, is a Hurewicz fibration. Every Hurewicz fibration is a Serre fibration. Any product, pullback, countable inverse composition, or retract of a fibration is a fibration. (This last statement applies to both kinds of fibrations.)*

For covering spaces, the lift of each path is unique. For general fibrations, the lift of each path turns out to be unique up to homotopy, but is not unique on the nose.

Every map $p\colon E \to B$ can be replaced by a homotopy-equivalent Hurewicz fibration. Simply replace E by the "dual mapping cylinder" $E \times_B B^I$, and take the map from this new version of E to B by the formula

$$(e, \gamma) \mapsto \gamma(1).$$

As before, this new map is equivalent to the old map p, in the sense that the following diagram commutes, and the vertical maps are homotopy equivalences:

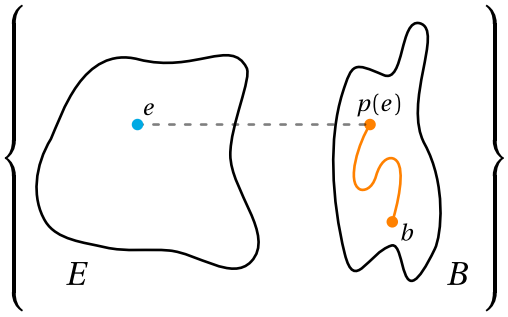
$$\begin{array}{ccc} E & \xrightarrow{\ p\ } & B \\ \downarrow{\scriptstyle\sim} & & \| \\ E \times_B B^I & \xrightarrow{\text{fibration}} & B. \end{array}$$

For each $b \in B$, the **homotopy fiber** $F_b p$ is the fiber of this new map at the point b:

$$F_b p = E \times_B B^I \times_B \{b\}.$$

It is the space of points $e \in E$ with paths from $p(e) \in B$ to $b \in B$, as shown below. If B is a based space, we denote the homotopy fiber over the basepoint by Fp:

$$Fp = E \times_B B^I \times_B \{*\}.$$



Lemma 1.2.17. *If p is a Hurewicz fibration then each inclusion $p^{-1}(b) \to F_b p$ is a homotopy equivalence. If p is a Serre fibration then each inclusion $p^{-1}(b) \to F_b p$ is a weak homotopy equivalence (see Section 1.4).*

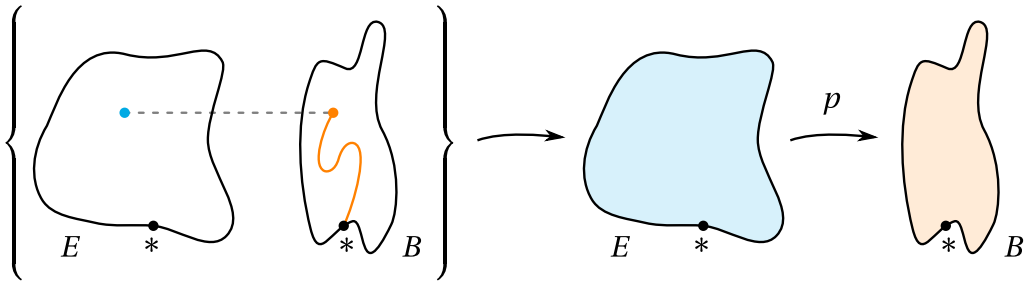
Definition 1.2.18. A **fiber sequence** is any diagram of based spaces of the form

$$Fp \longrightarrow E \xrightarrow{p} B,$$

or anything weakly equivalent to such a diagram. In particular, if $E \to B$ is a fibration of either kind, then

$$p^{-1}(*) \longrightarrow E \xrightarrow{p} B$$

is a fiber sequence.



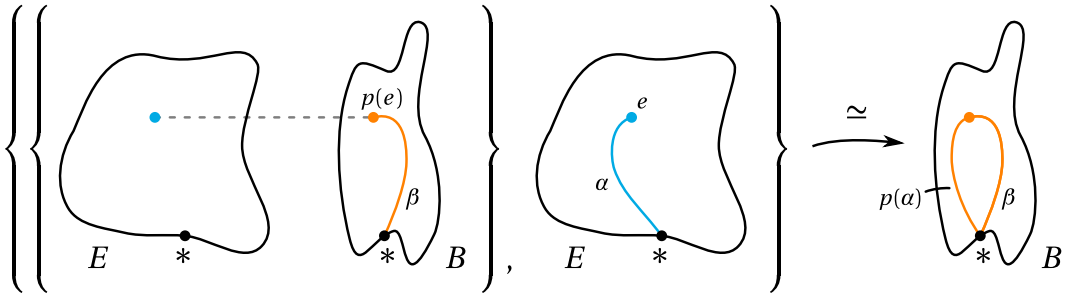
Let's assume that $E \to B$ is a map of based spaces, and when we take homotopy fiber, we always take it over the basepoint.[5] Then if we replace $Fp \to E$ by a fibration and take its

[5]However our notion of "fibration" will remain unbased. There is a based notion of "fibration," but it rarely comes up in practice.

fiber, the result is equivalent to the space of based loops in B,

$$\Omega B = \mathrm{Map}_*(S^1, B).$$

The argument for this is illustrated just below.



Repeating this process indefinitely produces the fiber Puppe sequence

$$\cdots \longrightarrow \Omega^2 E \xrightarrow{\Omega^2 p} \Omega^2 B \longrightarrow \Omega F p \longrightarrow \Omega E \xrightarrow{-\Omega p} \Omega B \longrightarrow F p \longrightarrow E \xrightarrow{p} B.$$

Proposition 1.2.19. *For any based space Z, taking based homotopy classes of maps from Z into the Puppe sequence produces a long exact sequence of pointed sets*

$$\cdots \longrightarrow [Z, \Omega F p]_* \longrightarrow [Z, \Omega E]_* \longrightarrow [Z, \Omega B]_* \longrightarrow [Z, F p]_* \longrightarrow [Z, E]_* \longrightarrow [Z, B]_*.$$

Remark 1.2.20. Starting at the term $[Z, \Omega B]_*$, the sets are all groups and the maps are group homomorphisms. The group operation concatenates the loops in B:

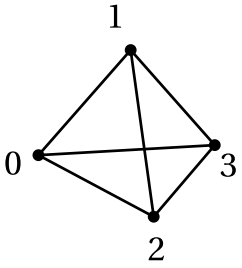
$$Z \xrightarrow{(f,g)} \Omega B \times \Omega B \xrightarrow{\text{concatenate}} \Omega B.$$

Starting at $[Z, \Omega^2 B]_*$, these group structures are all abelian.

Remark 1.2.21. The long exact sequences of Proposition 1.2.13 and Proposition 1.2.19 are to higher algebra what the snake lemma and zig-zag lemma are to homological algebra. They are the basic tool that we use to build our long exact sequences.

1.3 Homology and cohomology

1.3.1 Homology

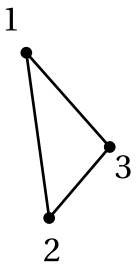


Let Δ^n be the convex hull of the standard basis vectors in $\mathbb{R}^{n+1}$:

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \ : \ \sum_i t_i = 1, \ t_i \geq 0 \ \forall i \right\}$$

The case of $n = 3$ is pictured to the left.

The ith face of Δ^n is the subspace of points in which $t_i = 0$, which we identify with Δ^{n-1} in the obvious way (preserving the order of the coordinates). The 0th face of Δ^3 is pictured to the right.



Given a continuous map $\sigma\colon \Delta^n \to X$, we define its ith face $d_i\sigma\colon \Delta^{n-1} \to X$ to be the restriction of σ to this subspace.

Let G be any abelian group. (The most common examples are $\mathbb{Z}$, $\mathbb{Z}/p$, and $\mathbb{Q}$.) A **singular n-chain** in X is a formal linear combination of continuous maps $\sigma\colon \Delta^n \to X$, with coefficients in G. In other words, a singular n-chain is an element of the group

$$C_n(X;G) = \bigoplus_{\Delta^n \to X} G.$$

We define the boundary map

$$\partial_n\colon C_n(X;G) \to C_{n-1}(X;G)$$

by sending each simplex σ to the alternating sum of its faces,

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i d_i\sigma,$$

and extending to $C_n(X;G)$ by linearity. These form a **chain complex**, meaning that $\partial_{n-1}\partial_n = 0$ for all values of n. The nth **homology** of this chain complex is the abelian group

$$H_n(X;G) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

When $G = \mathbb{Z}$ we simply write the chains as $C_n(X)$ and the homology as $H_n(X)$.

A pair (X,A) is a space X with a subspace $A \subseteq X$. A map of pairs $(X,A) \to (Y,B)$ is a map $f : X \to Y$ such that $f(A) \subseteq B$. A homotopy of maps of pairs $(X,A) \to (Y,B)$ is a homotopy $h\colon X \times I \to Y$ such that $h(A \times I) \subseteq B$.

We define the relative chains of (X,A) with G coefficients to be the quotient

$$C_n(X,A;G) = C_n(X;G)/C_n(A;G) \cong \bigoplus_{\sigma\colon \Delta^n \to X,\ \sigma(\Delta^n)\not\subseteq A} G.$$

This forms a chain complex. We denote its homology by $H_n(X,A;G)$, or just $H_n(X,A)$ when $G = \mathbb{Z}$. When $A = \varnothing$, this coincides with the "absolute" homology group $H_n(X)$ that we already defined. When $A = *$, we call $H_n(X,*;G)$ the **reduced homology** with G coefficients.

Homology satisfies the following properties, called the **Eilenberg-Steenrod axioms**.

Proposition 1.3.1 (Homotopy)**.** *$H_n(-,-;G)$ defines a functor from pairs of spaces to abelian groups. Homotopic maps of pairs induce the same map on homology.*

For a map of pairs $f : (X,A) \to (Y,B)$, we let $f_*\colon H_n(X,A;G) \to H_n(Y,B;G)$ denote the induced map on homology. It takes each chain in X, which is a linear combination of simplices $\alpha\colon \Delta^n \to X$, and to the linear combination with the same coefficients of the simplices $(f \circ \alpha)\colon \Delta^n \to Y$.

Proposition 1.3.2 (Exactness)**.** *For each pair there is a long exact sequence*

$$\cdots \xrightarrow{\partial} H_n(A;G) \xrightarrow{i_*} H_n(X;G) \xrightarrow{j_*} H_n(X,A;G) \xrightarrow{\partial} H_{n-1}(A;G) \xrightarrow{i_*} \cdots$$

ending at $H_0(X,A;G)$. The maps i_* and j_* are induced by the maps of pairs

$$(A,\varnothing) \xrightarrow{i} (X,\varnothing) \xrightarrow{j} (X,A).$$

Every map in this sequence is natural. In other words, each map of pairs $f : (X,A) \to (Y,B)$ gives a commuting diagram as follows.

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\partial} & H_n(A;G) & \xrightarrow{i_*} & H_n(X;G) & \xrightarrow{j_*} & H_n(X,A;G) & \xrightarrow{\partial} & H_{n-1}(A;G) & \xrightarrow{i_*} & \cdots \\
 & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\
\cdots & \xrightarrow{\partial} & H_n(B;G) & \xrightarrow{i_*} & H_n(Y;G) & \xrightarrow{j_*} & H_n(Y,B;G) & \xrightarrow{\partial} & H_{n-1}(B;G) & \xrightarrow{i_*} & \cdots
\end{array}$$

Proposition 1.3.3 (Excision)**.** *If $A \to X$ is a cofibration then the collapse map $(X,A) \to (X/A,*)$ induces an isomorphism on homology.*[6]

Proposition 1.3.4 (Additivity)**.** *If $X = \coprod_\alpha X_\alpha$ is a decomposition into subspaces that are unions of path components, then the map*

$$\bigoplus_\alpha H_k(X_\alpha) \longrightarrow H_k(X)$$

is an isomorphism.

[6] This is a consequence of the excision theorem given in [Hat02, Thm 2.20].

Proposition 1.3.5 (Dimension)**.** *$H_0(*;G) \cong G$, and $H_k(*;G) \cong 0$ for all $k \neq 0$.*

Remark 1.3.6. It follows that every cofiber sequence $A \to X \to Cf$ induces a long exact sequence on reduced homology groups.

Amazingly, the Eilenberg-Steenrod axioms determine $H_*(X,A;G)$ completely, at least on pairs (X,A) where X is a CW complex and A is a subcomplex. For instance, it follows from the exactness axiom that $H_k(S^n,*;G)$ is G when $k=n$ and 0 otherwise. We can explain how every other space is determined as well, but first we need to recall what a natural isomorphism is.

Definition 1.3.7. If F and G are both functors from the category **C** to the category **D**, a **natural transformation** $\eta\colon F \Rightarrow G$ consists of a morphism $\eta(X)\colon F(X) \to G(X)$ in the category **D** for each object X in **C**, such that every morphism $f : X \to Y$ in **C** gives a commuting diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \eta(X)}\downarrow & & \downarrow{\scriptstyle \eta(Y)} \\ G(X) & \xrightarrow{G(f)} & G(Y). \end{array}$$

For instance, in Proposition 1.3.2, we are claiming that the maps i_*, j_*, and δ are all natural transformations. (Think about this. What are the functors in each case?)

A **natural isomorphism** is a natural transformation in which each of the maps $F(X) \to G(X)$ is an isomorphism. In other words, F and G do the same thing to each object (up to isomorphism), they do the same thing to each morphism as well.

Now that we've recalled this language, we can talk about a natural isomorphism between two functors h_n and h'_n from CW pairs to abelian groups. This is a choice of isomorphism $h_n(X,A) \cong h'_n(X,A)$ such that every map of pairs $f : (X,A) \to (Y,B)$ gives a commuting square

$$\begin{array}{ccc} h_n(X,A) & \xrightarrow{f_*} & h_n(Y,B) \\ \cong\updownarrow & & \updownarrow\cong \\ h'_n(X,A) & \xrightarrow{f_*} & h'_n(Y,B). \end{array}$$

So again, not only do they give isomorphic groups, but along this isomorphism, the two functors send each continuous map of spaces, to the same homomorphism of abelian groups.

Now we are ready to state how homology is completely determined by the group G. An **ordinary homology theory** with coefficient group G is any sequence of functors $h_n(X,A)$ from CW pairs to abelian groups, along with "boundary" or "snake" natural

transformations $\partial : h_n(X,A) \to h_{n-1}(A)$, satisfying the above axioms. For the dimension axiom, we fix the choice of isomorphism $h_0(*) \cong G$.

We define a natural isomorphism of homology theories to be a sequence of natural isomorphisms $h_n \cong h'_n$ for all $n \geq 0$, commuting with the boundary maps $\partial : h_n(X,A) \to h_{n-1}(A)$, and that on $h_0(*)$ is the composite of the chosen identifications with G:

$$h_0(*) \cong G \cong h'_0(*).$$

Theorem 1.3.8. *Any two ordinary homology theories h and h' with coefficients in the same group G are naturally isomorphic, and the natural isomorphism between them is unique.*

See e.g. [May99]. If we don't specify the isomorphism on $h_0(*)$, then we get a natural isomorphism $h \cong h'$ for each automorphism of the group G.

The proof of Theorem 1.3.8 proceeds by showing that each ordinary homology theory h is isomorphic to **cellular homology**. Cellular homology for a CW complex X is constructed by taking any existing ordinary homology theory (such as singular homology), and defining a new chain complex by

$$C_n^{CW}(X;G) = h_n(X^{(n)}, X^{(n-1)}; G) \cong \bigoplus_{n\text{-cells}} G,$$

with boundary maps given by

$$h_n(X^{(n)}, X^{(n-1)}; G) \xrightarrow{\partial} h_{n-1}(X^{(n-1)}; G) \xrightarrow{j_*} h_{n-1}(X^{(n-1)}, X^{(n-2)}; G).$$

Similarly $C_n^{CW}(X,A;G)$ is defined as the quotient $C_n^{CW}(X;G)/C_n^{CW}(A;G)$, which is a direct sum of one copy of G for every n-cell of X that is not in A, and boundary maps inherited from $C_n^{CW}(X;G)$. The homology of this complex is the cellular homology $H_n^{CW}(X,A;G)$.

We then show that the starting homology theory h is isomorphic in a unique way to cellular homology H^{CW}. However, the definition of H^{CW} can be phrased in a way that is independent of which homology theory we used to construct it. Therefore any two ordinary homology theories h and h' are uniquely isomorphic to each other, because each one is uniquely isomorphic to H^{CW}.

If X is any well-based space then the cofiber sequence $X \to CX \to CX/X \simeq \Sigma X$ gives a long exact sequence on homology. Since the homology of $(CX, *)$ vanishes in every degree, we therefore get a natural isomorphism

$$H_{n+1}(\Sigma X, *; G) \cong H_n(X, *; G).$$

For abelian groups A and B, take any presentation of A, i.e. a short exact sequence

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0 \tag{1.3.9}$$

in which P_0 and P_1 are free abelian groups, and define

$$\mathrm{Tor}(A,B) := \ker(P_1 \otimes B \to P_0 \otimes B).$$

This Tor group is independent of the choice of presentation $P_1 \to P_0$, and we have $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$. If A or B is a free abelian group then $\mathrm{Tor}(A,B) = 0$; otherwise the Tor group describes their "common torsion."

Theorem 1.3.10 (Universal coefficient theorem for homology)**.** *There is a natural short exact sequence*

$$0 \longrightarrow H_n(X;\mathbb{Z}) \otimes G \longrightarrow H_n(X;G) \longrightarrow \mathrm{Tor}(H_{n-1}(X;\mathbb{Z}),G) \longrightarrow 0$$

and similar sequences with the reduced and relative homology groups. The sequence splits, but not naturally.

Here and elsewhere in this book, the word "natural" means that every map in the statement of the theorem is a natural transformation. In the last statement of the theorem, we are saying that there are maps of the form

$$H_n(X;G) \to H_n(X;\mathbb{Z}) \otimes G$$

that split the sequence, for each space X separately, but these maps don't form a natural transformation. They don't commute with f_*!

Theorem 1.3.11 (Künneth theorem for homology)**.** *For any pair of spaces X and Y there is a natural short exact sequence*

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \xrightarrow{\times} H_n(X \times Y) \longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(X), H_j(Y)) \longrightarrow 0.$$

that splits, but not naturally. For based spaces we also get

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(X,*) \otimes H_j(Y,*) \xrightarrow{\times} H_n(X \wedge Y,*) \longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(X,*), H_j(Y,*)) \longrightarrow 0.$$

For field coefficients, there are isomorphisms

$$\bigoplus_{i+j=n} H_i(X;k) \otimes_k H_j(Y;k) \xrightarrow{\times} H_n(X \times Y;k),$$

$$\bigoplus_{i+j=n} H_i(X,*;k) \otimes_k H_j(Y,*;k) \xrightarrow{\times} H_n(X \wedge Y,*;k).$$

The maps labeled $\times$ are called the **cross product** maps on homology. We spell out the definition in the case of the Cartesian product, since the one for the smash product is similar. We first construct an isomorphism of chain complexes

$$C_*^{CW}(X;\mathbb{Z}) \otimes C_*^{CW}(Y;\mathbb{Z}) \cong C_*^{CW}(X \times Y;\mathbb{Z}).$$

Here the tensor product of chain complexes on the left is defined by tensoring all of the levels together and adding their degrees, so that at level n it looks like this:

$$\bigoplus_{i+j=n} C_i^{CW}(X;\mathbb{Z}) \otimes C_j^{CW}(Y;\mathbb{Z}) \cong C_n^{CW}(X \times Y;\mathbb{Z}).$$

The isomorphism comes from the observation that the n-cells in $X \times Y$ are given by pairs of an i-cell in X and j-cell in Y such that $i+j=n$. This isomorphism commutes with the differential, where the differential on the tensor product of chain complexes is given as usual by the Leibniz rule

$$\partial(\alpha \otimes \beta) = \partial\alpha \otimes \beta + (-1)^i \alpha \otimes \partial\beta, \qquad \alpha \in C_i(X),\ \beta \in C_j(Y).$$

(See also Exercise 4.4.1.) Finally, any map of chain complexes of this form gives a map on homology

$$\bigoplus_{i+j=n} H_i(X;\mathbb{Z}) \otimes H_j(Y;\mathbb{Z}) \xrightarrow{\times} H_n(X \times Y;\mathbb{Z}),$$

by taking pairs of cycles $[\alpha],[\beta]$ to the image of the cycle $\alpha \otimes \beta$ in $C_n(X \times Y)$. This defines the cross product on homology.[7]

If X has a multiplication $\mu\colon X \times X \to X$ that is associative and unital up to homotopy, we define the **Pontryagin product** on $H_*(X)$ as the composite

$$H_i(X) \otimes H_j(X) \xrightarrow{\times} H_{i+j}(X \times X) \xrightarrow{\mu_*} H_{i+j}(X). \tag{1.3.12}$$

This makes $H_*(X)$ into a graded ring.

Example 1.3.13. The n-torus $T^n = (S^1)^{\times n}$ has such a multiplication, which in each coordinate is the usual group structure on S^1. As a ring, $H_*(T^n)$ is isomorphic to $\Lambda_*(\mathbb{Z}^n) = \Lambda_*[x_1,\dots,x_n]$, an exterior algebra with n generators that are all in degree 1. In other words, it is the ring of non-commutative polynomials on these generators, modulo the two-sided ideal generated by the relations $x_i^2 = 0$ and $x_i x_j = -x_j x_i$.

The based loop space ΩX of any space X has a multiplication as well by concatenating the loops. When $n \geq 2$, $H_*(\Omega S^n)$ is isomorphic as a ring to $\mathbb{Z}[x]$, a polynomial ring with a single generator x in degree $(n-1)$.

1.3.2 Cohomology

We define the **cohomology** of X with G coefficients by applying $\mathrm{Hom}(-,G)$ to the chain complex $C_n(X) = C_n(X;\mathbb{Z})$ to get

$$C^n(X;G) := \mathrm{Hom}(C_n(X), G).$$

[7] This same cross product can also be defined using simplicial homology, but instead of an isomorphism $C_*(X) \otimes C_*(Y) \cong C_*(X \times Y)$, we only get a map called the "Eilenberg–Zilber map." See Proposition 9.2.7.

As n varies, this forms a cochain complex, which is just a chain complex whose differential raises degree. The coboundary $\delta(\alpha)$ of a homomorphism $\alpha\colon C_n(X) \to G$ is defined to be the composite $\alpha \circ \partial : C_{n+1}(X) \to G$.

Taking the kernel of this "coboundary map" δ modulo the image of δ gives the cohomology groups $H^n(X;G)$. Doing the same to the chain complexes

$$C^n(X,A;G) = \mathrm{Hom}(C_n(X,A),G) = \mathrm{Hom}(C_n(X)/C_n(A),G)$$

gives the relative cohomology groups $H^n(X,A;G)$.

Cohomology satisfies a dual version of the above axioms for homology.

Proposition 1.3.14 (Homotopy)**.** *$H^n(-,-;G)$ defines a contravariant functor from pairs to abelian groups. Homotopic maps induce the same map on cohomology.*

The word **contravariant** means that cohomology is a functor that reverses the direction of maps. So each map $f : X \to Y$ is sent to a map going the other way $H^n(Y) \to H^n(X)$, and this assignment respects composition of maps and identity maps.

For a map of pairs $f : (X,A) \to (Y,B)$, we let $f^*\colon H^n(Y,B;G) \to H^n(X,A;G)$ denote the induced map on cohomology.

Proposition 1.3.15 (Exactness)**.** *For each pair there is a long exact sequence*

$$\cdots \xleftarrow{\delta} H^n(A;G) \xleftarrow{i^*} H^n(X;G) \xleftarrow{j^*} H^n(X,A;G) \xleftarrow{\delta} H^{n-1}(A;G) \xleftarrow{i^*} \cdots$$

ending at $H^0(X,A;G)$. The maps i^* and j^* are induced by the maps of pairs $(A,\varnothing) \xrightarrow{i} (X,\varnothing) \xrightarrow{j} (X,A)$. Every map in this sequence is natural; in other words, each map of pairs $f : (X,A) \to (Y,B)$ gives a commuting diagram as follows.

$$\begin{array}{ccccccccccc}
\cdots & \xleftarrow{\delta} & H^n(A;G) & \xleftarrow{i^*} & H^n(X;G) & \xleftarrow{j^*} & H^n(X,A;G) & \xleftarrow{\delta} & H^{n-1}(A;G) & \xleftarrow{i^*} & \cdots \\
 & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & & \\
\cdots & \xleftarrow{\delta} & H^n(B;G) & \xleftarrow{i^*} & H^n(Y;G) & \xleftarrow{j^*} & H^n(Y,B;G) & \xleftarrow{\delta} & H^{n-1}(B;G) & \xleftarrow{i^*} & \cdots
\end{array}$$

Proposition 1.3.16 (Excision)**.** *If $A \to X$ is a cofibration then the collapse map $(X,A) \to (X/A,*)$ induces an isomorphism on cohomology.*

Proposition 1.3.17 (Additivity)**.** *If $X = \coprod_\alpha X_\alpha$ is a decomposition into subspaces that are unions of path components, then the map*

$$H^k(X) \longrightarrow \prod_\alpha H^k(X_\alpha)$$

is an isomorphism.

Proposition 1.3.18 (Dimension). *$H^0(*;G) \cong G$, and $H^k(*;G) \cong 0$ for all $k \neq 0$.*

An **ordinary cohomology theory** with coefficient group G is any sequence of functors $h^n(X,A)$ from CW pairs to abelian groups, along with "boundary" or "snake" natural transformations $\delta : h^{n-1}(A) \to h^n(X,A)$, satisfying the above axioms. As before, in the dimension axiom, we fix the choice of isomorphism $h^0(*) \cong G$.

As before, any two such theories on CW complexes are naturally isomorphic in a unique way, because they are isomorphic to cellular cohomology $H^n_{CW}(-)$, which is the cohomology of the cochain complex

$$C^n_{CW}(X;G) := \mathrm{Hom}(C^{CW}_n(X), G).$$

We again get, as consequence of homotopy and exactness, a suspension isomorphism

$$H^{n+1}(\Sigma X, *; G) \cong H^n(X, *; G).$$

For abelian groups A and B, take any presentation of A as above

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0,$$

and define

$$\mathrm{Ext}(A,B) := \mathrm{Hom}(P_1,B)/\mathrm{im}\,\mathrm{Hom}(P_0,B).$$

Theorem 1.3.19 (Universal coefficient theorem for cohomology). *There is a natural short exact sequence*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X;G) \longrightarrow \mathrm{Hom}(H_n(X), G) \longrightarrow 0.$$

and similar sequences with the reduced and relative cohomology groups. The sequence splits, but not naturally.

We say that a space Y is **finite type** if its chain complex is equivalent to one that is a finitely generated free abelian group at each level. So any CW complex with finitely many cells in each dimension will be of finite type.

Theorem 1.3.20 (Künneth theorem for cohomology). *For any space X and any finite type space Y, there is a natural short exact sequence*

$$0 \longrightarrow \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y) \xrightarrow{\times} H^n(X \times Y) \longrightarrow \bigoplus_{i+j=n+1} \mathrm{Tor}(H^i(X), H^j(Y)) \longrightarrow 0.$$

that splits, but not naturally. For based spaces we also get

$$0 \longrightarrow \bigoplus_{i+j=n} H^i(X,*) \otimes H^j(Y,*) \xrightarrow{\times} H^n(X \wedge Y, *) \longrightarrow \bigoplus_{i+j=n+1} \mathrm{Tor}(H^i(X,*), H^j(Y,*)) \longrightarrow 0.$$

For field coefficients, there are isomorphisms

$$\bigoplus_{i+j=n} H^i(X;k) \otimes_k H^j(Y;k) \xrightarrow{\times} H^n(X \times Y;k).$$

$$\bigoplus_{i+j=n} H^i(X,*;k) \otimes_k H^j(Y,*;k) \xrightarrow{\times} H^n(X \wedge Y,*;k).$$

Again the maps labeled $\times$ are called the **cross product** on cohomology. They are defined for all X and Y, but the theorem only holds if Y is finite type. As before, we define this cross product by forming an isomorphism of cochain complexes

$$\bigoplus_{i+j=n} C^i_{CW}(X;\mathbb{Z}) \otimes C^j_{CW}(Y;\mathbb{Z}) \cong C^n_{CW}(X \times Y;\mathbb{Z}),$$

and concluding that every pair of cocycles on the left goes to a cocycle on the right, giving a map on cohomology

$$\bigoplus_{i+j=n} H^i(X;\mathbb{Z}) \otimes H^j(Y;\mathbb{Z}) \xrightarrow{\times} H^n(X \times Y;\mathbb{Z}).$$

For any space X, we define the **cup product** as the composite of the cross product and the diagonal map $\Delta: X \to X \times X$:

$$H^i(X) \otimes H^j(X) \xrightarrow{\times} H^{i+j}(X \times X) \xrightarrow{\Delta^*} H^{i+j}(X).$$

It is defined on reduced cohomology similarly. This makes the cohomology of any space $H^*(X)$, and also with ring coefficients $H^*(X;R)$, into a graded ring in a natural way. The cross product map $H^i(X) \otimes H^j(Y) \to H^n(X \times Y)$ is always a ring homomorphism.

1.3.3 Poincaré duality

There is another product of this flavor called the **cap product**:

$$H^i(X) \otimes H_j(X) \xrightarrow{\cap} H_{j-i}(X).$$

We won't define it here since we won't need to use it directly; see for instance [Hat02, Sec 3.3]. The way to remember how the cap product $\cap$ interacts with the cup product $\cup$ is that $\cup$ makes H^* into a ring, and $\cap$ makes H_* into a module over that ring.

A n-dimensional **manifold** M is a Hausdorff space that can be covered with countably many open sets, each homeomorphic to an open subset of $\mathbb{R}^n$. A manifold is **closed** if it is compact.[8]

[8] And has no boundary. But we just defined manifolds without boundary, so this is automatic. A manifold with boundary would be locally homeomorphic to $\mathbb{R}^{n-1} \times [0,\infty)$, and the points identified with $\mathbb{R}^{n-1} \times \{0\}$ would be the boundary points.

By the stronger form of the excision theorem in [Hat02, Thm 2.20], the homology groups $H_n(M, M-\{x\})$ are all isomorphic to $H_n(D^n, D^n-\{0\}) \cong \mathbb{Z}$. An **orientation** of M is a choice of generator of $H_n(M, M-\{x\})$ for every point $x \in M$, such that whenever x and y lie in the same disc D and are connected by the embedded curve γ, the chosen generators agree along the isomorphisms

$$H_n(M, M-\{x\}) \xleftarrow{\cong} H_n(M, M-\gamma) \xrightarrow{\cong} H_n(M, M-\{y\}).$$

We might try to define such a collection of elements by taking one element of $H_n(M)$, which gives an element in every group $H_n(M, M-\{x\})$. It turns out that every orientation of M arises this way:

Proposition 1.3.21. *When M is connected, either $H_n(M) = 0$ and no orientation of M exists, or $H_n(M) = \mathbb{Z}$ and the two generators give the two possible orientations of M.*

Therefore we represent an orientation by a class $[M] \in H_n(M)$ called the **fundamental class**.

Theorem 1.3.22 (Poincaré duality)**.** *If M is a closed connected n-dimensional manifold, with orientation $[M]$, then cap product with $[M]$ gives an isomorphism*

$$H^i(M; \mathbb{Z}) \xrightarrow{\cong} H_{n-i}(M; \mathbb{Z}).$$

Even when M fails to have an orientation, there is still a mod 2 fundamental class $[M] \in H_n(M; \mathbb{Z}/2)$ that goes to the generator of every group $H_n(M, M-\{x\}) \cong \mathbb{Z}/2$. Capping with this mod 2 fundamental class $[M]$ gives isomorphisms

$$H^i(M; \mathbb{Z}/2) \xrightarrow{\cong} H_{n-i}(M; \mathbb{Z}/2).$$

1.4 Homotopy groups and homotopy theory

1.4.1 Homotopy groups and relative homotopy groups

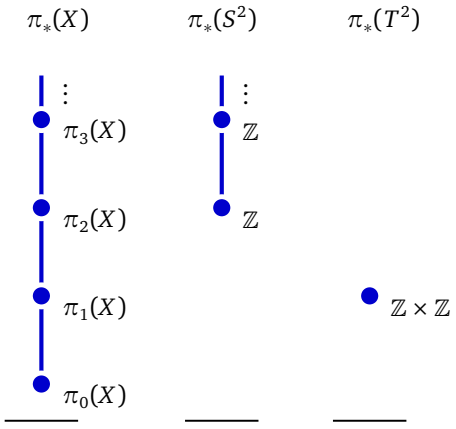
For any based space X, the nth **homotopy group** $\pi_n(X, *)$, often just written as $\pi_n(X)$, is the set of homotopy classes of based maps $S^n \to X$. We sometimes draw these groups as columns of blue dots, as shown below.

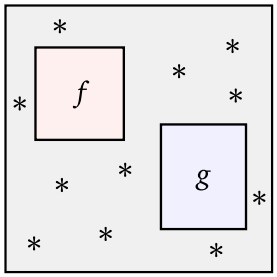
It is often more conventient to define these as homotopy classes of maps of pairs

$$(I^n, \partial I^n) \to (X, *),$$

where $I^n = I \times \cdots \times I$ is the n-dimensional cube and ∂I^n is its boundary.

When $n = 0$, $\pi_0(X)$ is the set of based maps $S^0 \to X$ up to homotopy. The basepoint of S^0 must go to the basepoint of X, but the other point can go anywhere, so we get the set of path components of X. When $n = 1$, $\pi_1(X)$ is the set of based loops in X up to homotopy.





$f + g$

When $n \geq 1$, we can make $\pi_n(X)$ into a group as follows. Let's say that an embedding $I^n \to I^n$ is **rectilinear** if it is an n-fold product of linear embeddings $i(x) = ax + b$ with $a > 0$. Intuitively, it is an embedding that scales and translates each coordinate, but does not otherwise rotate or stretch.

We add two elements $f, g \in \pi_n(X)$ by picking any two disjoint rectilinear embeddings $i, j : (0,1)^n \to I^n$, and defining the map $f + g$ to be $f \circ i^{-1}$ inside the image of i, $g \circ j^{-1}$ in the image of j, and the constant map to the basepoint $*$ at all other points. This definition varies continuously as we vary the embeddings.

When $n \geq 2$, all pairs of disjoint rectilinear embeddings are homotopic, so this gives a well-defined operation on $\pi_n(X)$ that is commutative. It is also associative and unital, making $\pi_n(X)$ into an abelian group.

When $n = 1$, the space of pairs of disjoint embeddings is not connected. In other words, when we compose paths, the order matters. Therefore $\pi_1(X)$ is a not-necessarily-abelian group, the **fundamental group** of X.

Proposition 1.4.1. *In each path component of X, the group $\pi_n(X)$ does not depend on the choice of basepoint, up to isomorphism.*

To be slightly more careful, each choice of path γ between two basepoints x_0 and x_1 gives a change-of-basepoint isomorphism $\pi_n(X, x_0) \cong \pi_n(X, x_1)$, which depends on the choice of path. In Figure 1.4.7 we illustrate this in the special case of a path from x_0 to itself, which can give a nontrivial isomorphism $\pi_n(X, x_0) \cong \pi_n(X, x_0)$.

The construction of $\pi_n(X)$ is a functor: any basepoint-preserving map $f : X \to Y$ induces a map $f_* : \pi_n(X) \to \pi_n(Y)$ by sending $\alpha : S^n \to X$ to the composition $f \circ \alpha : S^n \to Y$.

Proposition 1.4.2. *Any two based maps $(X,*) \rightrightarrows (Y,*)$ with a basepoint-preserving homotopy between them induce the same map $\pi_n(X,*) \to \pi_n(Y,*)$. If $f : (X,*) \to (Y,*)$ has a homotopy inverse (not necessarily respecting the basepoint!) then $f_* : \pi_n(X,*) \to \pi_n(Y,*)$ is an isomorphism.*

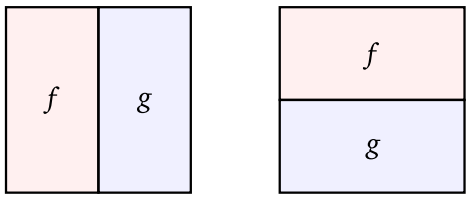
Theorem 1.4.3 (van Kampen)**.** *If X is a union of path-connected open sets (or path-connected subcomplexes) A_α containing the basepoint, and if the double intersections $A_\alpha \cap A_\beta$ and triple intersections $A_\alpha \cap A_\beta \cap A_\gamma$ are path-connected, then $\pi_1(X)$ is the free product of the groups $\pi_1(A_\alpha)$, modulo the relation that for each loop $[\gamma] \in \pi_1(A_\alpha \cap A_\beta)$, its images in $\pi_1(A_\alpha)$ and $\pi_1(A_\beta)$ are identified.*

The connectedness assumptions can be removed from the van Kampen theorem if one is willing to take a colimit of "groupoids" instead of a colimit of groups. See e.g. [May99].

Corollary 1.4.4. *For a connected CW complex X, $\pi_1(X)$ is computed by taking the graph $X^{(1)}$, contracting a spanning tree of this graph to a point, then taking a generator for each remaining 1-cell in $X^{(1)}$ and a relation for each 2-cell in $X^{(2)}$.*

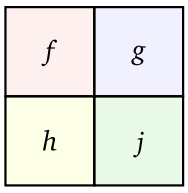
There is no such neat formula for π_n in general, making these groups hard to compute for finite complexes, even for spheres.

The proof that $\pi_2(X)$ is abelian can be done in a more formal way using the Eckmann-Hilton argument. We observe that the "horizontal composition" and the "vertical composition" of two maps f and g, illustrated below, give two operations on the set $\pi_2(X)$.



$f \square g$ $\qquad$ $f \triangle g$

Each of these operations has an identity element, given by the constant map to the basepoint, and the two operations interchange in the sense that the following picture is well-defined:



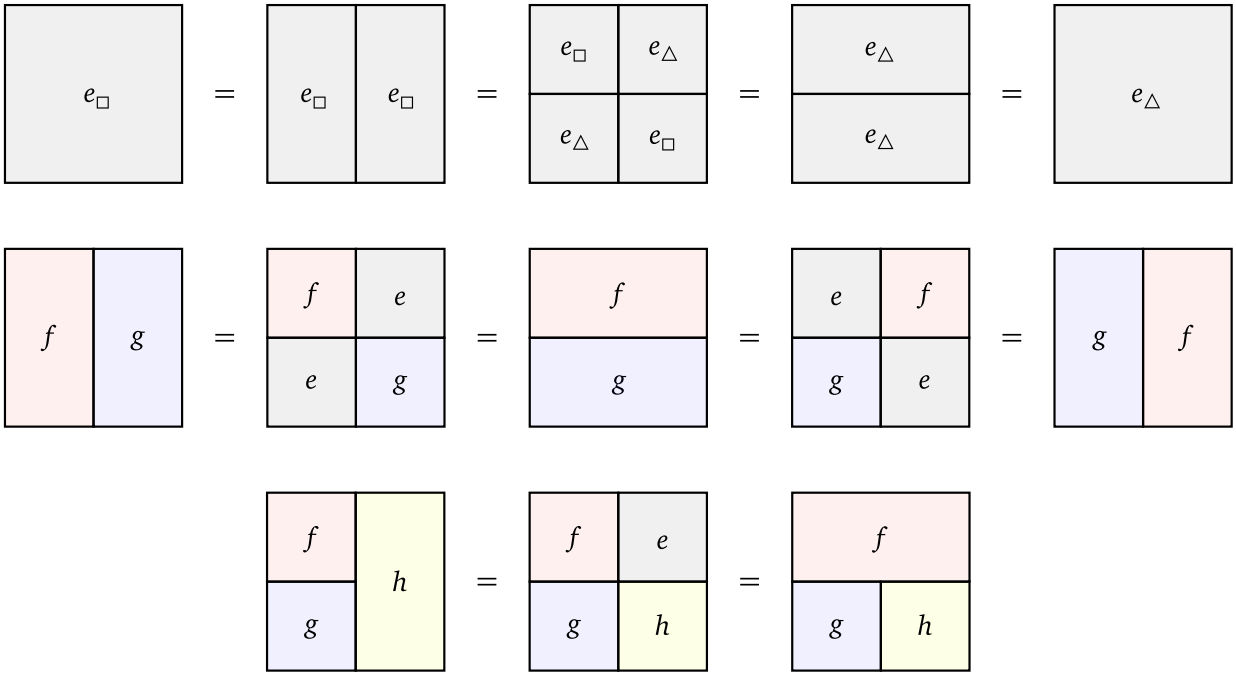
$$(f \square g) \triangle (h \square j) = (f \triangle h) \square (g \triangle j)$$

Lemma 1.4.5 (Eckmann-Hilton argument)**.** *Suppose S is a set with two binary operations*

$$\square : S \times S \to S, \quad \triangle : S \times S \to S,$$

each operation has an identity element $e_\square$ and $e_\triangle$, and that the two operations satisfy the interchange law illustrated just above. Then the two operations coincide ($\square = \triangle$), they have the same identity element, and this single operation is associative and commutative as well.

We promised not to give proofs in this chapter, but this one can be illustrated in just a few pictures, and the argument is so irresistible that we have to at least draw the pictures:

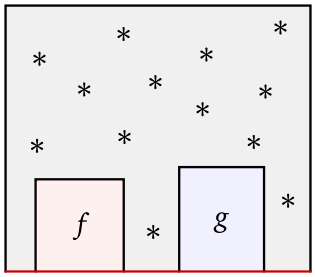


Returning to the subject of homotopy groups, for any subspace $A \subseteq X$ containing the basepoint we define the nth **relative homotopy group** $\pi_n(X,A)$ to be homotopy classes of maps of triples $(D^n, S^{n-1}, *) \to (X, A, *)$. Again, it is more convenient to define these as maps of triples

$$(I^n, \partial I^n, J^{n-1}) \to (X, A, *)$$

where J^{n-1} is the subset of boundary ∂I^n in which the interior of the bottom face has been removed. The group operation here is parametrized by nice embeddings that touch the bottom face of I^n, as shown.

$f + g$ in $\pi_n(X,A)$



So there are only $(n-1)$ different directions to compose the maps, instead of n directions

as before. Therefore $\pi_n(X,A)$ is an abelian group when $n \geq 3$, a group when $n = 2$, a set when $n = 1$, and is undefined when $n = 0$. Note that $\pi_n(X,\{*\}) \cong \pi_n(X)$.

More generally, for any based map $f : A \to X$ we can define $\pi_n(X,A)$ to be the set of commuting squares of based maps of the form

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\subseteq} & D^n \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & X, \end{array}$$

up to based homotopies of the vertical maps that make the square commute at each time in the homotopy.

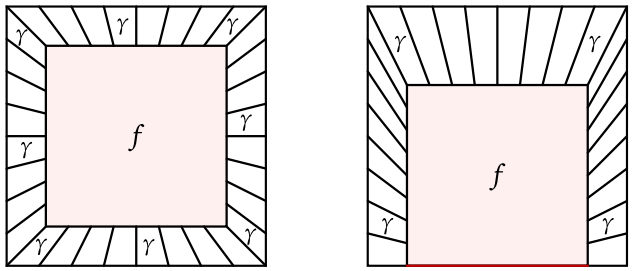
Theorem 1.4.6. *There is a natural long exact sequence of sets*

$$\cdots \xrightarrow{\partial} \pi_n(A) \xrightarrow{f_*} \pi_n(X) \longrightarrow \pi_n(X,A) \xrightarrow{\partial} \pi_{n-1}(A) \xrightarrow{f_*} \cdots$$

As in Proposition 1.2.13, "exact" means that the image of each map coincides with the preimage of the basepoint under the next map in the sequence. The maps in this sequence come from the maps of pairs

$$(A,*) \longrightarrow (X,*) \longrightarrow (X,A),$$

and the operation that restricts each map $(D^n, S^{n-1}) \longrightarrow (X,A)$ to a map $S^{n-1} \to A$. The maps in the sequence are group homomorphisms up until $\pi_1(A) \to \pi_1(X)$, after which $\pi_1(X,A)$ is a set. There is an action of $\pi_1(A)$ on every term in this long exact sequence that is illustrated below.



$\gamma \cdot f$ in $\pi_n(A)$ or $\pi_n(X)$ $\qquad$ $\gamma \cdot f$ in $\pi_n(X,A)$

Figure 1.4.7

One might call this action "conjugation," since the resulting action of $\pi_1(A)$ on itself is in fact the conjugation action

$$g \cdot a := gag^{-1}.$$

Every map of the long exact sequence commutes with this $\pi_1(A)$-conjugation, and so every map in the sequence for $n \geq 2$ is a map of $\mathbb{Z}[\pi_1(A)]$-modules.

Remark 1.4.8. More generally, for any maps of spaces $B \to A \to X$ there is a natural long exact sequence of sets

$$\cdots \xrightarrow{\partial} \pi_n(A,B) \xrightarrow{f_*} \pi_n(X,B) \longrightarrow \pi_n(X,A) \xrightarrow{\partial} \pi_{n-1}(A,B) \xrightarrow{f_*} \cdots$$

which is a long exact sequence of groups up until $\pi_2(X,A)$.

Theorem 1.4.9. *If $p: E \to B$ is a Serre fibration with fiber F, there is natural long exact sequence of sets*

$$\cdots \xrightarrow{\partial} \pi_n(F) \longrightarrow \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \cdots$$

which is a long exact sequence of groups up until $\pi_1(B)$. Furthermore, the image of $\pi_1(B)$ in the set $\pi_0(F)$ may be identified with the cosets of $p_*\pi_1(E)$ in the group $\pi_1(B)$.

This is proven by combining the exact sequence of Theorem 1.4.6 with the isomorphism $\pi_n(E,F) \cong \pi_n(B)$ that is induced by the map of pairs $(E,F) \to (B,*)$.

When $E \to B$ is a covering space, F is discrete, and this sequence becomes

$$\pi_n(E) \xrightarrow{\cong} \pi_n(B) \qquad \forall n \geq 2, \qquad\qquad 0 \longrightarrow \pi_1(E) \xrightarrow{p_*} \pi_1(B) \longrightarrow F \longrightarrow *.$$

As before, the exact sequence on the right is an exact sequence of sets. Without loss of generality both E and B are path-connected; then we can set $G = \pi_1(E)$, $H = p_*\pi_1(E)$, and then the set F is in bijection with the cosets G/H. When H is normal, G/H is still a group, and it is isomorphic to the group of deck transformations of the covering space. (A deck transformation is a homeomorphism $E \cong E$ commuting with the map to B.)

Let X be a connected CW complex. (This is equivalent to being a path-connected CW complex.) A **universal cover** of X is a path-connected covering space $p: \widetilde{X} \to X$ with $\pi_1(\widetilde{X}) = 0$. Universal covers exist and are unique, in the sense that any two must have a homeomorphism between them that commutes with the map to X. By the previous results, $\pi_1(X)$ can be identified with both the fiber $p^{-1}(x_0)$ and with the deck transformation group of the universal cover $\widetilde{X}$.

Remark 1.4.10. Along these identifications, the fiber of $p: \widetilde{X} \to X$ is $G = \pi_1(X)$. There are two actions of G on this fiber: one is through deck transformations, and the other is through path-lifting. In general these are not the same action! One is the action of G on itself by multiplication on the left, while the other is by multiplication on the *right*!

1.4.2 Cells and connectivity

A map of CW complexes $f: X \to Y$ is **cellular** if $f(X^{(n)}) \subseteq Y^{(n)}$ for all n.

Proposition 1.4.11 (Cellular approximation)**.** *Every map of CW complexes is homotopic to a cellular map.*

Corollary 1.4.12. $\pi_i(S^n) = 0$ *for* $i < n$*. More generally, attaching an* n*-cell does not affect* π_i *for* $i < n-1$*.*

Definition 1.4.13. A map $A \to X$ is **n-connected** if it is surjective on π_n, and an isomorphism on π_i for $0 \le i < n$, for every choice of basepoint in A. (It is only necessary to check for one basepoint in each path component.) Equivalently, a map is n-connected if it is surjective on π_0, and the relative groups $\pi_i(X,A)$ vanish for $1 \le i \le n$.

A space is n-connected if the map $* \to X$ is n-connected. Equivalently, $\pi_i(X)$ vanishes for $0 \le i \le n$. A 0-connected space is path-connected, and a 1-connected space is simply-connected.

A **weak homotopy equivalence**, or just weak equivalence, is a map $A \to X$ inducing isomorphisms on π_n for all $n \ge 0$ and all choices of basepoint in A. Equivalently, it is n-connected for all $n \ge 0$. Two spaces X and Y are **weakly equivalent** if they are connected by a zig-zag of weak equivalences:

$$X \xrightarrow{\sim} X_1 \xleftarrow{\sim} X_2 \xrightarrow{\sim} \cdots \xleftarrow{\sim} Y$$

We abbreviate this as $X \simeq Y$. In general, it may not be possible to simplify this zig-zag to a single weak equivalence $X \to Y$.

Every homotopy equivalence is a weak equivalence, but not vice-versa. So homotopy equivalent spaces are weakly equivalent, but not the other way around.

Example 1.4.14. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ with the discrete topology. Let $\mathbb{N}^*$ be the one-point compactification, and let $\mathbb{N}_+$ be $\mathbb{N}$ with a disjoint basepoint, which is still a discrete space. Then there is a weak equivalence $\mathbb{N}_+ \to \mathbb{N}^*$, but there is no weak equivalence $\mathbb{N}^* \to \mathbb{N}_+$, and the two spaces are not homotopy equivalent.

Proposition 1.4.15 (Cell attachment)**.** *For any map* $A \to X$ *and any element* $\alpha \in \pi_n(X,A)$*, one can attach an* n*-cell to* A *to get a factorization* $A \to A' \to X$*, such that*

$$\pi_i(X,A) \to \pi_i(X,A')$$

is an isomorphism for $i < n$*, and surjective for* $i = n$*, with kernel containing* α*.*

Informally, this means that attaching an n-cell to A makes

$$\begin{aligned} \pi_n(A) &\to \pi_n(X) && \text{more surjective,} \\ \pi_{n-1}(A) &\to \pi_{n-1}(X) && \text{more injective, and} \\ \pi_k(A) &\to \pi_k(X) && \text{unchanged for } k < n-1. \end{aligned}$$

See also Remark 2.6.21.

Corollary 1.4.16. *$A \to X$ is n-connected iff it is weakly equivalent to a relative CW complex $A \to X'$ in which every cell has dimension $\geq n+1$.*

Corollary 1.4.17 (Cofibrant replacement)**.** *Every map $A \to X$ factors into a relative CW complex $A \to B$ and a weak equivalence $B \to X$.*

This replacement can be done in a functorial way, so each commuting square as shown on the left gives a commuting diagram as on the right.

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ A' & \longrightarrow & X' \end{array} \qquad\qquad \begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & X' \end{array}$$

In particular, there is a functor $Q\colon \mathbf{Top} \to \mathbf{Top}$ taking each space X to a CW complex QX, with a weak equivalence $QX \xrightarrow{\sim} X$. As X varies, these form a natural transformation from Q to the identity functor $\mathrm{id}\colon \mathbf{Top} \to \mathbf{Top}$.

Theorem 1.4.18 (Whitehead)**.** *Every weak equivalence between CW complexes is a homotopy equivalence. If $A \subseteq X$ is the inclusion of a weakly equivalent subcomplex, then X deformation retracts onto A.*

Whitehead's theorem tells us that the study of all spaces up to weak equivalence is essentially the same thing as the study of CW complexes up to homotopy equivalence. This is great because homotopy equivalences are far better behaved. For instance, a homotopy equivalence is always represented by a single map $X \to Y$, there is no need to use zig-zags.

Cell attachment is also used to build the **Postnikov tower**. This is a sequence of path-connected spaces and maps of the form

$$X \longrightarrow \cdots \longrightarrow P_n X \longrightarrow P_{n-1}X \longrightarrow \cdots \longrightarrow P_2 X \longrightarrow P_1 X \longrightarrow P_0 X = *,$$

such that $X \to P_n X$ is an isomorphism on π_i for $i \leq n$, and the remaining homotopy groups of $P_n X$ vanish. (Since the spaces are all path-connected, the choice of basepoint does not matter.) So $P_1 X$ only has π_1, while $P_2 X$ only has π_1 and π_2, and so on. This tower can be defined for every based path-connected space X in such a way that each P_n is a functor, and each map in the above diagram is a natural transformation.

Finally, we relate homotopy groups back to homology and cohomology groups.

Definition 1.4.19. The **Hurewicz map** $\pi_n(X) \to H_n(X)$ takes each element $S^n \to X$ to the image of some fixed generator of $H_n(S^n)$ in $H_n(X)$. We similarly define the relative version $\pi_n(X,A) \to H_n(X,A)$ using a fixed generator of $H_n(D^n, S^{n-1})$.

The Hurewicz map is a natural transformation and a homomorphism when $n \geq 1$, while the relative version is a homomorphism when $n \geq 2$.

Proposition 1.4.20. *[Hat02, Prop 4.21] Any weak equivalence $f : X \to Y$ induces an isomorphism on singular homology and cohomology, with any coefficients.*

Proposition 1.4.21 (Hurewicz)**.** *The Hurewicz map induces*

- *an isomorphism $\pi_1(X)^{\text{ab}} \cong H_1(X)$ if X is path-connected,*
- *an isomorphism $\pi_n(X) \cong H_n(X)$ if X is $(n-1)$-connected and $n \geq 2$, and*
- *an isomorphism $\pi_n(X,A) \cong H_n(X,A)$ if (X,A) is $(n-1)$-connected, $n \geq 2$, and A is 1-connected.*

As a result, for simply-connected spaces, the lowest nonzero homotopy group of a space or map is controlled by the lowest nonzero homology group. Above that lowest level, homology says much less about homotopy, and the homotopy groups are much more mysterious.

Corollary 1.4.22 (Whitehead)**.**

- *A map of simply-connected CW complexes $X \to Y$ is a homotopy equivalence iff $H_*(X) \to H_*(Y)$ is an isomorphism.*
- *A map of connected CW complexes $X \to Y$ is a homotopy equivalence iff $\pi_1(X) \to \pi_1(Y)$ and $H_*(\widetilde{X}) \to H_*(\widetilde{Y})$ are both isomorphisms. (Here $\widetilde{X}$ and $\widetilde{Y}$ are the universal covers.)*

In other words, homology and the fundamental group are powerful enough to detect homotopy equivalences. We can reduce even further to homology with field coefficients:

Proposition 1.4.23. *A map of spaces, or more generally a map of chain complexes, gives an isomorphism on homology with $\mathbb{Z}$ coefficients iff it gives an isomorphism with $\mathbb{Q}$ coefficients and with $\mathbb{Z}/p$ coefficients for every prime number p.*

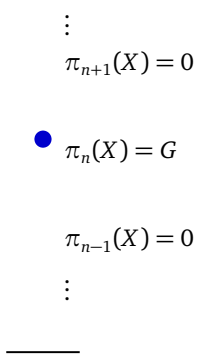
So to detect homotopy equivalences, we only need the fundamental group, covering spaces, and homology with *field coefficients*.

1.4.3 Eilenberg–Mac Lane spaces and the Yoneda lemma

Definition 1.4.24. An **Eilenberg–Mac Lane space** $K(G,n)$ is a CW complex with only one nonzero homotopy group, $\pi_n \cong G$. We fix the identification between $\pi_n(K(G,n))$ and G.

Lemma 1.4.25. *Eilenberg–Mac Lane spaces are unique up to homotopy equivalence. The homotopy equivalence is also unique up to homotopy, if we insist that it respects the chosen identification $\pi_n \cong G$.*

More generally, for any homomorphism of abelian groups $\varphi : G \to H$, there is a unique homotopy class of based maps $K(G,n) \to K(H,n)$ whose induced map on π_n is the homomorphism φ. So the set of based homotopy classes $[K(G,n), K(H,n)]_*$ is identified with the set of group homomorphisms $G \to H$.



Corollary 1.4.26. *We have canonical homotopy equivalences*

$$K(G,n) \simeq \Omega K(G,n+1).$$

Theorem 1.4.27 (Brown representability)**.** *For each abelian group G and integer $n \geq 0$, there is a natural isomorphism*

$$H^n(X,A;G) \cong [X/A, K(G,n)]_*$$

where (X,A) ranges over all pairs of a CW complex X and a subcomplex A.

(See Theorem 2.5.25 for a proof.) Note that each side is a contravariant functor from the category of CW-pairs and continuous maps of pairs to the category of abelian groups.

Example 1.4.28. The circle is an Eilenberg–Mac Lane space, $S^1 = K(\mathbb{Z},1)$, so the Brown representability theorem tells us that $H^1(X,A) \cong [X/A, S^1]_*$. In particular,

$$H^1(X) \cong [X_+, S^1]_* \cong [X, S^1].$$

So first cohomology is just homotopy classes of maps to the circle. Intuitively, the cohomology class tells us how many times each 1-cycle in X wraps around S^1. Since this is a natural isomorphism, for each map $f : X \to Y$, the induced map on cohomology corresponds to the map $[Y, S^1] \to [X, S^1]$ that pre-composes with f.

In Proposition 1.3.15 we saw that every cofiber sequence $A \to X \to Cf$ gives a long exact sequence on cohomology. We could prove this again using the Brown representability theorem: take the long exact sequence in Proposition 1.2.13 with $Z = K(G,n)$, and use the isomorphisms

$$[\Sigma X, K(G,n)]_* \cong [X, \Omega K(G,n)]_* \cong [X, K(G,n-1)]_*$$

to link them together for different values of n.

Brown representability is powerful when combined with the Yoneda Lemma, which we recall next. For any category $\mathbf{C}$, let $\mathbf{C}^{\mathrm{op}}$ be the **opposite category**, whose morphisms are just the morphisms of $\mathbf{C}$ in reverse:

$$\mathbf{C}^{\mathrm{op}}(X,Y) := \mathbf{C}(Y,X).$$

We like to use this because contravariant functor out of $\mathbf{C}$ is the same thing as a functor out of $\mathbf{C}^{\mathrm{op}}$. Now given any object $X \in \mathrm{ob}\,\mathbf{C}$, the sets

$$F(Y) = \mathbf{C}(Y,X)$$

define a contravariant functor from $\mathbf{C}$ to sets, in other words a functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$. We call this the **functor represented by** X. Let $G\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ be any other functor from $\mathbf{C}^{\mathrm{op}}$ to sets.

Lemma 1.4.29 (Yoneda)**.** *Natural transformations $\eta\colon F \Rightarrow G$ correspond bijectively to elements of the set $G(X)$. The bijection takes each transformation η to the element $\eta(\mathrm{id}_X)$.*

Corollary 1.4.30. *If F is represented by X and G is represented by Y, natural transformations $\eta\colon F \Rightarrow G$ correspond to morphisms $X \to Y$ in $\mathbf{C}$. Natural isomorphisms $F \cong G$ correspond to isomorphisms $X \cong Y$ in $\mathbf{C}$.*

If you've never seen this before, you should think of the following fact from linear algebra: if V is a one-dimensional vector space spanned by the vector v, and W is any other vector space, then linear transformations $T\colon V \to W$ correspond to elements of W, the bijection given by sending each transformation T to the vector $T(v) \in W$.

The proof of the Yoneda Lemma is identical to the proof of this fact from linear algebra. The vector space V corresponds to the representable functor $F(Y) = \mathbf{C}(Y,X)$, the vector v is the identity element $\mathrm{id}_x \in \mathbf{C}(X,X)$, the morphisms of $\mathbf{C}^{\mathrm{op}}$ are the "scalars" that act on this vector to give all other "vectors" in $F(Y)$ for $Y \in \mathbf{C}$, and the linear transformation T corresponds to the natural transformation η. Once you know where the generator id_x goes, everything else is determined by "multiplying the result by scalars."

We apply the Yoneda Lemma to the category $h\mathbf{CW}$, whose objects are CW complexes and whose morphisms are homotopy classes of maps. By the Brown representability theorem, cohomology $H^n(-;G)$ is the functor $h\mathbf{CW}^{\mathrm{op}} \to \mathbf{Ab}$ represented by $K(G,n)$:

$$H^n(X;G) \cong H^n(X_+,*;G) \cong [X_+,K(G,n)]_* \cong [X,K(G,n)].$$

Therefore, any natural transformation from cohomology to cohomology must arise from a map of Eilenberg–Mac Lane spaces in the homotopy category $h\mathbf{CW}$.

Example 1.4.31. We have a bijection

$$[X, S^1] \times [X, S^1] \cong [X, S^1 \times S^1],$$

so the functor $H^1(X) \times H^1(X)$ is represented by the product $S^1 \times S^1$. We also know that $H^2(X)$ is represented by $K(\mathbb{Z}, 2) \simeq \mathbb{CP}^\infty$. Therefore by the Yoneda Lemma, the cup product $H^1(X) \times H^1(X) \to H^2(X)$ must arise from some map of spaces $c : S^1 \times S^1 \to \mathbb{CP}^\infty$. Once we know what c is, the cup product is the map

$$[X, S^1] \times [X, S^1] \cong [X, S^1 \times S^1] \xrightarrow{c \circ -} [X, \mathbb{CP}^\infty].$$

More generally, for any ring R and any $m, n \geq 0$ the cup product

$$H^m(-; R) \times H^n(-; R) \xrightarrow{\cup} H^{m+n}(-; R)$$

is a natural transformation from the functor represented by $K(R, m) \times K(R, n)$ to the functor represented by $K(R, m+n)$. By the Yoneda lemma, this also must come from some map of spaces

$$c : K(R, m) \times K(R, n) \longrightarrow K(R, m+n).$$

This is a convenient point of view. It's often much easier to manipulate maps of spaces than it is to manipulate the algebraic definition of the cup product.

1.5 Basic homotopy colimits and limits

By Proposition 1.3.2 and Proposition 1.3.3, whenever $f : A \to X$ is a cofibration, the short exact sequence of spaces

$$A \longrightarrow X \longrightarrow X/A,$$

gives a long exact sequence on homology. If $A \to X$ is not a cofibration, this fails to be true, because the quotient X/A does not have the "correct homotopy type" for the quotient of X by A. However, this can be fixed: we can take the homotopy cofiber $C^{(u)}f = C^{(u)}A \cup_A X$ instead. The short exact sequence of spaces

$$A \longrightarrow X \longrightarrow C^{(u)}f$$

always gives a long exact sequence on homology, for any map f. Furthermore, when f is a cofibration these are the same sequence, because $C^{(u)}f \simeq X/A$. In other words, the quotient X/A may or may not have the correct homotopy type, but the homotopy cofiber $C^{(u)}f$ always has the correct homotopy type that X/A is "supposed to have."

The same phenomenon happens for other kinds of colimits. There is a "correct homotopy type" for every colimit. When the colimit happens to have this correct homotopy type, we can calculate its homology from the homology of the pieces $X(i)$.

Unfortunately, the colimit doesn't always have this correct homotopy type. However, it is always possible to fix this by making a thicker version of the colimit called the **homotopy colimit**, which is *always* correct. And the ordinary colimit is correct whenever it is equivalent to the homotopy colimit.

To be more specific, recall from Section 1.1 that the colimit of a diagram $X : \mathbf{I} \to \mathbf{Top}$ is constructed as the disjoint union of the spaces $X(i)$, modulo an identification $x \sim f(x)$ for each of the maps f in the diagram. In the homotopy colimit, this construction is thickened up by creating a *path* from x to $f(x)$, instead of gluing them directly together. If the diagram is sufficiently complicated, we also have to create additional paths between these paths.

In this section, we restrict our attention to what we call the three basic homotopy colimits:

- coproducts,
- homotopy pushouts, and
- homotopy sequential colimits.

These three cases are especially important, in part because all other homotopy colimits can be built out of them.[9] The reader interested in the case of a general homotopy colimit may skip ahead to Proposition 5.3.21 or Definition 7.4.23. The expository paper [Dug08] is also an excellent reference.

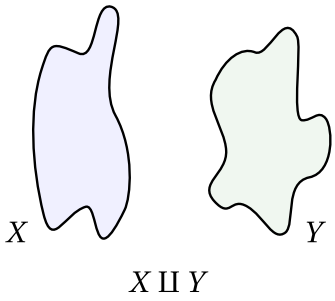
Remark 1.5.1. We are being deliberately vague about the meaning of the term "correct homotopy type." We will make this precise later by saying it is the *left-derived functor* of the colimit (see Definition 3.4.15 and Definition 3.5.10). For now, we just want to build intuition for the correct homotopy type, and connect it to the examples you might have seen in your algebraic topology classes.

1.5.1 Homotopy colimits of unbased spaces

As mentioned above, there are three basic homotopy colimits.

[9] In fact, all homotopy colimits can be built out of just coproducts and homotopy pushouts!

The first is the coproduct, or disjoint union, $\coprod_\alpha X_\alpha$. This is the colimit of a diagram of spaces $\{X_\alpha\}$ that has no nontrivial morphisms. It is also the homotopy colimit of $\{X_\alpha\}$. Indeed, there are no additional morphisms, so there are no homotopies to add!

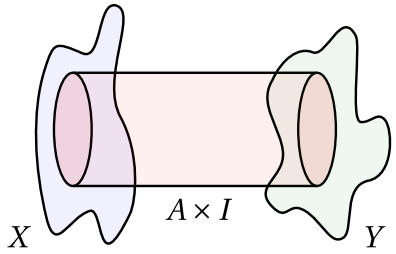


So in this case, the homotopy colimit and the actual colimit are the same thing. In other words, coproducts are always correct.

The next example is the pushout.

Definition 1.5.2. The **homotopy pushout** of X and Y along A is defined as the double mapping cylinder

$$X \cup_A^h Y := X \cup_{(A\times\{0\})} (A \times I) \cup_{(A\times\{1\})} Y.$$



Collapsing away the cylinder gives a map to the ordinary pushout $X \cup_A Y$.

Lemma 1.5.3. *This collapse map is a weak equivalence if*

- *X and Y are open subsets of a topological space and A is their intersection, or*
- *X and Y are CW complexes sharing a common subcomplex A, or more generally*
- *either $A \to X$ or $A \to Y$ is a cofibration.*

So the ordinary pushout is correct under any of these assumptions.

Example 1.5.4. The unbased homotopy cofiber $C^{(u)}f$ and unreduced suspension SX from Section 1.2 can be expressed as homotopy pushouts

$$C^{(u)}f = * \cup_A^h X, \qquad SX = * \cup_X^h *.$$

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ * & \longrightarrow & C^{(u)}f \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & SX \end{array}$$

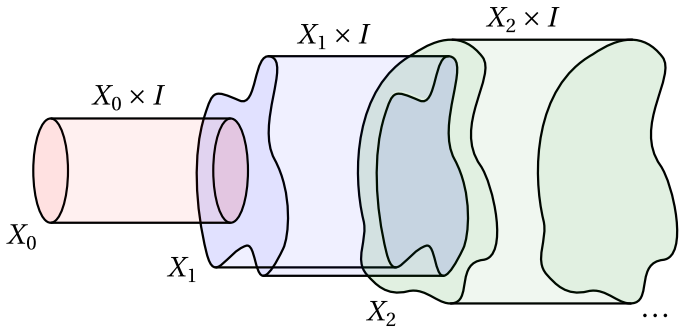
The last basic colimit is the colimit of a sequence of maps

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots.$$

We take the homotopy colimit in this case by forming successive mapping cylinders.

Definition 1.5.5. The **mapping telescope** or **sequential homotopy colimit** is formed by taking a cylinder on each space X_n and gluing the ends together along the maps f_n:

$$\operatorname*{hocolim}_{n\to\infty} X_n := \coprod_{n\geq 0} \big(X_n \times [0,1]\big) \Big/ \big((x_n, 1) \sim (f_{n+1}(x_n), 0)\big)$$



Collapsing down the telecope gives a map to the ordinary colimit, $\operatorname*{colim}_{n\to\infty} X_n$.

Lemma 1.5.6. *This collapse map is a weak equivalence if*

- *the X_n are all open subsets of a topological space, or*
- *the X_n are all subcomplexes of a CW complex, or more generally*
- *the maps $f_n : X_n \to X_{n+1}$ are closed inclusions.*[10]

What properties does the "correct" homotopy type have? We can give more detail in each of these three examples:

- For coproducts $\coprod_\alpha X_\alpha$, the correct homotopy type gives a direct sum on homology (Proposition 1.3.4), and on homotopy gives the homotopy groups of each of the spaces X_α, depending on where we pick our basepoint.

  On cohomology, and more generally on homotopy classes of maps $[-, Y]$, we get a product:

$$\left[\coprod_\alpha X_\alpha, Y\right] \cong \prod_\alpha [X, Y].$$

[10] This relies on the assumption that we are in CGWH spaces. In ordinary topological spaces, we would have to assume that the maps f_n are cofibrations.

- For pushouts $X \cup_A Y$, the correct homotopy type gives the Mayer-Vietoris exact sequence on homology

$$\cdots \xrightarrow{\partial} H_n(A) \xrightarrow{i} H_n(X) \oplus H_n(Y) \xrightarrow{j} H_n(X \cup_A^h Y) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i} \cdots$$

  where the map i is induced by the given maps $A \to X$ and $A \to Y$, and j is similar except that one of the two maps is negated. On cohomology we get the same sequence with maps reversed, while on $[-, Z]$ we get the weaker statement that the map to the fiber product

$$[X \cup_A^h Y, Z] \twoheadrightarrow [X, Z] \times_{[A,Z]} [Y, Z]$$

  is surjective. On the fundamental group π_1, the van Kampen theorem applies. In the case where A, X, and Y are connected, this gives $\pi_1(X \cup_A^h Y)$ as the free product $\pi_1(X) * \pi_1(Y)$ modulo relations from $\pi_1(A)$. On π_n, we get the homotopy excision or Blakers-Massey theorem, which says that the map

$$\pi_n(X, A) \to \pi_n(X \cup_A^h Y, Y)$$

  is an isomorphism in a range, depending on the connectivity of the maps $A \to X$ and $A \to Y$ [Hat02, Thm 4.23].

- For sequential colimits, the correct homotopy type is the one in which the colimit passes to the homology and homotopy groups:

$$H_k\left(\operatorname*{hocolim}_{n\to\infty} X_n\right) \cong \operatorname*{colim}_{n\to\infty} H_k(X_n), \qquad \pi_k\left(\operatorname*{hocolim}_{n\to\infty} X_n\right) \cong \operatorname*{colim}_{n\to\infty} \pi_k(X_n).$$

  More generally, the colimit commutes with $[Y, -]$ if Y is finite CW, see Exercise 1.7.31. On the other hand, for cohomology we get a $\lim^1$ exact sequence

$$0 \longrightarrow \lim_{n\to\infty}{}^1 H^{k-1}(X_n) \longrightarrow H^k\left(\operatorname*{hocolim}_{n\to\infty} X_n\right) \longrightarrow \lim_{n\to\infty} H^k(X_n) \longrightarrow 0. \tag{1.5.7}$$

We recall the definition of $\lim^1$, the "derived functor" of the limit:

Definition 1.5.8. For any inverse system of abelian groups

$$\cdots \longrightarrow A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0$$

we define the map of products

$$\prod_n A_n \xrightarrow{1-f} \prod_n A_n$$

by sending the infinite sequence $(a_n)_{n\geq 0}$ to the sequence $(a_n - f_{n+1}(a_{n+1}))_{n\geq 0}$. Essentially, this is the identity map, minus the map that applies f_n to each A_n. The limit of the original system of abelian groups is the kernel of $(1-f)$. We define $\lim^1$ to be its cokernel:

$$0 \longrightarrow \lim_{n\to\infty} A_n \longrightarrow \prod_n A_n \xrightarrow{1-f} \prod_n A_n \longrightarrow \lim_{n\to\infty}{}^1 A_n \longrightarrow 0.$$

It is well known that $\lim^1_{n\to\infty} A_n$ vanishes:

- when the maps f_n are isomorphisms for all sufficiently large n, or more generally
- when the maps f_n are surjective for all sufficiently large n, or more generally
- when for each m, the image of the map $A_n \to A_m$ becomes constant for all sufficiently large n (the *Mittag-Leffler condition*).

The following is a fundamental theorem about homotopy colimits. For the three basic homotopy colimits, it can be deduced from our description of how the homotopy colimit interacts with homology and homotopy groups.

Theorem 1.5.9. *Any map of diagrams $X \to Y$ that gives a weak equivalence*

$$X(i) \xrightarrow{\sim} Y(i)$$

for each i, gives a weak equivalence of homotopy colimits

$$\operatorname*{hocolim}_{i\in\mathbf{I}} X(i) \xrightarrow{\sim} \operatorname*{hocolim}_{i\in\mathbf{I}} Y(i).$$

Corollary 1.5.10. *The smash product of spaces $(X,Y) \mapsto X \wedge Y$ preserves weak homotopy equivalences, if X and Y are well-based topological spaces.*

In other words, if $X \to X'$ and $Y \to Y'$ are weak equivalences of well-based spaces, then $X \wedge Y \to X' \wedge Y'$ is a weak equivalence. This corollary follows from Remark 1.2.9, and the fact that X and Y are well-based.

Remark 1.5.11. Recall that the homotopy category of spaces $h\mathbf{Top}$ is the one in which the morphisms are homotopy classes of maps $[X,Y]$. This category does *not* have pushouts or sequential colimits. So, the homotopy colimit is *not* the colimit of the diagram in the homotopy category.

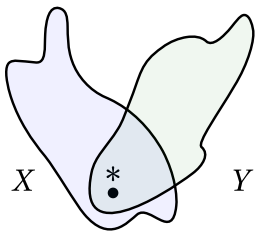
1.5.2 Homotopy colimits of based spaces

Suppose now that we have a diagram X in which the spaces $X(i)$ are based, and the maps preserve the basepoints.

Definition 1.5.12. The **based homotopy colimit** is defined by first making the spaces $X(i)$ well-based, in other words, making $* \to X(i)$ is a cofibration. Then we take the quotient of the unbased homotopy colimit by the subspace of all the basepoints and the paths between them:

$$\operatorname{hocolim}^{(b)} X(i) = \big(\operatorname{hocolim}^{(u)} X(i)\big)\big/\big(\operatorname{hocolim}^{(u)} *\big).$$

If the spaces $X(i)$ are not well-based, then the above construction is not the homotopy colimit, because it may have the wrong homotopy type.

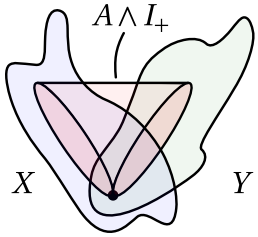


Theorem 1.5.9 applies to based homotopy colimits as well – they preserve all weak equivalences of diagrams.

In particular, the based homotopy coproduct is the wedge sum $\bigvee_\alpha X_\alpha$, assuming X_α is well-based. The special case of two spaces $X \vee Y$ is is illustrated to the right.

The based homotopy pushout is

$$X \cup_{(A\times\{0\})} \left(\frac{A\times I}{\{*\}\times I}\right) \cup_{(A\times\{1\})} Y, \tag{1.5.13}$$



assuming A, X, and Y are well-based. This is also illustrated to the right – it is the double mapping cylinder with the subspace $\{*\}\times I$ collapsed to a point.

Similarly, the based mapping telescope is

$$\coprod_{n\geq 0}\left(\frac{X_n\times[0,1]}{\{*\}\times[0,1]}\right)\Big/\big((x_n,1)\sim(f_{n+1}(x_n),0)\big). \tag{1.5.14}$$

If you have a diagram of based spaces, you have the option of taking the based homotopy colimit and getting a based space, or taking the unbased homotopy colimit and getting an unbased space. In general, these are not equivalent to each other. For instance, the disjoint union and the wedge sum are not equivalent:

$$X \amalg Y \not\simeq X \vee Y.$$

However, *in the special case* of a pushout or a sequential colimit, the unbased and based versions of the homotopy colimit do give the same homotopy type. So it does not matter which one you use.

Lemma 1.5.15.
- *If A, X, and Y are well-based then the collapse map from the unbased homotopy pushout of Definition 1.5.2 to the based homotopy pushout in (1.5.13) is a homotopy equivalence.*
- *The collapse map from the unbased mapping telescope of Definition 1.5.5 to the based mapping telescope in (1.5.14) is always a weak equivalence.*[11] *(Exercise 1.7.32)*

Example 1.5.16. The homotopy cofiber Cf and reduced suspension ΣX from Section 1.2 are the based homotopy pushouts of the same diagrams from Example 1.5.4:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ * & \longrightarrow & Cf \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X. \end{array}$$

Remark 1.5.17. It is an unfortunate fact that most authors use the same notation for the based and unbased version of homotopy colimits. We sometimes use decorations (u) to distinguish the unbased ones, as we did with $C^{(u)}f$, but for the most part we follow historical convention and use the same notation for both. We are at least partly justified in doing this by Lemma 1.5.15.

1.5.3 Homotopy limits

The dual of a homotopy colimit is a **homotopy limit**. Since the universal property of a limit is stated using maps *into* $X(i)$, we add homotopies to this limit by using path spaces $X(i)^I = \mathrm{Map}(I, X(i))$ instead of cylinders $X(i) \times I$. As with colimits, we have three basic homotopy limits:

- products,
- homotopy pullbacks, and
- homotopy sequential limits.

The first of these is the product $\prod_\alpha X_\alpha$. As before, this is already a homotopy limit, so products of spaces are always correct.[12]

The next case is the pullback. Suppose we have a pullback diagram of spaces

$$\begin{array}{ccc} & & X \\ & & \downarrow f \\ Y & \xrightarrow{g} & B. \end{array}$$

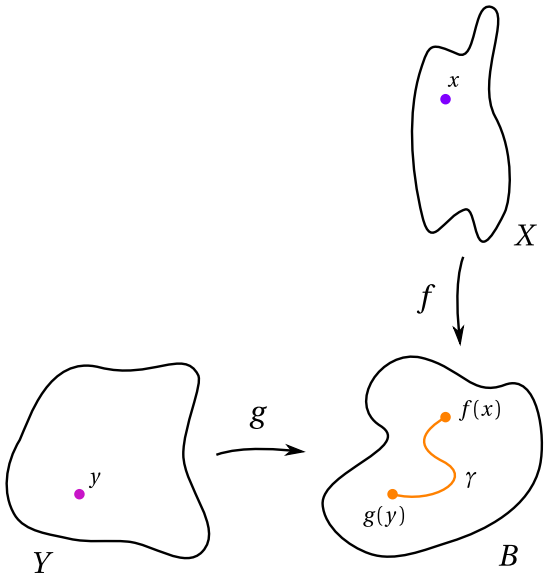
[11] This is isn't true if we move away from (CGWH) spaces. If phrased for general spaces, we also need the X_n to be well-based.

[12] It is a bit of a surprise that in spectra, infinite products are not always correct – the factors have to be made into Ω-spectra first.

Definition 1.5.18. The **homotopy pullback** is the homotopy fiber product

$$\begin{aligned} X \times_B^h Y &= X \times_B B^I \times_B Y \\ &= \{\, x \in X,\ y \in Y, \gamma \colon I \to B \ :\ f(x) = \gamma(0),\ g(y) = \gamma(1) \,\}. \end{aligned}$$

Here the two maps $B^I \rightrightarrows B$ are the evaluation at the two endpoints of I. In other words, a point in the homotopy pullback is a choice of two points $x \in X$ and $y \in Y$, and a path connecting their images in B.



The inclusion of constant paths $B \to B^I$ gives a map from the ordinary fiber product $X \times_B Y$ to homotopy pullback $X \times_B^h Y$.

Lemma 1.5.19. *The inclusion $X \times_B Y \to X \times_B^h Y$ is a weak equivalence if $X \to B$ or $Y \to B$ is a Serre fibration.*

Example 1.5.20. The homotopy fiber Fp and based loopspace ΩX from Section 1.2 can be expressed as homotopy pullbacks

$$Fp = * \times_B^h E, \qquad \Omega X = * \times_X^h *.$$

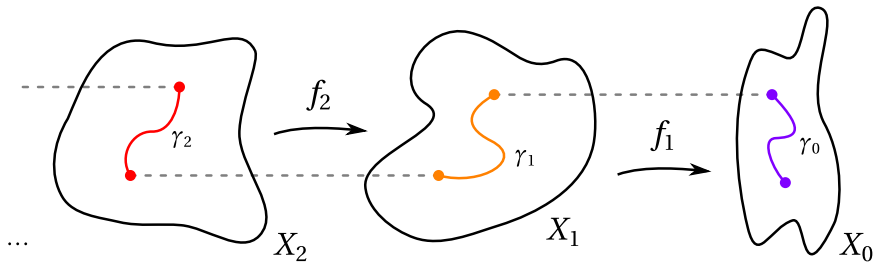
$$\begin{array}{ccc} Fp & \longrightarrow & E \\ \downarrow & & \downarrow p \\ * & \longrightarrow & B \end{array} \qquad \begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \end{array}$$

Definition 1.5.21. The **mapping microscope** or **sequential homotopy limit** of a sequence of maps

$$\cdots \longrightarrow X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

is the infinite homotopy fiber product

$$\begin{aligned}\operatorname*{holim}_{n\to\infty} X_n &= \cdots \times_{X_2} X_2^I \times_{X_1} X_1^I \times_{X_0} X_0^I \\ &= \{\, \gamma_n \colon I \to X_n \quad \forall n \geq 0 \;:\; f_n(\gamma_n(1)) = \gamma_{n-1}(0) \,\}\,.\end{aligned}$$



The ordinary limit includes into this homotopy limit, as the subspace in which all of the paths are constant.

Lemma 1.5.22. *This inclusion map is a weak equivalence if all of the maps f_n are Serre fibrations.*

In each of these cases, the homotopy limit captures the correct homotopy type:

- For products, the correct homotopy type is the one that gives a product on homotopy groups, and more generally on $[Y,-]$:

$$\left[Y, \prod_\alpha X_\alpha\right] \cong \prod_\alpha [Y,X].$$

  On homology, we get the Künneth formula (Theorem 1.3.11) for finite products.

- For pullbacks, the correct homotopy type is the one in which we get the homotopy Mayer-Vietoris sequence

$$\cdots \longrightarrow \pi_n(X \times_B^h Y) \longrightarrow \pi_n(X) \oplus \pi_n(Y) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(X \times_B^h Y) \longrightarrow \cdots,$$

  see Exercise 1.7.22. More generally, we get a surjective map

$$[Z, X \times_B^h Y] \twoheadrightarrow [Z,X] \times_{[Z,B]} [Z,Y]$$

  and similarly with based homotopy classes of maps. On homology and cohomology, we get the Eilenberg-Moore spectral sequence, and when $Y = *$ we get the Serre spectral sequence (see **??**).

- For sequential homotopy limits, the correct homotopy type is the one in which we get a $\lim^1$ exact sequence

$$0 \longrightarrow \lim_{n\to\infty}{}^1\, \pi_{k+1}(X_n) \longrightarrow \pi_k\left(\operatorname*{holim}_{n\to\infty} X_n\right) \longrightarrow \lim_{n\to\infty} \pi_k(X_n) \longrightarrow 0. \tag{1.5.23}$$

  See Definition 1.5.8 for a recollection of $\lim^1$.

  The homology of the homotopy limit has no relation to the homology of the spaces X_n in general. The most we can say is that if the maps $f_n\colon X_n \to X_{n-1}$ have connectivity increasing to ∞, then the sequential limit commutes with homology and cohomology. So,

$$H_k\left(\operatorname*{holim}_{n\to\infty} X_n\right) \cong \lim_{n\to\infty} H_k(X_n) \quad \text{and} \quad H^k\left(\operatorname*{holim}_{n\to\infty} X_n\right) \cong \operatorname*{colim}_{n\to\infty} H^k(X_n)$$

  if the connectivity of f_n goes to ∞.

The dual of Theorem 1.5.9 is the following.

Theorem 1.5.24. *Any map of diagrams $X \to Y$ that gives a weak equivalence*

$$X(i) \xrightarrow{\sim} Y(i)$$

for each i, gives a weak equivalence of homotopy limits

$$\operatorname*{holim}_{i\in\mathbf{I}} X(i) \xrightarrow{\sim} \operatorname*{holim}_{i\in\mathbf{I}} Y(i).$$

Homotopy limits for based spaces are defined in exactly the same way as homotopy limits for unbased spaces. If the spaces $X(i)$ are all based and the maps preserve the basepoints, then the homotopy limit that we already constructed has a canonical choice of basepoint. For instance, in the product it is the element that is the basepoint in every coordinate, while in the homotopy pullback it is given by the basepoint of X, the basepoint of Y, and the constant path at the basepoint in B.

The reason for this is given in the next section: the forgetful functor $\mathbf{Top}_* \to \mathbf{Top}$ is a right adjoint, so it commutes with limits. It turns out that this also means it preserves homotopy limits, at least up to equivalence (Exercise 3.6.29). On the other hand, $\mathbf{Top}_* \to \mathbf{Top}$ is not a left adjoint, and it doesn't preserve colimits or homotopy colimits.

1.6 Adjunctions

Recall from Section 1.1 that there is a bijection between continuous maps

$$K \times X \to Y \quad \text{and} \quad K \to \operatorname{Map}(X, Y).$$

This is an example of an "adjunction" from category theory.

Definition 1.6.1. An **adjunction** between categories **C** and **D** is a pair of functors

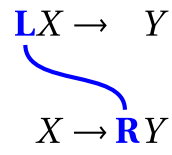
$$\mathbf{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{D},$$

and for each object X in **C** and Y in **D**, a bijection between the sets of morphisms

$$\{f : LX \to Y\} \leftrightarrow \{\widetilde{f} : X \to RY\},$$

that are natural in X and Y. In other words we have bijections $\mathbf{D}(LX, Y) \cong \mathbf{C}(X, RY)$ that are natural in X and Y.

The functor L is called the **left adjoint** and R is called the **right adjoint**. The maps f and $\widetilde{f}$ are **adjunct**. We often say that $(L \dashv R)$, or that (L, R) is an **adjoint pair**.



Adjunctions are a simple manipulation: whenever you want, you can pull the L off the source of a map and stick an R to the target, without gaining or losing information.

The naturality means that if we have another map $g : X' \to X$, then the maps

$$LX' \xrightarrow{L(g)} LX \xrightarrow{f} Y$$
$$X' \xrightarrow{g} X \xrightarrow{\widetilde{f}} RY$$

are also adjunct. In other words, $\widetilde{(f \circ L(g))} = \widetilde{(f)} \circ g$. We also have a similar condition for composing with maps $h : Y \to Y'$.

Remark 1.6.2. You can remember the naturality statement by thinking of the adjunction as giving you a natural notion of "a map across categories" from $X \in \mathbf{C}$ to $Y \in \mathbf{D}$. A map in this sense from X to Y is defined by either a map $LX \to Y$ in **D** or a map $X \to RY$ in **C**. They mean the same thing.

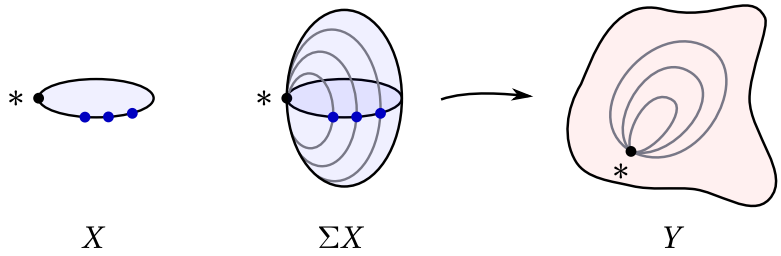
We can take such a "map across categories" from X to Y and compose it with an arrow $X' \to X$ in **C**. This has two possible interpretations, but the naturality statement says that those two interpretations produce the same "map across categories" from X' to Y. We will expand on this point of view in Section 10.6.1.

Example 1.6.3. In unbased spaces **Top**, we have an adjoint pair in which $K \times -$ is the left adjoint and $\mathrm{Map}(K, -)$ is the right adjoint.

In based spaces $\mathbf{Top}_*$, we have an adjoint pair in which $K \wedge -$ is the left adjoint and $\mathrm{Map}_*(K, -)$ is the right adjoint.

Corollary 1.6.4. *Suspension and loops are adjoint: based maps $\Sigma X \to Y$ are in canonical correspondence with based maps $X \to \Omega Y$.*

This means any time we have a map $\Sigma X \to Y$, we can immediately substitute for it a map $X \to \Omega Y$. This particular case can be visualized explicitly: each point in $x \in X$ gives a loop through ΣX, that traverses the segment $I \times \{x\}$ inside $I \times X$. Giving a based map $\Sigma X \to Y$ is the same thing as sending each of these loops to a based loop in Y, in a continuous way. That's the same thing as a based map $X \to \Omega Y$.



Corollary 1.6.5. *For every based space B we have bijections*[13]

$$\pi_n(B) \cong \pi_{n-1}(\Omega B) \cong \cdots \cong \pi_1(\Omega^{n-1}B) \cong \pi_0(\Omega^n B).$$

This could also be deduced from Proposition 1.2.19, or from Theorem 1.4.9 by taking the map $* \to B$ and making it into a fibration. Its homotopy fiber is ΩB.

Example 1.6.6. The forgetful functor from based spaces to unbased spaces

$$\mathbf{Top}_* \to \mathbf{Top}$$

is a right adjoint. Its left adjoint is the functor that adds a disjoint basepoint, $(-)_+$. In other words, based maps $X_+ \to Y$ correspond to unbased maps $X \to Y$.

Example 1.6.7. The forgetful functor from unbased spaces to sets

$$\mathbf{Top} \to \mathbf{Set}$$

is a right adjoint. Its left adjoint is the functor that gives each set S the discrete topology. If X is a space, then continuous maps $S \to X$ with this topology correspond to maps of sets $S \to X$.

Example 1.6.8. The forgetful functor from abelian groups to sets

$$\mathbf{Ab} \to \mathbf{Set}$$

is a right adjoint. Its left adjoint is the free abelian group functor, taking each set S to the direct sum $\bigoplus_{s \in S} \mathbb{Z}$. Maps of abelian groups

$$\bigoplus_{s \in S} \mathbb{Z} \to A$$

correspond to maps of sets $S \to A$.

[13] In fact, these are all group isomorphisms. The group structure in each case can be defined using either π_k when $k \geq 1$, or the loop-concatention operation on $\Omega^{n-k}B$ when $k \leq n-1$. These give the same group structure by the Eckmann-Hilton argument (Lemma 1.4.5).

Theorem 1.6.9. *Left adjoints preserve colimits. Given any left adjoint $L\colon \mathbf{C} \to \mathbf{D}$ and diagram $X\colon \mathbf{I} \to \mathbf{C}$, we have a canonical isomorphism*

$$\operatorname*{colim}_{i\in\mathbf{I}} L(X(i)) \cong L\left(\operatorname*{colim}_{i\in\mathbf{I}} X(i)\right).$$

Similarly, right adjoints preserve limits:

$$\lim_{i\in\mathbf{I}} R(X(i)) \cong R\left(\lim_{i\in\mathbf{I}} X(i)\right).$$

To explain a little more what this canonical map is, if $F\colon \mathbf{C} \to \mathbf{D}$ is any functor, there is a canonical map of the form

$$\operatorname*{colim}_{i\in\mathbf{I}} F(X(i)) \longrightarrow F\left(\operatorname*{colim}_{i\in\mathbf{I}} X(i)\right) \tag{1.6.10}$$

whenever these two colimits exist. To see why, for each i we apply F to the map from $X(i)$ to the colimit of X, giving $F(X(i)) \to F(\operatorname{colim} X)$. As i varies, these assemble into a map from the colimit of the $F(X(i))$ to $F(\operatorname{colim} X)$, as claimed. If the colimits always exist, then we can think of this even more simply as a natural transformation

$$\operatorname{colim} \circ F \Rightarrow F \circ \operatorname{colim}.$$

Usually this canonical map isn't an isomorphism. We say that F **preserves colimits** when it is an isomorphism. Similar considerations give a canonical map going the other way

$$F\left(\lim_{i\in\mathbf{I}} X(i)\right) \longrightarrow \lim_{i\in\mathbf{I}} F(X(i)), \tag{1.6.11}$$

or more simply

$$F \circ \lim \Rightarrow \lim \circ F,$$

and we say that F **preserves limits** when this is an isomorphism. You can remember which way each map goes because "it's easy to map out of a colimit, and it's easy to map into a limit."

Example 1.6.12. Theorem 1.6.9 explains why limits are "the same" in the categories **Top**, $\mathbf{Top}_*$, **Ab**, and **Set**, but colimits are different. Because to go from one of these categories to another "without changing" your object, you use a forgetful functor, for instance the forgetful functor $\mathbf{Ab} \to \mathbf{Set}$. This forgetful functor is a right adjoint, so it must preserve limits! On the other hand, it's not a left adjoint, as we can see from the fact that it doesn't preserve colimits.

The left adjoints between these categories do commute with colimits, but each of them changes the underlying set. For instance, the disjoint basepoint functor $(-)_+$ from **Top** to $\mathbf{Top}_*$ adds an extra point. *This* functor is the one that commutes with colimits. In other words, a based colimit of a diagram of the form $X(i)_+$ is the same as the colimit of the $X(i)$, with one extra basepoint added.

Example 1.6.13. Taking Cartesian product $K \times -$ commutes with colimits of unbased spaces:

$$K \times \operatorname{colim}^{(u)} X(i) \cong \operatorname{colim}^{(u)} (K \times X(i)).$$

Taking smash product $K \wedge -$ commutes with colimits of based spaces:

$$K \wedge \operatorname{colim}^{(b)} X(i) \cong \operatorname{colim}^{(b)} (K \wedge X(i)).$$

Dually, taking the mapping space $\mathrm{Map}(K,-)$ or $\mathrm{Map}_*(K,-)$ commutes with limits. So for instance the based loopspace functor commutes with limits of based spaces:

$$\Omega\left(\lim X(i)\right) \cong \lim \Omega(X(i)).$$

Theorem 1.6.14. *Left adjoints are unique. If $L_1, L_2 \colon \mathbf{C} \to \mathbf{D}$ are two functors that are both left adjoint to the same functor $R \colon \mathbf{D} \to \mathbf{C}$, then there is a unique natural isomorphism $L_1 \cong L_2$ that agrees with the identifications*

$$\mathbf{D}(L_1 X, Y) \cong \mathbf{C}(X, RY) \cong \mathbf{D}(L_2 X, Y).$$

The dual statement also hold for right adjoints.

This is a deeply useful result because it allows us to *define* something as a left adjoint, and know that the definition is unique. See Exercise 1.7.36 for a proof.

The **unit** of an adjunction $(L \dashv R)$ is the map

$$\eta \colon X \to RLX$$

adjunct to the identity map $LX \to LX$. The **counit** is the map

$$\epsilon \colon LRY \to Y$$

adjunct to the identity $RY \to RY$. These are natural transformations, satisfying the "triangle identities"

$$\begin{array}{ccc} LX & \xrightarrow{L\eta} & LRLX \\ & \searrow^{=} & \downarrow{\scriptstyle \epsilon L} \\ & & LX, \end{array} \qquad \begin{array}{ccc} RY & \xrightarrow{\eta R} & RLRY \\ & \searrow^{=} & \downarrow{\scriptstyle R\epsilon} \\ & & RY. \end{array}$$

If you know the unit and counit, you can reconstruct the adjunction: the map $f : LX \to Y$ is adjunct to the composite

$$X \xrightarrow{\eta} RLX \xrightarrow{Rf} RY,$$

and $\widetilde{f} : X \to RY$ is adjunct to the composite

$$LX \xrightarrow{L\widetilde{f}} LRY \xrightarrow{\epsilon} Y.$$

Going back and forth gives the same map back! See [Rie17, Ch. 4] for an extensive discussion.

1.7 Exercises

1.7.1. Verify that the formulas (1.1.6) and (1.1.5) for the colimit and limit of a diagram of spaces, have the required universal properties.

1.7.2. (a) Show that the limit of a diagram of the form

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

is isomorphic to the first term X_0. Therefore the limit is not very interesting, only the colimit is interesting.

(b) Explain in a similar way the trivial colimits and limits that appear in Figure 1.1.4.

(c) How would you generalize these examples? (Recall that an object a in a category $\mathbf{I}$ is **initial** if a admits a unique map to each object $b \in \mathbf{I}$, and a is **terminal** if it admits a unique map *from* each object $b \in \mathbf{I}$.)

1.7.3. Suppose G is a group and X is a space with a continuous left action by G. The space of orbits X_G is the quotient of X by the relation $x \sim gx$ for every $x \in X$ and $g \in G$. Explain how the orbit space is an example of a colimit. What is the corresponding limit?

1.7.4. Suppose $X : \mathbf{I} \to \mathbf{Top}_*$ is a diagram of based spaces and basepoint-preserving maps between them. By forgetting the basepoints, it also defines a diagram of unbased spaces $X : \mathbf{I} \to \mathbf{Top}$.

(a) Explain why the colimit of X could change when the universal property is phrased in the category of based spaces $\mathbf{Top}_*$, instead of the category of unbased spaces $\mathbf{Top}$.

(b) Check that the coproduct in based spaces is the wedge sum, as claimed, and not the disjoint union. This shows that in general the based and unbased colimits are not homeomorphic to each other.

(c) Prove that the unbased colimit and based colimit agree if $\mathbf{I}$ is **connected**, meaning every pair of objects is related by a finite zig-zag of morphisms

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \cdots \longleftarrow \bullet.$$

In particular, the definition of a pushout and of a sequential colimit has the same formula in based spaces that it had in unbased spaces.

(d) Prove that limits always have the same formula in both based spaces and unbased spaces.

1.7.5. A topological space X is *discrete* if its points are open; equivalently, if every subset is open.

(a) Prove that the colimit of any diagram of discrete spaces $X(i)$ is discrete. In particular, this is true for sequential colimits.

(b) Show by example that the limit of a diagram of discrete spaces $X(i)$ may not be discrete. In other words, "limits can create a topology."

You might consider the 2-adic integers, defined as the limit of the sets $\mathbb{Z}/(2^n\mathbb{Z})$ along the surjective homomorphisms

$$\mathbb{Z}/(2^n\mathbb{Z}) \xrightarrow{\cdot 1} \mathbb{Z}/(2^{n-1}\mathbb{Z})$$

that multiply by 1, i.e. that send $a \mapsto a$ for each $a \in \mathbb{Z}/(2^n\mathbb{Z})$.

1.7.6. Let **Ab** denote the category of abelian groups and group homomorphisms. If $A\colon \mathbf{I} \to \mathbf{Ab}$ is a diagram of abelian groups and homomorphisms, show that its colimit is given by the same formula as (1.1.6), except the disjoint unions are replaced by direct sums:

$$\operatorname*{colim}_{\mathbf{I}} A(i) = \left(\bigoplus_{i \in \operatorname{ob}\mathbf{I}} A(i)\right) \Big/ \big(a \sim f(a) \quad \forall f : i \to j,\ a \in A(i)\big).$$

On the other hand, its limit is given by the same formula as for spaces, (1.1.5).

1.7.7. Suppose we have a sequential diagram in an arbitrary category **C**

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \cdots$$

and that the diagram stabilizes, i.e. the maps f_i are isomorphisms from X_n onwards. Prove that the colimit of this diagram is isomorphic to X_n.

1.7.8. The system of abelian groups

$$\mathbb{Z} \xrightarrow{\cdot 1} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \cdots$$

does not stabilize. Prove that its colimit in abelian groups is isomorphic to the rational numbers $\mathbb{Q}$.

1.7.9. Recall that the forgetful functors $\mathbf{Top}_* \to \mathbf{Top} \to \mathbf{Set}$ and $\mathbf{Ab} \to \mathbf{Set}$ are right adjoints. Use this to explain why the limits are preserved in Exercises 1.7.4 and 1.7.6 but not in Exercise 1.7.5.

1.7.10. For several exercises now we have been using the sequential category that we denote

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots$$

This is a convenient illustration, but there is an issue: the picture does not show all of the morphisms in the category. Where is the composite of $0 \to 1$ and $1 \to 2$?

By convention, when we draw a category in this way as a directed graph, what we mean is the category generated by the directed graph. In this exercise, we make this notion precise.

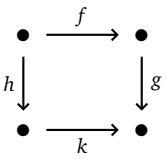
(a) Suppose we are given a directed graph, i.e. a set of objects V and a set of arrows E such that each arrow f has a chosen source $s(f)$ and target $t(f)$. We define a category $\mathbf{I}$ with the same objects V, whose morphisms from x to y are words $f_n \cdots f_2 f_1$, such that

$$s(f_0) = x, \quad t(f_i) = s(f_{i+1}), \quad t(f_n) = y.$$

When $x = y$ we also allow the empty word from x to x. Prove that this is a category. We call it the **category generated by the graph** (V, E, s, t).

(b) If we take the directed graph shown at the start of the exercise, prove that the category it generates has a unique morphism $m \to n$ when $m \leq n$ and no morphisms $m \to n$ when $m > n$.

(c) Warning! Prove that the category generated by the graph



is *not* the same as the category that encodes commuting squares. To get a commuting square we would have to add the relation $gf = kh$.

1.7.11. (a) If $\mathbf{I}$ is generated by the directed graph (V, E, s, t) as in the previous exercise, explain why when we take a colimit of a diagram $X : \mathbf{I} \to \mathbf{Top}$, we only have to mod out by the generating morphisms, not all of the morphisms. This is why in a sequential colimit, it is enough to identify each space X_n with its image in the next space X_{n+1}.

(b) What is the corresponding fact for limits?

1.7.12. In this exercise we show that the smash product in $\mathbf{Top}_*$ is associative up to canonical isomorphism. This is more subtle than the Cartesian product in **Top**, and it requires us to work in CGWH topological spaces. If we had used ordinary topological spaces, the claim would be false! (See [MS06, Thm 1.7.1] for a counterexample.)

(a) For three based spaces X, Y, and Z, define the three-fold smash product $X \wedge Y \wedge Z$ as the quotient of $X \times Y \times Z$ by the subspace of all triples (x, y, z) in

which at least one coordinate is the basepoint. Check that we have bijections of the underlying sets

$$(X \wedge Y) \wedge Z \cong (X \wedge Y \wedge Z) \cong X \wedge (Y \wedge Z). \tag{1.7.1}$$

(b) Recall from [Str09, Prop 2.20] that in CGWH spaces, the product of two quotient maps is a quotient map.
Use this to prove that the map from $X \times Y \times Z$ to each of the spaces in (1.7.1) is a quotient map. Therefore the topologies agree, so we have a homeomorphism $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$.

1.7.13. Let $X, Y, Z \in \mathbf{Top}_*$ be based topological spaces.

(a) Let $\mathbf{Set}(X, Y)$ be the collection of set maps (not necessarily continuous) from X to Y. Explain why there is a bijection between set maps $Z \to \mathbf{Set}(X, Y)$ and set maps $Z \times X \to Y$.

(b) The defining feature of the mapping space topology is that this correspondence passes to continuous maps $Z \to \mathrm{Map}(X, Y)$ and continuous maps $Z \times X \to Y$. Prove that along this correspondence, the based continuous maps $Z \to \mathrm{Map}_*(X, Y)$ landing in the subspace $\mathrm{Map}_*(X, Y)$ of based continuous maps, correspond to the continuous maps $Z \times X \to Y$ that factor through the smash product $Z \wedge X$ (Definition 1.1.16).

(c) Explain why a homotopy of maps $X \rightrightarrows Y$ preserving the basepoint is the same thing as a map $X \wedge I_+ \to Y$.

1.7.14. Suppose $f : A \to X$ is a cofibration, as in Definition 1.2.2.

(a) Prove that there is a closed neighborhood C of $i(A) \subseteq X$ that has a retract back to A. (Hint: Apply the homotopy extension property when $Y = (A \times I) \cup_{A \times \{0\}} (X \times \{0\})$, and let C be the preimage of $A \times [1/2, 1]$ intersected with $X \times \{1\}$.)

(b) Explain why, once we know that A is a subspace of X, the neighborhood C actually *deformation* retracts onto A. This is why cofibrations are sometimes called "neighborhood deformation retract (NDR) pairs."

1.7.15. Explain how the definition of a fibration is similar to the definition of a cofibration. We informally say that the definitions are dual to each other.

1.7.16. Prove that for any map $f : A \to X$, the inclusion of $A \times \{0\}$ into the mapping cylinder

$$A \times I \cup_{A \times \{1\}} X$$

is a cofibration. Dually, for any map $p : E \to B$, prove that the projection

$$E \times_B B^I \to B, \qquad \{\, (e, \gamma) \in E \times B^I \;:\; p(e) = \gamma(0) \,\} \mapsto \gamma(1)$$

is a fibration.

1.7.17. Recall that $h\mathbf{CW}$ is defined to have objects the CW complexes and morphisms the *homotopy classes* of maps. Prove that every homotopy equivalence is an isomorphism in $h\mathbf{CW}$. Conversely, prove that any functor $\mathbf{CW} \to \mathbf{D}$ sending homotopy equivalences to isomorphisms must factor through a functor $h\mathbf{CW} \to \mathbf{D}$. In other words, inverting homotopy equivalences is the same thing as passing to homotopy classes of maps. (See also Proposition 3.1.27).

1.7.18. Prove that the pushout or pullback of an isomorphism is an isomorphism. Prove that the pushout of a cofibration is a cofibration. Prove that the pullback of a fibration is a fibration. (We define the pushout or pullback of a map in Definition 1.1.10.)

1.7.19. Prove that in the definition of a fibration, the dotted lift is in general not unique, but it is unique up to homotopy. (The homotopy should be through lifts that make the diagram commute!)

1.7.20. If Cf is the based homotopy cofiber of $f : A \to X$, and everything is well-based, prove that the based homotopy cofiber of $X \to Cf$ is equivalent to ΣA. Dually, prove that the homotopy fiber of $Fp \to E$ of a map $p : E \to B$ is equivalent to ΩB.

1.7.21. Give another example of two spaces that are weakly equivalent but not homotopy equivalent, different from the one in Example 1.4.14. (Hint: Recall the topologist's sine curve. What is it weakly equivalent to?)

1.7.22. Prove the following claim from Section 1.5. Suppose the commuting square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$$

is a homotopy pullback square. In other words, the induced map from A to the homotopy pullback

$$A \to B \times_D C \to B \times^h_D C$$

is a weak equivalence. Prove that for each choice of basepoint $a \in A$ with images $b \in B$, $c \in C$, and $d \in D$, there is a homotopy Mayer-Vietoris sequence

$$\cdots \longrightarrow \pi_{n+1}(D,d) \xrightarrow{\partial} \pi_n(A,a) \xrightarrow{(f_*,h_*)} \pi_n(B,b) \oplus \pi_n(C,c) \xrightarrow{k_*-g_*} \pi_n(D,d) \longrightarrow \cdots$$

(Hint: You might try computing the homotopy fiber of $A \to B \times C$.)

1.7.23. Prove that $E \to B$ is both a Serre fibration and a weak equivalence, if and only if a lift exists for every commuting square of the following form.

$$\begin{array}{ccc} S^{n-1} \subseteq & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ D^n & \longrightarrow & B \end{array}$$

1.7.24. Prove the claim that if the connectivity of the maps $f_n : X_n \to X_{n-1}$ increases to ∞, then the formation of the homotopy limit commutes with homology and cohomology. You might want to use the $\lim^1$ exact sequence for homotopy groups.

1.7.25. (a) Suppose $f : A \to X$ is a cofibration and Y is any space. Composing with f gives a continuous map

$$(-) \circ f : \mathrm{Map}(X, Y) \to \mathrm{Map}(A, Y).$$

Prove that this map is a fibration.

(b) Suppose $p : E \to B$ is a fibration and Y is any space. Composing with p gives a continuous map

$$p \circ (-) : \mathrm{Map}(Y, E) \to \mathrm{Map}(Y, B).$$

Prove that this map is a fibration.

1.7.26. Use the Yoneda Lemma (Lemma 1.4.29) to prove the homeomorphism (1.1.17)

$$\mathrm{Map}(K \times X, Y) \cong \mathrm{Map}(K, \mathrm{Map}(X, Y))$$

from the bijection on the underlying sets. (Hint: What functor is represented by each side? Can you show that the two functors are isomorphic?)

1.7.27. Use the Yoneda Lemma to prove the following homeomorphisms in the category of (CGWH) topological spaces.

(a) $\mathrm{Map}\left(\coprod_\alpha X_\alpha, Y\right) \cong \prod_\alpha \mathrm{Map}(X_\alpha, Y)$.

(b) $\mathrm{Map}\left(X, \prod_\alpha Y_\alpha\right) \cong \prod_\alpha \mathrm{Map}(X, Y_\alpha)$.

(c) If A is a subspace of Y, then $\mathrm{Map}(X, A)$ is a subspace of $\mathrm{Map}(X, Y)$.

(d) If $X/\sim$ is a quotient of X, then $\mathrm{Map}(X/\sim, Y)$ is a subspace of $\mathrm{Map}(X, Y)$.

(For (d) you will have to recall from [Str09, Prop 2.20] that if K is any space, then $K \times (X/\sim)$ is a quotient of $K \times X$.)

1.7.28. Use the previous exercise to deduce the following homeomorphisms for spaces of based maps, recalling that $\mathrm{Map}_*(X, Y)$ is always defined as a subspace of $\mathrm{Map}(X, Y)$.

(a) $\mathrm{Map}_*\left(\bigvee_\alpha X_\alpha, Y\right) \cong \prod_\alpha \mathrm{Map}_*(X_\alpha, Y)$.

(b) $\mathrm{Map}_*\left(X, \prod_\alpha Y_\alpha\right) \cong \prod_\alpha \mathrm{Map}_*(X, Y_\alpha)$.

(c) If A is a subspace of Y, then $\mathrm{Map}_*(X, A)$ is a subspace of $\mathrm{Map}_*(X, Y)$.

(d) If $X/\sim$ is a quotient of X, then $\mathrm{Map}_*(X/\sim, Y)$ is a subspace of $\mathrm{Map}_*(X, Y)$.

(e) $\mathrm{Map}_*(K \wedge X, Y) \cong \mathrm{Map}_*(K, \mathrm{Map}_*(X, Y))$.

1.7.29. Prove the following strengthening of the Yoneda Lemma, in the category of (CGWH) topological spaces. Let $\mathbf{Top}(-,-)$ denote the set of continuous maps, as opposed to the space of continuous maps. Suppose that X and Y are (CGWH) topological spaces, and that we have a natural isomorphism $\mathbf{Top}(Z,X) \cong \mathbf{Top}(Z,Y)$ for all *compact Hausdorff* spaces Z, not for all spaces Z. Prove that X and Y are still homeomorphic.

1.7.30. Suppose Y and X are based spaces. Recall that $[Y,X]$ refers to unbased homotopy classes of maps, and $[Y,X]_*$ refers to based homotopy classes of maps. In this exercise we describe the precise relationship between these two sets.

(a) Define the based space $Y \cup_* I$ by gluing the basepoint $* \in Y$ to the point $0 \in I$, and then take the basepoint of this new space to be $1 \in I$. There is an obvious collapse map of based spaces $Y \cup_* I \to Y$. If Y is well-based, prove that this is a basepoint-preserving homotopy equivalence, and therefore has a based homotopy inverse $Y \xrightarrow{\sim} Y \cup_* I$.

(b) If Y is well-based and X is path-connected, show that the forgetful map $[Y,X]_* \to [Y,X]$ is surjective.

(c) If Y is well-based, show that there is an action of $\pi_1(X)$ on $[Y,X]_*$, sending the map $f : Y \to X$ and $\alpha : I \to I$ to the composite

$$Y \simeq Y \cup_* I \xrightarrow{f \cup \alpha} X.$$

Then show that $[Y,X]_* \to [Y,X]$ is injective iff this action is trivial.

(d) We say that X is a strict H-space if there is a multiplication $\mu : X \times X \to X$ such that $\mu(*,x) = x$ and $\mu(x,*) = x$. (If X is well-based, this is equivalent to asking that $\mu(*,-)$ and $\mu(-,*)$ are only homotopic to the identity.) Prove in this case that the action of $\pi_1(X)$ on $[Y,X]_*$ is trivial.

So if X and Y are well-based and X is a connected H-space, we have $[Y,X]_* \cong [Y,X]$.

1.7.31. Suppose we have an infinite sequence of closed inclusions (of CGWH spaces)

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots$$

and Y is any compact topological space.

(a) Prove that every map $Y \to \operatorname{colim}_{n\to\infty} X_n$ factors through some finite stage X_n. (Recall that all spaces in this book are assumed weak Hausdorff, and this implies they are T_1. In other words, their points are closed.)

(b) Prove that homotopy classes of maps commute with the colimit,

$$\left[Y, \operatorname*{colim}_{n\to\infty} X_n\right] \cong \operatorname*{colim}_{n\to\infty} [Y, X_n].$$

Similarly if Y and all X_n are based we get $[Y, \operatorname*{colim}_{n\to\infty} X_n]_* \cong \operatorname*{colim}_{n\to\infty} [Y, X_n]_*$.

(c) Use Exercise 1.7.29 to get a homeomorphism

$$\operatorname{Map}\left(Y, \operatorname*{colim}_{n\to\infty} X_n\right) \cong \operatorname*{colim}_{n\to\infty} \operatorname{Map}(Y, X_n)$$

and in the based case

$$\operatorname{Map}_*\left(Y, \operatorname*{colim}_{n\to\infty} X_n\right) \cong \operatorname*{colim}_{n\to\infty} \operatorname{Map}_*(Y, X_n).$$

In particular, based loopspace commutes with the colimit,

$$\Omega\left(\operatorname*{colim}_{n\to\infty} X_n\right) \cong \operatorname*{colim}_{n\to\infty} \Omega X_n.$$

(d) Show by counterexample that these fail if Y is not compact.

1.7.32. As in the previous exercise, suppose we have an infinite sequence of maps, not necessarily closed inclusions this time

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots$$

and Y is any compact topological space.

(a) Explain how the unbased mapping telescope is the colimit of a sequence of closed inclusions $T_n \subseteq T_{n+1}$, where each T_n deformation retracts onto X_n. Conclude that

$$\left[Y, \operatorname*{hocolim}^{(u)}_{n\to\infty} X_n\right] \cong \operatorname*{colim}_{n\to\infty} [Y, X_n].$$

(b) Explain how the based mapping telescope is also a colimit along closed inclusions. Use this to prove

$$\left[Y, \operatorname*{hocolim}^{(b)}_{n\to\infty} X_n\right]_* \cong \operatorname*{colim}_{n\to\infty} [Y, X_n]_*$$

for any based compact space Y and based spaces X_n.

(c) Show that there is a canonical weak equivalence

$$\operatorname*{hocolim}^{(b)}_{n\to\infty} \Omega X_n \xrightarrow{\sim} \Omega\left(\operatorname*{hocolim}^{(b)}_{n\to\infty} X_n\right).$$

(d) (*) Construct an isomorphism

$$\left[Y, \operatorname*{hocolim}^{(u)}_{n\to\infty} X_n\right]_* \cong \operatorname*{colim}_{n\to\infty} [Y, X_n]_*$$

for any well-based compact space Y and based spaces X_n. You will have to drag the basepoint of Y around, as in Exercise 1.7.30.

Conclude that the collapse map from the unbased mapping telescope to the based mapping telescope is always a weak equivalence, proving part of Lemma 1.5.15.

1.7.33. Similarly to the previous exercises, but easier, assume we have a sequence of maps

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots$$

and Z is any topological space, not necessarily compact.

(a) Prove that there is a homeomorphism

$$\mathrm{Map}\left(\operatorname*{colim}_{n\to\infty} X_n, Z\right) \cong \lim_{n\to\infty} \mathrm{Map}(X_n, Z)$$

and in the based case

$$\mathrm{Map}_*\left(\operatorname*{colim}_{n\to\infty} X_n, Z\right) \cong \lim_{n\to\infty} \mathrm{Map}_*(X_n, Z).$$

You might find Exercises 1.7.27 and 1.7.28 helpful, or you may prefer to apply the Yoneda Lemma directly.

(b) Explain why this does not in general give a bijection on homotopy classes of maps

$$\left[\operatorname*{colim}_{n\to\infty} X_n, Z\right] \ncong \lim_{n\to\infty} [X_n, Z].$$

(It does give a $\lim^1$ sequence if we take based maps and Z is a loop space, so that the maps form an abelian group. In particular, this gives the $\lim^1$ sequence for cohomology of a homotopy colimit.)

1.7.34. Prove that we can form a mapping telescope of any diagram $X_0 \to X_1 \to \cdots$ using only coproducts and homotopy pushouts. (Hint: Try taking only the even terms X_{2n} in one coproduct, and only the odd terms X_{2n+1} in another.)

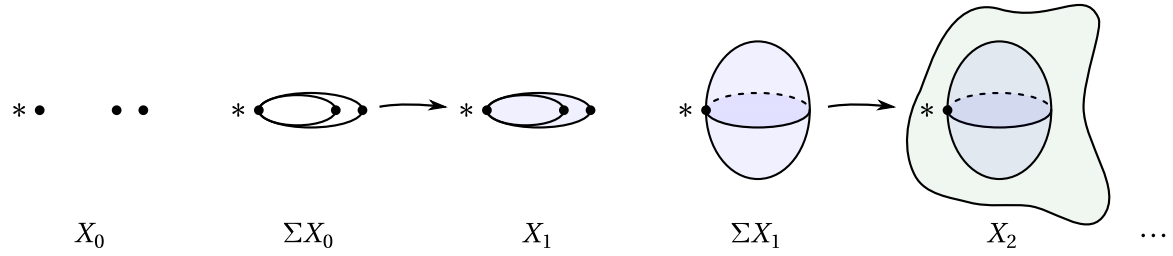
1.7.35. Adapt Example 1.6.13 from strict colimits to the homotopy colimits defined in Section 1.5. In other words, prove that $K \times -$ commutes with unbased homotopy colimits, and $K \wedge -$ commutes with based homotopy colimits.

1.7.36. Prove that left adjoints are unique (Theorem 1.6.14). (This follows almost immediately from the Yoneda Lemma, Lemma 1.4.29, which you've hopefully been able to practice in the last few exercises.)

Chapter 2

Spectra

In this chapter we begin the study of spectra and prove their fundamental properties. Spectra are like topological spaces that behave more algebraically, like an abelian group or a chain complex. In fact, the definition of a spectrum resembles the definition of a chain complex, except that the levels are topological spaces X_n, and instead of boundary maps we have "bonding maps" $\Sigma X_n \to X_{n+1}$ as shown.



Since spectra are like chain complexes, and the study of spectra is like homological algebra, spectra are easier to work with than spaces. Also, the calculations we make with spectra approximate the ones that we wanted to do with topological spaces. For instance, when X and Y are highly connected topological spaces, their wedge sum $X \vee Y$ and a product $X \times Y$ are equivalent in some range of degrees. In spectra, they are just equivalent!

We study spectra up to a thing called "stable equivalence." This is just like how in homotopy theory, we study spaces up to homotopy equivalence, or weak homotopy equivalence. It is possible to invert the stable equivalences and form a category called the stable homotopy category $\mathrm{Ho}\,\mathbf{Sp}$. However, we will save that until the next chapter, because we have to study spectra directly for a little while first.

2.1 Definition and basic examples

2.1.1 Spectra and stable homotopy groups

Definition 2.1.1. A **spectrum** X is a sequence of based spaces $\{X_n\}_{n\geq 0}$ and bonding maps (or structure maps) $\xi_n \colon \Sigma X_n \longrightarrow X_{n+1}$, which are continuous maps of based spaces.

A **map of spectra** $f : X \to Y$ is a sequence of continuous based maps $f_n \colon X_n \to Y_n$ such that each square of the following form commutes.

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\xi_n} & X_{n+1} \\ {\scriptstyle \Sigma f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma Y_n & \xrightarrow{\upsilon_n} & Y_{n+1} \end{array}$$

It is important to say right away that we only care about the *limiting behavior* of the spaces X_n as $n \to \infty$. To capture this idea, we define the homotopy groups of a spectrum as a colimit (direct limit) of the homotopy groups of the spaces X_n. Because of the suspensions, the homotopy groups shift in degree as we go from each level to the next.

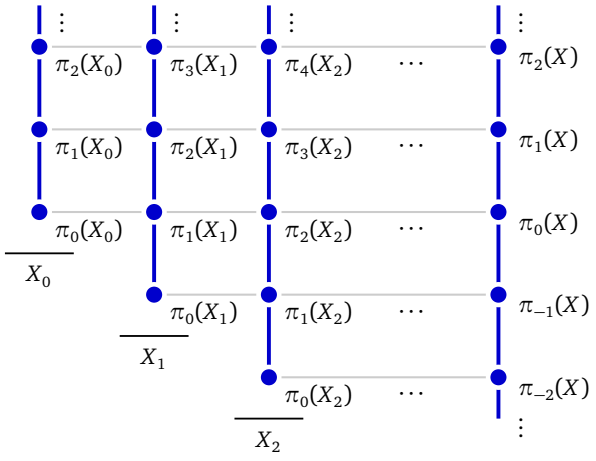
Definition 2.1.2. If X is a spectrum, for each $k \in \mathbb{Z}$, the kth **stable homotopy group** $\pi_k(X)$ is defined as the colimit of the system of abelian groups

$$\cdots \longrightarrow \pi_{k+n}(X_n) \xrightarrow{\sigma} \pi_{k+n+1}(\Sigma X_n) \xrightarrow{\xi_n} \pi_{k+n+1}(X_{n+1}) \longrightarrow \cdots$$

where each homotopy group is taken at the basepoint. The operation σ suspends each map $S^{k+n} \to X_n$ to produce a map $S^{k+n+1} \to \Sigma X_n$. When k is negative, the system is only defined for $n \geq |k|$. (See Exercise 1.7.6 for a discussion of colimits of abelian groups.)

Remark 2.1.3. In the literature, spectra of this form are sometimes called **sequential spectra** or **prespectra**. In this book, we will call them sequential spectra when we want to distinguish them from the diagram spectra that occur in Chapter 6.

Intuitively, what is going on in this definition is that every sphere in X_n becomes a sphere in X_{n+1} of *one dimension higher*. The stable homotopy groups are counting these spheres up to homotopy, in the limit where $n \to \infty$. The dimension of the sphere is therefore going to ∞, but the difference between this dimension and the level of the spectrum stays at a constant value $k \in \mathbb{Z}$.



We'll sometimes draw the homotopy groups of each level X_n as blue dots, connected by vertical blue lines. The bonding maps of the spectrum X create maps between these homotopy groups that shift degree by one. We depict this by drawing the levels of a spectrum in staggered formation, so that the maps between the homotopy groups go straight to the right. We then draw the stable homotopy groups all the way on the right-hand side.

Remark 2.1.4. We might say that an **element** of X of degree k is a map $\phi : S^{k+n} \to X_n$, up to the relation that ϕ is identified with $\sigma(\phi)$. This is the correct notion of "degree k points" in the spectrum X. The kth homotopy group $\pi_k(X)$ is the set of degree k elements up to homotopy.

Before proceeding further, it's helpful to fix some conventions. There are many homeomorphic models for the n-sphere, but we'll define it as the one-point compactification of $\mathbb{R}^n$:

$$S^n := \mathbb{R}^n \cup \{\infty\}.$$

Let $X \wedge Y$ denote the smash product (Definition 1.1.16). We define the reduced suspension to be $\Sigma X = X \wedge S^1$, with the S^1 on the right-hand side of X. Similarly, the n-fold suspension is $\Sigma^n X = X \wedge S^n$. We'll always use this convention for the suspensions that occur in the bonding maps of a spectrum.

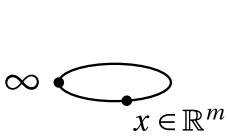
The smash product of two spheres is a sphere: $S^m \wedge S^n \cong S^{m+n}$. To see this, we recall that the smash product of two spaces is, away from the basepoint, just the Cartesian product:

$$(X \wedge Y) - \{*\} \cong (X - \{*\}) \times (Y - \{*\}).$$

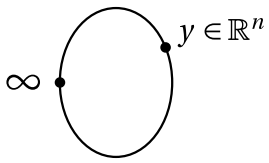
Therefore $S^m \wedge S^n$ minus its basepoint is identified with the product $\mathbb{R}^m \times \mathbb{R}^n$.

Definition 2.1.5. The canonical isomorphism $S^m \wedge S^n \cong S^{m+n}$ is the one that, after removing the basepoints, becomes the concatenation map

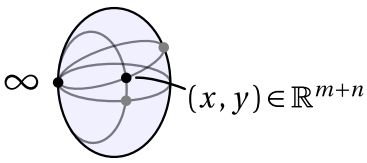
$$\begin{array}{ccc} \mathbb{R}^m \times \mathbb{R}^n & \xrightarrow{\cong} & \mathbb{R}^{m+n} \\ (x_1, \ldots, x_m), (y_1, \ldots, y_n) & \longmapsto & (x_1, \ldots, x_m, y_1, \ldots, y_n). \end{array}$$



$S^m = \mathbb{R}^m \cup \{\infty\}$



$S^n = \mathbb{R}^n \cup \{\infty\}$



$S^{m+n} = \mathbb{R}^{m+n} \cup \{\infty\}$

We can now add more precision to the definition of the stable homotopy groups in Definition 2.1.2. We define σ to be the operation that takes each map $\phi : S^{k+n} \to X_n$ to the composite map

$$S^{k+n+1} \xrightarrow{\cong} S^{k+n} \wedge S^1 \xrightarrow{\phi \wedge \mathrm{id}} X_n \wedge S^1, \qquad \sigma(\phi)$$

where the isomorphism is the canonical isomorphism of Definition 2.1.5.

2.1.2 Suspension and desuspension spectra

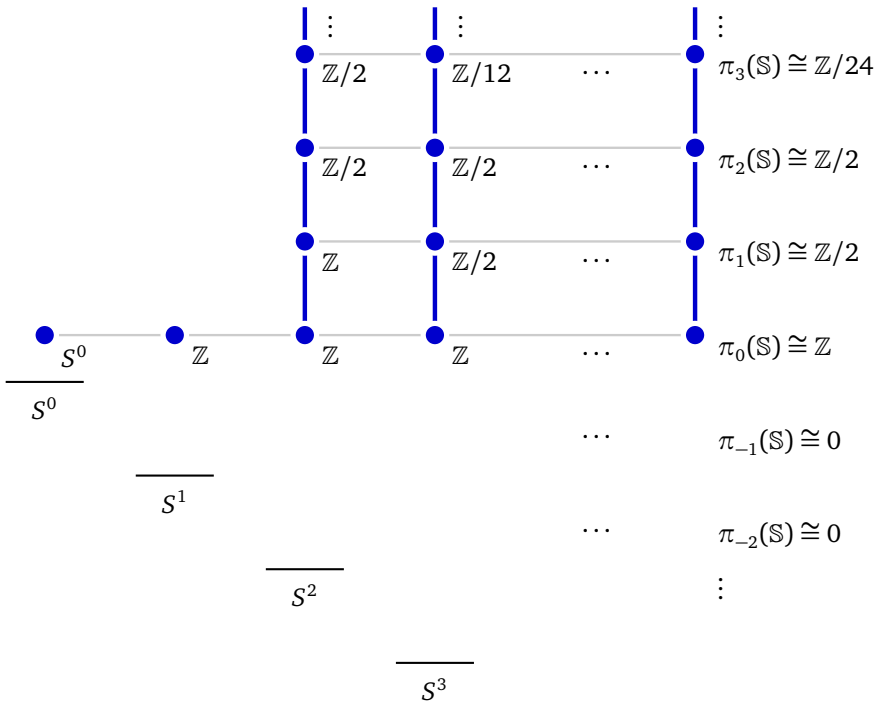
Example 2.1.6. The **sphere spectrum** $\mathbb{S}$ has nth level S^n and bonding maps the canonical homeomorphisms

$$S^n \wedge S^1 \cong S^{n+1}.$$

Following Definition 2.1.5, this isomorphism is given away from the basepoint by the formula

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R} &\xrightarrow{\cong} \mathbb{R}^{n+1} \\ (x_1, \ldots, x_n), (y_1) &\longmapsto (x_1, \ldots, x_n, y_1). \end{aligned}$$

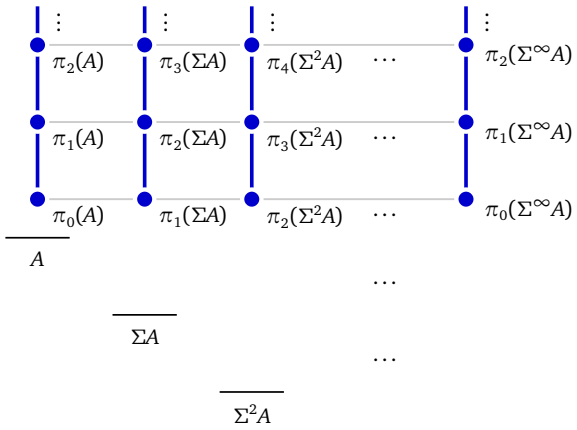
The first few stable homotopy groups of $\mathbb{S}$ are shown below.



The negative homotopy groups are all zero, since $\pi_{k+n}(S^n) = 0$ when $k < 0$ (Corollary 1.4.12). But, *we don't know all of its positive homotopy groups!* Computing them is

one of the central problems of homotopy theory, and to this day, it continues to be an open problem. It's strange, but in homotopy theory, we work frequently with groups that we cannot compute. It adds an air of mystery and adventure to the subject.

Example 2.1.7. If A is any based space, the **suspension spectrum** $\Sigma^\infty A$ is a spectrum whose nth level is $\Sigma^n A = A \wedge S^n$.



The bonding map is the canonical homeomorphism $A \wedge S^n \wedge S^1 \cong A \wedge S^{n+1}$. Its first two homotopy groups (assuming A is well-based) are:

$$\pi_0(\Sigma^\infty A) \cong H_0(A, *; \mathbb{Z}),$$
$$\pi_1(\Sigma^\infty A) \cong H_0(A, *; \mathbb{Z}/2) \oplus H_1(A, *; \mathbb{Z}),$$

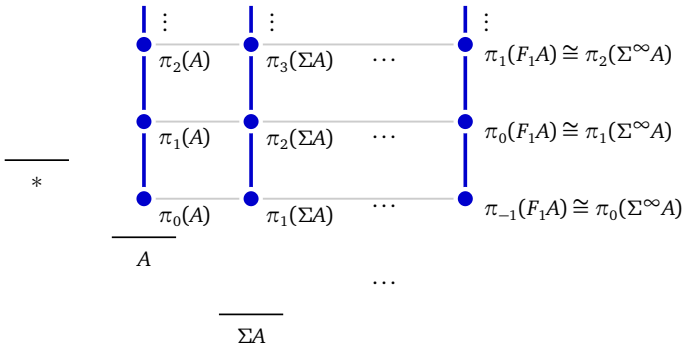
and the negative homotopy groups are zero. (See Exercise 2.7.45.) Note that the sphere spectrum $\mathbb{S}$ is the suspension spectrum $\Sigma^\infty S^0$.

If B is an unbased space, we define its **unbased suspension spectrum** by adding a disjoint basepoint and then taking the suspension spectrum. The resulting spectrum is denoted either $\Sigma^\infty B_+$ or $\Sigma^\infty_+ B$. So we have $\mathbb{S} = \Sigma^\infty_+(*)$, and

$$\pi_0(\Sigma^\infty_+ B) \cong H_0(B; \mathbb{Z}), \qquad \pi_1(\Sigma^\infty_+ B) \cong H_0(B; \mathbb{Z}/2) \oplus H_1(B; \mathbb{Z}).$$

Returning to the case where A is a based space, suppose we replace A by its d-fold suspension $\Sigma^d A$ for some $d \geq 0$. Then the suspension spectrum $\Sigma^\infty(\Sigma^d A)$ has the same homotopy groups as $\Sigma^\infty A$, but shifted up d slots. In spectra, we can give an inverse to this operation:

Example 2.1.8. For any integer $d \geq 0$, the **shift desuspension spectrum**, also called the **free spectrum** $F_d A$, is a spectrum whose nth level is the one-point space $*$ when $n < d$, and $\Sigma^{n-d} A$ when $n \geq d$. The bonding maps are again canonical homeomorphisms. The spectrum $F_1 A$ is illustrated to the right.



The spectrum $F_d A$ is also sometimes denoted $\Sigma^{-d} A$ or $\Sigma^{\infty-d} A$. Its homotopy groups are those of $\Sigma^\infty A$, but shifted down d slots.

Example 2.1.9. As a special case, the $(-d)$-sphere spectrum $\mathbb{S}^{-d} := F_d S^0$ is the spectrum that at level n is the sphere S^{n-d}, so long as n is large enough that $n-d \geq 0$. For positive d we define the d-sphere spectrum as $\mathbb{S}^d = F_0 S^d = \Sigma^\infty S^d$.

The category of spectra is delightful – it allows us to formally define negative-dimensional objects, without having access to any actual negative-dimensional geometry. The $(-d)$-sphere is an object that, after suspending d times, becomes the 0-sphere. This is philosophically similar to how the integers $\mathbb{Z}$ are defined from the natural numbers $\mathbb{N}$: an integer is an object that, if you add a large enough natural number to it, becomes a natural number.

2.1.3 Stable equivalences

Let **Sp** denote the category of spectra defined by Definition 2.1.1. For each $k \in \mathbb{Z}$, the stable homotopy group π_k forms a functor from spectra to abelian groups,

$$\mathbf{Sp} \xrightarrow{\pi_k} \mathbf{Ab}.$$

To spell this out: each map of spectra $f : X \to Y$ induces a map of colimit systems

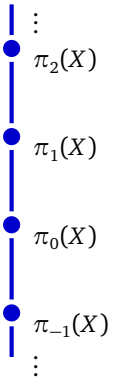
$$\begin{array}{ccccccc}
\cdots \longrightarrow & \pi_{k+n}(X_n) & \xrightarrow{\sigma} & \pi_{k+n+1}(\Sigma X_n) & \xrightarrow{\xi_n} & \pi_{k+n+1}(X_{n+1}) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle \Sigma f_n} & & \downarrow{\scriptstyle f_{n+1}} & \\
\cdots \longrightarrow & \pi_{k+n}(Y_n) & \xrightarrow{\sigma} & \pi_{k+n+1}(\Sigma Y_n) & \xrightarrow{\upsilon_n} & \pi_{k+n+1}(Y_{n+1}) & \longrightarrow \cdots
\end{array}$$

which in turn gives a map on the colimits, $f_* \colon \pi_k(X) \to \pi_k(Y)$. It is easy to check this rule respects identity maps of spectra, and composition of maps of spectra. This is what we mean when we say that π_* is a functor.

Definition 2.1.10.

- A **stable equivalence**, or **π_*-isomorphism**, is a map of spectra $X \to Y$ inducing an isomorphism $\pi_k(X) \xrightarrow{\cong} \pi_k(Y)$ for all $k \in \mathbb{Z}$.
- A **level equivalence** is a map of spectra $X \to Y$ that at each level $X_n \to Y_n$ is a weak homotopy equivalence. So it induces isomorphisms on the homotopy groups $\pi_{k+n}(X_n)$ at all basepoints of X_n.
- A **homotopy equivalence** is a map of spectra $f : X \to Y$ such that there exists another map $g : Y \to X$ and homotopies of maps of spectra $g \circ f \sim \mathrm{id}_X$, $f \circ g \sim \mathrm{id}_Y$. (See Definition 2.3.11.)
- Spectra X and Y are **stably equivalent** if they are connected by any zig-zag of stable equivalences. Level equivalent and homotopy equivalent spectra are defined similarly.

Stable homotopy theory is the study of spectra up to stable equivalence. We consider two stably equivalent spectra to be "essentially the same." This is similar to homological algebra, where we consider chain complexes up to quasi-isomorphism, i.e. maps inducing isomorphisms on homology. In the pictures we been drawing, this means that we focus on the blue line at the far right. It is a mathematical object that can have homotopy groups in every degree.



We have the implications

homotopy equivalence $\Rightarrow$ level equivalence $\Rightarrow$ stable equivalence,

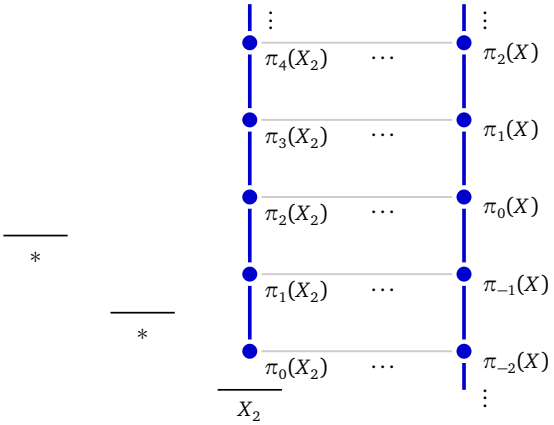
so homotopy equivalences and level equivalences are stricter than stable equivalences. For certain classes of nice spectra, the implications can be reversed, see Lemma 2.2.5 and Corollary 2.6.17.

Example 2.1.11. If X is any spectrum, we can cut off everything before level d to make a new spectrum X'. To be precise,

$$X'_n = \begin{cases} * & \text{if} \quad n < d \\ X_n & \text{if} \quad n \geq d, \end{cases}$$

and bonding maps the same as X, starting at level d.

The inclusion $X' \subseteq X$ is a stable equivalence. In other words, we can always throw away finitely many of the levels of X, without changing its stable homotopy type.



Example 2.1.12. The **zero spectrum** $*$ has every level the one-point space $*$, and every bonding map is the unique map

$$\Sigma(*) \cong * \longrightarrow *.$$

Every spectrum X admits a unique maps of spectra $* \to X \to *$. We say X is **weakly contractible** if one (equivalently both) of these maps are stable equivalences. This is also equivalent to having vanishing homotopy groups, $\pi_k(X) \cong 0$ for all $k \in \mathbb{Z}$.

Given two spectra X and Y, the **zero map** $X \to Y$ is the unique map of spectra that factors through $*$. At level n it is the map $X_n \to Y_n$ sending everything to the basepoint.

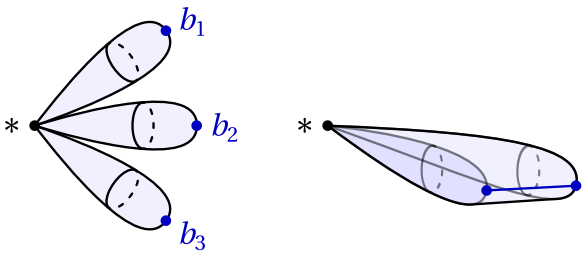
Warning 2.1.13. The bonding maps matter! Given any sequence $\{X_n\}$, we can form the spectrum X in which the bonding maps are all zero,

$$\Sigma X_n \longrightarrow * \longrightarrow X_{n+1}.$$

Then X is weakly contractible, i.e. $X \to *$ is a stable equivalence.

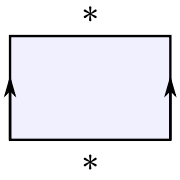
2.1.4 Thom spectra

If B is an unbased space, the n-fold suspension $\Sigma^n_+ B = \Sigma^n B_+ = B_+ \wedge S^n$ can be visualized as a collection of n-spheres, one for each $b \in B$, that share a common basepoint.

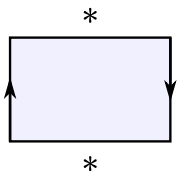


$\Sigma^n_+\{b_1, b_2, b_3\}$ $\Sigma^n_+ I$

For example, the one-fold suspension of $B = S^1$ can be written as a square with edges identified as shown, and the top and bottom edges collapsed to the basepoint $*$. Each point in B gives a circle, which is a single vertical line in this picture, but the circles are glued together continuously, giving the whole square. The resulting space $\Sigma_+ S^1$ is homeomorphic to S^2 with two points identified, and homotopy equivalent to $S^1 \vee S^2$.



One might imagine trying to modify this suspension by spinning the spheres around as we rove around B. For instance, the one-fold suspension of $B = S^1$ can be twisted like a Möbius strip. The resulting "twisted suspension" of S^1 is written as a square with edges identified as shown, and is homeomorphic to the projective plane $\mathbb{RP}^2$.



To do this more generally, we recall that a **vector bundle** over B is a continuous map $p : E \to B$ whose fibers $p^{-1}(b)$ are vector spaces, in a "continuous" way. More precisely, for each point $b \in B$ there is an open set U containing b such that the restricted map $p^{-1}(U) \to U$ is identified with the map $U \times \mathbb{R}^n \to U$, $(x, v) \mapsto x$, and this identification commutes with with addition and scalar multiplication in each of the fibers. See e.g. [Ste51, MS74, Coh98, Hat03] for more about the theory of vector bundles.

Intuitively, a vector bundle is a collection of vector spaces, one for each point of B, but as we move around B they are allowed to twist around, like a Möbius strip.

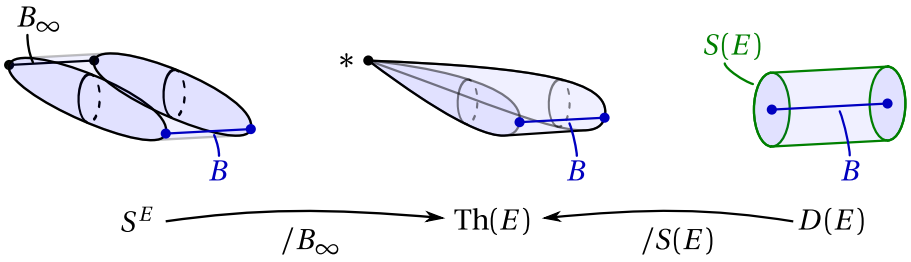
Given any vector bundle, we can define a space that is like the n-fold suspension of B, except that the spheres twist according to the bundle E. To construct it, we first take the one-point compactification of the fibers, giving a map $S^E \to B$. In other words, we take the disjoint union of E and a space $B_\infty \cong B$ of "points at infinity," and topologize the result so that now the map $E \cup B_\infty \to B$ is locally identified with a projection of the form $U \times S^n \to U$, extending the original identification with $U \times \mathbb{R}^n$. Then we collapse B_∞ to a point, so that there is only one point at infinity.

Definition 2.1.14. We call the result of this construction the **Thom space** of the bundle $E \to B$, and denote it by $\mathrm{Th}(E)$, $\Sigma^E B$, or B^E.

So $\mathrm{Th}(E)$ is just the vector bundle E with a single additional point, which we can reach by going to ∞ in *any* of the fibers of $E \to B$. If B is compact, then $\mathrm{Th}(E)$ is the one-point compactification of E, but if B is non-compact then this is not quite true, because you can only reach the new point by going to ∞ along the fibers, not along the base.

The following is a straightforward exercise for the reader familiar with vector bundles. Recall that a metric on E is a continuously varying inner product on the fibers.

Lemma 2.1.15. *There is a homeomorphism* $\mathrm{Th}(E) \cong D(E)/S(E)$*, where* $D(E) \subseteq E$ *is the unit disc bundle and* $S(E) \subseteq D(E)$ *is the unit sphere bundle, for any fixed choice of metric on* E*.*



Example 2.1.16. Let ϵ^n denote the n-dimensional trivial bundle $B \times \mathbb{R}^n \to B$. Its Thom space is

$$(B \times S^n)/(B \times \{\infty\}) \cong (B \times D^n)/(B \times S^{n-1}) \cong \Sigma^n_+ B.$$

In other words, the Thom space of a trivial bundle is just the n-fold suspension of B_+.

Recall that if E is a vector bundle, we let $E \oplus \epsilon^1$ denote the product $E \times \mathbb{R}$. This is a vector bundle which in each fiber is the direct sum of the fibers of E and ϵ^1, hence the notation. We observe that this has the effect of suspension,

$$\mathrm{Th}(E \oplus \epsilon^1) \cong \mathrm{Th}(E \times \mathbb{R}) \cong \Sigma \mathrm{Th}(E), \tag{2.1.17}$$

see Exercise 2.7.5. Therefore, if we make a spectrum by formally de-suspending $\mathrm{Th}(E)$, this should correspond to *subtracting off* a trivial bundle.

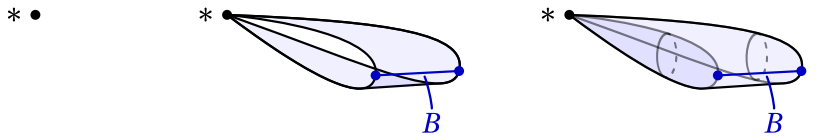
Example 2.1.18. Suppose we have a virtual bundle over a finite-dimensional CW complex B. That is, we have a formal difference of vector bundles $E - E'$. We may always embed E' in a trivial bundle $\epsilon^q = B \times \mathbb{R}^q$ and let E'' be its orthogonal complement, and get

$$E - E' = E + E'' - E' - E'' = (E + E'') - \epsilon^q.$$

Substituting $E \oplus E''$ for E, we may assume without loss of generality that our virtual bundle ζ is of the form $E - \epsilon^q$. In other words, the bundle that is subtracted is a trivial bundle. We define the **Thom spectrum** of $\zeta = E - \epsilon^q$ as

$$\mathrm{Th}(\zeta) := F_q(\mathrm{Th}(E)).$$

That is, we take the Thom space for E, and then take its shift desuspension spectrum. This spectrum is also denoted $\Sigma^\zeta B$ or B^ζ. It is left to Exercise 2.7.6 to prove that $\mathrm{Th}(\zeta)$ is, up to stable equivalence, independent of the choice of presentation of ζ.



$\mathrm{Th}(\zeta)_{q-1} = *$ $\quad$ $\mathrm{Th}(\zeta)_q = \mathrm{Th}(E)$ $\quad$ $\mathrm{Th}(\zeta)_{q+1} = \Sigma\mathrm{Th}(E)$

Intuitively, ζ is a mathematical object that is not a vector bundle, but if we add enough trivial line bundles to it, it becomes a vector bundle. We can visualize the definition of its Thom spectrum as an iterative process: start with the virtual bundle ζ, and at each step, add a trivial line bundle. If the resulting virtual bundle $\zeta \oplus \epsilon^n$ cannot be represented by a vector bundle, take the zero space $*$. If $\zeta \oplus \epsilon^n$ can be represented by a vector bundle $E \oplus \epsilon^{n-q}$, take its Thom space.

Put another way, the "Thom space" $\mathrm{Th}(\zeta)$ does not exist as a geometric object, but it does exist if suspended enough times, and this is enough to define a spectrum out of it.

Example 2.1.19. Suppose M is a closed manifold with tangent bundle τ. The virtual bundle $-\tau$ can be presented as $\nu - \epsilon^q$, where ν is the normal bundle of some embedding $M \to \mathbb{R}^q$. Therefore its Thom spectrum is

$$\mathrm{Th}(-\tau) = \Sigma^{-\tau} M = M^{-\tau} := F_q(\mathrm{Th}(\nu)).$$

Example 2.1.20. There is a universal Thom spectrum MO whose base is the space BO classifying all stable real vector bundles. Since BO is not finite-dimensional, we have to modify the above construction a bit and use the space $BO(n)$ at spectrum level n, so that as $n \to \infty$ we get BO as the base space.

In more detail, let $BO(n)$ denote the space of n-dimensional subspaces of $\mathbb{R}^\infty$, and let $\gamma^n \to BO(n)$ be the tautological bundle whose fiber over each point consists of all the vectors in that n-dimensional subspace.

We define the spectrum MO by letting $MO_n = \mathrm{Th}(\gamma^n)$. To define the bonding maps, we take the maps $BO(n) \to BO(n+1)$ that add a fixed line to each subspace. The pullback of γ^{n+1} along this map is $\gamma^n \oplus \epsilon^1$, so we get maps

$$\Sigma MO_n = \Sigma \mathrm{Th}(\gamma^n) \cong \mathrm{Th}(\gamma^n \oplus \epsilon^1) \to \mathrm{Th}(\gamma^{n+1}) = MO_{n+1}.$$

These define the bonding maps of MO. Intuitively, MO is a suspension spectrum of $BO(n)$ with $n \to \infty$, but the spheres are twisted according to the vector bundles γ^n.

The Pontryagin-Thom isomorphism says that for each $d \geq 0$, the stable homotopy group $\pi_d(MO)$ is isomorphic to the group of d-dimensional closed manifolds M^d up to cobordism. See [Wes96, Kup17, Mil94, Sto68, Coh20] for more details.

2.2 Ω-spectra and infinite loop spaces

Recall from Corollary 1.6.4 that the reduced suspension $\Sigma\colon \mathbf{Top}_* \to \mathbf{Top}_*$ and the based loopspace $\Omega\colon \mathbf{Top}_* \to \mathbf{Top}_*$ are adjoint. This is a fancy way of saying that continuous based maps $\Sigma X \to Y$ are the same thing as continuous based maps $X \to \Omega Y$.

Applying this to Definition 2.1.1, we see that a spectrum is equivalently described as a sequence of based spaces X_n and based maps

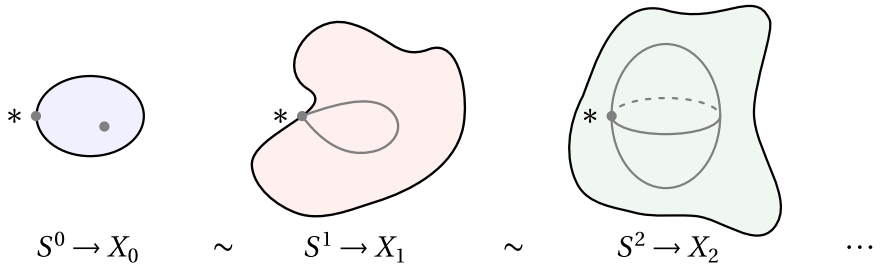
$$\widetilde{\xi}_n : X_n \to \Omega X_{n+1}.$$

In other words, each point in X_n creates a loop in the next space X_{n+1}. We call $\widetilde{\xi}_n$ the **adjunct bonding map**.

Definition 2.2.1. A **fibrant spectrum** or **Ω-spectrum** is a spectrum X in which the adjunct bonding maps $X_n \to \Omega X_{n+1}$ are weak homotopy equivalences.

The zeroth space X_0 of an Ω-spectrum is also called an **infinite loop space**.

Intuitively, this means that not only does a sphere in X_n create a sphere in X_{n+1} of one dimension higher, but having a sphere in X_n is *equivalent* to having a sphere in X_{n+1} of one dimension higher.



When $\Sigma X_n \to X_{n+1}$ is a weak equivalence, its adjunct $X_n \to \Omega X_{n+1}$ is usually not a weak equivalence, and vice-versa. For instance, the adjunct of $S^2 \cong S^2$ is a map $S^1 \to \Omega S^2$ that is not a weak equivalence. Having a two-sphere in S^1 is *not* equivalent to having a three-sphere in S^2.

So the sphere spectrum $\mathbb{S}$ is not an Ω-spectrum. In fact, almost none of the suspension spectra and Thom spectra of the previous section are Ω-spectra.

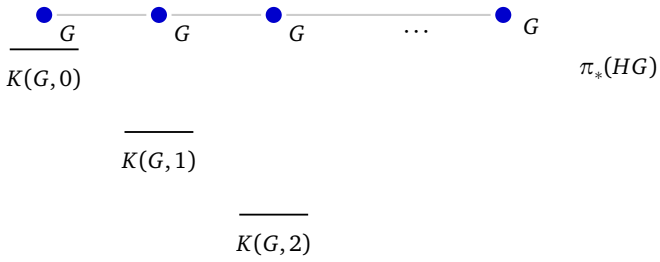
Example 2.2.2. Let G be an abelian group. Recall that an Eilenberg–Mac Lane space $K(G,n)$ is a CW complex whose nth homotopy group is identified with G and all other homotopy groups are zero (Definition 1.4.24). Define an **Eilenberg–Mac Lane spectrum** HG by

$$(HG)_n = K(G,n),$$

and bonding maps the canonical equivalences $K(G,n) \simeq \Omega K(G,n+1)$.

By construction, this is an Ω-spectrum with only one stable homtopy group,

$$\pi_k(HG) = \begin{cases} G & k = 0 \\ 0 & k \neq 0, \end{cases}$$



just like the Eilenberg–Mac Lane spaces $K(G,n)$. In the exercises we will see that this property defines HG uniquely, up to stable equivalence. Since $(HG)_0 = G$, we conclude that every abelian group is an infinite loop space.

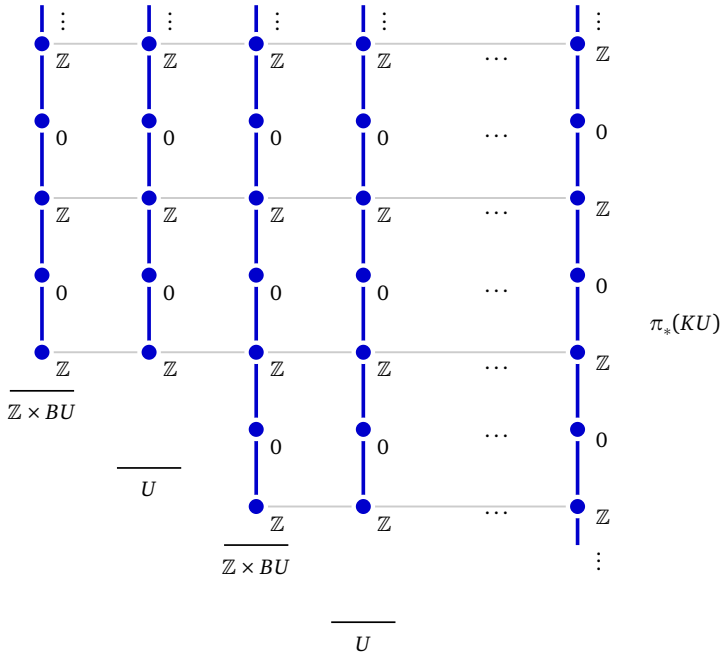
Example 2.2.3. Let $U = \operatorname{colim}_{n\to\infty} U(n)$ be the infinite unitary group and $BU = \operatorname{colim}_{n\to\infty} BU(n)$ its classifying space. The space $\mathbb{Z} \times BU$ classifies stable complex vector bundles, and the Bott periodicity theorem [Bot59] tells us that

$$\Omega(\mathbb{Z} \times BU) \simeq U, \qquad \Omega U \simeq \mathbb{Z} \times BU.$$

The **complex K-theory spectrum** KU has every even level $KU_{2n} = \mathbb{Z} \times BU$, every odd level $KU_{2n+1} = U$, and adjunct bonding maps given just above. The stable homotopy groups follow a periodic pattern, alternating between $\mathbb{Z}$ and 0:

$$\pi_k(KU) = \begin{cases} \mathbb{Z} & k = 2n \\ 0 & k = 2n+1. \end{cases}$$

Therefore $\mathbb{Z} \times BU$ is also an infinite loop space.



Lemma 2.2.4. *When X is an Ω-spectrum, the map from each group $\pi_{k+n}(X_n)$ to the colimit $\pi_k(X)$ is always an isomorphism.*

Proof. The definition of $\pi_k(X)$ can be rearranged to the colimit of

$$\ldots \longrightarrow \pi_{k+n}(X_n) \xrightarrow{\widetilde{\xi}_n} \pi_{k+n}(\Omega X_{n+1}) \xrightarrow{\cong} \pi_{k+n+1}(X_{n+1}) \longrightarrow \ldots$$

Here the isomorphism comes from the adjunction between Σ and Ω, see Corollary 1.6.5. If X is an Ω-spectrum then all the maps of this system are isomorphisms, hence the map from any one term to the colimit is also an isomorphism. □

Lemma 2.2.5. *Between Ω-spectra, every stable equivalence is a level equivalence.*

Proof. By the previous lemma, if $f : X \to Y$ is a stable equivalence of Ω-spectra, then for any $n \geq 0$, the map $\pi_*(X_n) \to \pi_*(Y_n)$ induced by f_n is an isomorphism. However, this is only the homotopy group at the basepoint, not at all basepoints, so it does not immediately follow that f_n is a weak equivalence.

We do get to conclude that $f_{n+1} : X_{n+1} \to Y_{n+1}$ is a weak equivalence on the path component of the basepoint. It follows that $\Omega f_{n+1} : \Omega X_{n+1} \to \Omega Y_{n+1}$ is a weak equivalence. Since X and Y are Ω-spectra, this is equivalent to $f_n : X_n \to Y_n$ so f_n is indeed a weak equivalence for every $n \geq 0$. □

We next show that every spectrum is stably equivalent to an Ω-spectrum. We first do the sphere spectrum as a special case.

Example 2.2.6. The fibrant sphere spectrum $f\mathbb{S}$ has nth level equal to the colimit of the spaces

$$S^n \xrightarrow{\widetilde{\xi}_n} \Omega S^{n+1} \xrightarrow{\Omega\widetilde{\xi}_{n+1}} \Omega^2 S^{n+2} \xrightarrow{\Omega^2\widetilde{\xi}_{n+2}} \Omega^3 S^{n+3} \xrightarrow{\Omega^3\widetilde{\xi}_{n+3}} \dots$$

where each map is the adjunct of the identity map of a sphere. We call this colimit $\Omega^\infty\Sigma^\infty S^n$. It is also sometimes called QS^n after Quillen. Intuitively, $\Omega^\infty\Sigma^\infty S^n$ is the space of based maps $S^k \to S^{n+k}$, but stabilized so that $k \to \infty$. It is also the space of "elements" of degree n in $\mathbb{S}$, in the sense of Remark 2.1.4.

(We could have also taken the homotopy colimit, or mapping telescope, instead of the strict colimit. The homotopy colimit is equivalent to the strict colimit, because of Lemma 1.5.6 and the fact that the maps $\Omega^k\widetilde{\xi}_{n+k}$ are closed inclusions.)

To define the adjunct bonding map, we use Section 1.7 Exercise 1.7.31 to rewrite $\Omega^\infty\Sigma^\infty S^{n+1}$ as the colimit along the bottom row of the following diagram. We form a homeomorphism between this colimit and $\Omega^\infty\Sigma^\infty S^n$ using identity maps:

$$\begin{array}{ccccccccc}
S^n & \xrightarrow{\widetilde{\xi}_n} & \Omega S^{n+1} & \xrightarrow{\Omega\widetilde{\xi}_{n+1}} & \Omega^2 S^{n+2} & \xrightarrow{\Omega^2\widetilde{\xi}_{n+2}} & \dots & \longrightarrow & \Omega^\infty\Sigma^\infty S^n \\
\downarrow{\scriptstyle \widetilde{\xi}_n} & \swarrow{\scriptstyle \mathrm{id}} & & \swarrow{\scriptstyle \mathrm{id}} & & \swarrow{\scriptstyle \mathrm{id}} & & & \downarrow{\scriptstyle \cong} \\
\Omega S^{n+1} & \xrightarrow[\Omega\widetilde{\xi}_{n+1}]{} & \Omega^2 S^{n+2} & \xrightarrow[\Omega^2\widetilde{\xi}_{n+2}]{} & \Omega^3 S^{n+3} & \xrightarrow[\Omega^3\widetilde{\xi}_{n+3}]{} & \dots & \longrightarrow & \Omega\Omega^\infty\Sigma^\infty S^{n+1}
\end{array}$$

This defines the adjunct bonding maps for $f\mathbb{S}$. The inclusion of the first term of each colimit system gives maps $S^n \to \Omega^\infty\Sigma^\infty S^n$, commuting with these bonding maps, giving a map of spectra $\mathbb{S} \to f\mathbb{S}$. We will see shortly that it is a stable equivalence.

Remark 2.2.7. In the previous example, it is possible to follow several different conventions when defining the maps $\Omega^k\widetilde{\xi}_{n+k}$. We adopt the following one. The k-fold loop space is defined as $\Omega^k X := \mathrm{Map}_*(S^k, X)$. When we take Ω^k of a map $X_{n+k} \to \Omega X_{n+k+1}$, the extra loop in the target goes to the *right* of the existing k loops. In other words, the map

$$\Omega^k X_{n+k} \xrightarrow{\Omega^k\widetilde{\xi}_{n+k}} \Omega^{k+1} X_{n+k+1}$$

takes $\left(\phi : S^k \to X_{n+k}\right) \in \Omega^k X_{n+k}$ to the composite

$$S^{k+1} \xrightarrow{\cong} S^k \wedge S^1 \xrightarrow{\phi\wedge\mathrm{id}} X_{n+k} \wedge S^1 \xrightarrow{\xi_{n+k}} X_{n+k+1}.$$
$$\sigma(\phi)$$

Example 2.2.8. Similarly to the last example, we may define a fibrant suspension spectrum of a space A by setting level n to be $\Omega^\infty\Sigma^\infty\Sigma^n A$, the colimit of

$$\Sigma^n A \xrightarrow{\widetilde{\xi}_n} \Omega\Sigma^{n+1} A \xrightarrow{\Omega\widetilde{\xi}_{n+1}} \Omega^2\Sigma^{n+2} A \xrightarrow{\Omega^2\widetilde{\xi}_{n+2}} \Omega^3\Sigma^{n+3} A \xrightarrow{\Omega^3\widetilde{\xi}_{n+3}} \dots$$

This is stably equivalent to $\Sigma^\infty A$.

We can generalize the construction of the last two examples to any spectrum X. The only problem is that the colimit is not always a homotopy colimit (Section 1.5), so it may have the wrong homotopy type. The following proof fixes this by using the homotopy colimit instead.

Proposition 2.2.9. *To each spectrum X, we may assign an Ω-spectrum RX and a stable equivalence $X \to RX$ in a natural way.*

We sometimes call RX a **fibrant replacement** of X. The naturality part of the statement is that R is a functor and the maps $X \to RX$ define a natural transformation. So for each map $X \to Y$, the following square commutes.

$$\begin{array}{ccc} X & \longrightarrow & RX \\ \downarrow & & \downarrow \\ Y & \longrightarrow & RY \end{array}$$

Proof. We define $(RX)_n$ to be the based homotopy colimit of the spaces

$$X_n \xrightarrow{\widetilde{\xi}_n} \Omega X_{n+1} \xrightarrow{\Omega\widetilde{\xi}_{n+1}} \Omega^2 X_{n+2} \xrightarrow{\Omega^2\widetilde{\xi}_{n+2}} \Omega^3 X_{n+3} \longrightarrow \cdots$$

where $\Omega^k\widetilde{\xi}_{n+k}$ is defined as in Remark 2.2.7. We map $X_n \to (RX)_n$ by including the first term into the colimit. The commuting grid of maps

$$\begin{array}{ccccccccc} X_n & \xrightarrow{\widetilde{\xi}_n} & \Omega X_{n+1} & \xrightarrow{\Omega\widetilde{\xi}_{n+1}} & \Omega^2 X_{n+2} & \xrightarrow{\Omega^2\widetilde{\xi}_{n+2}} & \Omega^3 X_{n+3} & \xrightarrow{\Omega^3\widetilde{\xi}_{n+3}} & \cdots \\ \downarrow{\scriptstyle \widetilde{\xi}_n} & \swarrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle \Omega\widetilde{\xi}_{n+1}} & \swarrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle \Omega^2\widetilde{\xi}_{n+2}} & \swarrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle \Omega^3\widetilde{\xi}_{n+3}} & & \\ \Omega X_{n+1} & \xrightarrow[\Omega\widetilde{\xi}_{n+1}]{} & \Omega^2 X_{n+2} & \xrightarrow[\Omega^2\widetilde{\xi}_{n+2}]{} & \Omega^3 X_{n+3} & \xrightarrow[\Omega^3\widetilde{\xi}_{n+3}]{} & \Omega^3 X_{n+4} & \xrightarrow[\Omega^4\widetilde{\xi}_{n+4}]{} & \cdots \end{array}$$

induces a map of homotopy colimits. It is homotopic to the map induced by the dashed arrows, and is therefore a weak equivalence. Composing with the weak equivalence from Exercise 1.7.32 gives a composite map

$$\operatorname*{hocolim}_{m\to\infty} \Omega^m X_{n+m} \xrightarrow{\sim} \operatorname*{hocolim}_{m\to\infty} \Omega^{1+m} X_{n+1+m} \xrightarrow{\sim} \Omega\left(\operatorname*{hocolim}_{m\to\infty} \Omega^m X_{n+1+m}\right).$$

We define the adjunct bonding map $(RX)_n \xrightarrow{\sim} \Omega(RX)_{n+1}$ to be this composite. Since it is a weak equivalence, RX is an Ω-spectrum.

It remains to show that the map $X \to RX$ induces isomorphisms on the stable homotopy

groups. To do this we form the following commuting grid:

$$\begin{array}{ccccccc}
\pi_{k+n}(X_n) & \longrightarrow & \pi_{k+n}(\Omega X_{n+1}) & \longrightarrow & \pi_{k+n}(\Omega^2 X_{n+2}) & \longrightarrow \cdots \longrightarrow & \pi_{k+n}(RX_n) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow\cong \\
\pi_{k+n+1}(X_{n+1}) & \longrightarrow & \pi_{k+n+1}(\Omega X_{n+2}) & \longrightarrow & \pi_{k+n+1}(\Omega^2 X_{n+3}) & \longrightarrow \cdots \longrightarrow & \pi_{k+n+1}(RX_{n+1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow\cong \\
\pi_{k+n+2}(X_{n+2}) & \longrightarrow & \pi_{k+n+2}(\Omega X_{n+3}) & \longrightarrow & \pi_{k+n+2}(\Omega^2 X_{n+4}) & \longrightarrow \cdots \longrightarrow & \pi_{k+n+2}(RX_{n+2}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow\cong \\
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow\cong \\
\pi_k(X) & \xrightarrow{\cong} & A_1 & \xrightarrow{\cong} & A_2 & \xrightarrow{\cong} \cdots \xrightarrow{\cong} & \pi_k(RX).
\end{array}$$

The horizontal maps are induced by $\Omega^m \widetilde{\xi}_{n+m}$, and the vertical maps are induced by $\Omega^m \widetilde{\xi}_{n+m}$, composed with suspension-loop adjunction $\pi_a(\Omega\Omega^m X_b) \cong \pi_{a+1}(\Omega^m X_b)$. We emphasize that we use the *left-hand* loop in $\Omega^{m+1} = \Omega\Omega^m$ for this adjunction; this is necessary to ensure that the diagram commutes.

Along the left-hand column, this gives the colimit system defining the stable homotopy groups of X. In each row, it gives the colimit system defining the homotopy groups of a mapping telescope, and therefore gives the homotopy groups of each level $(RX)_{n+k}$. The induced vertical maps between these define the stable homotopy groups of RX itself, so $\pi_k(RX)$ is the colimit of the entire grid.

Since RX is an Ω-spectrum, the maps along the right-hand side of this diagram are isomorphisms. The maps along the bottom edge are isomorphisms as well. To see this, we use the suspension-loop adjunction to create diagonal maps in the grid pointing downwards and to the left, that subdivide the grid into commuting triangles. These diagonal maps induce inverses to the maps along the bottom edge, which are therefore isomorphisms.

Since the colimits in both directions commute, $\pi_k(RX)$ is the colimit of a system of isomorphisms starting with $\pi_k(X)$. Therefore the map $\pi_k(X) \to \pi_k(RX)$ is an isomorphism. □

Remark 2.2.10. In the literature one sometimes finds **strict Ω-spectra** in which the adjunct maps $X_n \to \Omega X_{n+1}$ are homeomorphisms, see e.g. [LMSM86, EKMM97]. For instance, the fibrant sphere spectrum of Example 2.2.6 is a strict Ω-spectrum. In these cases, X_0 is quite literally a loop space of a loop space of a loop space... and so on ad infinitum. Every spectrum is equivalent to a strict Ω-spectrum, but the replacement requires two steps: we first make the maps $\widetilde{\xi}_n$ into closed inclusions, then we take the strict colimit of the maps $\Omega^m \widetilde{\xi}_{n+m}$, as in Example 2.2.6. The resulting spectrum is equivalent to the Ω-spectrum RX from the proof above.

As in the Whitehead theorem for spaces (Theorem 1.4.18), the results of this section imply that studying spectra up to stable equivalence is the same thing as studying Ω-spectra up to level equivalence. Sometimes this is a good idea. On the other hand, sometimes it's a better idea to leave our spectra alone. For instance, suspension spectra are much easier to think about than their fibrant replacements.

Definition 2.2.11. If X is any spectrum, its **infinite loop space** $\Omega^\infty X$ is the 0th space of any Ω-spectrum RX receiving a stable equivalence from X.

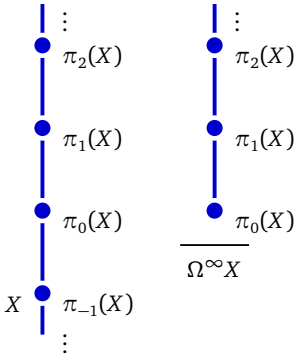
For example, the infinite loop space of the sphere spectrum is

$$\Omega^\infty \mathbb{S} \simeq \Omega^\infty \Sigma^\infty S^0 = \operatorname*{colim}_{n\to\infty} \Omega^n S^n.$$

By the above results,

$$\pi_k(\Omega^\infty X) \cong \pi_k(X), \quad k \geq 0.$$

So the infinite loop space captures everything about X that happens at π_0 and above. It can be visualized by taking the bi-infinite sequence of homotopy groups of a spectrum, and cutting it off below 0 to form a space again.



Remark 2.2.12. The infinite loop space of X is unique up to weak equivalence. We will prove this in Example 3.5.9 after characterizing $\Omega^\infty X$ as the right-derived 0th space of X.

2.3 Operations on spectra

We're ready to define the basic operations on spectra, just as we did for spaces in Section 1.1.

Definition 2.3.1. Given spectra X and Y, their **wedge sum** or **coproduct** $X \vee Y$ is the spectrum whose nth level is the wedge sum $X_n \vee Y_n$ and whose bonding maps are

$$\Sigma(X_n \vee Y_n) \xleftarrow{\cong} (\Sigma X_n) \vee (\Sigma Y_n) \xrightarrow{\xi_n \vee \upsilon_n} X_{n+1} \vee Y_{n+1}.$$

The map on the left is a canonical isomorphism. It arises from the inclusions $\Sigma X_n \to \Sigma(X_n \vee Y_n)$ and $\Sigma Y_n \to \Sigma(X_n \vee Y_n)$. It is an isomorphism because Σ is a left adjoint, so it commutes with coproducts (Theorem 1.6.9).

Similarly, the **product** $X \times Y$ is the spectrum whose nth level is the product $X_n \times Y_n$ and whose bonding maps are

$$\Sigma(X_n \times Y_n) \longrightarrow (\Sigma X_n) \times (\Sigma Y_n) \xrightarrow{\xi_n \times \upsilon_n} X_{n+1} \times Y_{n+1}.$$

The map on the left arises from the projections $\Sigma(X_n \times Y_n) \to \Sigma X_n$ and $\Sigma(X_n \times Y_n) \to \Sigma Y_n$. Alternatively, the adjunct bonding map is the composite

$$X_n \times Y_n \xrightarrow{\widetilde{\xi}_n \times \widetilde{v}_n} (\Omega X_{n+1}) \times (\Omega Y_{n+1}) \xleftarrow{\cong} \Omega(X_{n+1} \times Y_{n+1}),$$

where the second map is an isomorphism because Ω is a right adjoint and therefore commutes with products (Theorem 1.6.9).

Example 2.3.2. A wedge sum of suspension spectra is a suspension spectrum,

$$\Sigma^\infty A \vee \Sigma^\infty B \simeq \Sigma^\infty (A \vee B), \qquad \Sigma^\infty_+ A \vee \Sigma^\infty_+ B \simeq \Sigma^\infty_+ (A \amalg B).$$

A product of Eilenberg–Mac Lane spectra is an Eilenberg–Mac Lane spectrum,

$$HG_1 \times HG_2 \cong H(G_1 \times G_2).$$

Definition 2.3.3. Given a spectrum X, a **subspectrum** $A \subseteq X$ is a sequence of subspaces $A_n \subseteq X_n$ such that $\xi_n(A_n) \subseteq A_{n+1}$. This makes the spaces A_n into a spectrum as well.

Definition 2.3.4. For any integer $d \in \mathbb{Z}$, the d-fold **shift operator** sends each spectrum X to the spectrum $\mathrm{sh}^d X$ whose nth level is the $(n+d)$th level of X,

$$(\mathrm{sh}^d X)_n = X_{d+n},$$

with the same bonding maps as X. When $n + d < 0$, we define $(\mathrm{sh}^d X)_n = *$. Clearly sh^d has the effect of shifting the homotopy groups,

$$\pi_k(\mathrm{sh}^d X) = \pi_{k-d}(X).$$

Example 2.3.5. The d-sphere $\mathbb{S}^d$ of Example 2.1.9 can be re-interpreted as $\mathrm{sh}^d \mathbb{S}$. Shifting a suspension spectrum $\Sigma^\infty A$ makes a suspension spectrum $\Sigma^\infty \Sigma^d A$ if $d \geq 0$, and a desuspension spectrum $F_d A$ if $d \leq 0$. Shifting an Eilenberg–Mac Lane spectrum creates a spectrum $\mathrm{sh}^d HG$ in which the only nonzero homotopy group is $\pi_d = G$.

Definition 2.3.6. If X is a spectrum and K is a based space, we form the **tensor** or **smash product** $K \wedge X$ by smashing K with every level of X. So $K \wedge X$ is the spectrum whose nth level is $K \wedge X_n$ and whose bonding maps are

$$K \wedge X_n \wedge S^1 \xrightarrow{\mathrm{id} \wedge \xi_n} K \wedge X_{n+1}.$$

We call this the tensor because it distributes over direct sums, see Exercise 2.7.10.

As a special case, the **reduced suspension** is $\Sigma X := S^1 \wedge X$. This is just the reduced suspension on each spectrum level, but with the new S^1 is on the left of X, while the

bonding maps put an S^1 on the right. If one is less careful and puts them on the same side, it is necessary to apply a shuffle map $S^1 \wedge S^1 \cong S^1 \wedge S^1$ when defining the bonding maps.[1]

We sometimes put the "extra" suspension in bold, to emphasize which one is written on the left of X, and not used in the bonding maps. So in $\Sigma\mathbf{\Sigma}X_n$, we put the bold suspension on the left in the expression $S^1 \wedge X \wedge S^1$, and the un-bolded one on the right.

Example 2.3.7. The suspension spectrum $\Sigma^\infty A$ is isomorphic to the tensor $A \wedge \mathbb{S}$.

Definition 2.3.8. If X is a spectrum and K is a based space, we form the **cotensor** or **function spectrum** $F(K,X)$ by applying $\mathrm{Map}_*(K,-)$ to every spectrum level of X. So $F(K,X)$ is the spectrum whose nth level is the space of based maps $\mathrm{Map}_*(K,X_n)$. The bonding maps are

$$\mathrm{Map}_*(K,X_n) \xrightarrow{\mathrm{Map}_*(K,\widetilde{\xi}_n)} \mathrm{Map}_*(K,\Omega X_{n+1}) \xleftarrow{\cong} \Omega\mathrm{Map}_*(K,X_{n+1}).$$

where the last isomorphism is deduced from (1.1.18). More concretely, we take each map $\big(\phi : K \to X_n\big)$ to the map $S^1 \to \mathrm{Map}_*(K,X_{n+1})$ adjoint to the composite

$$K \wedge S^1 \xrightarrow{\phi\wedge\mathrm{id}} X_n \wedge S^1 \xrightarrow{\xi_n} X_{n+1}.$$

As a special case, the **based loops** are $\Omega X := F(S^1,X)$. This is just the based loops on each spectrum level. We again have to be careful to keep the "extra" Ω separate from the Ω that appears when we define the bonding maps. Since we always write Ω to the left of X_n, this means that the bonding maps involve a shuffle map:

$$\mathbf{\Omega}X_n \xrightarrow{\mathbf{\Omega}\widetilde{\xi}_n} \mathbf{\Omega}\Omega X_{n+1} \cong \Omega\mathbf{\Omega}X_{n+1}.$$

It is an exercise to check that the operations $K \wedge -$ and $F(K,-)$ are adjoint functors on spectra. (Exercise 2.7.15.)

Warning 2.3.9. The group A_1 from the end of the proof of Proposition 2.2.9 is *not* canonically identified with $\pi_k(\mathbf{\Omega}X)$. The two colimit systems differ by shuffles applied to the spheres that map to X_{k+n}. The colimits are still isomorphic, but choosing an isomorphism requires us to make choices about how to add in shuffle maps. See also Exercise 2.7.19c.

Example 2.3.10. For each based finite CW complex X, its **Spanier-Whitehead dual** is the function spectrum $F(X,\mathbb{S})$. This is formally like the linear dual of a vector space, $V^* = \mathrm{Hom}_k(V,k)$, only using the sphere spectrum as the ground ring. It is a theorem of Spanier and Whitehead that when M is a smooth closed manifold, the Thom spectrum $\Sigma^{-\tau}M$ from Example 2.1.19 is stably equivalent to the dual $F(M_+,\mathbb{S})$, see Theorem 4.2.18.

[1] If you don't apply the shuffle maps, you do get a well-defined spectrum, it's just hard to relate it to anything else because it's not a special case of the more general construction $K \wedge X$.

Definition 2.3.11. A **homotopy** of maps of spectra $X \to Y$ is a map $h\colon I_+ \wedge X \to Y$, where $\wedge$ is the tensor from Definition 2.3.6, or equivalently $X \to F(I_+, Y)$, where F is the cotensor from Definition 2.3.8. This is the same as asking for homotopies of based maps $X_n \to Y_n$ that agree along the bonding maps.

Definition 2.3.12. For spectra X and Y, we define the **mapping space** $\mathrm{Map}_*(X,Y)$ as the space of all tuples of maps $f_n\colon X_n \to Y_n$ that commute with the bonding maps of X and Y:

$$\mathrm{Map}_*(X,Y) = \left\{ f_n \in \mathrm{Map}_*(X_n, Y_n)\ \forall n\ :\ f_{n+1} \circ \xi_n = \upsilon_n \circ (\Sigma f_n) \right\} \subseteq \prod_{n\geq 0} \mathrm{Map}_*(X_n, Y_n)$$

This can also be written as an equalizer (i.e. the limit of a diagram of the following form):

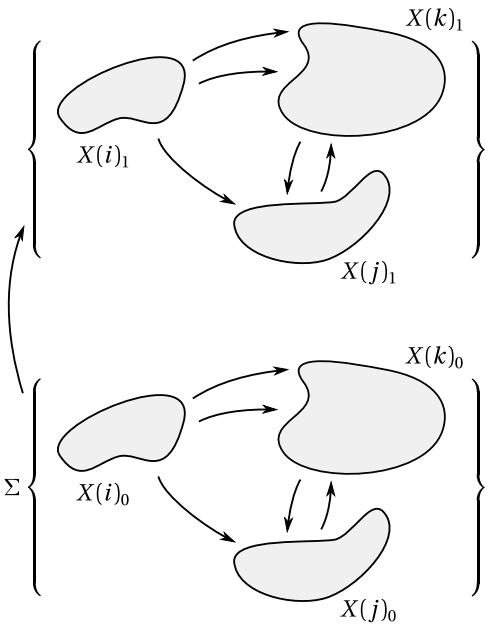
$$\mathrm{Map}_*(X,Y) \longrightarrow \prod_{n\geq 0} \mathrm{Map}_*(X_n, Y_n) \rightrightarrows \prod_{m\geq 0} \mathrm{Map}_*(\Sigma X_m, Y_{m+1}).$$

Of the two parallel maps, one composes with the bonding map for X, and the other suspends and then composes with the bonding map for Y:

$$\mathrm{Map}_*(X_n, Y_n) \longrightarrow \mathrm{Map}_*(\Sigma X_{n-1}, Y_n), \qquad \mathrm{Map}_*(X_n, Y_n) \longrightarrow \mathrm{Map}_*(\Sigma X_n, Y_{n+1}).$$
$$\vdots$$

It is an exercise to check that a point in $\mathrm{Map}(X,Y)$ is precisely a map of spectra $X \to Y$, and a path in $\mathrm{Map}(X,Y)$ is a homotopy. More generally, a map $K \to \mathrm{Map}(X,Y)$ corresponds to both a map from the tensor $K \wedge X \to Y$ and a map to the cotensor $X \to F(K,Y)$ (Exercise 2.7.15).

Let $\mathbf{I}$ be a small category. As in Section 1.1, we say that a **diagram of spectra** indexed by $\mathbf{I}$ is a functor $X\colon \mathbf{I} \to \mathbf{Sp}$. In particular, we get a spectrum $X(i)$ for each object $i \in \mathrm{ob}\,\mathbf{I}$ and a map $X(i) \to X(j)$ for each morphism $i \to j$. This is the same thing as, for each $n \geq 0$, a diagram of spaces $X(i)_n$, along with bonding maps $\Sigma X(i)_n \to X(i)_{n+1}$ commuting every map of the diagram $X(i)_n \to X(j)_n$, as illustrated on the right.



Definition 2.3.13. The **colimit** of the diagram $X\colon \mathbf{I} \to \mathbf{Sp}$ is formed by taking the based colimit from (1.1.15) at each spectrum level n. The structure map is

$$\Sigma \operatorname*{colim}_{\mathbf{I}}{}^{(b)}\, X(i)_n \xleftarrow{\ \cong\ } \operatorname*{colim}_{\mathbf{I}}{}^{(b)}\, \Sigma X(i)_n \xrightarrow{\operatorname{colim}^{(b)}_{\mathbf{I}}\ \xi(i)_n} \operatorname*{colim}_{\mathbf{I}}{}^{(b)}\, X_{n+1},$$

where the isomorphism arises because Σ is a left adjoint and so it commutes with colimits, as in Example 1.6.13. Usually we drop the (b) decoration.

Example 2.3.14. The wedge sum of spectra $X \vee Y$ from Definition 2.3.1 is an example of a colimit. For another example, given two maps of spectra $A \to X$ and $A \to Y$, the **pushout** $X \cup_A Y$ is formed by taking the pushout at each spectrum level, $X_n \cup_{A_n} Y_n$. The bonding maps are defined using the homeomorphism

$$\Sigma(X_n \cup_{A_n} Y_n) \cong (\Sigma X_n) \cup_{(\Sigma A_n)} (\Sigma Y_n).$$

Definition 2.3.15. The **limit** of the diagram $X : \mathbf{I} \to \mathbf{Sp}$ is formed by taking the limit (1.1.5) at each spectrum level. The adjunct structure map is

$$\lim_{\mathbf{I}} X(i)_n \xrightarrow{\lim \widetilde{\xi}(i)_n} \lim_{\mathbf{I}} \Omega X(i)_{n+1} \xleftarrow{\cong} \Omega \lim_{\mathbf{I}} X(i)_{n+1},$$

where the isomorphism arises because Ω is a right adjoint and so it commutes with limits.

Example 2.3.16. The product of spectra $X \times Y$ from Definition 2.3.1 is an example of a limit. For another example, given two maps of spectra $X \to B$ and $Y \to B$, the **pullback** $X \times_B Y$ is formed by taking the pullback at each spectrum level, $X_n \times_{B_n} Y_n$. The bonding maps are defined using the homeomorphism

$$(\Omega X_n) \times_{(\Omega B_n)} (\Omega Y_n) \cong \Omega(X_n \times_{B_n} Y_n).$$

It is an exercise to check that the spectra defined in Definition 2.3.13 and Definition 2.3.13 have the universal property of the colimit and limit, respectively, in the category of spectra (see Exercise 2.7.13).

Definition 2.3.17. For any diagram of spectra $X : \mathbf{I} \to \mathbf{Sp}$, the **homotopy colimit** is formed by taking the based homotopy colimit at each spectrum level n. The bonding map is

$$\Sigma \operatorname{hocolim}^{(b)}_{\mathbf{I}} X(i)_n \xleftarrow{\cong} \operatorname{hocolim}^{(b)}_{\mathbf{I}} \Sigma X(i)_n \xrightarrow{\operatorname{hocolim}^{(b)}_{\mathbf{I}} \xi(i)_n} \operatorname{hocolim}^{(b)}_{\mathbf{I}} X_{n+1},$$

where the isomorphism can be constructed directly for the examples of the homotopy colimits we have encountered so far. We usually drop the (b) decoration.

Since we have not introduced general homotopy colimits yet, we focus on the basic examples. The **homotopy pushout** of a diagram of spectra

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \\ Y & & \end{array}$$

is formed at each level as the based mapping cylinder,

$$(X \cup^h_A Y)_n = X_n \cup^h_A Y_n = X_n \cup_{(A_n \wedge \{0\}_+)} (A_n \wedge I_+) \cup_{(A_n \wedge \{1\}_+)} Y_n.$$

The homotopy colimit of a sequence of maps of spectra

$$X(0) \xrightarrow{f_1} X(1) \xrightarrow{f_2} X(2) \longrightarrow \dots$$

is formed at each level as the based mapping telescope,

$$\left(\operatorname*{hocolim}^{(b)}_{\mathbf{I}} X(i)\right)_n = \bigvee_{k\geq 0} \big(X(k)_n \wedge I_+\big)/\big((x_k, 1) \sim (f_{k+1}(x_k), 0)\big).$$

Remark 2.3.18. One might worry, because of Lemma 1.5.15, that we need to assume the levels of these spectra are well-based spaces, so that the based homotopy colimit has the correct homotopy type. It is a minor miracle that this isn't necessary: in spectra, this model of the homotopy colimit always has the correct homotopy type, up to *stable* equivalence. This is proven in Theorem A.2.25.

Example 2.3.19. For any map of spectra $f : X \to Y$, the homotopy pushout $* \cup^h_X Y$ is called the **homotopy cofiber** Cf. At each spectrum level, it is the based mapping cone from Section 1.2,

$$(Cf)_n = C(f_n) = CX_n \cup_{X_n} Y_n.$$

This can also be described as $(I \wedge X_n) \cup_{X_n} Y_n$, if consider the interval I to be a based space with basepoint 0. The bonding maps arise from commuting Σ with the mapping cone.

Definition 2.3.20. For any diagram of spectra $X : \mathbf{I} \to \mathbf{Sp}$, the **homotopy limit** is formed by replacing all the $X(i)$ by Ω-spectra $RX(i)$, then taking the homotopy limit at each spectrum level n. The adjunct bonding map is

$$\operatorname*{holim}_{\mathbf{I}} (RX(i))_n \xrightarrow{\sim} \operatorname*{holim}_{\mathbf{I}} \Omega(RX(i))_{n+1} \xleftarrow{\cong} \Omega \operatorname*{holim}_{\mathbf{I}} (RX(i))_{n+1},$$

where again the isomorphism can be constructed directly.

Warning 2.3.21. We could define a spectrum by just taking the homotopy limit at each spectrum level:

$$\operatorname*{holim}_{\mathbf{I}} X(i)_n \xrightarrow{\lim \widetilde{\xi}(i)_n} \operatorname*{holim}_{\mathbf{I}} \Omega X(i)_{n+1} \xleftarrow{\cong} \Omega \operatorname*{holim}_{\mathbf{I}} X(i)_{n+1}.$$

However, this is only the homotopy limit in spectra in the following cases:

- The spectra $X(i)$ are all Ω-spectra, or
- the category $\mathbf{I}$ is homotopy finite, meaning it has finitely many objects and finitely many strings of composable non-identity morphisms.

If neither of these conditions hold, then this is not the correct way to define the homotopy limit. The issue is that *stable* equivalences $X(i) \to Y(i)$ don't necessarily give a stable equivalence on this spectrum. In particular, an infinite product of spectra sometimes has the wrong homotopy type. (See Exercise 2.7.26.)

As a special case, the **homotopy pullback** of a diagram of spectra

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ Y & \longrightarrow & B \end{array}$$

is formed at each level as the homotopy fiber product,

$$(X \times_B^h Y)_n = X_n \times_B^h Y_n = X_n \times_{B_n} B_n^I \times_{B_n} Y_n.$$

This will always have the correct homotopy type.

Example 2.3.22. For any map of spectra $f : X \to Y$, the homotopy pullback $* \times_Y^h X$ is called the **homotopy fiber** Ff. At each spectrum level, it is the pullback

$$(Ff)_n = F(f_n) = X_n \times_{Y_n} PY_n$$

where $PY_n = \mathrm{Map}_*(I, Y_n)$ is the **based path space**, the space of paths in Y_n whose endpoint $0 \in I$ is sent to the basepoint of Y_n. So a point in Ff_n is a point in X_n, and a path from its image in Y_n to the basepoint of Y_n.

Definition 2.3.23. For spectra X and Y, define the **handicrafted smash product** $X \wedge Y$ by

$$(X \wedge Y)_n = X_{p_n} \wedge Y_{q_n},$$

for any sequence of pairs of the form (p_n, q_n) with the following conditions: $(p_0, q_0) = (0, 0)$, at every stage either p_n or q_n increases by one, and both p_n and q_n increase without bound as $n \to \infty$. The bonding maps are as follows:

$$X_p \wedge Y_q \wedge S^1 \xrightarrow{\mathrm{id} \wedge v_q} X_p \wedge Y_{q+1}$$

$$X_p \wedge Y_q \wedge S^1 \xrightarrow{\cong} X_p \wedge S^1 \wedge Y_q \xrightarrow{(-1)^q} X_p \wedge S^1 \wedge Y_q \xrightarrow{\xi_p \wedge \mathrm{id}} X_{p+1} \wedge Y_q,$$

where $(-1)^q$ is the map $S^1 \to S^1$ that flips the circle if q is odd.

Remark 2.3.24. This version of the smash product is not well-defined – it depends on the choice of sequence (p_n, q_n). Any two sequences turn out to give stably equivalent results, but the proof of this fact is complicated, and will not be used in this book. Instead, in Chapter 6 we will pass to an equivalent model of spectra where this smash product becomes well-defined and independent of choices.

The sign in Definition 2.3.23 can be explained as follows. Consider all possible smash products of the levels of X and Y:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
| & & | & & | & & \\
X_0 \wedge Y_2 & - & X_1 \wedge Y_2 & - & X_2 \wedge Y_2 & - & \dots \\
| & & | & & | & & \\
X_0 \wedge Y_1 & - & X_1 \wedge Y_1 & - & X_2 \wedge Y_1 & - & \dots \\
| & & | & & | & & \\
X_0 \wedge Y_0 & - & X_1 \wedge Y_0 & - & X_2 \wedge Y_0 & - & \dots
\end{array}$$

The suspension of each space in this grid maps to the space above it, and also to the space to its right. However, if we traverse the grid using different choices of sequence (p_n, q_n), the different routes do not commute, for reasons of applying the suspensions in a different order. The sign convention fixes this, making the two routes commute up to homotopy.

As a result, if we take the maps on homotopy groups induced by these bonding maps, the resulting grid of homotopy groups commutes. (The $(-)$ decorations denote those maps where the flip $S^1 \to S^1$ is applied in the definition of the bonding map. For $k < 0$, the grid is only defined once we move sufficiently far up or to the right.)

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \text{colim} = \pi_k(X \wedge Y) \\
\uparrow{\scriptstyle +} & & \uparrow{\scriptstyle +} & & \uparrow{\scriptstyle +} & & \\
\pi_{k+3}(X_0 \wedge Y_3) & \xrightarrow{-} & \pi_{k+4}(X_1 \wedge Y_3) & \xrightarrow{-} & \pi_{k+5}(X_2 \wedge Y_3) & \xrightarrow{-} & \cdots \\
\uparrow{\scriptstyle +} & & \uparrow{\scriptstyle +} & & \uparrow{\scriptstyle +} & & \\
\pi_{k+2}(X_0 \wedge Y_2) & \xrightarrow{+} & \pi_{k+3}(X_1 \wedge Y_2) & \xrightarrow{+} & \pi_{k+4}(X_2 \wedge Y_2) & \xrightarrow{+} & \cdots \\
\uparrow{\scriptstyle +} & & \uparrow{\scriptstyle +} & & \uparrow{\scriptstyle +} & & \\
\pi_{k+1}(X_0 \wedge Y_1) & \xrightarrow{-} & \pi_{k+2}(X_1 \wedge Y_1) & \xrightarrow{-} & \pi_{k+3}(X_2 \wedge Y_1) & \xrightarrow{-} & \cdots \\
\uparrow{\scriptstyle +} & & \uparrow{\scriptstyle +} & & \uparrow{\scriptstyle +} & & \\
\pi_k(X_0 \wedge Y_0) & \xrightarrow{+} & \pi_{k+1}(X_1 \wedge Y_0) & \xrightarrow{+} & \pi_{k+2}(X_2 \wedge Y_0) & \xrightarrow{+} & \cdots
\end{array}$$

The choice of sequence (p_n, q_n) gives a path through this grid that is cofinal, meaning any other term in the grid maps to something in this path. It follows that $\pi_k(X \wedge Y)$ is isomorphic to the colimit of this grid of abelian groups.

Example 2.3.25. The smash product of a spectrum X with a suspension spectrum $\Sigma^\infty K$ is stably equivalent to the tensor,

$$(\Sigma^\infty K) \wedge X \simeq K \wedge X.$$

The smash product of two suspension spectra is also equivalent to a suspension spectrum,

$$(\Sigma^\infty A) \wedge (\Sigma^\infty B) \simeq \Sigma^\infty (A \wedge B).$$

More generally, the smash product of two shift desuspensions is equivalent to the shift desuspension,

$$(F_d A) \wedge (F_e B) \simeq F_{d+e}(A \wedge B).$$

Remark 2.3.26. The constructions in this section are all functors, meaning that they can also be applied to maps of spectra or maps of diagrams of spectra. For instance, a map $X \to X'$ will also give a map on the suspensions $\Sigma X \to \Sigma X'$, and a pair of maps $X \to X'$, $Y \to Y'$ gives a map $X \vee Y \to X' \vee Y'$. The definition of these maps is usually obvious, so we don't spell them out explicitly.

2.4 Stability theorems

Homotopy theorists often find that spectra are easier to work with than spaces. The reason for this is "stability," a property similar to excision for homology. Stability can be formulated in a few different ways:

- Σ and Ω are inverses up to stable equivalence.
- Cofiber and fiber sequences of spectra are the same, up to stable equivalence.
- Homotopy pushout and homotopy pullback squares of spectra are the same, up to stable equivalence.

In this section, we prove these claims and deduce some standard corollaries.

2.4.1 Suspension and loops

Recall from Definition 2.3.6 and Definition 2.3.8 that we take a suspension $\mathbf{\Sigma}X$ of a spectrum X by suspending each level of X separately, and similarly we form $\mathbf{\Omega}X$ by taking the loopspace of each spectrum level separately.

Proposition 2.4.1. *For any spectrum X there are natural isomorphisms*

$$\pi_{k+1}(\mathbf{\Sigma}X) \cong \pi_k(X) \cong \pi_{k-1}(\mathbf{\Omega}X).$$

Proof. We take the colimit defining $\pi_k(X)$ in Definition 2.1.2, and restrict to the terms of the form $\pi_{k+2n+1}(\Sigma X_{2n}) = \pi_{k+2n+1}(X_{2n} \wedge S^1)$. This gives the same colimit by Exercise 2.7.11.

We similarly take the colimit defining $\pi_{k+1}(S^1 \wedge X)$ and restrict to the terms $\pi_{k+1+2n}(S^1 \wedge X_{2n})$. We form an isomorphism between the colimit systems by the symmetry isomorphism $X_{2n} \wedge S^1 \cong S^1 \wedge X_{2n}$.

To check this isomorphism commutes with the maps of the colimit system, it suffices to show the following diagram commutes up to homotopy.

$$\begin{array}{ccccc} X_{2n} \wedge S^1 \wedge S^1 \wedge S^1 & \xrightarrow{\xi_{2n} \wedge \mathrm{id} \wedge \mathrm{id}} & X_{2n+1} \wedge S^1 \wedge S^1 & \xrightarrow{\xi_{2n+1} \wedge \mathrm{id}} & X_{2n+2} \wedge S^1 \\ \downarrow \cong & & & & \downarrow \cong \\ S^1 \wedge X_{2n} \wedge S^1 \wedge S^1 & \xrightarrow{\mathrm{id} \wedge \xi_{2n} \wedge \mathrm{id}} & S^1 \wedge X_{2n+1} \wedge S^1 & \xrightarrow{\mathrm{id} \wedge \xi_{2n+1}} & S^1 \wedge X_{2n+2}. \end{array}$$

The two branches do not strictly agree, because different copies of S^1 are "fed" into the bonding maps ξ. However, the two branches differ by a self-map of $S^3 = S^1 \wedge S^1 \wedge S^1$ that applies a 3-cycle to the copies of S^1. As a map $S^3 \to S^3$, this has degree 1, so it is homotopic to the identity. Therefore the diagram commutes up to homotopy, which is enough to conclude it commutes on homotopy groups.

The isomorphism $\pi_k(X) \cong \pi_{k-1}(\mathbf{\Omega} X)$ is constructed in the same way, using the version of the colimit system from Lemma 2.2.4. □

Remark 2.4.2. One can use this to prove that that $\mathbf{\Sigma} X \simeq \mathrm{sh} X$. See Exercise 2.7.19.

Remark 2.4.3. There is only one natural isomorphism $\pi_{k+1}(\mathbf{\Sigma} X) \cong \pi_k(X)$, up to a factor of ± 1. See Exercise 2.7.20.

Corollary 2.4.4. *For a map of spectra $f : X \to Y$, the following are equivalent:*

- *$f : X \to Y$ is a stable equivalence.*
- *$\Sigma f : \Sigma X \to \Sigma Y$ is a stable equivalence.*
- *$\Omega f : \Omega X \to \Omega Y$ is a stable equivalence.*

Proof. The isomorphisms of Proposition 2.4.1 are natural, meaning that f induces commuting diagrams

$$\begin{array}{ccccc} \pi_{k+1}(\mathbf{\Sigma} X) & \xleftarrow{\cong} & \pi_k(X) & \xleftarrow{\cong} & \pi_{k-1}(\mathbf{\Omega} X) \\ \downarrow (\Sigma f)_* & & \downarrow f_* & & \downarrow (\Omega f)_* \\ \pi_{k+1}(\mathbf{\Sigma} Y) & \xleftarrow{\cong} & \pi_k(Y) & \xleftarrow{\cong} & \pi_{k-1}(\mathbf{\Omega} Y). \end{array}$$

Therefore f_* is an isomorphism for all k iff $(\Sigma f)_*$ is an isomorphism for all k, iff $(\Omega f)_*$ is an isomorphism for all k. □

Remark 2.4.5. If $f : A \to B$ is a weak equivalence of spaces, then $\Omega f : \Omega A \to \Omega B$ is also weak equivalence, but $\Sigma f : \Sigma A \to \Sigma B$ is only a weak equivalence if A and B are well-based.

So it is surprising that Corollary 2.4.4 doesn't require us to assume that the levels X_n and Y_n are well-based. Essentially, this happens because the suspensions are already baked into the definition of $\pi_k(X)$.

Corollary 2.4.6. *There are natural stable equivalences $X \to \Omega\Sigma X$ and $\Sigma\Omega X \to X$.*

In fact, these are the unit and counit maps of the adjunction (Σ, Ω) on the category of spectra.

Proof. If we compose together the two isomorphisms from Proposition 2.4.1

$$\pi_k(X) \cong \pi_{k+1}(\mathbf{\Sigma} X) \cong \pi_k(\mathbf{\Omega\Sigma} X),$$

we see that it takes each map $S^{k+2n} \to X_{2n}$ to its suspension $\Sigma S^{k+2n} \to \Sigma X_{2n}$, then its adjunct $S^{k+2n} \to \Omega\Sigma X_{2n}$. This is the same as the composite

$$S^{k+2n} \longrightarrow X_{2n} \longrightarrow \Omega\Sigma X_{2n}$$

where the second map is the unit of the adjunction (Σ, Ω). Therefore the unit map is a stable equivalence. The proof for the counit map is the same but composes the isomorphisms in a different order, $\pi_k(X) \cong \pi_{k-1}(\mathbf{\Omega} X) \cong \pi_k(\mathbf{\Sigma\Omega} X)$. □

Therefore Σ is invertible up to stable equivalence, and its inverse is Ω.

2.4.2 Cofiber and fiber sequences

Next we show that cofiber and fiber sequences of spectra coincide. We first give the two definitions.

Definition 2.4.7. Informally, a **cofiber sequence** is anything of the form

$$X \xrightarrow{f} Y \longrightarrow Cf,$$

where Cf is the homotopy cofiber from Example 2.3.19. A **fiber sequence** is anything of the form

$$Fg \longrightarrow Y \xrightarrow{g} Z,$$

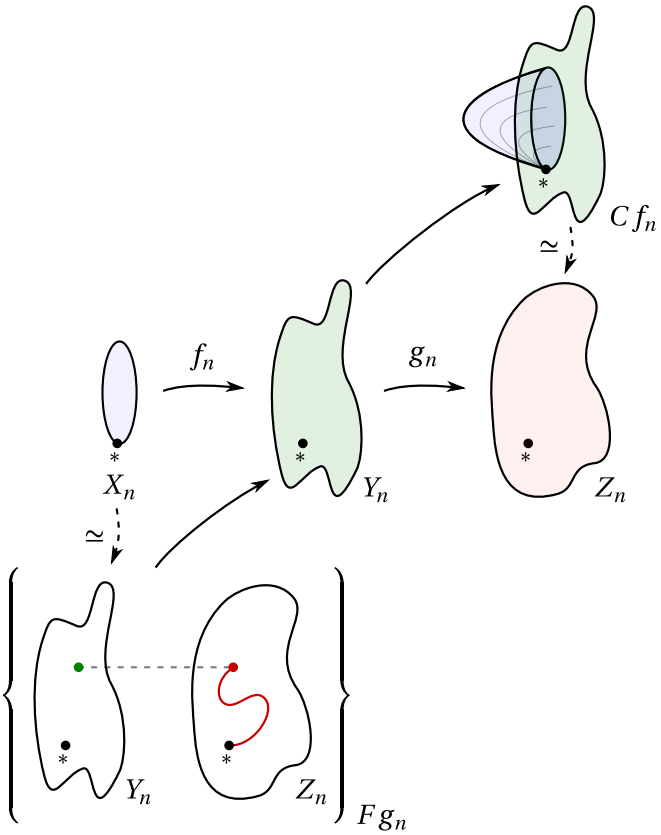
where Fg is the homotopy fiber from Example 2.3.22.

To define these more formally, consider three spectra X, Y, Z, two maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

and a homotopy $h\colon X \wedge I \to Z$ from the zero map to $g \circ f$. This homotopy induces maps

$$h \cup g\colon Cf \longrightarrow Z, \qquad f \times h\colon X \longrightarrow Fg.$$



We say that (X, Y, Z, f, g, h) is a **cofiber sequence** if $Cf \to Z$ is a stable equivalence, and a **fiber sequence** if $X \to Fg$ is a stable equivalence.

A **map of cofiber or fiber sequences** consists of three maps $X \to X'$, $Y \to Y'$, and $Z \to Z'$ commuting with f, g, and h.

Remark 2.4.8. There are many other equivalent ways to define cofiber and fiber sequences. For instance, we could ask for any stable equivalence $Cf \to Z$, instead of asking for one that comes from a homotopy. See Exercise 2.7.23 for another equivalent definition.

One benefit of our definition is that both notions have the same data, only the condition is different. This makes it easier to prove that cofiber and fiber sequences coincide (Proposition 2.4.14).

Example 2.4.9. If $A \to B \to B/A$ is a cofiber sequence of based spaces in the sense of Definition 1.2.12, then the suspension spectra

$$\Sigma^\infty A \longrightarrow \Sigma^\infty B \longrightarrow \Sigma^\infty (B/A)$$

form a cofiber sequence of spectra. On the other hand, if $X \to Y \to Z$ is a fiber sequence of spectra, then the infinite loop spaces (Definition 2.2.11)

$$\Omega^\infty X \longrightarrow \Omega^\infty Y \longrightarrow \Omega^\infty Z$$

form a fiber sequence of based spaces (Definition 1.2.18). See Exercise 2.7.35.

Lemma 2.4.10. *For each fiber sequence of spectra we may construct a long exact sequence*

$$\dots \longrightarrow \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{g_*} \pi_k(Z) \xrightarrow{\partial} \pi_{k-1}(X) \longrightarrow \dots$$

so that maps of fiber sequences give maps of long exact sequences.

Proof. Without loss of generality, $X \to Fg$ is an isomorphism. Then for each n, the fiber sequence of spaces $X_n \to Y_n \to Z_n$ gives a three-term exact sequence

$$\pi_{k+n}(X_n) \xrightarrow{f_*} \pi_{k+n}(Y_n) \xrightarrow{g_*} \pi_{k+n}(Z_n).$$

Sequential colimits preserve exact sequences, so this passes to a three-term sequence of stable homotopy groups

$$\pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{g_*} \pi_k(Z)$$

that is exact at $\pi_k(Y)$. Finally, just as in Proposition 1.2.19, we can continue to take homotopy fibers to get two more fiber sequences

$$\Omega Z \xrightarrow{\partial} X \xrightarrow{f} Y, \qquad\qquad \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{\partial} X.$$

Repeating the same argument with these two sequences and identifying $\pi_k(\Omega Z) \cong \pi_{k+1}(Z)$ shows that the desired sequence is exact. (Strictly speaking, we get a sequence in which some signs are negated, but this does not change exactness.) □

Lemma 2.4.11. *For each cofiber sequence of spectra we may construct a long exact sequence*

$$\dots \longrightarrow \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{g_*} \pi_k(Z) \xrightarrow{\partial} \pi_{k-1}(X) \longrightarrow \dots$$

so that maps of cofiber sequences give maps of long exact sequences.

Remark 2.4.12. This lemma, unlike the previous one, is not true for spaces. It only holds for highly connected spaces in a range, by the homotopy excision theorem.

Proof. Again we may assume that $Cf \to Z$ is an isomorphism. As in the previous lemma, it suffices to prove that

$$\pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{g_*} \pi_k(Cf)$$

is exact in the middle. It is clear that the composite is zero. Going the other way, suppose $\alpha \in \pi_k(Y)$ and $g_*(\alpha) = 0$ in $\pi_k(Cf)$. An element that is zero in the colimit must attain zero at some finite stage, so there is a representative $\alpha_n \in \pi_{k+n}(Y_n)$ such that $(g_n)_*(\alpha_n) = 0$ in $\pi_{k+n}(Cf_n)$. Concretely, this means the composite

$$S^{k+n} \xrightarrow{\alpha_n} Y_n \xrightarrow{g_n} Cf_n$$

is nullhomotopic. It therefore extends to a map $\overline{\alpha}_n \colon CS^{k+n} \to Cf_n$. Now define β_{n+1} to be the composite

$$S^{k+n+1} \xleftarrow{\cong} CS^{k+n} \cup_{S^{k+n}} CS^{k+n} \xrightarrow{C\alpha_n \cup \overline{\alpha}_n} CY_n \cup_{Y_n} Cf_n \xrightarrow{\sim} \Sigma X_n \xrightarrow{\xi_n} X_{n+1}$$

where the second-to-last map is the usual identification in the Puppe sequence that collapses CY_n to a point. Let $\beta \in \pi_k(X)$ be the corresponding element in the stable homotopy of X. The proof will be complete once we show that $f_*(\beta) = \pm\alpha$.

To prove this, consider the map $\varphi \colon Cf_n \to CY_n$ that applies f_n to the cone on X_n. This map fits into a commuting diagram

$$\begin{array}{ccccccc} CS^{k+n} \cup_{S^{k+n}} CS^{k+n} & \xrightarrow{C\alpha_n \cup \overline{\alpha}_n} & CY_n \cup_{Y_n} Cf_n & \xrightarrow{\sim} & \Sigma X_n & \xrightarrow{\xi_n} & X_{n+1} \\ & & \downarrow{\scriptstyle \mathrm{id} \cup \varphi} & & \downarrow{\scriptstyle \Sigma f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ & & CY_n \cup_{Y_n} CY_n & \xrightarrow{\sim} & \Sigma Y_n & \xrightarrow{\upsilon_n} & Y_{n+1} \end{array}$$

in which both $\sim$ maps collapse the first CY_n. The composite along the top route represents $\pm f_*(\beta_{n+1})$.[2]

Now modify the maps $\sim$ by collapsing the right-hand cone instead. This produces new maps homotopic to the original ones, and after this change, the composite along the bottom route is $\pm(\upsilon_n \circ \Sigma\alpha_n) = \pm\alpha_{n+1}$. Therefore $f_*(\beta) = \pm\alpha$. □

Lemma 2.4.13. *For each map of spectra $f : X \to Y$ there is a natural stable equivalence $\epsilon \colon \Sigma Ff \to Cf$, or equivalently $\widetilde{\epsilon} \colon Ff \to \Omega Cf$.*

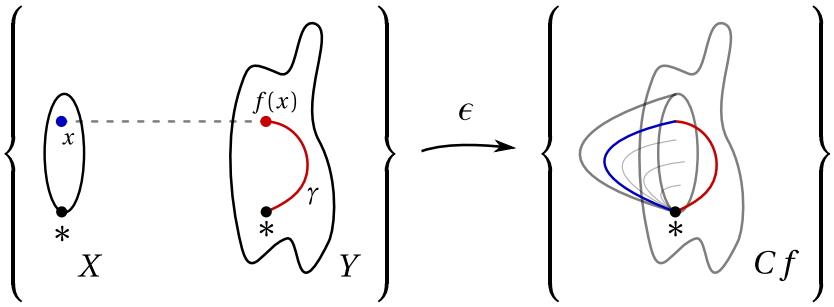
Proof. Inside this proof, we use the definition $S^1 = I/\{0,1\}$, so that the reduced suspension ΣX is a quotient of $I \times X$. We define $\epsilon \colon \Sigma Ff \to Cf$ by the formula

$$\epsilon : \Sigma(X \times_Y PY) \longrightarrow Y \cup_X CX$$

[2]The $\pm$ appears because we have not specified how to make the identifications $CX/X \cong \Sigma X$, and the two obvious choices differ by a flip in the suspension coordinate. Therefore, if we choose not to specify which of these identifications we use, we only get a well-defined element in the homotopy group up to sign. This is good enough for this proof though!

$$\epsilon(t,x,\gamma) = \begin{cases} \gamma(2t) & t \le 1/2 \\ (x, 2-2t) & t \ge 1/2. \end{cases}$$

In other words, given a point $x \in X_n$ and a path $\gamma : I \to Y_n$ from $*$ to $f(x)$, we create a loop in Cf_n by composing the path γ in Y_n with the path that traverses $I \times \{x\}$ in the cone CX_n.



We check that the following squares of spectra commute up to homotopy, where the top row is the suspension of the fiber Puppe sequence for f, and the bottom row is the cofiber Puppe sequence for f, and the vertical maps on the left are the counit maps from Corollary 2.4.6.

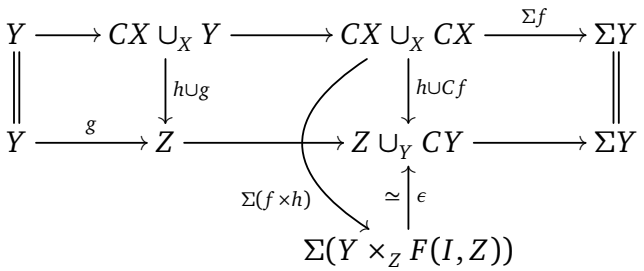
$$\begin{array}{ccccccccc} \Sigma\Omega X & \xrightarrow{\Sigma\Omega f} & \Sigma\Omega Y & \longrightarrow & \Sigma(X \times_Y PY) & \longrightarrow & \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y \\ \downarrow & & \downarrow & & \downarrow \epsilon & & \downarrow \text{flip} & & \downarrow \text{flip} \\ X & \xrightarrow{f} & Y & \longrightarrow & Y \cup_X CX & \longrightarrow & \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y \end{array}$$

The top row forms an exact sequence on stable homotopy groups, by Lemma 2.4.10 and Proposition 2.4.1. The bottom row forms an exact sequence on stable homotopy groups by Lemma 2.4.11. The two vertical maps to the left of ϵ are π_*-isomorphisms by Corollary 2.4.6, while the two maps to the right are π_*-isomorphisms by direct inspection. By the five-lemma, ϵ is a π_*-isomorphism as well. □

Proposition 2.4.14. *The data (X, Y, Z, f, g, h) describes a cofiber sequence iff it describes a fiber sequence.*

Proof. Consider the following diagram, in which the rows are taken from the based cofiber Puppe sequences for $f : X \to Y$ and $g : Y \to Z$, respectively. In particular, the map $CX \cup_X Y \to CX \cup_X CX$ is obtained by collapsing Y to a point, which gives ΣX, and then

rewriting ΣX as the union of two cones.



We check that given vertical maps form squares that commute up to homotopy, and that the composite $\epsilon \circ \Sigma(f \times h)$ is equal to $h \cup Cf$. Since each row describes a long exact sequence of homotopy groups, we get the implications

$$\begin{aligned}(X, Y, Z, f, g, h) \text{ is a cofiber sequence} &\iff h \cup g \text{ is a stable equivalence} \\ &\iff h \cup Cf \text{ is a stable equivalence} \\ &\iff \Sigma(f \times h) \text{ is a stable equivalence} \\ &\iff f \times h \text{ is a stable equivalence} \\ &\iff (X, Y, Z, f, g, h) \text{ is a fiber sequence.}\end{aligned}$$

□

2.4.3 Pushout and pullback squares

Definition 2.4.15. Given a commuting square of spectra

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D, \end{array}$$

we say it is a **homotopy pushout square** or a **coCartesian square** if the induced map

$$B \cup_A^h C \to D$$

is a stable equivalence. It is a **homotopy pullback square** or a **Cartesian square** if the induced map

$$A \to B \times_D^h C$$

is a stable equivalence. The following is left as an exercise (Exercise 2.7.21).

Lemma 2.4.16. *A commuting square of spectra is a homotopy pushout iff the induced map of homotopy cofibers $Cf \to Cg$ is a stable equivalence. It is a homotopy pullback iff the induced map of homotopy fibers $Ff \to Fg$ is a stable equivalence.*

Of couse, since the definition of homotopy pushout is symmetric, the homotopy cofibers and fibers could be taken along the vertical maps h and k instead. Combining Lemma 2.4.13 and Lemma 2.4.16 gives:

Corollary 2.4.17. *A commuting square of spectra is homotopy pushout iff it is homotopy pullback.*

This finishes the proof of the stability theorems. We conclude with some corollaries.

2.4.4 Wedge sums are equivalent to products

Let X and Y be spectra. The inclusions $X_n \vee Y_n \subseteq X_n \times Y_n$ give a map of spectra

$$X \vee Y \longrightarrow X \times Y. \tag{2.4.18}$$

Composing with the inclusions of X and Y into $X \vee Y$, and the projection of the product $X \times Y$ onto each of its factors, we get maps on homotopy groups

$$\pi_k(X) \oplus \pi_k(Y) \longrightarrow \pi_k(X \vee Y) \longrightarrow \pi_k(X \times Y) \longrightarrow \pi_k(X) \times \pi_k(Y). \tag{2.4.19}$$

Proposition 2.4.20. *All three maps in* (2.4.19) *are isomorphisms. Therefore the inclusion* $X \vee Y \to X \times Y$ *from* (2.4.18) *is a stable equivalence.*

Proof. The last map is left to Exercise 2.7.26. It suffices to prove the first map is an isomorphism, since the composite of all three maps clearly is. Observe that

$$C(X \to X \vee Y) \to Y$$

is a level equivalence of spectra, and therefore there is a cofiber sequence of the form

$$X \longrightarrow X \vee Y \longrightarrow Y.$$

The second of these maps is split by the inclusion $Y \to X \vee Y$, and therefore the associated long exact sequence is split. We conclude that $\pi_k(X) \oplus \pi_k(Y) \to \pi_k(X \vee Y)$ is an isomorphism. □

2.4.5 Retracts are summands

Suppose that X is a spectrum, and A is a retract of X, in the rather weak sense that we can find maps

$$A \xrightarrow{i} X \xrightarrow{p} A'$$

whose composite is a stable equivalence. Then A is a summand of X,

$$X \simeq A \vee (?).$$

To make this precise, let Fp be the homotopy fiber of p, and $h\colon I \wedge Fp \to A'$ the canonical homotopy from the composite $Fp \xrightarrow{j} X \xrightarrow{p} A'$ to the zero map.

Proposition 2.4.21. *The map $j \vee i\colon Fp \vee A \longrightarrow X$ is a stable equivalence.*

Proof. We form the map of cofiber/fiber sequences

$$\begin{array}{ccccc} Fp & \longrightarrow & Fp \vee A & \longrightarrow & (I \wedge Fp) \vee A \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle j\vee i} & & \downarrow{\scriptstyle h\vee(p\circ i)} \\ Fp & \xrightarrow{\ j\ } & X & \xrightarrow{\ p\ } & A'. \end{array}$$

Both squares commute. In addition, the nullhomotopy h for the top composite is carried to the nullhomotopy for the bottom composite. We therefore get a commuting map of long exact sequences of homotopy groups.[3] Since $I \wedge Fp$ is contractible, the two outside vertical maps are π_*-isomorphisms, therefore so is the map in the middle. □

The complementary summand could also be described as Ci, so that $X \simeq A \vee Ci$, see Exercise 2.7.24. Using Proposition 2.4.20, we conclude:

Corollary 2.4.22. *If X contains A as a retract, then*

$$\pi_*(X) \cong \pi_*(A) \oplus \pi_*(Fp) \cong \pi_*(A) \oplus \pi_*(Ci).$$

2.4.6 Smash products and cofiber sequences

Finally we explain how the smash product interacts with cofiber sequences.

Lemma 2.4.23. *If $X \to Y \to Cf$ is a cofiber sequence of spaces or spectra, smashing with a space or spectrum W produces another cofiber sequence*

$$X \wedge W \to Y \wedge W \to Cf \wedge W.$$

Dually, the cotensors $F(-,-)$ also preserve cofiber and fiber sequences, see Exercise 2.7.31.

[3] In general, checking the nullhomotopy is necessary to ensure that the map of long exact sequences commutes with the connecting homomorphism. In this particular case, the connecting homomorphisms are zero, so the diagram would commute either way.

Proof. For spectra, this quickly reduces to checking the statement for spaces. It suffices to check that the operation $-\wedge W$ commutes with the formation of the based homotopy cofiber Cf. This occurs because $-\wedge W$ is a left adjoint, so it preserves pushouts (Example 1.6.13), and it also commutes with smash products because the smash product is associative:

$$\begin{aligned} C(X\wedge W\to Y\wedge W) &= (I\wedge X\wedge W)\cup_{(X\wedge W)}(Y\wedge W)\\ &\cong \big((I\wedge X)\cup_X Y\big)\wedge W\\ &= C(X\to Y)\wedge W. \end{aligned}$$

See also Exercise 1.7.35. □

Remark 2.4.24. If $X\to Y\to Z$ is a cofiber sequence and Z is only equivalent to Cf, not isomorphic, then we might want

$$X\wedge W\to Y\wedge W\to Z\wedge W$$

to be a cofiber sequence too. This will work, but we have to be a little careful – we should make sure that $Z\wedge W$ is also equivalent to $Cf\wedge W$. This will happen so long as all the spaces or spectrum levels are well-based.

The statement of Lemma 2.4.23 doesn't require anything to be well-based to work. This has the following surprising consequence:

Corollary 2.4.25. *If $f: X\to Y$ is a stable equivalence of spectra, with no conditions on basepoints, and K is a based CW complex, then $K\wedge X\to K\wedge Y$ is also a stable equivalence.*

Proof. Let $K^{(n)}$ be the nth level of the skeletal filtration of K. Inductively, we show that

$$\mathrm{id}\wedge f: K^{(n)}\wedge X\to K^{(n)}\wedge Y$$

is a stable equivalence. For $n=0$, this is a wedge of copies of f, so it follows from Exercise 2.7.30. For larger n, we take the cofiber sequence

$$K^{(n-1)}\xrightarrow{i} K^{(n)}\longrightarrow Ci\cong\bigvee_\alpha S^n$$

and smash with X and Y to get two cofiber sequences, using Lemma 2.4.23:

$$\begin{array}{ccccc} K^{(n-1)}\wedge X & \xrightarrow{i\wedge\mathrm{id}} & K^{(n)}\wedge X & \longrightarrow & \bigvee_\alpha \Sigma^n X\\ \sim\downarrow{\scriptstyle \mathrm{id}\wedge f} & & \downarrow{\scriptstyle \mathrm{id}\wedge f} & & \sim\downarrow{\scriptstyle \Sigma^n f}\\ K^{(n-1)}\wedge Y & \xrightarrow{i\wedge\mathrm{id}} & K^{(n)}\wedge Y & \longrightarrow & \bigvee_\alpha \Sigma^n Y. \end{array}$$

This induces a map of long exact sequences by Lemma 2.4.11. Since the outside maps are stable equivalences, so is the map in the middle.

Finally, by Exercise 2.7.28 the stable homotopy groups of $K\wedge X$ are the colimit of the stable homotopy groups of $K^{(n)}\wedge X$. Since a colimit of isomorphisms is an isomorphism, the induced map $\pi_*(K\wedge X)\to\pi_*(K\wedge Y)$ is an isomorphism. □

2.5 Extraordinary homology and cohomology

2.5.1 Extraordinary homology

Recall that a CW pair (X,A) is a choice of CW complex X and subcomplex $A \subseteq X$. A map of pairs $(X,A) \to (Y,B)$ is a continuous map $X \to Y$ sending A into B.

Definition 2.5.1. An **extraordinary homology theory** is any sequence of functors $E_k(-,-)$ from CW pairs to abelian groups, along with a natural "boundary map"

$$E_k(X,A) \xrightarrow{\partial} E_{k-1}(A,\emptyset),$$

satisfying the first four Eilenberg-Steenrod axioms (Propositions 1.3.1–1.3.4), but not necessarily the fifth axiom (Proposition 1.3.5). The **coefficient group** is the graded abelian group $E_*(*,\emptyset)$.

For instance, bordism theory $\mathfrak{N}_k(X)$ from Example 0.1.1 is an extraordinary homology theory (which we only described for pairs of the form $(X,\emptyset)$). Stable homotopy groups $\pi_k(\Sigma^\infty_+ X)$ are another extraordinary homology theory.

Definition 2.5.2. For an unbased space X, the **unreduced E-homology** of X is

$$E_k(X) = E_k(X,\emptyset).$$

For a based space X, the **reduced E-homology** of X is

$$E_k(X) = E_k(X,*)$$

Reduced homology is sometimes denoted $\widetilde{E}_k(X)$ to distinguish it from unreduced homology, but it is more common to use the same notation for both, using the word "unreduced" or "reduced" to indicate which one we mean.

Lemma 2.5.3. *There is a canonical splitting for based spaces X*

$$E_k(X,\emptyset) \cong E_k(X,*) \oplus E_k(*,\emptyset).$$

Therefore, the unreduced and reduced groups always differ by $E_k(*,\emptyset)$.

Proof. This follows from the exactness axiom and the fact that the map $E_k(*,\emptyset) \to E_k(X,\emptyset)$ is split by the map from X back to a point. □

By the excision axiom, $E_*(X,A) \cong E_*(X/A,*)$, so an extraordinary homology theory $E_*(-,-)$ can be completely recovered from just the reduced E-homology functor $E_*(-)$ on based spaces, or the unreduced E-homology functor $E_*(-)$ on unbased spaces. The details are left to Exercise 2.7.3.

Lemma 2.5.4. *The reduced homology groups have suspension isomorphisms*

$$E_k(X,*) \cong E_{k+n}(\Sigma^n X,*)$$

for all $n \geq 0$ and all values of k.

Proof. Using Lemma 2.5.3, we get a long exact sequence on reduced homology for any cofiber sequence of based CW complexes $A \to X \to X/A$. We apply this to the cofiber sequence $X \to CX \to \Sigma X$ to get suspension isomorphisms $E_{k+1}(\Sigma X,*) \cong E_k(X,*)$. □

Remark 2.5.5. In Definition 2.5.1, it does not matter whether the functors E_k are defined only for $k \geq 0$ or for all $k \in \mathbb{Z}$. Any theory that is only defined for $k \geq 0$ can be canonically extended to negative k using the above two lemmas. See Exercise 2.7.2.

Remark 2.5.6. The **direct limit axiom** for an extraordinary homology theory E_* states that if X is a CW complex, the natural map from the colimit over all the finite subcomplexes of X,

$$\operatorname*{colim}_{K \subseteq X \text{ finite}} E_n(K) \xrightarrow{\cong} E_n(X),$$

is an isomorphism. This holds whether we use reduced or unreduced E-homology. This axiom actually follows from the other axioms given in Propositions 1.3.1–1.3.4. See Exercise 2.7.4.

Every spectrum E represents an extraordinary homology theory E_* in the following way.

Proposition 2.5.7. *For any spectrum E, the E-homology groups*

$$E_k(X,A) := \pi_k((X/A) \wedge E)$$

form an extraordinary homology theory, whose coefficient group is $\pi_*(E)$.

Here the smash product is the tensor spectrum from Definition 2.3.6. Recalling that $X/\emptyset = X_+$, the unreduced homology groups in this theory are given by $\pi_k(X_+ \wedge E)$, and the reduced groups are given by $\pi_k(X \wedge E)$. By Corollary 2.4.25, a stable equivalence $E \to E'$ gives an isomorphism on the extraordinary homology groups.

Proof. For simplicity we focus on the reduced homology groups of based CW complexes – the proof can be easily adapted to pairs. Let $A \to X \to X/A$ be a cofiber sequence of based CW complexes. Since $A \to X$ is a cofibration, X/A is homotopy equivalent to Cf, the homotopy cofiber of $A \to X$. Taking the smash product with E creates a sequence of spectra

$$A \wedge E \longrightarrow X \wedge E \longrightarrow Cf \wedge E. \tag{2.5.8}$$

By Lemma 2.4.23, this is a cofiber sequence. The long exact sequence and boundary map for E_* therefore come from Lemma 2.4.11. The additivity axiom follows from Exercise 2.7.30 at the end of this section. The homotopy and excision axioms are immediate. □

Example 2.5.9. The sphere spectrum therefore represents an extraordinary homology theory, called "stable homotopy." The reduced version of this theory is

$$\pi_k^S(X) := \pi_k(\Sigma^\infty X),$$

while the unreduced version is $\pi_k(\Sigma_+^\infty X)$. In particular, we get additivity and long exact sequences on stable homotopy groups, unlike the case of ordinary homotopy groups $\pi_k(X)$. The coefficient group is the stable homotopy groups of spheres,

$$\pi_*(\mathbb{S}) = \quad \cdots \quad 0 \quad 0 \quad 0 \quad \mathbb{Z} \quad \mathbb{Z}/2 \quad \mathbb{Z}/2 \quad \mathbb{Z}/24 \quad \cdots$$

Example 2.5.10. The Eilenberg–Mac Lane spectrum HG also represents a homology theory. Its coefficient group is

$$\pi_*(HG) = \quad \cdots \quad 0 \quad 0 \quad 0 \quad G \quad 0 \quad 0 \quad 0 \quad \cdots$$

Since these coefficients are concentrated in degree 0, the dimension axiom is satisfied, so we have defined an ordinary homology theory with G coefficients. By the uniqueness of ordinary homology (Theorem 1.3.8), we conclude:

Corollary 2.5.11. *There is a canonical natural isomorphism*

$$\pi_k((X/A) \wedge HG) \cong H_k(X, A; G)$$

for CW pairs (X, A) and abelian groups G.

Remark 2.5.12. The spectral definition of homology should remind you of the definition of homology with G coefficients:

$$\begin{array}{lccc} \pi_k(& X_+ & \wedge & E \ ) \\ H_k(& C_*(X) & \otimes & G \ ) \end{array}$$

The smash product with E plays the role of tensoring with G. We have gone from taking homology with coefficients in an ordinary abelian group G to homology with exotic coefficients in a spectrum E. You might even want to write "$X_+ \otimes E$" to strengthen this analogy.

We highlight a theorem that we will prove later in the book, due to G.W. Whitehead [Whi62]. It states that every extraordinary homology theory arises from the recipe given in Proposition 2.5.7. In other words, spectra are a more powerful version of "homological algebra" that can describe *every* extraordinary homology theory.

Theorem 2.5.13 (Whitehead representability)**.** *If h_* is any extraordinary homology theory, there is a spectrum E and a natural isomorphism of homology theories $E_* \cong h_*$, where E_* is defined as in Proposition 2.5.7. Furthermore E is unique up to stable equivalence.*

For example, bordism theory $\mathfrak{N}_k(X)$ is isomorphic to $MO_k(X)$, where MO is the Thom spectrum from Example 2.1.20, see also Example 2.5.37. We will give the proof of Theorem 2.5.13 in Theorem 4.2.23.

Remark 2.5.14. In the definition of an extraordinary homology theory, suppose we take one of the boundary homomorphisms $\partial : E_k(X,A) \to E_{k-1}(A,\emptyset)$ and change it by a factor of (-1). Then we get a new theory, technically, but it is isomorphic to the old theory. (Take the identity map on E_{k-1} and below, and the negation map $(-1)\cdot \mathrm{id}$ on E_k and above.) For this reason, it is not necessary to be especially careful about signs when defining ∂. Different choices will give the same theory in the end, up to isomorphism.

2.5.2 Extraordinary cohomology

We next turn to extraordinary cohomology, by considering contravariant functors, in other words functors that reverse the direction of maps.

Definition 2.5.15. An **extraordinary cohomology theory** is any sequence of *contravariant* functors $E^k(-,-)$ from CW pairs to abelian groups, along with natural boundary maps

$$E^k(X,A) \xleftarrow{\delta} E^{k-1}(A,\emptyset),$$

satisfying the first four Eilenberg-Steenrod axioms (Propositions 1.3.14–1.3.17), but not necessarily the fifth axiom (Proposition 1.3.18). The coefficient group is the graded abelian group $E^*(*,\emptyset)$.

As before, this has both a reduced and an unreduced version, but they are equivalent. Without loss of generality, the groups are defined for all $k \in \mathbb{Z}$.

Remark 2.5.16. There is no direct limit axiom for extraordinary cohomology. Instead, there is a $\lim^1$ exact sequence for sequential limits, as in (1.5.7). See Exercise 3.6.15.

Proposition 2.5.17. *For any Ω-spectrum E, the E-cohomology groups*

$$\begin{aligned} E^k(X,A) &:= \pi_{-k}(F(X/A,E)) \\ &\cong [X/A, E_k]_* \qquad \text{for } k \geq 0, \\ &\quad [X/A, \Omega^{|k|}E_0]_* \quad \text{for } k < 0, \end{aligned}$$

form an extraordinary cohomology theory, whose coefficient group is $\pi_{-*}(E)$.

Here $F(X/A,E)$ is the cotensor or function spectrum from Definition 2.3.8. The proof of Proposition 2.5.17 is left to Exercise 2.7.32 at the end of the chapter.

Note that, compared to Proposition 2.5.7, the condition on E was made stronger, but we also get a description in terms of the levels of the spectrum E_k, which wasn't possible for homology. You might say that this version of the theorem is the more natural one.

Example 2.5.18. Taking the fibrant sphere spectrum $f\mathbb{S}$ from Example 2.2.6, we get a cohomology theory called **stable cohomotopy**, whose reduced version is

$$\pi_S^k(X) := \operatorname*{colim}_{n\to\infty} [\Sigma^n X, S^{k+n}]_*.$$

Its coefficient group is $\pi_{-*}(\mathbb{S})$. So it has the stable homotopy groups of spheres in negative degrees, and is zero in positive degrees.

Example 2.5.19. The Eilenberg–Mac Lane spectrum HG also represents a cohomology theory, whose coefficients are simply G in degree 0. Again by the uniqueness of ordinary cohomology, we conclude:

Corollary 2.5.20. *There is a canonical natural isomorphism*

$$\pi_{-k}(F(X/A, HG)) \cong H^k(X, A; G)$$

for CW pairs (X, A) and abelian groups G.

Remark 2.5.21. The spectral definition of cohomology should remind you of the definition of cohomology with G coefficients:

$$\begin{aligned} \pi_{-k}(&\quad F \quad (\quad X_+ \quad , \quad E \quad)\ ) \\ H_{-k}(&\ \mathrm{Hom}\ (\ C_*(X)\ ,\ G\ )\ ) \end{aligned}$$

The function spectrum into E plays the role of taking cochains with coefficients in G. You might even want to write "$\mathrm{Hom}(X_+, E)$" to strengthen this analogy.

When E is an Ω-spectrum, the reduced homology and cohomology groups it represents can be drawn as follows. Note that for cohomology each functor E^k corresponds to a single spectrum level, whereas for homology each spectrum level corresponds to a single sphere S^k.

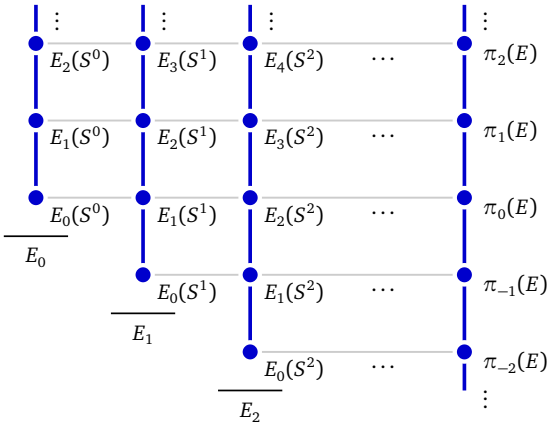


Figure 2.5.22: An Ω-spectrum representing a homology theory. All homology groups are reduced.

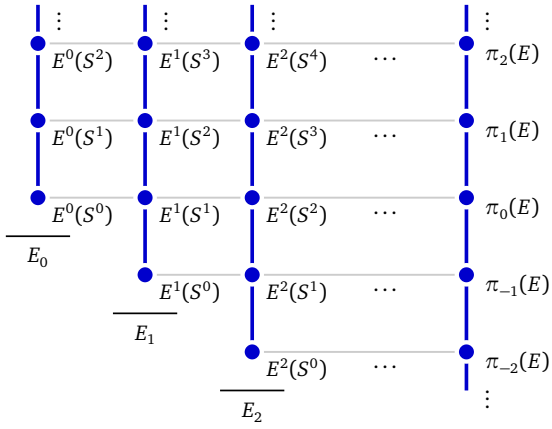


Figure 2.5.23: The same Ω-spectrum representing a cohomology theory. All cohomology groups are reduced.

Remark 2.5.24. The minus signs on the homotopy groups in Proposition 2.5.17 arise because cohomology is traditionally graded with the boundary maps going up in degree. This convention is called *cohomological grading*. If one is willing to make the opposite convention of *homological grading*, the minus sign goes away, and Figure 2.5.23 changes to become Figure 2.5.22. However, if we do this then we must define the functors $E^k(-)$ for all $k \leq 0$, and not all $k \geq 0$, because the functors that determine the theory are now the ones with negative k.

The converse of Proposition 2.5.17 is the following classical theorem of Brown [Bro62, Bro65], that every extraordinary cohomology theory can be constructed this way.

Theorem 2.5.25 (Brown representability)**.** *If h^* is any extraordinary cohomology theory, then there is an Ω-spectrum E and a natural isomorphism of cohomology theories $E^* \cong h^*$, where E^* is defined as in Proposition 2.5.17. Furthermore E is unique up to stable equivalence.*

Proof. We give a moderately detailed sketch – a version with much more detail can be found in e.g. [Hat02, Thm 4E.1], [May99]. By the excision axiom, it is enough to focus on the reduced groups $h^k(X,*)$. It suffices to build for each $k \geq 0$ a CW complex E_k and an isomorphism $h^k(X,*) \cong [X, E_k]_*$. By the Yoneda Lemma (Lemma 1.4.29), the space E_k is unique up to homotopy equivalence, and the suspension isomorphisms in the cohomology theory are represented by weak equivalences $E_{k-1} \simeq \Omega E_k$, making these spaces into an Ω-spectrum.

The main part of the argument is therefore actually building the spaces E_k. We know E_k will be a loopspace, and we have the adjunction $[X, \Omega E_{k+1}]_* \cong [\Sigma X, E_{k+1}]$, where ΣX is always connected. Therefore it suffices to build the connected component of the basepoint of E_{k+1}, and to prove it represents $h^{k+1}(-,*)$ on the category of *connected* based CW complexes.

Simplifying notation, we are given a "set-valued cohomology theory," in other words a functor F from connected based CW complexes to sets, such that

- [Homotopy axiom] F takes homotopic maps to the same map of sets,
- [Wedge axiom] F takes wedge sums to products, in particular $F(*) = *$, and
- [Mayer-Vietoris axiom] for each homotopy pushout $X \cup_A Y$, any pair of elements in $F(X)$ and $F(Y)$ that agree when restricted to $F(A)$, must come from some element of $F(X \cup_A Y)$. (This element might not be unique.)

Our goal is to build a connected based CW complex C and a cohomology class $\alpha \in F(C)$ such that the map

$$\begin{aligned} [X, C]_* &\longrightarrow F(X) \\ f &\longmapsto f^*(\alpha) \end{aligned} \tag{2.5.26}$$

is an isomorphism for all connected based CW complexes X.

Inductively, we can build a C and $\alpha \in F(C)$ such that (2.5.26) is an isomorphism for $X = S^i$, $1 \leq i < n$, and surjective when $X = S^n$. In the inductive step, we kill the kernel of (2.5.26) for $X = S^{n-1}$ by attaching n-cells, using the Mayer-Vietoris axiom to extend α to the result. When X is a sphere, the pinch maps $S^i \to S^i \vee S^i$ make the maps (2.5.26) into group homomorphisms, so killing this kernel has the effect of making the map injective. We then make (2.5.26) surjective for $X = S^n$ by taking a wedge sum with n-spheres, using the wedge axiom to extend α to the result. These steps don't affect (2.5.26) for lower-dimensional spheres because a map that attaches n-cells is $(n-1)$-connected (Proposition 1.4.15).

At the end, we have produced a space C and $\alpha \in F(C)$ such that (2.5.26) is an isomorphism whenever X is a sphere. For arbitrary X, if $x \in F(X)$ is any cohomology class, we can apply the above induction again, starting with $X \vee C$, and attaching cells. This creates a new space C_x, and a cohomology class $\alpha_x \in F(C_x)$ restricting to $x \in F(X)$ and $\alpha \in F(C)$, such that pulling back α_x gives isomorphisms $[S^n, C_x]_* \cong F(S^n)$ for every $n \geq 1$. The inclusion $C \to C_x$ is therefore a weak equivalence. By Theorem 1.4.18, C_x deformation retracts to C. Composing the inclusion of X with this retraction produces a map $f : X \to C$ such that $f^*(\alpha) = x$. This shows that (2.5.26) is surjective for any connected based CW complex X.

For any pair of maps $f, g : X \rightrightarrows C$ giving the same class $x \in F(X)$, we consider the based double mapping cylinder $(X \wedge I_+) \vee_{X \wedge \{0,1\}_+} C$. Since $f^*(\alpha) = g^*(\alpha) = x$, the Mayer-Vietoris axiom produces a cohomology class on this cylinder that restricts to $\alpha \in F(C)$. We once again apply our inductive procedure, starting with this double mapping cylinder and producing a space $C_{f,g}$ that deformation retracts to C. The inclusion of $X \wedge I_+ \to C_{f,g}$ composed with this retraction produces a homotopy between f and g. Therefore (2.5.26) is injective for any connected based CW complex X. □

The above proof can be improved, at the expense of additional headaches involving inverse limits, to a statement that only requires the cohomology theory h^* to be defined on *finite* complexes [Ada71, Thm 1.6].

Theorem 2.5.27 (Brown representability for finite complexes)**.** *If h^* is any extraordinary cohomology theory, defined only on finite CW complexes, then there is an Ω-spectrum E and a natural isomorphism of cohomology theories $E^* \cong h^*$. Furthermore E is unique up to stable equivalence.*

2.5.3 More examples

We finish this section with several more interesting examples of extraordinary homology and cohomology theories.

Example 2.5.28. Complex K-theory from Example 0.1.2 is an unreduced cohomology theory, represented by the spectrum KU defined in Example 2.2.3. In particular, for unbased spaces X the theory gives

$$KU^0(X) := KU^0(X, \varnothing) \cong \pi_0(F(X_+, KU)) \cong [X, \mathbb{Z} \times BU].$$

There is also a connective version $ku^*(X)$, represented by a spectrum ku in which $\pi_{2k} = \mathbb{Z}$ for all $k \geq 0$, and all other homotopy groups are zero.

Example 2.5.29. Similarly, we may take **real K-theory**, the K-theory of real vector bundles. This is represented by an Ω-spectrum KO whose infinite loop space is $\mathbb{Z} \times BO$, the classifying space for stable real vector bundles:

$$KO^0(X) := KO^0(X, \emptyset) \cong \pi_0(F(X_+, KO)) \cong [X, \mathbb{Z} \times BO].$$

There is also a connective version $ko^*(X)$, represented by ko, with the same homotopy groups in the nonnegative degrees only.

Example 2.5.30. Let p be a prime. We define **sphere spectrum mod p** by

$$\mathbb{S}/p = F_n M(\mathbb{Z}/p, n)$$

where the Moore space $M(\mathbb{Z}/p, n)$ is the homotopy cofiber of the degree p map

$$S^n \xrightarrow{p} S^n \longrightarrow M(\mathbb{Z}/p, n).$$

The value of n is irrelevant, as it does not affect $\mathbb{S}/p$ up to stable equivalence. We can think of $\mathbb{S}/p$ informally as the cofiber of a "multiplication by p" map of the sphere spectrum,

$$\mathbb{S} \xrightarrow{p} \mathbb{S} \longrightarrow \mathbb{S}/p.$$

We define **stable homotopy mod p** to be the reduced homology theory represented by this spectrum:

$$\pi_k^S(X; \mathbb{Z}/p) := \pi_k(X \wedge \mathbb{S}/p).$$

The cofiber sequence defining $M(\mathbb{Z}/p, n)$ gives a long exact sequence

$$\cdots \longrightarrow \pi_k^S(X) \xrightarrow{\cdot p} \pi_k^S(X) \longrightarrow \pi_k^S(X; \mathbb{Z}/p) \longrightarrow \pi_{k-1}^S(X) \xrightarrow{\cdot p} \pi_{k-1}^S(X) \longrightarrow \cdots$$

just like the long exact sequence for homology with mod p coefficients. Alternatively, this can be presented by a short exact sequence

$$0 \longrightarrow \pi_k^S(X) \otimes \mathbb{Z}/p \longrightarrow \pi_k^S(X; \mathbb{Z}/p) \longrightarrow \mathrm{Tor}(\pi_{k-1}^S(X), \mathbb{Z}/p) \longrightarrow 0.$$

Example 2.5.31. We define the **rational sphere spectrum** by

$$\mathbb{S}_{\mathbb{Q}} = F_n M(\mathbb{Q}, n)$$

where the Moore space $M(\mathbb{Q}, n)$ is the homotopy colimit of the maps

$$S^n \xrightarrow{1} S^n \xrightarrow{2} S^n \xrightarrow{3} S^n \longrightarrow \cdots \longrightarrow M(\mathbb{Q}, n).$$

Rational stable homotopy is the reduced homology theory represented by this spectrum:

$$\pi_k^S(X; \mathbb{Q}) := \pi_k(X \wedge \mathbb{S}_{\mathbb{Q}}).$$

The homotopy colimit defining $M(\mathbb{Q}, n)$, together with Exercise 2.7.28, can be used to show that $\pi_k^S(X; \mathbb{Q}) \cong \pi_k^S(X) \otimes \mathbb{Q}$. It turns out that $\mathbb{S}_{\mathbb{Q}} \simeq H\mathbb{Q}$ (see Exercise 2.7.41), so rational stable homotopy is isomorphic to rational homology,

$$\pi_k^S(X) \otimes \mathbb{Q} \cong H_k(X, *; \mathbb{Q}).$$

Example 2.5.32. We define the **sphere spectrum localized at** p by

$$\mathbb{S}_{(p)} = F_n M(\mathbb{Z}_{(p)}, n)$$

where the Moore space $M(\mathbb{Z}_{(p)}, n)$ is the homotopy colimit of the maps in Example 2.5.31, except we only take those maps whose degrees are not multiples of p. We define **stable homotopy localized at** p by taking the associated homology theory, giving

$$\pi_k^S(X; \mathbb{Z}_{(p)}) \cong \pi_k^S(X) \otimes \mathbb{Z}_{(p)}.$$

Example 2.5.33. We can similarly define the **sphere spectrum completed at** p to be the homotopy inverse limit of $\mathbb{S}/(p^k)$ as $k \to \infty$. This can be defined more precisely as

$$\mathbb{S}_p^\wedge = \mathrm{sh}^n F(M(\mathbb{Z}/(p^\infty), n), f\mathbb{S}),$$

where $f\mathbb{S}$ is the Ω-sphere spectrum of Example 2.2.6, F is the cotensor or function spectrum of Definition 2.3.8, and $M(\mathbb{Z}/(p^\infty), n)$ is the Moore space whose homology is the Prüfer group $\mathbb{Z}/(p^\infty) = \operatorname{colim}_{k\to\infty} \mathbb{Z}/(p^k)$ in degree n.

For finite CW complexes X, the associated homology theory is

$$\pi_k^S(X)_p^\wedge \cong \pi_k^S(X) \otimes \mathbb{Z}_p^\wedge,$$

where $\mathbb{Z}_p^\wedge = \lim_{k\to\infty} \mathbb{Z}/(p^k)$ is the p-adic integers.[4]

[4]For infinite CW complexes X, it's a little more natural to take the p-completion of the suspension spectrum $(\Sigma^\infty X)_p^\wedge$, rather than taking the smash product $X \wedge (\mathbb{S}_p^\wedge)$.

Example 2.5.34. The previous four examples can be applied to any spectrum E. We define

$$\begin{array}{rcccll} E/p & = & E \wedge \mathbb{S}/p & \simeq & \mathrm{sh}^{-n} M(\mathbb{Z}/p, n) \wedge E & E \text{ mod } p, \\ E_{(p)} & = & E \wedge \mathbb{S}_{(p)} & \simeq & \mathrm{sh}^{-n} M(\mathbb{Z}_{(p)}, n) \wedge E & E \text{ localized at } p, \\ E_{\mathbb{Q}} & = & E \wedge \mathbb{S}_{\mathbb{Q}} & \simeq & \mathrm{sh}^{-n} M(\mathbb{Q}, n) \wedge E & \text{the rationalization of } E, \text{ and} \\ E^{\wedge}_{p} & = & F(\mathbb{S}/(p^{\infty}), RE) & \simeq & \mathrm{sh}^{n} F(M(\mathbb{Z}/(p^{\infty}), n), RE) & \text{the } p\text{-completion of } E. \end{array}$$

We call homology theory associated to E/p the "E-homology mod p," $E_*(X; \mathbb{Z}/p)$. The cofiber sequence defining $M(\mathbb{Z}/p, n)$ gives an exact sequence

$$0 \longrightarrow E_k(X) \otimes \mathbb{Z}/p \longrightarrow E_k(X; \mathbb{Z}/p) \longrightarrow \mathrm{Tor}(E_{k-1}(X), \mathbb{Z}/p) \longrightarrow 0.$$

The next two spectra define the p-local E-homology $E_*(X; \mathbb{Z}_{(p)})$ and rational E-homology $E_*(X; \mathbb{Q})$. Using Exercise 2.7.28 we get isomorphisms

$$E_k(X; \mathbb{Z}_{(p)}) \cong E_k(X) \otimes \mathbb{Z}_{(p)}, \qquad E_k(X; \mathbb{Q}) \cong E_k(X) \otimes \mathbb{Q}.$$

Applying these constructions to $E = H\mathbb{Z}$ recovers the usual notion of homology with coefficients in $\mathbb{Z}/p$, $\mathbb{Z}_{(p)}$, or $\mathbb{Q}$ (Exercise 2.7.42).

The last construction defines (on finite CW complexes) p-complete E-homology, which fits into a short exact sequence

$$0 \longrightarrow \mathrm{Ext}(\mathbb{Z}/(p^{\infty}), E_k(X)) \longrightarrow E_k(X)^{\wedge}_{p} \longrightarrow \mathrm{Hom}(\mathbb{Z}/(p^{\infty}), E_{k-1}(X)) \longrightarrow 0.$$

When the groups $E_*(X)$ are finitely generated, this simplifies to

$$E_k(X)^{\wedge}_{p} \cong E_k(X) \otimes \mathbb{Z}^{\wedge}_{p}.$$

On infinite complexes, it is a little better to complete the smash product $(X \wedge E)^{\wedge}_{p}$, rather than taking the smash product $X \wedge (E^{\wedge}_{p})$. (These agree when X is finite CW.)

Example 2.5.35. The complex K-theory spectrum KU from Example 2.2.3 and Example 2.5.28 becomes simpler after rationalizing – it becomes a wedge sum of shifted copies of $H\mathbb{Q}$:

$$KU_{\mathbb{Q}} \simeq \bigvee_{n \in \mathbb{Z}} \Sigma^{2n} H\mathbb{Q}.$$

In fact, this is true for every spectrum E: the rationalization $E_{\mathbb{Q}}$ is a wedge sum of shifted copies of $H\mathbb{Q}$. In this case, we conclude that the topological K-theory of CW complexes is rationally nothing more than shifted copies of ordinary homology:

$$KU^0(X) \otimes \mathbb{Q} \cong \bigoplus_{n \geq 0} H^{2n}(X; \mathbb{Q}).$$

This isomorphism is called the **Chern character**.

Example 2.5.36. Suppose we only localize the spectrum KU at a prime p, rather than rationalizing. The resulting spectrum $KU_{(p)}$ decomposes into a wedge sum of shifted copies of a single spectrum L called the **Adams summand**:

$$KU_{(p)} \simeq L \vee \Sigma^2 L \vee \ldots \vee \Sigma^{2p-4} L.$$

Its homotopy groups are:

$$\pi_n(L) = \begin{cases} \mathbb{Z}_{(p)} & \text{if } n \equiv 0 \mod 2p-2, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, when $p \neq 2$ the p-localized real K-theory spectrum $KO_{(p)}$ splits into half as many summands

$$KO_{(p)} \simeq L \vee \Sigma^4 L \vee \ldots \vee \Sigma^{2p-6} L.$$

More generally, as soon as 2 is inverted, KU is equivalent to $KO \vee \Sigma^2 KO$.

Example 2.5.37. Bordism theory $\mathfrak{N}_k(X)$ from Example 0.1.1 is an unreduced homology theory, represented by the Thom spectrum MO from Example 2.1.20:

$$\mathfrak{N}_k(X) \cong \pi_k(X_+ \wedge MO).$$

The proof of this uses the Pontryagin-Thom isomorphism, see [Wes96, Kup17, Mil94, Sto68, Coh20]. The spectrum MO has homotopy groups the underlying abelian group of the graded $\mathbb{Z}/2$-algebra

$$\pi_*(MO) \cong (\mathbb{Z}/2)[\ x_n : n \neq 2^k - 1\ ] = (\mathbb{Z}/2)[x_2, x_4, x_5, x_6, x_8, \ldots], \qquad |x_n| = n.$$

In particular, the first few homotopy groups are $\mathbb{Z}/2, 0, \mathbb{Z}/2, 0, \mathbb{Z}/2^2, \ldots$. As a spectrum, it is a very nontrivial theorem that MO is equivalent to a wedge of Eilenberg–Mac Lane spectra

$$MO \simeq \bigvee_\alpha \Sigma^{n_\alpha} H(\mathbb{Z}/2),$$

one for each $\mathbb{Z}/2$ summand in the graded abelian group $\pi_*(MO)$.[5] It follows that the bordism groups of a space X are calculated as

$$\mathfrak{N}_k(X) \cong \pi_*(MO) \otimes_{\mathbb{Z}/2} H_*(X; \mathbb{Z}/2).$$

In summary, bordism is determined by mod 2 homology.

[5] Equivalently, it is a product of the same Eilenberg–Mac Lane spectra. See Exercise 3.6.25.

Example 2.5.38. There is similarly a theory of **complex cobordism** $MU_*(X)$, see [Rav86, Mil94]. It is represented by a spectrum MU, that is built just as MO is built in Example 2.1.20, but with complex vector bundles. The universal classifying space is $BU(n)$, with tautological bundle $\gamma^n \to BU(n)$. Since the the compactification of $\mathbb{C}$ is a two-sphere, the structure maps have the form

$$\Sigma^2 \mathrm{Th}(\gamma^n) \to \mathrm{Th}(\gamma^{n+1}). \tag{2.5.39}$$

We therefore let the Thom space for γ^n be spectrum level $2n$, rather than spectrum level n. We interpolate between these using suspensions

$$MU_{2n} = \mathrm{Th}(\gamma^n), \qquad MU_{2n+1} = \Sigma \mathrm{Th}(\gamma^n),, \qquad MU_{2n+2} = \mathrm{Th}(\gamma^{n+1}), \qquad \cdots$$

and use the map (2.5.39) to define the bonding map $\Sigma MU_{2n+1} \to MU_{2n+2}$. Again, intuitively, MU is a twisted suspension spectrum of the classifying space $BU = \operatorname{colim}_{n\to\infty} BU(n)$.

The coefficient group is the underlying abelian group of the graded $\mathbb{Z}$-algebra

$$\pi_*(MU) \cong \mathbb{Z}[x_1, x_2, x_3, \ldots], \qquad |x_n| = 2n.$$

Example 2.5.40. The spectrum MU does not split quite as nicely as MO, but after localizing at a prime p, it splits into shifted copies of a single spectrum BP called the **Brown-Peterson spectrum**. Its homotopy groups are the underlying abelian group of the graded ring

$$\pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \ldots], \qquad |v_n| = 2(p^n - 1).$$

There are a number of other interesting spectra, distilled out of MU, that play an important role in computations in stable homotopy theory. They are derived from BP by killing some elements of π_* and inverting others:

$$\begin{aligned} \pi_*(BP\langle n\rangle) &= \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n] \\ \pi_*(E(n)) &= \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n, v_n^{-1}] \\ \pi_*(K(n)) &= (\mathbb{Z}/p)[v_n, v_n^{-1}] \end{aligned}$$

The last one in particular, **Morava K-theory**, is very computable, because it acts like homology with field coefficients. As n varies, it interpolates between rational homology $K(0) = H\mathbb{Q}$ and mod p homology $K(\infty) = H\mathbb{Z}/p$. The first theory $K(1)$ is also equivalent to L/p, where L is the Adams summand of Example 2.5.36.

These spectra allow us to pick out phenomena in $\pi_*(\mathbb{S})$ that occur at one prime p and one "frequency" $2(p^n - 1)$, like separating light into colors using a prism. For this reason, the study of stable homotopy category from this point of view is called **chromatic homotopy theory**. We will discuss this a little more in Section 3.3.3, but see e.g. [Rav86, Rav92, Lur10, Gui24] for much more in this direction.

Remark 2.5.41. The analog of the universal coefficient theorem for extraordinary homology is called the Atiyah-Hirzebruch spectral sequence, which for unreduced homology is written

$$H_p(X; E_q(*)) \Rightarrow E_{p+q}(X),$$

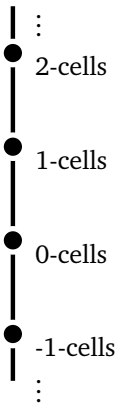
and for cohomology

$$H^p(X; E^q(*)) \Rightarrow E^{p+q}(X).$$

See **??**. The much more impressive Adams spectral sequence makes it possible to go from cohomology of a spectrum to homotopy groups, at least after localizing or completing at a prime p – see **??**.

2.6 Cellular and CW spectra

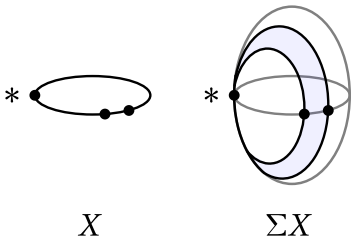
CW spectra are essentially CW complexes with negative-dimensional cells. Every spectrum is stably equivalent to a CW spectrum, so we can think of spectra as complexes with negative-dimensional cells. When thinking this way, we draw the cells as black dots and connect them with lines as shown.



This is a satisfying intuition, and it's useful on a technical level too. We can use it to prove versions of the Whitehead theorem, cellular approximation, Postnikov towers, and the Hurewicz theorem for spectra.

2.6.1 Definition

Recall from Definition 1.1.11 that a cell complex X is a space that is built by repeatedly attaching discs, of varying dimensions. If X is a based space, we ask that the basepoint be one of the cells. A CW complex is a cell complex in which we only attach n-cells to cells of lower dimension.



Lemma 2.6.1. *If X is a based cell complex, then ΣX is a based cell complex in a natural way, with one $(n+1)$-cell for each n-cell of X other than the basepoint.*

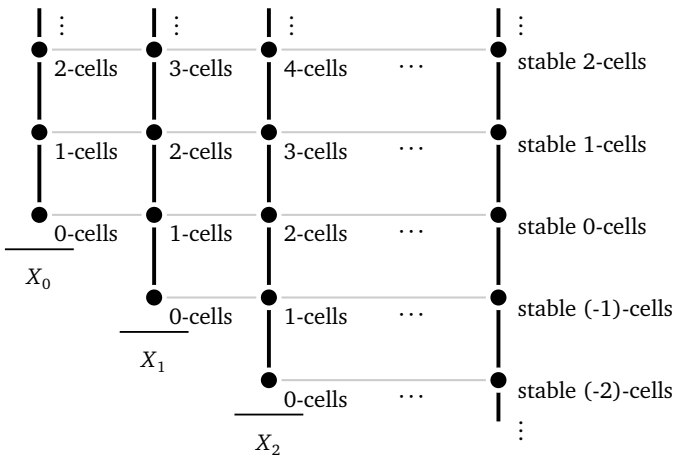
Proof. For each n-cell $D^n \to X$, we write the map as a map of based spaces $D^n_+ \to X$ and then take the reduced suspension, giving $\Sigma D^n_+ \to \Sigma X$. The space ΣD^n_+ is a quotient of

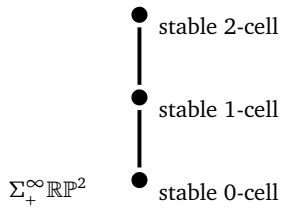
$I \times D^n \cong D^{n+1}$ by an equivalence relation along its boundary. We therefore get a map $D^{n+1} \to \Sigma X$, in other words an $(n+1)$-cell in ΣX. It is now straightforward to check that ΣX is homeomorphic to the disjoint union of these cells modulo their attaching maps, making ΣX into a cell complex. □

Definition 2.6.2. The spectrum X is a **cellular spectrum** if each space X_n can be given the structure of a based cell complex, so that each bonding map $\Sigma X_n \to X_{n+1}$ is the inclusion of a subcomplex. A **CW spectrum** is the same except that each X_n is a CW complex.

If X is a cellular spectrum and $k \in \mathbb{Z}$, a **stable k-cell** in X is a non-basepoint $(k+n)$-cell in X_n for any $n \geq 0$. This is identified with the corresponding cell of dimension $(k+n+1)$ in $\Sigma X_n \subseteq X_{n+1}$, and so on as $n \to \infty$.

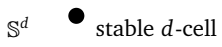
As we did with the stable homotopy groups, we can depict these stable cells by drawing the levels of the spectrum in a staggered formation. Then each cell creates a cell immediately to its right, and the limit of this process at the far right is the set of stable cells of X.





Example 2.6.3. If K is a based cell or CW complex then its suspension spectrum $\Sigma^\infty K$ is a cellular or CW spectrum, respectively. It has a stable k-cell for every k-cell of K, other than the basepoint. More generally, the shift desuspension spectrum $F_n K$ is cellular, with a stable $(k-n)$-cell for every k-cell of K, other than the basepoint.

Example 2.6.4. As a special case of the previous example, for any $d \in \mathbb{Z}$, the d-sphere spectrum $\mathbb{S}^d$ from Example 2.1.9 is characterized by the fact that it has a single stable d-cell.



Example 2.6.5. If B is an unbased cell or CW complex and ζ is a virtual bundle over B then the Thom spectrum $\mathrm{Th}(\zeta)$ constructed in Example 2.1.18 is a cellular or CW spectrum, respectively. The Thom spectrum MO from Example 2.1.20 and MU from Example 2.5.38 are both CW spectra.

Example 2.6.6. When building the Eilenberg–Mac Lane spectrum HG from Example 2.2.2, we may choose each level $K(G,n)$ to be a CW complex, and iteratively replace $K(G,n+1)$ with the mapping cylinder of the map $\Sigma K(G,n) \to K(G,n+1)$, so as to make the entire spectrum into a CW spectrum.

You should picture cellular spectra as built by an iterative process. At the first step, we start with the zero spectrum, and attach cells (of all dimensions) to spectrum level 0 to make X_0. As we do this, we're also attaching the suspensions of these cells to all the higher spectrum levels. At the end, we've made the suspension spectrum $\Sigma^\infty X_0$.

X_0	cells attached at level 0
ΣX_0	Σ(cells attached at level 0)
$\Sigma^2 X_0$	Σ^2(cells attached at level 0)
$\vdots$	$\vdots$

Then, we attach cells to spectrum level 1, and all of their suspensions to the higher levels. At level 1, this makes a space X_1 that looks like ΣX_0 with more cells attached, and we get the suspensions of this space at all the higher levels:

X_0	cells attached at level 0	
X_1	Σ(cells attached at level 0)	cells attached at level 1
ΣX_1	Σ^2(cells attached at level 0)	Σ(cells attached at level 1)
$\Sigma^2 X_1$	Σ^3(cells attached at level 0)	Σ^2(cells attached at level 1)
$\vdots$	$\vdots$	$\vdots$

At level 2, attach more cells to ΣX_1 to make X_2, and their suspensions, and so on.

X_0	cells attached at level 0		
X_1	Σ(cells attached at level 0)	cells attached at level 1	
X_2	Σ^2(cells attached at level 0)	Σ(cells attached at level 1)	cells attached at level 2
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Clearly, we can do this whole process relative to some fixed starting spectrum A. We attach cells to A_0 to form X_0, and also attach their suspensions to the higher levels of A, giving the spectrum in the second column just below. Then we attach cells to $A_1 \cup_{\Sigma A_0} \Sigma X_0$ to form X_1, and their suspensions to the higher levels, giving the third column below, and so on:

to start	after one step	after two steps	after three steps
A_0	X_0	X_0	X_0
A_1	$A_1 \cup_{\Sigma A_0} \Sigma X_0$	X_1	X_1
A_2	$A_2 \cup_{\Sigma^2 A_0} \Sigma^2 X_0$	$A_2 \cup_{\Sigma A_1} \Sigma X_1$	X_2
A_3	$A_3 \cup_{\Sigma^3 A_0} \Sigma^3 X_0$	$A_3 \cup_{\Sigma^2 A_1} \Sigma^2 X_1$	$A_3 \cup_{\Sigma A_2} \Sigma X_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

A map of spectra $A \to X$ built in this way is called a "relative cellular spectrum." Writing this down more precisely:

Definition 2.6.7. For a map of spectra $f : A \to X$ the nth **relative bonding map** is the map

$$A_n \cup_{\Sigma A_{n-1}} \Sigma X_{n-1} \to X_n \tag{2.6.8}$$

given by f_n and ξ_{n-1}. For $n = 0$, it is just the map $f_0 \colon A_0 \to X_0$.

The map f is a **relative cellular spectrum** if each of the relative bonding maps is a relative cell complex of spaces. A relative CW spectrum is defined the same way.

Of course, $* \to X$ is a relative cellular spectrum when X is a cellular spectrum.

2.6.2 Stable cell attachments

This iterative process can be described more holistically. When we attach a cell to one spectrum level of A, and all of its suspensions to the higher levels, we are really attaching a free spectrum $F_n D_+^{k+n}$ to A.

To explain this further, recall that for a unbased space K, the shift desuspension spectrum $F_n K_+$ is the one-point space at every level until n, then K_+ at level n, then suspensions of K_+ after that:

$$F_n K_+ = \{* \quad * \quad \dots \quad * \quad K_+ \quad \Sigma(K_+) \quad \Sigma^2(K_+) \quad \dots\}$$

We call this the **free spectrum** on the space K at level n, for the following reason:

Lemma 2.6.9. *A map of spectra $F_n K_+ \to X$ is the same data as a map of unbased spaces $K \to X_n$.*

In other words, $F_n(-)_+$ is the left adjoint of the operation ev_n that takes every spectrum X to its nth space X_n and forgets the basepoint of X_n (see Exercise 2.7.17).

Definition 2.6.10. For any spectrum A and any map $\varphi : S^{(k-1)+n} \to A_n$, we **attach a k-cell** to A along φ by taking the pushout spectrum

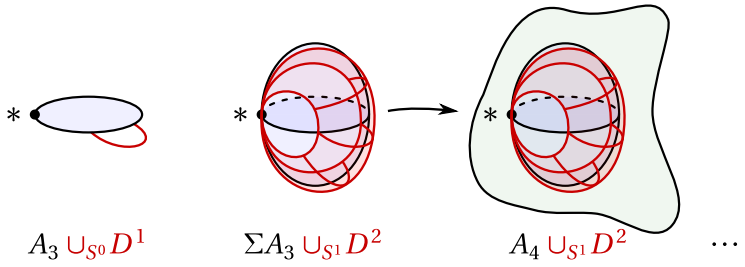
$$A' = (F_n D_+^{k+n}) \cup_{F_n(S_+^{(k-1)+n})} A.$$

At each spectrum level $m \geq n$, using Lemma 2.6.1, this gives the pushout

$$A'_m = (\Sigma^{m-n} D_+^{k+n}) \cup_{\Sigma^{m-n}(S_+^{(k-1)+n})} A_m \cong D^{k+m} \cup_{S^{(k-1)+m}} A_m.$$

In other words, it is A_m with a $(k+m)$-cell attached. So, this operation attaches a single stable k-cell (in the sense of Definition 2.6.2) to the spectrum A.

We illustrate just below the process of attaching a stable (-2)-cell to A, by attaching a 1-cell to spectrum level 3, and all of its suspensions to the higher spectrum levels.



Proposition 2.6.11. *The map $f : A \to X$ is a relative cellular spectrum if and only if it is a countable composition*

$$A = X^{(-1)} \longrightarrow X^{(0)} \longrightarrow X^{(1)} \longrightarrow X^{(2)} \longrightarrow \ldots \longrightarrow X^{(j-1)} \longrightarrow X^{(j)} \longrightarrow \ldots \longrightarrow X$$

in which each map $X^{(j-1)} \to X^{(j)}$ attaches an arbitrary number of cells of varying dimensions:

$$\begin{array}{ccc} \bigvee_i F_{n_i} S_+^{(k_i-1)+n_i} & \longrightarrow & \bigvee_i F_{n_i} D_+^{k_i+n_i} \\ \downarrow & & \downarrow \\ X^{(j-1)} & \longrightarrow & X^{(j)}. \end{array}$$

Proof. This is mainly an exercise in bookkeeping. It is clear that any f of this form attaches cells and their suspensions as described earlier in this section, though it may attach cells to the levels in different orders. Still, it is enough to see that the relative bonding maps are all cell complexes, so f is a relative cellular spectrum. Conversely, if f is a relative cellular spectrum, the procedure described just before Definition 2.6.7 expresses f as a countable composition of stable cell attachments. Strictly speaking, it is a countable composition *of countable compositions* of cell attachments, but this can be re-indexed by attaching each cell earlier in the process, to make a single countable composition. See Exercise 2.7.29. □

Recall from Corollary 1.4.17 that any map of topological spaces $A \to X$ factors into a relative CW complex $A \to B$, followed by a weak equivalence $B \to X$. As a consequence, every topological space X is weakly equivalent to a CW complex QX. We now prove the same fact for spectra.

Theorem 2.6.12. *If $f : A \to X$ is any map of spectra, it can be factored into a relative CW spectrum $A \to B$ and a level equivalence $B \to X$.*

Corollary 2.6.13. *Every spectrum X is level equivalent to a CW spectrum QX, and stably equivalent to a CW Ω-spectrum QRX. (That is, QRX is a CW spectrum and also an Ω-spectrum.)*

Proof of Theorem 2.6.12. By Corollary 1.4.17, the map $A_0 \to X_0$ can be factored into a CW complex followed by an equivalence,

$$A_0 \longrightarrow B_0 \xrightarrow{\sim} X_0. \tag{2.6.14}$$

The reduced suspension of (2.6.14) fits in with the structure maps of A and X in the following way:

$$\begin{array}{ccccc} \Sigma A_0 & \longrightarrow & \Sigma B_0 & \longrightarrow & \Sigma X_0 \\ \downarrow & & \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_1 \cup_{\Sigma A_0} \Sigma B_0 & - - \to & X_1 \end{array}$$

The dotted-arrow map is induced from our original map $f_1 : A_1 \to X_1$ and the suspension of the map $B_0 \to X_0$ that we constructed in the previous step. We may factor it into a relative cell complex followed by a weak equivalence, giving a new space we call B_1:

$$A_1 \cup_{\Sigma A_0} \Sigma B_0 \longrightarrow B_1 \xrightarrow{\sim} X_1$$

Notice that B_1 is equipped with compatible maps coming in from A_1 and ΣB_0, and a map going out to X_1. By construction, the relative bonding map $A_1 \cup_{\Sigma A_0} \Sigma B_0 \to B_1$ is a cell complex and $B_1 \to X_1$ is a weak equivalence.

Repeating this procedure for each spectrum level gives a spectrum B and the maps $A \to B$ and $B \to X$ with the desired properties. □

Remark 2.6.15. The factorization in Theorem 2.6.12, just as the one in Corollary 1.4.17, can be defined naturally, so that it is a functor. In particular, we get a functor $Q : \mathbf{Sp} \to \mathbf{Sp}$ that replaces each spectrum X by a CW spectrum QX, and a natural level equivalence $QX \to X$. See also Section 5.1.

Next we prove the Whitehead theorem for spectra.

Proposition 2.6.16. *If $f : X \to Y$ is a level equivalence of cellular spectra, then it is a homotopy equivalence, in the sense that there is a map $g : Y \to X$ and homotopies from $f \circ g$ and $g \circ f$ to the identity.*

Corollary 2.6.17 (Whitehead theorem for spectra)**.** *If X and Y are cellular Ω-spectra then $f : X \to Y$ is a stable equivalence iff it is a homotopy equivalence.*

Proof of Proposition 2.6.16. Replace Y with the mapping cylinder $M = (X \wedge I_+) \cup_X Y$. It suffices to show that M deformation retracts onto X as a spectrum. Since f is a level equivalence and both X_n and Y_n are cell complexes, M_n is a cell complex that deformation retracts onto the subcomplex X_n.

The complex M_n also contains ΣM_{n-1} as a subcomplex, and the intersection with X_n is exactly ΣX_{n-1}. We therefore have inclusions of cell complexes

$$X_n \longrightarrow X_n \cup_{\Sigma X_{n-1}} \Sigma M_{n-1} \longrightarrow M_n. \tag{2.6.18}$$

The second map here is the relative bonding map for $X \to M$ from Definition 2.6.7.

The first map of (2.6.18) is a homotopy equivalence, since ΣM_{n-1} deformation retracts onto ΣX_{n-1}. The composite is also a homotopy equivalence. Therefore the second inclusion is a homotopy equivalence as well. Therefore M_n deformation retracts onto the pushout $X_n \cup_{\Sigma X_{n-1}} \Sigma M_{n-1}$, by the Whitehead theorem for spaces, specifically the second form of the theorem mentioned in Theorem 1.4.18.

Composing $(n+1)$ of these deformation retractions together gives a deformation retraction of M_n onto X_n. Specifically, the composite

$$M_n \to X_n \cup_{\Sigma X_{n-1}} \Sigma M_{n-1} \to X_n \cup_{\Sigma^2 X_{n-2}} \Sigma^2 M_{n-2} \to \cdots \to X_n \cup_{\Sigma^n X_0} \Sigma^n M_0 \to X_n$$

can be deformed back to the identity of M_n, by performing the homotopy from the right-map to the identity, composing this with the remaining maps, and then repeating the process with the remaining homotopies until we end with the identity of M_n. We parametrize this so that the homotopy at the far right happens from time $1/2$ to 1, the next one happens from time $1/4$ to $1/2$, and so on, until the homotopy at the far left which happens from time $1/2^{n+1}$ to $1/2^n$. We take the constant homotopy at the identity map of M_n from time 0 to $1/2^{n+1}$.

This gives a homotopy $h\colon M_n \wedge I_+ \to M_n$ at each spectrum level n, and our choices of parametrization ensure that they agree along the bonding maps:

$$\begin{array}{ccc} \Sigma M_{n-1} & \xrightarrow{\xi_{n-1}} & M_n \\ {\scriptstyle \Sigma h_t}\downarrow & & \downarrow{\scriptstyle h_t} \\ \Sigma M_{n-1} & \xrightarrow{\xi_{n-1}} & M_n \end{array}$$

In particular, from time 0 to $1/2^{n+1}$, both vertical maps are the identity. From time $1/2^{n+1}$ to $1/2^n$, the right-hand map deforms onto a subspace containing ΣM_{n-1} and so is the identity on ΣM_{n-1}, while the left-hand map is the identity. For all remaining times, the right-hand map has been directly defined so as to agree with the left-hand map on ΣM_{n-1}, so the diagram commutes. Therefore we have a homotopy of maps of spectra $M \to M$, from the identity of M to a map that retracts onto X. □

Remark 2.6.19. We will later give a second proof of this using the abstract theory of model categories. See also Exercise 3.6.8.

Remark 2.6.20. The Whitehead theorem is named after J.H.C. Whitehead, while Whitehead representability is named after G.W. Whitehead.

2.6.3 Relative homotopy groups

To get Postnikov towers and the Hurewicz theorem, we will have to make cellular approximations of spectra in a different way, that does not proceed one level at a time. As before, we attach cells to kill the relative homotopy groups. But, we're going to think of the relative homotopy groups of the spectrum as a whole, not on each level separately.

Recall from Chapter 1 that for a map $f : A \to X$ of topological spaces, the relative homotopy group $\pi_n(X,A)$ describes commuting squares of based maps

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\subseteq} & D^n \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & X \end{array}$$

up to homotopies of the vertical maps. Each element $\alpha \in \pi_n(X,A)$ therefore gives a way to attach an n-cell to A, and to extend the map $A \to X$ to the new, larger version of A that has this cell attached. By Proposition 1.4.15, this has the effect of killing $\alpha \in \pi_n(X,A)$ without changing $\pi_{<n}(X,A)$.

Remark 2.6.21. By the long exact sequence on homotopy groups, this means that attaching an n-cell changes $\pi_*(A)$ in a way that makes

$$\begin{array}{rl} \pi_n(A) \to \pi_n(X) & \text{more surjective,} \\ \pi_{n-1}(A) \to \pi_{n-1}(X) & \text{more injective, and} \\ \pi_k(A) \to \pi_k(X) & \text{unchanged for } k < n-1. \end{array}$$

Applying this process iteratively, we can make A larger and larger, and agree up to π_n with X for progressively larger values of n.

We want to do the same process in spectra. It is surprisingly straightforward; we just have to define relative homotopy groups of spectra as the colimit of relative homotopy of the levels, and attach cells to spectra using Definition 2.6.10.

Definition 2.6.22. If $f : A \to X$ is a map of spectra, and $k \in \mathbb{Z}$, we define the kth **relative stable homotopy group** $\pi_k(X,A)$ as the colimit of the system

$$\ldots \longrightarrow \pi_{k+n}(X_n,A_n) \xrightarrow{\sigma} \pi_{k+n+1}(\Sigma X_n, \Sigma A_n) \xrightarrow{\xi_n} \pi_{k+n+1}(X_{n+1},A_{n+1}) \longrightarrow \ldots$$

The map σ is defined as in Definition 2.1.2. To make the definitions agree as nicely as possible, we define $\pi_m(X_n,A_n)$ to mean maps of pairs $(D^m,S^{m-1}) \to (X_n,A_n)$, where S^{m-1} is the one-point compactification of $\mathbb{R}^{m-1}$ and D^m is the one-point compactification of $[0,\infty) \times \mathbb{R}^{m-1}$.

Proposition 2.6.23. *For any map of spectra $f : A \to X$ there is a long exact sequence*

$$\ldots \longrightarrow \pi_k(A) \xrightarrow{f_*} \pi_k(X) \longrightarrow \pi_k(X,A) \xrightarrow{\partial} \pi_{k-1}(A) \longrightarrow \ldots$$

and a natural isomorphism $\pi_k(X,A) \cong \pi_k(Cf)$.

Proof. This follows by taking the colimit of the corresponding exact sequences at each spectrum level (Theorem 1.4.6). See Exercise 2.7.14. □

Each element of $\pi_k(X,A)$ can be represented by a commuting diagram of based spaces as shown, for any sufficiently large n.

$$\begin{array}{ccc} S^{(k-1)+n} & \longrightarrow & D^{k+n} \\ {\scriptstyle \partial\alpha_n}\downarrow & & \downarrow{\scriptstyle \alpha_n} \\ A_n & \xrightarrow{f_n} & X_n \end{array} \tag{2.6.24}$$

Thinking of this as a diagram of unbased spaces, we can therefore attach a k-cell to A along $\partial\alpha_n$, as in Definition 2.6.10. It is easy to see that up to stable equivalence, the result only depends on $\partial\alpha \in \pi_{k-1}(A)$, and not on n.

Proposition 2.6.25. *For any map of spectra $f : A \to X$, any integer $k \in \mathbb{Z}$, and any element $\alpha \in \pi_k(X,A)$, this produces a factorization $A \to A' \to X$ such that*

$$\pi_i(X,A) \to \pi_i(X,A')$$

is an isomorphism for $i < k$, and surjective for $i = k$, with kernel containing α.

Proof. By Proposition 1.4.15, this happens at each spectrum level n for sufficiently large n. In other words, the map $\pi_{i+n}(X_n,A_n) \to \pi_{i+n}(X_n,A'_n)$ is an isomorphism for $i < k$, and surjective for $i = k$, with kernel containing α_n. Taking the colimit over n gives the result. □

Remark 2.6.26. We can do the same with any collection of elements $\alpha \in \pi_k(X,A)$, or even the entire group. This has the effect of killing $\pi_k(X,A)$ while preserving all of the lower homotopy groups.

As discussed in Remark 2.6.21, attaching stable k-cells to A makes

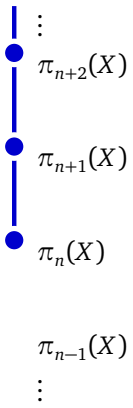
$$\begin{aligned} \pi_k(A) &\to \pi_k(X) && \text{more surjective,} \\ \pi_{k-1}(A) &\to \pi_{k-1}(X) && \text{more injective,} \end{aligned}$$

and doesn't affect the lower homotopy groups. Proceeding up one value of k at a time, we can therefore make the map $A \to X$ an isomorphism on homotopy groups, one group at a time.

But we have to start somewhere. In spectra, there could be homotopy groups in infinitely many negative degrees, which would make it impossible to start.

Definition 2.6.27. A spectrum X is **$(n-1)$-connected** or **n-connective** if $\pi_i(X) = 0$ for $i < n$. Similarly a map of spectra $A \to X$ is $(n-1)$-connected if $\pi_i(X,A) = 0$ for $i < n$.

A (-1)-connected spectrum X is also called **connective**. A spectrum is **bounded below** if it is $(n-1)$-connected for some $n \in \mathbb{Z}$.



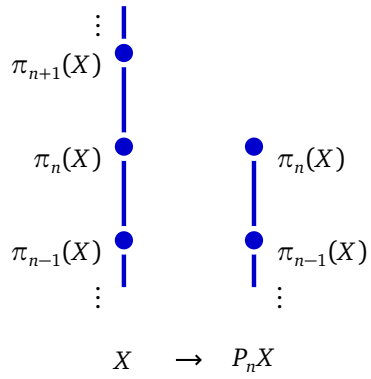
Example 2.6.28. Every suspension spectrum $\Sigma^\infty K$ and Eilenberg–Mac Lane spectrum HG is connective. Every Thom spectrum $\mathrm{Th}(\zeta)$ is bounded below by the dimension of the virtual bundle ζ. In particular, MO and MU are connective. The complex K-theory spectra KU and KO are not bounded below.

Repeated application of Proposition 2.6.25 and Proposition 2.6.23 gives:

Proposition 2.6.29. *The spectrum X is $(n-1)$-connected iff it is stably equivalent to a CW spectrum with stable cells in dimension n and above.*

The map $A \to X$ is $(n-1)$-connected iff it is stably equivalent to a relative CW spectrum with stable cells in dimension n and above.

This gives a second way to replace spectra by CW spectra, different than Theorem 2.6.12.



Definition 2.6.30. For any spectrum X and any integer n, we construct the **nth Postnikov stage** $X \to P_nX$ as follows. We take the map $X \to *$ and attach cells to X to kill the relative homotopy groups $\pi_k(*, X)$ for all $k \geq n+1$, just as we attached cells to A for a map $A \to X$ in Proposition 2.6.25. To be more precise, we attach all possible stable $(n+1)$-cells to the map $X \to *$, then all possible $(n+2)$-cells, and so on. In other words, we attach a stable cell for every diagram of spectra of the form

$$\begin{array}{ccc} S^{(k-1)+m} & \longrightarrow & D^{k+m} \\ {\scriptstyle \partial\alpha_m}\downarrow & & \downarrow \\ X_m & \longrightarrow & * \end{array}$$

with $k = n+1$, then we do this again with $k = n+2$, and so on. This produces a relative CW spectrum $X \to P_nX$.

By Proposition 2.6.25, the map $X \to P_nX$ is an isomorphism on homotopy groups up to degree n, while above degree n the homotopy groups of P_nX are zero.

Example 2.6.31. The 0th Postnikov stage of the sphere spectrum $P_0\mathbb{S}$ has only a $\mathbb{Z}$ in degree zero. By uniqueness of Eilenberg–Mac Lane spectra (Exercise 2.7.39), it is therefore stably equivalent to $H\mathbb{Z}$. We therefore have constructed a map (up to stable equivalences) $\mathbb{S} \to H\mathbb{Z}$, giving an isomorphism on π_0. We could also construct such a map more simply by taking $S^0 \to K(\mathbb{Z}, 0)$ by taking the non-basepoint to the point $1 \in \mathbb{Z}$, and then taking its suspensions.

Example 2.6.32. More generally, if X is any connective spectrum, the 0th Postnikov stage is a map of spectra $X \to H(\pi_0 X)$ that is an isomorphism on π_0. So for instance we get a map $MO \to H\mathbb{Z}/2$, and $MU \to H\mathbb{Z}$.

Lemma 2.6.33. *For each $n \in \mathbb{Z}$ there is a canonical map $P_{n+1}X \to P_nX$ giving an isomorphism on homotopy groups up to degree n.*

Proof. Each of the cells that we attach to form $P_{n+1}X$ is also a cell that we used for P_nX, since we took *all possible* cells when killing π_{n+2} and higher. In other words, as we construct the complex P_nX inductively, at each stage it contains the corresponding stage for $P_{n+1}X$ as a subcomplex. Either way, this gives the map $P_{n+1}X \to P_nX$, and since it commutes with the map in from X, it gives an isomorphism on the lower homotopy groups. □

Definition 2.6.34. The above construction and lemma define the **Postnikov tower** of X, a bi-infinite tower of the following form.

$$X \longrightarrow \ldots \longrightarrow P_2X \longrightarrow P_1X \longrightarrow P_0X \longrightarrow P_{-1}X \longrightarrow P_{-2}X \longrightarrow \ldots$$

The cofiber or fiber of each map $P_nX \to P_{n-1}X$ has homotopy concentrated in a single degree. By Exercise 2.7.39, it is therefore a shift of an Eilenberg–Mac Lane spectrum. We therefore get a fiber Puppe sequence of spectra

$$\Sigma^n H\pi_n(X) \longrightarrow P_nX \longrightarrow P_{n-1}X \longrightarrow \Sigma^{n+1} H\pi_n(X).$$

Sometimes the Postnikov tower is bottomless. There is a lowest nontrivial stage precisely when X is bounded below.

Remark 2.6.35. The map $P_{n-1}X \to \Sigma^{n+1} H\pi_n(X)$ is sometimes called the nth **k-invariant** of the Postnikov tower. Once we define cohomology of spectra (Definition 3.2.18), we can reinterpret this as a class in $H^{n+1}(P_{n-1}X; \pi_n(X))$. The tower can be constructed step-by-step if you know these invariants. It is remarkable that this always works in spectra, as a result of the stability theorems – in topological spaces, these k-invariants don't always exist, they only exist for a Postnikov tower of *principal* fibrations. See [Hat02, Thm 4.69].

Example 2.6.36. The last three stages of the Postnikov tower for the sphere spectrum have the following homotopy groups:

$\vdots$	$\vdots$	$\vdots$	$\vdots$
π_3	0	0	0
π_2	$\mathbb{Z}/2$	0	0
π_1	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
π_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
	$P_2\mathbb{S}$	$P_1\mathbb{S}$	$P_0\mathbb{S}$

As already mentioned, $P_0\mathbb{S}$ is stably equivalent to $H\mathbb{Z}$.

We also form the **tower of connective covers** or **Whitehead tower** of X by defining $X\langle n\rangle$ to be the homotopy fiber of $X \to P_{n-1}X$. This gives a bi-infinite tower mapping to X

$$\ldots \longrightarrow X\langle 2\rangle \longrightarrow X\langle 1\rangle \longrightarrow X\langle 0\rangle \longrightarrow X\langle -1\rangle \longrightarrow X\langle -2\rangle \longrightarrow \ldots \longrightarrow X,$$

where $X\langle n\rangle$ is n-connective and the map $X\langle n\rangle \to X$ is an isomorphism on $\pi_{\geq n}$. If X is bounded below, then this tower is eventually the constant tower at X.

2.6.4 Homology of spectra

Now that cellular spectra are in place, we can take their homology groups, and prove the Hurewicz theorem. There are actually two equivalent ways to define the kth homology group $H_k(X; G)$ of the spectrum X:

Definition 2.6.37.

- As in Definition 2.1.2, define $H_k(X;G)$ as the colimit of the system

$$\ldots \longrightarrow H_{k+n}(X_n, *; G) \xrightarrow[\cong]{\sigma} H_{k+n+1}(\Sigma X_n, *; G) \xrightarrow{\xi_n} H_{k+n+1}(X_{n+1}, *; G) \longrightarrow \ldots$$

  where the map σ is the suspension isomorphism. We assume the levels of X are well-based so that σ is an isomorphism.

- As in Proposition 2.5.7, define

$$H_k(X;G) = \pi_k(X \wedge HG)$$

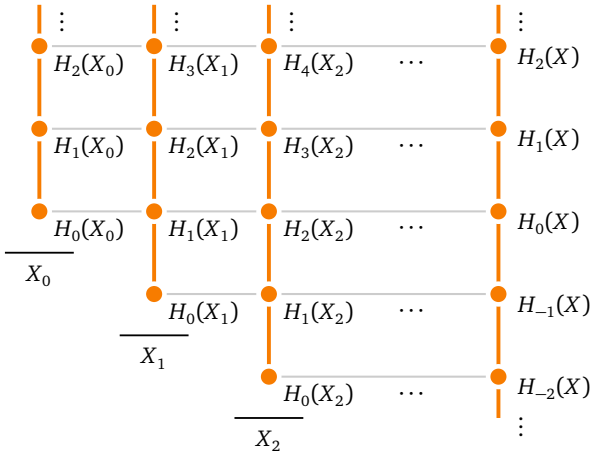
  where the $\wedge$ is the handicrafted smash product of Definition 2.3.23. We assume that the levels of X and HG are CW complexes, so that $X_p \wedge (HG)_q$ has the correct homotopy type.

More generally, for any extraordinary homology theory E we define the E-homology of X as either

$$E_k(X) \cong \operatorname*{colim}_{n\to\infty} E_{k+n}(X_n, *), \quad \text{or} \quad E_k(X) = \pi_k(X \wedge E).$$

See also Definition 3.2.18 and Example 3.2.22 for the definition of cohomology of spectra.

The first definition of homology is illustrated to the right. As for homotopy groups, what we really care about is the limiting behavior, so we focus on the single line of homology groups at the far right. The rest of the figure is just a presentation of those groups.



Proposition 2.6.38. *The above two definitions of $H_k(X;G)$, and more generally of $E_k(X)$, give isomorphic homology groups. Furthermore, any stable equivalence $X \to Y$ induces an isomorphism on homology $E_k(X) \to E_k(Y)$ for all $k \in \mathbb{Z}$.*

Proof. The homotopy groups of the handicrafted smash product are given by the colimit of the following grid.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle +} & & \uparrow{\scriptstyle +} & & \uparrow{\scriptstyle +} & & \\
\pi_{k+2}(X_0 \wedge E_2) & \xrightarrow{+} & \pi_{k+3}(X_1 \wedge E_2) & \xrightarrow{+} & \pi_{k+4}(X_2 \wedge E_2) & \xrightarrow{+} & \dots \\
\uparrow{\scriptstyle +} & & \uparrow{\scriptstyle +} & & \uparrow{\scriptstyle +} & & \\
\pi_{k+1}(X_0 \wedge E_1) & \xrightarrow{-} & \pi_{k+2}(X_1 \wedge E_1) & \xrightarrow{-} & \pi_{k+3}(X_2 \wedge E_1) & \xrightarrow{-} & \dots \\
\uparrow{\scriptstyle +} & & \uparrow{\scriptstyle +} & & \uparrow{\scriptstyle +} & & \\
\pi_{k}(X_0 \wedge E_0) & \xrightarrow{+} & \pi_{k+1}(X_1 \wedge E_0) & \xrightarrow{+} & \pi_{k+2}(X_2 \wedge E_0) & \xrightarrow{+} & \dots
\end{array}$$

The colimit along the ith column is $\pi_{k+i}(X_i \wedge E) \cong E_{k+i}(X_i, *)$. Passing to the $(i+1)$st column induces a map of the form

$$E_{k+i}(X_i, *) \cong E_{k+i+1}(\Sigma X_i, *) \to E_{k+i+1}(X_{i+1}, *),$$

where the first map is a natural isomorphism $\pi_{k+i}(X_i \wedge E) \cong \pi_{k+i+1}(\Sigma X_i \wedge E)$. This isomorphism must agree up to sign with the suspension isomorphism for the homology theory E_*, by Exercise 2.7.20. (In order to argue this, we have to observe that both constructions can be done on any spectrum Y, giving an isomorphism $\pi_k(Y) \cong \pi_{k+1}(\Sigma Y)$, and then specialized back to $Y = X_i \wedge E$.)

Since signs don't change the colimit, we conclude that the colimit along the top row is isomorphic to the first definition of homology from Definition 2.6.37. But this agrees with the colimit of the entire grid of abelian groups, which is the second definition of homology from Definition 2.6.37. Therefore the two definitions are isomorphic.

To prove that homology respects stable equivalences, we take colimits along the rows instead. The colimit along the jth row is isomorphic to $\pi_{k+j}(E_j \wedge X)$. Any stable equivalence $X \to Y$ induces isomorphisms on these groups by Corollary 2.4.25, and therefore an isomorphism on the colimit of the entire grid. □

Example 2.6.39. The homology of a suspension spectrum is just the homology of the underlying space:

$$H_n(\Sigma^\infty_+ A) \cong H_n(A).$$

More generally, if $\zeta \to B$ is an oriented virtual bundle of dimension d then we get a "virtual Thom isomorphism"

$$H_n(\mathrm{Th}(\zeta)) \cong H_{n-d}(B).$$

See Exercise 2.7.7.

Warning 2.6.40. The homology of a spectrum X is *not* the same as the homology of its infinite loopspace $\Omega^\infty X$, even though they have the same homotopy:

$$\pi_n(\Omega^\infty X) \cong \pi_n(X), \qquad H_n(\Omega^\infty X) \ncong H_n(X) \text{ or } H_n(X, *).$$

For instance, if X is a shifted rational Eilenberg–Mac Lane spectrum

$$X = \mathrm{sh}^{2n} H\mathbb{Q} \simeq \Sigma^{2n} H\mathbb{Q},$$

its homology is just a $\mathbb{Q}$ in degree 2n, but the homology of its infinite loop space $K(\mathbb{Q}, 2n)$ has a $\mathbb{Q}$ in every degree that is a multiple of $2n$:

	(0)	$\cdots$	$(2n-1)$	$(2n)$	$(2n+1)$	$\cdots$	$(4n-1)$	$(4n)$	$(4n+1)$	$\cdots$	$(6n)$	$\cdots$
$H_*(X) =$	0	$\cdots$	0	$\mathbb{Q}$	0	$\cdots$	0	0	0	$\cdots$	0	$\cdots$
$H_*(\Omega^\infty X) =$	$\mathbb{Z}$	$\cdots$	0	$\mathbb{Q}$	0	$\cdots$	0	$\mathbb{Q}$	0	$\cdots$	$\mathbb{Q}$	$\cdots$

On the other hand, if $X = \mathrm{sh}^{2n+1} H\mathbb{Q}$, the infinite loop space $K(\mathbb{Q}, 2n+1)$ would have the same reduced homology as X, just a $\mathbb{Q}$ in degree $2n+1$:

	(0)	$\cdots$	$(2n)$	$(2n+1)$	$(2n+2)$	$\cdots$
$H_*(X) =$	0	$\cdots$	0	$\mathbb{Q}$	0	$\cdots$
$H_*(\Omega^\infty X) =$	$\mathbb{Z}$	$\cdots$	0	$\mathbb{Q}$	0	$\cdots$

In general, $H_*(X;\mathbb{Q})$ is the *indecomposables* of $H_*(\Omega^\infty X;\mathbb{Q})$ as an algebra, or the *primitives* of $H_*(\Omega^\infty X;\mathbb{Q})$ as a coalgebra. See **??** for more details.

Remark 2.6.41. If X is a CW spectrum, there is a third way to define the homology of X. Define the cellular chain complex $C_*(X)$ by taking $C_k(X)$ to be the free abelian group on the stable k-cells of X, for all $k \in \mathbb{Z}$. This is a colimit of the cellular chain complexes of the levels X_n:

$$C_k(X) = \operatorname*{colim}_{n\to\infty} C_{k+n}(X_n, *).$$

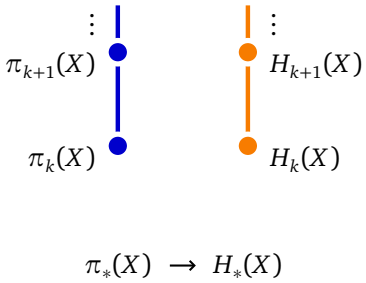
The cellular boundary maps of the spaces X_n give well-defined boundary maps on $C_*(X)$, and we define $H_*(X;G)$ to be the homology of $C_*(X) \otimes G$. Since this is defined using the colimit of the chain complexes $C_*(X_n)$, on homology we get the colimit of the homology of the levels, so this is isomorphic to the first definition in Definition 2.6.37.

Definition 2.6.42. For any spectrum X and $k \in \mathbb{Z}$, we define the Hurewicz map $\pi_k(X) \to H_k(X)$ by applying the space-level Hurewicz maps of Definition 1.4.19 to the colimit system in Definition 2.6.37. By Exercise 2.7.40, we can also obtain this by taking any map of spectra $\mathbb{S} \to H\mathbb{Z}$ that induces the identity map $\mathbb{Z} \to \mathbb{Z}$ on π_0, and smashing with X to get

$$\pi_k(X \wedge \mathbb{S}) \longrightarrow \pi_k(X \wedge H\mathbb{Z}).$$

Theorem 2.6.43 (Hurewicz theorem for spectra)**.** *If the spectrum X is $(k-1)$-connected then $\pi_k(X) \to H_k(X)$ is an isomorphism.*

Proof. By Proposition 2.6.29, X admits a stable equivalence from a spectrum Y that has stable cells only in degree k and above. By Proposition 2.6.38, the map $Y \to X$ induces an isomorphism on homology, so we just have to show that $\pi_k(Y) \to H_k(Y)$ is an isomorphism. Since Y has only stable cells of degree k and higher, Y_n only has cells (other than the basepoint) of dimension $(k+n)$ and higher. By the Hurewicz theorem for spaces, Proposition 1.4.21, $\pi_{k+n}(Y_n) \to H_{k+n}(Y_n, *)$ is an isomorphism. Taking the colimit of these isomorphisms gives the desired isomorphism. □



Corollary 2.6.44. *If X and Y are bounded below then the map $f : X \to Y$ is a stable equivalence iff it induces an isomorphism on homology with $\mathbb{Z}$ coefficients.*

So we can use ordinary homology to tell whether a map is a stable equivalence. This is great because homology is often easier to compute than stable homotopy.

2.7 Exercises

2.7.1. Modify the definition of a spectrum by allowing levels X_n for all $n \in \mathbb{Z}$. Explain why the theory of such spectra is equivalent to the theory of spectra presented in this chapter.

2.7.2. Similarly to the last problem, prove that extraordinary homology theories E_n defined with $n \in \mathbb{Z}$ can be recovered up to isomorphism from their restrictions to $n \geq 0$.

2.7.3. In this chapter we defined extraordinary homology theories as functors on CW pairs. One can similarly define reduced homology theories as functors on based CW complexes, or unreduced theories on unbased CW complexes. Write out the variant of the Eilenberg-Steenrod axioms in each of these cases. Show how to go back and forth between these versions without changing the homology theory, up to isomorphism.

2.7.4. (*) Suppose h_* is an extraordinary homology theory on based CW complexes, defined as in Exercise 2.7.3. In this exercise we show that h_* also satisfies the reduced version of the direct limit axiom from Remark 2.5.6.

(a) Define a new homology theory c_* on each based CW complex X by the formula

$$c_n(X) = \operatorname*{colim}_{K \subseteq X \text{ finite}} h_n(K).$$

Here the colimit runs over all subspaces of X that contain the basepoint and that are homeomorphic to a finite CW complex. This may seem huge, but it is not: if we take any CW complex structure on X and restrict to those K that are finite subcomplexes of the chosen structure, we get the same colimit.

Prove that c_* is a functor on the category of based CW complexes and continuous based maps, and that it satisfies the axioms for an extraordinary homology theory.

(b) There is a canonical natural transformation $c_* \to h_*$. Observe that this is an isomorphism on every finite based CW complex. Prove that it is also an isomorphism on any n-dimensional based CW complex by induction on n. You might find the usual cofiber sequence helpful:

$$X^{(n-1)} \longrightarrow X^{(n)} \longrightarrow \bigvee S^n.$$

(c) Given an arbitrary based CW complex X, write it as a mapping telescope of n-dimensional CW complexes $X^{(n)}$, and use the cofiber sequence

$$\bigvee_{n\geq 0} X^{(n)} \longrightarrow \operatorname*{hocolim}_{n\to\infty} X^{(n)} \longrightarrow \bigvee_{n\geq 0} \Sigma X^{(n)}$$

to show that $c_* \to h_*$ is an isomorphism on X as well. Conclude that h_* satisfies the direct limit axiom.

2.7.5. (a) Prove that adding a trivial line bundle to a vector bundle E has the effect of suspending its Thom spaces, as in (2.1.17).

(b) More generally, given two vector bundles $E_1 \to B_1$ and $E_2 \to B_2$, form the external product bundle by taking their Cartesian product

$$E_1 \times E_2 \to B_1 \times B_2.$$

Prove that this goes to the smash of the Thom spaces,

$$\mathrm{Th}(E_1 \times E_2) \cong \mathrm{Th}(E_1) \wedge \mathrm{Th}(E_2).$$

2.7.6. A virtual bundle ζ over a finite-dimensional CW complex B can be presented in several equivalent ways. If

$$\zeta = E_1 - \epsilon^m = E_2 - \epsilon^n$$

are two presentations of the same virtual bundle, then for some k we have an isomorphism of bundles

$$E_1 \oplus \epsilon^{n+k} \cong E_2 \oplus \epsilon^{m+k}.$$

(Here $\epsilon^n = B \times \mathbb{R}^n$ is a trivial bundle.) Verify that for any two such presentations, the resulting constructions of the Thom spectrum $\mathrm{Th}(\zeta)$ are stably equivalent. Therefore $\mathrm{Th}(\zeta)$ is a well-defined spectrum up to stable equivalence.

2.7.7. The Thom isomorphism says that if $E \to B$ is an orientable vector bundle of dimension d, then

$$H_{n+d}(\mathrm{Th}(E), *) \cong H_n(B).$$

See e.g. [Coh98, Hat03]. Use this to prove a similar result for the Thom *spectrum* of a virtual bundle $\zeta = E - \epsilon^q$, as defined in Example 2.1.18, if E is orientable. (You might let p be the dimension of E, so that $d = p - q$. See Example 2.6.39.)

2.7.8. Suppose B is a CW complex which may not be finite-dimensional. Then a virtual bundle on B is defined a little differently, as a choice of virtual bundle $\zeta^{(n)} = E^{(n)} - \epsilon^{q_n}$ on each skeleton $B^{(n)}$, such that $\zeta^{(n)}|_{B^{(n-1)}}$ is isomorphic to $\zeta^{(n-1)} = E^{(n-1)} - \epsilon^{q_{n-1}}$. Note that the dimension p_n of the vector bundle $E^{(n)}$ may go to infinity as $n \to \infty$.

(All together this is classified by a single map $B \to \mathbb{Z} \times BO$. The $\mathbb{Z}$ summand gives the dimension of the bundle (which may be negative), and the map to BO factors through some $BO(p_n)$ on each skeleton $B^{(n)}$, giving the vector bundle $E^{(n)}$.)

Combine the constructions of Example 2.1.18 and Example 2.1.20 to define the Thom spectrum of such a virtual bundle ζ over an arbitrary cell complex B.

2.7.9. (a) Prove that a product of Ω-spectra $\prod_\alpha X_\alpha$ is again an Ω-spectrum. More generally, show that any limit of Ω-spectra is an Ω-spectrum.

(b) Prove that if E is an Ω-spectrum and A is a CW complex then $F(A, E)$ is also an Ω-spectrum. Conclude that the homotopy fiber of any map of Ω-spectra is an Ω-spectrum.

2.7.10. Prove that $X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z)$. Here X could be a based space or a spectrum, and Y and Z could likewise both be based spaces, or both be spectra.

2.7.11. Consider a system of abelian groups

$$\cdots \longrightarrow A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \longrightarrow \cdots$$

Define a new system by taking $B_1 = A_{i_1}$ for some i_1, $B_2 = A_{i_2}$ for some $i_2 > i_1$, and so on. Define the maps $B_k \to B_{k+1}$ by composing the maps f_i. This is usually called a **cofinal subsystem**. Prove that

$$\operatorname*{colim}_i A_i \cong \operatorname*{colim}_k B_k.$$

Explain how this is being implicitly used in the proof of Proposition 2.4.1.

2.7.12. Consider the functor ev_0 that sends a spectrum X to its 0th space X_0, in the category of based spaces. Prove that the suspension spectrum functor Σ^∞, from based spaces to spectra, is the left adjoint. How does this change if we think of ev_0 as a functor to unbased spaces instead?

2.7.13. (a) Prove that the coproduct $X \vee Y$ satisfies the universal property of the coproduct in the category of spectra. More generally, prove that the colimit of spectra satisfies the universal property of the colimit.

(b) Prove that the two bonding maps we defined for the product spectrum $X \times Y$ actually agree.

(c) Prove that the product $X \times Y$ satisfies the universal property of the product in the category of spectra. More generally, prove that the limit of spectra satisfies the universal property of the limit.

2.7.14. Prove that the relative stable homotopy groups $\pi_k(X,A)$ from Definition 2.6.22 fit into a natural long exact sequence

$$\ldots \longrightarrow \pi_k(A) \longrightarrow \pi_k(X) \longrightarrow \pi_k(X,A) \longrightarrow \pi_{k-1}(A) \longrightarrow \ldots$$

and more generally for two maps $A \to B \to X$ a long exact sequence

$$\ldots \longrightarrow \pi_k(B,A) \longrightarrow \pi_k(X,A) \longrightarrow \pi_k(X,B) \longrightarrow \pi_{k-1}(B,A) \longrightarrow \ldots$$

Define a natural map $\pi_k(X,A) \to \pi_k(Cf)$ and prove it is an isomorphism. (You may find Lemma 2.4.11 and its proof helpful.)

2.7.15. Recall the tensor $K \wedge -$ from Definition 2.3.6, the cotensor $F(K,-)$ from Definition 2.3.8, and the mapping space $\mathrm{Map}_*(-,-)$ from Definition 2.3.12.

(a) Prove that for any based space K, the functors $K \wedge -$ and $F(K,-)$ on spectra form an adjoint pair. (The idea is to deduce this from the same statement for based spaces.)

(b) Prove that for any spectrum X, the functors $- \wedge X$ and $\mathrm{Map}_*(X,-)$ form an adjoint pair between spaces and spectra. In summary, for spaces K and spectra X and Y, we get bijections

$$K \to \mathrm{Map}_*(X,Y) \quad \leftrightarrow \quad K \wedge X \to Y \quad \leftrightarrow \quad X \to F(K,Y).$$

(c) Use the Yoneda Lemma (Lemma 1.4.29) to upgrade these to homeomorphisms of topological spaces

$$\mathrm{Map}_*(K, \mathrm{Map}_*(X,Y)) \cong \mathrm{Map}_*(K \wedge X, Y) \cong \mathrm{Map}_*(X, F(K,Y)).$$

2.7.16. Prove that the un-shift functor $\mathrm{sh}^{-1}(-)$ and the shift functor $\mathrm{sh}(-)$ form an adjoint pair. It is easy to see that, like Σ and Ω, they are inverses up to stable equivalence.

2.7.17. (a) Show that the free spectrum functor $F_n \colon \mathbf{Top}_* \to \mathbf{Sp}$ is the left adjoint of the forgetful map $\mathrm{ev}_n \colon \mathbf{Sp} \to \mathbf{Top}_*$ that sends each spectrum X to its nth level X_n. As a special case, Σ^∞ is the left adjoint of ev_0.

(b) Show the same for the functor $F_n(-)_+ : \mathbf{Top} \to \mathbf{Sp}$ and the forgetful map to *unbased* spaces $\mathbf{Sp} \to \mathbf{Top}$.

2.7.18. Suppose X and Y are spectra. A **weak map** $f : X \rightsquigarrow Y$ consists of based maps $f_n : X_n \to Y_n$ and *homotopies* $\Sigma X_n \wedge I_+ \to Y_{n+1}$, from $f_{n+1} \circ \xi_n$ to $\upsilon_n \circ \Sigma f_n$, for each $n \geq 0$.

Show that a weak map gives a zig-zag $X \leftarrow X' \to Y$ of actual maps of spectra, where $X' \to X$ is a level equivalence and the maps $X'_n \to Y_n$ are in the homotopy class of f_n. (Hint: Take X'_n to be the mapping telescope of $\Sigma^n X_0 \to \Sigma^{n-1} X_1 \to \ldots \to X_n$.)

2.7.19. (*) Let X be any spectrum. By Proposition 2.4.1, the spectra ΣX and $\mathrm{sh} X$ (see Definition 2.3.6 and Definition 2.3.4) have isomorphic homotopy groups.

(a) Find a zig-zag of stable equivalences between ΣX and $\mathrm{sh} X$. You may find Exercise 2.7.18 helpful.

(b) Use this to conclude that ΩX and $\mathrm{sh}^{-1} X$ are also stably equivalent. In particular, $F_k S^0$ is stably equivalent to $\Omega^k \mathbb{S}$.

(c) Take the homotopy colimit of the stable equivalences $X \to \Omega\, \mathrm{sh} X$, to get a second construction of RX from Proposition 2.2.9. It is not homeomorphic to the first construction, only equivalent, but along the bottom of the last grid in the proof, we now get the groups $\pi_{k+m}(\Omega^m X)$ instead of A_m.

2.7.20. Show that for any fixed value of $k \in \mathbb{Z}$, the isomorphism of $\pi_{k+1}(\Sigma X) \cong \pi_k(X)$ from Proposition 2.4.1 is unique up to a sign. You might find it helpful to note that each element of $\alpha \in \pi_k(X)$ gives a map $\mathbb{Z} \cong \pi_k(\mathbb{S}^k) \to \pi_k(X)$, and so you can compare the isomorphism for X to the same isomorphism for $\mathbb{S}^k$.

Once we have set up the stable homotopy category in the next chapter, this exercise can be done more easily, as an application of the Yoneda Lemma.

2.7.21. Prove Lemma 2.4.16, that a square of spectra is a homotopy pushout iff the induced map of homotopy cofibers is a stable equivalence, and homotopy pullback iff the induced map of homotopy fibers is an equivalence.

2.7.22. Prove that for any homotopy pushout/pullback square of spectra as in Definition 2.4.15, there is a homotopy Mayer-Vietoris sequence as in Exercise 1.7.22:

$$\ldots \longrightarrow \pi_k(A) \xrightarrow{(f_*, h_*)} \pi_k(B) \oplus \pi_k(C) \xrightarrow{k_* - g_*} \pi_k(D) \xrightarrow{\partial} \pi_{k-1}(A) \longrightarrow \ldots$$

2.7.23. Consider the following alternate definition of cofiber sequence of spectra. A cofiber sequence of spectra $X \to Y \to Z$ is a homotopy pushout square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ C & \longrightarrow & Z \end{array}$$

where C is weakly contractible, i.e. $C \to *$ is a stable equivalence. Prove that every cofiber sequence in our sense is a cofiber sequence in this sense. Conversely, prove that every cofiber sequence in this sense can be changed up to stable equivalence to be a cofiber sequence in our sense. (You may find it helpful to make the spectra into CW Ω-spectra and to use the Whitehead theorem.)

2.7.24. Dualize the proof of Proposition 2.4.21 to get the following: when the composite $A \xrightarrow{i} X \xrightarrow{p} A'$ is a stable equivalence, the canonical map $X \to Ci \times A'$ is also a stable equivalence. Conclude that the composite $Fp \to X \to Ci$ is a stable equivalence. So when we have a retract in spectra, the complementary piece is both the cofiber of the inclusion, and the fiber of the projection.

2.7.25. Define the free loop spectrum LX of a spectrum X by the cotensor

$$LX = F(S^1_+, X).$$

(a) Verify that on each spectrum level this takes the space of unbased loops.

(b) Show that LX is stably equivalent to the product $X \times \Omega X$.

(c) Show that $L(\Sigma^\infty_+ A)$ is not equivalent to $\Sigma^\infty_+ LA$ for any nonempty unbased space A.

2.7.26. (a) Fill in the missing step of Proposition 2.4.20 by proving that homotopy groups preserve finite products:

$$\pi_k\left(\prod_{i=1}^n X(i)\right) \cong \prod_{i=1}^n \pi_k(X(i)),$$

for any spectra $X(1)$, ..., $X(n)$ and $k \in \mathbb{Z}$.

(b) Prove that stable homotopy groups preserve infinite products of Ω-spectra:

$$\pi_k\left(\prod_\alpha X(\alpha)\right) \cong \prod_\alpha \pi_k(X(\alpha)),$$

for any collection of Ω-spectra $\{X(\alpha)\}_{\alpha \in A}$ and $k \in \mathbb{Z}$.

(c) Show that stable homotopy groups do not preserve infinite products in general. (This is what we mean by infinite products having the wrong homotopy type.) (You might consider taking a product of infinitely many copies of a colimit system that does not stabilize, such as the one in Exercise 2.7.41.)

2.7.27. Dualize the proof of Corollary 2.4.25 and prove that if K is a *finite* CW complex, then the cotensor $F(K, -)$ preserves all stable equivalences. What goes wrong if K is infinite? (Consider Exercise 2.7.26.)

2.7.28. Use Exercise 1.7.31 to prove that stable homotopy groups commute with sequential homotopy colimits of spectra:

$$\pi_k\left(\operatorname*{colim}_{n\to\infty} X(n)\right) \cong \operatorname*{colim}_{n\to\infty} \pi_k(X(n)).$$

They also commute with the strict colimit, provided the maps $X(n) \to X(n+1)$ are closed inclusions at each spectrum level.

2.7.29. Recall from Exercise 1.7.31 that if Y is compact and

$$X^{(-1)} \longrightarrow X^{(0)} \longrightarrow X^{(1)} \longrightarrow X^{(2)} \longrightarrow \ldots \longrightarrow X$$

is a sequence of closed inclusions with colimit X, then any map $Y \to X$ factors through some $X^{(j)}$.

(a) Prove that the composite of two relative cell complexes is a relative cell complex. (Either in spaces or in spectra; the argument is the same.)

(b) Prove that the composite of countably many relative cell complexes is a relative cell complex. (This is also true for larger transfinite compositions, we would just need more set theory to define and work with them.)

2.7.30. (a) Suppose S is a compact (CGWH) space, and $\bigvee_\alpha K(\alpha)$ is a wedge sum of based (CGWH) spaces. Prove that any map

$$S \longrightarrow \bigvee_{\alpha\in A} K(\alpha)$$

must factor through a finite wedge, i.e. $\bigvee_{\alpha\in F} K(\alpha)$ for some finite subset $F \subseteq A$. (This is similar to Exercise 1.7.31.)

(b) Prove that stable homotopy groups preserve infinite coproducts:

$$\pi_k\left(\bigvee_\alpha X(\alpha)\right) \cong \bigoplus_\alpha \pi_k(X(\alpha)),$$

for any collection of spectra $\{X(\alpha)\}_{\alpha\in A}$ and $k \in \mathbb{Z}$. This finishes the proof of Proposition 2.5.7, that spectra define extraordinary homology theories.

2.7.31. (a) Prove that for any spectrum E, the cotensor $F(-,E)$ sends coproducts of based spaces $\bigvee_\alpha X_\alpha$ to products of spectra. More generally, it sends colimits to limits.

(b) Prove also that the cotensor $F(-,E)$ sends cofiber sequences of CW complexes $A \xrightarrow{f} X \to Cf$ to fiber sequences of spectra.

(c) Switching to the other slot, prove that the cotensor $F(A,-)$ sends fiber sequences of spectra $Ff \to E \to B$ to fiber sequences of spectra.

We have to be a little careful in this exercise – if we have a cofiber sequence of spaces $A \to X \to C$, where C is only *equivalent* to the cofiber, not isomorphic, then $F(-,E)$ may not respect that equivalence, so applying $F(-,E)$ may not give a fiber sequence. This can be corrected by taking the right-derived functor of $F(A,E)$, in other words, by making A a CW complex and making E an Ω-spectrum before feeding them into F. See Definition 3.4.15.

2.7.32. Let E be any Ω-spectrum. Prove that $\pi_{-*}(F(X/A,E))$ satisfies the Eilenberg-Steenrod axioms for a cohomology theory (Proposition 2.5.17). Explain how the proof fails if E is not an Ω-spectrum. (You may find it helpful to use Exercises 2.7.9, 2.7.26, and 2.7.31.)

2.7.33. Prove that the homology theory defined in Proposition 2.5.7 extends to all pairs (X,A) in which $A \to X$ is a cofibration, and preserves all weak equivalences of such pairs.

On the other hand, the cohomology theory in Proposition 2.5.7 does not extend this way, because weak equivalences $X \to X'$ do not always induce weak equivalences on $\mathrm{Map}(-,Y)$. Instead, we extend it by replacing an arbitrary pair (X,A) by an equivalent CW pair (QX,QA) and then feeding it into the earlier definition. In other words, we have to right-derive cohomology to extend it to all spaces (see Definition 3.4.15).

2.7.34. Suppose X and Y are based CW complexes.

(a) Prove that $\Sigma^\infty X_+$ is stably equivalent to the wedge sum $\mathbb{S} \vee \Sigma^\infty X$.

(b) Prove that $\Sigma^\infty(X \times Y)$ is stably equivalent to the wedge sum

$$\Sigma^\infty X \vee \Sigma^\infty Y \vee \Sigma^\infty(X \wedge Y).$$

(c) How does this generalize to $\Sigma^\infty(X_1 \times \cdots \times X_n)$?

2.7.35. (a) Prove that the suspension spectrum of a cofiber sequence of based spaces is a cofiber sequence of spectra.

(b) Show by counterexample that the suspension spectrum of a fiber sequence of spaces may not be a fiber sequence of spectra. (Hint: Use Exercise 2.7.34.)

(c) Check that the right adjoint ev_0 preserves fiber sequences of Ω-spectra. Does it also preserve cofiber sequences?

2.7.36. Check that $H_*(\Sigma^\infty A) \cong H_*(A,*)$ when A is well-based.

2.7.37. Explain why a cofiber/fiber sequence of spectra $X \to Y \to Z$ gives a long exact sequence on homology:

$$\cdots \longrightarrow H_n(X) \longrightarrow H_n(Y) \longrightarrow H_n(Z) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

Does this work for extraordinary homology as well?

2.7.38. A spectrum X is n-connected if $\pi_k(X) = 0$ for $k \leq n$; it is n-connective if $\pi_k(X) = 0$ for $k < n$. Suppose we have a cofiber/fiber sequence $X \to Y \to Z$.

(a) If X and Z are n-connected, prove that Y is n-connected.

(b) If X and Y are n-connected, what does this imply about Z? What if Y and Z are n-connected?

2.7.39. (Uniqueness of Eilenberg–Mac Lane spectra) Fix an abelian group G. Suppose Y is any spectrum such that $\pi_0(Y) \cong G$ and all other homotopy groups are zero. Let X be a particular spectrum with this property, which we build from the zero spectrum $*$ by attaching a stable 0-cell for each generator of G and a stable 1-cell for each relation, then attaching cells to kill all of the homotopy groups above π_0, as in Definition 2.6.30.

Prove that there is a stable equivalence $X \xrightarrow{\sim} Y$, and conclude that Eilenberg–Mac Lane spectra for G are unique up to stable equivalence. (Hint: You will have to define this equivalence one cell at a time. This is the same method that we use to prove that the spaces $K(G,n)$ are unique in Lemma 1.4.25.)

2.7.40. (a) Define a map of spectra $i\colon \mathbb{S} \to H\mathbb{Z}$ that on π_0 is the identity map $\mathbb{Z} \to \mathbb{Z}$.

(b) Use i to define canonical generators $g_k \in H_k(S^k)$, if $H_k(A)$ is defined to be $\pi_k(A \wedge H\mathbb{Z})$. We therefore get a Hurewicz homomorphism $\pi_k(A) \to H_k(A)$ for each based space A as in Definition 1.4.19, sending each $\phi\colon S^k \to A$ to the homology class $\phi_*(g_k)$.

(c) Prove that the Hurewicz homomorphism defined by these generators g_k agrees with the composite along the bottom of the following diagram.

$$\begin{array}{ccccccc}
\pi_k(S^k) & \longrightarrow & \pi_k(S^k \wedge \mathbb{S}) & \xrightarrow{\mathrm{id}\wedge i} & \pi_k(S \wedge H\mathbb{Z}) & = & H_k(S^k) \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi\wedge\mathrm{id}} & & \downarrow{\scriptstyle \phi\wedge\mathrm{id}} & & \downarrow{\scriptstyle \phi_*} \\
\pi_k(A) & \longrightarrow & \pi_k(A \wedge \mathbb{S}) & \xrightarrow{\mathrm{id}\wedge i} & \pi_k(A \wedge H\mathbb{Z}) & = & H_k(A).
\end{array}$$

(d) Show that for any spectrum X, the Hurewicz homomorphism applied to every spectrum level of X gives a map $\pi_k(X) \to H_k(X)$ which agrees up to sign (a factor of ± 1) with the map

$$(\mathrm{id} \wedge i)\colon \pi_k(X \wedge \mathbb{S}) \to \pi_k(X \wedge H\mathbb{Z}).$$

You might find the proof of Proposition 2.6.38 helpful.

This shows that the two definitions of the Hurewicz map in Definition 2.6.42 agree.

2.7.41. (a) Let A be any abelian group and let $A \xrightarrow{n} A$ be the map that multiplies by $n \in \mathbb{Z}$. Prove that the colimit of the maps

$$A \xrightarrow{1} A \xrightarrow{2} A \xrightarrow{3} \cdots$$

is the rationalization $A \otimes \mathbb{Q}$. (This gets easy if you know that $- \otimes \mathbb{Q}$ commutes with colimits.)

(b) Recall that the rational homotopy groups of spheres $\pi_k(S^n) \otimes \mathbb{Q}$ are

$$\pi_n(S^n) \otimes \mathbb{Q} = \mathbb{Q}, \qquad \pi_{4n-1}(S^{2n}) \otimes \mathbb{Q} = \mathbb{Q},$$

and zero otherwise. Use this to prove that the Moore space $M(\mathbb{Q}, n)$ from Example 2.5.31 is an Eilenberg–Mac Lane space $K(\mathbb{Q}, n)$ when n is odd.

(c) Using Exercise 2.7.39, prove that the rational sphere $\mathbb{S}_{\mathbb{Q}} := F_n M(\mathbb{Q}, n)$ is equivalent to the Eilenberg–Mac Lane spectrum $H\mathbb{Q}$.

(d) For any prime p, explain why the same proof will not show that $\mathbb{S}_{(p)}$ is equivalent to $H\mathbb{Z}_{(p)}$. (In fact, these are not equivalent to each other.)

2.7.42. (a) Prove that the mod p version of ordinary homology, in the sense of Example 2.5.34, agrees with homology with $\mathbb{Z}/p$ coefficients. In other words, show that $H\mathbb{Z} \wedge \mathbb{S}/p \simeq H(\mathbb{Z}/p)$.

(b) Similarly, prove that the rationalization of $H\mathbb{Z}$ agrees with $H\mathbb{Q}$.

(c) Show that the map of spectra $\mathbb{S} \to H\mathbb{Z}$ of Exercise 2.7.40 is an equivalence after rationalization, but not before.

2.7.43. Prove that the following conditions are equivalent. We say the spectrum X is **finite** if these conditions hold.

(1) X is stably equivalent to a cellular spectrum with finitely many stable cells.

(2) X is stably equivalent to a free spectrum $F_k A$ on a finite cell complex A.

(3) X is bounded below (Definition 2.6.27) and the direct sum of the homology groups $\bigoplus_k H_k(X; \mathbb{Z})$ is finitely generated.

(For (3) $\Rightarrow$ (1), you'll have to use the long exact sequence on homology, and the fact that any subgroup of $\mathbb{Z}^{\oplus n}$ is isomorphic to $\mathbb{Z}^{\oplus k}$ for some k.)

2.7.44. If you know about simplicial sets, try defining spectra in simplicial sets, using $\Delta[1]/\partial\Delta[1]$ as the circle. Define a geometric realization functor to spectra in topological spaces, and a singular complex functor coming back. Prove these are inverses up to stable equivalence. (If you don't know about simplicial sets, we'll talk about them in more detail in Chapter 7).

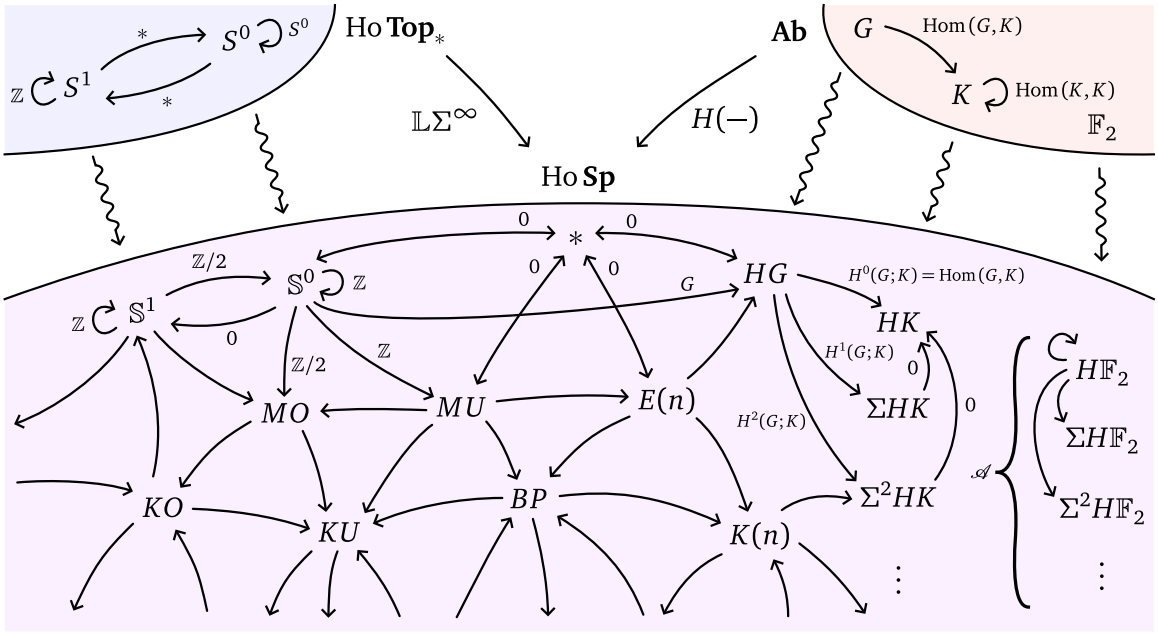
2.7.45. Use the Hurewicz theorem to verify the calculations of π_0 and π_1 of suspension spectra from Example 2.1.7:

$$\pi_0(\Sigma^\infty A) \cong H_0(A, *; \mathbb{Z}),$$
$$\pi_1(\Sigma^\infty A) \cong H_0(A, *; \mathbb{Z}/2) \oplus H_1(A, *; \mathbb{Z}).$$

(Hint: Prove this first when A is connected. Then try using a technique like the one you used in part (a) of Exercise 2.7.34 to extend to the non-connected case.)

Chapter 3

The stable homotopy category



The most fundamental problem in homotopy theory is to distinguish spaces up to homotopy equivalence, and to count the maps between them up to homotopy. This can be neatly organized by saying that there is a homotopy category $h\mathbf{CW}$ of CW complexes and homotopy classes of maps (defined in Section 1.2.1), and our most basic goal in life is to better understand this category.

But now we're doing stable homotopy theory, which is this new form of homological algebra based on spectra. So now the most basic goal is to classify spectra up to stable equivalence, and to count the maps between them, up to the correct equivalence relation. What should the correct equivalence relation be?

It turns out that homotopies of maps of spectra, in the most naïve sense, is not the right notion. Rather, we should invert the stable equivalences in the category **Sp** in some universal way, forming a new category called the **stable homotopy category** Ho **Sp**. Then we should count the maps in this new category. For instance, the maps in Ho **Sp** from $\mathbb{S}$ to $\mathbb{S}$ are given by the integers $\mathbb{Z}$. More generally, the maps from $\mathbb{S}^n = \Sigma^n\mathbb{S}$ to $\mathbb{S}$ are given by $\pi_n(\mathbb{S})$, the stable homotopy groups of spheres. We know a lot about these groups, but it is still an open problem to compute them for every $n \geq 0$.

This process of going from **Sp** to Ho **Sp** is called **localization**. Localization is when we take take any class of morphisms W in a category **C** and invert them, making them into isomorphisms. The result is a new category $\mathbf{C}\langle W^{-1}\rangle$ with the same objects as **C**, but different morphisms. There's more than just the inverses of the maps in W; for instance, we can also compose those inverses with the existing maps in **C**.

If we do this same procedure to the category **Top** of all topological spaces, we can make a category Ho **Top** that universally inverts the weak homotopy equivalences. This turns out to be equivalent to $h\mathbf{CW}$, the category of CW complexes and homotopy classes of maps. So we could rephrase the most fundamental question of homotopy theory as: understand the category Ho **Top**. (Or we could do the same thing with based spaces and based CW complexes, giving a category Ho $\mathbf{Top}_*$.)

In summary, the most fundamental problem in stable homotopy theory is to count the isomorphism classes of objects, and the maps between them, in the stable homotopy category Ho **Sp**. In this chapter we give the most basic tools for starting this problem. We show how to define Ho **Sp**, which in and of itself is not so easy, and how to begin counting the maps between objects. We also define the "E-local" version Ho $\mathbf{Sp}_E$ for any homology theory E, which is often simpler and easier to compute.

We limit ourselves here to the basic computation techniques that use long exact sequences. The more powerful techniques for counting the maps Ho **Sp** and Ho $\mathbf{Sp}_E$ are based on spectral sequences, which we discuss in **??**.

We also discuss how to define functors and operations on the stable homotopy category Ho **Sp**. The easiest version of this is that any functor on **Sp** that preserves stable equivalences defines a functor on Ho **Sp** automatically. On the other hand, if F is a functor on **Sp** that only preserves stable equivalences on some subcategory of "nice" spectra, then it often has a **left-derived functor** $\mathbb{L}F$ or **right-derived functor** $\mathbb{R}F$. The derived functor is unique up to equivalence, and it agrees with F on the "nice" spectra, but it has the added feature that it preserves stable equivalences everywhere.

For example, we have to left-derive the suspension spectrum functor $\Sigma^\infty : \mathbf{Top}_* \to \mathbf{Sp}$ in order to get a functor of homotopy categories $\mathbb{L}\Sigma^\infty :$ Ho $\mathbf{Top}_* \to$ Ho **Sp**, as illustrated above.

We describe derived functors in this chapter following the point of view of the very influ-

ential book by Dwyer, Hirschhorn, Kan, and Smith [DHKS04], that only uses categories with weak equivalences. This is both simpler and more general than the older notion of a derived functor. (The older notion uses "model categories," which we will cover later in Chapter 5.)

These derived functors allow us to take things like sums, products, suspension spectra, and homotopy colimits in the stable homotopy category Ho **Sp**, starting from the same operations on **Sp**, and that is what really gets this party going.

3.1 Three equivalent definitions

The first task is to define the stable homotopy category Ho **Sp**, following the philosophy that it should be the category that takes **Sp** and universally inverts the stable equivalences. In this section we give three different ways to define it: spectra and zig-zags of maps between them, CW Ω-spectra and homotopy classes of maps between them, and CW spectra with "eventually-defined" maps between them up to eventually-defined homotopy. We also introduce the notion of an "equivalence of categories," to explain in what sense these three approaches are equivalent to each other.

3.1.1 First definition: zig-zags

Our first definition of Ho **Sp** will have the same objects as **Sp**, but the maps will be zig-zags of maps of spectra.

Definition 3.1.1. The stable homotopy category $\mathrm{Ho}\,\mathbf{Sp} = \mathbf{Sp}\langle s^{-1}\rangle$ has an object for every spectrum X. The morphisms from X to Y, denoted $[X,Y]_s$, are equivalence classes of zig-zags of maps from X to Y. We require that the backwards-pointing maps are all stable equivalences. For instance:

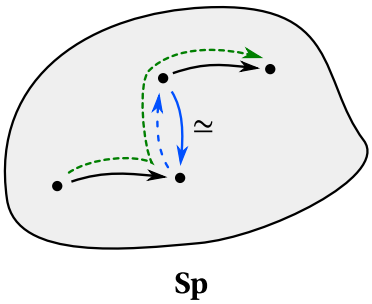
$$X \longrightarrow X_1 \xleftarrow{\sim} X_2 \xleftarrow{\sim} X_3 \longrightarrow X_4 \xleftarrow{\sim} X_5 \longrightarrow X_6 \xleftarrow{\sim} Y$$

Two zig-zags give the same morphism if they can be related by the following moves:

- compose two maps pointing in the same direction,
- delete any identity map, or
- cancel out any pair of the form $\xrightarrow{f}\xleftarrow{f}$ or $\xleftarrow{f}\xrightarrow{f}$.

We compose two zig-zags by concatenating them. The identity is the empty zig-zag of maps from X to X, equivalently the one-term zig-zag $\left(\xrightarrow{\mathrm{id}_X}\right)$. We let $\delta\colon \mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}$ denote the functor that takes each map f to the one-term zig-zag $\left(\xrightarrow{f}\right)$.

Intuitively, Ho **Sp** is the category that turns the stable equivalences into isomorphisms, and does as little else as possible. To see why this gives zig-zags, consider the figure to the right. The blob represents the category of spectra. The blue morphism in the center is a stable equivalence, so we give it a formal inverse, drawn as a blue dashed arrow.



However, once this morphism exists, we can compose it with existing morphisms. So for instance we can get the zig-zag that follows the green dashed arrow along the top of the figure.

Example 3.1.2. We can define a "degree n" map $\mathbb{S} \to \mathbb{S}$ in the stable homotopy category by taking the following zig-zag.

$$\begin{array}{ccccc}
\mathbb{S} & \xleftarrow{\sim} & F_1 S^1 & \longrightarrow & \mathbb{S} \\
\| & & & & \| \\
S^0 & \longleftarrow & * & \longrightarrow & S^0 \\
S^1 & \xleftarrow{\cong} & S^1 & \xrightarrow{n} & S^1 \\
S^2 & \xleftarrow{\cong} & S^2 & \xrightarrow{n} & S^2 \\
S^3 & \xleftarrow{\cong} & S^3 & \xrightarrow{n} & S^3 \\
\vdots & & \vdots & & \vdots
\end{array}$$

We couldn't have done this in **Sp**, because there is no map of spaces $S^0 \to S^0$ that suspends to give the degree n map $S^1 \to S^1$, unless $n = 0$ or 1. It only defines a morphism in Ho **Sp**.

In fact, every map $\mathbb{S} \to \mathbb{S}$ in the stable homotopy category is of this form, for some $n \in \mathbb{Z}$, by Example 3.1.36. This is hard to prove right now though. You would have to show that any zig-zag from $\mathbb{S}$ to $\mathbb{S}$ can be simplified to the one above.

Example 3.1.3. The Hopf map $\eta\colon S^3 \to S^2$ passes to a map in the stable homotopy category $\mathbb{S}^1 \to \mathbb{S}^0$. Informally, this is because suspension is invertible. Formally, we take

the following zig-zag.

$$\begin{array}{ccccccc}
\mathbb{S}^1 & \xleftarrow{\sim} & F_2S^3 & \xrightarrow{F_2\eta} & F_2S^2 & \xrightarrow{\sim} & \mathbb{S}^0 \\
\hline\hline
S^1 & \longleftarrow & * & \xrightarrow{0} & * & \longrightarrow & S^0 \\
S^2 & \longleftarrow & * & \xrightarrow{0} & * & \longrightarrow & S^1 \\
S^3 & \xleftarrow{\cong} & S^3 & \xrightarrow{\eta} & S^2 & \xrightarrow{\cong} & S^2 \\
S^4 & \xleftarrow{\cong} & S^4 & \xrightarrow{\Sigma\eta} & S^3 & \xrightarrow{\cong} & S^3 \\
S^5 & \xleftarrow{\cong} & S^5 & \xrightarrow{\Sigma^2\eta} & S^4 & \xrightarrow{\cong} & S^4 \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Again, η does not define a morphism in the category of spectra **Sp**, only the homotopy category Ho **Sp**. The homotopy category allows us to cut off a few levels from $\mathbb{S}^1$ and then define the map on the rest.

Example 3.1.4. For any spectrum X, we can define a degree n map $X \xrightarrow{n} X$ in many ways, including the following:

$$X \xrightarrow{(\mathrm{id},\dots,\mathrm{id})} \textstyle\prod_{i=1}^n X \xleftarrow{\sim} \bigvee_{i=1}^n X \xrightarrow{(\mathrm{id},\dots,\mathrm{id})} X$$

$$X \xleftarrow{\sim} \mathrm{sh}^{-1}\,\Sigma X \xrightarrow{S^1 \xrightarrow{n} S^1} \mathrm{sh}^{-1}\,\Sigma X \xleftarrow{\sim} X$$

$$X \xleftarrow{\sim} \mathrm{sh}^{1}\,\Omega X \xrightarrow{S^1 \xrightarrow{n} S^1} \mathrm{sh}^{1}\,\Omega X \xleftarrow{\sim} X.$$

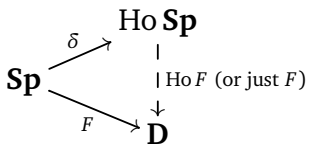
These all define the same map in the stable homotopy category by Exercise 3.6.11.

Example 3.1.5. If $f : X \to Y$ is a stable equivalence, then $\delta(f) = \left(\xrightarrow{f}\right)$ is an isomorphism in Ho **Sp**. Its inverse is the zig-zag $\left(\xleftarrow{f}\right)$:

$$\left(\xrightarrow{f}\right)\left(\xleftarrow{f}\right) = \left(\xrightarrow{f}\xleftarrow{f}\right) = (\mathrm{id}_X), \qquad \left(\xleftarrow{f}\right)\left(\xrightarrow{f}\right) = \left(\xleftarrow{f}\xrightarrow{f}\right) = (\mathrm{id}_Y).$$

In fact, Ho **Sp** is the universal category in which the stable equivalences in **Sp** become isomorphisms.

Proposition 3.1.6. *Suppose $F : \mathbf{Sp} \to \mathbf{D}$ is any functor taking stable equivalences to isomorphisms. Then there is a unique functor making this triangle commute:*



(3.1.7)

Proof. We define the functor Ho F by sending X to $F(X)$, and sending each zig-zag to the composite of F of the morphisms (inverted when they point backwards). This gives the desired commuting triangle. Any other functor making the triangle commute would have to agree with Ho F on objects and one-term zig-zags. But then it follows that it agrees on every zig-zag. □

Example 3.1.8. By definition, each stable equivalence $X \to Y$ gives an isomorphism on the stable homotopy groups $\pi_k(-)$. By Proposition 3.1.6, π_k therefore defines a functor on the stable homotopy category, Ho **Sp** → **Ab**. By Proposition 2.6.38, homology $H_k(-; G)$ also defines a functor Ho **Sp** → **Ab**.

It is clear that the proof of Proposition 3.1.6 has nothing to do with spectra.

Definition 3.1.9. Let **C** be any category, and let W be any collection of morphisms that is closed under composition. We call each morphism in W a "weak equivalence." The **homotopy category** Ho $\mathbf{C} = \mathbf{C}\langle W^{-1}\rangle$ has the same objects as **C**, and the morphisms are equivalence classes of zig-zags with backwards maps in W, as in Definition 3.1.1.

We also call this the **Gabriel-Zisman localization** of **C** by W. As we mentioned in the chapter introduction, this is called a "localization" of **C** because it is obtained from **C** by formally inverting some of the morphisms, turning them into isomorphisms.

Some authors use square brackets to denote this localization, but we use angle brackets because we want to distinguish it from the Dwyer-Kan localization $\mathbf{C}[W^{-1}]$ that we define in Section 10.2.

Considering one-term zig-zags, we see there is a functor

$$\delta : \mathbf{C} \longrightarrow \mathbf{C}\langle W^{-1}\rangle.$$

Theorem 3.1.10. *[GZ67, 1.1] Any functor $F : \mathbf{C} \to \mathbf{D}$ that sends weak equivalences to isomorphisms, must factor uniquely through $\mathbf{C}\langle W^{-1}\rangle$.*

Definition 3.1.11. The homotopy category of unbased spaces Ho **Top** = **Top**$\langle w^{-1}\rangle$ is formed from **Top** by inverting the weak homotopy equivalences. A morphism in Ho **Top** is a zig-zag of maps of unbased spaces, where the maps pointing to the left are weak homotopy equivalences.

Definition 3.1.12. The homotopy category of based spaces $\mathrm{Ho}\,\mathbf{Top}_* = \mathbf{Top}_*\langle w^{-1}\rangle$ is formed from $\mathbf{Top}_*$ by inverting the weak homotopy equivalences. A morphism in $\mathrm{Ho}\,\mathbf{Top}_*$ is a zig-zag of based maps of based spaces, where the maps pointing to the left are weak homotopy equivalences.

Example 3.1.13. Singular homology $H_k(-;G)$ preserves weak equivalences. By Theorem 3.1.10, it therefore defines a functor on the homotopy category $\mathrm{Ho}\,\mathbf{Top} \to \mathbf{Ab}$.

Remark 3.1.14. There is a size issue in these constructions: for any two objects X and Y in $\mathbf{C}$, the collection of morphisms from X to Y in $\mathrm{Ho}\,\mathbf{C}$ may be so large that it is not a set, but rather a proper class. This is addressed by allowing the homotopy category to have sets lying in a larger universe. In the examples we actually encounter, such as $\mathrm{Ho}\,\mathbf{Sp}$, the morphisms do in fact form a set, so this workaround isn't necessary.

We end with a lemma about the homotopy category that will be useful when comparing it to other models. Recall that two maps of spectra $f, g : X \rightrightarrows Y$ are homotopic if they extend to a map of spectra $h : I_+ \wedge X \to Y$.

Lemma 3.1.15. *If f and g are homotopic, then $\delta(f) = \delta(g)$ in* $\mathrm{Ho}\,\mathbf{Sp}$. *The same is true for spaces: homotopic maps of spaces give the same morphism in* $\mathrm{Ho}\,\mathbf{Top}$.

Proof. Let $i_0, i_1 : X \rightrightarrows I_+ \wedge X$ be the inclusion of the top and bottom of the cylinder, and let $p : I_+ \wedge X \to S^0 \wedge X \cong X$ be the projection. Then in $\mathrm{Ho}\,\mathbf{Sp}$,

$$\left(\xrightarrow{f}\right) = \left(\xrightarrow{i_0}\xrightarrow{h}\right) = \left(\xrightarrow{i_0}\xrightarrow{p}\xleftarrow{p}\xrightarrow{h}\right) = \left(\xleftarrow{p}\xrightarrow{h}\right)$$

$$= \left(\xrightarrow{i_1}\xrightarrow{p}\xleftarrow{p}\xrightarrow{h}\right) = \left(\xrightarrow{i_1}\xrightarrow{h}\right) = \left(\xrightarrow{g}\right).$$

□

Definition 3.1.16. For spectra X and Y, we let

- $[X, Y]_s$ denote the set of maps from X to Y in $\mathrm{Ho}\,\mathbf{Sp}$, i.e. zig-zags up to the equivalence relation of Definition 3.1.1, and
- $[X, Y]_h$ denote maps of spectra $X \to Y$ up to homotopy.

Remark 3.1.17. By Lemma 3.1.15, there is a well-defined map $[X, Y]_h \to [X, Y]_s$. But this is often not a bijection. For instance, $[\mathbb{S}, \mathbb{S}]_h = \{0, 1\}$, but we will prove later that $[\mathbb{S}, \mathbb{S}]_s \cong \mathbb{Z}$.

3.1.2 Equivalences of categories and a second definition

In this section we give several definitions of $\mathrm{Ho}\,\mathbf{Sp}$ that are equivalent to the one in Definition 3.1.1. We first recall what an equivalence of categories is.

Definition 3.1.18. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is an **isomorphism of categories** if it is a bijection on objects and on morphisms. The functor F is an **equivalence of categories** if it is a bijection on isomorphism classes of objects

$$\mathrm{ob}\,\mathbf{C}/\mathrm{isom} \xrightarrow{\cong} \mathrm{ob}\,\mathbf{D}/\mathrm{isom} \tag{3.1.19}$$

and a bijection on morphisms between any pair of objects,

$$\mathbf{C}(X,Y) \xrightarrow{\cong} \mathbf{D}(F(X),F(Y)). \tag{3.1.20}$$

Implicit in this definition is the fact that any functor F induces a function (3.1.19) on isomorphism classes of objects. This is because every functor preserves isomorphisms,

$$X \cong Y \;\Rightarrow\; F(X) \cong F(Y).$$

We say $F\colon \mathbf{C} \to \mathbf{D}$ is **fully faithful** if (3.1.20) is a bijection for each pair of objects $X, Y \in \mathrm{ob}\,\mathbf{C}$. Informally, this means that $\mathbf{C}$ and $\mathbf{D}$ have "the same morphisms," but not necessarily the same objects. However, by Exercise 3.6.1, this condition implies that (3.1.19) is injective.

We say that F is **essentially surjective** if every $Z \in \mathrm{ob}\,\mathbf{D}$ is isomorphic to $F(X)$ for some $X \in \mathrm{ob}\,\mathbf{C}$. Of course, this is the same thing as saying that (3.1.19) is surjective. Putting this all together:

Lemma 3.1.21. *F is an equivalence of categories iff it is both fully faithful and essentially surjective.*

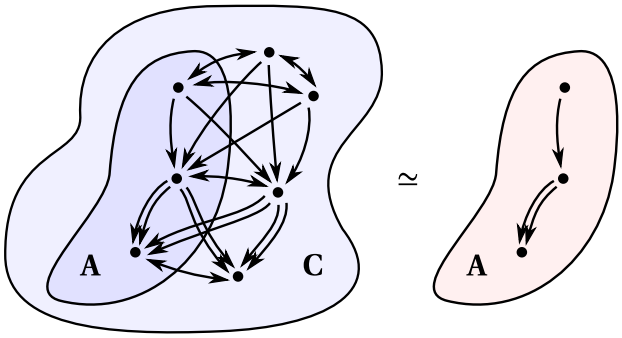
The distinction between "isomorphism" and "equivalence" of categories is very much like the difference between a homeomorphism and a homotopy equivalence of topological spaces. A homeomorphism $X \to Y$ gives a bijection on the underlying set. A homotopy equivalence only gives a bijection on the path components and the homotopy groups. The following standard lemma takes this analogy further.

Lemma 3.1.22. *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor. If there exists a second functor $G\colon \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $GFX \cong X$ and $FGY \cong Y$ for all $X \in \mathrm{ob}\,\mathbf{C}$ and $Y \in \mathrm{ob}\,\mathbf{D}$, then F is an equivalence of categories. The converse holds as well, modulo issues of size and the axiom of choice.*

Example 3.1.23. Let $\mathbf{A} \subseteq \mathbf{C}$ be a full subcategory. That is, we select some objects from $\mathbf{C}$, and take all morphisms between them:

$$\mathbf{A}(X, Y) := \mathbf{C}(X, Y).$$

The inclusion $\mathbf{A} \subseteq \mathbf{C}$ is fully faithful by definition. So, it is an equivalence of categories as soon as $\mathbf{A}$ contains at least one object in each isomorphism class. In this case the inverse functor $G\colon \mathbf{C} \to \mathbf{A}$ from Lemma 3.1.22 can be chosen to be the identity on $\mathbf{A}$, so that it is a "deformation retract" of categories.



Example 3.1.24. If $\mathbf{A}$ is a **skeleton** of $\mathbf{C}$, in other words a full subcategory containing exactly one object in each isomorphism class, then $\mathbf{A} \subseteq \mathbf{C}$ is an equivalence of categories.

To give a common example from linear algebra, let $\mathbf{C}$ be the category of finite-dimensional real vector spaces, and $\mathbf{A} \subseteq \mathbf{C}$ be the full subcategory consisting of the vector space $\mathbb{R}^n$ for each $n \geq 0$. Then $\mathbf{A}$ is a skeleton of $\mathbf{C}$, and is therefore equivalent to $\mathbf{C}$. Choosing a deformation retract of $\mathbf{C}$ onto $\mathbf{A}$ amounts to choosing a basis for each finite-dimensional vector space.

Example 3.1.25. One of the main theorems about covering spaces is best stated as an equivalence of categories. Suppose B is a connected CW complex, or more generally a path-connected, locally path-connected, and semilocally-simply-connected space. Then there is a category $\mathbf{Cov}(B)$ whose objects are covering spaces $p\colon E \to B$ and whose maps are continuous maps $E \to E'$ commuting with the map to B. There is a functor from this category to the category of sets with right action of $G = \pi_1(B, b_0)$, given by sending each covering space $p\colon E \to B$ to its fiber $p^{-1}(b_0)$, with G acting through path lifting.

The theorem states that this is an equivalence of categories. So covering spaces *up to isomorphism* are the same thing as G-sets up to isomorphism, and the set of maps between two covering spaces is the set of maps between their corresponding G-sets.

Definition 3.1.26. Define $\mathrm{Ho}\,\mathbf{CW} = \mathbf{CW}\langle w^{-1}\rangle$ to have objects the CW complexes, and maps the zig-zags of CW complexes as in Definition 3.1.1, where the backwards maps are weak homotopy equivalences. Define $h\mathbf{CW}$ to have objects the CW complexes, and morphisms the homotopy classes of maps.

By Lemma 3.1.15, homotopic maps give the same zig-zag, so we have functors

$$h\mathbf{CW} \longrightarrow \mathrm{Ho}\,\mathbf{CW} = \mathbf{CW}\langle w^{-1}\rangle \longrightarrow \mathrm{Ho}\,\mathbf{Top} = \mathbf{Top}\langle w^{-1}\rangle.$$

Proposition 3.1.27. *These functors are equivalences of categories.*

Proof. $h\mathbf{CW}$ and $\mathrm{Ho}\,\mathbf{CW}$ have the same objects, so we just have to show they have the same morphisms. The easiest way to do this is to show that $h\mathbf{CW}$ satisfies the universal property of $\mathrm{Ho}\,\mathbf{CW}$: given any map $F : \mathbf{CW} \to \mathbf{D}$ taking weak homotopy equivalences to isomorphisms, it factors uniquely through $h\mathbf{CW}$.

By the Whitehead theorem (Theorem 1.4.18), weak homotopy equivalences between CW complexes are homotopy equivalences. Therefore $\mathbf{CW} \to h\mathbf{CW}$ does in fact take weak homotopy equivalences to isomorphisms. Furthermore, if $F : \mathbf{CW} \to \mathbf{D}$ takes weak homotopy equivalences to isomorphisms, then for homotopic maps f, g we have

$$F(f) = F(h) \circ F(i_0) = F(h) \circ F(p)^{-1} = F(h) \circ F(i_1) = F(g)$$

where p, i_0, and i_1 are defined as in Lemma 3.1.15. Therefore F factors through $h\mathbf{CW}$ in a unique way. This verifies the universal property, so $h\mathbf{CW} \cong \mathrm{Ho}\,\mathbf{CW}$.

For the second part of the proof, we know that $\mathrm{Ho}\,\mathbf{CW} \to \mathrm{Ho}\,\mathbf{Top}$ is essentially surjective, because every space is weakly equivalent to a CW complex. So we just have to show these categories have the same morphisms between a given pair of objects. In other words, for two CW complexes K and L, we have to show we get a bijection between zig-zags of CW complexes, and zig-zags of all spaces.

Let $Q(-)$ be a functor that replaces spaces by equivalent CW complexes. To each zig-zag of arbitrary spaces between K and L, we apply Q and get a commuting diagram

$$\begin{array}{ccccccccccccc}
K & \longrightarrow & X_1 & \overset{\sim}{\longleftarrow} & X_2 & \longrightarrow & X_3 & \overset{\sim}{\longleftarrow} & X_4 & \longrightarrow & X_5 & \longrightarrow & L \\
\uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} \\
QK & \longrightarrow & QX_1 & \overset{\sim}{\longleftarrow} & QX_2 & \longrightarrow & QX_3 & \overset{\sim}{\longleftarrow} & QX_4 & \longrightarrow & QX_5 & \longrightarrow & QL.
\end{array}$$

We define the inverse function $\mathrm{Ho}\,\mathbf{Top}(K, L) \to \mathrm{Ho}\,\mathbf{CW}(K, L)$ by taking the zig-zag along the top to the zig-zag along the bottom and sides. This is an inverse to the map $\mathrm{Ho}\,\mathbf{CW}(K, L) \to \mathrm{Ho}\,\mathbf{Top}(K, L)$ by Exercise 3.6.6, so we have a bijection on the morphism sets. □

Now we can apply this same reasoning to the category of spectra.

Definition 3.1.28. Let

- $\mathbf{Sp}^{CW} \subseteq \mathbf{Sp}$ be the full subcategory of CW spectra,

- $\mathbf{Sp}^{\Omega} \subseteq \mathbf{Sp}$ be the full subcategory of Ω-spectra, and
- $\mathbf{Sp}^{CW,\Omega} \subseteq \mathbf{Sp}$ be the full subcategory of CW Ω-spectra.

We take the homotopy category of each of these by inverting the stable equivalences. We also let $h\mathbf{Sp}^{CW,\Omega}$ have the same objects as $\mathbf{Sp}^{CW,\Omega}$, but the morphisms are maps of spectra up to homotopy, $[X, Y]_h$.

Proposition 3.1.29. *The inclusions of categories*

$$\begin{array}{ccc} \mathbf{Sp}^{CW,\Omega} & \longrightarrow & \mathbf{Sp}^{\Omega} \\ \downarrow & & \downarrow \\ \mathbf{Sp}^{CW} & \longrightarrow & \mathbf{Sp} \end{array}$$

induce equivalences of homotopy categories

$$\begin{array}{ccccc} h\mathbf{Sp}^{CW,\Omega} & \xleftrightarrow{\cong} & \mathrm{Ho}\,\mathbf{Sp}^{CW,\Omega} & \xrightarrow{\sim} & \mathrm{Ho}\,\mathbf{Sp}^{\Omega} \\ & & \downarrow \sim & & \downarrow \sim \\ & & \mathrm{Ho}\,\mathbf{Sp}^{CW} & \xrightarrow{\sim} & \mathrm{Ho}\,\mathbf{Sp}. \end{array}$$

Proof. The proof is the same as in Proposition 3.1.27. For the first part, we use the Whitehead theorem (Corollary 2.6.17) to show that stable equivalences go to isomorphisms in $h\mathbf{Sp}^{CW,\Omega}$. For the second part, we use the CW replacement functor Q constructed in Theorem 2.6.12 and the Ω-spectrum replacement functor R from Proposition 2.2.9. □

Proposition 3.1.29 tells us that if X and Y are CW Ω-spectra, then the map

$$[X, Y]_h \to [X, Y]_s$$

is a bijection. This is not true for arbitrary X and Y, as mentioned in Remark 3.1.17. See Exercise 3.6.5.

Proposition 3.1.29 gives us four more models for the stable homotopy category. The most interesting of these is the one on the far left – we restrict our attention to CW Ω-spectra, but now we just take single maps (not zig-zags) up to homotopy. This is our second definition of the stable homotopy category.

Remark 3.1.30. We can give an explicit inverse to the inclusion $\mathrm{Ho}\,\mathbf{Sp}^{CW} \subseteq \mathrm{Ho}\,\mathbf{Sp}$. Let $Q\colon \mathbf{Sp} \to \mathbf{Sp}^{CW}$ be the functor that replaces each spectrum X by a stably equivalent CW spectrum. Since Q is stably equivalent to the identity, composing Q with the inclusion $\mathrm{Ho}\,\mathbf{Sp}^{CW} \subseteq \mathrm{Ho}\,\mathbf{Sp}$ (in either order) is isomorphic to the identity. Therefore Q gives the inverse equivalence of categories $\mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}^{CW}$.

We can similarly use R to go back to Ω-spectra. So we have a diagram of equivalences of categories

$$\begin{array}{ccccc} h\mathbf{Sp}^{CW,\Omega} & \xleftrightarrow{\cong} & \mathrm{Ho}\,\mathbf{Sp}^{CW,\Omega} & \underset{\sim}{\overset{Q}{\rightleftarrows}} & \mathrm{Ho}\,\mathbf{Sp}^{\Omega} \\ & & \downarrow \sim & & \sim \downarrow \uparrow R \\ & & \mathrm{Ho}\,\mathbf{Sp}^{CW} & \underset{Q}{\overset{\sim}{\rightleftarrows}} & \mathrm{Ho}\,\mathbf{Sp} \end{array}$$

We conclude that there is a bijection for all spectra X and Y

$$[X,Y]_s \cong [QRX, QRY]_h.$$

Example 3.1.31. The degree n map of the sphere from Example 3.1.2 can be described in $h\mathbf{Sp}^{CW,\Omega}$ by taking the map $\Omega^\infty\Sigma^\infty S^0 \to \Omega^\infty\Sigma^\infty S^0$ that applies the degree n map to every sphere in the colimit system after S^0. With a fair amount of effort, this can be extended to the rest of the fibrant sphere spectrum $f\mathbb{S}$ from Example 2.2.6. We then have to apply the CW replacement functor Q to both sides, and we arrive at a map in $h\mathbf{Sp}^{CW,\Omega}$.

Phew! This example demonstrates that it is difficult to define maps of CW Ω-spectra directly.

3.1.3 Third definition: cells now, maps later

We have now seen two definitions of the stable homotopy category. The first definition takes zig-zags of maps up to an equivalence relation. This is convenient because it is easy to name a single morphism explicitly. But it is inconvenient because it is very hard to count how many morphisms there are from X to Y. Literally any object in the entire category could show up in the zig-zag!

The second definition takes CW Ω-spectra and homotopy classes of maps. In principle, this makes it easier to count how many morphisms there are. In practice, though, this is not actually very useful. CW Ω-spectra are too complicated.

We will now give a third definition that captures the best of both worlds.

Definition 3.1.32. Let X be a CW spectrum. A **cofinal subspectrum** $X' \subseteq X$ consists of a subcomplex $X'_n \subseteq X_n$ on each level, such that $\xi_n(\Sigma X'_n) \subseteq X'_{n+1}$, and such that every stable cell of X has a representative in X'.

Example 3.1.33. The sphere spectrum $\mathbb{S}$ has the shift desuspension $F_n S^n$ as a cofinal subspectrum, for each $n \geq 0$. The suspension spectrum $\Sigma^\infty\mathbb{RP}^2$ has a stable 1-cell e^1 and a stable 2-cell e^2. We could form a cofinal spectrum by taking e^1 at spectrum level 15, where it is a 16-cell, and e^2 at spectrum level 18, where it is a 20-cell attached to the 19-cell representing e^1.

Definition 3.1.34. Suppose X is a CW spectrum and Y is a spectrum. An **eventually-defined map** $X \to Y$ is a cofinal subspectrum $X' \subseteq X$ and a map of spectra $X' \to Y$. In other words, the map is defined on each stable cell, but not right away – we may wait several levels before the map becomes defined.

Given two eventually-defined maps $(X', f : X' \to Y)$ and $(X'', g : X'' \to Y)$, an eventually-defined homotopy is a third cofinal subspectrum $X''' \subseteq X' \cap X''$ and a homotopy from f to g on X'''.

Let $[X, Y]_e$ denote the set of eventually-defined maps up to eventually-defined homotopy.

The above definition was famously summarized by Adams as "cells now, maps later." See [Ada74].

Definition 3.1.35. The category $e\mathbf{Sp}^{CW}$ has an object for each CW spectrum. The morphisms from X to Y are $[X, Y]_e$, the eventually-defined maps up to eventually-defined homotopy. In particular, we can always restrict f to a smaller cofinal subspectrum $X'' \subseteq X'$ without changing the resulting morphism in $[X, Y]_e$.

We compose two morphisms

$$(X' \subseteq X, f : X' \to Y), \quad (Y' \subseteq Y, g : Y' \to Z)$$

by taking any cofinal $X'' \subseteq X'$ whose image is contained in Y', and taking the composite $g \circ f$ on X''. Such an X'' exists because each stable cell of X has image in Y contained in finitely many stable cells, and hence is eventually contained in Y'. It is straightforward to check that this rule is well-defined on homotopy classes of maps.

Example 3.1.36. The maps $[\mathbb{S}, \mathbb{S}]_e$ are easily computed. The sphere has just one stable 0-cell, so a cofinal subspectrum must be of the form $F_n S^n$ for some n. The set of maps $[S^n, S^n]_*$ is $\mathbb{Z}$ so long as $n \geq 1$. It follows that $[\mathbb{S}, \mathbb{S}]_e \cong \mathbb{Z}$.

Similarly, using the fact that $[S^{n+1}, S^n]_* \cong \mathbb{Z}/2$ for $n \geq 3$, we can compute that $[\mathbb{S}^1, \mathbb{S}^0]_e \cong \mathbb{Z}/2$. The nonzero element is the Hopf map η, defined as in Example 3.1.3 by waiting until spectrum level two and then taking the Hopf map $S^3 \to S^2$.

We will prove that $e\mathbf{Sp}^{CW}$ is equivalent to the other models of the homotopy category, extending the diagram from Remark 3.1.30 to the following.

$$\begin{array}{ccccc} h\mathbf{Sp}^{CW,\Omega} & \xleftarrow{\cong} & \mathrm{Ho}\,\mathbf{Sp}^{CW,\Omega} & \underset{\sim}{\overset{Q}{\rightleftarrows}} & \mathrm{Ho}\,\mathbf{Sp}^{\Omega} \\ \downarrow \sim & & \downarrow \sim & & \sim \downarrow \circlearrowright R \\ e\mathbf{Sp}^{CW} & \xleftarrow{\cong} & \mathrm{Ho}\,\mathbf{Sp}^{CW} & \underset{Q}{\overset{\sim}{\rightleftarrows}} & \mathrm{Ho}\,\mathbf{Sp} \end{array} \tag{3.1.37}$$

Lemma 3.1.38. *The inclusion of a cofinal subspectrum $X' \to X$ is a stable equivalence.*

Proof. To show that $\pi_k(X') \to \pi_k(X)$ is surjective, we take any element and represent it by a map $S^{k+n} \to X_n$. Since S^{k+n} is compact, the image of this map is contained in a finite subcomplex. If n is sufficiently large then all the cells of this complex are contained in X'_n, so the map lifts to X'. The proof of injectivity is the same argument with homotopies $S^{k+n} \wedge I_+ \to X_n$. □

Proposition 3.1.39. *Every stable equivalence of CW spectra $f : X \to Y$ is an isomorphism in $e\mathbf{Sp}^{CW}$; it has an eventually-defined homotopy inverse.*

Proof. We factor f into the mapping cylinder $X \to M$ and then the homotopy equivalence of spectra $M \to Y$. The fully-defined homotopy equivalence $M \simeq Y$ is clearly an isomorphism in $e\mathbf{Sp}^{CW}$. So, we just need to show that $X \to M$ is an isomorphism as well.

We do this by defining an eventually-defined deformation retract of M onto X. We define this on a cofinal subspectrum of $M \wedge I_+$, working one stable cell of M at a time. To be more specific, we order the stable cells according to which spectrum level they first appear, and then according to their dimension, as in the proof of Theorem 2.6.12. So when it is time to define the retract on $e^k \times I$ for a given stable k-cell e^k of M, attached at level m and represented by a disc $D^{k+n} \to M_n$ for all $n \geq m$, we can assume that the retract has been defined on every cell in the image of the attaching map of e^k.

We are required to define a map of pairs

$$(D^{k+n} \times I, D^{k+n} \times 0) \to (M_n, X_n)$$

that is already specified on the top $D^{k+n} \times \{1\}$ (as the identity of M_n) and on the sides $\partial D^{k+n} \times I$ (by the previously defined cells), when n is sufficiently large. Since the top and sides of $D^{k+n} \times I$ are homeomorphic to D^{k+n}, we get a class $\alpha_n \in \pi_{k+n}(M_n, X_n)$, and the extension exists if this class is zero. Since $\pi_k(M, X) = 0$, the class α_n is zero for sufficiently large n. Therefore, after possibly waiting longer to define the deformation retract on this cell, the required extension exists. Repeating for each cell, we get the desired eventually-defined deformation retract. □

Theorem 3.1.40. *There is an isomorphism of categories $e\mathbf{Sp}^{CW} \cong \mathrm{Ho}\,\mathbf{Sp}^{CW}$.*

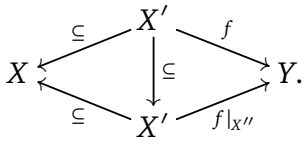
Proof. We define a functor $F : e\mathbf{Sp}^{CW} \longrightarrow \mathrm{Ho}\,\mathbf{Sp}^{CW}$, in other words maps

$$[X, Y]_e \to [X, Y]_s,$$

by taking the eventually-defined map $(X' \subseteq X, f : X' \to Y)$ to the zig-zag

$$X \xleftarrow{\subseteq} X' \xrightarrow{f} Y.$$

Passing to a further cofinal subspectrum $X'' \subseteq X'$ gives the same morphism in $[X, Y]_s$ by examining the commuting diagram



If we change f by a homotopy, we also get the same morphism in $[X, Y]_s$ by Lemma 3.1.15.

We check that this respects compositions of maps, so that it defines a functor. Given a second morphism $Y \longrightarrow Z$ defined on $Y' \subset Y$, $F(g \circ f)$ becomes the top branch of the diagram

$$\begin{array}{ccccc} X'' & \longrightarrow & Y' & \longrightarrow & Z \\ \downarrow \sim & & \downarrow \sim & & \\ X & \xleftarrow[\sim]{} & X' & \longrightarrow & Y \end{array}$$

while $F(g) \circ F(f)$ is the bottom branch. Since the diagram commutes in $\mathbf{Sp}^{CW}$, it also commutes as a diagram in $\mathrm{Ho}\,\mathbf{Sp}^{CW}$, so these two routes give the same morphism of $\mathrm{Ho}\,\mathbf{Sp}^{CW}$.

To show that F is an isomorphism, we define its inverse $G\colon \mathrm{Ho}\,\mathbf{Sp}^{CW} \to e\mathbf{Sp}^{CW}$. There is an obvious functor $\mathbf{Sp}^{CW} \to e\mathbf{Sp}^{CW}$. By Proposition 3.1.39, this functor takes stable equivalences to isomorphisms. Therefore it factors through the homotopy category $\mathrm{Ho}\,\mathbf{Sp}^{CW}$, giving our desired functor $G\colon \mathrm{Ho}\,\mathbf{Sp}^{CW} \to e\mathbf{Sp}^{CW}$.

The composite $G \circ F$ takes each eventually-defined map $(X' \subseteq X, f : X' \to Y)$ to the inverse of $X' \subseteq X$, composed with $f : X' \to Y$. The inverse of the inclusion is the eventually-defined map $X \to X'$ that is just the identity map on X'. Composing this with f gives the original map back, $(X' \subseteq X, f : X' \to Y)$, so $G \circ F$ is the identity functor.

The composite $F \circ G$ only has to be calculated on $\mathbf{Sp}^{CW}$, by the universal property of $\mathrm{Ho}\,\mathbf{Sp}^{CW}$. For each map of spectra $f : X \to Y$, it is sent to the eventually-defined map $(X = X, f : X \to Y)$, and then back to $f : X \to Y$ again. So $F \circ G$ is the identity functor. □

In summary, all of the models of the stable homotopy category from the last three sections are equivalent to each other. We are free to use whichever one is the most convenient at the moment:

$$[X, Y]_s \cong [X, Y]_h \text{ when } X \text{ and } Y \text{ are both CW } \Omega\text{-spectra, and}$$
$$[X, Y]_s \cong [X, Y]_e \text{ when } X \text{ and } Y \text{ are both CW spectra.}$$

It will be helpful to weaken these hypotheses a little more. Notice that eventually-defined maps $[X, Y]_e$ make sense if X is CW, but Y is arbitrary. In this setting we have three maps

$$[X, Y]_h \to [X, Y]_e \to [X, Y]_s.$$

Proposition 3.1.41. *If X is CW and Y is an Ω-spectrum, both of the maps $[X,Y]_h \to [X,Y]_e \to [X,Y]_s$ are bijections.*

Proof. Let QY be the CW replacement of Y. The level equivalence $QY \to Y$ gives a commuting diagram

$$\begin{array}{ccccc} [X,QY]_h & \longrightarrow & [X,QY]_e & \xrightarrow{\cong} & [X,QY]_s \\ \downarrow & & \downarrow{\scriptstyle (2)} & & \downarrow{\scriptstyle \cong} \\ [X,Y]_h & \xrightarrow{(1)} & [X,Y]_e & \longrightarrow & [X,Y]_s. \end{array}$$

The isomorphism in the top row is by Theorem 3.1.40, while the isomorphism on the right is because $QY \to Y$ is an isomorphism in $\mathrm{Ho}\,\mathbf{Sp}$. To finish proving the claim, it is enough to show that the maps (1) and (2) are isomorphisms.

To show that (1) is surjective, we show each eventually-defined map $f : X' \to Y$, defined on a cofinal subspectrum $X' \subseteq X$, is homotopic to an actual map $g : X \to Y$. It suffices to define a map of spectra

$$X \wedge \{0\}_+ \cup_{X' \wedge \{0\}_+} X' \wedge I_+ \longrightarrow Y$$

that agrees with f along $X' \wedge \{1\}_+$. As in the proof of Proposition 3.1.39, we define this extension one cell of X at a time. Suppose we have a stable k-cell in X, first appearing in X as as $D^{k+m} \to X_m$. For each $n \geq m$, this gives a cell

$$D^{k+n} \to \Sigma^{n-m} D^{k+m}_+ \to X_n.$$

Since X' is cofinal in X, there is a first value of n on which f is defined on this cell,

$$\Sigma^{n-m} D^{k+m}_+ \to Y_n.$$

This rearranges to a map $D^{k+m} \to \Omega^{n-m} Y_n$, and our goal is to modify this map up to homotopy to a map that lifts to Y_m. Furthermore, this homotopy has already been defined on the boundary ∂D^{k+m}, because we have already defined the homotopy on the lower-dimensional cells that are attached at spectrum level m, and on all cells attached at previous levels of the spectrum. All together, this defines a class in $\pi_{k+m}(\Omega^{n-m} Y_n, Y_m)$. Since Y is an Ω-spectrum, this relative homotopy group is zero, so the desired extension exists.

This proves that (1) is surjective. The proof of injectivity is similar: we define a map

$$X \wedge (I \times \{0\})_+ \cup_{X' \wedge (I \times \{0\})_+} X' \wedge (I \times I)_+ \longrightarrow Y$$

that is already defined on the top and sides of $I \times I$. Again, we do this one cell of X at a time, and the required extensions exist because Y is an Ω-spectrum. The proof that (2) is an isomorphism is similar but somewhat easier, and so we leave it to Exercise 3.6.9. □

Proposition 3.1.42. *If X is CW, then $[X,Y]_e \to [X,Y]_s$ is a bijection.*

Proof. This follows from the fact that the map (2) in the previous proof is an isomorphism. The proof (Exercise 3.6.9) does not require us to assume that Y is an Ω-spectrum. □

Remark 3.1.43. Throughout this section and the last, we have used CW spectra for simplicity. In fact, all of the definitions and results also work for the larger category of cellular spectra. In particular, $[X,Y]_s \cong [X,Y]_h$ if X is cellular and Y is an Ω-spectrum. We will give another proof of this using model categories in Theorem 5.2.12 and Theorem 5.2.27.

3.2 Fundamental properties

We have now given three equivalent models for the stable homotopy category:

- All spectra, with maps $[X,Y]_s$ defined as zig-zags,
- CW spectra, with homotopy classes of eventually-defined maps $[X,Y]_e$, and
- CW Ω-spectra, with homotopy classes of actual maps $[X,Y]_h$.

These are equivalent categories. So they have the same objects up to isomorphism, and the same morphisms between those objects. In fact, we have bijections

$$[X,Y]_s \cong [X,Y]_h \text{ when } X \text{ is CW and } Y \text{ is an } \Omega\text{-spectrum, and}$$
$$[X,Y]_s \cong [X,Y]_e \text{ when } X \text{ is CW.}$$

Now that we have finally finished the *definition* of the stable homotopy category, we can start proving its fundamental properties. These properties are somewhat parallel to the stability theorems and their corollaries from Section 2.4.

3.2.1 Suspension, coproducts, and additive structure

Proposition 3.2.1. *Σ and Ω define inverse equivalences of categories*

$$\mathrm{Ho}\,\mathbf{Sp} \underset{\Omega}{\overset{\Sigma}{\rightleftarrows}} \mathrm{Ho}\,\mathbf{Sp}.$$

In particular,

$$[\Sigma X, \Sigma Y]_s \cong [X,Y]_s \cong [\Omega X, \Omega Y]_s.$$

Proof. Both suspension and loopspace preserve all stable equivalences by Corollary 2.4.4. Therefore by the universal property of the homotopy category (Proposition 3.1.6), they descend to functors on the homotopy category. The natural isomorphisms from Corollary 2.4.6 relate each composite back to the identity in the homotopy category. □

For the next proposition, let $\{X(\alpha)\}_{\alpha\in A}$ be any collection of spectra and let Y be any spectrum.

Proposition 3.2.2. *The canonical map that restricts the sum to each summand*

$$\left[\bigvee_\alpha X(\alpha), Y\right]_s \to \prod_\alpha [X(\alpha), Y]_s$$

is a bijection. Furthermore the canonical map that projects the product to each factor

$$\left[Y, \prod_\alpha X(\alpha)\right]_s \to \prod_\alpha [Y, X(\alpha)]_s$$

is a bijection if the product is finite, or if every $X(\alpha)$ is an Ω-spectrum.

Proof. For the first claim, we replace each $X(\alpha)$ by the CW spectrum $QX(\alpha)$. Since $QX(\alpha)$ is stably equivalent to $X(\alpha)$, and wedge sums preserve stable equivalences (Exercise 2.7.30), this gives an isomorphism on $[-,-]_s$. (See also Exercise 3.6.4.) Similarly we replace Y by the Ω-spectrum RY.

Once we have done this, the wedge sum $\bigvee_\alpha QX(\alpha)$ is a CW spectrum and RY is an Ω-spectrum, so by Proposition 3.1.41, the maps in the stable category agree with homotopy classes of maps $[-,-]_h$. We therefore have the following isomorphisms.

$$\begin{array}{ccc}
[\bigvee_\alpha QX(\alpha), RY]_h & \longrightarrow & \prod_\alpha [QX(\alpha), RY]_h \\
\downarrow\cong & & \downarrow\cong \\
[\bigvee_\alpha QX(\alpha), RY]_s & \longrightarrow & \prod_\alpha [QX(\alpha), RY]_s \\
\downarrow\cong & & \downarrow\cong \\
[\bigvee_\alpha X(\alpha), Y]_s & \longrightarrow & \prod_\alpha [X(\alpha), Y]_s.
\end{array}$$

The top horizontal map in this diagram is an isomorphism, by the universal property of the coproduct applied to maps and to homotopies. Therefore all of the horizontal maps are isomorphisms, which proves the first claim.

The same argument applies to products, except that we use Exercise 2.7.26 to show that replacing the $X(\alpha)$ by Ω-spectra induces an equivalence on the product. □

Remark 3.2.3. Proposition 3.2.2 tells us that the wedge sum $\bigvee$ is the coproduct *both* in the category of spectra **Sp** and in the homotopy category Ho **Sp**. On the other hand, the product $\prod$ in spectra is only the product in the homotopy category if it is a finite product, or if the inputs are Ω-spectra. To get the product in Ho **Sp** in general, we will have to replace the $X(\alpha)$ by Ω-spectra first, and then take their product.

Corollary 3.2.4. *The zero spectrum $*$ is a **zero object** in* Ho **Sp***: for any spectrum X, there is a unique map $X \to *$ and a unique map $* \to X$ in* Ho **Sp**.

Proof. The proof of Proposition 3.2.2 works perfectly well for empty coproducts and empty products, both of which give $*$. □

Theorem 3.2.5. *The stable homotopy category* Ho **Sp** *is **additive**:*

- *the sets $[X, Y]_s$ are abelian groups,*
- *the composition maps $[X, Y]_s \times [Y, Z]_s \to [X, Z]_s$ are bilinear,*
- Ho **Sp** *has all finite coproducts and products, and a zero object $*$, and*
- *the canonical map $X \vee Y \to X \times Y$ is an isomorphism.*

Proof. The statements about coproducts, products, and the zero object are already handled by the above results and Proposition 2.4.20.

To define the abelian group structure, for each spectrum X we pick another spectrum X' and a stable equivalence $X \simeq \Sigma X'$. For instance, X' could be ΩX, or $\mathrm{sh}^{-1} X$.

We also pick a map of spaces $p : S^1 \to S^1 \vee S^1$ that pinches the circle in the middle, or any other map whose degree is $(1, 1)$. By Exercise 2.7.10, this gives a "pinch map" in the stable homotopy category

$$\begin{aligned} S^1 \wedge X' &\longrightarrow (S^1 \vee S^1) \wedge X' \\ &\cong (S^1 \wedge X') \vee (S^1 \wedge X') \\ \Sigma X' &\longrightarrow \Sigma X' \vee \Sigma X' \\ X &\longrightarrow X \vee X. \end{aligned}$$

By abuse of notation, we also let $p : X \to X \vee X$ denote this pinch map. *This is a map in the homotopy category* Ho **Sp**, not an actual map of spectra.

Suppose $f, g \in [X, Y]_s$ are two morphisms in the homotopy category, in other words, zig-zags of morphisms of spectra. We can form a map $(f, g) : X \vee X \to Y$ in the homotopy category, because $\vee$ is the coproduct, by Proposition 3.2.2. Concretely, this map applies the zig-zags to each summand separately to end at $Y \vee Y$, then maps each of these to Y by id_Y.

We define the sum $(f+g) \in [X,Y]_s$ by composing (f,g) with p:

$$X \xrightarrow{p} X \vee X \xrightarrow{(f,g)} Y. \qquad X \xrightarrow{f+g} Y$$

The usual proof that $\pi_1(X)$ is a group applies here and shows that the zero map $X \to * \to Y$ is the identity for $+$, and that the inverse of a map f is formed by flipping the suspension coordinate of $\Sigma X' \simeq X$. See Exercise 3.6.10.

To show this rule is well-defined and abelian, it will be helpful to write it a different way. The diagram of spaces

$$\begin{array}{ccc} & & S^1 \vee S^1 \\ & \overset{p}{\nearrow} & \downarrow \subseteq \\ S^1 & \xrightarrow[(\mathrm{id}_{S^1},\mathrm{id}_{S^1})]{} & S^1 \times S^1 \end{array}$$

commutes up to homotopy. It follows using Lemma 3.1.15 that the diagram of zig-zags of maps of spectra

$$\begin{array}{ccc} & & X \vee X \\ & \overset{p}{\nearrow} & \downarrow \subseteq \\ X & \xrightarrow[(\mathrm{id}_X,\mathrm{id}_X)]{} & X \times X \end{array}$$

commutes in Ho **Sp**. Therefore our addition rule $f+g$ is equal to either route through the following commuting diagram in Ho **Sp**.

$$\begin{array}{ccccccc} & & X \vee X & \xrightarrow{f \vee g} & Y \vee Y & \xrightarrow{(\mathrm{id}_Y,\mathrm{id}_Y)} & Y \\ & & \downarrow \cong & & \downarrow \cong & & \\ X & \xrightarrow{(\mathrm{id}_X,\mathrm{id}_X)} & X \times X & \xrightarrow{f \times g} & Y \times Y & & \end{array} \tag{3.2.6}$$

It is a straightforward diagram-chase to check this is bilinear, in other words $h \circ (f+g) = (h \circ f) + (h \circ g)$, and similarly $(f+g) \circ h = (f \circ h) + (g \circ h)$; see Exercise 3.6.10.

To show that $+$ is commutative, we let $f \triangle g$ denote any rule for adding two maps together in Ho **Sp**, such that composition is bilinear. (It could even be the rule $+$ that we just defined.) Given two more maps $f', g' \colon X \rightrightarrows Y$, the map $(f \triangle f') + (g \triangle g')$ is given by the composite

$$\begin{array}{ccccccc} & & X \vee X & \xrightarrow{(f \triangle f') \vee (g \triangle g')} & Y \vee Y & \xrightarrow{(\mathrm{id}_Y,\mathrm{id}_Y)} & Y \\ & & \downarrow \sim & & \downarrow \sim & & \\ X & \xrightarrow{(\mathrm{id}_X,\mathrm{id}_X)} & X \times X & \xrightarrow{(f \triangle f') \times (g \triangle g')} & Y \times Y & & \end{array}$$

We have the equality $(f \triangle f') \vee (g \triangle g') = (f \vee g) \triangle (f' \vee g')$, because both give $f \triangle f'$ when restricted to the first copy of X, and $g \triangle g'$ when restricted to the second copy of X. By

the bilinearity of $\triangle$, we can compose this with the remaining maps of the diagram and see that the result is $(f+g)\triangle(f'+g')$. All together, this proves the interchange law

$$(f\triangle f')+(g\triangle g')=(f+g)\triangle(f'+g').$$

Therefore by the Eckmann-Hilton argument (Lemma 1.4.5), the two operations $+$ and $\triangle$ are equal to each other, and are abelian. □

The last step of the proof of Theorem 3.2.5 shows:

Proposition 3.2.7. *The abelian group structure in Theorem 3.2.5 is unique.*

Example 3.2.8. The isomorphism $[\mathbb{S},\mathbb{S}]_s \cong \mathbb{Z}$ is an isomorphism of abelian groups, because the rule from Theorem 3.2.5 for adding two maps $\mathbb{S}\to\mathbb{S}$ agrees with the usual rule for addition in $\pi_n(S^n)\cong\mathbb{Z}$.

Example 3.2.9. We can describe the addition $f+g$ in yet another way. For each spectrum Y, we pick another spectrum Y' and a stable equivalence $Y\simeq\Omega Y'$. Then the composition map $\Omega Y'\times\Omega Y'\to\Omega Y'$ gives a map in the stable homotopy category $Y\times Y\to Y$ that we call c. We add f and g by the rule

$$X\xrightarrow{(f,g)}Y\times Y\xrightarrow{c}Y.$$

To prove this agrees with the operation of Theorem 3.2.5, we could prove it is bilinear and then use Proposition 3.2.7, but it is a little faster to argue that the following square of spectra commutes:

$$\begin{array}{ccc} Y\vee Y & \xrightarrow{(\mathrm{id}_Y,\mathrm{id}_Y)} & Y \\ \cong\downarrow & \nearrow_{c} & \\ Y\times Y & & \end{array}$$

Sticking this onto the right-hand side of (3.2.6), we see that this rule agrees with the previous one.

Example 3.2.10. The category of of abelian groups **Ab** is also additive, as is the category ${}_R\mathbf{Mod}$ of left modules over a ring R. In both cases, the morphisms form abelian groups, and the finite coproducts and finite products are isomorphic to each other.

It is common to write finite coproducts in an additive category with the notation

$$X_1\oplus\ldots\oplus X_n,$$

also so we will sometimes use this notation when working in the stable homotopy category.[1] We can also denote maps between direct sums

$$X_1\oplus\ldots\oplus X_n\to Y_1\oplus\ldots\oplus Y_m$$

[1]Note that the point-set category of spectra, **Sp**, is not additive. So we have to continue to use the notation $\vee$ or $\times$ if we are not working in the homotopy category.

using matrix notation, every entry of the matrix being an element of one of the abelian groups $[X_j, Y_i]_s$. The composition of maps is given by matrix multiplication.

Example 3.2.11. $[\mathbb{S} \oplus \mathbb{S}, \mathbb{S} \oplus \mathbb{S}]_s$ is isomorphic to the abelian group of 2×2 matrices over $\mathbb{Z}$. The composition is by matrix multiplication. See also Exercise 3.6.12.

3.2.2 Computing the morphism sets $[X, Y]_s$

Recall from the beginning of Section 3.2 that we have bijections

$$[X, Y]_s \cong [X, Y]_h \text{ when } X \text{ is CW and } Y \text{ is an } \Omega\text{-spectrum, and}$$
$$[X, Y]_s \cong [X, Y]_e \text{ when } X \text{ is CW.}$$

These are bijections of abelian groups, if we write X as a suspension and use the usual formula for addition in $[\Sigma X', Y]_*$ to add two maps together. The description as eventually-defined maps $[X, Y]_e$ is useful for actually computing these sets, at least when X only has a few cells.

To move to larger examples, recall that for a based space A, the free spectrum $F_k A$ has the property that maps of spectra $F_k A \to Y$ are the same as maps of based spaces $A \to Y_k$, see Lemma 2.6.9. The same applies to homotopies, so we get

Lemma 3.2.12. *There is a canonical bijection* $[F_k A, Y]_h \cong [A, Y_k]_*$.

Corollary 3.2.13. *We get the following bijections:*

$$\begin{aligned}
[F_k A, Y]_s &\cong [A, Y_k]_* && \text{when } A \text{ is CW and } Y \text{ is an } \Omega\text{-spectrum,} \\
[F_k A, Y]_s &\cong \operatorname*{colim}_{n\to\infty} [\Sigma^n A, Y_{k+n}]_* && \text{when } A \text{ is finite CW and } Y \text{ is any spectrum, and} \\
[F_k A, F_\ell B]_s &\cong \operatorname*{colim}_{n\to\infty} [\Sigma^{n-k} A, \Sigma^{n-\ell} B]_* && \text{when } A \text{ is finite CW and } B \text{ is any space.}
\end{aligned}$$

Proof. The first bijection is because $[F_k A, Y]_s \cong [F_k A, Y]_h$ by Proposition 3.1.41. For the second bijection, we replace Y by the Ω-spectrum RY. By Exercise 1.7.32, we have

$$[A, RY_k]_* \cong \operatorname*{colim}_{n\to\infty} [A, \Omega^n Y_{k+n}]_* \cong \operatorname*{colim}_{n\to\infty} [\Sigma^n A, Y_{k+n}]_*.$$

Alternatively, we could use Proposition 3.1.41 and argue that because A has only finitely many cells, any eventually-defined map becomes actually defined at level n for sufficiently large n. The same argument gives the third bijection as well. □

Remark 3.2.14. Recall that a spectrum is **finite** if it is stably equivalent to $F_k A$ for a finite CW complex A. The subcategory of finite spectra $\mathrm{Ho}\,\mathbf{Sp}^{\mathrm{fin}} \subseteq \mathrm{Ho}\,\mathbf{Sp}$ is called the **Spanier-Whitehead category**. By Corollary 3.2.13, the morphisms of this category are stabilized maps of spaces, $\operatorname*{colim}_{n\to\infty} [\Sigma^{n-k} A, \Sigma^{n-\ell} B]_*$. See Proposition 4.2.12 for more details.

Recall from Example 2.1.9 that the d-dimensional sphere in spectra is

$$\mathbb{S}^d = \mathrm{sh}^d \mathbb{S} = \begin{cases} F_0 S^d \cong \Sigma^d \mathbb{S} & \text{when } d \geq 0, \\ F_{|d|} S^0 \simeq \Omega^{|d|} \mathbb{S} & \text{when } d \leq 0. \end{cases}$$

These are equivalent by Exercise 2.7.19.

Corollary 3.2.15. *For any spectrum X, there is a natural isomorphism*

$$[\mathbb{S}^k, X]_s \cong \operatorname*{colim}_{n\to\infty} \pi_{k+n}(X_n) = \pi_k(X).$$

This is an isomorphism of abelian groups, using the pinching definition of addition from Theorem 3.2.5. A category theorist would describe this result by saying that $\pi_k \colon \mathrm{Ho}\,\mathbf{Sp} \to \mathbf{Ab}$ is "co-represented" by $\mathbb{S}^k$.

Example 3.2.16. Note that for any X we have $[\mathbb{S}, X]_h = \pi_0(X_0)$. This is isomorphic to $\pi_0(X)$ if X is an Ω-spectrum. This is consistent with the claim that $[\mathbb{S}, X]_s \cong [\mathbb{S}, X]_h$ if $\mathbb{S}$ is CW and X is an Ω-spectrum.

Example 3.2.17. If X is any spectrum and G is an abelian group, then

$$[X, \Sigma^k HG]_s \cong H^k(X; G).$$

This is really a definition rather than a theorem, because we haven't defined the cohomology of spectra yet! We may do so now:

Definition 3.2.18. For any spectrum E, we define the E-cohomology of each spectrum X by the formula

$$E^k(X) := [X, \Sigma^k E]_s.$$

This generalizes Proposition 2.5.17 from spaces to spectra. Moreover, when X is a suspension spectrum of a well-based space A, this new definition of cohomology of spectra gives back the reduced cohomology of the space A. So for ordinary cohomology this gives:

$$[\Sigma^\infty A, \Sigma^k HG]_s \cong H^k(A, *; G).$$

Example 3.2.19. Since maps between finite wedges/products are given by products of mapping groups, we get

$$[\mathbb{S}^2 \vee \mathbb{S}^3, Y]_s \cong [\mathbb{S}^2, Y]_s \times [\mathbb{S}^3, Y]_s \cong \pi_2(Y) \times \pi_3(Y),$$

$$[X, Y \vee Z]_s \cong [X, Y \times Z]_s \cong [X, Y]_s \times [X, Z]_s.$$

Every spectrum is equivalent to a cellular spectrum, and every cellular spectrum is built by piecing together free spectra. So to go from Corollary 3.2.13 to any spectrum X, we only have to know how to glue these descriptions together. The following two results are the essential pieces to this process.

Proposition 3.2.20. *If (X, Y, Z, f, g, h) is a cofiber/fiber sequence of spectra and W is any fourth spectrum then there are long exact sequences*

$$\ldots \longrightarrow [W, X]_s \xrightarrow{f\circ -} [W, Y]_s \xrightarrow{g\circ -} [W, Z]_s \xrightarrow{\partial\circ -} [W, \Sigma X]_s \longrightarrow \ldots$$

$$\ldots \longrightarrow [\Sigma X, W]_s \xrightarrow{-\circ\partial} [Z, W]_s \xrightarrow{-\circ g} [Y, W]_s \xrightarrow{-\circ f} [X, W]_s \longrightarrow \ldots$$

Proof. For the first case, since X, Y, and Z form a fiber sequence, we can make Y and Z into Ω-spectra and replace X by the homotopy fiber, which is also an Ω-spectrum. Then we can make W a CW spectrum, so that $[-,-]_s$ is the same as $[-,-]_h$.

After these replacements, the proof is identical to the proof of the long exact sequence from Proposition 1.2.19: the sequence is exact at $[W, Y]_s$ because a nullhomotopy of the composite $W \to Y \to Z$ is exactly the data you need to extend the map to the homotopy fiber X.

For the second case, we instead make X and Y CW, replace Z by the homotopy cofiber, which is also CW, and make W an Ω-spectrum, so that $[-,-]_s$ is the same as $[-,-]_h$. Then we apply the proof of Proposition 1.2.13: the sequence is exact at $[Y, W]_s$ because a nullhomotopy of the composite $X \to Y \to W$ is exactly the data you need to extend the map to the homotopy cofiber Z. □

Proposition 3.2.21. *If X is the colimit of a sequence of relative cellular or CW spectra*

$$* \longrightarrow X^{(0)} \longrightarrow X^{(1)} \longrightarrow \ldots \longrightarrow \operatorname*{colim}_{n\to\infty} X^{(n)} = X,$$

and Y is another spectrum, then there is a short exact sequence

$$0 \longrightarrow \lim_{n\to\infty}{}^1\, [\Sigma X^{(n)}, Y]_s \longrightarrow [X, Y]_s \longrightarrow \lim_{n\to\infty} [X^{(n)}, Y]_s \longrightarrow 0.$$

In particular, if the maps $[X^{(n)}, Y]_s \to [X^{(n-1)}, Y]_s$ are surjective for sufficiently large n, then $[X, Y]_s \cong \lim_{n\to\infty} [X^{(n)}, Y]_s$.

Proof. This is left to Exercise 3.6.14. □

Example 3.2.22. If X is a spectrum and $E^*(-)$ is any reduced extraordinary cohomology theory, we may give a formula for the spectrum cohomology $E^k(X) = [X, \Sigma^k E]_s$ in terms of the levels of X:

$$0 \longrightarrow \lim_{n\to\infty}{}^1\, E^{k+n-1}(X_n) \longrightarrow E^k(X) \longrightarrow \lim_{n\to\infty} E^{k+n}(X_n) \longrightarrow 0.$$

This follows by applying Proposition 3.2.21 to the filtration of X in which we attach the cells one spectrum level at a time. Note the similarity to (1.5.7), except that the degrees of the cohomology groups are increasing as we increase the spectrum level, because we are using the suspension isomorphisms to go from each level to the next.

We also recall (Definition 2.6.37) that E-homology is also given in terms of the levels of X by the simpler formula

$$E_k(X) \cong \operatorname*{colim}_{n\to\infty} E_{k+n}(X_n).$$

Example 3.2.23. The previous example lets us compute maps between Eilenberg–Mac Lane spectra $[HG_1, \Sigma^k HG_2]_s$ as the stable cohomology of Eilenberg–Mac Lane spaces,

$$[HG_1, \Sigma^k HG_2]_s \cong \lim_{n\to\infty} H^{k+n}(K(G_1,n); G_2) \cong \lim_{n\to\infty} [K(G_1,n), K(G_2,k+n)]_*.$$

In particular, $[HG_1, HG_2]_s \cong \mathrm{Hom}(G_1, G_2)$, and the graded abelian group $[H\mathbb{Z}/2, \Sigma^k H\mathbb{Z}/2]_s$ is isomorphic to the Steenrod algebra $\mathscr{A}$. This is a graded algebra whose kth level is the abelian group of stable operations on mod 2 cohomology

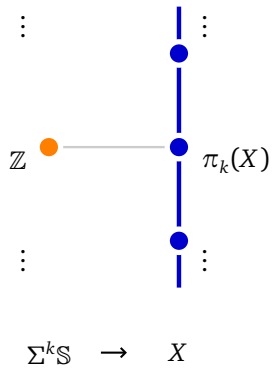
$$H^n(-;\mathbb{Z}/2) \Rightarrow H^{k+n}(-;\mathbb{Z}/2), \quad n \gg 0.$$

It is generated by operations Sq^k called the **Steenrod squares**, subject to relations called the Adem relations. See [MT68] for more details.

In a broad sense, the above results tell us that if we know the cells of X, and the homotopy groups of Y, then we have a good shot at computing $[X, Y]_s$. For instance, if X has a single cell, it is a shift of the sphere spectrum, so we get

$$[\Sigma^k \mathbb{S}, Y]_s \cong \pi_k(Y).$$

If X has just a few cells, we can use the cofiber sequence from Proposition 3.2.20 to patch together these copies of $\pi_k(Y)$ to get $[X, Y]_s$.



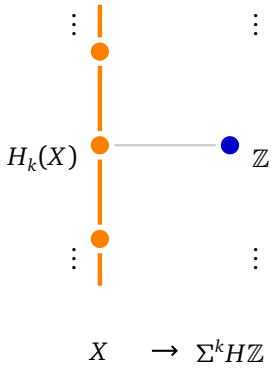
Example 3.2.24. The suspension spectrum $\Sigma^\infty \mathbb{RP}^2$ can be defined as the homotopy cofiber of the degree two map $\mathbb{S}^1 \to \mathbb{S}^1$. We therefore get a long exact sequence

$$\ldots \longrightarrow \pi_2(W) \xrightarrow{2} \pi_2(W) \longrightarrow [\Sigma^\infty \mathbb{RP}^2, W]_s \longrightarrow \pi_1(W) \xrightarrow{2} \pi_1(W) \longrightarrow \ldots$$

On the other hand, suppose Y has a single nonzero homotopy group.

Lemma 3.2.25. *If $\pi_n(Y) = 0$ for $n \neq k$, then $[X, Y]_s \cong H^k(X; \pi_k(Y))$.*

Proof. Y is stably equivalent to the Eilenberg–Mac Lane spectrum $\Sigma^k H(\pi_k(Y))$ by Exercise 2.7.39. The conclusion follows from the definition in Example 3.2.17. □



For instance,

$$[X, \Sigma^k H\mathbb{Z}]_s \cong H^k(X).$$

Using the Postnikov tower of Y from Definition 2.6.34, we can stitch these results together to get $[X, Y]_s$ if Y has just a few nonzero homotopy groups.

Both of these approaches can be further formalized into a spectral sequence. In fact, they both give the same spectral sequence. It starts from $H^*(X; \pi_*(Y))$ and converges to $[X, Y]_s$. See **??** for the construction. An important special case is when X is a suspension spectrum, where this is called the cohomology Atiyah-Hirzebruch spectral sequence, see **??**.

Intuitively, this means that the maps from X to Y are generated by the levels in which X has homology *and* Y has homotopy. To be more visual, think of each nonzero homotopy group of Y as a metal bar, and each cell or homology group of X as a plastic hook that can grab onto that bar, creating elements in $[X, Y]_s$ by the formula in Lemma 3.2.25. If there is only one degree where this happens, then this formula describes the set $[X, Y]_s$ completely. If not, then there may be some cancellation coming from the exact sequences in Proposition 3.2.20 and Proposition 3.2.21.

We illustrate this in Figure 3.2.26, using orange dots to represent the homology of X (the plastic hooks), and blue dots to represent the homotopy of Y (the metal bars). When there's only one degree where a hook can grab a bar, the maps $X \to Y$ in the stable homotopy category are given by the formula in Lemma 3.2.25. When there's more than one degree, the answer is more complicated.

3.2.3 Symmetric monoidal structure

The stable homotopy category Ho **Sp** is an example of a symmetric monoidal category. This will be discussed further in Chapter 4. For now, we'll just mention that there is a product on this category called the smash product, $\wedge\colon \mathrm{Ho}\,\mathbf{Sp} \times \mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}$. You can get a rough feel for what the smash product does from the following facts:

- The smash product preserves wedge sums, cofiber sequences, homotopy pushouts, and sequential compositions in each variable separately. So if Y is built out of cells, then $X \wedge Y$ is built in the same way out of X smashed with each of those cells.
- We have $\mathbb{S}^m \wedge \mathbb{S}^n \cong \mathbb{S}^{m+n}$. More generally, each m-cell in X and n-cell in Y becomes

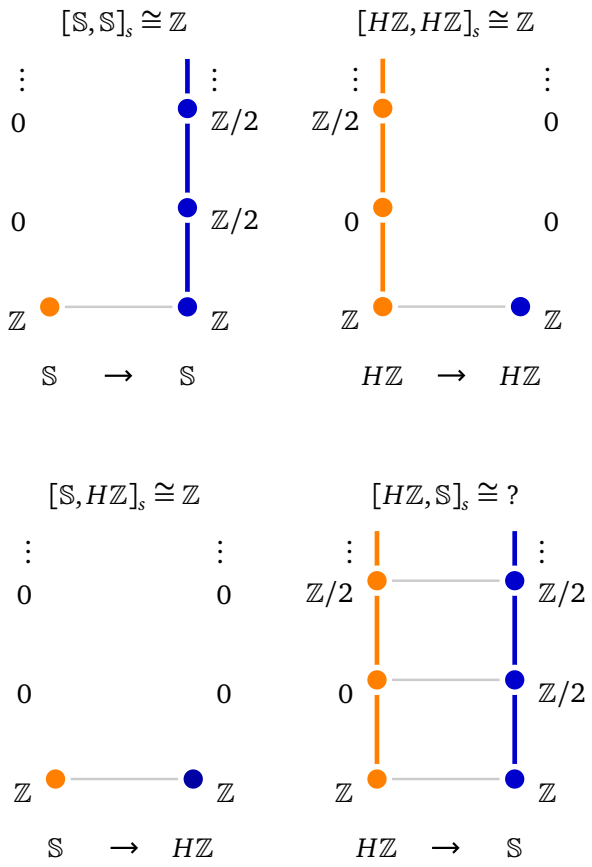


Figure 3.2.26

an $(m+n)$-cell in $X \wedge Y$. Morally, this is enough to tell you the homotopy type of the smash product of every pair of spectra.

- We have

$$X \wedge \mathbb{S} \cong X, \qquad X \wedge Y \cong Y \wedge X, \qquad (X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z).$$

- The map induced on morphisms by the smash product is bilinear, so it induces maps out of the tensor product

$$[X,Y]_s \otimes [W,Z]_s \longrightarrow [X \wedge W, Y \wedge Z]_s.$$

In particular, the smash product puts a ring structure on $[\mathbb{S},\mathbb{S}]_s \cong \mathbb{Z}$, that is the usual ring structure on the integers. We'll develop these and many more properties in Chapter 4.

3.3 The E-local stable homotopy category

We have finished introducing the stable homotopy category Ho **Sp**: its objects are spectra, and its morphisms are maps of spectra that have been "localized." That is, we have inverted the stable homotopy equivalences, turning them into isomorphisms. The morphisms in the resulting category are called $[X, Y]_s$, and they have some very pleasant properties. For instance, each of the sets $[X, Y]_s$ is now an abelian group, and the composition is bilinear.

However, outside of a few special cases, it is very difficult to compute the groups $[X, Y]_s$. For instance, the groups $[\mathbb{S}^n, \mathbb{S}^0]_s$ are the stable homotopy groups of spheres $\pi_n(\mathbb{S})$, and computing these is an open problem!

When a problem is impossible to solve, we usually proceed by making it simpler. In the case of Ho **Sp**, we can do this by localizing again, turning even more of the morphsims into isomorphisms. Now, which morphisms should we invert? If we choose them at random, it'll be hard to say anything about the resulting category. But if we're smart and pick a well-behaved set of morphisms, then our life gets much easier.

The class of morphisms that should we invert is called the "E-local equivalences" w_E. The resulting category $\mathrm{Ho}\,\mathbf{Sp}_E = \mathbf{Sp}\langle w_E^{-1}\rangle$ is called the E-local stable homotopy category, or just the E-local category for short. (So for instance if you've ever heard someone discuss the $K(n)$-local category, this is what they are talking about.) In many cases the E-local stable homotopy category $\mathrm{Ho}\,\mathbf{Sp}_E$ is completely understood. In others, it is the subject of active research, bringing us closer to Ho **Sp** by working one step at a time.

3.3.1 Local equivalences and local spectra

Let E be any spectrum. Let $E_*(-)$ denote the corresponding extraordinary homology theory, which is defined on spectra X by the formula from Definition 2.6.37:

$$E_k(X) = \pi_k(E \wedge X) \cong \operatorname*{colim}_{n\to\infty} E_{k+n}(X_n).$$

Definition 3.3.1. A map of spectra $X \to Y$ is an **E-local equivalence** if it induces an isomorphism $E_*(X) \cong E_*(Y)$. Let w_E denote the class of morphisms in **Sp** that are E-local equivalences.

The **E-local stable homotopy category** is constructed just like the stable homotopy category in Definition 3.1.1, except that we invert all of the E-local equivalences:

$$\mathrm{Ho}\,\mathbf{Sp}_E := \mathbf{Sp}\langle w_E^{-1}\rangle \cong \mathrm{Ho}\,\mathbf{Sp}\langle w_E^{-1}\rangle.$$

Example 3.3.2. Think through the following simple examples.

- An $\mathbb{S}$-local equivalence is just a stable equivalence, so $\mathrm{Ho}\,\mathbf{Sp}_{\mathbb{S}} = \mathrm{Ho}\,\mathbf{Sp}$.

- If $*$ is the zero spectrum, then a $*$-local equivalence is any map of spectra whatsoever. We can compute from this that the local category $\mathrm{Ho}\,\mathbf{Sp}_*$ has all objects isomorphic, and only the zero morphism between any pair of objects. In other words, it is equivalent to the one-point category (which we also denote $*$).
- A map of spectra is an $\mathbb{S}$-local equivalence iff it is an $\mathbb{S} \vee \mathbb{S}$-local equivalence. More generally, a map is an E-local equivalence iff it is an $E \vee E$-local equivalence. (Can you see why?) This illustrates that different choices of E will sometimes give the same local category $\mathrm{Ho}\,\mathbf{Sp}_E$.
- Let $\mathbb{S}_{\mathbb{Q}}$ be the rational sphere from Example 2.5.31, which is an Eilenberg–Mac Lane spectrum $H\mathbb{Q}$. An $\mathbb{S}_{\mathbb{Q}}$-local equivalence is any map that gives an equivalence on the rational stable homotopy groups $\pi_*(-) \otimes \mathbb{Q}$, which are the same thing as the rational homology groups $H_*(-;\mathbb{Q})$. Such a map is called a **rational equivalence** of spectra. We call its local category $\mathrm{Ho}\,\mathbf{Sp}_{\mathbb{Q}}$, the rational stable homotopy category.

Now you might expect that we're going to have to redo the last two sections for this new category $\mathrm{Ho}\,\mathbf{Sp}_E$, proving the fundamental properties one step at a time. But a miracle occurs: this new category $\mathrm{Ho}\,\mathbf{Sp}_E$ sits *inside* the original category $\mathrm{Ho}\,\mathbf{Sp}$, and so it inherits most of its properties for free! The subcategory that we want consists of the "E-local" spectra:

Definition 3.3.3. A spectrum Z is **E-local** if for every E-local equivalence $f : X \to Y$, composition with f gives a bijection on the set of maps in the stable homotopy category[2]

$$[Y, Z]_s \xrightarrow{\cong} [X, Z]_s.$$

Think of an E-local spectrum as one that can't tell the difference between E-local equivalences and stable equivalences. They look the same!

Example 3.3.4. All spectra are $\mathbb{S}$-local. The only $*$-local spectra are the ones that are weakly contractible (Example 2.1.12).

Lemma 3.3.5. *A spectrum Z is $H\mathbb{Q}$-local iff the homotopy groups $\pi_*(Z)$ are all $\mathbb{Q}$-modules. We call such a spectrum a **rational spectrum**.*

Proof. If Z is $H\mathbb{Q}$-local, then taking $X \to Y$ to be the map $\mathbb{S}^k \xrightarrow{n} \mathbb{S}^k$, we see that the map $\pi_k(Z) \xrightarrow{n} \pi_k(Z)$ is an isomorphism for all $k \in \mathbb{Z}$ and $n > 0$. Therefore the abelian groups $\pi_k(Z)$ are uniquely divisible, so they must be rational vector spaces in a unique way.

To go the other way we will have to use model categories. We will see in Example 5.7.3 that the rational equivalences $X \to Y$ are "generated" by maps of the form $\mathbb{S}^k \xrightarrow{n} \mathbb{S}^k$, so

[2] Some authors also require that E be an Ω-spectrum. We prefer to keep that requirement out, because it makes the definition invariant under stable equivalence.

that if $[-, Z]_s$ sends each of these maps to a bijection, then it must send every rational equivalence to a bijection. □

Lemma 3.3.6. *Between E-local spectra, a map $W \to Z$ is an E-local equivalence iff it is a stable equivalence.*

Proof. This follows from the definition of an E-local spectrum, and is left to Exercise 3.6.20. □

So in particular, a map between rational spectra is a stable equivalence whenever it is a rational equivalence.

The theory of E-local spectra took off in the 1970s when Bousfield was able to prove the following general result:[3]

Theorem 3.3.7 (Bousfield)**.** *There exists a functor $L_E\colon \mathbf{Sp} \to \mathbf{Sp}$ that sends every spectrum X to an E-local spectrum $L_E X$, and a natural transformation $X \to L_E X$ that is always an E-local equivalence.*

Example 3.3.8. When $E = \mathbb{S}_{\mathbb{Q}} \simeq H\mathbb{Q}$, then the localization $L_{H\mathbb{Q}}$ can be constructed as the mapping telescope from Example 2.5.34 or Exercise 3.6.35. We sometimes abbreviate this as $L_{\mathbb{Q}}$:

$$L_{\mathbb{Q}}X = L_{H\mathbb{Q}}X = \operatorname*{hocolim}_{n\to\infty}\left(X \xrightarrow{2} X \xrightarrow{3} X \xrightarrow{4} X \dots\right)$$

You can check directly that there is a natural map $X \to L_{H\mathbb{Q}}X$ that is a rational equivalence, and that $L_{H\mathbb{Q}}X$ has rational homotopy groups, so it fits the definition of $L_{H\mathbb{Q}}$.

For a general spectrum E, we give a proof of Theorem 3.3.7 in Section 5.7, using the technology of model categories. In this section we give some examples and explore some consequences, to give a feel for why this result is so important, before we go through the trouble of proving it carefully.

Corollary 3.3.9. *The E-localization is unique up to weak equivalence: if L_E and L'_E are any two functors with the property described in Theorem 3.3.7, then there is a zig-zag of natural weak equivalences from L_E to L'_E.*

Proof. Left to Exercise 3.6.20. □

Corollary 3.3.10. *The composite of the two functors*

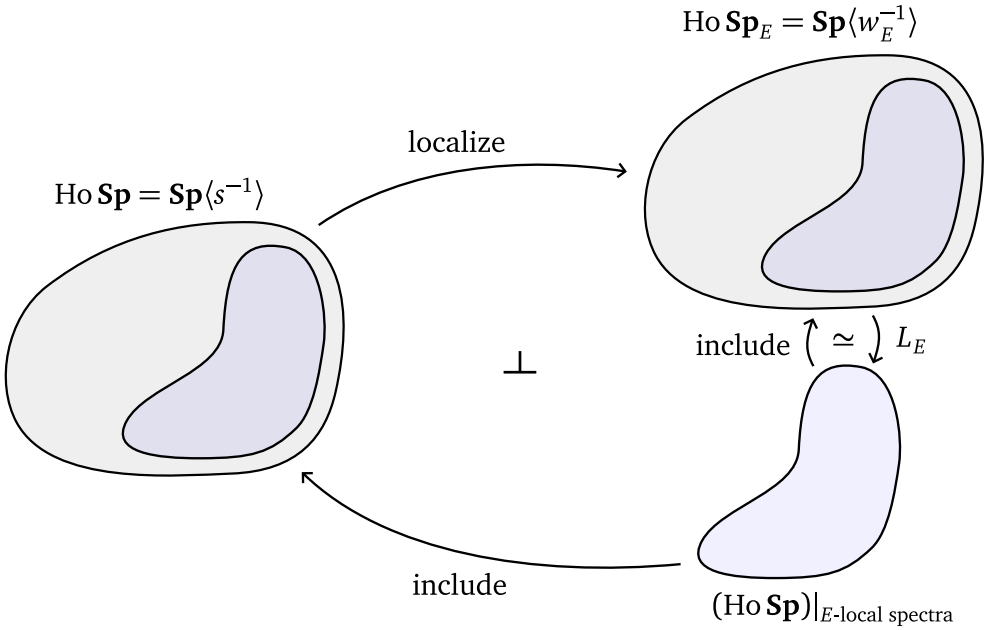
$$(\operatorname{Ho}\mathbf{Sp})|_{E\text{-local spectra}} \xrightarrow{\text{include}} \operatorname{Ho}\mathbf{Sp} = \mathbf{Sp}\langle s^{-1}\rangle \xrightarrow{\delta} \mathbf{Sp}\langle w_E^{-1}\rangle$$

is an equivalence of categories.

[3] This was known earlier for some particular choices of E, but Bousfield was the one who figured out how to get it to work for any E. This was one of the first big applications of model categories in homotopy theory, a subject that we'll return to in Chapter 5.

Proof. The inverse functor is given by L_E. It is a nice exercise to prove this is an inverse using Theorem 3.3.7, and is left to Exercise 3.6.20. □

In other words, $\mathrm{Ho}\,\mathbf{Sp}|_{E\text{-local spectra}}$ and $\mathbf{Sp}\langle w_E^{-1}\rangle$ have the same objects up to isomorphism, and the same morphisms between those objects. So we are entitled to call either one the E-local stable homotopy category $\mathrm{Ho}\,\mathbf{Sp}_E$. In fact, a little more is true: the maps between these two (equivalent) categories and the original stable homotopy category $\mathrm{Ho}\,\mathbf{Sp}$ form an adjunction, as illustrated below. This is an example of a "reflexive localization" – see Exercise 3.6.21 for more details.



This is remarkable. We can get the same E-local category by taking either a *subcategory* of $\mathrm{Ho}\,\mathbf{Sp}$, or a *localization* $\mathrm{Ho}\,\mathbf{Sp}\langle w_E^{-1}\rangle$. By analogy, when A is a retract of the topological space X, we can get A by restricting to a subspace, or by taking the quotient along the retraction $X \to A$. There are two ways to get there!

Example 3.3.11. The category $\mathbf{Sp}\langle w_{\mathbb{Q}}^{-1}\rangle$ consists of *all* spectra, and the morphisms are zig-zags in which the backwards maps are rational equivalences. The category $\mathrm{Ho}\,\mathbf{Sp}|_{H\mathbb{Q}\text{-local spectra}}$ consists only of *rational* spectra, and the morphisms are zig-zags in which any spectrum can appear (not just rational spectra), and the backwards maps are stable equivalences. (Although in fact you can turn any such zig-zag into another zig-zag consisting only of rational spectra, by applying $L_{\mathbb{Q}}$, as in the proof of Proposition 3.1.27.)

These two categories don't have the same objects, but they have the same *isomorphism* classes of objects, and the same morphisms between each pair of objects. That is the sense in which they are both giving the same "rational stable homotopy category."

3.3.2 Computing the morphism sets $[X, Y]_E$

The entire point of defining the E-local stable homotopy category is that the morphism sets $[X, Y]_E$ should be easier to compute than the morphism sets $[X, Y]_s$. So... how do we compute them?

In this section we briefly give the analogues of the results from Section 3.2.2. The idea is that if you know about the cohomology of X and the homotopy of $L_E Y$, then you have a good shot at getting the set $[X, Y]_E$. As an application, we will compute the morphisms in the rational stable homotopy category $\mathrm{Ho}\,\mathbf{Sp}_{\mathbb{Q}}$.

Proposition 3.3.12. *There are bijections that are natural in X and Y*

$$[X, Y]_E \cong [L_E X, L_E Y]_s \cong [X, L_E Y]_s,$$

so that the morphisms $[X, Y]_E$ can always be computed as the morphisms from X into the localization $L_E Y$.

Proof. The first bijection is from Corollary 3.3.10. The second bijection is from the definition of an E-local spectrum (Definition 3.3.3). □

Proposition 3.3.13. *Suspension Σ and loopspace Ω define functors $\mathrm{Ho}\,\mathbf{Sp}_E \to \mathrm{Ho}\,\mathbf{Sp}_E$ that are inverses up to natural isomorphism.*

Proof. You can prove this by either observing that the maps $X \to \Omega\Sigma X$ and $\Sigma\Omega X \to X$ are stable equivalences and therefore E-local equivalences, or by observing that Σ and Ω preserve the E-local spectra. □

Lemma 3.3.14. *Any finite product or homotopy pullback of E-local spectra is again E-local. Any infinite product of E-local Ω-spectra is an E-local Ω-spectrum.*

Proof. This follows from the fact that $[X, -]_s$ preserves finite products, preserves infinite products of Ω-spectra, and sends homotopy pullback squares to Mayer–Vietoris long exact sequences by Proposition 3.2.20 and the technique of Exercise 1.7.22. □

Lemma 3.3.15. *The E-localization functor L_E preserves homotopy pushout/pullback squares of spectra. It also preserves finite coproducts/products of spectra and cofiber/fiber sequences of spectra.*

Proof. Given a homotopy pullback $B \times^h_C D$, we get a map to the pullback of the localizations

$$B \times^h_C D \longrightarrow (L_E B) \times^h_{L_E C} (L_E D), \tag{3.3.16}$$

see Exercise 3.6.28 for more details. Since each of the maps $B \to L_E B$, $C \to L_E C$, and $D \to L_E D$ is an isomorphism on E-homology, so is the map (3.3.16) by Mayer–Vietoris

and the 5-lemma. Therefore the map (3.3.16) is an E-local equivalence. But the target is E-local by Lemma 3.3.14. It follows that the target must be equivalent to the E-localization of the source, by Exercise 3.6.20(b). Therefore the E-localization of the pullback is in fact the pullback of the E-localizations.

The localization of zero is always zero, $L_E(*) \simeq *$, and so it follows from the pullback statement that L_E also preserves finite products. □

Proposition 3.3.17. *The E-local stable homotopy category has all coproducts and products.*

- *The coproduct of the spectra $X(\alpha)$ is the wedge sum $\bigvee_\alpha X(\alpha)$, just as in Proposition 3.2.2.*
- *The product of the spectra $X(\alpha)$ is the product $\prod_\alpha X(\alpha)$, provided that either the product is finite, or that every $X(\alpha)$ is an E-local Ω-spectrum.*

Proof. For the coproduct we just use the bijection $[X,Y]_E \cong [X, L_E Y]_s$ and the fact that the wedge sum was the coproduct in the stable homotopy category. Or, we follow the proof of Proposition 3.2.2, but when we replace the target Y by an Ω-spectrum RY, we replace it by the E-local Ω-spectrum $R(L_E Y)$ instead. For the product, either of the two assumptions allow us to replace the $X(\alpha)$ by E-local Ω-spectra without changing the maps $[Y, \prod_\alpha X(\alpha)]_E$. Then the product is again an E-local Ω-spectrum by Lemma 3.3.14, so that the proof of Proposition 3.2.2 applies again. □

Corollary 3.3.18. *The E-local stable homotopy category* $\mathrm{Ho}\,\mathbf{Sp}_E$ *is an additive category, with zero object $*$, and sum given by the wedge sum or product $X \vee Y \simeq X \times Y$.*

Corollary 3.3.19. *If (X,Y,Z,f,g,h) is a cofiber/fiber sequence of spectra and W is any fourth spectrum then there are long exact sequences*

$$\dots \longrightarrow [W,X]_E \xrightarrow{f\circ -} [W,Y]_E \xrightarrow{g\circ -} [W,Z]_E \xrightarrow{\partial\circ -} [W,\Sigma X]_E \longrightarrow \dots$$

$$\dots \longrightarrow [\Sigma X,W]_E \xrightarrow{-\circ\partial} [Z,W]_E \xrightarrow{-\circ g} [Y,W]_E \xrightarrow{-\circ f} [X,W]_E \longrightarrow \dots$$

Corollary 3.3.20. *If X is the colimit of a sequence of relative cellular or CW spectra*

$$* \longrightarrow X^{(0)} \longrightarrow X^{(1)} \longrightarrow \dots \longrightarrow \operatorname*{colim}_{n\to\infty} X^{(n)} = X,$$

and Y is another spectrum, then there is a short exact sequence

$$0 \longrightarrow \lim_{n\to\infty}{}^1\, [\Sigma X^{(n)}, Y]_E \longrightarrow [X,Y]_E \longrightarrow \lim_{n\to\infty} [X^{(n)},Y]_E \longrightarrow 0.$$

In particular, if the maps $[X^{(n)},Y]_E \to [X^{(n-1)},Y]_E$ are surjective for sufficiently large n, then $[X,Y]_E \cong \lim_{n\to\infty} [X^{(n)},Y]_E$.

In summary, we have all of the computational tools we need to get $[X, Y]_E$ from the basic building blocks of $H^*(X)$ and $\pi_*(L_E Y)$.

Example 3.3.21. Let's compute the maps in the rational stable homotopy category $[X, Y]_{\mathbb{Q}}$. The only ingredient we need is that the rational stable homotopy groups of spheres $\pi_n(\mathbb{S}) \otimes \mathbb{Q}$ are given by $\mathbb{Q}$ when $n = 0$, and 0 otherwise.

By Proposition 3.3.12, $[X, Y]_{\mathbb{Q}}$ is the same as $[X_{\mathbb{Q}}, Y_{\mathbb{Q}}]_s$. First we show that the rationalization $X_{\mathbb{Q}}$ must be equivalent to a wedge of rational spheres, $\bigvee_\alpha \mathbb{S}^{n_\alpha}_{\mathbb{Q}}$. Pick a basis for each rational vector space $\pi_n(X)$. These basis elements give a map from a wedge of spheres $\bigvee_\alpha \mathbb{S}^{n_\alpha} \to X$, which is an isomorphism on rational homotopy groups. Applying the rationalization functor, we get a commuting square

$$\begin{array}{ccccc} & & \bigvee_\alpha \mathbb{S}^{n_\alpha} & \longrightarrow & X \\ & & \downarrow & & \downarrow \\ \bigvee_\alpha \mathbb{S}^{n_\alpha}_{\mathbb{Q}} & \xrightarrow{\sim} & \left(\bigvee_\alpha \mathbb{S}^{n_\alpha}\right)_{\mathbb{Q}} & \xrightarrow{\sim} & X_{\mathbb{Q}}. \end{array}$$

The bottom-left horizontal map is a stable equivalence, by looking at the homotopy groups. The maps in the commuting square are all rational equivalences, but the two spectra on the bottom are rational (in other words $H\mathbb{Q}$-local), so by Lemma 3.3.6, the other bottom horizontal map is also a stable equivalence.

Now that we know that $X_{\mathbb{Q}}$ is a wedge of rational spheres, the maps $[X_{\mathbb{Q}}, Y_{\mathbb{Q}}]_s$ split apart as a product of copies of

$$[\mathbb{S}^{n_\alpha}_{\mathbb{Q}}, Y_{\mathbb{Q}}]_s \cong [\mathbb{S}^{n_\alpha}, Y_{\mathbb{Q}}]_s \cong \pi_{n_\alpha}(Y_{\mathbb{Q}}),$$

using the fact that $Y_{\mathbb{Q}}$ is $H\mathbb{Q}$-local. Taking the product of these maps for all values of α returns the abelian group

$$\mathrm{Hom}(\pi_*(X), \pi_*(Y)) = \prod_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{Q}}(\pi_n(X), \pi_n(Y)).$$

Let $\mathbf{GrVect}_{\mathbb{Q}}$ denote the category of $\mathbb{Z}$-graded rational vector spaces.

Theorem 3.3.22. *The functor* $\pi_*\colon \mathrm{Ho}\,\mathbf{Sp}_{\mathbb{Q}} \to \mathbf{GrVect}_{\mathbb{Q}}$ *is an equivalence of categories.*

Proof. We have already checked it is fully faithful. To see it is essentially surjective, we notice that every graded rational vector space is isomorphic to the image of some wedge of rational spheres $\mathbb{S}_{\mathbb{Q}}$ under this functor. □

In summary, rational spectra are nothing more than their homotopy groups, which are graded rational vector spaces.[4] In particular, we know all of the morphism sets

$$[\mathbb{S}^n, \mathbb{S}^0]_{\mathbb{Q}} = \begin{cases} \mathbb{Q} & \text{when } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Throwing out information made the computation much easier!

3.3.3 More examples of E-localization

In this subsection we give without proof some additional examples of localizations and their properties.

Example 3.3.23. Recall from Example 2.5.32 that the p-local sphere spectrum $\mathbb{S}_{(p)}$ is defined by the mapping telescope

$$L_{(p)}\mathbb{S} = L_{\mathbb{S}_{(p)}}\mathbb{S} = \operatorname*{hocolim}_{n\to\infty} \left(\mathbb{S} \xrightarrow{q_1} \mathbb{S} \xrightarrow{q_2} \mathbb{S} \xrightarrow{q_3} \mathbb{S} \dots\right)$$

in which the q_i are integers that are not divisible by p, and that contain every prime factor other than p infinitely many times. (There are several ways to do this, but they are all equivalent!) If we take $E = \mathbb{S}_{(p)}$, then its homology theory for spectra is just $\pi_*(X) \otimes \mathbb{Z}_{(p)}$, in other words stable homotopy but will all primes other than p inverted.

The $\mathbb{S}_{(p)}$-local spectra are usually just called **p-local spectra**. They are the spectra with the property that the canonical map $\pi_*(X) \otimes \mathbb{Z}_{(p)} \to \pi_*(X)$ is an isomorphism. Equivalently, for every prime $q \neq p$, the map that multiplies by q is an isomorphism on $\pi_*(X)$. The localization $L_{(p)}$ is the mapping telescope as before applied to X, as in Example 2.5.34:

$$L_{(p)}X = L_{\mathbb{S}_{(p)}}X = \operatorname*{hocolim}_{n\to\infty} \left(X \xrightarrow{q_1} X \xrightarrow{q_2} X \xrightarrow{q_3} X \dots\right)$$

Localizing at a prime p can really simplify some things. For instance, complex cobordism MU splits into copies of the Brown–Peterson spectrum BP after localizing at p (Example 2.5.40).

The p-local stable homotopy category $\mathrm{Ho}\,\mathbf{Sp}_{(p)}$ has all of the information in $\mathrm{Ho}\,\mathbf{Sp}$ at the prime p, so it is easier. However, it has *all* of the information at the prime p, and that's still a lot of information. If you knew the p-local stable homotopy category $\mathrm{Ho}\,\mathbf{Sp}_{(p)}$ for

[4] To be more precise, the *homotopy category* of rational spectra is nothing more than graded rational vector spaces. We warn the reader that this does not extend to more sophisticated structures. For instance, if we are interested in diagrams of rational spectra, or rational ring spectra, these structures cannot be defined purely in terms of the homotopy category, so they could have more information. And in fact they do: they correspond to diagrams or algebras in rational *chain complexes*, not just in rational vector spaces.

every prime p, you would know the stable homotopy category; see Theorem 3.3.25 below. So we don't completely understand the p-local stable homotopy category $\mathrm{Ho}\,\mathbf{Sp}_{(p)}$. (The argument we ran for $\mathrm{Ho}\,\mathbf{Sp}_{\mathbb{Q}}$ doesn't work because we don't know the groups $\pi_*(\mathbb{S})_{(p)}$!)

Example 3.3.24. Recall that the sphere spectrum mod p, $\mathbb{S}/p$, is the cofiber of the map $\mathbb{S} \xrightarrow{p} \mathbb{S}$. A $\mathbb{S}/p$-local equivalence is called a **p-adic equivalence** of spectra. It is a map that gives an equivalence on "mod p homotopy groups"

$$\pi_*(X;\mathbb{Z}/p) := \pi_*(X \wedge \mathbb{S}/p),$$

which are assembled from the kernel and cokernel of the map $\pi_*(X) \xrightarrow{p} \pi_*(X)$ as in Example 2.5.30.

The $\mathbb{S}/p$-localization is usually called the **p-completion** $L_{\mathbb{S}/p}X = X^\wedge_p$. The p-completion is, informally, the homotopy inverse limit

$$X^\wedge_p \simeq \operatorname*{holim}_{n\to\infty} X/(p^n).$$

More precisely, we can define it as in Example 2.5.34 as the cotensor

$$X^\wedge_p = \mathrm{sh}^n F(M(\mathbb{Z}/(p^\infty), n), X)$$

where $M(\mathbb{Z}/(p^\infty), n)$ is the Moore space for the Prüfer p-group $\mathbb{Z}/(p^\infty)$.

The p-completion is very similar to the p-localization, because both are trying to isolate what happens "at the prime p." They are both successful at doing this, although they are not quite the same as one another, as the following theorem illustrates.

Theorem 3.3.25 (Fracture theorems)**.** *For any spectrum X, the following commuting squares are homotopy pullback squares:*

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_{(p)} \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & \left(\prod_p X_{(p)}\right)_{\mathbb{Q}} \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & \prod_p X^\wedge_p \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & \left(\prod_p X^\wedge_p\right)_{\mathbb{Q}} \end{array} \qquad \begin{array}{ccc} X_{(p)} & \longrightarrow & X^\wedge_p \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & \left(X^\wedge_p\right)_{\mathbb{Q}} \end{array}$$

In the first two squares, the product is over all prime numbers p.

We don't prove these theorems here, because we don't use them and the proofs would take us too far afield. They follow by applying [MP12, 8.1.3, 13.1.4, and 13.1.4 again] to each level of the associated Ω-spectra.

So the p-localization and p-completion can both be used to reconstruct X by working "one prime at a time," but they aren't the same. Conceptually, the reason for this is that

they isolate the behavior of X at p in different ways. The p-localization approaches this information "from the left," by dividing out all primes that aren't p. The p-completion approaches this information "from the right," by modding out by successively larger powers of p and taking an inverse limit. This is why the p-completion $X_p^\wedge$ is one step further away than the p-localization $X_{(p)}$, in the sense described in Theorem 3.3.25.

On the other hand, in practice the p-completion $X_p^\wedge$ is the easier one to compute. This is because mod p information is much more computable then localized-at-p information. For instance, we will see that the Adams spectral sequence gives information about $X_p^\wedge$, rather than $X_{(p)}$. And this is not distressing, because the fracture square in Theorem 3.3.25 allows us to turn this back into information about $X_{(p)}$ if we really want to.

Example 3.3.26. Chromatic homotopy theory studies intermediate localizations between $\mathrm{Ho}\,\mathbf{Sp}_{(p)}$ and $\mathrm{Ho}\,\mathbf{Sp}_{\mathbb{Q}}$, with the goal of bridging from $\mathrm{Ho}\,\mathbf{Sp}_{\mathbb{Q}}$ (which we completely understand) to $\mathrm{Ho}\,\mathbf{Sp}_{(p)}$ (which is equivalent to understanding the entire stable category, if we can do it for every prime p).

The next stage after rationalization is to consider localization at

- p-local complex K-theory $KU_{(p)} = KU \wedge \mathbb{S}_{(p)}$, or
- the Adams summand L from Example 2.5.36, or
- the theory $E(1)$ described in Example 2.5.40, whose homotopy groups are $\mathbb{Z}_{(p)}[v_1, v_1^{-1}]$ with $|v_1| = 2(p-1)$.

These turn out to give the same localization, the $E(1)$-local stable homotopy category. It is actually fairly well understood. For instance, we know all of the homotopy groups of the sphere spectrum here (see e.g. [Gui24, 5.9 and 5.13]), and these groups capture interesting behavior in the full stable homotopy groups of spheres $\pi_*(\mathbb{S})$ that happen at a certain "periodicity."

We could also localize at $K(1) = L/p$, the first Morava K-theory. The resulting $K(1)$-local stable homotopy category is even easier than the $E(1)$-local category. In fact, it is the stepping stone that allows us to pass from rationalization to $E(1)$-localization, because there is a fracture square of the form

$$\begin{array}{ccc} L_{E(1)}X & \longrightarrow & L_{K(1)}X \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & \left(L_{K(1)}X\right)_{\mathbb{Q}} \end{array}$$

just as in Theorem 3.3.25. So one computes the $E(1)$-localization by doing the $K(1)$-localization first, because it is simpler. The story continues with a sequence of spectra

$E(2)$, $E(3)$, ... and $K(2)$, $K(3)$, ... with fracture squares [Gui24, 9.1]

$$\begin{array}{ccc} L_{E(n)}X & \longrightarrow & L_{K(n)}X \\ \downarrow & & \downarrow \\ L_{E(n-1)}X & \longrightarrow & L_{E(n-1)}(L_{K(n)}X), \end{array}$$

such that the inverse limit gives the p-localization,

$$X_{(p)} \simeq \operatorname*{holim}_{n\to\infty} L_{E(n)}X,$$

when X is a finite CW spectrum [Rav92, 7.5.7].

In principle, this means that computing the morphisms in the $K(n)$-local category, and in particular the homotopy groups of the $K(n)$-local sphere $L_{K(n)}\mathbb{S}$, is the key to computing successively better approximations to the stable homotopy category. Again, see e.g. [Rav86, Rav92, Lur10, Gui24] for much more in this direction.

3.4 Homotopy functors and derived functors

Now that we have proven the fundamental properties of $\mathrm{Ho}\,\mathbf{Sp}$, it's time to start defining more operations on this category, such as the wedge sum, the product, the cofiber, and so on. We have already developed these operations in the category $\mathbf{Sp}$, but we need two more concepts before we can lift them to the homotopy category: homotopy functors and derived functors.

3.4.1 Homotopy functors

We begin with a list of operations on the category of spectra $\mathbf{Sp}$ that pass immediately to functors on the homotopy category $\mathrm{Ho}\,\mathbf{Sp}$:

- homotopy groups $\pi_k\colon \mathrm{Ho}\,\mathbf{Sp} \to \mathbf{Ab}$,
- homology groups $E_k\colon \mathrm{Ho}\,\mathbf{Sp} \to \mathbf{Ab}$,
- cohomology groups $E^k\colon \mathrm{Ho}\,\mathbf{Sp}^{\mathrm{op}} \to \mathbf{Ab}$,
- suspension and loops $\Sigma, \Omega\colon \mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}$,
- wedge sums $\bigvee\colon \mathrm{Ho}\,\mathbf{Sp}^{\times A} \to \mathrm{Ho}\,\mathbf{Sp}$,
- finite products $\prod\colon \mathrm{Ho}\,\mathbf{Sp}^{\times n} \to \mathrm{Ho}\,\mathbf{Sp}$,

- infinite products of Ω-spectra $\prod\colon (\mathrm{Ho}\,\mathbf{Sp}^{\Omega})^{\times A} \to \mathrm{Ho}\,\mathbf{Sp}$,
- shift $\mathrm{sh}^d\colon \mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}$,
- passage to the E-local category $\mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}_E$,
- the E-localization L_E as a functor $\mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}$, or as a functor $\mathrm{Ho}\,\mathbf{Sp}_E \to \mathrm{Ho}\,\mathbf{Sp}$,
- homotopy cofiber and fiber $C, F\colon \mathrm{Ho}(\mathbf{Sp}^{(\bullet\to\bullet)}) \to \mathrm{Ho}\,\mathbf{Sp}$,
- smashing with a cell complex $-\wedge K\colon \mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}$, and
- maps from a finite complex $F(K,-)\colon \mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}$.

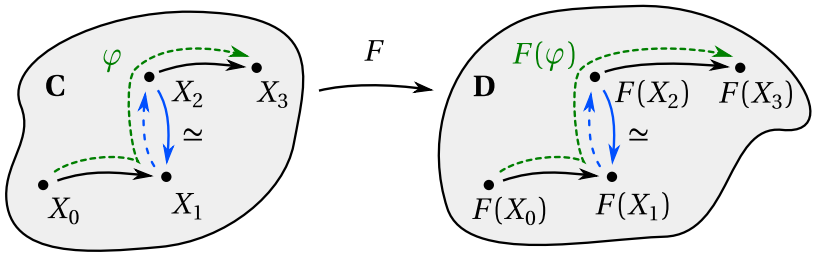
We define these using the universal property of the homotopy category from Theorem 3.1.10. Any functor $\mathbf{Sp} \to \mathbf{D}$ that sends stable equivalences to isomorphisms, must factor through $\mathrm{Ho}\,\mathbf{Sp}$. In most of the above cases, the target category is itself a homotopy category $\mathrm{Ho}\,\mathbf{D}$. When this happens, it's better to phrase the universal property this way:

Definition 3.4.1. If $\mathbf{C}$ and $\mathbf{D}$ are categories with weak equivalences, a functor $F\colon \mathbf{C} \to \mathbf{D}$ is **homotopical**, or a **homotopy functor**, if it takes each weak equivalence in $\mathbf{C}$ to a weak equivalence in $\mathbf{D}$.

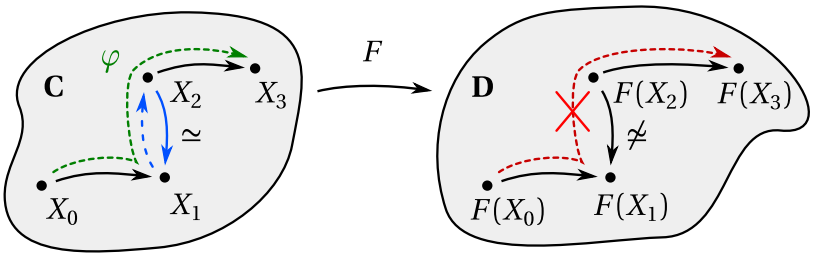
Lemma 3.4.2. *If F is a homotopy functor then it extends in a unique way to functor of homotopy categories making the following square commute.*

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\quad F \quad} & \mathbf{D} \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta} \\ \mathrm{Ho}\,\mathbf{C} & \xrightarrow[\mathrm{Ho}F\ (\text{or just } F)]{} & \mathrm{Ho}\,\mathbf{D} \end{array}$$

Proof. This follows immediately from the universal property of $\mathrm{Ho}\,\mathbf{C}$ in Theorem 3.1.10. The composite $\mathbf{C} \to \mathbf{D} \to \mathrm{Ho}\,\mathbf{D}$ sends weak equivalences to isomorphisms, so it factors through $\mathrm{Ho}\,\mathbf{C}$. Concretely, this factorization takes each zig-zag in $\mathrm{Ho}\,\mathbf{C}$ and applies F to every object and map in the zig-zag.



We need F to be homotopical for this to work. Otherwise, a backwards-pointing weak equivalence may turn into a backwards map that is not a weak equivalence, so we fail to get a valid zig-zag.



□

By a standard abuse of notation, we simply say F instead of $\mathrm{Ho}\,F$. In this case F is playing "double duty" by defining a functor both on $\mathbf{C}$ and on $\mathrm{Ho}\,\mathbf{C}$.

Example 3.4.3. The reduced suspension functor $\Sigma\colon \mathbf{Sp} \to \mathbf{Sp}$ preserves stable equivalences by Corollary 2.4.4. By Lemma 3.4.2, it therefore gives a functor $\mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}$. We already observed this in Proposition 3.2.1.

Example 3.4.4. Smashing with a CW complex K is a homotopy functor by Corollary 2.4.25. Taking maps from a finite complex K, $F(K,-)$, is a homotopy functor by Exercise 2.7.27. Therefore these both define functors on the stable homotopy category, $\mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}$.

Example 3.4.5. Let $\mathbf{Ch}$ be the category of unbounded ($\mathbb{Z}$-graded) chain complexes of abelian groups. The weak equivalences are the quasi-isomorphisms, maps $C \to D$ inducing isomorphisms on homology $H_*(C) \xrightarrow{\cong} H_*(D)$. The homotopy category $\mathrm{Ho}\,\mathbf{Ch}$ is also called the **derived category** $\mathscr{D}(\mathbb{Z})$.

Taking tensor product with the rationals

$$(-)\otimes\mathbb{Q}\colon \mathbf{Ch} \to \mathbf{Ch}$$

preserves quasi-isomorphisms, because $\mathbb{Q}$ is flat. Therefore it induces a functor

$$(-)\otimes\mathbb{Q}\colon \mathscr{D}(\mathbb{Z}) \to \mathscr{D}(\mathbb{Z}).$$

Of course, the same is true for the category of nonnegatively graded chain complexes $\mathbf{Ch}_{\geq 0}$ and its homotopy category $\mathscr{D}_{\geq 0}(\mathbb{Z})$.

Example 3.4.6. The wedge sum of spectra preserves equivalences by Exercise 2.7.30. But this isn't a functor from spectra to spectra – it is a functor from *pairs* of spectra to spectra.

Let $\mathbf{Sp}\times\mathbf{Sp}$ be the product category whose objects are pairs of spectra (X,Y) and whose maps are pairs of maps $(f\colon X\to X', g\colon Y\to Y')$. The weak equivalences are those pairs where both f and g are stable equivalences. By Exercise 3.6.17, inverting these gives the product of homotopy categories,

$$\mathrm{Ho}(\mathbf{Sp}\times\mathbf{Sp}) \cong (\mathrm{Ho}\,\mathbf{Sp})\times(\mathrm{Ho}\,\mathbf{Sp}).$$

The wedge sum is a functor $\vee\colon \mathbf{Sp} \times \mathbf{Sp} \to \mathbf{Sp}$, that sends all pairs of equivalences to equivalences. By Lemma 3.4.2 it therefore passes to a functor

$$\vee\colon \operatorname{Ho}\mathbf{Sp} \times \operatorname{Ho}\mathbf{Sp} \longrightarrow \operatorname{Ho}\mathbf{Sp}.$$

Alternatively, we could deduce this from its universal property (Proposition 3.2.2).

Example 3.4.7. If A is any set, the A-indexed wedge sum

$$\bigvee\colon \operatorname{Ho}\mathbf{Sp}^{\times A} \to \operatorname{Ho}\mathbf{Sp}$$

is defined on the homotopy category as in the previous example. For the product, however, we either have to restrict to A finite, or restrict the spectra to $\mathbf{Sp}^{\Omega}$. This ensures that the product preserves weak equivalences, by Exercise 2.7.26.

Example 3.4.8. Cohomology is a contravariant functor. So it is a functor on the opposite category, where the morphisms are reversed:

$$E^k\colon \mathbf{Sp}^{\mathrm{op}} \to \mathbf{Ab}.$$

An equivalence in $\mathbf{Sp}^{\mathrm{op}}$ is just the reverse of an equivalence in $\mathbf{Sp}$. By Exercise 3.6.18, inverting these gives the opposite of the homotopy category:

$$\operatorname{Ho}(\mathbf{Sp}^{\mathrm{op}}) = (\operatorname{Ho}\mathbf{Sp})^{\mathrm{op}}.$$

Therefore cohomology defines a functor $E^k\colon (\operatorname{Ho}\mathbf{Sp})^{\mathrm{op}} \to \mathbf{Ab}$.

Definition 3.4.9. For any small category $\mathbf{I}$, let $\mathbf{Sp}^{\mathbf{I}}$ be the category of $\mathbf{I}$-diagrams of spectra. A map of diagrams $X(-) \to Y(-)$ is a **pointwise equivalence** if for every $i \in \operatorname{ob}\mathbf{I}$ the map of spectra $X(i) \to Y(i)$ is a stable equivalence. Inverting the pointwise equivalences of diagrams gives a homotopy category $\operatorname{Ho}\mathbf{Sp}^{\mathbf{I}}$.

Warning 3.4.10. In general, this homotopy category of $\mathbf{I}$-diagrams of spectra is *not* the same as $\mathbf{I}$-diagrams in $\operatorname{Ho}\mathbf{Sp}$:

$$\operatorname{Ho}(\mathbf{Sp}^{\mathbf{I}}) \xrightarrow{\ncong} (\operatorname{Ho}\mathbf{Sp})^{\mathbf{I}}.$$

This only really happens if $\mathbf{I}$ is discrete: products of categories give products of homotopy categories (Exercise 3.6.17). Beyond this case, the above functor is not an isomorphism of categories, because not every diagram in $\operatorname{Ho}\mathbf{Sp}$ lifts to a diagram in $\mathbf{Sp}$, and if it does, the lift may not be unique.

We usually prefer to use $\operatorname{Ho}(\mathbf{Sp}^{\mathbf{I}})$. In other words, consider diagrams of actual spectra. Then, invert the equivalences between such diagrams.

Example 3.4.11. Taking $I = \{\bullet \to \bullet\}$, we get the category of arrows of spectra. An object is a map $X \to Y$ and a morphism is a commuting square. The homotopy cofiber C from Example 2.3.19 defines a functor

$$C\colon \mathbf{Sp}^{(\bullet\to\bullet)} \to \mathbf{Sp}.$$

By the long exact sequence of Lemma 2.4.11, any weak equivalence of arrows gives a weak equivalence of homotopy cofibers:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & Cf \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ X' & \xrightarrow{g} & Y' & \longrightarrow & Cg. \end{array}$$

Therefore C is a homotopy functor. By Lemma 3.4.2 it therefore passes to a functor

$$C\colon \operatorname{Ho}(\mathbf{Sp}^{(\bullet\to\bullet)}) \longrightarrow \operatorname{Ho}\mathbf{Sp}.$$

The same is true of the homotopy fiber F, using the long exact sequence of Lemma 2.4.10.

Remark 3.4.12. The homotopy cofiber Cf can also be defined if f is only a map in the homotopy category, see Exercise 3.6.19. However, it does not define a functor $(\operatorname{Ho}\mathbf{Sp})^{(\bullet\to\bullet)} \to \operatorname{Ho}\mathbf{Sp}$, only a function on the objects of the two categories. This makes $\operatorname{Ho}\mathbf{Sp}$ into a **triangulated category**, see e.g. [Nee01, HPS97]. When using this language, we call each sequence of maps of the form

$$X \xrightarrow{f} Y \longrightarrow Cf \longrightarrow \Sigma X$$

a **distinguished triangle**. Since cofiber and fiber sequences coincide, these can equivalently be described as sequences of the form

$$\Omega Z \longrightarrow Fp \longrightarrow W \xrightarrow{p} Z$$

modulo a funny issue about signs, see [May01] for more details.

This point of view is appealing for some applications, but it has serious shortcomings when we want to build anything more elaborate than a cone. We can't form more general kinds of homotopy colimits, for instance. It is often better to take diagrams of *actual* spectra, and *then* pass to the homotopy category of such diagrams. Use $\operatorname{Ho}(\mathbf{Sp}^{\mathbf{I}})$, not $(\operatorname{Ho}\mathbf{Sp})^{\mathbf{I}}$.

3.4.2 Derived functors

The list in Section 3.4.1 is missing several important operations. For instance, there ought to be a suspension spectrum functor

$$\Sigma^\infty\colon \operatorname{Ho}\mathbf{Top}_* \to \operatorname{Ho}\mathbf{Sp},$$

an infinite product functor

$$\prod\colon \operatorname{Ho}\mathbf{Sp}^{\times A} \to \operatorname{Ho}\mathbf{Sp},$$

and so on.

The problem is that these functors don't preserve equivalences. For instance, the infinite product $\prod\colon \mathbf{Sp}^{\times A} \to \mathbf{Sp}$ is does not send A-indexed families of stable equivalences to stable equivalences (see Proposition 3.2.2 and Exercise 2.7.26). Similarly, Σ^∞ does not send weak equivalences of based spaces to stable equivalences in general, see Exercise 3.6.22.

However, the infinite product does preserve stable equivalences, if we start with a family of Ω-spectra. And every spectrum is equivalent to an Ω-spectrum. So, if we first replace everything by an equivalent Ω-spectrum, then take the product

$$\prod_{\alpha\in A} RX(\alpha),$$

we get a well-defined functor on the homotopy category! We call this the **right-derived product**

$$\mathbb{R}(\textstyle\prod)\colon \operatorname{Ho}\mathbf{Sp}^{\times A} \longrightarrow \operatorname{Ho}\mathbf{Sp}.$$

To give another example, recall from Corollary 1.5.10 that the smash product $\wedge$ defines a homotopy functor on well-based spaces

$$\mathbf{Top}_*^{wb} \times \mathbf{Top}_*^{wb} \to \mathbf{Top}_*^{wb},$$

but not on all based spaces.

But this is fine! Every space X is equivalent to a space QX that is a CW complex, and therefore well-based. So, just replace our spaces by CW complexes, and *then* take their smash product. We call this the **left-derived smash product**:

$$X \wedge^{\mathbb{L}} Y := QX \wedge QY.$$

This preserves all weak homotopy equivalences, so it gives a functor on the homotopy category,

$$\wedge^{\mathbb{L}}\colon \operatorname{Ho}\mathbf{Top}_* \times \operatorname{Ho}\mathbf{Top}_* \longrightarrow \operatorname{Ho}\mathbf{Top}_*.$$

By the same token, the suspension spectrum Σ^∞ preserves equivalences if we make the input well-based (Example 3.4.23), so we get the **left-derived suspension spectrum**

$$\mathbb{L}\Sigma^\infty X := \Sigma^\infty(QX),$$

which gives a functor $\operatorname{Ho}\mathbf{Top}_* \to \operatorname{Ho}\mathbf{Sp}$.

If you have seen homological algebra, this is not so different from taking a projective resolution to compute Tor groups, or an injective resolution to compute Ext groups. Here's a way to put them under a common umbrella. (In this definition and throughout this section, we closely follow the philosophy of the landmark book [DHKS04].)

Definition 3.4.13. Suppose **C** is a category with a class of morphisms W called the weak equivalences. We say that W satisfies the **2-out-of-3 property** if for any two maps f and g whose composite $g \circ f$ is defined, if two of the maps f, g, and $g \circ f$ are in W, then so is the third. A **category with weak equivalences** is any category **C** with any class of morphisms W satisfying the 2-out-of-3 property.

Example 3.4.14 ([DHKS04, §40])**.** Isomorphisms always have the 2-out-of-3 property. More generally, if $F : \mathbf{C} \to \mathbf{D}$ is any functor, the collection of maps in **C** that go to isomorphisms in **D**, has the 2-out-of-3 property. Applying this to the functor that takes all of the homotopy groups of a spectrum

$$\pi_* : \mathbf{Sp} \to \prod_{k \in \mathbb{Z}} \mathbf{Ab},$$

we see that the stable equivalences in **Sp** satisfy 2-out-of-3.

Definition 3.4.15. Suppose **C** and **D** are categories with weak equivalences, and $F : \mathbf{C} \to \mathbf{D}$ is any functor.

A **left deformation** of F is

- a functor $Q : \mathbf{C} \to \mathbf{C}$,
- a natural transformation $Q \Rightarrow \mathrm{id}$, and
- a full subcategory $\mathbf{A} \subseteq \mathbf{C}$ containing $Q(\mathbf{C})$,

such that

- $QX \xrightarrow{\sim} X$ is a weak equivalence for all X in **C**, and
- F is homotopical (preserves weak equivalences) in the subcategory $\mathbf{A} \subseteq \mathbf{C}$.

We call $\mathbb{L}F = F \circ Q$ the **left-derived functor** of F.[5]

Similarly, a **right deformation** of F is

- a functor $R : \mathbf{C} \to \mathbf{C}$,
- a natural transformation $\mathrm{id} \Rightarrow R$, and

[5] In many sources "left-derived functor" is taken to mean the functor of homotopy categories that has the universal property given in Exercise 3.6.33. In this book, we push for the idea that "left-derived functor" should really mean a functor produced by a left deformation as above. This is a strengthening of the definition, but it occurs in all of the examples that we care about, and it is much more useful than the weaker definition. It has the rightful claim to the title of "left-derived functor."

- a full subcategory $\mathbf{A} \subseteq \mathbf{C}$ containing $R(\mathbf{C})$,

such that

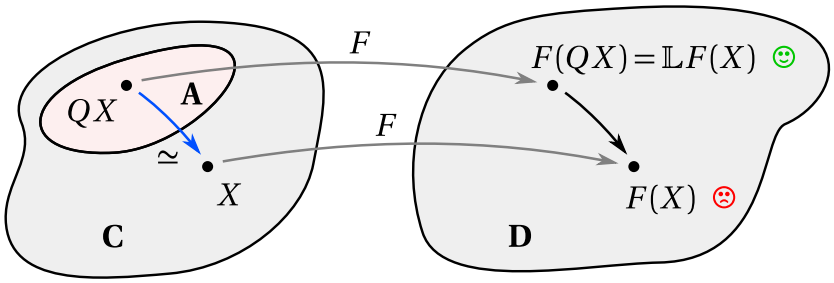
- $X \xrightarrow{\sim} RX$ is a weak equivalence for all X in $\mathbf{C}$, and
- F is homotopical in $\mathbf{A}$.

We call $\mathbb{R}F = F \circ R$ the **right-derived functor** of F.

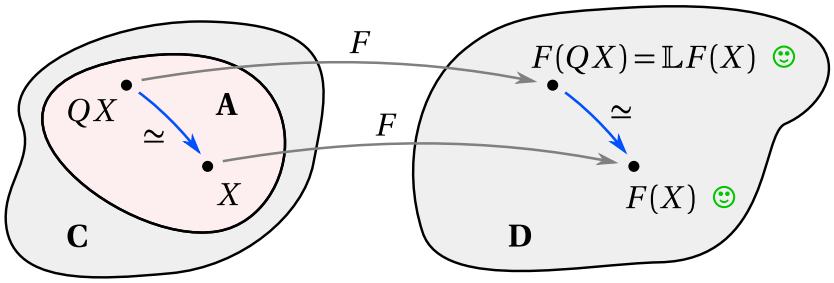
To summarize, the left-derived functor $\mathbb{L}F$ is formed by sampling the behavior of F on $\mathbf{A}$, and extending that behavior to the rest of $\mathbf{C}$. We replace each object X by something equivalent in $\mathbf{A}$, then apply F to it.

Remark 3.4.16. We call Q a "deformation" because it replaces each object X by an object QX that is almost the same as X. (It's almost the same because $QX \to X$ is a weak equivalence.) We call $\mathbb{L}F$ the "derived functor" because it is formed by taking F and changing the input *just a little bit*, so that the output is better behaved.

Think of the objects in $\mathbf{A}$ as the "good" objects, because that's where F preserves weak equivalences. So if $X \in \mathbf{A}$ then $F(X)$ is "good" as well. If $X \notin \mathbf{A}$ then $F(X)$ is possibly bad. (Though we're not sure. It's like a fruit that might be rotten, and you don't really want to bite into it to find out.)



We want to keep F as it is on the good objects, but change it on the bad objects, so that it is good everywhere. This is precisely what the left-derived functor $\mathbb{L}F$ does – it is "good" all the time, and is essentially the same as F when F is behaving well, but it fixes up the times that F behaves badly. The next two lemmas make this precise.



Lemma 3.4.17. *If $X \in \mathbf{A}$, then $\mathbb{L}F(X)$ is equivalent to $F(X)$.*

Proof. The map $QX \to X$ is a weak equivalence of objects in $\mathbf{A}$, so applying F gives a weak equivalence $F(QX) \xrightarrow{\sim} F(X)$. □

Lemma 3.4.18. *If F has a left- or right-derived functor, then the derived functor preserves weak equivalences. It therefore induces a map of homotopy categories* $\operatorname{Ho}\mathbf{C} \to \operatorname{Ho}\mathbf{D}$.

Proof. We do the left-derived case. Since Q is a functor and $QX \to X$ is a natural transformation, for any weak equivalence $X \xrightarrow{\sim} Y$, we get a commuting diagram

$$\begin{array}{ccc} QX & \xrightarrow{\sim} & X \\ \downarrow & & \downarrow \sim \\ QY & \xrightarrow{\sim} & Y. \end{array}$$

Applying the 2-out-of-3 property, twice, we see that $QX \to Y$ is a weak equivalence, and therefore $QX \to QY$ is a weak equivalence. (In fact, we've just shown that Q is homotopical – it preserves weak equivalences.)

Since QX and QY both lie in $\mathbf{A}$, $F(QX) \to F(QY)$ is a weak equivalence in $\mathbf{D}$. Therefore $\mathbb{L}F = F \circ Q$ preserves weak equivalences. □

Example 3.4.19. The simplest example is when F is already a homotopy functor. Then its left-derived functor is just F itself. We see this by taking

- $Q\colon \mathbf{C} \to \mathbf{C}$ to be the identity functor,
- $Q \Rightarrow \mathrm{id}$ to be the identity natural transformation, and
- $\mathbf{A} = \mathbf{C}$.

Similarly, the right-derived functor of F is just F itself. This preserves equivalences, so we get

$$F\colon \operatorname{Ho}\mathbf{C} \longrightarrow \operatorname{Ho}\mathbf{D}.$$

Example 3.4.20. For any set A, the A-indexed product of spectra $\prod\colon \mathbf{Sp}^{\times A} \to \mathbf{Sp}$ has a right deformation:

- $R\colon \mathbf{Sp}^{\times A} \to \mathbf{Sp}^{\times A}$ replaces every spectrum in the tuple with an Ω-spectrum, in other words it applies R from Proposition 2.2.9 to each slot,
- $\mathrm{id} \Rightarrow R$ is in each slot the natural map from $X(\alpha)$ to its replacement $RX(\alpha)$, and
- $\mathbf{A}$ is the category of A-tuples of Ω-spectra.

The map $(X(\alpha))_{\alpha\in A} \to (RX(\alpha))_{\alpha\in A}$ is an equivalence of A-tuples, because it is a stable equivalence in each slot. We know by Exercise 2.7.26 that the infinite product preserves equivalences on $\mathbf{A}$. Therefore the right-derived functor is

$$\mathbb{R}\left(\prod_{\alpha\in A}\right)X(\alpha) = \prod_{\alpha\in A} RX(\alpha).$$

We emphasize the two conclusions again: one, the right-derived product preserves equivalences, so it gives a map of homotopy categories

$$\mathbb{R}(\textstyle\prod)\colon \operatorname{Ho}\mathbf{Sp}^{\times A} \longrightarrow \operatorname{Ho}\mathbf{Sp}.$$

Two, the right-derived product is equivalent to the product if the product is nice. We only changed the product in the bad cases. So the right-derived product is exactly the functor we want.

3.4.3 Examples of left-derived functors

Example 3.4.21. The smash product of two spaces $\wedge\colon \mathbf{Top}_* \times \mathbf{Top}_* \to \mathbf{Top}_*$ has a left deformation:

- $Q\colon \mathbf{Top}_*\times\mathbf{Top}_* \to \mathbf{Top}_*\times\mathbf{Top}_*$ replaces every pair of spaces (X,Y) by CW complexes (QX,QY),
- $Q \Rightarrow \mathrm{id}$ is in each slot the natural map from QX to X, and
- $\mathbf{A}$ is the category of pairs of well-based spaces.

The map $(QX,QY)\to(X,Y)$ is an equivalence of pairs. We know by Corollary 1.5.10 that the smash product preserves equivalences on $\mathbf{A}$. Therefore the left-derived functor is

$$X \wedge^{\mathbb{L}} Y := QX \wedge QY.$$

This preserves all weak homotopy equivalences, so it gives a functor on the homotopy category,

$$\wedge^{\mathbb{L}}\colon \operatorname{Ho}\mathbf{Top}_* \times \operatorname{Ho}\mathbf{Top}_* \longrightarrow \operatorname{Ho}\mathbf{Top}_*.$$

Example 3.4.22. To give an analogous example from homological algebra, as in Example 3.4.5, let $\mathbf{Ch}$ be the category of unbounded chain complexes and let $\mathscr{D}(\mathbb{Z})$ be its homotopy category. The tensor product

$$\otimes\colon \mathbf{Ch}\times\mathbf{Ch}\to\mathbf{Ch}$$

does not preserve quasi-isomorphisms. However, it does preserve quasi-isomorphisms if at least one of the chain complexes has free abelian groups in every degree. Letting $QC \xrightarrow{\sim} C$ denote a replacement by a levelwise free complex, we therefore get a left-derived functor

$$C \otimes^{\mathbb{L}} D := QC \otimes^{\mathbb{L}} QD,$$

giving a product on the derived category

$$\otimes^{\mathbb{L}} \colon \mathscr{D}(\mathbb{Z}) \times \mathscr{D}(\mathbb{Z}) \to \mathscr{D}(\mathbb{Z}).$$

The homology of this chain complex has the same Künneth theorem as the homology of a product of spaces (Theorem 1.3.11). Taking a derived tensor product of one-term chain complexes A and B and taking homology gives the Tor groups $\mathrm{Tor}^i(A,B)$.

Example 3.4.23. The suspension spectrum functor $\Sigma^\infty : \mathbf{Top}_* \to \mathbf{Sp}$ has a left deformation. As in Example 3.4.21, we take Q to be CW replacement and $\mathbf{A} \subseteq \mathbf{Top}_*$ to be well-based spaces.

We have to check that Σ^∞ preserves equivalences on $\mathbf{A}$. If $K \to L$ is a weak equivalence of spaces, the map $K \wedge S^n \to L \wedge S^n$ is an equivalence by Corollary 1.5.10, so $\Sigma^\infty K \to \Sigma^\infty L$ is a level equivalence of spectra. Therefore Σ^∞ sends weak equivalences in $\mathbf{A}$ to stable equivalences in $\mathbf{Sp}$.

Therefore we get a left-derived suspension spectrum functor

$$\mathbb{L}\Sigma^\infty K := \Sigma^\infty QK,$$

which defines a map of homotopy categories

$$\mathbb{L}\Sigma^\infty \colon \mathrm{Ho}\,\mathbf{Top}_* \to \mathrm{Ho}\,\mathbf{Sp}.$$

Example 3.4.24. The tensor $\wedge \colon \mathbf{Top}_* \times \mathbf{Sp} \to \mathbf{Sp}$ is interesting. To derive $K \wedge X$, it is enough to replace K by a CW complex, by Corollary 2.4.25. On the other hand, we could decide to replace both K and X by CW objects. This gives two different left-derived functors that are equivalent to each other:

$$QK \wedge X \simeq QK \wedge QX.$$

We will see in Proposition 3.5.2 that this always happens – any two left-derived functors of F are equivalent.

We now collect the previous examples together, and add a few more that follow the same argument.

Proposition 3.4.25. *The following functors can be left-derived:*

- smash product $\wedge\colon \mathbf{Top}_* \times \mathbf{Top}_* \to \mathbf{Top}_*$,
- smash product with a well-based space $K \wedge (-)\colon \mathbf{Top}_* \to \mathbf{Top}_*$,
- reduced suspension $\Sigma\colon \mathbf{Top}_* \to \mathbf{Top}_*$,
- suspension spectrum $\Sigma^\infty\colon \mathbf{Top}_* \to \mathbf{Sp}$,
- free spectrum $F_n\colon \mathbf{Top}_* \to \mathbf{Sp}$,
- tensor $\wedge\colon \mathbf{Top}_* \times \mathbf{Sp} \to \mathbf{Sp}$, and
- handicrafted smash product $\wedge\colon \mathbf{Sp} \times \mathbf{Sp} \to \mathbf{Sp}$.

In every case, the functor can be derived by making the spaces CW complexes, and the spectra into CW spectra. See Exercise 3.6.23.

Example 3.4.26. Let $\mathbf{Top}^{\{\bullet\leftarrow\bullet\rightarrow\bullet\}}$ be the category of diagrams of the form

$$X \longleftarrow A \longrightarrow Y. \tag{3.4.27}$$

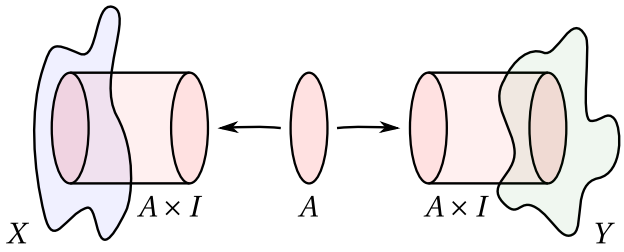
A weak equivalence of diagrams is a map that gives an equivalence on each space separately:

$$\begin{array}{ccccc} X & \longleftarrow & A & \longrightarrow & Y \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ X' & \longleftarrow & A' & \longrightarrow & Y'. \end{array}$$

The colimit, or pushout of the diagram, defines a functor $\mathbf{Top}^{\{\bullet\leftarrow\bullet\rightarrow\bullet\}} \to \mathbf{Top}$. We claim that its left-derived functor is the homotopy pushout $X \cup_A (A \times I) \cup_A Y$, from Definition 1.5.2.

To see this, let $\mathbf{A}$ be the subcategory of diagrams in which the maps $A \to X$ and $A \to Y$ are cofibrations. By Lemma 1.5.3 and Theorem 1.5.9, weak equivalences between diagrams of this form go to weak equivalences of pushouts. Let Q be the operation that replaces (3.4.27) with the diagram

$$X \cup_A (A \times [0,1/2]) \longleftarrow A \times \{1/2\} \longrightarrow (A \times [1/2,1]) \cup_A Y. \tag{3.4.28}$$



By Exercise 1.7.16, the maps of this diagram are cofibrations, so Q lands in $\mathbf{A}$. Collapsing away the cylinders gives a weak equivalence back to the original diagram (3.4.27).

Therefore, the left-derived pushout is the pushout of (3.4.28), which is the homotopy pushout $X \cup_A (A \times I) \cup_A Y$.

The same applies to pushouts of based spaces and spectra, and to the mapping telescope $\mathbf{Top}^{\{\bullet\to\bullet\to\dots\}} \to \mathbf{Top}$. See Exercises 3.6.26 and 3.6.27.

More generally, the colimit of any diagram of spaces can be left-derived, and the left-derived functor is equivalent to the homotopy colimit:

$$\mathbb{L}\text{colim} \simeq \text{hocolim}.$$

This isn't a theorem, it's a *definition*.

Definition 3.4.29. Suppose $\mathbf{I}$ is any category, and $\mathbf{C}$ is any category with weak equivalences. A map of diagrams in $\mathbf{C}^{\mathbf{I}}$ is a weak equivalence if it is a **pointwise equivalence** or **level equivalence**, meaning that $X(i) \to Y(i)$ is a weak equivalence in $\mathbf{C}$ for every $i \in \text{ob}\,\mathbf{I}$. The **homotopy colimit**

$$\text{hocolim}\colon \mathbf{C}^{\mathbf{I}} \to \mathbf{C}$$

is defined to be the left-derived functor of $\text{colim}\colon \mathbf{C}^{\mathbf{I}} \to \mathbf{C}$.

This does not exist in general. For starters, the colimit functor has to exist. But even then, it may fail to have a left-derived functor.

We will see later that the homotopy colimit exists when $\mathbf{I}$ is any category and $\mathbf{C}$ is the category of spaces, based spaces or spectra (Proposition 5.3.21, Example 5.4.4, and Exercise 5.8.40). Furthermore, for spectra, the homotopy colimit can be computed by taking the homotopy colimit in based spaces at each spectrum level. This is proven in Exercise 5.8.40, but the reader is invited to verify this in the special case of a coproduct, a pushout, and a sequential colimit. See Exercises 3.6.26 and 3.6.27.

Remark 3.4.30. The reduced suspension of spaces $\Sigma\colon \mathbf{Top}_* \to \mathbf{Top}_*$ does not preserve all equivalences, see Exercise 3.6.22. So it has to be left-derived. It is a bit of a miracle that the reduced suspension of spectra $\Sigma\colon \mathbf{Sp} \to \mathbf{Sp}$ preserves all equivalences (Corollary 2.4.4), and doesn't have to be left-derived. It has to do with the fact that the definition of π_* of a spectrum has Σ baked into it.

3.4.4 Examples of right-derived functors

We now turn to right-derived functors. We have already seen that the infinite product can be right-derived in Example 3.4.20.

Example 3.4.31. The evaluation functor $\text{ev}_n\colon \mathbf{Sp} \to \mathbf{Top}_*$ has a right deformation. We take R to be Ω-spectrum replacement and $\mathbf{A} \subseteq \mathbf{Sp}$ to be the Ω-spectra. Since a stable

equivalence of Ω-spectra is a level equivalence (Lemma 2.2.5), every stable equivalence in **A** gives an equivalence after ev_n.

Therefore we get a right-derived evaluation functor

$$\mathbb{R}\mathrm{ev}_n := \mathrm{ev}_n RX,$$

which defines a map of homotopy categories

$$\mathbb{R}\mathrm{ev}_n \colon \mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Top}_*.$$

In the special case $n = 0$, $\mathbb{R}\mathrm{ev}_0$ is also called Ω^∞ (Definition 2.2.11).

We can also right-derive the mapping space $\mathrm{Map}(X,Y)$ from Definition 2.3.12, by replacing X by a CW spectrum QX, and replacing Y by an Ω-spectrum RY. This really is a right deformation because the maps $QX \to X$ and $Y \to RY$ give a single map $(X,Y) \to (QX, RY)$ in the category $\mathbf{Sp}^{\mathrm{op}} \times \mathbf{Sp}$.

Lemma 3.4.32. *On the subcategory of pairs (X,Y) in which X is CW and Y is an Ω-spectrum, the mapping space* $\mathrm{Map}(-,-)$ *preserves all equivalences.*

Proof. The key observation is that

$$\pi_k(\mathrm{Map}(X,Y)) \cong \pi_0(\mathrm{Map}(S^k \wedge X, Y)) \cong [\Sigma^k X, Y]_h,$$

using Exercise 2.7.15. By Proposition 3.1.41, this sends stable equivalences $X \to X'$ and $Y \to Y'$ to bijections, so long as X and X' are CW and Y and Y' are Ω-spectra. (It is also enough to take π_k at the basepoint of $\mathrm{Map}(X,Y)$, because $\mathrm{Map}(X,Y) \simeq \Omega\mathrm{Map}(X, \mathrm{sh}^1 Y)$ is a loop space, so its homotopy groups at all basepoints are isomorphic.) □

Proposition 3.4.33. *The following functors can be right-derived:*

- evaluation $\mathrm{ev}_n \colon \mathbf{Sp} \to \mathbf{Top}_*$,
- mapping spaces $\mathrm{Map}(-,-) \colon \mathbf{Top}^{\mathrm{op}} \times \mathbf{Top} \to \mathbf{Top}$,
- based mapping spaces $\mathrm{Map}_*(-,-) \colon \mathbf{Top}_*^{\mathrm{op}} \times \mathbf{Top}_* \to \mathbf{Top}_*$,
- cotensors $F(-,-) \colon \mathbf{Top}_*^{\mathrm{op}} \times \mathbf{Sp} \to \mathbf{Sp}$,
- cotensor with a fixed CW complex, $F(K,-) \colon \mathbf{Sp} \to \mathbf{Sp}$, and
- mapping spaces $\mathrm{Map}_*(-,-) \colon \mathbf{Sp}^{\mathrm{op}} \times \mathbf{Sp} \to \mathbf{Sp}$.

In every case, we use Ω-spectra, and if we are taking maps out of some object then we make it a CW object.

Example 3.4.34. This example is the dual of Example 3.4.26. Let $\mathbf{Top}^{\{\bullet\to\bullet\leftarrow\bullet\}}$ be the category of diagrams of the form

$$X \longrightarrow B \longleftarrow Y. \tag{3.4.35}$$

A weak equivalence of diagrams is a map that gives an equivalence on each space separately:

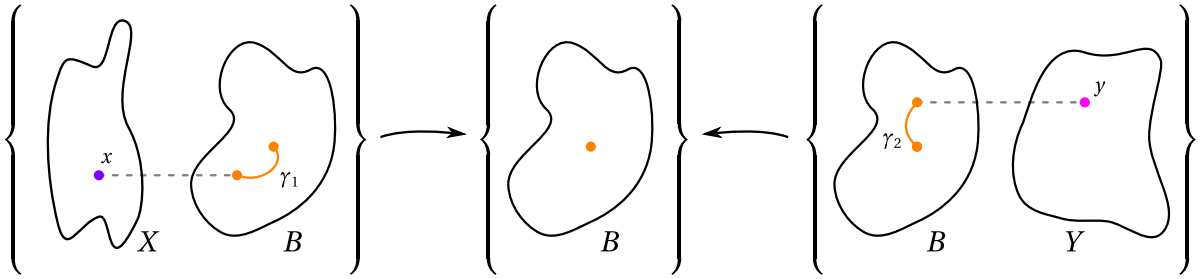
$$\begin{array}{ccccc} X & \longrightarrow & B & \longleftarrow & Y \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ X' & \longrightarrow & B' & \longleftarrow & Y'. \end{array}$$

The limit, or pullback of the diagram, defines a functor $\mathbf{Top}^{\{\bullet\to\bullet\leftarrow\bullet\}} \to \mathbf{Top}$. We claim that its right-derived functor is the homotopy pullback $X \times_B B^I \times_B Y$, from Definition 1.5.18.

To see this, let $\mathbf{A}$ be the subcategory of diagrams in which the maps $X \to B$ and $Y \to B$ are fibrations. By Lemma 1.5.19 and Theorem 1.5.24, weak equivalences between diagrams of this form go to weak equivalences of pullbacks. Let R be the operation that replaces (3.4.27) with the diagram

$$X \times_B B^{[0,1/2]} \longrightarrow B \longleftarrow B^{[1/2,1]} \times_B Y, \tag{3.4.36}$$

where the fiber products are taken over 0 and 1, and the maps to the B in the middle evaluate at $1/2$.



By Exercise 1.7.16, the maps of this diagram are fibrations, so R lands in $\mathbf{A}$. We get a weak equivalence from the original diagram (3.4.35) by including the subspace in which the paths are constant. Therefore, the right-derived pullback is the pullback of (3.4.36), which is the homotopy pullback $X \times_B B^I \times_B Y$

The same applies to pullbacks of based spaces and spectra, and to sequential limits. See Exercise 3.6.28.

More generally, the limit of a diagram of any shape can be right-derived, and the right-derived functor is equivalent to the homotopy limit:

$$\mathbb{R}\lim \simeq \operatorname{holim}.$$

As before, this isn't a theorem, it's a *definition*.

Definition 3.4.37. Suppose **I** is any category, and **C** is any category with weak equivalences. As in Definition 3.4.29, we take the weak equivalences in $\mathbf{C}^{\mathbf{I}}$ to be the pointwise equivalences of diagrams. The **homotopy limit**

$$\operatorname{holim}\colon \mathbf{C}^{\mathbf{I}} \to \mathbf{C}$$

is defined to be the right-derived functor of $\lim\colon \mathbf{C}^{\mathbf{I}} \to \mathbf{C}$.

Again, this does not exist in general, but we will prove later that it exists for any category **I** when **C** is the category of spaces, based spaces or spectra (Corollary 7.5.28 and Corollary 8.5.17). For spaces, this is constructed in Definition 7.5.27. For spectra, the homotopy limit is computed by making the terms of the diagram into Ω-spectra, and then taking the homotopy limit of the spaces at each spectrum level. See Exercise 3.6.28 for some special cases.

Example 3.4.38. There is a hom functor on unbounded chain complexes

$$\operatorname{Hom}(-,-)\colon \mathbf{Ch}^{\mathrm{op}} \times \mathbf{Ch} \to \mathbf{Ch}.$$

It can be right-derived by making the first chain complex projective, or the second chain complex injective. Either way, applying this to one-term chain complexes and taking homology of the result gives the Ext groups $\operatorname{Ext}^i(A,B)$.

Example 3.4.39. We have already seen that the identity functor $(\mathbf{Sp}, s) \to (\mathbf{Sp}, w_E)$ is homotopical, if s denotes the class of stable equivalences and w_E the E-local equivalences from Definition 3.3.1. This is consistent with the fact that we have a natural "projection" map from the stable category $\operatorname{Ho}\mathbf{Sp}$ to the E-local stable category $\operatorname{Ho}\mathbf{Sp}_E$.

The identity functor going the other way $(\mathbf{Sp}, w_E) \to (\mathbf{Sp}, s)$ is not homotopical, but it can be right-derived. We let **A** be the subcategory of E-local spectra, and $R\colon \mathbf{Sp} \to \mathbf{A}$ the E-localization functor L_E from Theorem 3.3.7. The right-derived identity functor is therefore Bousfield's functor L_E! And indeed this does send all E-local equivalences to stable equivalences.

Example 3.4.40. For any set A, the A-indexed product of spectra $\prod\colon \mathbf{Sp}^{\times A} \to \mathbf{Sp}$ has a right deformation, even if we use the *E-local equivalences* in **Sp** instead of the stable equivalences. The replacement functor $\mathbf{Sp}^{\times A} \to \mathbf{Sp}^{\times A}$ replaces every spectrum in the tuple with an E-local Ω-spectrum, in other words it applies $R \circ L_E$ to each slot, where L_E is the functor from Theorem 3.3.7. Once all the spectra in the tuple are E-local, any E-local equivalence between them is a stable equivalence by Lemma 3.3.6, so the product will now preserve all E-local equivalences. The right-derived product in this setting is therefore

$$\mathbb{R}\left(\prod_{\alpha\in A}\right) X(\alpha) = \prod_{\alpha\in A} R(L_E X(\alpha)).$$

This illustrates an interesting point: the right-derived functor $\mathbb{R}F$ depends not just on F, but also on the class of weak equivalences! Changing the weak equivalences can change the output of $\mathbb{R}F$.

3.5 Advanced properties of derived functors*

In this section we collect more useful results about derived functors whose proofs are a little more abstract than the previous ones. As in the previous section, many of these results and their proofs come from [DHKS04].

3.5.1 Uniqueness of derived functors

Derived functors are supposed to be unique. However, they can't be unique on the nose. Different choices of left deformation Q will lead to *weakly equivalent* functors $F \circ Q$, not isomorphic ones. So to capture this uniqueness, we have to consider functors up to weak equivalence.

Definition 3.5.1. Let $\mathbf{C}$ and $\mathbf{D}$ be categories with weak equivalences (satisfying the 2-out-of-3 property). The **functor category** $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ has an object for each functor $F : \mathbf{C} \to \mathbf{D}$, and a morphism for each natural transformation $F \to G$.

A map of functors $F \to G$ in $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ is considered to be a weak equivalence if it is a **pointwise equivalence** in the sense of Definition 3.4.29. In other words, for each $X \in \mathbf{C}$, the map $F(X) \to G(X)$ is a weak equivalence in $\mathbf{D}$.

This gives the functor category $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ a class of weak equivalences, so we can take its homotopy category $\mathrm{Ho}\,\mathrm{Fun}(\mathbf{C}, \mathbf{D})$. Note that a map in this homotopy category is a zig-zag of functors, e.g.

$$F \longrightarrow F_1 \xleftarrow{\sim} F_2 \xleftarrow{\sim} F_3 \longrightarrow F_4 \longrightarrow G.$$

As in Definition 3.4.1, a **homotopy functor** is a functor $F : \mathbf{C} \to \mathbf{D}$ that sends weak equivalences $X \xrightarrow{\sim} Y$, to weak equivalences $F(X) \xrightarrow{\sim} F(Y)$.

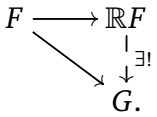
Proposition 3.5.2. *If F has a left-derived functor $\mathbb{L}F$, then in* $\mathrm{Ho}\,\mathrm{Fun}(\mathbf{C}, \mathbf{D})$*, the functor $\mathbb{L}F$ is terminal among homotopy functors mapping to F. Dually, a right-derived functor $\mathbb{R}F$ is initial among homotopy functors receiving a map from F.*

In other words, if G is any other homotopy functor, and $G \to F$ is any zig-zag of natural transformations from G to F, then there is a unique zig-zag $G \to \mathbb{L}F$ making this triangle commute:

$$\begin{array}{ccc} G & & \\ {\scriptstyle \exists!}\downarrow & \searrow & \\ \mathbb{L}F & \longrightarrow & F. \end{array}$$

So $\mathbb{L}F$ is the "closest homotopy functor to F" on the left.

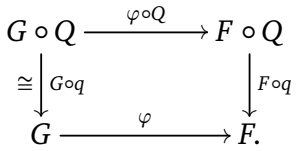
Dually, if F has a right-derived functor $\mathbb{R}F$, then any zig-zag from F to a homotopy functor G gives a unique zig-zag $\mathbb{R}F \to G$ making this triangle commute:



So $\mathbb{R}F$ is the "closest homotopy functor to F" on the right.

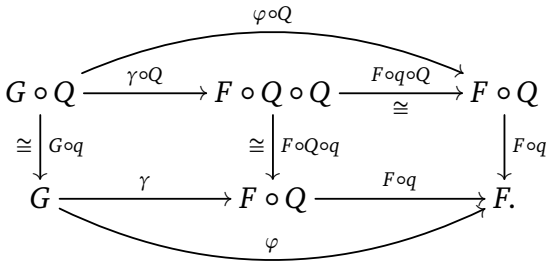
Proof. We do the left-derived case in detail. The right-derived case is the same except we reverse all the arrows.

Let $q : QX \to X$ denote the weak equivalence from Q back to the identity, and let $\varphi : G \to F$ denote the zig-zag from G to F. Since this is a zig-zag of natural transformations, on the map $q : QX \to X$ it gives a commuting square in the homotopy category of functors,



The maps $G(QX) \to G(X)$ are weak equivalences, so the left-hand vertical map in the above square is an isomorphism in the homotopy category. The left-hand vertical and top horizontal map give the desired zig-zag from G to $\mathbb{L}F$.

To show this is unique, let $\gamma : G \to F \circ Q$ be any other zig-zag that when composed with $F \circ q : F \circ Q \to F$ gives $\varphi : G \to F$. Applying this zig-zag to the map $q : QX \to X$ produces the following commuting diagram in the homotopy category of functors.



The vertical isomorphisms are because G and $F \circ Q$ are homotopy functors. For the isomorphism in the top row, $q \circ Q : QQX \to QX$ is a weak equivalence of objects in $\mathbf{A}$, so F sends it to a weak equivalence, hence an isomorphism in the homotopy category.

As a map in the homotopy category, γ is therefore equal to a composite of $\varphi \circ Q$ with some fixed isomorphisms. The map γ is therefore determined by φ, so it is unique.[6] □

[6]This version of the argument is inspired by the proof in [Goo03, 1.8], see also [Rie17, 6.4.11] and [Lur09, 5.2.7.4(3)].

Corollary 3.5.3. *Any two left-derived functors of F are canonically equivalent by a zig-zag of equivalences of functors. The same is true for right-derived functors.*

There is a similar universal property that $\mathbb{L}F$ enjoys as a functor on $\mathrm{Ho}\,\mathbf{C}$, see Exercise 3.6.33.

Corollary 3.5.4. *The two maps $QQX \rightrightarrows QX$ that cancel either the first or second copy of Q, give the same map in* $\mathrm{Ho}\,\mathbf{C}$.

Proof. Take $F = \mathrm{id}_{\mathbf{C}}$ to be the identity functor and $G = QQ$. The two maps $QQX \rightrightarrows QX$ are natural transformations $G \to \mathbb{L}F$ that commute with the map back to F, so by Proposition 3.5.2 they are equal in the homotopy category. □

Example 3.5.5. The smash product of spaces $\wedge\colon \mathbf{Top}_* \times \mathbf{Top}_* \to \mathbf{Top}_*$ can be left-derived by replacing the inputs (X, Y) by CW complexes. It can also be derived by "whiskering" the inputs: attach one end of an interval I to the basepoint of X, and let the other end of I be the new basepoint. This space $I \cup_* X$ is well-based, and collapsing I gives an equivalence back to X. By Corollary 3.5.3, these two derived smash products are equivalent to each other:

$$QX \wedge QY \simeq (I \cup_* X) \wedge (I \cup_* Y).$$

A similar argument applies when smashing a space K with a spectrum X, though whiskering a spectrum is a little harder to do because of the bonding maps – you have to iteratively replace each map $\Sigma X_n \to X_{n+1}$ by its mapping cylinder.

Example 3.5.6. As in Example 3.4.24, the tensor $\wedge\colon \mathbf{Top}_* \times \mathbf{Sp} \to \mathbf{Sp}$ can be derived by making both K and X CW, or by making just K CW. These are equivalent by Corollary 3.5.3. This is also easy to see directly: since $QK \wedge -$ preserves all equivalences, we have an equivalence

$$QK \wedge QX \simeq QK \wedge X.$$

Example 3.5.7. In Example 3.4.26, we left-derived the pushout $X \cup_A Y$ by replacing the maps $A \to X$ and $A \to Y$ by their mapping cylinders. We could, instead, replace them by relative CW complexes $A \to X' \simeq X$ and $A \to Y' \simeq Y$ using Corollary 1.4.17. This gives a different model for the left-derived pushout, but by Corollary 3.5.3, the two models are canonically equivalent to each other.

Example 3.5.8. The hom functor on chain complexes $\mathrm{Hom}\colon \mathbf{Ch}^{\mathrm{op}} \times \mathbf{Ch} \to \mathbf{Ch}$ can be right-derived by replacing the source by a projective chain complex, or the target by an injective one. By Corollary 3.5.3, these produce equivalent results. We conclude the standard result in homological algebra that $\mathrm{Ext}^i(A, B)$ can be computed by taking a projective resolution of A, or by taking an injective resolution of B, before applying Hom and taking the cohomology of the result.

Example 3.5.9. In Definition 2.2.11 we define the infinite loop space $\Omega^\infty X$ of a spectrum X to be the right-derived functor of ev_0 on X. By Corollary 3.5.3, this is independent of the method we use to replace X by an Ω-spectrum.

3.5.2 Correct homotopy types

We can now explain what we meant by the "correct" homotopy type in Section 1.5.

Definition 3.5.10. Suppose F has a left-derived functor $\mathbb{L}F$. We say that F is **correct** on X, or that "F has the correct homotopy type on X," if $\mathbb{L}F(X) \to F(X)$ is a weak equivalence. Correct for right-derived functors is defined similarly.

Since derived functors are unique, this doesn't depend on how we deform F.

Remark 3.5.11. F preserves all equivalences whenever it is correct. Therefore, if we take all values of X for which F is correct, we get the largest possible choice of subcategory $\mathbf{A}$ that we can use when deriving F.

Example 3.5.12. The smash product of spaces $X \wedge Y$ is correct if X and Y are CW complexes. It is also correct if X and Y are well-based. The tensor $K \wedge X$ is correct if K is a CW complex. It is also correct if K and all the spectrum levels X_n are well-based.

The pushout $X \cup_A Y$ is correct if one of the maps $A \to X$ or $A \to Y$ is a cofibration (Lemma 1.5.3). The sequential colimit is correct if the maps of the system are closed inclusions (Lemma 1.5.6). The pullback $X \times_B Y$ is correct if one of the maps $X \to B$ or $Y \to B$ is a fibration (Lemma 1.5.19). The sequential limit is correct if the maps of the system are fibrations (Lemma 1.5.22).

The coproduct of spectra $\bigvee_\alpha X(\alpha)$ is always correct. The product of spectra $\prod_\alpha X(\alpha)$ is correct if it is a finite product, or if the inputs are Ω-spectra.

3.5.3 Composites of derived functors

Suppose we have functors on categories with weak equivalences that can be composed,

$$\mathbf{C}_1 \xrightarrow{F_1} \mathbf{C}_2 \xrightarrow{F_2} \mathbf{C}_3,$$

and that

- F_1 has a left-deformation Q_1 landing in $\mathbf{A}_1 \subseteq \mathbf{C}_1$,
- F_2 has a left-deformation Q_2 landing in $\mathbf{A}_2 \subseteq \mathbf{C}_2$, and

- the deformations are compatible in the sense that $F_1(\mathbf{A}_1) \subseteq \mathbf{A}_2$.

Lemma 3.5.13. *Under these hypotheses, Q_1 is also a left deformation for the composite $F_2 \circ F_1$, and we have a weak equivalence of functors*

$$\mathbb{L}(F_2 \circ F_1) \simeq (\mathbb{L}F_2) \circ (\mathbb{L}F_1).$$

A similar statement holds for composites of right-derived functors.

Proof. This is much easier to prove than it is to state. Any weak equivalence in $\mathbf{A}_1$ goes to a weak equivalence in $F_1(\mathbf{A}_1) \subseteq \mathbf{A}_2$, and therefore goes to a weak equivalence in $F_2(\mathbf{A}_2) \subseteq \mathbf{C}_3$. Therefore Q_1 is a left deformation of $F_2 \circ F_1$. Since F_2 preserves equivalences on $\mathbf{A}_2$, we get an equivalence

$$(\mathbb{L}F_2)(\mathbb{L}F_1)(X) = F_2Q_2F_1Q_1X \xrightarrow{\sim} F_2F_1Q_1X = \mathbb{L}(F_2 \circ F_1)(X).$$

For right-derived functors the same argument applies. The equivalence looks like:

$$\mathbb{R}(G_2 \circ G_1)(X) = G_2G_1R_1X \xrightarrow{\sim} G_2R_2G_1R_1X = (\mathbb{R}G_2)(\mathbb{R}G_1)(X).$$

□

Remark 3.5.14. The above can be summarized as "left-derived functors compose, *if* the deformations are compatible with each other." They do not compose in general if $F_1(\mathbf{A}_1) \not\subseteq \mathbf{A}_2$. See Example 3.5.19 for a counterexample.

Example 3.5.15. The suspension spectrum and disjoint basepoint functors

$$\mathbf{Top} \xrightarrow{(-)_+} \mathbf{Top}_* \xrightarrow{\Sigma^\infty} \mathbf{Sp}$$

satisfy the hypotheses of Lemma 3.5.13. Therefore the left-derived functors compose:

$$\mathrm{Ho}\,\mathbf{Top} \xrightarrow{(-)_+} \mathrm{Ho}\,\mathbf{Top}_* \xrightarrow{\mathbb{L}\Sigma^\infty} \mathrm{Ho}\,\mathbf{Sp}, \qquad \mathbb{L}(\Sigma^\infty_+) : \mathrm{Ho}\,\mathbf{Top} \to \mathrm{Ho}\,\mathbf{Sp}$$

The same applies to the right adjoints (Exercise 3.6.34).

Example 3.5.16. The suspension functor commutes with suspension spectrum, up to isomorphism:

$$\begin{array}{ccc} \mathbf{Top}_* & \xrightarrow{\Sigma^\infty} & \mathbf{Sp} \\ {\scriptstyle\Sigma}\downarrow & & \downarrow{\scriptstyle\Sigma} \\ \mathbf{Top}_* & \xrightarrow{\Sigma^\infty} & \mathbf{Sp} \end{array}$$

Thinking of this as two different factorizations of $\Sigma^\infty \circ \Sigma \cong \Sigma \circ \Sigma^\infty$, the hypotheses of Lemma 3.5.13 are satisfied, so we get

$$(\mathbb{L}\Sigma^\infty)(\mathbb{L}\Sigma) \simeq \mathbb{L}(\Sigma^\infty \circ \Sigma) \cong \mathbb{L}(\Sigma \circ \Sigma^\infty) \simeq (\mathbb{L}\Sigma)(\mathbb{L}\Sigma^\infty).$$

In other words, the corresponding square of left-derived functors on the homotopy category also commutes, up to isomorphism:

$$\begin{array}{ccc} \mathrm{Ho}\,\mathbf{Top}_* & \xrightarrow{\mathbb{L}\Sigma^\infty} & \mathrm{Ho}\,\mathbf{Sp} \\ {\scriptstyle \mathbb{L}\Sigma}\downarrow & & \downarrow{\scriptstyle \mathbb{L}\Sigma=\Sigma} \\ \mathrm{Ho}\,\mathbf{Top}_* & \xrightarrow{\mathbb{L}\Sigma^\infty} & \mathrm{Ho}\,\mathbf{Sp} \end{array}$$

Example 3.5.17. As in the previous exercise, the loopspace functor commutes with evaluation

$$\begin{array}{ccc} \mathbf{Top}_* & \xleftarrow{\mathrm{ev}_0} & \mathbf{Sp} \\ {\scriptstyle \Omega}\downarrow & & \downarrow{\scriptstyle \Omega} \\ \mathbf{Top}_* & \xleftarrow{\mathrm{ev}_0} & \mathbf{Sp}, \end{array}$$

giving a commuting square of right-derived functors

$$\begin{array}{ccc} \mathrm{Ho}\,\mathbf{Top}_* & \xrightarrow{\Omega^\infty} & \mathrm{Ho}\,\mathbf{Sp} \\ {\scriptstyle \Omega}\downarrow & & \downarrow{\scriptstyle \Omega} \\ \mathrm{Ho}\,\mathbf{Top}_* & \xrightarrow{\Omega^\infty} & \mathrm{Ho}\,\mathbf{Sp} \end{array}$$

Example 3.5.18. The smash product of based spaces is associative, $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$. We can draw this as a commuting square (up to isomorphism) of functors

$$\begin{array}{ccc} \mathbf{Top}_* \times \mathbf{Top}_* \times \mathbf{Top}_* & \xrightarrow{\mathrm{id}\times\wedge} & \mathbf{Top}_* \times \mathbf{Top}_* \\ {\scriptstyle \wedge\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \wedge} \\ \mathbf{Top}_* \times \mathbf{Top}_* & \xrightarrow{\wedge} & \mathbf{Top}_*. \end{array}$$

The hypotheses of Lemma 3.5.13 are satisfied, so we get a commuting square of left-derived smash products

$$\begin{array}{ccc} \mathrm{Ho}\,\mathbf{Top}_* \times \mathrm{Ho}\,\mathbf{Top}_* \times \mathrm{Ho}\,\mathbf{Top}_* & \xrightarrow{\mathrm{id}\times\wedge^{\mathbb{L}}} & \mathrm{Ho}\,\mathbf{Top}_* \times \mathrm{Ho}\,\mathbf{Top}_* \\ {\scriptstyle \wedge^{\mathbb{L}}\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \wedge^{\mathbb{L}}} \\ \mathrm{Ho}\,\mathbf{Top}_* \times \mathrm{Ho}\,\mathbf{Top}_* & \xrightarrow{\wedge^{\mathbb{L}}} & \mathrm{Ho}\,\mathbf{Top}_* \end{array}$$

giving an isomorphism on the homotopy category

$$(X \wedge^{\mathbb{L}} Y) \wedge^{\mathbb{L}} Z) \cong X \wedge^{\mathbb{L}} (Y \wedge^{\mathbb{L}} Z).$$

If we proceed this way with the unit isomorphisms $S^0 \wedge X \cong X$ and symmetry isomorphisms $X \wedge Y \cong Y \wedge X$, we can prove that $\mathrm{Ho}\,\mathbf{Top}_*$ is a symmetric monoidal category. See Lemma 4.1.8. The argument will apply to $\mathrm{Ho}\,\mathbf{Sp}$ as well, as soon as we actually have a smash product of spectra that has isomorphisms $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$, $\mathbb{S} \wedge X \cong X$, and $X \wedge Y \cong Y \wedge X$. See Section 6.2.

Example 3.5.19. The colimit of a diagram of spaces $X : \mathbf{I} \to \mathbf{Top}$ can be defined in two stages, by first forming the coequalizer diagram

$$\coprod_{i \to j} X(i) \rightrightarrows \coprod_i X(i)$$

and then taking the coequalizer of it. In other words, a colimit can always be built out of coproducts and coequalizers:

$$\mathbf{Top}^{\mathbf{I}} \xrightarrow{\text{coproduct}} \mathbf{Top}^{\{\bullet \rightrightarrows \bullet\}} \xrightarrow{\text{coequalizer}} \mathbf{Top}, \qquad \mathbf{Top}^{\mathbf{I}} \xrightarrow{\text{colimit}} \mathbf{Top}$$

However, the *homotopy* colimit is not the *homotopy* coequalizer of the same diagram. We do *not* have an agreement up to isomorphism between the following functors:

$$\mathrm{Ho}(\mathbf{Top}^{\mathbf{I}}) \xrightarrow{\mathbb{L}(\text{coproduct})} \mathrm{Ho}(\mathbf{Top}^{\{\bullet \rightrightarrows \bullet\}}) \xrightarrow{\mathbb{L}(\text{coequalizer})} \mathrm{Ho}\,\mathbf{Top}, \qquad \mathrm{Ho}(\mathbf{Top}^{\mathbf{I}}) \xrightarrow{\mathbb{L}(\text{colimit})} \mathrm{Ho}\,\mathbf{Top}.$$

We deduce from this that the conditions of Lemma 3.5.13 are not satisfied in this example.

3.5.4 Adjunctions between derived functors

Another common situation is that left and right deformable functors come in adjoint pairs. When this happens, their derived functors also form an adjoint pair:

Proposition 3.5.20. *Suppose that $(F \dashv G)$ is a pair of adjoint functors*

$$F : \mathbf{C} \to \mathbf{D}, \qquad G : \mathbf{D} \to \mathbf{C},$$

that F has a left deformation, and that G has right deformation. Then the derived functors

$$\mathbb{L}F : \mathrm{Ho}\,\mathbf{C} \to \mathrm{Ho}\,\mathbf{D}, \qquad \mathbb{R}G : \mathrm{Ho}\,\mathbf{D} \to \mathrm{Ho}\,\mathbf{C}$$

are adjoint as well.

Proof. Recall that an adjunction can be described by a pair of natural transformations

$$\eta\colon \mathrm{id}_{\mathbf{C}} \to GF, \qquad \epsilon\colon FG \to \mathrm{id}_{\mathbf{D}},$$

satisfying the "triangle identities"

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow^{=} & \downarrow{\scriptstyle \epsilon F} \\ & & F, \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow^{=} & \downarrow{\scriptstyle G\epsilon} \\ & & G. \end{array}$$

We define maps in the homotopy category of functors

$$\widetilde{\eta}\colon \mathrm{id}_{\mathbf{C}} \to \mathbb{R}G\mathbb{L}F = G \circ R \circ F \circ Q, \qquad \widetilde{\epsilon}\colon F \circ Q \circ G \circ R = \mathbb{L}F\mathbb{R}G \to \mathrm{id}_{\mathbf{D}},$$

satisfying the same triangle identities in the homotopy category. We define them as the zig-zags

$$\mathrm{id}_{\mathbf{C}} \xleftarrow{q} Q \xrightarrow{\eta Q} GFQ \xrightarrow{GrFQ} GRFQ, \qquad FQGR \xrightarrow{FqGR} FGR \xrightarrow{\epsilon R} R \xleftarrow{r} \mathrm{id}_{\mathbf{D}},$$

or in shorthand notation

$$\mathrm{id}_{\mathbf{C}} \xleftarrow{q} Q \xrightarrow{\eta} GFQ \xrightarrow{r} GRFQ, \qquad FQGR \xrightarrow{q} FGR \xrightarrow{\epsilon} R \xleftarrow{r} \mathrm{id}_{\mathbf{D}}.$$

We check that the following two diagrams commute in the homotopy category of functors, which verifies the triangle identities in the homotopy category. The two possible descriptions of the map $FQQ \to FQ$ are equal by Corollary 3.5.4, and similarly for $GR \to GRR$ in the second diagram.

$$\begin{array}{ccccc} FQQ & \xrightarrow{\eta} & FQGFQ & \xrightarrow{r} & FQGRFQ \\ {\scriptstyle \sim}\downarrow{\scriptstyle 1q=q1} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\ FQ & \xrightarrow{\eta} & FGFQ & \xrightarrow{r} & FGRFQ \\ & \searrow^{=} & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} \\ & & FQ & \xrightarrow[\sim]{r} & RFQ \end{array} \qquad \begin{array}{ccccc} QGR & \xrightarrow{\eta} & GFQGR & \xrightarrow{r} & GRFQGR \\ {\scriptstyle \sim}\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\ GR & \xrightarrow{\eta} & GFGR & \xrightarrow{r} & GRFGR \\ & \searrow^{=} & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} \\ & & GR & \xrightarrow[\sim]{1r=r1} & GRR \end{array}$$

□

Example 3.5.21. This gives the following adjunctions of homotopy categories.

Left adjoint	**Right adjoint**
Left-derived suspension $\mathbb{L}\Sigma\colon \operatorname{Ho}\mathbf{Top}_* \to \operatorname{Ho}\mathbf{Top}_*$	Loopspace $\Omega\colon \operatorname{Ho}\mathbf{Top}_* \to \operatorname{Ho}\mathbf{Top}_*$
Disjoint basepoint $(-)_+\colon \operatorname{Ho}\mathbf{Top} \to \operatorname{Ho}\mathbf{Top}_*$	Forget basepoint $U\colon \operatorname{Ho}\mathbf{Top}_* \to \operatorname{Ho}\mathbf{Top}$
Suspension $\Sigma\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Sp}$	Loopspace $\Omega\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Sp}$
Left-derived suspension spectrum $\mathbb{L}\Sigma^\infty\colon \operatorname{Ho}\mathbf{Top}_* \to \operatorname{Ho}\mathbf{Sp}$ $\mathbb{L}\Sigma^\infty_+\colon \operatorname{Ho}\mathbf{Top} \to \operatorname{Ho}\mathbf{Sp}$	Right-derived 0th space $\Omega^\infty = \mathbb{R}\mathrm{ev}_0\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Top}_*$ $U\Omega^\infty = \mathbb{R}(U\mathrm{ev}_0)\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Top}$
Left-derived free spectrum $\mathbb{L}F_n\colon \operatorname{Ho}\mathbf{Top}_* \to \operatorname{Ho}\mathbf{Sp}$ $\mathbb{L}F_n(-)_+\colon \operatorname{Ho}\mathbf{Top} \to \operatorname{Ho}\mathbf{Sp}$	Right-derived nth space $\mathbb{R}\mathrm{ev}_n\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Top}_*$ $\mathbb{R}(U\mathrm{ev}_n)\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Top}$

Left adjoint	**Right adjoint**
Tensor $K \wedge (-)\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Sp}$ (K is a CW complex)	Right-derived cotensor $\mathbb{R}F(K,-)\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Sp}$
Left-derived tensor $(-) \wedge^{\mathbb{L}} X\colon \operatorname{Ho}\mathbf{Top}_* \to \operatorname{Ho}\mathbf{Sp}$ (X is a CW spectrum)	Right-derived mapping space $\mathbb{R}\mathrm{Map}_*(X,-)\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Sp}$

The last two can be summarized like this: for any space K and spectra X and Y, there are bijections

$$[X, \mathbb{R}F(K,Y)]_s \cong [K \wedge^{\mathbb{L}} X, Y]_s \cong [K, \mathbb{R}\mathrm{Map}_*(X,Y)]_*. \tag{3.5.22}$$

In other words, the tensor has two right adjoints, depending on which variable we focus on, and the left-derived tensor has the corresponding right-derived functors as right adjoints. In fact, this always happens when we have a functor with two right adjoints, and compatible deformations. See Remark 3.5.25 for more details.

Example 3.5.23. The universal property of a coproduct can be stated as follows: a map $\coprod_\alpha X_\alpha \to Y$ is the same thing as a map of tuples $(X_\alpha) \to (Y)$, to the constant tuple that is Y in every slot. Therefore, the coproduct is the left adjoint of the functor that creates

constant tuples.

These adjunctions pass to the homotopy category of spaces and spectra:

Left adjoint	**Right adjoint**
Disjoint union $\coprod\colon \mathrm{Ho}\,\mathbf{Top}^{\times A} \to \mathrm{Ho}\,\mathbf{Top}$	Constant tuple $\mathrm{Ho}\,\mathbf{Top} \to \mathrm{Ho}\,\mathbf{Top}^{\times A}$
Wedge sum $\bigvee\colon \mathrm{Ho}\,\mathbf{Top}_*^{\times A} \to \mathrm{Ho}\,\mathbf{Top}_*$	Constant tuple $\mathrm{Ho}\,\mathbf{Top}_* \to \mathrm{Ho}\,\mathbf{Top}_*^{\times A}$
Wedge sum $\bigvee\colon \mathrm{Ho}\,\mathbf{Sp}^{\times A} \to \mathrm{Ho}\,\mathbf{Sp}$	Constant tuple $\mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}^{\times A}$

Dually, the product is the right adjoint of the functor that creates constant tuples: a map $Y \to \prod_\alpha X_\alpha$ is the same thing as a map of tuples $(Y) \to (X_\alpha)$. Passing to the homotopy category:

Left adjoint	**Right adjoint**
Constant tuple $\mathrm{Ho}\,\mathbf{Top} \to \mathrm{Ho}\,\mathbf{Top}^{\times A}$	Product $\prod\colon \mathrm{Ho}\,\mathbf{Top}^{\times A} \to \mathrm{Ho}\,\mathbf{Top}$
Constant tuple $\mathrm{Ho}\,\mathbf{Top}_* \to \mathrm{Ho}\,\mathbf{Top}_*^{\times A}$	Product $\prod\colon \mathrm{Ho}\,\mathbf{Top}_*^{\times A} \to \mathrm{Ho}\,\mathbf{Top}_*$
Constant tuple $\mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}^{\times A}$	Right-derived product $\mathbb{R}\prod\colon \mathrm{Ho}\,\mathbf{Sp}^{\times A} \to \mathrm{Ho}\,\mathbf{Sp}$

This recovers Proposition 3.2.2 by a different method!

Example 3.5.24. More generally, in any category $\mathbf{C}$ with weak equivalences such that the homotopy colimit exists, the homotopy colimit is the left-derived functor of the colimit $\mathrm{colim}\colon \mathbf{C}^{\mathbf{I}} \to \mathbf{C}$, whose right adjoint is the functor that sends each object of $\mathbf{C}$ to the constant diagram on that object (Definition 3.4.29). Clearly the constant diagram functor preserves equivalences, so it is its own right-derived functor. Therefore we get an adjunction of homotopy categories as shown, generalizing the earlier examples of coproducts and products:

Left adjoint	Right adjoint
Homotopy colimit $\operatorname{hocolim}\colon \operatorname{Ho}(\mathbf{C}^{\mathbf{I}}) \to \operatorname{Ho}\mathbf{C}$	Constant diagram $\operatorname{const}\colon \operatorname{Ho}\mathbf{C} \to \operatorname{Ho}(\mathbf{C}^{\mathbf{I}})$
Constant diagram $\operatorname{const}\colon \operatorname{Ho}\mathbf{C} \to \operatorname{Ho}(\mathbf{C}^{\mathbf{I}})$	Homotopy limit $\operatorname{holim}\colon \operatorname{Ho}(\mathbf{C}^{\mathbf{I}}) \to \operatorname{Ho}\mathbf{C}$.

As in Warning 3.4.10, it is very important here that we use $\operatorname{Ho}(\mathbf{C}^{\mathbf{I}})$, and not $(\operatorname{Ho}\mathbf{C})^{\mathbf{I}}$. We work with the homotopy category of diagrams, *not* diagrams in the homotopy category.

Remark 3.5.25. The tensor has a right adjoint in each variable, the cotensor and the mapping space

$$\begin{aligned} \wedge &\colon \mathbf{Top}_* \times \mathbf{Sp} \to \mathbf{Sp} \\ F &\colon \mathbf{Top}_*^{\mathrm{op}} \times \mathbf{Sp} \to \mathbf{Sp} \\ \operatorname{Map}_* &\colon \mathbf{Sp}^{\mathrm{op}} \times \mathbf{Sp} \to \mathbf{Top}_*. \end{aligned}$$

This is an example of a **two-variable adjunction**: a collection of three functors

$$\begin{aligned} F(-,-) &\colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}, \\ G(-,-) &\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{E} \to \mathbf{D}, \\ H(-,-) &\colon \mathbf{D}^{\mathrm{op}} \times \mathbf{E} \to \mathbf{C} \end{aligned}$$

and natural isomorphisms of functors $\mathbf{C}^{\mathrm{op}} \times \mathbf{D}^{\mathrm{op}} \times \mathbf{E} \to \mathbf{Set}$,

$$\mathbf{C}(c, H(d,e)) \cong \mathbf{E}(F(c,d), e) \cong \mathbf{D}(d, G(c,e)).$$

In other words, G is the adjoint of F in one slot and H is the adjoint of F in the other slot.

Two-variable adjunctions can be deformed as well. Any time we pick

- a left deformation $Pc \xrightarrow{\sim} c$ landing in $\mathbf{A}_1 \subseteq \mathbf{C}$,
- a left deformation $Qd \xrightarrow{\sim} d$ landing in $\mathbf{A}_2 \subseteq \mathbf{D}$, and
- a right deformation $e \xrightarrow{\sim} Re$ landing in $\mathbf{B} \subseteq \mathbf{E}$,

such that

- $(Pc, Qd) \xrightarrow{\sim} (c,d)$ is a left deformation of F,
- $(c,e) \xrightarrow{\sim} (Pc, Re)$ is a right deformation of G, and
- $(d,e) \xrightarrow{\sim} (Qd, Re)$ is a right deformation of H,

then the derived functors

$$\begin{aligned}\mathbb{L}F(-,-)\colon\ &\operatorname{Ho}\mathbf{C}\times\operatorname{Ho}\mathbf{D}\to\operatorname{Ho}\mathbf{E},\\ \mathbb{R}G(-,-)\colon\ &\operatorname{Ho}\mathbf{C}^{\mathrm{op}}\times\operatorname{Ho}\mathbf{E}\to\operatorname{Ho}\mathbf{D},\\ \mathbb{R}H(-,-)\colon\ &\operatorname{Ho}\mathbf{D}^{\mathrm{op}}\times\operatorname{Ho}\mathbf{E}\to\operatorname{Ho}\mathbf{C}\end{aligned}$$

also form a two-variable adjunction on the homotopy categories. So we get natural bijections for $(c,d,e)\in\operatorname{Ho}\mathbf{C}^{\mathrm{op}}\times\operatorname{Ho}\mathbf{D}\times\operatorname{Ho}\mathbf{E}$,

$$[c,\mathbb{R}H(d,e)]_{\operatorname{Ho}\mathbf{C}}\cong[\mathbb{L}F(c,d),e]_{\operatorname{Ho}\mathbf{E}}\cong[d,\mathbb{R}G(c,e)]_{\operatorname{Ho}\mathbf{D}}.$$

It is not so hard to see that such isomorphisms exist by applying Proposition 3.5.20 for each fixed value of c and d. However, to prove that they are natural in c and d involves no small amount of diagram-chasing. We omit the full proof.

3.6 Exercises

3.6.1. (a) Suppose $F\colon\mathbf{C}\to\mathbf{D}$ is a functor. Prove that F gives a well-defined function from isomorphism classes in $\mathbf{C}$ to isomorphism classes in $\mathbf{D}$, as in (3.1.19).

(b) Suppose that F is fully faithful. Prove that this map of isomorphism classes is injective.

3.6.2. Modify the universal property for the homotopy category by allowing the triangle (3.1.7) to commute up to natural isomorphism. Show that the homotopy category constructed in Definition 3.1.9 (or Definition 3.1.1) also satisfies this property, and that the homotopy category in this sense is unique up to equivalence of categories, rather than isomorphism of categories.

3.6.3. Prove Lemma 3.1.22, showing that the two notions of "equivalence of categories" are the same.

3.6.4. Suppose $f\colon X\to Y$ is a stable equivalence of spectra. Explain why composition with f gives isomorphisms

$$[W,X]_s\xrightarrow{\cong}[W,Y]_s,\qquad[Y,W]_s\xrightarrow{\cong}[X,W]_s.$$

3.6.5. (a) Define $h\mathbf{Top}$ by taking all topological spaces and homotopy classes of maps. Explain why there is a functor $h\mathbf{Top}\to\operatorname{Ho}\mathbf{Top}$, and why it is not an equivalence of categories. In Proposition 3.1.27 we proved that $h\mathbf{CW}\simeq\operatorname{Ho}\mathbf{CW}$; how does this proof fail for $h\mathbf{Top}$ and $\operatorname{Ho}\mathbf{Top}$?

(b) Similarly, let $h\mathbf{Sp}$ denote the category of spectra and homotopy classes of maps between them, $[X,Y]_h$. Explain why there is a functor $h\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}$, and why this functor is *not* an equivalence of categories.

3.6.6. Finish the proof of Proposition 3.1.27 by showing that the given maps $\mathrm{Ho}\,\mathbf{Top}(K,L) \to \mathrm{Ho}\,\mathbf{CW}(K,L)$ and $\mathrm{Ho}\,\mathbf{CW}(K,L) \to \mathrm{Ho}\,\mathbf{Top}(K,L)$ are inverses of each other. You may find it helpful to think of the commuting diagram at the end of the proof as a commuting diagram in $\mathrm{Ho}\,\mathbf{Top}$ or $\mathrm{Ho}\,\mathbf{CW}$, instead of using the equivalence relation on zig-zags directly.

3.6.7. Let $f : X \to Y$ be any map of spectra. Explain why the image in the homotopy category $\delta(f) = \left(\xrightarrow{f}\right)$ is an isomorphism iff f is a stable equivalence. In other words, the stable equivalences in spectra are **saturated**. (You might have to use Proposition 3.1.41).

3.6.8. Give a second proof of the Whitehead theorem for spectra (Proposition 2.6.16) by defining the deformation retract of M to X one stable cell at a time, as in the proof of Proposition 3.1.39.

3.6.9. Finish the proof of Proposition 3.1.41 by showing that the map (2) is an isomorphism. (You won't need to assume that Y is an Ω-spectrum.)

3.6.10. Finish the proof of Theorem 3.2.5 by proving the composition is bilinear, that the zero map is an identity element for $+$, and that flipping the suspension coordinate gives inverse elements.

3.6.11. Use Proposition 3.2.7 to show that the different models for the degree n map $X \xrightarrow{n} X$ in Example 3.1.4 give the same map in $[X,X]_s$.

3.6.12. Compute the set of endomorphisms of $\Sigma^\infty_+ S^1 \simeq \mathbb{S}^0 \vee \mathbb{S}^1$ in the stable homotopy category, using that $\pi_0(\mathbb{S}) = \mathbb{Z}$, $\pi_1(\mathbb{S}) = \mathbb{Z}/2$, and $\pi_{-1}(\mathbb{S}) = 0$. Describe this set as a ring, with multiplication coming from composition.

3.6.13. If Y is an Ω-spectrum, prove that the map $F_n S^n \to F_0 S^0$ adjoint to the identity of S^n induces a bijection $[F_0 S^0, Y]_h \to [F_n S^n, Y]_h$. (You can do this directly without resorting to Proposition 3.1.41.)

3.6.14. (a) Recall the $\lim^1$ exact sequence for spaces from Section 1.5, equation (1.5.23):

$$0 \longrightarrow \lim_{n\to\infty}{}^1\ \pi_{k+1}(X_n) \longrightarrow \pi_k\left(\operatorname*{holim}_{n\to\infty} X_n\right) \longrightarrow \lim_{n\to\infty}\ \pi_k(X_n) \longrightarrow 0.$$

Use this to prove the $\lim^1$ exact sequence for spectra (Proposition 3.2.21).

(b) Dually, if Y is the homotopy limit of a sequence of Ω-spectra

$$Y = \operatorname*{holim}_{n\to\infty} Y^{(n)} \longrightarrow \ldots \longrightarrow Y^{(1)} \longrightarrow Y^{(0)} \longrightarrow *,$$

and X is another spectrum, prove that there is also a short exact sequence

$$0 \longrightarrow \lim_{n\to\infty}{}^1\, [\Sigma X, Y^{(n)}]_s \longrightarrow [X,Y]_s \longrightarrow \lim_{n\to\infty} [X, Y^{(n)}]_s \longrightarrow 0.$$

3.6.15. Use the $\lim^1$ sequence from Proposition 3.2.21 and the isomorphism $[X, \Sigma^k E]_s \cong E^k(X)$ from Definition 3.2.18 to show that for any extraordinary cohomology theory E and any sequence of spaces X_n, we have a $\lim^1$ exact sequence

$$0 \longrightarrow \lim_{n\to\infty}{}^1\, E^{k-1}(X_n) \longrightarrow E^k\left(\operatorname*{hocolim}_{n\to\infty} X_n\right) \longrightarrow \lim_{n\to\infty} E^k(X_n) \longrightarrow 0.$$

Note the similarity to (1.5.7) for ordinary homology, and the difference with (3.2.22), since the X_n simply form a diagram of spaces, not a spectrum.

3.6.16. In Definition 2.6.34 we gave, for each spectrum X, an explicit definition of a tower of maps $X \to P_n X$ that are isomorphisms on π_k for $k \le n$, and such that $\pi_k(P_n X) = 0$ for $k \ge n+1$.

Prove that any other tower of spectra $T_n X$ with the same properties is stably equivalent to this tower, by a commuting diagram in the stable homotopy category as shown.

$$\begin{array}{ccccccccc} & & \cdots & \longrightarrow & P_{n+1}X & \longrightarrow & P_nX & \longrightarrow & \cdots \\ & \nearrow & & & \downarrow\sim & & \downarrow\sim & & \\ X & & & & & & & & \\ & \searrow & & & & & & & \\ & & \cdots & \longrightarrow & T_{n+1}X & \longrightarrow & T_nX & \longrightarrow & \cdots \end{array}$$

(Hint: Use Lemma 3.2.12 repeatedly.) This also implies that the tower of connective covers of X are unique up to equivalence.

3.6.17. (a) If $\mathbf{C}$ and $\mathbf{D}$ are categories with weak equivalences, prove that

$$\mathrm{Ho}(\mathbf{C} \times \mathbf{D}) \cong (\mathrm{Ho}\,\mathbf{C}) \times (\mathrm{Ho}\,\mathbf{D}).$$

For instance, you might prove this by showing that $(\mathrm{Ho}\,\mathbf{C}) \times (\mathrm{Ho}\,\mathbf{D})$ has the correct universal property.

(b) The same claim follows for a product of finitely many categories by induction. Give a counterexample showing that the claim is false for an infinite product.

3.6.18. Recall that the opposite category $\mathbf{C}^{\mathrm{op}}$ has the same objects as $\mathbf{C}$, but the morphisms are reversed, $\mathbf{C}^{\mathrm{op}}(X,Y) = \mathbf{C}(Y,X)$. Prove that the homotopy category commutes with the opposite category,

$$\mathrm{Ho}(\mathbf{C}^{\mathrm{op}}) \cong (\mathrm{Ho}\,\mathbf{C})^{\mathrm{op}}.$$

3.6.19. (a) Prove that when two maps of spectra $X \rightrightarrows Y$ are homotopic, their homotopy cofibers are equivalent.

(b) For any element of $[X, Y]_s$, define its homotopy cofiber by representing it by a single map $QRX \to QRY$ and taking its homotopy cofiber. Explain why part (a) is needed for this to be well-defined.

(c) Explain why for any isomorphism of maps in $\mathrm{Ho}\,\mathbf{Sp}$, in other words every commuting diagram in the homotopy category

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ W & \longrightarrow & Z, \end{array}$$

there is an equivalence between the cofibers of $X \to Y$ and $W \to Z$, as we have just defined them.

3.6.20. In this exercise we prove some of the consequences of the existence of the E-localization functor L_E from Theorem 3.3.7. Recall the definition of an E-local spectrum from Definition 3.3.3.

(a) Prove Lemma 3.3.6: a map of E-local spectra is an E-local equivalence iff it is a stable equivalence. (You don't need the functor L_E for this one.)

(b) Prove Corollary 3.3.9: any two different E-localization functors L_E and L'_E must be weakly equivalent. In fact, for any single spectrum X, if there is a single E-local equivalence $X \to Y$ such that Y is E-local, then Y must be equivalent to $L_E X$. (You might find Proposition 3.5.2 or its proof helpful.)

(c) Prove Corollary 3.3.10: the subcategory of $\mathrm{Ho}\,\mathbf{Sp}$ on the E-local spectra is equivalent to the category $\mathbf{Sp}\langle w_E^{-1}\rangle$ in which we inverted the E-local equivalences. (Use the functor L_E!)

(d) Extend the previous part by showing that in fact there is an adjunction between $\mathrm{Ho}\,\mathbf{Sp}$ and $\mathrm{Ho}\,\mathbf{Sp}_E$, that is an equivalence between the subcategory of E-local spectra and $\mathrm{Ho}\,\mathbf{Sp}_E$.

3.6.21. (*) A **reflexive localization** of a category $\mathbf{C}$ is a functor $F : \mathbf{C} \to \mathbf{D}$ such that

- F is a Gabriel-Zisman localization, in other words F is the composite of a functor $\delta : \mathbf{C} \to \mathbf{C}\langle W^{-1}\rangle$ constructed as in Definition 3.1.9 followed by an equivalence of categories $\mathbf{C}\langle W^{-1}\rangle \simeq \mathbf{D}$, and
- F has a fully faithful right adjoint functor $G : \mathbf{D} \to \mathbf{C}$, whose image we call the "local" objects in $\mathbf{C}$.

Without loss of generality, we may as well take W to consist of all the morphisms of $\mathbf{C}$ that go to isomorphisms in $\mathbf{D}$.

(a) Explain why when both of these assumptions hold, the localized category $\mathbf{C}\langle W^{-1}\rangle$ is equivalent to a full subcategory of $\mathbf{C}$, and there is a functor $L\colon \mathbf{C} \to \mathbf{C}$ landing in this subcategory, with a natural transformation $\mathrm{id} \to L$ such that $X \to LX$ is in W.

(b) Prove that if only the first assumption holds, and the right adjoint G exists, then G must be fully faithful.

(c) Prove that if only the second assumption holds, then in fact the first assumption must hold as well. So any functor F with a fully-faithful right adjoint *must* be a localization!

3.6.22. (a) Prove that $\Sigma(-)$ does not preserve all weak equivalences of based spaces. (Hint: Consider the subspace $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}_{>0}\} \subset \mathbb{R}$, or equivalently the one-point compactification of $\mathbb{Z}_{>0}$. Its suspension is an infinite shrinking wedge of circles.)

(b) Prove that $\Sigma^\infty(-)$ does not send all weak equivalences of based spaces to stable equivalences of spectra.

3.6.23. Complete the list of left-derived functors from Proposition 3.4.25:

(a) Prove that the free spectrum functor $F_n\colon \mathbf{Top}_* \to \mathbf{Sp}$ can be left-derived.

(b) Prove that the handicrafted smash product $\wedge\colon \mathbf{Sp} \times \mathbf{Sp} \to \mathbf{Sp}$ can be left-derived.

3.6.24. We lift Exercises 2.7.26 and 2.7.30 to the homotopy category. Prove that the stable homotopy groups of the left-derived coproduct $\mathbb{L}\bigvee$ and right-derived product $\mathbb{R}\prod$ are, respectively, the coproduct and the product of the homotopy groups:

$$\pi_n\left(\mathbb{L}\bigvee_\alpha X_\alpha\right) \cong \bigoplus_\alpha \pi_n(X_\alpha),$$
$$\pi_n\left(\mathbb{R}\prod_\alpha X_\alpha\right) \cong \prod_\alpha \pi_n(X_\alpha).$$

3.6.25. Suppose $\{X_\alpha\}_{\alpha\in A}$ is any set of spectra. There is always a map from the wedge to the (right-derived) product

$$\bigvee_{\alpha\in A} X_\alpha \longrightarrow \mathbb{R}\prod_{\alpha\in A} X_\alpha.$$

Prove that this map is a stable equivalence of spectra (in other words an isomorphism in $\mathrm{Ho}\,\mathbf{Sp}$) iff for every $n \in \mathbb{Z}$, finitely many of the spectra X_α have $\pi_n(X_\alpha) \neq 0$.

3.6.26. (a) Extend Example 3.4.26 to based spaces. Show that the pushout operation

$$\mathbf{Top}_*^{\{\bullet\leftarrow\bullet\rightarrow\bullet\}} \to \mathbf{Top}_*$$

can be left-deformed. Its left-derived functor is obtained by *first* making the spaces A, X, and Y well-based, *then* taking the homotopy pushout.

(b) Extend Example 3.4.26 to spectra. Show that the pushout operation

$$\mathbf{Sp}^{\{\bullet\leftarrow\bullet\rightarrow\bullet\}} \to \mathbf{Sp}$$

can be left-deformed. Its left-derived functor is the homotopy pushout. (No assumptions are needed on the basepoints.)

3.6.27. (a) As in Example 3.4.26, show that the sequential colimit functor

$$\operatorname*{colim}_{n\to\infty} : \mathbf{Top}^{\{\bullet\rightarrow\bullet\rightarrow\cdots\}} \to \mathbf{Top}$$

is left-deformable, and its left-derived functor is the mapping telescope of Definition 1.5.5.

(b) As in Exercise 3.6.26, extend this to based spaces and spectra. No assumptions are needed on the basepoints.

3.6.28. (a) Extend Example 3.4.34 to based spaces and spectra. Show that the pullback is right-deformable, and its right-derived functor is the homotopy pullback. No assumptions are needed on the basepoints.

(b) Show that the sequential limit functor

$$\lim_{n\to\infty} : \mathbf{Top}^{\{\cdots\rightarrow\bullet\rightarrow\bullet\}} \to \mathbf{Top}$$

is right-deformable, and its right-derived functor is the mapping microscope of Definition 1.5.21.

(c) The previous part works equally well for based spaces. However, for spectra, show that the right-derived functor of

$$\lim_{n\to\infty} : \mathbf{Sp}^{\{\cdots\rightarrow\bullet\rightarrow\bullet\}} \to \mathbf{Sp}$$

is obtained by *first* making the spectra $X(i)$ into Ω-spectra, *then* taking the homotopy limit at each spectrum level. Therefore the "homotopy limit in spectra" has to be constructed by making the inputs into Ω-spectra first.

3.6.29. Let $\mathbf{C}$ and $\mathbf{D}$ be categories with weak equivalences, let $\mathbf{I}$ be a fixed small category, and assume that the colimit functors $\mathbf{C}^{\mathbf{I}} \to \mathbf{C}$ and $\mathbf{D}^{\mathbf{I}} \to \mathbf{D}$ for $\mathbf{I}$-shaped diagrams exist and are left-deformable. In each case we call the left-derived functor the *homotopy colimit*, hocolim $= \mathbb{L}$colim, following Definition 3.4.29.

(a) Let $F : \mathbf{C} \to \mathbf{D}$ be both a left adjoint, and a left-deformable functor. Use Proposition 3.5.20 to prove that there is a weak equivalence $(\text{hocolim} \circ \mathbb{L}F) \simeq (\mathbb{L}F \circ \text{hocolim})$, and it is the unique map such that the following square commutes in the homotopy category of functors:

$$\begin{array}{ccc} \text{hocolim} \circ \mathbb{L}F & \xrightarrow{\sim} & \mathbb{L}F \circ \text{hocolim} \\ \downarrow & & \downarrow \\ \text{colim} \circ F & \xrightarrow{\cong} & F \circ \text{colim}. \end{array}$$

This is a derived version of Theorem 1.6.9, that left adjoints preserve colimits.

(b) Formulate and prove the result that right-deformable right adjoints preserve homotopy limits.

3.6.30. We lift Exercise 2.7.35 to the homotopy category.

(a) Prove that the left-derived suspension spectrum functor $\mathbb{L}\Sigma^\infty\colon \operatorname{Ho}\mathbf{Top}_* \to \operatorname{Ho}\mathbf{Sp}$ sends cofiber sequences of spaces to cofiber/fiber sequences of spectra.

(b) Prove that the right-derived evaluation functor $\Omega^\infty = \mathbb{R}\mathrm{ev}_0\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Top}_*$ sends cofiber/fiber sequences of spectra to fiber sequences of spaces.

3.6.31. Give an example of a functor F on a category with weak equivalences (satisfying the 2-out-of-3 property), such that:

(a) F has a left-derived functor but no right-derived functor. (With proof.)

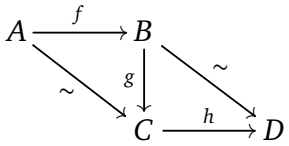
(b) F has neither a left- nor a right-derived functor.

(c) F has a left- and a right-derived functor, and the composite $\mathbb{L}F \to F \to \mathbb{R}F$ is not an equivalence.

(d) F has a left- and a right-derived functor, and the composite $\mathbb{L}F \to F \to \mathbb{R}F$ is an equivalence, but F is not homotopical.

(Hint: Let $F\colon \mathbf{I} \to \mathbf{Top}$, where $\mathbf{Top}$ has the usual weak equivalences, and $\mathbf{I}$ is as small as possible.)

3.6.32. Let $\mathbf{C}$ be a category and W a class of morphisms in $\mathbf{C}$. We say W satisfies the **2-out-of-6 property** if for any three composable maps f, g, h, if $g \circ f$ and $h \circ g$ are in W as shown below, then f, g, h, and $h \circ g \circ f$ are also in W.



(a) Prove that 2 out 6 implies 2-out-of-3. So it is a stronger condition.

(b) As in Example 3.4.14, prove that if $W = F^{-1}(\text{isomorphisms})$ for some functor F, then W has 2-out-of-6. Conclude that the class of stable equivalences in $\mathbf{Sp}$ has 2-out-of-6.

(c) Show that any class of weak equivalences W can be enlarged to have 2-out-of-6, without changing the resulting homotopy category $\mathbf{C}\langle W^{-1}\rangle$.

3.6.33. Suppose F has a left-derived functor. Adapt the proof of Proposition 3.5.2 to show that $\mathbb{L}F$ is terminal among functors $G\colon \operatorname{Ho}\mathbf{C} \to \operatorname{Ho}\mathbf{D}$ with a natural transformation

$G \Rightarrow F$ as functors $\mathbf{C} \to \mathrm{Ho}\,\mathbf{D}$. In other words, given any other such G, there is a unique natural transformation $G \Rightarrow \mathbb{L}F$ on $\mathrm{Ho}\,\mathbf{C}$, that on $\mathbf{C}$ commutes with the given natural transformations to F. We also say that $\mathbb{L}F$ is the *right Kan extension* of F along $\delta : \mathbf{C} \to \mathrm{Ho}\,\mathbf{C}$.

3.6.34. Prove that the evaluation functor and underlying unbased space

$$\mathbf{Sp} \xrightarrow{\mathrm{ev}_n} \mathbf{Top}_* \xrightarrow{U} \mathbf{Top}$$

satisfy the hypotheses of Lemma 3.5.13, and therefore their right-derived functors compose, as in Example 3.5.15.

3.6.35. In this exercise we give a less ad-hoc definition of the rationalization from Example 2.5.34. Let X be any spectrum, and let us attempt to define the rationalization $X_{\mathbb{Q}}$ as the the homotopy colimit of $X \xrightarrow{2} X \xrightarrow{3} X \xrightarrow{4} X \ldots$, where n denotes the identity map of X added to itself n times.

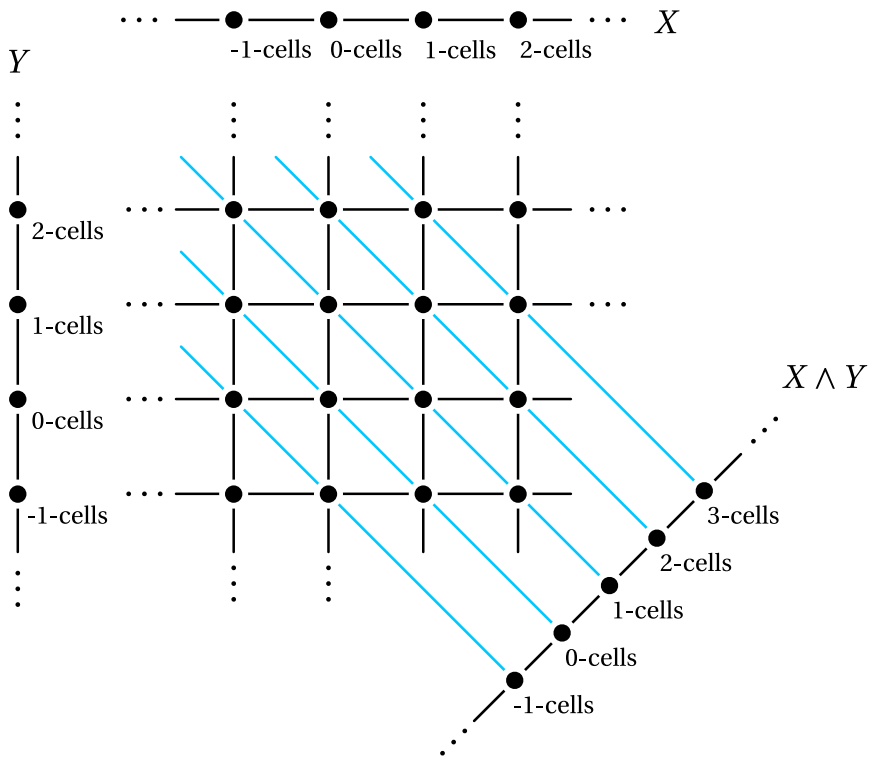
(a) Explain the issue with this definition. How would you fix it?

(b) Explain why your revised definition is equivalent to the rationalization $\mathrm{sh}^{-n} M(\mathbb{Q}, n) \wedge X$ that we saw earlier in Example 2.5.34.

Chapter 4

Properties of the smash product

The smash product of spectra, denoted $\wedge$, plays the role of the tensor product $\otimes$ in higher algebra. The fundamental intuition about $X \wedge Y$ is that it has a stable $(m+n)$-cell for each choice of a stable m-cell of X and a stable n-cell of Y:



This is enough to get a feel for what $X \wedge Y$ looks like, at least up to stable equivalence.

We defined the smash product already in Definition 2.3.23, by picking a sequence of values of p and q and defining $(X \wedge Y)_{p+q} = X_p \wedge Y_q$. Morally, this ought to make spectra

into a **symmetric monoidal category**, meaning we have isomorphisms

$$(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z), \qquad X \wedge Y \cong Y \wedge X, \qquad X \wedge \mathbb{S} \cong X.$$

The problem is, the construction we gave depends on choices, and different choices give spectra that are not even isomorphic, only equivalent. As a result, it is impossible to prove the above isomorphisms in the category of spectra **Sp**. We can only give such isomorphisms in the homotopy category Ho **Sp**.

In Chapter 6 we will give an explicit solution to this problem, modifying the category of spectra **Sp** to the category of **symmetric spectra** $\mathbf{Sp}^\Sigma$, and making $\mathbf{Sp}^\Sigma$ into a symmetric monoidal category. Unfortunately, doing this in detail requires a long detour through the theory of model categories, so that we can show that symmetric spectra are equivalent to sequential spectra, and prove that the smash product on $\mathbf{Sp}^\Sigma$ preserves stable equivalences.

It's important not to get lost in the theoretical machinery, and to remember what's important, why we care about the smash product. So in this chapter we give the propaganda for $\wedge$. We treat the smash product of spectra as a black box, assuming it exists, and showing what you can do with it: define rings and modules, and take duals and traces. It makes our "brave new" form of homological algebra much richer, by allowing us to study all sorts of multiplicative structures.

We also open up the box a little bit and define the smash product explicitly in the **Spanier-Whitehead category**, the subcategory of Ho **Sp** consisting of only the finite spectra. Even here the definition is a little complicated, signaling the difficulties that lie ahead when we define the smash product for all spectra.

We also show how to lift the Poincaré duality theorem to a theorem about spectra called Atiyah duality, and generalize it to a result called Spanier-Whitehead duality. We get as a corollary a Poincaré duality theorem that holds with coefficients in any extraordinary homology theory E.

4.1 Symmetric monoidal categories

The most important thing to know about the smash product is that it makes the stable homotopy category Ho **Sp** into a symmetric monoidal category.

4.1.1 Definition of a symmetric monoidal category

Definition 4.1.1. A **symmetric monoidal category** is a category $\mathbf{C}$, a functor

$$\otimes : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C},$$

a unit object $I \in \operatorname{ob}\mathbf{C}$, and natural isomorphisms

$$\alpha\colon (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z), \qquad \gamma\colon X \otimes Y \cong Y \otimes X, \qquad \rho\colon X \otimes I \cong X,$$

that are coherent (see Definition 4.1.4). We often drop the notation for these isomorphisms and just say "the canonical isomorphism," because they tend to be obvious in most examples.

Example 4.1.2. We list several symmetric monoidal categories, with their tensor and unit object.[1]

- $\mathbf{Set}, \amalg, \varnothing$
- $\mathbf{Set}, \times, *$
- $\mathbf{Top}, \amalg, \emptyset$
- $\mathbf{Top}, \times, *$
- $\mathbf{Top}_*, \vee, *$
- $\mathbf{Top}_*, \times, *$
- $\mathbf{Top}_*, \wedge, S^0$
- $\mathbf{Sp}, \vee, *$
- $\mathbf{Sp}, \times, *$
- $\mathbf{Ab}, \oplus, 0$
- $\mathbf{Ab}, \otimes, \mathbb{Z}$
- $\mathbf{Mod}_k, \oplus, 0$
- $\mathbf{Mod}_k, \otimes_k, k$
- $\mathbf{GrMod}_k, \oplus, 0$
- $\mathbf{GrMod}_k, \otimes_k, k[0]$
- $\mathbf{Ch}(k), \oplus, 0$
- $\mathbf{Ch}_{\geq 0}(k), \otimes_k, k[0]$
- $\mathbf{Ch}(k), \otimes_k, k[0]$
- $\operatorname{Ho}\mathbf{Top}, \amalg, \emptyset$
- $\operatorname{Ho}\mathbf{Top}, \times, *$
- $\operatorname{Ho}\mathbf{Top}_*, \vee^{\mathbb{L}}, *$
- $\operatorname{Ho}\mathbf{Top}_*, \times, *$
- $\operatorname{Ho}\mathbf{Top}_*, \wedge^{\mathbb{L}}, S^0$
- $\operatorname{Ho}\mathbf{Sp}_*, \vee, *$
- $\mathscr{D}(k), \oplus, 0$
- $\mathscr{D}(k)_{\geq 0}, \otimes_k^{\mathbb{L}}, k[0]$
- $\mathscr{D}(k), \otimes_k^{\mathbb{L}}, k[0]$

Here k is any field or commutative ring, and $k[0]$ refers to the chain complex or graded module that has just a k in degree 0. The category $\mathbf{GrMod}_k$ means graded modules, and $\mathscr{D}(k)$ is the derived category, in other words the homotopy category of chain complexes of k-modules $\operatorname{Ho}\mathbf{Ch}(k)$.

So for instance, in $\mathbf{Ab}$ we have the isomorphism $X \otimes \mathbb{Z} \cong X$, while in $\mathbf{Top}_*$ we have the isomorphism $X \wedge S^0 \cong X$. In this sense the 0-sphere S^0 is like the integers $\mathbb{Z}$, because each one is an identity element for its respective product.

Notice that the same category can have several different symmetric monoidal structures. A symmetric monoidal category is a category with an extra *structure* placed on top of it, not just a *property* of the category.

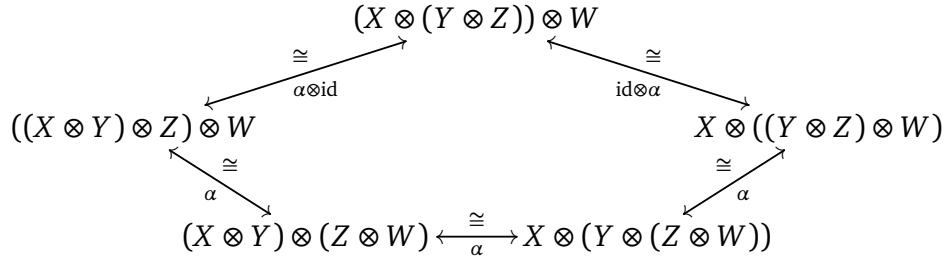
Remark 4.1.3. The choice of isomorphisms in Definition 4.1.1 are also part of the structure. We could keep the same product, but change the *isomorphisms*, and get a different symmetric monoidal category! For instance, in graded k-modules $\mathbf{GrMod}_k$, we could

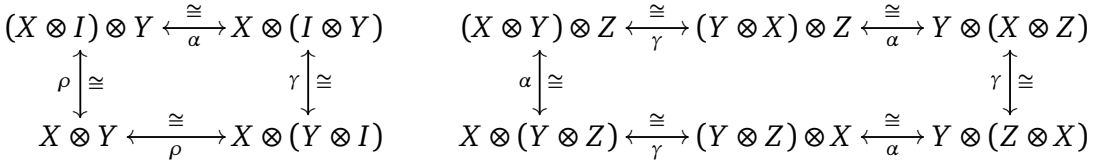
[1] For the category $(\mathbf{Top}_*, \wedge, S^0)$ it is necessary to use CGWH topological spaces, and not ordinary topological spaces, so that the smash product is associative up to isomorphism. See Exercise 1.7.12.

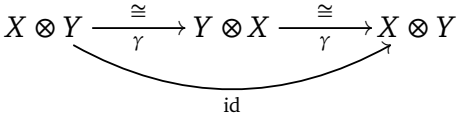
define the symmetry isomorphism $X \otimes Y \cong Y \otimes X$ by sending each tensor of homogeneous terms $a \otimes b$ to either $b \otimes a$, or to $(-1)^{|a||b|}(b \otimes a)$. The first choice is the most obvious isomorphism, but the second choice is the one that obeys the Koszul sign rule.

When we pass to chain complexes **Ch**, we have to make the second choice here and send $a \otimes b$ to $(-1)^{|a||b|}(b \otimes a)$. If we didn't, the isomorphism $X \otimes Y \cong Y \otimes X$ would fail to be a map of chain complexes! See Exercise 4.4.1.

Definition 4.1.4. In Definition 4.1.1, we should explain what we mean by "coherent." If we take any string of distinct objects $X_1, \ldots, X_n$ and apply strings of the above isomorphisms to rearrange the objects, add and remove units, and regroup parentheses, if we ever come back to the same expression, the composite map must be the identity. For example, the following composites have to be the identity:







In fact, these diagrams suffice. As soon as they all commute, any other diagram of the same form must commute.

Theorem 4.1.5 (Mac Lane, Kelly)**.** *If the above diagrams commute, then any formal diagram formed from distinct objects $X_1, \ldots, X_n$ by applying instances of the isomorphisms α, γ and ρ must commute.*

In practical terms, this coherence property means that we are allowed to drop the parentheses and just write $X \otimes Y \otimes Z$. We also don't have a choose an ordering when we multiply distinct terms together:

$$X \otimes Y \otimes Z \cong Z \otimes X \otimes Y.$$

However, we *do* have to keep track of the ordering when we have repeated terms in the tensor product. In particular, the swap map $\gamma : X \otimes X \cong X \otimes X$ is often not the identity. In **Top**, $\times$ it's the self-map of the space $X \times X$ that switches the two coordinates. In $\mathbf{Ch}_{\geq 0}(\mathbb{Z}), \otimes$, it switches the tensors around and applies minus signs whenever we switch terms of odd degree.

So for example, if we use the swap maps to form an isomorphism

$$X \otimes X \otimes X \otimes Y \otimes Y \cong X \otimes X \otimes X \otimes Y \otimes Y,$$

then we don't have to keep track of exactly which instances of the swap map γ we applied, but we do have to keep track of the resulting permutation $\sigma \in \Sigma_3 \times \Sigma_2$. The permutation σ determines which isomorphism we get.

Example 4.1.6. If **C** is a category with all finite products $\times$ and terminal object $*$, then $(\mathbf{C}, \times, *)$ is a symmetric monoidal category in a canonical way (Exercise 4.4.2). We say **C** is **cartesian monoidal** in this case. For instance, **Set**, **Top**, $\mathbf{Top}_*$, and **Ab** are all cartesian monoidal.

Dually, if **C** has all finite coproducts and an initial object $\emptyset$, then $(\mathbf{C}, \amalg, \emptyset)$ is a symmetric monoidal category – in this case **C** is **cocartesian monoidal**.

Remark 4.1.7. Often a category **C** has two symmetric monoidal structures $\oplus$ and $\otimes$, and the tensor distributes over the direct sum:

$$A \otimes (B_1 \oplus B_2) \cong (A \otimes B_1) \oplus (A \otimes B_2).$$

For instance, if **C** has coproducts and $\otimes$ preserves coproducts in each variable, then we get isomorphisms as above. We'll frequently encounter the situation that **C** is an **additive category** with direct sum $\oplus$ (see Theorem 3.2.5), and also a symmetric monoidal category under some tensor product $\otimes$. In this situation, we ask that the tensor preserves finite sums in each variable, so that we get isomorphisms as above. See Exercise 4.4.4 for more details.

Several of the examples in Example 4.1.2 are homotopy categories. To explain how this works, suppose $(\mathbf{C}, \otimes, I)$ is a symmetric monoidal category, and **C** has a class of weak equivalences W satisfying the 2-out-of-3 property (Definition 3.4.13), so that we can form the homotopy category $\mathrm{Ho}\,\mathbf{C} = \mathbf{C}\langle W^{-1} \rangle$.

Lemma 4.1.8. *Suppose there is a full subcategory* $\mathbf{A} \subseteq \mathbf{C}$ *such that*

- $\otimes$ *preserves weak equivalences between objects in* **A**,
- *if* $X, Y \in \mathbf{A}$ *then* $X \otimes Y \in \mathbf{A}$,

- *the unit I is also in* $\mathbf{A}$,[2]
- *there is a functor* $Q\colon \mathbf{C} \to \mathbf{C}$ *landing in* $\mathbf{A}$, *and*
- *there is a natural weak equivalence* $q\colon QX \xrightarrow{\sim} X$.

Then the left-derived tensor $X \otimes^{\mathbb{L}} Y = QX \otimes QY$ *and the unit* $I \in \mathbf{C}$ *make* $\operatorname{Ho}\mathbf{C}$ *into a symmetric monoidal category.*

For instance, these conditions are satisfied in $(\mathbf{Top}_*, \wedge, S^0)$, where $\mathbf{A}$ is the subcategory of based CW complexes. So $(\operatorname{Ho}\mathbf{Top}_*, \wedge^{\mathbb{L}}, S^0)$ is a symmetric monoidal category.

These conditions are also satisfied in $(\mathbf{Ch}_{\geq 0}(\mathbb{Z}), \otimes, \mathbb{Z}[0])$, where $\mathbf{A}$ is the subcategory of levelwise projective chain complexes. So the derived category $(\mathscr{D}_{\geq 0}(\mathbb{Z}), \otimes^{\mathbb{L}}, \mathbb{Z}[0])$ is also a symmetric monoidal category.

Proof. As in the proof of Lemma 3.4.18, the left-derived tensor $X \otimes^{\mathbb{L}} Y = QX \otimes QY$ preserves equivalences, because Q preserves equivalences and $\otimes$ preserves equivalences between objects in $\mathbf{A}$. Therefore it defines a functor

$$\otimes^{\mathbb{L}}\colon \operatorname{Ho}\mathbf{C} \times \operatorname{Ho}\mathbf{C} \to \operatorname{Ho}\mathbf{C}.$$

We define the associativity isomorphism by removing extra Qs and then using the associativity isomorphism for $\otimes$:

$$\begin{array}{ccccc}
(X \otimes^{\mathbb{L}} Y) \otimes^{\mathbb{L}} Z & = & Q(QX \otimes QY) \otimes QZ & \qquad & QX \otimes Q(QY \otimes QZ) & = & X \otimes^{\mathbb{L}} (Y \otimes^{\mathbb{L}} Z) \\
 & & \sim \big\downarrow{\scriptstyle q\otimes 1} & & \sim \big\downarrow{\scriptstyle 1\otimes q} & & \\
 & & (QX \otimes QY) \otimes QZ & \xleftrightarrow{\cong} & QX \otimes (QY \otimes QZ) & &
\end{array}$$

The vertical maps are equivalences because q is an equivalence between objects in $\mathbf{A}$, so $\otimes$ sends this to an equivalence. Therefore it is an isomorphism in the homotopy category $\operatorname{Ho}\mathbf{C}$. The composite of these maps is therefore an isomorphism in the homotopy category. (See also Example 3.5.18, which constructs the same isomorphism in a different way.)

We construct the other two isomorphisms in a similar way. It is an exercise to see that their coherence follows from the coherence of the same isomorphisms in $\mathbf{C}$. □

Remark 4.1.9. On the equivalent subcategory $\operatorname{Ho}\mathbf{A} \subseteq \operatorname{Ho}\mathbf{C}$, this symmetric monoidal structure is isomorphic to a simpler one: just use the product $X \otimes Y$. It passes to the homotopy category $\operatorname{Ho}\mathbf{A}$ because it preserves all equivalences. The natural isomorphism $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ in $\mathbf{A}$ also gives a natural isomorphism in $\operatorname{Ho}\mathbf{A}$, and similarly for the other isomorphisms.

[2] It's actually enough if the map $QI \xrightarrow{\sim} I$ induces an equivalence $QI \otimes QX \xrightarrow{\sim} I \otimes QX \cong QX$.

All of the examples in the right-hand column of Example 4.1.2 arise this way. For instance, the homotopy category of based spaces $\mathrm{Ho}\,\mathbf{Top}_*$ is symmetric monoidal under the left-derived smash product $X \wedge^{\mathbb{L}} Y = QX \wedge QY$, where QX is a cell complex equivalent to X. If we restrict attention to cell complexes, or well-based spaces, we can just use the smash product $X \wedge Y$ instead of the left-derived smash product.

Example 4.1.10 (Smash product as a black box, part 1)**.** As we've already discussed, in this chapter, we develop the smash product of spectra as a black box. Here's what's written on the label of that box:

- There is a category of spectra $\widehat{\mathbf{Sp}}$, which is not the same as the category of sequential spectra $\mathbf{Sp}$, but has all of the same constructions, such as suspension spectra, wedge sums, cofibers, and so on. There is an equivalence of homotopy categories $\mathrm{Ho}\,\widehat{\mathbf{Sp}} \simeq \mathrm{Ho}\,\mathbf{Sp}$.
- $\widehat{\mathbf{Sp}}$ is a symmetric monoidal category. The tensor is called the smash product $\wedge$. We will not define it explicitly here, but along the equivalence $\mathrm{Ho}\,\widehat{\mathbf{Sp}} \simeq \mathrm{Ho}\,\mathbf{Sp}$, it is equivalent to the handicrafted smash product of Definition 2.3.23.
- The unit of $\widehat{\mathbf{Sp}}$ is the sphere spectrum $\mathbb{S}$. More generally, smashing with any suspension spectrum $\Sigma^\infty K$ gives the tensor from Definition 2.3.6:

  $$(\Sigma^\infty K) \wedge X \cong K \wedge X.$$

  In particular, the smash product of two suspension spectra is a suspension spectrum,

  $$(\Sigma^\infty A) \wedge (\Sigma^\infty B) \cong \Sigma^\infty (A \wedge B).$$

  These claims are stronger than those in Example 2.3.25 because these maps are *isomorphisms* in the category $\widehat{\mathbf{Sp}}$, not just stable equivalences.
- The smash product preserves colimits in each variable. It follows that the smash product preserves wedge sums, homotopy pushouts, and mapping telescopes in each variable as well (Exercise 4.4.20).
- We have isomorphisms

  $$F_d A \wedge F_e B \cong F_{d+e}(A \wedge B).$$

  It follows that the smash product of cellular spectra is a cellular spectrum (Exercise 4.4.21).
- Not only does the smash product preserve cellular spectra, it also preserves all stable equivalences between them. So the hypotheses of Lemma 4.1.8 are satisfied, making $\mathrm{Ho}\,\widehat{\mathbf{Sp}}$ into a symmetric monoidal category. The unit is again $\mathbb{S}$, and the product is the left-derived smash product

  $$X \wedge^{\mathbb{L}} Y = QX \wedge QY,$$

where Q is any replacement by a cellular spectrum, as in Theorem 2.6.12. We also get that the smash product preserves cofiber/fiber sequences in each variable (Exercise 4.4.22).

In summary, we get two symmetric monoidal categories $(\widehat{\mathbf{Sp}}, \wedge, \mathbb{S})$ and $(\mathrm{Ho}\,\widehat{\mathbf{Sp}}, \wedge^{\mathbb{L}}, \mathbb{S})$.

The fact that the smash product preserves cofiber sequences in each slot is often captured by saying that $\mathrm{Ho}\,\widehat{\mathbf{Sp}}$ is a **tensor triangulated category**. See [HPS97] for more details on this point of view.

Remark 4.1.11. By Exercise 4.4.13, the symmetric monoidal structure on $\mathrm{Ho}\,\widehat{\mathbf{Sp}}$ transfers over to give a symmetric monoidal structure on $\mathrm{Ho}\,\mathbf{Sp}$.

4.1.2 Rings and modules

Symmetric monoidal categories are a setting in which we can talk abstractly about rings and modules over them.

Definition 4.1.12. Let $\mathbf{C}, \otimes, I$ be a symmetric monoidal category. A **monoid** (or **algebra**) in $\mathbf{C}$ is an object R together with a multiplication map and a unit map

$$\mu : R \otimes R \longrightarrow R, \qquad \eta : I \longrightarrow R,$$

such that the following two diagrams commute.

$$\begin{array}{ccc} R\otimes R\otimes R & \xrightarrow{\mu\otimes 1} & R\otimes R \\ {\scriptstyle 1\otimes\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ R\otimes R & \xrightarrow[\mu]{} & R \end{array} \qquad \begin{array}{ccccc} I\otimes R & \xrightarrow{\eta\otimes 1} & R\otimes R & \xleftarrow{1\otimes\eta} & R\otimes I \\ & \searrow^{\cong} & \downarrow{\scriptstyle \mu} & \swarrow_{\cong} & \\ & & R. & & \end{array}$$

You should think of these as capturing the idea that $(ab)c = a(bc)$, and $1a = a = a1$.

Definition 4.1.13. A **left module** in over R is another object M together with an action map

$$\alpha : R \otimes M \longrightarrow M,$$

such that the following two diagrams commute.

$$\begin{array}{ccc} R\otimes R\otimes M & \xrightarrow{\mu\otimes 1} & R\otimes M \\ {\scriptstyle 1\otimes\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ R\otimes M & \xrightarrow[\alpha]{} & M \end{array} \qquad \begin{array}{ccc} I\otimes M & \xrightarrow{\eta\otimes 1} & R\otimes M \\ & \searrow^{\cong} & \downarrow{\scriptstyle \alpha} \\ & & M. \end{array}$$

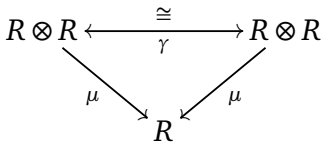
Again, this captures the idea that $(ab)m = a(bm)$, and $1m = m$. A **right module** is defined the same way, except that the R is on the right, so we get $m(ab) = (ma)b$ instead of $(mb)a$.

We typically only consider rings and modules in the "multiplicative" examples of symmetric monoidal categories. In the "additive" examples, these concepts turn out to not give any interesting examples. See Exercise 4.4.7.

Example 4.1.14. We consider monoids and modules in several examples.

- In **Set**, $\times, *$, a monoid object is the same thing as a monoid G. A left module over G is the same thing as a set with left G-action (Definition 5.3.8).
- In **Ab**, $\otimes, \mathbb{Z}$, a monoid object is the same thing as a ring R. The fact that the multiplication is a map $R \otimes R \to R$ makes it bilinear in each slot, so that the axioms of a ring are satisfied. (This is a very concise way of remembering the axioms for a ring!) A left module over R is the same thing as a left module in the usual sense.
- In **Top**, $\times, *$, a monoid object is the same thing as a topological monoid G. A left module over G is the same thing as a space with a continuous left G-action (Definition 5.3.8).
- In $\mathbf{Top}_*, \wedge, S^0$, a monoid object is the same thing as a based topological monoid G. These are spaces with associative multiplication, and two special points, the basepoint 0 and the unit point 1. We have $0 * a = 0$ and $1 * a = a$ for every $a \in G$.
- In $\mathbf{Mod}_k, \otimes_k, k$, a monoid is the same thing as a k-algebra.
- In $\mathbf{GrMod}_{\mathbb{Z}}, \otimes, \mathbb{Z}[0]$, a monoid is the same thing as a graded ring, and a module is a graded module.
- In $\mathbf{Ch}_{\geq 0}(\mathbb{Z}), \otimes, \mathbb{Z}[0]$, a monoid is the same thing as a differential graded algebra (DGA).
- In the classical homotopy category $h\mathbf{Top}, \times, *$, a monoid is the same thing as an associative H-space – a space that has a multiplication $X \times X \to X$ that is associative and unital up to homotopy.

Definition 4.1.15. A monoid R in $\mathbf{C}$ is **commutative** if the following diagram commutes.



Lemma 4.1.16. *If R is commutative then every left module over R is also a right module and vice-versa.*

Example 4.1.17.

- In **Set**, $\times, *$, this gives commutative monoids.
- In **Ab**, $\otimes, \mathbb{Z}$, this gives commutative rings.

- In $\mathbf{GrMod}_{\mathbb{Z}}, \otimes, \mathbb{Z}[0]$, this gives commutative graded rings. Whether the commutativity is $ab = ba$ or $ab = (-1)^{|a||b|}(ba)$ depends on which symmetric monoidal structure we choose.
- In $\mathbf{Ch}_{\geq 0}(k), \otimes, k[0]$, this gives commutative differential graded algebras (CDGAs). Again, we have to use the structure in which $ab = (-1)^{|a||b|}(ba)$, otherwise the ring multiplication doesn't give a map of chain complexes.

Definition 4.1.18. A **ring spectrum** is a monoid object in $(\widehat{\mathbf{Sp}}, \wedge, \mathbb{S})$. So it is a spectrum R with multiplication and unit maps

$$\mu : R \wedge R \longrightarrow R, \qquad \eta : \mathbb{S} \longrightarrow R,$$

such that the diagrams in Definition 4.1.12 commute. A **commutative ring spectrum** is a commutative monoid object in $(\widehat{\mathbf{Sp}}, \wedge, \mathbb{S})$, and a **module spectrum** is a module object over R.

Example 4.1.19. The sphere spectrum $\mathbb{S}$ is the unit of the smash product, and is therefore a commutative ring spectrum, whose multiplication and unit maps

$$\mu : \mathbb{S} \wedge \mathbb{S} \cong \mathbb{S}, \qquad \eta : \mathbb{S} = \mathbb{S}$$

are the canonical isomorphism and the identity map, respectively. This is true more generally: in any symmetric monoidal category $\mathbf{C}$ the unit I is a commuative monoid in $\mathbf{C}$. See Exercise 4.4.9.

Remark 4.1.20. It used to be that the term "ring spectrum" referred to a monoid object in the homotopy category $(\mathrm{Ho}\,\widehat{\mathbf{Sp}}, \wedge^{\mathbb{L}}, \mathbb{S})$, and a "highly structured ring spectrum" was a monoid in $(\widehat{\mathbf{Sp}}, \wedge, \mathbb{S})$. However, this terminology has fallen out of favor. Monoids in the actual category of spectra turn out to be way more useful than monoids in the homotopy category. So we don't spend much time talking about monoids in the homotopy category anymore.

4.1.3 Symmetric monoidal functors

A symmetric monoidal functor is a map of symmetric monoidal categories that respects the tensor product. There are different levels to this:

Definition 4.1.21. Suppose $(\mathbf{C}, \otimes_{\mathbf{C}}, I_{\mathbf{C}})$ and $(\mathbf{D}, \otimes_{\mathbf{D}}, I_{\mathbf{D}})$ are two symmetric monoidal categories. A **lax symmetric monoidal functor** is

- A functor $F : \mathbf{C} \to \mathbf{D}$,
- A natural map $m : F(X) \otimes_{\mathbf{D}} F(Y) \to F(X \otimes_{\mathbf{C}} Y)$, and

- a map $i : I_{\mathbf{D}} \to F(I_{\mathbf{C}})$,

that are coherent in the sense of Definition 4.1.26 below. A **strong symmetric monoidal functor** is the same thing, except that the maps m and i are isomorphisms:

$$F(X) \otimes_{\mathbf{D}} F(Y) \cong F(X \otimes_{\mathbf{C}} Y), \qquad I_{\mathbf{D}} \cong F(I_{\mathbf{C}}).$$

Example 4.1.22. The forgetful functor $(\mathbf{Top}, \times, *) \to (\mathbf{Set}, \times, *)$ is strong symmetric monoidal. The product of spaces is the product on the underyling set.

Its left adjoint takes each set and gives it the discrete topology – this is also strong symmetric monoidal, because a product of discrete spaces is discrete.

Example 4.1.23. The forgetful functor $(\mathbf{Top}_*, \wedge, S^0) \to (\mathbf{Top}, \times, *)$ is lax symmetric monoidal. In other words, for based spaces X and Y, there is a canonical map $X \times Y \to X \wedge Y$, but it isn't an isomorphism.

Its left adjoint adds a disjoint basepoint $(-)_+$. This is strong symmetric monoidal, since

$$X_+ \wedge Y_+ \cong (X \times Y)_+, \qquad S^0 \cong (*)_+.$$

Example 4.1.24. The forgetful functor $(\mathbf{Ab}, \otimes, *) \to (\mathbf{Set}, \times, *)$ is lax symmetric monoidal. In other words, for abelian groups A and B, there is a canonical map of sets $A \times B \to A \otimes B$, but it isn't an isomorphism.

The left adjoint is the free abelian group functor $(\mathbf{Set}, \times, *) \to (\mathbf{Ab}, \otimes, *)$. It takes the set S to the free abelian group $\mathbb{Z}^{\oplus S} = \bigoplus_{s \in S} \mathbb{Z}$. This is strong symmetric monoidal, using the fact that tensor distributes over direct sum:

$$\mathbb{Z}^{\oplus S} \otimes \mathbb{Z}^{\oplus T} \cong \mathbb{Z}^{\oplus (S \times T)}, \qquad \mathbb{Z} \cong \mathbb{Z}^{\oplus \{*\}}.$$

Remark 4.1.25. There is a pattern to these examples: we have an adjoint pair $(F \dashv G)$ in which F is strong symmetric monoidal and G is at least lax symmetric monoidal. In fact, this is always true – if F is strong symmetric monoidal and has a right adjoint G, then G must be lax symmetric monoidal. See Exercise 4.4.11.

Definition 4.1.26. For a symmetric monoidal functor, coherence means that if we take any expression $F(X_1) \otimes \cdots \otimes F(X_n)$ (with the X_i distinct) and apply composites of the above maps, any two such composites that end at the same expression, must be the same map. As for symmetric monoidal categories, finitely many diagrams suffice. If the following diagrams commute, then the isomorphisms are coherent:

$$\begin{array}{ccccc}
(F(X) \otimes F(Y)) \otimes F(Z) & \xrightarrow{m \otimes \mathrm{id}} & F(X \otimes Y) \otimes F(Z) & \xrightarrow{m} & F((X \otimes Y) \otimes Z) \\
\cong \uparrow \downarrow \alpha & & & & \cong \uparrow \downarrow \alpha \\
F(X) \otimes (F(Y) \otimes F(Z)) & \xrightarrow{\mathrm{id} \otimes m} & F(X) \otimes F(Y \otimes Z) & \xrightarrow{m} & F(X \otimes (Y \otimes Z))
\end{array}$$

$$\begin{array}{ccc} F(X)\otimes I_{\mathbf{D}} & \xrightarrow{i} & F(X)\otimes F(I_{\mathbf{C}}) \\ \cong\uparrow\rho & & \downarrow m \\ F(X) & \xleftarrow[\cong]{\rho} & F(X\otimes I_{\mathbf{C}}) \end{array} \qquad \begin{array}{ccc} F(X)\otimes F(Y) & \xleftarrow[\cong]{\gamma} & F(Y)\otimes F(X) \\ \downarrow m & & \downarrow m \\ F(X\otimes Y) & \xleftarrow[\cong]{\gamma} & F(Y\otimes X) \end{array}$$

For strong symmetric monoidal functors, you can remember this by saying that, up to canonical isomorphism, there's only one way to tensor together a bunch of objects in $\mathbf{C}$ and apply F to them. Whether you apply F to them first, then tensor, or whether you tensor them first, then apply F, even if you add units in the middle for some reason, everything is identified up to canonical isomorphism.

One reason we care about monoidal functors is that they preserve monoids (rings) and modules over them.

Proposition 4.1.27. *If $F : \mathbf{C} \to \mathbf{D}$ is lax symmetric monoidal, and R is a monoid in $\mathbf{C}$, then $F(R)$ is a monoid in $\mathbf{D}$.*

Similarly, F takes modules over R to modules over $F(R)$, and takes commutative monoids to commutative monoids.

Proof. This is fairly easy – the multiplication on $F(R)$ is defined to be

$$F(R)\otimes_{\mathbf{D}} F(R) \xrightarrow{m} F(R\otimes_{\mathbf{C}} R) \xrightarrow{F(\mu)} F(R),$$

and the unit map is

$$I_{\mathbf{D}} \xrightarrow{i} F(I_{\mathbf{C}}) \xrightarrow{F(\eta)} F(R).$$

The required diagrams commute by coherence and the fact that they commuted in $\mathbf{C}$. See Exercise 4.4.15. The case of modules and commutative monoids is similar. □

Example 4.1.28.

- The forgetful functor $(\mathbf{Top}, \times, *) \to (\mathbf{Set}, \times, *)$ takes topological monoids to ordinary monoids.
- By the adjunction between $(\mathbf{Top}_*, \wedge, S^0)$ and $(\mathbf{Top}, \times, *)$, every monoid in the based sense is also a monoid in the unbased sense, and if G is a monoid in the unbased sense then G_+ is a monoid in the based sense.
- The free abelian group functor $(\mathbf{Set}, \times, *) \to (\mathbf{Ab}, \otimes, *)$ takes each monoid or group G to the ring $\mathbb{Z}[G]$, often called the **monoid ring** or **group ring**.

Example 4.1.29. Homology is lax symmetric monoidal, either as a functor

$$(\mathbf{Top}, \times, *) \to (\mathbf{GrAb}, \otimes, \mathbb{Z}[0])$$

or as a functor

$$(\mathbf{Ch}(\mathbb{Z}), \otimes, \mathbb{Z}[0]) \to (\mathbf{GrAb}, \otimes, \mathbb{Z}[0]).$$

In particular, we get maps $H_*(X) \otimes H_*(Y) \to H_*(X \times Y)$, that aren't isomorphisms in general (Theorem 1.3.11). Therefore, if X is a topological monoid, this makes $H_*(X)$ into a graded ring. The product is the **Pontryagin product** we defined in (1.3.12).

Example 4.1.30. For any symmetric monoidal category $\mathbf{C}$ satisfying the assumptions of Lemma 4.1.8, the map to the homotopy category $\mathbf{C} \to \mathrm{Ho}\,\mathbf{C}$ is lax symmetric monoidal. Specifically, the maps $q : QX \to X$ give the natural map

$$X \otimes^{\mathbb{L}} Y = QX \otimes QY \longrightarrow X \otimes Y.$$

Therefore, every monoid in $\mathbf{C}$ becomes a monoid in the homotopy category $\mathrm{Ho}\,\mathbf{C}$.

In particular, every topological monoid in **Top** is also an associative H-space (a monoid in $\mathrm{Ho}\,\mathbf{Top}$). Every ring spectrum in $\widehat{\mathbf{Sp}}$ is also a monoid in the stable homotopy category $\mathrm{Ho}\,\widehat{\mathbf{Sp}}$.

Example 4.1.31 (Smash product as a black box, part 2)**.** We extend the black box from Example 4.1.10 by claiming that the following three functors are symmetric monoidal.

- First we consider the suspension spectrum functor, either from based or unbased spaces:

$$(\mathbf{Top}, \times, *) \xrightarrow{(-)_+} (\mathbf{Top}, \times, *) \xrightarrow{\Sigma^\infty} (\widehat{\mathbf{Sp}}, \wedge, \mathbb{S}).$$
$$\Sigma^\infty_+$$

  We claim that in both cases this functor is strong symmetric monoidal. In particular, we have isomorphisms

$$(\Sigma^\infty_+ X) \wedge (\Sigma^\infty_+ Y) \cong \Sigma^\infty_+ (X \times Y),^{3} \qquad \mathbb{S} \cong \Sigma^\infty_+ (*).$$

  If G is a group or monoid, the suspension spectrum $\Sigma^\infty_+ G$ is therefore a ring spectrum. This is sometimes called the **spherical group ring** and denoted $\mathbb{S}[G]$ to emphasize the analogy with $\mathbb{Z}[G]$. Concretely, the multiplication is by

$$(\Sigma^\infty_+ G) \wedge (\Sigma^\infty_+ G) \cong \Sigma^\infty_+ (G \times G) \xrightarrow{\Sigma^\infty_+(\mu)} \Sigma^\infty_+ G$$

  and the unit is by

$$\mathbb{S} \cong \Sigma^\infty_+ (*) \xrightarrow{\Sigma^\infty_+(\eta)} \Sigma^\infty_+ G.$$

  If G is commutative, then $\Sigma^\infty_+ G$ is a commutative ring spectrum.

[3] We already asserted the existence of this isomorphism in Example 4.1.10, but the claim here is stronger. Not only does this isomorphism exist, but along this isomorphism, the associativity and commutativity isomorphisms for the smash product of spectra agree with those same isomorphisms for topological spaces. We need this stronger claim to prove that $\Sigma^\infty_+ G$ is a ring – for instance, when checking that its associativity in $\widehat{\mathbf{Sp}}$ follows from the associativity of G in **Top**.

- Next, we claim that the Eilenberg–Mac Lane spectrum

$$(\mathbf{Ab}, \otimes, \mathbb{Z}) \xrightarrow{H(-)} (\widehat{\mathbf{Sp}}, \wedge, \mathbb{S})$$

  is lax symmetric monoidal. So, there are natural maps

$$HA \wedge HB \longrightarrow H(A \otimes B), \qquad \mathbb{S} \longrightarrow H\mathbb{Z}.$$

  Once we believe this claim, Proposition 4.1.27 tells us that for each ring R, its Eilenberg–Mac Lane spectrum HR is a ring spectrum. Spelling it out, the multiplication is

$$HR \wedge HR \longrightarrow H(R \otimes R) \xrightarrow{H(\mu)} HR$$

  and the unit is

$$\mathbb{S} \longrightarrow H\mathbb{Z} \xrightarrow{H(\eta)} HR.$$

  Furthermore, if R is commutative then HR is a commutative ring spectrum.

- Finally, we claim that the stable homotopy groups π_* are lax symmetric monoidal as a functor on the homotopy category of spectra:

$$(\mathrm{Ho}\,\widehat{\mathbf{Sp}}, \wedge^{\mathbb{L}}, \mathbb{S}) \xrightarrow{\pi_*} (\mathbf{GrAb}, \otimes, \mathbb{Z}[0]).$$

  For this we use the symmetric monoidal structure on **GrAb** in which the flip of $a \otimes b$ is $(-1)^{|a||b|} b \otimes a$.

  It follows that π_* is also lax symmetric monoidal as a functor on the category of spectra. Either way, there are natural maps

$$\pi_*(X) \otimes \pi_*(Y) \longrightarrow \pi_*(X \wedge Y), \qquad \mathbb{Z} \longrightarrow \pi_0(\mathbb{S}).$$

  It also follows that if R is a ring spectrum, then $\pi_*(R)$ is a graded ring. If R is commutative then $\pi_*(R)$ is graded-commutative, in other words commutative but using the Koszul sign rule.

  In particular, $\pi_*(HR) = R[0]$ gives our original ring back, and the stable homotopy groups of spheres $\pi_*(\mathbb{S})$ are a graded-commutative ring.

Remark 4.1.32. When $F : \mathbf{C} \to \mathbf{D}$ is a strong symmetric monoidal functor, it is helpful to think of **C** and **D** as "commutative rings" in categories, and F as a ring homomorphism, making **D** into an "algebra" over **C**.

In particular, **D** is a "module" over **C**, where objects in **C** act on the category **D** by the formula

$$X \cdot Y := F(X) \otimes_{\mathbf{D}} Y, \qquad X \in \mathbf{C},\ Y \in \mathbf{D}.$$

In particular, the suspension spectrum functor Σ^∞_+ is strong symmetric monoidal, so we can think of spectra as an "algebra" or "module" over unbased spaces. An unbased space A acts on spectra by sending each spectrum X to

$$A_+ \wedge X \cong (\Sigma^\infty_+ A) \wedge X.$$

This has all of the properties that you would expect for a commutative ring acting on a module, including associativity (up to isomorphism).

4.1.4 Closed symmetric monoidal categories

Definition 4.1.33. A **closed symmetric monoidal category C** is a symmetric monoidal category in which $X \otimes -$ always has a right adjoint $\mathrm{Hom}(X, -)$. In other words, a map $X \otimes Y \to Z$ is the same thing as a map $Y \to \mathrm{Hom}(X, Z)$.

We call $\mathrm{Hom}(-,-)$ the **internal hom** of **C**. Since maps of functors $X \otimes (-) \to X' \otimes (-)$ give maps of adjoints $\mathrm{Hom}(X', -) \to \mathrm{Hom}(X, -)$, this is a functor in both slots, not just the second slot. It therefore defines a functor

$$\mathrm{Hom}\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{C}$$

and we have natural bijections

$$\mathbf{C}(X, \mathrm{Hom}(Y, Z)) \cong \mathbf{C}(X \otimes Y, Z) \cong \mathbf{C}(Y, \mathrm{Hom}(X, Z)).$$

Example 4.1.34.

- The category of k-vector spaces, or more generally k-modules $(\mathbf{Mod}_k, \otimes_k, k)$ for a commutative ring k, is closed symmetric monoidal. The internal hom is $\mathrm{Hom}_k(X, Y)$, the set of k-linear maps from X to Y, made into a module by adding and scaling maps pointwise (on each input separately). We get the usual tensor-hom adjunction

$$X \otimes_k Y \to Z \qquad \longleftrightarrow \qquad X \to \mathrm{Hom}_k(Y, Z).$$

- The category $(\mathbf{Set}, \times, *)$ is closed symmetric monoidal using the set of maps $X \to Y$.
- The category $(\mathbf{Top}, \times, *)$ of (CGWH) topological spaces is closed symmetric monoidal using the space $\mathrm{Map}(X, Y)$ of continuous maps from X to Y.[4]
- The category $(\mathbf{Top}_*, \wedge, S^0)$ is closed symmetric monoidal using the space $\mathrm{Map}_*(X, Y)$ of continuous based maps from X to Y.

[4] This is another place where we needed to work with CGWH spaces. In "plain vanilla" topological spaces we don't get the adjunction between the product $\times$ and the mapping space.

- The category $(\mathbf{GrMod}_k, \otimes_k, k[0])$ is closed symmetric monoidal using the set of graded maps $\mathrm{Hom}_k(X,Y)$, whose dth level consists of maps from X to Y that shifts the grading up by d, in other words $X_n \to Y_{n+d}$. The categories of chain complexes have similarly-defined internal homs.
- We will see that $\widehat{\mathbf{Sp}}$ and $\mathrm{Ho}\,\widehat{\mathbf{Sp}}$ are also closed symmetric monoidal. The internal hom is called the **function spectrum** $F(X,Y)$. So we have bijections of maps of spectra
$$X \wedge Y \to Z \qquad \leftrightarrow \qquad X \to F(Y,Z)$$
and of maps in the homotopy category
$$QX \wedge QY \to RZ \qquad \leftrightarrow \qquad QX \to F(QY,RZ).$$

When $\mathbf{C}$ is a closed symmetric monoidal category, we get lots of additional structure for free. For instance, since $X \otimes -$ is a left adjoint, it preserves all colimits!

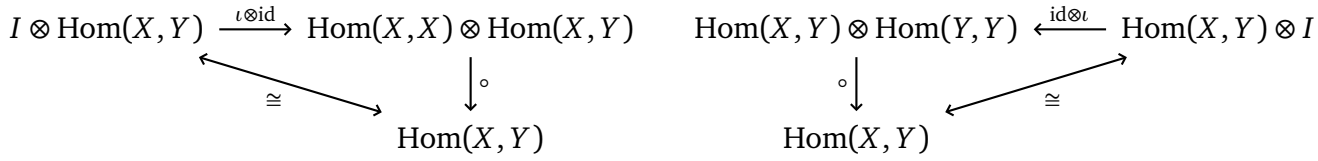
We can also deduce that the internal hom satisfies identities
$$\mathrm{Hom}(I,X) \cong X, \qquad \mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y,Z)).$$
See Exercise 4.4.17. We next define evaluation and composition.

Definition 4.1.35. For any two objects X and Y in a closed symmetric monoidal category $\mathbf{C}$, we define the **evaluation map**
$$X \otimes \mathrm{Hom}(X,Y) \xrightarrow{\mathrm{ev}} Y \tag{4.1.36}$$
as the adjunct of the identity map $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Y)$. In practice, this is the map that "plugs X in" to each function in $\mathrm{Hom}(X,Y)$. We can also think of it more formally as the counit of the adjunction between $X \otimes -$ and $\mathrm{Hom}(X,-)$.

For any three objects X, Y, and Z, we define the **composition map**
$$\mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(Y,Z) \xrightarrow{\circ} \mathrm{Hom}(X,Z) \tag{4.1.37}$$
as the adjunct to the composite
$$X \otimes \mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(Y,Z) \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} Y \otimes \mathrm{Hom}(Y,Z) \xrightarrow{\mathrm{ev}} Z.$$

We also define the **identity map** $\iota : I \to \mathrm{Hom}(X,X)$ to be the adjunct of the canonical isomorphism $X \otimes I \cong X$.

Proposition 4.1.38. *The composition maps are associative and unital in the sense that the following diagrams commute.*
$$\begin{array}{ccc} \mathrm{Hom}(W,X) \otimes \mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(Y,Z) & \xrightarrow{\circ\otimes\mathrm{id}} & \mathrm{Hom}(W,Y) \otimes \mathrm{Hom}(Y,Z) \\ \downarrow{\scriptstyle \mathrm{id}\otimes\circ} & & \downarrow{\scriptstyle \circ} \\ \mathrm{Hom}(W,X) \otimes \mathrm{Hom}(X,Z) & \xrightarrow{\circ} & \mathrm{Hom}(W,Z), \end{array}$$



Proof. We do just the first diagram since the other two are similar. It suffices to prove that both branches are equal to $\circ^{(3)}$, the adjunct of the map

$$W \otimes \mathrm{Hom}(W,X) \otimes \mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(Y,Z) \longrightarrow Z$$

that applies evaluation three times. Recall that when $\widetilde{f} : A \to RB$ is adjunct to $f : LA \to B$, then the composite $LA \xrightarrow{L\widetilde{f}} LRB \xrightarrow{\epsilon} B$ is equal to f. Applying this to the definition of the composition map, we get that this square commutes:

$$\begin{array}{ccc} W \otimes \mathrm{Hom}(W,X) \otimes \mathrm{Hom}(X,Y) & \xrightarrow{\mathrm{id}\otimes\circ} & W \otimes \mathrm{Hom}(W,Y) \\ \downarrow{\scriptstyle \mathrm{ev}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{ev}} \\ X \otimes \mathrm{Hom}(X,Y) & \xrightarrow{\mathrm{ev}} & Y. \end{array}$$

Multiplying by $\mathrm{Hom}(Y,Z)$ gives the commuting diagram

$$\begin{array}{ccc} W \otimes \mathrm{Hom}(W,X) \otimes \mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(Y,Z) & \xrightarrow{\mathrm{id}\otimes\circ\otimes\mathrm{id}} & W \otimes \mathrm{Hom}(W,Y) \otimes \mathrm{Hom}(Y,Z) \\ \downarrow{\scriptstyle \mathrm{ev}\otimes\mathrm{id}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{ev}\otimes\mathrm{id}} \\ X \otimes \mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(Y,Z) & \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} & Y \otimes \mathrm{Hom}(Y,Z) \\ & & \downarrow{\scriptstyle \mathrm{ev}} \\ & & Z. \end{array}$$

Since the adjunction is natural, we therefore get the commuting triangle

$$\begin{array}{ccc} \mathrm{Hom}(W,X) \otimes \mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(Y,Z) & \xrightarrow{\circ\otimes\mathrm{id}} & \mathrm{Hom}(W,Y) \otimes \mathrm{Hom}(Y,Z) \\ & {\scriptstyle \circ^{(3)}}\searrow & \downarrow{\scriptstyle \circ} \\ & & \mathrm{Hom}(W,Z). \end{array}$$

That does half of the proof. For the other half, consider the following diagram. The right-hand square commutes by the same argument with adjuncts, and the left-hand square commutes because morphisms in each slot of $\otimes$ commute with each other.

$$\begin{array}{ccccc} W \otimes \mathrm{Hom}(W,X) \otimes \mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(Y,Z) & \xrightarrow{\mathrm{ev}\otimes\mathrm{id}\otimes\mathrm{id}} & X \otimes \mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(Y,Z) & \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} & Y \otimes \mathrm{Hom}(Y,Z) \\ \downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{id}\otimes\circ} & & \downarrow{\scriptstyle \mathrm{id}\otimes\circ} & & \downarrow{\scriptstyle \mathrm{ev}} \\ W \otimes \mathrm{Hom}(W,X) \otimes \mathrm{Hom}(X,Z) & \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} & X \otimes \mathrm{Hom}(X,Z) & \xrightarrow{\mathrm{ev}} & Z \end{array}$$

Again since the adjunction is natural, this gives the commuting triangle

$$\begin{array}{ccc} \mathrm{Hom}(W,X)\otimes\mathrm{Hom}(X,Y)\otimes\mathrm{Hom}(Y,Z) & & \\ \downarrow{\scriptstyle \mathrm{id}\otimes\circ} & \searrow^{\circ^{(3)}} & \\ \mathrm{Hom}(W,X)\otimes\mathrm{Hom}(X,Z) & \xrightarrow[\circ]{} & \mathrm{Hom}(W,Z), \end{array}$$

which finishes the proof that the entire square commutes. □

This proves that **C** is **enriched** over itself. In other words, we can think of **C** as having mapping objects $\mathrm{Hom}(X,Y)$ that are objects of **C**, instead of the original morphisms $\mathbf{C}(X,Y)$ which only form a set. These "**C**-valued homomorphisms" then have a composition that is associative and has identity maps, as spelled out in Proposition 4.1.38.

It turns out that taking the set of maps in from the unit I recovers the original category. In other words, we have bijections of sets

$$\mathbf{C}(I,\mathrm{Hom}(X,Y))\cong\mathbf{C}(X,Y),$$

and along these bijections, the composition maps (4.1.37) agree with the composition in the original category **C**. This justifies our choice to refer to both of them as "the composition in **C**." Similarly, the identity in $\mathrm{Hom}(X,X)$ agrees with the identity in $\mathbf{C}(X,X)$.

Finally, we also get an "external" tensor product on the internal homs.

Definition 4.1.39. For any four objects W, X, Y, and Z in a closed symmetric monoidal category **C**, we define the **external tensor**

$$\mathrm{Hom}(X,Y)\otimes\mathrm{Hom}(W,Z)\xrightarrow{\overline{\otimes}}\mathrm{Hom}(X\otimes W,Y\otimes Z) \tag{4.1.40}$$

as the adjunct of the map

$$X\otimes\mathrm{Hom}(X,Y)\otimes W\otimes\mathrm{Hom}(W,Z)\xrightarrow{\mathrm{ev}\otimes\mathrm{ev}}Y\otimes Z.$$

Taking $W=I$ gives as a special case the **assembly map**

$$\mathrm{Hom}(X,Y)\otimes Z\xrightarrow{a}\mathrm{Hom}(X,Y\otimes Z) \tag{4.1.41}$$

which is adjunct to the evaluation map $X\otimes\mathrm{Hom}(X,Y)\otimes Z\to Y\otimes Z$.

It turns out that the external tensor also commutes with the composition map in the same way that the tensor on the original category **C** commutes with composition, namely that

$$(g\otimes g')\circ(f\otimes f')=(g\circ f)\otimes(g'\circ f').$$

The proof is a long diagram chase.

Example 4.1.42. In $\mathbf{Mod}_k$ these operations are the obvious ones on the modules $\mathrm{Hom}_k(X,Y)$, and similarly for **Top** these are the obvious operations on the mapping spaces $\mathrm{Map}(X,Y)$.

Lemma 4.1.43. *Suppose* $\mathbf{C}$ *is a closed symmetric monoidal category satisfying the assumptions of Lemma 4.1.8. Suppose in addition there is a full subcategory* $\mathbf{B} \subseteq \mathbf{C}$ *such that*

- Hom *preserves weak equivalences on* $\mathbf{A}^{\mathrm{op}} \times \mathbf{B}$*,*
- *if* $X \in \mathbf{A}$ *and* $Y \in \mathbf{B}$ *then* $\mathrm{Hom}(X,Y) \in \mathbf{B}$*,*
- *there is a functor* $R\colon \mathbf{C} \to \mathbf{C}$ *landing in* $\mathbf{B}$*, and*
- *there is a natural weak equivalence* $r : X \xrightarrow{\sim} RX$*.*

Then the right-derived hom $\mathbb{R}\mathrm{Hom}(X,Y) = \mathrm{Hom}(QX,RY)$ *makes the homotopy category* $\mathrm{Ho}\,\mathbf{C}$ *into a closed symmetric monoidal category.*

For instance, these conditions are satisfied in $(\mathbf{Top}_*, \wedge, S^0)$ and $(\mathbf{Ch}_{\geq 0}(\mathbb{Z}), \otimes, \mathbb{Z}[0])$, where $\mathbf{B}$ is the entire category.

Proof. We only have to show that for each X separately, $\mathbb{R}\mathrm{Hom}(X,-)$ is a right adjoint to $X \otimes^{\mathbb{L}} (-)$. This follows from Proposition 3.5.20 applied to the functors $QX \otimes (-)$ and $\mathrm{Hom}(QX,-)$. □

Example 4.1.44. This makes the derived category of chain complexes $\mathscr{D}(k)_{\geq 0}$ or $\mathscr{D}(k)$ into a closed symmetric monoidal category. The tensor product is computed by taking projective resolutions and tensoring, exactly the kind of manipulation that computes Tor groups. The internal hom takes projective resolution of the source and then takes the chain complex of maps, exactly the kind of manipulation that computes Ext groups.

Example 4.1.45 (Smash product as a black box, part 3)**.** Continuing from Example 4.1.10 and Example 4.1.31, we assert that

- $\widehat{\mathbf{Sp}}$ is a closed symmetric monoidal category, with internal hom called the **function spectrum** $F(X,Y)$. So maps $X \to F(Y,Z)$ correspond to maps $X \wedge Y \to Z$, and we have natural isomorphisms of spectra (not just equivalences!)

$$F(X \wedge Y, Z) \cong F(X, F(Y,Z)), \qquad F(\mathbb{S}, X) \cong X.$$

- More generally, for any based space K, it follows from the isomorphism $(\Sigma^\infty K) \wedge X \cong K \wedge X$ that functions out of a suspension spectrum are the same thing as the cotensor from Definition 2.3.8:

$$F(\Sigma^\infty K, X) \cong F(K, X).$$

- The conditions of Lemma 4.1.43 are satisfied, with **B** the subcategory of Ω-spectra. So we have a right-derived function spectrum

$$\mathbb{R}F(X,Y) = F(QX,RY),$$

  where R is any replacement by an Ω-spectrum, such as the one in Proposition 2.2.9.

It follows from Lemma 4.1.43 that the stable homotopy category Ho $\widehat{\mathbf{Sp}}$ is closed symmetric monoidal. So there are natural bijections

$$[X \wedge^{\mathbb{L}} Y, Z]_s \cong [X, \mathbb{R}F(Y,Z)]_s. \tag{4.1.46}$$

The reader might want to verify that this is consistent with the isomorphisms we got in (3.5.22) when K is a based space:

$$[X, \mathbb{R}F(K,Y)]_s \cong [K \wedge^{\mathbb{L}} X, Y]_s \cong [K, \mathbb{R}\mathrm{Map}_*(X,Y)]_*.$$

It also follows with a good deal of additional work (Exercise 4.4.24) that if $X \to Y \to Z$ is a cofiber/fiber sequence and W is a spectrum, then we have cofiber/fiber sequences

$$\mathbb{R}F(Z,W) \to \mathbb{R}F(Y,W) \to \mathbb{R}F(X,W), \qquad \mathbb{R}F(W,X) \to \mathbb{R}F(W,Y) \to \mathbb{R}F(W,Z).$$

4.2 Duality and the Spanier-Whitehead category

In each symmetric monoidal category **C** there is a class of especially well-behaved objects called the dualizable objects. Intuitively, these are the objects that are "finite enough."

Dualizable objects have many nice properties. For instance, the assembly map

$$\mathrm{Hom}(X,Y) \otimes Z \longrightarrow \mathrm{Hom}(X, Y \otimes Z)$$

is an isomorphism if either X or Z is dualizable, and the "double dual" map

$$X \longrightarrow \mathrm{Hom}(\mathrm{Hom}(X,I),I)$$

is an isomorphism if X is dualizable. Setting $X^* = \mathrm{Hom}(X,I)$, we therefore have

$$X^* = \mathrm{Hom}(X,I), \qquad X \cong \mathrm{Hom}(X^*,I).$$

Dualizability makes several more manipulations possible. For instance, every map $X \to Y$ induces a dual map $Y^* \to X^*$ going the other way. Any self-map of a dualizable object X has a "trace" that is a morphism $I \to I$.

Because dualizable objects are simpler than general objects, it's also sometimes easier to define the subcategory of **C** on the dualizable objects. This is especially true for the homotopy category of spectra Ho **Sp**: the subcategory of dualizable objects was first defined by Spanier and Whitehead in the 1950s, before the smash product was defined on the larger category Ho **Sp** by Boardman and Adams, and long before the smash product was defined on **Sp** by Hovey–Shipley–Smith and Elmendorf–Kriz–Mandell–May in the 1990s.

4.2.1 Duality in a symmetric monoidal category

Definition 4.2.1. Let $(\mathbf{C}, \otimes, I)$ be a symmetric monoidal category. An object X is **dualizable** if the functor $X \otimes (-)$ has a right adjoint of the form $X^* \otimes (-)$, for some other object X^* called the **dual** of X.

In other words, X is dualizable if there exists a dual object X^*, and also evaluation and coevaluation maps

$$e : X \otimes X^* \longrightarrow I, \qquad c : I \longrightarrow X^* \otimes X,$$

such that the following two composites are identity maps:

$$\begin{aligned} X \cong X \otimes I \xrightarrow{1\otimes c} X \otimes X^* \otimes X \xrightarrow{e\otimes 1} I \otimes X \cong X, \\ X^* \cong I \otimes X^* \xrightarrow{c\otimes 1} X^* \otimes X \otimes X^* \xrightarrow{1\otimes e} X^* \otimes I \cong X^*. \end{aligned} \tag{4.2.2}$$

The conditions in (4.2.2) are called the **triangle identitites** for c and e. Note that c and e are the counit and the unit, respectively, of the adjunction between $X \otimes (-)$ and $X^* \otimes (-)$.

Example 4.2.3. In k-vector spaces $\mathbf{C} = \mathbf{Vect}_k$, a vector space V is dualizable iff it is finite-dimensional. Its dual $V^* = \mathrm{Hom}_k(V, k)$ is the usual k-linear dual.

The evaluation map $V \otimes V^* \to k$ is the usual map that plugs a vector v into a linear functional $f(-)$ to get a scalar $f(v)$.

The coevaluation map $k \to V^* \otimes V$ is the map that sends 1 to the sum $\sum_i e_i^* \otimes e_i$, where $\{e_1, \ldots, e_n\}$ is any basis for V. Along the isomorphism $V^* \otimes V \cong \mathrm{Hom}_k(V, V)$, this corresponds to the identity transformation $V \to V$.

The previous example is typical.

Proposition 4.2.4. *In any closed symmetric monoidal category* **C***, the dual of X, if it exists, must be isomorphic to the mapping object to the unit,*

$$DX \cong \mathrm{Hom}(X, I).$$

Proof. The dual X^* is defined so that $X^* \otimes -$ is the right adjoint of $X \otimes -$. But right adjoints are unique up to isomorphism, and we already have a right adjoint $\mathrm{Hom}(X,-)$. So the two adjoints must be isomorphic:

$$X^* \otimes Y \cong \mathrm{Hom}(X,Y).$$

Plugging in $Y = I$ gives $X^* \cong \mathrm{Hom}(X,I)$. □

It also follows from this argument that the evaluation map e is exactly the map $X \otimes \mathrm{Hom}(X,I) \to I$ that we constructed in (4.1.36).

Definition 4.2.5. In any closed symmetric monoidal category $\mathbf{C}$, we call $DX = \mathrm{Hom}(X,I)$ the **functional dual** of X. It always exists, but it only satisfies the properties of the dual if X is dualizable.

Example 4.2.6. If V is an infinite-dimensional vector space, it has a functional dual $DV = \mathrm{Hom}_k(V,k)$, but $D(DV)$ is not canonically isomorphic to V again.

Proposition 4.2.7. *If X is an object in a closed symmetric monoidal category $\mathbf{C}$, the following are equivalent:*

1. *X is dualizable.*
2. *The assembly map $DX \otimes X \longrightarrow \mathrm{Hom}(X,X)$ from (4.1.41) is an isomorphism.*
3. *The assembly map $DX \otimes Y \longrightarrow \mathrm{Hom}(X,Y)$ from (4.1.41) is an isomorphism for all Y.*

Proof. (1) ⇒ (3) If X is dualizable, the operation $\mathrm{Hom}(X,-)$ is naturally isomorphic to $DX \otimes (-)$. The assembly map is exactly this isomorphism, so it is an isomorphism for all Y.

(3) ⇒ (2) Obvious.

(2) ⇒ (1) We define the coevaluation map c to be the composite

$$I \xrightarrow{\mathrm{id}} \mathrm{Hom}(X,X) \cong DX \otimes X$$

and the evaluation map e to be the map

$$X \otimes DX \cong X \otimes \mathrm{Hom}(X,I) \xrightarrow{\mathrm{ev}} I.$$

It is then an exercise to verify the triangle identities for c and e. □

Example 4.2.8. In vector spaces $\mathbf{C} = (\mathbf{Vect}_k, \otimes_k, k)$, the map $DV \otimes_k V \to \mathrm{Hom}_k(V,V)$ is injective, and its image is the endomorphisms of finite rank. It is an isomorphism iff V is finite-dimensional. Hence by Proposition 4.2.7, V is dualizable iff it is finite-dimensional.

Example 4.2.9. If k is instead a commutative ring and $\mathbf{C} = \mathbf{Mod}_k$ is the category of k-modules with tensor product $\otimes_k$, a k-module M is dualizable iff it is finitely generated and projective over k.

Indeed, assume that M is dualizable. Then $\mathrm{Hom}_k(M,-) \cong DM \otimes_k (-)$ is right-exact, and hence M is projective. Since $DM \otimes_k M \to \mathrm{Hom}_k(M,M)$ is an isomorphism, it hits the identity map of M, expressing this identity map as a finite sum of maps, each of which has image in the span of a single element of M. It follows that M is finitely generated.

Conversely, if M is finitely generated and projective then it is a retract of $k^{\oplus n}$ for some $n \geq 0$. We can show the assembly map $DM \otimes_k M \to \mathrm{Hom}_k(M,M)$ is a retract of the same assembly map for $k^{\oplus n}$, and is therefore an isomorphism. Therefore M is dualizable by Proposition 4.2.7.

Example 4.2.10. Passing the previous example to bounded or unbounded chain complexes $\mathbf{C} = \mathbf{Ch}_k$, a chain complex is dualizable iff it is bounded (only finitely many nonzero terms) and finitely generated projective in each degree. Note that the internal hom is given by the product

$$\mathrm{Map}(A_*, B_*)_n = \prod_k \mathrm{Hom}_R(A_n, B_{n+k})$$

with boundary map

$$\partial_{\mathrm{Map}(A,B)}(f) = \partial_B \circ f - (-1)^{|f|} f \circ \partial_A,$$

so that

$$(DA)_n = \mathrm{Map}(A, \underline{R})_n = \mathrm{Hom}(A_{-n}, R).$$

Essentially, the dual flips the chain complex over and dualizes each level separately.

If we pass to the homotopy category $\mathscr{D}(k) = \mathrm{Ho}\,\mathbf{Ch}_k$, a chain complex is dualizable iff it is **perfect**, meaning that it is quasi-isomorphic to a bounded complex of finitely generated projective modules. The dual is computed by making the complex into a bounded complex of finitely generated projective modules and then doing the above procedure.

Example 4.2.11 (Smash product as a black box, part 4)**.** We add to our black box the claim that a spectrum is dualizable if and only if it is stably equivalent to a finite spectrum. That is, a spectrum with finitely many stable cells.

4.2.2 The Spanier-Whitehead category

It's much easier to define the smash product on the subcategory of dualizable spectra in $\mathrm{Ho}\,\mathbf{Sp}$ than the entire category. It builds intuition for what happens in general, so let's start there first.

Recall from Remark 3.2.14 that the **Spanier-Whitehead category** is the full subcategory $\mathrm{Ho}\,\mathbf{Sp}_{\mathrm{fin}} \subseteq \mathrm{Ho}\,\mathbf{Sp}$ on the finite spectra. A finite spectrum is one that has finitely many stable

cells – equivalently, it is a shift of a suspension spectrum $\Sigma^\infty A$ of a finite CW complex A. See Exercise 2.7.43.

So, we can describe the Spanier-Whitehead category as having an object (A,k) for every finite based CW complex A and integer $k \in \mathbb{Z}$. This corresponds to the spectrum

$$\mathrm{sh}^k \Sigma^\infty A \cong \begin{cases} F_0 \Sigma^k A & \text{when } k \geq 0, \\ F_{|k|} A & \text{when } k \leq 0. \end{cases}$$

In particular, (S^0, k) corresponds to the shifted sphere spectrum $\mathbb{S}^k = \mathrm{sh}^k \mathbb{S}$.

By Corollary 3.2.13, the morphisms are computed as a colimit of homotopy classes of based maps,

$$\operatorname*{colim}_{n\to\infty} [\Sigma^{k+n} A, \Sigma^{\ell+n} B]_*.$$

Here the colimit system is defined as soon as n is large enough that both $k+n$ and $\ell+n$ are nonnegative. When $k = \ell$, it is classical to refer to this colimit as $\{A, B\}$, the abelian group of "stable maps" from the CW complex A to the CW complex B. Collecting this all together:

Proposition 4.2.12. *The Spanier-Whitehead category* $\mathrm{Ho}\,\mathbf{Sp}_{\mathrm{fin}}$ *is equivalent to the category whose objects are pairs* (A,k) *with* A *a finite based CW complex and* $k \in \mathbb{Z}$*. The morphisms from* (A,k) *to* (B,ℓ) *are the colimit*

$$\operatorname*{colim}_{n\to\infty} [\Sigma^{k+n} A, \Sigma^{\ell+n} B]_*$$

Definition 4.2.13. We define the symmetric monoidal structure on $\mathrm{Ho}\,\mathbf{Sp}_{\mathrm{fin}}$ by taking smash product of the CW complexes and adding the integers:

$$(A,k) \wedge (C,r) := (A \wedge C, k+r).$$

This extends to the morphisms by taking each pair of maps

$$f : \Sigma^{k+n_1} A \longrightarrow \Sigma^{\ell+n_1} B, \qquad g : \Sigma^{r+n_2} C \longrightarrow \Sigma^{s+n_2} D$$

to the map

$$\begin{array}{ccc} \Sigma^{k+r+n_1+n_2} A \wedge C & & \Sigma^{\ell+s+n_1+n_2} B \wedge D \\ \cong \downarrow (-1)^{rn_1} & & \cong \uparrow (-1)^{sn_1} \\ \Sigma^{k+n_1} A \wedge \Sigma^{r+n_2} C & \xrightarrow{f\wedge g} & \Sigma^{\ell+n_1} B \wedge \Sigma^{s+n_2} D \end{array}$$

The minus signs indicate that we should apply a map of the specified degree to the sphere $S^{k+r+n_1+n_2}$ or $S^{\ell+s+n_1+n_2}$. This follows the Koszul sign rule, and guarantees that this rule is independent of the choice of n_1 and n_2. See Exercise 4.4.25.

We define the associativity and unit isomorphisms

$$((A,k)\wedge(B,\ell))\wedge(C,m)\cong(A,k)\wedge((B,\ell)\wedge(C,m)),\qquad (A,k)\wedge(S^0,0)\cong(A,k)$$

in the obvious way. For the symmetry isomorphism $(A,k)\wedge(C,r)\cong(C,r)\wedge(A,k)$, we use another Koszul sign rule

$$\Sigma^{k+r}A\wedge C\xrightarrow[\cong]{(-1)^{kr}}\Sigma^{r+k}C\wedge A.$$

Proposition 4.2.14. *This defines a symmetric monoidal structure on the Spanier-Whitehead category* $\mathrm{Ho}\,\mathbf{Sp}_{\mathrm{fin}}$.

Proof. It is a straightforward exercise to check that the isomorphisms we defined are natural isomorphisms (Exercise 4.4.26). The diagrams in Definition 4.1.4 all commute because the signs we introduce are the signs of permutations for the sphere coordinates, and when we go around each diagram, the composite permutation is the identity, so the product of the signs is $+1$. □

It is a consequence of Theorem 3.2.5 that the Spanier-Whitehead category is additive: the morphisms $(A,k)\to(B,\ell)$ are abelian groups, there is a zero object, and finite coproducts and products exist and are isomorphic to each other. It follows from Exercise 4.4.4 that the smash product distributes over the sums of maps, so that $f\wedge(g_1+g_2)=(f\wedge g_1)+(f\wedge g_2)$.

Lemma 4.2.15. *Dualization commutes with finite sums and shifts:*

$$D(X\vee Y)\simeq DX\vee DY,\qquad D(\Sigma^k X)\simeq\Sigma^{-k}DX.$$

This is left to Exercise 4.4.27. As a special case, the dual of the n-sphere is the $(-n)$-sphere:

$$D(\mathbb{S}^n)\simeq D(S^0,n)\simeq(S^0,-n)\simeq\mathbb{S}^{-n}.$$

It turns out that much more is true: every finite spectrum is dualizable.

Proposition 4.2.16 (Spanier-Whitehead duality)**.** *Every object* (A,k) *in the Spanier-Whitehead category is dualizable.*

Intuitively, the dual of a finite cellular spectrum X is a spectrum DX with a stable $(-k)$-cell for every stable k-cell of X. To give a more specific construction in the Spanier-Whitehead category, the dual of (A,k) is computed by embedding A into a sphere S^{n+1}, letting B be a finite CW complex equivalent to the complement $S^{n+1}\setminus A$, and taking the dual to be $(B,n-k)$. The proof of Proposition 4.2.16 is pleasantly geometric, but goes outside the scope of our immediate focus, so it will be deferred to **??**.

Corollary 4.2.17. *The Spanier-Whitehead category is closed symmetric monoidal – there is an internal hom that is right adjoint to the smash product.*

Proof. If X and Y are objects of the Spanier-Whitehead category and X^* is the dual of X, we *define* $\mathrm{Hom}(X,Y)$ to be $X^* \wedge Y$. It is then immediate that $\mathrm{Hom}(X,-)$ is a right adjoint to $X \wedge -$ as required. □

In the special case that $A = M_+$ where M is a closed smooth manifold, if we embed M_+ into S^n and take its complement B, it is possible to show that $SB \simeq N/\partial N$, where N is a tubular neighborhood of M in $\mathbb{R}^n$ and ∂N is its boundary (Exercise 4.4.28). This result is commonly called Atiyah duality.

Theorem 4.2.18 (Atiyah duality)**.** *If M is a closed smooth manifold, then the dual of $(M_+, 0)$ in the Spanier-Whitehead category is $(N/\partial N, -n)$, where N is any tubular neighborhood of a smooth embedding $M \to \mathbb{R}^n$.*

Example 4.2.19. If $M = S^1$, we can embed it into $\mathbb{R}^2$ in the standard way, and its neighborhood N is an annulus. It is easy to see that $N/\partial N$ is homotopy equivalent to $S^1 \vee S^2$. We therefore get a duality

$$D(S^1_+, 0) \simeq (S^1 \vee S^2, -2).$$

We could have calculated this directly. By Exercise 2.7.34, we have an isomorphism in the Spanier-Whitehead category

$$(S^1_+, 0) \cong \mathbb{S}^1 \vee \mathbb{S}^0$$

where $\mathbb{S}^k = (S^0, k)$ is the shifted sphere spectrum. Therefore by Lemma 4.2.15,

$$D(S^1_+, 0) \cong D(\mathbb{S}^1 \vee \mathbb{S}^0) \cong \mathbb{S}^{-1} \vee \mathbb{S}^0 \cong (S^1 \vee S^2, -2).$$

4.2.3 Consequences of duality

Let us now return to the larger stable homotopy category $\mathrm{Ho}\,\mathbf{Sp}$. Recall that in Example 4.1.10, Example 4.1.31, Example 4.1.45, and Example 4.2.11 we gave some black-boxed properties of the smash product on this larger category, or really a category $\mathrm{Ho}\,\widehat{\mathbf{Sp}}$ that is equivalent to it. In particular, $\mathrm{Ho}\,\widehat{\mathbf{Sp}}$ is a closed symmetric monoidal category, under the left-derived smash product $X \wedge^{\mathbb{L}} Y = QX \wedge QY$ and right-derived function spectrum $\mathbb{R}F(X,Y) = F(QX, RY)$, and a spectrum is dualizable if and only if it is finite.

Lemma 4.2.20. *If X is a finite (dualizable) spectrum, then the homology of DX is isomorphic to the cohomology of X,*

$$E_q(DX) \cong E^{-q}(X).$$

Proof. The left-hand side is defined to be $[\mathbb{S}^q, DX \wedge^{\mathbb{L}} E]_s$, maps in the stable category from $\mathbb{S}^q$ to the left-derived smash product $DX \wedge^{\mathbb{L}} E$. Since $DX \wedge^{\mathbb{L}} (-)$ is the right adjoint of $X \wedge^{\mathbb{L}} (-)$, by uniqueness of right adjoints, we must have

$$\begin{aligned}[\mathbb{S}^q, DX \wedge^{\mathbb{L}} E]_s &\cong [\mathbb{S}^q, \mathbb{R}F(X,E)]_s \\ &\cong [\mathbb{S}^q \wedge^{\mathbb{L}} X, E]_s \\ &\cong [\mathrm{sh}^q X, E]_s \\ &\cong E^{-q}(X).\end{aligned}$$

The second-to-last isomorphism uses Exercise 4.4.19. □

Of course, since DX is also dualizable and $DDX \simeq X$, we also get

$$E^q(DX) \cong E_{-q}(X).$$

We can now use this observation to give the stable-homotopy proof of Poincaré duality.

Theorem 4.2.21 (Poincaré duality, cf. Theorem 1.3.22)**.** *If M is a closed smooth d-dimensional manifold, there is an isomorphism*

$$H^i(M;\mathbb{Z}) \cong H_{d-i}(M;\widetilde{\mathbb{Z}})$$

where $\widetilde{\mathbb{Z}}$ is the local coefficient system whose fiber is $\mathbb{Z}$ and that twists according to the orientation of M. In case M is orientable, this simplifies to

$$H^i(M;\mathbb{Z}) \cong H_{d-i}(M;\mathbb{Z}).$$

In the general, non-orientable case we also get isomorphism with $\mathbb{Z}/2$ coefficients

$$H^i(M;\mathbb{Z}/2) \cong H_{d-i}(M;\mathbb{Z}/2).$$

Proof. By Lemma 4.2.20, the cohomology of M is the homology of its Spanier-Whitehead dual:

$$H^i(M;\mathbb{Z}) \cong H_{-i}(DM;\mathbb{Z}).$$

By Theorem 4.2.18, this Spanier-Whitehead dual is a shift of $N/\partial N$, where N is a tubular neighborhood of M in $\mathbb{R}^n$:

$$H_{-i}(DM;\mathbb{Z}) \cong H_{n-i}(N/\partial N;\mathbb{Z}).$$

The space $N/\partial N$ can be identified with the Thom space $\mathrm{Th}(\nu)$ of the normal bundle ν of M in $\mathbb{R}^n$, which has dimension $(n-d)$. By the Thom isomorphism of Example 2.6.39, if this bundle is orientable then

$$H_{n-i}(N/\partial N;\mathbb{Z}) \cong H_{n-i}(\mathrm{Th}(\nu);\mathbb{Z}) \cong H_{d-i}(M;\mathbb{Z}),$$

and more generally it gives $H_{d-i}(M;\widetilde{\mathbb{Z}})$ where $\widetilde{\mathbb{Z}}$ twists according to the orientation of the normal bundle of M. This is the same as the orientation of the tangent bundle of M, and the conclusion follows. The isomorphism with $\mathbb{Z}/2$ coefficients follows in the same way. □

This proof has the advantage that it generalizes to coefficients in any *extraordinary* homology theory E. We say that the normal bundle of $M \to \mathbb{R}^n$ is **E-orientable** if there is an isomorphism

$$E_{(n-d)+q}(\mathrm{Th}(\nu)) \cong E_q(M).^{5}$$

Theorem 4.2.22. *If the normal bundle of M is E-orientable,*

$$E^i(M) \cong E_{d-i}(M).$$

The proof is the same as that of Theorem 4.2.21.

We can also now give the proof of Theorem 2.5.13:

Theorem 4.2.23 (Whitehead representability)**.** *If h_* is any extraordinary homology theory then there is a spectrum E and a natural isomorphism of homology theories $E_* \cong h_*$, where E_* is defined as in Proposition 2.5.7. Furthermore E is unique up to stable equivalence.*

Proof. Given an extraordinary homology theory h_*, define

$$h^n(X) := h_{-n}(F(X,\mathbb{S})).$$

This is a cohomology theory on finite spaces, not all spaces, because h_* doesn't take coproducts to products. By Theorem 2.5.27, this is represented by a (unique) spectrum E. Then we get for finite spaces

$$\begin{aligned} h_n(X) &\cong h_n(DDX) \\ &\cong h^{-n}(DX) \\ &= [S^n, \mathbb{R}F(DX,E)] \\ &\cong [S^n, DDX \wedge^{\mathbb{L}} E] \\ &\cong [S^n, X \wedge^{\mathbb{L}} E] \\ &\cong E_n(X). \end{aligned}$$

These isomorphisms are natural in the finite CW complex X, so $h_* \cong E_*$ on all finite CW complexes. By the direct limit axiom (Remark 2.5.6), we therefore get $h_* \cong E_*$ on the larger category of all CW complexes. □

[5] Actually, the definition of orientability should be a little stricter than this – it should say that the bundle ν looks trivial after taking a smash product with E. We would need parametrized spectra to state this precisely, see [ABG+14]. In the end, this also gives the isomorphism $E_{(n-d)+q}(\mathrm{Th}(\nu)) \cong E_q(M)$.

4.2.4 Traces

Definition 4.2.24. Let $(\mathbf{C}, \otimes, I)$ be a symmetric monoidal category. Let X be any dualizable object, with coevaluation and evaluation maps c and e as in Definition 4.2.1, and $f : X \to X$ any endomorphism of X. The **trace** of f is the map $\mathrm{tr}(f) \colon I \to I$ given by the composite

$$I \xrightarrow{c} X^* \otimes X \xrightarrow{1\otimes f} X^* \otimes X \xleftrightarrow{\cong} X \otimes X^* \xrightarrow{e} I. \tag{4.2.25}$$

The **Euler characteristic** $\chi(X) \colon I \to I$ is the trace of the identity map of X.

Example 4.2.26. In k-vector spaces $\mathbf{C} = \mathbf{Vect}_k$, the trace of a map $f : V \to V$ is a k-linear map $k \to k$, which we can interpret simply as an element of k. Under this interpretation, $\mathrm{tr}(f) \in k$ is exactly the usual definition of trace, the sum of the diagonal entries of any matrix that represents f. The Euler characteristic of V is its dimension $\dim V \in \mathbb{Z}$, mapped forward to k.

Example 4.2.27. Suppose $\mathbf{C} = \mathbf{Mod}_k$ is the category of modules over the commutative ring k. By Example 4.2.9, a module M is dualizable iff it is finitely generated and projective, so we have $M \oplus N \cong k^n$ for some other module N.

If $f : M \to M$ is a k-linear map, its trace is computed by forming the map $(f, 0) \colon k^n \to k^n$ that is f on M and 0 on N. Then we take the matrix for $(f, 0)$ in any basis of k^n and add up its diagonal entries. The result does not depend on n, N, or the choice of basis!

In particular, the Euler characteristic of M is the trace of the idempotent map $k^n \to k^n$ that projects to M and includes back into k^n. This trace is the "dimension" of M, but it is not always an integer!

Example 4.2.28. Suppose $\mathbf{C} = \mathbf{Ch}_k$ is unbounded chain complexes over k. If C_* is a bounded complex of finitely generated projective modules and $f : C_* \to C_*$ is a self-map, its trace is the alternating sum

$$\mathrm{tr}(f) = \sum_n (-1)^n \mathrm{tr}(f_n \colon C_n \to C_n),$$

where the trace of f_n is computed as in the previous exercise. This expression comes from writing out (4.2.25) explicitly. The Koszul sign rule in the definition of $\gamma \colon X \otimes Y \cong Y \otimes X$ is responsible for the presence of signs here. See Exercise 4.4.29.

In particular, the Euler characteristic of C_* is the alternating sum of the ranks of its levels. If C_* is the cellular chain complex of some finite CW complex X, this agrees with the Euler characteristic $\chi(X)$.

Example 4.2.29. Suppose $\mathbf{C} = \mathrm{Ho}\,\mathbf{Ch}_k$ is the derived category. Then the trace of a map is computed as in Example 4.2.28, only we change C_* up to quasi-isomorphism first into a bounded complex of finitely generated projective modules.

Remark 4.2.30. If k is a principal ideal domain (PID), it is a theorem in algebra that the trace of a map of complexes $f : C_* \to C_*$ is also computed as the alternating sum of traces on homology,

$$\mathrm{tr}(f) = \sum_n (-1)^n \mathrm{tr}(f_n : H_n(C_*) \to H_n(C_*))$$

For chain complexes over $\mathbb{Z}$, we can further pass to $\mathbb{Q}$ when computing this trace without changing the answer. This captures the familiar theorem that the Euler characteristic of a space $\chi(X)$ is the alternating sum of the ranks of the rational homology groups $H_n(X;\mathbb{Q})$.

Example 4.2.31. If X is a finite CW complex and $f : X \to X$ is a self-map, the trace of $C_*(f) : C_*(X) \to C_*(X)$ is equal to the alternating sum of traces on rational homology

$$L(f) = \sum_n (-1)^n \mathrm{tr}(f_n : H_n(X;\mathbb{Q}) \to H_n(X;\mathbb{Q})),$$

and is always an integer. We call this number the **Lefschetz number** of f.

It is a theorem of Lefschetz and Hopf that this number equals the number of fixed points of f, added together with appropriate weights. In particular, if there are no fixed points, the number must vanish:

Theorem 4.2.32 (Lefschetz Fixed Point Theorem)**.** *If $f : X \to X$ is a self-map of a finite CW complex, with no fixed points, then $L(f) = 0$.*

Equivalently, if $L(f) \neq 0$ then f *must* have a fixed point. This is an important classical theorem in topology that greatly generalizes the Brouwer fixed point theorem (any self-map of D^n must have a fixed point).

Example 4.2.33. If X is a finite CW complex then its suspension spectrum $\Sigma^\infty_+ X$ is a finite spectrum, and is therefore dualizable in the stable homotopy category $\mathrm{Ho}\,\widehat{\mathbf{Sp}}$. If $f : X \to X$ is a self-map, the trace of $\Sigma^\infty_+ f : \Sigma^\infty_+ X \to \Sigma^\infty_+ X$ in the stable homotopy category is a zig-zag $\mathbb{S} \to \mathbb{S}$, equivalently, an integer $\mathrm{tr}(f) \in \mathbb{Z} \cong [\mathbb{S},\mathbb{S}]_s$. The integer we get is exactly the Lefschetz number $L(f)$. See Exercise 4.4.30.

For much more information about traces in symmetric monoidal categories, see [PS14], or [DP80] for a more classical reference.

4.3 Summary of the black box

For the reader's convenience, we summarize all of the black-boxed properties of the smash product from this chapter. See Example 4.1.10, Example 4.1.31, Example 4.1.45, and Example 4.2.11 for more details.

- There is a category of spectra $\widehat{\mathbf{Sp}}$ that is a closed symmetric monoidal category, under the smash product $\wedge$, the function spectrum $F(-,-)$ and unit the sphere spectrum $\mathbb{S}$.
- Smashing with any suspension spectrum $\Sigma^\infty K$ gives the tensor from Definition 2.3.6:

  $$(\Sigma^\infty K) \wedge X \cong K \wedge X.$$

  We also have isomorphisms

  $$F_d A \wedge F_e B \cong F_{d+e}(A \wedge B).$$

- The smash product $X \wedge Y$ preserves stable equivalences, and is cellular, if X and Y are both cellular.
- The function spectrum $F(X,Y)$ preserves stable equivalences, and is an Ω-spectrum, whenever X is cellular and Y is an Ω-spectrum.
- The suspension spectrum functor $\Sigma^\infty(-)$ is strong symmetric monoidal.
- The Eilenberg–Mac Lane spectrum functor $H(-)$ is lax symmetric monoidal.
- The stable homotopy groups $\pi_*(-)$ are lax symmetric monoidal.
- A spectrum is dualizable if and only if it is a finite spectrum.

It follows that:

- Functions out of a suspension spectrum are the same thing as the cotensor from Definition 2.3.8:

  $$F(\Sigma^\infty K, X) \cong F(K, X).$$

- The derived smash product and derived function spectrum

  $$X \wedge^{\mathbb{L}} Y = QX \wedge QY, \qquad \mathbb{R}F(X,Y) = F(QX, RY)$$

  make the homotopy category $\mathrm{Ho}\,\widehat{\mathbf{Sp}}$ into a closed symmetric monoidal category.
- The derived smash product preserves cofiber/fiber sequences in each variable (Exercise 4.4.22).
- The derived function spectrum preserves cofiber/fiber sequences in each variable (Exercise 4.4.24).
- More generally, the derived smash product $X \wedge^{\mathbb{L}} Y$ preserves homotopy colimits in each variable, and the derived function spectrum $\mathbb{R}F(X,Y)$ sends homotopy colimits in the X variable or homotopy limits in the Y variable to homotopy limits. See for example Exercises 4.4.20 and 4.4.24.

4.4 Exercises

4.4.1. We define the tensor product of two chain complexes $A \otimes B$ by the formula

$$(A \otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j, \quad d(a \otimes b) = (da) \otimes b + (-1)^{|a|} a \otimes (db).$$

The differential is defined as above on each homogeneous piece $A_i \otimes B_j$. It extends uniquely to $(A \otimes B)_n$ since it is a homomorphism.

Note that the differential resembles the Leibniz rule from calculus, but with a sign that's introduced when we "switch d past a," consistent with the Koszul sign rule.

(a) Verify that this is indeed a chain complex. Explain what would go wrong without the $(-1)^{|a|}$ term.

(b) Define a symmetry isomorphism $\gamma : A \otimes B \to B \otimes A$ by

$$\gamma(a \otimes b) = (-1)^{|a||b|}(b \otimes a).$$

Verify that this is indeed a map of chain complexes, commuting with the differential. Again, explain what would happen without the $(-1)^{|a|||b|}$ term.

4.4.2. We say a symmetric monoidal category $\mathbf{C}$ is **cartesian monoidal** if its symmetric monoidal structure uses the categorical product $\times$.

(a) Show that any category $\mathbf{C}$ with finite products is cartesian monoidal. (When we assume that $\mathbf{C}$ has finite products, that also means the empty product exists, i.e. the terminal object $* \in \mathbf{C}$.)

(b) Dualize to cocartesian monoidal categories, in which the structure uses the coproduct $\amalg$.

4.4.3. Use Lemma 4.1.8 to show that the stable homotopy category $\mathrm{Ho}\widehat{\mathbf{Sp}}$ is a symmetric monoidal category under the left-derived wedge sum $\vee^{\mathbb{L}}$. Then explain how we could get this more easily from the previous exercise and Proposition 3.2.2.

(This is *not* the symmetric monoidal structure on spectra that comes from the smash product $\wedge$. This one is *additive*, the other one is *multiplicative*.)

4.4.4. Recall from Theorem 3.2.5 that an **additive** category $\mathbf{C}$ is one in which

- the sets $\mathbf{C}(X, Y)$ are abelian groups,
- the composition maps $\mathbf{C}(X, Y) \times \mathbf{C}(Y, Z) \to \mathbf{C}(X, Z)$ are bilinear,
- $\mathbf{C}$ has all finite coproducts and products, and a zero object $*$, and
- the canonical map $X \amalg Y \to X \times Y$ is an isomorphism.

In such a category we let $\oplus$ refer to the coproduct. An **additive symmetric monoidal category** is a category $\mathbf{C}$ that is both additive and symmetric monoidal, whose symmetric monoidal structure $\otimes$ preserves finite coproducts in each slot. In other words, $\otimes$ distributes over $\amalg$.

Prove that in any such category, the map on morphisms induced by the tensor product

$$\mathbf{C}(X,Y)\times\mathbf{C}(W,Z)\longrightarrow\mathbf{C}(X\otimes W, Y\otimes Z)$$

is a bilinear map of abelian groups. (You might want to use the definition of the addition in $\mathbf{C}(X,Y)$ given in the proof of Theorem 3.2.5.)

4.4.5. In this exercise we prove that "spectra do not have diagonal maps." To be more precise, first observe that in the category of based spaces, there are diagonal maps $\Delta\colon K\to K\wedge K$ that commute with all maps of based spaces $K\to L$. Therefore on suspension spectra there are diagonal maps

$$\Sigma^\infty K\xrightarrow{\Delta}\Sigma^\infty(K\wedge K)\ \cong\ (\Sigma^\infty K)\wedge(\Sigma^\infty K).$$

that commute with all maps $\Sigma^\infty K\to\Sigma^\infty L$ in the image of the functor Σ^∞. However, we claim that there is no way to extend this to a natural transformation in the stable homotopy category

$$X\xrightarrow{\eta}X\wedge X.$$

Indeed, even on suspension spectra, the transformation $\Sigma^\infty(K)\to(\Sigma^\infty K)\wedge(\Sigma^\infty K)$ constructed above does not commute with all maps $\Sigma^\infty K\to\Sigma^\infty K$ in the stable homotopy category. Prove this!

(It suffices to take $K=S^0$ and to consider the map $\mathbb{S}\to\mathbb{S}$ given by two times the identity. Exercise 4.4.4 is helpful for determining the effect of this map on $\mathbb{S}\wedge\mathbb{S}$.)

4.4.6. Suppose that E is a ring spectrum. Show that the cohomology of any *space* X inherits a cup product from E, by taking any two maps in the stable homotopy category

$$\Sigma^\infty X\xrightarrow{f}\Sigma^m E,\qquad \Sigma^\infty X\xrightarrow{g}\Sigma^n E$$

to the composite

$$\Sigma^\infty X\xrightarrow{\Delta}\Sigma^\infty(X\wedge X)\ \cong\ (\Sigma^\infty X)\wedge(\Sigma^\infty X)\xrightarrow{f\wedge g}\Sigma^m E\wedge\Sigma^n E\xrightarrow{\mu}\Sigma^{m+n}E.$$

The cohomology of a spectrum X does not have such a cup product – why?

4.4.7. Suppose $\mathbf{C}$ is a symmetric monoidal category under its coproduct $\amalg$.

- Show that every object X of $\mathbf{C}$ is a monoid object (with respect to $\amalg$) in a canonical way.

- For each pair of objects X and Y, show that the structure of an X-module on Y (with respect to $\amalg$) is the same thing as a morphism $X \to Y$.

This is why we only consider ring and module objects in the more "multiplicative" examples.

4.4.8. Suppose **C** is a *cartesian* monoidal category, in other words a symmetric monoidal category but the product is the categorical product $\times$. We say that G is a **group object** in **C** if it is a monoid and it also has an inversion map $i : G \to G$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{(i,1) \text{ or } (1,i)} & G \times G \\ \downarrow & & \downarrow{\scriptstyle \mu} \\ * & \xrightarrow{\eta} & G. \end{array}$$

Informally, this means that $gg^{-1} = 1 = g^{-1}g$.

(a) Identify the group objects in **Set**, **Top**, and the category **Diff** of smooth manifolds and smooth maps.

(b) Explain why this definition doesn't make sense in a general symmetric monoidal category **C**.[6]

4.4.9. Let $(\mathbf{C}, \otimes, I)$ be any symmetric monoidal category.

(a) Prove that the unit object I is always a commutative monoid in **C**. As a result, the sphere spectrum $\mathbb{S}$ must be a commutative ring spectrum.

(b) Prove that every object $X \in \mathbf{C}$ is an I-module in a canonical way. Therefore, every spectrum is an $\mathbb{S}$-module in a canonical way.

(It is intended that you use the coherence statement in Definition 4.1.4 liberally.)

4.4.10. Let $(\mathbf{C}, \otimes, I)$ be any symmetric monoidal category. Define the correct notion of a map of monoids $R \to S$ in **C**. When you are done, compare your definition against Definition 6.2.32. (No peeking!)

4.4.11. Suppose $(F \dashv G)$ is an adjoint pair of functors between the symmetric monoidal categories **C** and **D**. Show that natural transformations

$$F(K \otimes L) \to F(K) \otimes F(L)$$

[6] One thing we can do in any symmetric monoidal category **C** is prove that the category of commutative algebras $\mathrm{CAlg}(\mathbf{C})$ has coproducts (Lemma 6.5.27). Therefore the opposite category $\mathrm{CAlg}(\mathbf{C})^{\mathrm{op}}$ is cartesian monoidal, so we can take a group object there. This is the same thing as a **commutative Hopf algebra** in **C**.

correspond to natural transformations

$$G(X) \otimes G(Y) \to G(X \otimes Y),$$

and that maps $F(I_{\mathbf{C}}) \to I_{\mathbf{D}}$ correspond to maps $I_{\mathbf{C}} \to G(I_{\mathbf{D}})$. It turns out that the natural transformation and map for F are coherent iff the corresponding natural transformation and map for G are coherent. So, G is lax symmetric monoidal iff F is "oplax" symmetric monoidal. In practice, when this happens, usually F is strong symmetric monoidal.

4.4.12. Suppose $(\mathbf{C}, \otimes, I)$ is a symmetric monoidal category and $\otimes'$ is another functor $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$ that is naturally isomorphic to $\otimes$. Explain how there is a canonical symmetric monoidal structure on $\mathbf{C}$ that uses $\otimes'$ instead of $\otimes$.

4.4.13. Suppose (F, G) is an equivalence of categories $\mathbf{C} \simeq \mathbf{D}$. Show that if $(\mathbf{C}, \otimes, I)$ is a symmetric monoidal category, we can define a symmetric monoidal structure on $\mathbf{D}$ using

$$X \otimes_{\mathbf{D}} Y := F(G(X) \otimes_{\mathbf{C}} G(Y)), \qquad I_{\mathbf{D}} := F(I_{\mathbf{C}}).$$

4.4.14. In this exercise we describe a common trick for making symmetric monoidal structures a little stricter. A **permutative category** is a symmetric monoidal category $(\mathbf{C}, \otimes, I)$ in which the isomorphisms α, ρ, and $\rho \circ \gamma$ are *equalities*:

$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z), \qquad X \otimes I = X, \qquad I \otimes X = X.$$

The symmetry isomorphism γ is still allowed to be a nontrivial isomorphism.

(a) Define a variant of the category of finite sets $(\mathbf{Fin}, \amalg, \varnothing)$ which is a permutative category. Take the objects to be just the sets $\{1, \ldots, n\}$ for $n \geq 0$, and the tensor product concatenates the sets.

(b) Given any symmetric monoidal category $\mathbf{C}$, construct a category $\mathbf{C}'$ whose objects are finite lists $(X_1, \ldots, X_n)$ with $X_i \in \mathbf{C}$, and $n \geq 0$. The morphisms between two lists $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_m)$ are defined to be the set of morphisms between their tensor products

$$X_1 \otimes \cdots \otimes X_n \longrightarrow Y_1 \otimes \cdots \otimes Y_m.$$

(The empty tensor product is defined to be the unit object I.) Prove that $\mathbf{C}'$ is a permutative category, with tensor product given by concatenation of the lists, and unit given by the empty list.

(c) Prove that there is an equivalence of categories $\mathbf{C}' \to \mathbf{C}$ that is also a strong symmetric monoidal functor. In this sense, every symmetric monoidal category is equivalent to a permutative category.

4.4.15. Finish the proof of Proposition 4.1.27 by showing that the diagrams in Definition 4.1.12 commute for $F(R)$ if they commuted for R.

4.4.16. Suppose $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{E}$ are both lax monoidal functors. Explain why the composite $G \circ F\colon \mathbf{C} \to \mathbf{E}$ is also a lax monoidal functor.

4.4.17. Prove that in any closed symmetric monoidal category $\mathbf{C}$, we have isomorphisms

$$\mathrm{Hom}(I, X) \cong X, \qquad \mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)).$$

(Hint: Use the Yoneda Lemma: if $\mathbf{C}(-, A)$ and $\mathbf{C}(-, B)$ are isomorphic as functors then necessarily $A \cong B$.)

4.4.18. Suppose $\mathbf{C}$ is a closed symmetric monoidal category.

(a) Prove that the assembly map $\mathrm{Hom}(X, Y) \otimes Z \to \mathrm{Hom}(X, Y \otimes Z)$ is an isomorphism if X is dualizable. (Hint: recall that $\mathrm{Hom}(X, -)$ is naturally isomorphic to $X^* \otimes -$.)

(b) Prove that the double dual map $X \to DDX$ is an isomorphism if X is dualizable.

(c) Prove that the assembly map $\mathrm{Hom}(X, Y) \otimes Z \to \mathrm{Hom}(X, Y \otimes Z)$ is an isomorphism if Z is dualizable. (Hint: use the double dual map for Z.)

4.4.19. Using the claimed properties in Example 4.1.10, prove that for any integer $d \in \mathbb{Z}$, the derived smash product $\mathbb{S}^d \wedge^{\mathbb{L}} X$ is equivalent to the shift $\mathrm{sh}^d X$. (You might want to prove this for nonnegative d first, where $\mathrm{sh}^d X \simeq \Sigma^d X$, and then use the uniqueness of adjoints to argue the result for negative d.)

4.4.20. Using the claimed properties in Example 4.1.10, prove that the smash product preserves wedge sums, homotopy pushouts, and mapping telescopes in each variable.

4.4.21. Using the claimed properties in Example 4.1.10, prove that the smash product of cellular spectra is a cellular spectrum, as illustrated at the beginning of Chapter 4.

4.4.22. Using the claimed properties in Example 4.1.10, prove that for any cofiber/fiber sequence $X \to Y \to Z$ and spectrum W, the derived smash products

$$W \wedge^{\mathbb{L}} X \to W \wedge^{\mathbb{L}} Y \to W \wedge^{\mathbb{L}} Z$$

form a cofiber/fiber sequence as well.

4.4.23. Using Exercise 4.4.19 and the claimed properties in Example 4.1.45, prove that for any integer $d \in \mathbb{Z}$, the derived mapping space $\mathbb{R}F(\mathbb{S}^d, X)$ is equivalent to the negative shift $\mathrm{sh}^{-d} X$.

4.4.24. In this exercise we use the claimed properties of the function spectrum in Example 4.1.45 to show that the function spectrum preserves cofiber and fiber sequences in each slot.

(a) Using the fact that left adjoints preserve colimits and right adjoints preserve limits (Theorem 1.6.9), prove that the function spectrum F sends colimits in the first slot, or limits in the second slot, to limits.

(b) For a map of spectra $f : X \to Y$ with homotopy cofiber Cf and homotopy fiber Ff, prove that $F(Cf, W)$ is the homotopy fiber of $F(f, W)$, and that $F(W, Ff)$ is the homotopy fiber of $F(W, f)$.

(c) Prove that for any cofiber/fiber sequence $X \to Y \to Z$ and spectrum W, the derived maps into W

$$\mathbb{R}F(Z, W) \to \mathbb{R}F(Y, W) \to \mathbb{R}F(X, W)$$

and derived maps out of W

$$\mathbb{R}F(W, X) \to \mathbb{R}F(W, Y) \to \mathbb{R}F(W, Z)$$

form cofiber/fiber sequences as well.

4.4.25. Show that the rule for the smash product of two morphisms in Definition 4.2.13 is independent of n_1 and n_2, so that it gives a well-defined map

$$\operatorname*{colim}_{n_1 \to \infty} [\Sigma^{k+n_1} A, \Sigma^{\ell+n_1} C]_* \times \operatorname*{colim}_{n_2 \to \infty} [\Sigma^{r+n_2} B, \Sigma^{s+n_2} D]_* \longrightarrow \operatorname*{colim}_{n \to \infty} [\Sigma^{k+r+n} A \wedge C, \Sigma^{\ell+s+n} B \wedge D]_*.$$

4.4.26. Prove that the map $(A, k) \wedge (C, r) \cong (C, r) \wedge (A, k)$ from Definition 4.2.13 is natural, i.e. every diagram of the following form commutes:

$$\begin{array}{ccc} (A,k) \wedge (C,r) & \xrightarrow{f \wedge g} & (B,\ell) \wedge (D,s) \\ \cong \downarrow & & \downarrow \cong \\ (C,r) \wedge (A,k) & \xrightarrow{g \wedge f} & (D,s) \wedge (B,\ell) \end{array}$$

Optionally, do the same for the associativity and unit isomorphisms.

4.4.27. Prove Lemma 4.2.15. Essentially, you just have to check that sums and shifts can be pulled through the definition of dual pair from Definition 4.2.1.

4.4.28. Suppose M is a closed smooth manifold that is smoothly embedded into $\mathbb{R}^n$, and let B be its complement. Let N be a closed tubular neighborhood of M, and let B' be the closure of its complement. Use the map of cofiber sequences

$$\begin{array}{ccccccc} B' & \longrightarrow & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n/B' & \xleftarrow{\cong} & N/\partial N \\ \downarrow \sim & & \downarrow \sim & & \vdots \sim & & \\ B & \longrightarrow & CB & \longrightarrow & SB & & \end{array}$$

to show that the suspension SB is homotopy equivalent to $N/\partial N$.

4.4.29. Verify Example 4.2.28 by checking that the trace of a self-map of a complex is the alternating sum of traces on each level.

4.4.30. (a) Prove that a strong symmetric monoidal functor $F : \mathbf{C} \to \mathbf{D}$ preserves dualizable objects: if X is dualizable in $\mathbf{C}$ then $F(X)$ is dualizable in $\mathbf{D}$.

(b) Prove that F also preserves traces: along the isomorphism $I_{\mathbf{D}} \cong F(I_{\mathbf{C}})$, the map $F(\mathrm{tr}(f))$ agrees with the map $\mathrm{tr}(F(f))$, for any self-map $f : X \to X$.

(c) The rational homology functor $H_*(-;\mathbb{Q}) \colon \mathrm{Ho}\,\widehat{\mathbf{Sp}} \to \mathbf{GrMod}_{\mathbb{Q}}$ is strong symmetric monoidal. Use this to show that the trace of a finite complex in the stable homotopy category $\mathrm{Ho}\,\mathbf{Sp}$ agrees with the Lefschetz number $L(f)$ from Example 4.2.31.

Chapter 5

Spectra as a model category

What do spaces and spectra have in common? They are both categories with weak equivalences, so we can form a homotopy category $\mathrm{Ho}\,\mathbf{C}$, and talk about derived functors, as in Chapter 3.

There is another thing in common: they both have a notion of a "cell complex." A cell complex of spaces is built by taking successive pushouts along maps of the form $S^{n-1} \to D^n$. A cell complex of spectra is built in the same way from maps of the form $F_n S^{k-1}_+ \to F_n D^k_+$.

There is a third thing that they have in common: they both have a notion of "fibrant object." A fibrant topological space is... just a topological space. In other words, every space is fibrant. A fibrant spectrum, on the other hand, is an Ω-spectrum.

Finally, in both settings, maps in the homotopy category $\mathrm{Ho}\,\mathbf{C}(X, Y)$ are the same thing as homotopy classes of maps $X \to Y$, if X is a cell complex and Y is fibrant. So the cell complexes and the fibrant objects are the "nice" objects, for the purpose of counting maps in the homotopy category.

In this chapter, we introduce a formal framework that generalizes this: cofibrantly generated model categories. A **model category** is a category with three classes of maps, called the cofibrations, weak equivalences, and fibrations, that satisfy a few axioms. A **cofibrantly generated model category** also has a notion of a "cell complex" and an "acyclic cell complex" that generate the rest of the information. In particular, the cofibrations are the maps that are retracts of cell complexes, and the fibrations are the maps that have a lifting property along every acyclic cell complex.

Why do we need this terminology? For two reasons. First, we have already introduced a variant of spectra called "E-local spectra," which have the same objects but a different class of weak equivalences. Second, we will soon introduce two new kinds of spectra called **symmetric spectra** and **orthogonal spectra**. We sometimes refer to both of these as "diagram spectra."

This leaves us with many, many different categories of objects that could all be called "spectra." This presents a problem: once we have several models of spectra flying around, it becomes difficult to keep track of exactly what is true, and what we have proven, in each model separately.

This gets a lot easier if we develop a common language and a uniform set of notation to refer to concepts that exist across the different models. For instance, the smash product always preserves equivalences between "cofibrant" spectra. What exactly is a "cofibrant" spectrum? It's different in each case. But the same facts about cofibrant spectra hold in all three settings, so we don't have to memorize the theorem separately in each case.

When we use the same language to describe all of these settings, we can move more quickly and make fewer mistakes when jumping from one model to the other. And model categories are an excellent language for this purpose.

Model categories are also very useful on a technical level. We will use them to prove Theorem 3.3.7, and to prove many facts about the smash product of spectra in Chapter 6. What model categories do in both of these applications is break up the weak equivalences into "chunks." So, we can prove that a functor like the smash product $X \wedge Y$ preserves weak equivalences by checking it on one chunk at a time. In topological spaces, the chunks are maps of the form $D^n \times \{0\} \to D^n \times I$. It is very easy to show that maps like this go to weak equivalences. For spectra, the chunks are more complicated, because they also involve the truncation maps of the form $F_n S^n \to F_0 S^0$, which are stable equivalences. Still, this is a powerful technique, and it would be very difficult to study diagram spectra without it.

5.1 Generalized cell complexes and fibrations

Our treatment of model categories will be somewhat backwards. We'll start with the definition of a "cell complex" in an arbitrary category **C**. Then we'll show how to factor every map in **C** into a cell complex, followed by a map that has a lifting property with respect to cell complexes. This is the famous **small-object argument**. It was developed by Quillen when he invented model categories, and it's really the core idea that powers the whole theory.

5.1.1 I-cell complexes and I-fibrations

Let $\mathbf{C}$ be any category. Assume that every diagram in $\mathbf{C}$ has a colimit.[1] In particular, we're assuming that $\mathbf{C}$ has all coproducts, pushouts, and sequential colimits.

Let I be any set of maps in $\mathbf{C}$. For example, I could be the set $\{S^{n-1} \to D^n\}_{n\geq 0}$ in the category **Top**. We think of the maps in I as the "cells" in $\mathbf{C}$.

Definition 5.1.1. An **I-cell complex** is a sequential composition of pushouts of coproducts of maps in I. In other words, it is a composite

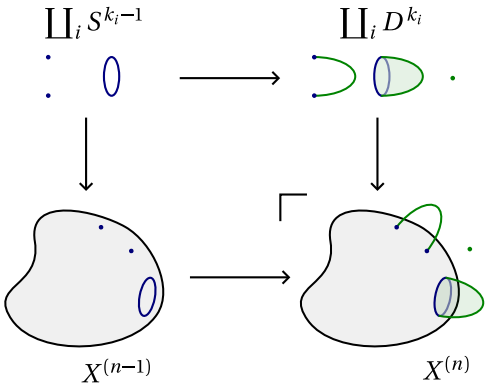
$$A = X^{(0)} \longrightarrow X^{(1)} \longrightarrow X^{(2)} \longrightarrow \ldots \longrightarrow X^{(n-1)} \longrightarrow X^{(n)} \longrightarrow \ldots \longrightarrow X = \operatorname*{colim}_{n\to\infty} X^{(n)}$$

where $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching a collection of cells $K_\alpha \to L_\alpha$ from the set I:

$$\begin{array}{ccc} \coprod_\alpha K_\alpha & \longrightarrow & \coprod_\alpha L_\alpha \\ \downarrow & & \downarrow \\ X^{(n-1)} & \longrightarrow & X^{(n)}. \end{array}$$

Example 5.1.2.

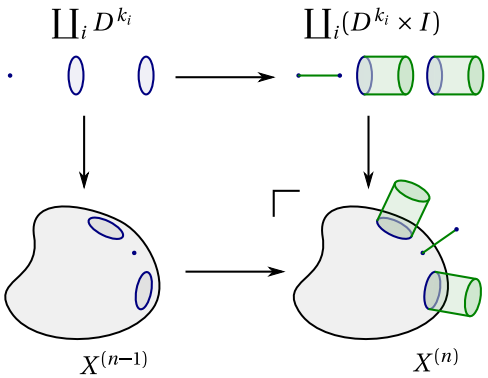
- If $\mathbf{C} = \mathbf{Top}$ and $I = \{S^{n-1} \to D^n\}_{n\geq 0}$, then an I-cell complex is the same thing as a relative cell complex in the usual sense (Definition 1.1.11).



- If $\mathbf{C} = \mathbf{Sp}$ and $I = \{F_n S^{k-1}_+ \to F_n D^k_+\}_{n,k\geq 0}$, then an I-cell complex is the same thing as a relative cellular spectrum (Definition 2.6.7 and Proposition 2.6.11).
- Suppose $\mathbf{C} = \mathbf{Ch}_{\geq 0}(\mathbb{Z})$ is nonnegatively-graded chain complexes of abelian groups, and I consists of all shifts of the map of chain complexes $(0 \to \mathbb{Z}) \to (\mathbb{Z} \to \mathbb{Z})$, where every instance of $\mathbb{Z} \to \mathbb{Z}$ that occurs is the identity map (see Example 5.2.11). Then an I-cell complex in $\mathbf{Ch}_{\geq 0}$ is a map $C_\bullet \to D_\bullet$ such that each $C_n \to D_n$ is injective, with quotient a free abelian group $\bigoplus \mathbb{Z}$.

[1] Recall from Section 1.1 that a diagram is defined to be a functor $\mathbf{I} \to \mathbf{C}$ where $\mathbf{I}$ is a small category. In other words, the objects and morphisms of $\mathbf{I}$ form a set, not a proper class.

- If $\mathbf{C} = \mathbf{Top}$ and $J = \{D^n \times \{0\} \to D^n \times I\}_{n\geq 0}$, then a J-cell complex is a map built by successive **elementary expansions**. In other words, we select a collection of discs in the space $D^n \to A$, extrude each one to form a cylinder $D^n \times I$, then repeat the process countably many times, to get a larger space X. In this case the inclusion $A \to X$ is both a relative cell complex and a weak equivalence.



Remark 5.1.3. It is common to use a more general definition of cell complex that involves transfinite compositions, which we give in Exercise 5.8.42. However, this extra generality is usually not necessary, and it takes longer to learn, so we begin with this countable definition for simplicity. In fact, for spaces and spectra, cell complexes in the transfinite sense *coincide* with cell complexes in this countable sense by Exercise 5.8.43, so there's no difference at all between the two notions.

The set I also determines a class of generalized "fibrations" in $\mathbf{C}$: those maps that have lifts along every map in I.

Definition 5.1.4. An **I-fibration** is a map $p : X \to Y$ such that, for any commuting square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array} \tag{5.1.5}$$

with i a map in I, a dashed lift exists, making both triangles commute. We call this condition the **right lifting property** with respect to I.

Example 5.1.6. If $\mathbf{C} = \mathbf{Top}$ and $J = \{D^n \times \{0\} \to D^n \times I\}_{n\geq 0}$, a J-fibration is a map in which every square of the form

$$\begin{array}{ccc} D^n \times \{0\} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ D^n \times I & \longrightarrow & Y \end{array} \tag{5.1.7}$$

has a lift. By definition, this is the same thing as a Serre fibration.

Example 5.1.8. If $\mathbf{C} = \mathbf{Top}$ and $I = \{S^{n-1} \to D^n\}_{n\geq 0}$, an I-fibration is a map of topological spaces $X \to Y$ such that every square

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ D^n & \longrightarrow & Y \end{array} \tag{5.1.9}$$

has a dashed lift. This implies that $\pi_n(Y,X) = 0$ for each $n \geq 0$, so the map $X \to Y$ is a weak equivalence. Furthermore, for any cell complex $A \to B$, any commuting square of the form (5.1.5) has a lift, by defining the lift inductively, one cell at a time. Therefore, in particular, every square of the form (5.1.7) has a lift, so $X \to Y$ is a Serre fibration.

Conversely, if $X \to Y$ is both a weak equivalence and a Serre fibration, then it is an I-fibration. You show this by first showing that the square (5.1.9) has a lift up to homotopy, then using the homotopy lifting property of the fibration to modify this to a strict lift. (See Exercise 1.7.23.)

In conclusion, the I-fibrations in **Top** are the maps that are both weak equivalences and Serre fibrations. We call these maps the **acyclic Serre fibrations**.

Example 5.1.10. If $\mathbf{C} = \mathbf{Sp}$ and $I = \{F_n S^{k-1}_+ \to F_n D^k_+\}_{n,k\geq 0}$, an I-fibration is a map of spectra $X \to Y$ such that every square

$$\begin{array}{ccc} F_n S^{k-1}_+ & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ F_n D^k_+ & \longrightarrow & Y \end{array}$$

has a dashed lift. Using the adjunction $(F_n(-)_+, \mathrm{ev}_n)$ between spectra and unbased spaces, this rearranges to finding a lift

$$\begin{array}{ccc} S^{k-1} & \longrightarrow & X_n \\ \downarrow & \nearrow & \downarrow p_n \\ D^k & \longrightarrow & Y_n. \end{array} \tag{5.1.11}$$

Therefore, $X \to Y$ is an I-fibration iff it is an acyclic Serre fibration (a weak equivalence and a Serre fibration) at each spectrum level.

5.1.2 The small-object argument

Recall that $\mathbf{C}$ is a category with all (small) colimits, and I is any set of maps in $\mathbf{C}$. For simplicity we continue to work in the countable case, and refer the reader to Exercise 5.8.42 for the more general transfinite case.

Definition 5.1.12. We say that I has the **countable smallness condition** if for each map $K \to L$ in the set I, and each I-cell complex

$$X^{(0)} \longrightarrow X^{(1)} \longrightarrow \cdots X^{(\infty)} = \operatorname*{colim}_{n\to\infty} X^{(n)},$$

every map $K \to X^{(\infty)}$ factors through some $X^{(n)}$.

For instance, the set $I = \{S^{n-1} \to D^n\}_{n\geq 0}$ in **Top** has the countable smallness condition, because S^{n-1} is compact (see Exercise 1.7.31).

Theorem 5.1.13 (Small-object argument, countable case)**.** *Suppose I satisfies the countable smallness condition. Then every map in* **C** *can be factored into an I-cell complex followed by an I-fibration:*

$$X \xrightarrow{I\text{-cell complex}} X^{(\infty)} \xrightarrow{I\text{-fibration}} Y.$$

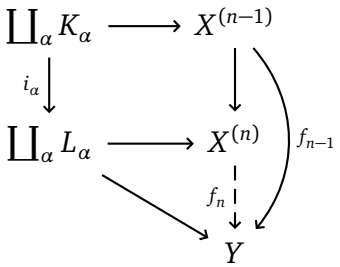
Furthermore, this factorization is functorial.

Proof. Let $f : X \to Y$ be any map. We will define $f_n : X^{(n)} \to Y$ for $n \geq 0$ by induction on n. When $n = 0$, we take $X^{(0)} = X$ and $f_0 = f$. Inductively, once we have constructed $X^{(n-1)}$ and f_{n-1}, we take the set of all possible commuting squares of the form

$$\begin{array}{ccc} K & \longrightarrow & X^{(n-1)} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f_{n-1}} \\ L & \longrightarrow & Y \end{array} \tag{5.1.14}$$

with $i \in I$. Note that I is assumed to be a *set*, not a proper class, so the collection of all such squares forms a set as well.

Take the coproduct of all the maps i, one for each such square, and then define $X^{(n)}$ by taking the pushout of $X^{(n-1)}$ along this coproduct:



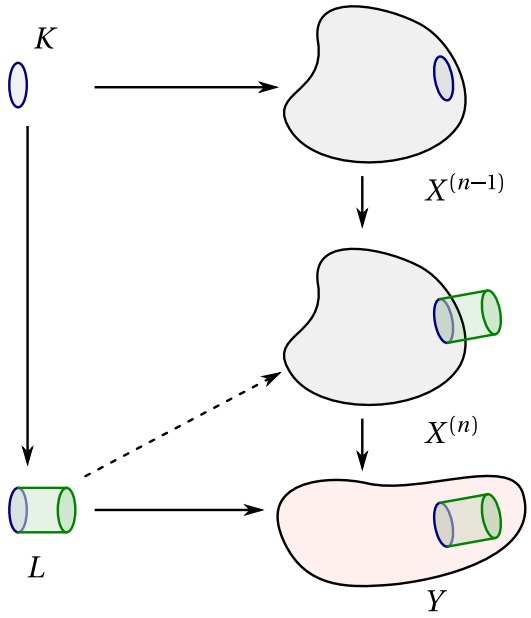
(5.1.15)

Finally, let $X^{(\infty)} = \operatorname*{colim}_{n\to\infty} X^{(n)}$ and f_∞ the induced map to Y.

We have now factored f into the I-cell complex $X \to X^{(\infty)}$, followed by f_∞, so it remains to prove that f_∞ is an I-fibration. Given any commuting square

$$\begin{array}{ccc} K & \longrightarrow & X^{(\infty)} \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f_\infty} \\ L & \longrightarrow & Y \end{array} \tag{5.1.16}$$

with $i \in I$, by the countable smallness condition, the map $K \to X^{(\infty)}$ factors through some finite stage $X^{(n-1)}$. By the construction of $X^{(n)}$, this map i is one of the maps we included in the coproduct in (5.1.15), so we get a map $L \to X^{(n)}$. Composing this with $X^{(n)} \to X^{(\infty)}$, we get the dashed map making (5.1.16) commute.



Finally we show that this can be made into a functor. Given a commuting square

$$\begin{array}{ccc} X & \xrightarrow{\xi} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{\upsilon} & Y', \end{array}$$

composing with ξ and υ turns each of the squares in (5.1.14) to a corresponding square for f'. This gives a rule for how to map each $X^{(n)}$ to $(X')^{(n)}$, and therefore a map of their colimits over n, commuting with everything else. This makes the factorization into a functor. □

The small-object argument is amazing. It miraculously produces a factorization that has *both* a cell complex and a fibration, even though intuitively, such a factorization does not seem possible. How could a complex built in this way possibly act like a fiber bundle?

The magic happens because we solved all possible lifting problems for $X^{(\infty)} \to Y$, by explicitly attaching a cell for every possible lifting problem. This doesn't work if we only do the cell attachments once, because for every lifting problem that we *solve*, many more new lifting problems are *created*. But, if we perform this procedure countably many times and take the colimit, every lifting problem that we create is solved at the next stage of the construction. That is the miracle of the small-object argument.

Example 5.1.17. If $\mathbf{C} = \mathbf{Top}$, the small-object argument for the set $I = \{S^{n-1} \to D^n\}_{n\geq 0}$ factors every map $X \to Y$ into a cell complex followed by a map that is *both* a weak equivalence and a Serre fibration:

$$X \xrightarrow{\text{cell complex}} X' \xrightarrow[\text{weak equivalence}]{\text{Serre fibration}} Y.$$

This is surprising, and much stronger than Corollary 1.4.17.

Example 5.1.18. If $\mathbf{C} = \mathbf{Top}$, the small-object argument for the set $J = \{D^n \times \{0\} \to D^n \times I\}_{n\geq 0}$ factors every map $X \to Y$ into a map that is both a cell complex and a weak equivalence, followed by a Serre fibration:

$$X \xrightarrow[\text{weak equivalence}]{\text{cell complex}} X' \xrightarrow{\text{Serre fibration}} Y.$$

Again, this is stronger than the construction that replaces the map $E \to B$ by the fibration $E \times_B B^I \to B$. The inclusion $E \to E \times_B B^I$ is a weak equivalence, but is not a cell complex in general.

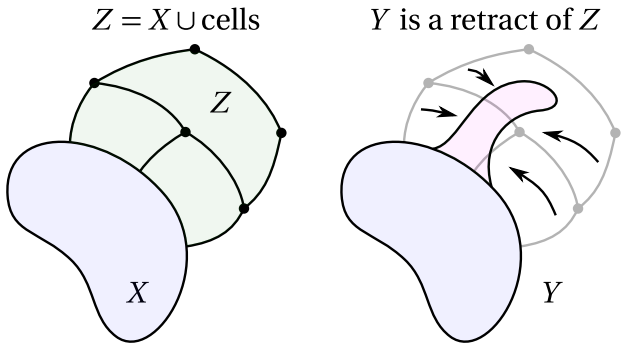
5.1.3 Quillen cofibrations

Definition 5.1.19. A map of spaces $X \to Y$ is a **Quillen cofibration** if it is a retract in the arrow category $\mathbf{Top}^{\{\bullet\to\bullet\}}$ of a relative cell complex.

More generally, in any category $\mathbf{C}$ that is closed under colimits, and any set of maps I satisfying Definition 5.1.12, an **I-cofibration** is any map that is a retract of an I-cell complex.

It is an exercise to show that, without loss of generality, the cell complex has the same domain X (Exercise 5.8.4). So $X \to Y$ is a Quillen cofibration if it is a topological inclusion, Y sits inside a larger space Z as a retract, and $X \to Z$ is a relative cell complex.

It follows from Lemma 1.2.5 that every Quillen cofibration is also a Hurewicz cofibration, in other words a map with the homotopy extension property.



Proposition 5.1.20. *Suppose I satisfies Definition 5.1.12. Then a map $A \to B$ is an I-cofibration iff it has the **left lifting property** with respect to every I-fibration. In other words, there is a lift for every commuting square*

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

in which $p : X \to Y$ is an I-fibration.

Proof. We first prove that every I-cofibration has this property. If the map $A \to B$ is actually in I, then the lift exists by the definition of I-fibration (Definition 5.1.4). By Exercise 5.8.6 below, the maps with the left lifting property are preserved by coproducts, pushouts, sequential compositions, and retracts. Therefore every retract of an I-cell complex has the left lifting property as well.

Conversely, suppose that $f : A \to B$ has the left lifting property. By the small object argument, f factors into an I-cell complex followed by an I-fibration. This allows us to form the following square.

$$\begin{array}{ccc} A & \xrightarrow{I\text{-cell complex}} & A' \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle I\text{-fibration}} \\ B & = & B \end{array}$$

It follows that f is a retract of the I-cell complex $A \to A'$, because this can be rewritten as the following retract diagram in the category $\mathbf{Top}^{\{\bullet\to\bullet\}}$:

$$\begin{array}{ccccc} A & = & A & = & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle I\text{-cell complex}} & & \downarrow{\scriptstyle f} \\ B & \dashrightarrow & A' & \xrightarrow{I\text{-fibration}} & B \end{array}$$

□

Corollary 5.1.21. *I-cofibrations are preserved by coproducts, pushouts, sequential compositions, and retracts.*

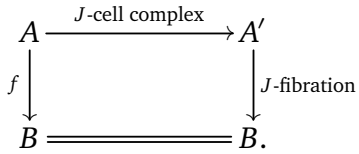
Proof. We just proved these are the maps with the left-lifting property with respect to the I-fibrations. Those are preserved by coproducts, pushouts, etc. by Exercise 5.8.6. □

Example 5.1.22. If $\mathbf{C} = \mathbf{Top}$ and $J = \{D^n \times \{0\} \to D^n \times I\}_{n\geq 0}$, a J-cofibration is a retract of a complex built out of elementary expansions, see Example 5.1.2. These coincide with the maps that have the left lifting property along every Serre fibration.

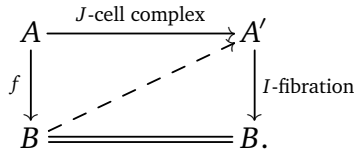
Proposition 5.1.23. *In* **Top***, with the sets of maps I and J used above, a map $A \to B$ is a J-cofibration iff it is both an I-cofibration and a weak equivalence.*

Proof. It is easy to see that a J-cofibration is both an I-cofibration and a weak equivalence, because the maps in J are all relative cell complexes and weak equivalences.

Conversely, suppose that $f : A \to B$ is both an I-cofibration and a weak equivalence. Then, as in the proof of Proposition 5.1.20, we can factor f into a J-cell complex and a J-fibration (a Serre fibration):



By the 2-out-of-3 property, since both f and the J-cell complex are weak equivalences, the Serre fibration on the right is also a weak equivalence. By Example 5.1.10, the map on the right is therefore an I-fibration. By Proposition 5.1.20, therefore, a lift exists:



Therefore, as in the proof of Proposition 5.1.20, f is a retract of a J-cell complex. In other words, it is a J-cofibration. □

Definition 5.1.24. An **acyclic Quillen cofibration** is a map in **Top** that satisfies either of the above two equivalent conditions:

- it is both a weak equivalence, and a retract of a cell complex, or
- it is a retract of a complex of elementary expansions.

In summary, Proposition 5.1.23 shows that equivalent subcomplexes can be expressed in terms of elementary expansions. For spaces, this is not such a big deal, because we don't have much difficulty in proving that things like the smash product $X \wedge Y$ preserve weak equivalences. However, the arguments we just performed were very formal, and can be applied to spectra as well. In that setting, this is a powerful technique.

5.2 Model categories

5.2.1 Definition and examples

Let us formalize what we have proven so far about **Top**. Let **C** be any category.

Definition 5.2.1. A **model structure** on **C** is a choice of three subcategories C, W, and F whose maps are called the "cofibrations," "weak equivalences," and "fibrations," respectively. We refer to maps in $W \cap C$ as "acyclic cofibrations," and maps in $W \cap F$ as "acyclic fibrations." These must satisfy four axioms:

- The weak equivalences W are closed under 2-out-of-3 (see Definition 3.4.13).
- Each of the three classes W, C, and F is closed under retracts, in the category of arrows $\mathbf{C}^{\{\bullet\to\bullet\}}$. In other words, for any commuting diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X & \longrightarrow & Y & \longrightarrow & X, \end{array}$$

  if $g \in C$ then $f \in C$. The same applies to W and to F.
- C has the left lifting property with respect to $W \cap F$, and $W \cap C$ has the left lifting property with respect to F.

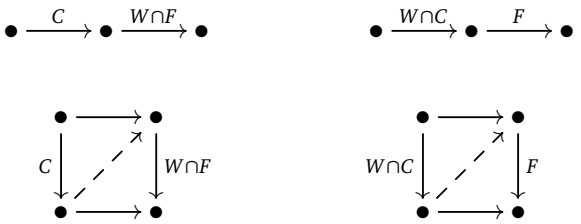
  In other words, given a commuting square in **C**

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y, \end{array} \tag{5.2.2}$$

  if $i \in C$ and $p \in W \cap F$, or if $i \in W \cap C$ and $p \in F$, a dashed lift exists.

- There is a functorial factorization of each map f into $f = p \circ i$, where i is a cofibration and p is an acyclic fibration. There is another functorial factorization in which i is an acyclic cofibration and p is a fibration.

The last two axioms can be summarized pictorially as follows.



Definition 5.2.3. We say **C** has all small colimits and limits if for every diagram in **C**, i.e. a functor from a small category **I** to **C**, the colimit and the limit of the diagram both exist.

Definition 5.2.4. A **model category** is a category **C** that has all small colimits and limits, plus a choice of model structure on **C**.

We can neatly organize the conclusions of Theorem 5.1.13, Proposition 5.1.20, and Proposition 5.1.23 by saying that **Top** is a model category:

Theorem 5.2.5 (Quillen)**.** *The category of unbased topological spaces* **Top** *has a model structure in which*

- *the cofibrations are the Quillen cofibrations,*
- *the weak equivalences are the weak homotopy equivalences, and*
- *the fibrations are the Serre fibrations.*

We would like to produce more examples. We can use the same technique: start with a category **C**, and pick a subcategory W, and two sets of maps I and J. We then declare that

- the cofibrations are the I-cofibrations (the retracts of the I-cell complexes),
- the weak equivalences are the maps in W, and
- the fibrations are the J-fibrations (the right lifting property with respect to J).

Proposition 5.2.6. *This defines a model structure on* **C***, if the following list of conditions is satisfied.*

1. W is closed under 2-out-of-3 and retracts.
2. I satisfies the countable smallness condition.

3. J satisfies the countable smallness condition.
4. J-cell complexes are in $W \cap (I\text{-cof})$.
5. $I\text{-fib} \subseteq W \cap (J\text{-fib})$.
6. Either $W \cap (I\text{-cof}) \subseteq J\text{-cof}$ or $W \cap (J\text{-fib}) \subseteq I\text{-fib}$.

Proof. If in the last point if we know that $W \cap (J\text{-fib}) \subseteq I\text{-fib}$, then the proofs of Theorem 5.1.13, Proposition 5.1.20, and Proposition 5.1.23 all apply, proving that we have a model structure.

If in the last point we have $W \cap (I\text{-cof}) \subseteq J\text{-cof}$, then we don't need Proposition 5.1.23 to prove that $W \cap (I\text{-cof}) \subseteq J\text{-cof}$. However, we do need to know that $W \cap (J\text{-fib}) \subseteq I\text{-fib}$. We show this by dualizing the proof of Proposition 5.1.23, see Exercise 5.8.5. The rest of the argument then proceeds as before. See also [Hov99, 2.1.19]. □

Remark 5.2.7. The above proof works equally well if we assume that I and J satisfy the "transfinite" version of the smallness condition given in Exercise 5.8.42.

Definition 5.2.8. A **cofibrantly generated** model structure on $\mathbf{C}$ is one produced by the method of Proposition 5.2.6, with either the countable smallness condition or the transfinite one from Exercise 5.8.42. In such a model structure, we always have

- $C = I\text{-cof}, \quad W \cap C = J\text{-cof},$
- $F = J\text{-fib}, \quad W \cap F = I\text{-fib}.$

So the cofibrations, acyclic cofibrations, fibrations, and acyclic fibrations are all neatly described in terms of the generators in I and J. For this reason, we call I the **generating cofibrations** and J the **generating acyclic cofibrations**.

Think of a cofibrantly generated model category as a category with "generalized cell complexes and fibrations." All of the nice properties of cell complexes and Serre fibrations in **Top** carry over to any cofibrantly generated model category.

Example 5.2.9. The category of based topological spaces $\mathbf{Top}_*$ has a cofibrantly generated model structure with

- C the Quillen cofibrations,
- W the weak homotopy equivalences,
- F the Serre fibrations,
- $I = \{S^{n-1}_+ \to D^n_+\}_{n \geq 0}$, and

- $J = \{(D^n \times \{0\})_+ \to (D^n \times I)_+\}_{n\geq 0}$.

Essentially, a map is a cofibration, weak equivalence, or fibration if it is such after forgetting about the basepoint. The generators have disjoint basepoints added to them. See Exercise 5.8.8.

Example 5.2.10. In this book, the objects of the categories **Top** and $\mathbf{Top}_*$ are compactly generated weak Hausdorff spaces. It turns out that if we drop the weak Hausdorff condition, or both conditions, then a model structure still exists, with essentially the same definition as above.

Example 5.2.11. For any ring R, the category $\mathbf{Ch}_{\geq 0}(R)$ of nonnegatively-graded chain complexes of R-modules has a cofibrantly generated model structure in which

- the cofibrations are the maps that in each degree are injective and with cokernel a projective R-module,
- the weak equivalences are the quasi-isomorphisms, i.e. the maps inducing isomorphisms on homology,
- the fibrations are the maps that are surjective in all positive degrees,
- I consists of all maps of chain complexes of the form $(0 \to R) \to (R \to R)$, and their shifts, and also $(0) \to (R)$ in degree zero, and
- J consists of all maps from the zero complex to a shift of the complex $(R \to R)$.

In the definitions of I and J, every unnamed map $R \to R$ is the identity map of R. Note that attaching a "cell" in I has the effect of adding a summand $\oplus R$ to one level of the chain complex, and also specifying its image under the boundary map, if we added it above level 0. The verification of the conditions in Proposition 5.2.6 is left to Exercise 5.8.10.

The larger category of unbounded chain complexes has a similar model structure with essentially the same generators I and J, where the weak equivalences are again the quasi-isomorphisms and the fibrations are the maps that are surjective in every degree. See Exercise 5.8.11.

The following is the main theorem of this chapter.

Theorem 5.2.12 (Bousfield-Friedlander)**.** *The category of sequential spectra* **Sp** *has the following two model structures.*

In the **level model structure**,

- the cofibrations are the retracts of the relative cellular spectra (Definition 2.6.7),

- the weak equivalences are the level equivalences $X_n \xrightarrow{\sim} Y_n$, and
- the fibrations are the level fibrations: the maps $X \to Y$ in which each level $X_n \to Y_n$ is a Serre fibration.

We establish this model structure in Theorem 5.3.40.

In the **stable model structure**,

- the cofibrations are the same, the retracts of the relative cellular spectra,
- the weak equivalences are the stable equivalences, and
- the fibrations are the stable fibrations: the maps $X \to Y$ in which each level $X_n \to Y_n$ is a Serre fibration and each square

$$\begin{array}{ccc} X_n & \longrightarrow & \Omega X_{n+1} \\ \downarrow & & \downarrow \\ Y_n & \longrightarrow & \Omega Y_{n+1} \end{array}$$

  is a homotopy pullback square.

We establish this model structure in Section 5.6.

For the level model structure, the generating cofibrations are the maps

$$I = \left\{ F_n S^{k-1}_+ \to F_n D^k_+ \ : \ n, k \geq 0 \right\}$$

from Example 5.1.10, and the generating acyclic cofibrations are the maps

$$J = \left\{ F_n(D^k \times \{0\})_+ \to F_n(D^k \times I)_+ \ : \ n, k \geq 0 \right\}.$$

In the stable model structure, we use the same set I (Definition 5.6.1), but a larger set J (Definition 5.6.3).

Remark 5.2.13. Not every model structure is cofibrantly generated. There is a **Strøm model structure** on **Top** in which

- the cofibrations are the classical ones, i.e. the maps with the homotopy extension property,
- the weak equivalences are the homotopy equivalences, and
- the fibrations are the Hurewicz fibrations.

This is not (at least in any known way) generated by sets of maps I and J, so it cannot be proven by the method we've developed thus far. Fortunately, all of the examples we actually care about are cofibrantly generated, so we won't spend much time thinking about model structures that are not cofibrantly generated.

Our goal in the remainder of the chapter is to develop the properties of model categories in general, and to prove Theorem 5.2.12, giving two model structures on spectra.

5.2.2 Cofibrant and fibrant objects

Let **C** be a model category. So it has a model structure, and also, it has all small colimits and limits. In particular, **C** has an initial object $\varnothing$, and a terminal object $*$. For any object X, there is a unique map $\varnothing \to X$, and a unique map $X \to *$.

Definition 5.2.14. We say that

- X is **cofibrant** if $\varnothing \to X$ is a cofibration.
- X is **fibrant** if $X \to *$ is a fibration.
- X is **bifibrant** if it is both cofibrant and fibrant.

You should think of the cofibrant and fibrant objects as the "good" or "well-behaved" objects.

Example 5.2.15. In **Top**, the initial object is the empty set, so a space X is cofibrant iff it is a retract of a cell complex.

The terminal object is the one-point space. It is easy to see that $X \to *$ is always a Serre fibration, so every space is fibrant.

Example 5.2.16. In $\mathbf{Ch}_{\geq 0}(R)$, the initial and terminal object are both the zero complex. A complex is cofibrant if it is a retract of a levelwise free complex. Equivalently, if it is levelwise projective. Every complex is fibrant.

Example 5.2.17. In **Sp**, the initial and terminal object are both the zero spectrum. In both model structures, a spectrum is cofibrant if it is a retract of a cellular spectrum. In the level model structure, every spectrum is fibrant. In the stable model structure, the fibrant spectra are precisely the Ω-spectra!

Definition 5.2.18. The **cofibrant replacement** of an object X is the object QX and map $QX \to X$ obtained by factoring the unique map $\varnothing \to X$ into a cofibration followed by an acyclic fibration:

$$\varnothing \longrightarrow QX \xrightarrow{\sim} X$$

Note that Q is a functor, and $QX \to X$ is a natural transformation.

Similarly, the **fibrant replacement** of an object X is the object RX and map $X \to RX$ obtained by factoring the unique map $X \to *$ into an acyclic cofibration followed by a fibration:

$$X \xrightarrow{\sim} RX \longrightarrow *.$$

Note that R is a functor, and $X \to RX$ is a natural transformation.

Lemma 5.2.19. *The cofibrant replacement and fibrant replacement functors are homotopical – they preserve weak equivalences.*

Proof. This follows quickly from the 2-out-of-3 property. □

Lemma 5.2.20. *Q preserves fibrant objects and R preserves cofibrant objects. So QRX and RQX are both bifibrant.*

Proof. If X is fibrant, then $X \to *$ is a fibration, and $QX \to X$ is an acyclic fibration. Composing them, we conclude that $QX \to *$ is a fibration. The argument for R is similar. □

Remark 5.2.21. In **Sp** with the stable model structure, these operations behave like the Q and R that we defined in Theorem 2.6.12 and Proposition 2.2.9. The difference is that:

- They have *better properties*: $QX \to X$ is now a fibration and $X \to RX$ is now a cofibration, in addition to these maps being weak equivalences.
- However, they are *less explicit*, especially compared to the construction in Proposition 2.2.9.

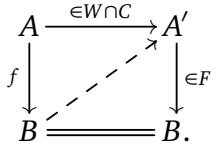
5.2.3 Fundamental properties of model categories

Lemma 5.2.22. *In any model category* **C** *the following statements hold:*

- f is an cofibration iff it has the left-lifting property with respect to all acyclic fibrations.
- f is an acyclic cofibration iff it has the left-lifting property with respect to all fibrations.
- f is an fibration iff it has the right-lifting property with respect to all acyclic cofibrations.
- f is an acyclic fibration iff it has the right-lifting property with respect to all cofibrations.

- Every isomorphism in **C** is a cofibration, a fibration, and a weak equivalence.
- The cofibrations are closed under pushouts, coproducts, sequential compositions, and retracts. The same applies to acyclic cofibrations.
- The fibrations are closed under pullbacks, products, sequential inverse compositions, and retracts. The same applies to acyclic fibrations.

Proof. The first four points are similar to the proof of Proposition 5.1.20. For example, if $f : A \to B$ is a map that has the left lifting property with respect to all fibrations, we factor f into an acyclic cofibration followed by a fibration. This gives a commuting square



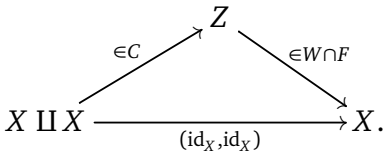
The lift exists because f has the left lifting property. But this shows that f is a retract of a map in $W \cap C$, and therefore $f \in W \cap C$.

An isomorphism clearly has the left lifting property and right lifting property with respect to every map. Therefore it is a cofibration, a fibration, and a weak equivalence.

The last two points are left as exercises, see Exercises 5.8.6 and 5.8.6. □

In any model category **C**, it is possible to talk about homotopies. We just have to define a suitable replacement for the cylinder $X \times I$ in topological spaces.

Definition 5.2.23. For any cofibrant object X, a **cylinder** on X is any object Z fitting into a diagram of the following form.



Example 5.2.24. In **Top**, the product $X \times I$ is an example of a cylinder, because if X is a cell complex, the inclusion of $X \times \{0\}$ and $X \times \{1\}$ into $X \times I$ is also a relative cell complex. The same conclusion follows for retracts of cell complexes. The projection $X \times I \to X$ is clearly a fibration as well.

We can similarly define a mapping cylinder for a map $f : X \to Y$ by factoring the map $(f, \mathrm{id}_Y) : X \amalg Y \to Y$ into a cofibration and an acyclic fibration. This idea is used to prove the following.

Suppose $F : \mathbf{C} \to \mathbf{D}$ is a functor from a model category **C** to a category **D**. Suppose that **D** has a class of weak equivalences (satisfying the 2-out-of-3 property).

Lemma 5.2.25 (Ken Brown's Lemma)**.** *Suppose F takes acyclic cofibrations $X \to Y$ between cofibrant objects $X, Y \in \mathbf{C}$ to weak equivalences $F(X) \xrightarrow{\sim} F(Y)$ in $\mathbf{D}$. Then F takes* all *weak equivalences of cofibrant objects to weak equivalences.*

Proof. The idea is to use the mapping cylinder. Given any weak equivalence of cofibrant objects $f : X \to Y$, factor $(f, \mathrm{id}_Y) \colon X \amalg Y \to Y$ into a cofibration and an acyclic fibration, and let M_f be the intermediate space. Since f is a weak equivalence and Y is cofibrant, the composition $X \to X \amalg Y \to M_f$ is both a weak equivalence and a cofibration. Similarly $Y \to M_f$ is an acyclic cofibration. So F sends both of these maps to weak equivalences.

The projection $M_f \to Y$ is a left inverse to $Y \to M_f$, and is therefore F sends it to a weak equivalence as well. The map f is now the composition $X \to M_f \to Y$ of maps that F sends to weak equivalences, so F sends $X \to Y$ to a weak equivalence. □

The dual form for fibrations has a dual proof, which we leave to Exercise 5.8.20:

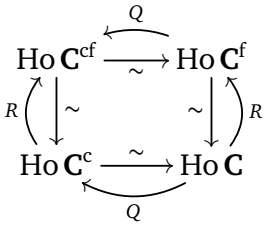
Lemma 5.2.26 (Ken Brown's Lemma, dual form)**.** *Suppose F takes acyclic fibrations $X \to Y$ between fibrant objects $X, Y \in \mathbf{C}$ to weak equivalences $F(X) \xrightarrow{\sim} F(Y)$ in $\mathbf{D}$. Then F takes* all *weak equivalences of fibrant objects to weak equivalences.*

Returning to ordinary cylinders, we say that two maps $f, g : X \rightrightarrows Y$ are **left homotopic** if there is some extension of $(f, g) \colon X \amalg X \to Y$ to $Z \to Y$, for some cylinder object $X \amalg X \to Z \to X$ in which $X \amalg X \to Z$ is a cofibration and $Z \to X$ is a weak equivalence. We will only consider this relation when X is cofibrant and Y is fibrant, in which case it does not depend on the choice of Z.

We skip the proof of the following, as it is rather long and introduces several more definitions that we will not need. See [Hov99, §1.2] for a full proof. Recall that $\mathrm{Ho}\,\mathbf{C} = \mathbf{C}\langle W^{-1}\rangle$ refers to the category in which the weak equivalences of $\mathbf{C}$ have been inverted.

Theorem 5.2.27 (Fundamental Theorem of Model Categories)**.** *If X is cofibrant and Y is fibrant, then the maps in the homotopy category $\mathrm{Ho}\,\mathbf{C}(X, Y)$ are the maps in the original category $\mathbf{C}(X, Y)$, modulo the relation of left homotopy.*

As in Proposition 3.1.29, since Q and R are functors that are weakly equivalent to the identity, we get equivalences of categories



where $\mathbf{C}^{cf} \subseteq \mathbf{C}$ denotes the bifibrant objects, $\mathbf{C}^{c}$ the cofibrant objects, and $\mathbf{C}^{f}$ the fibrant objects. As in Remark 3.1.30, we conclude that for any pair of objects $X, Y \in \mathbf{C}$, the maps in the homotopy category can be computed as

$$\begin{aligned} \operatorname{Ho}\mathbf{C}(X,Y) &\cong \mathbf{C}(QX, RY)/(\text{left homotopy}) \\ &\cong \mathbf{C}(QRX, QRY)/(\text{left homotopy}) \\ &\cong \mathbf{C}(RQX, RQY)/(\text{left homotopy}). \end{aligned}$$

Corollary 5.2.28 (Whitehead's Theorem for Model Categories)**.** *Between bifibrant objects, a map $f : X \to Y$ is a weak equivalence iff it is a homotopy equivalence.*

In other words, model categories are a formal setting where we get the same results as those in Section 3.1. Note that we have already proven the Fundamental Theorem and Whitehead's Theorem in **Sp**, in Proposition 3.1.41 and Corollary 2.6.17.

Remark 5.2.29. For the Quillen model structure on spaces, this recovers Proposition 3.1.27, that the homotopy category is equivalent to CW complexes and homotopy classes of maps. For Quillen's model structure on chain complexes, this recovers the fact that the derived category $\mathscr{D}_{\geq 0}(R) = \operatorname{Ho}\mathbf{Ch}_{\geq 0}(R)$ is equivalent to projective chain complexes and chain homotopy classes of maps.

We also recall another useful result from [Hov99, Thm 1.2.10].

Proposition 5.2.30. *Every model category is **saturated**. This means that a map $f : X \to Y$ is a weak equivalence iff it is an isomorphism in the homotopy category $\mathbf{C}\langle W^{-1}\rangle = \operatorname{Ho}\mathbf{C}$.*

It is true by definition that every weak equivalence becomes an isomorphism in $\operatorname{Ho}\mathbf{C}$, so what this is really saying is that we didn't somehow make some larger class of morphisms into isomorphisms in $\operatorname{Ho}\mathbf{C}$ as well.

It follows that the weak equivalences in any model category $\mathbf{C}$ also satisfy the **2-out-of-6 property**. See Exercise 3.6.32 for details.

5.2.4 Properness

Definition 5.2.31. A model category $\mathbf{C}$ is **left proper** if for every pushout square as shown, with i a cofibration and f a weak equivalence, the map g is also a weak equivalence.

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[j]{} & B \cup_A C \end{array}$$

Dually, **C** is **right proper** if for every pullback square as shown, with j a fibration and g a weak equivalence, the map f is also a weak equivalence.

$$\begin{array}{ccc} B \times_D C & \xrightarrow{i} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[j]{} & D \end{array}$$

We say **C** is **proper** if both of these conditions hold.

Example 5.2.32. The Quillen model structure on **Top** is proper – this is one of the basic facts about spaces that we recalled in Theorem 1.5.9 and Theorem 1.5.24.

Example 5.2.33. Both the level and stable model structures on **Sp** are proper – this can be deduced from the stability theorems of Section 2.4, see Exercise 5.8.7.

Lemma 5.2.34 (Gluing lemma)**.** *If* **C** *is left proper, any weak equivalence of pushout diagrams*

$$\begin{array}{ccccc} X & \xleftarrow{i} & A & \longrightarrow & Y \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ X' & \xleftarrow{i'} & A' & \longrightarrow & Y', \end{array}$$

each of which the maps i and i' are cofibrations, induces a weak equivalence of pushouts

$$X \cup_A Y \xrightarrow{\sim} X' \cup_{A'} Y'.$$

In fact, the gluing lemma is equivalent to **C** being left proper. We omit the proof because it is formal but quite long. The dual result for pullbacks along fibrations holds if **C** is right proper.

Remark 5.2.35. In *any* model category, the pushout preserves weak equivalences if $A \to X$ is a cofibration *and* all three objects A, X, and Y are cofibrant. The proof is left to Exercise 5.8.12. Dually, the pullback preserves equivalences if one of the maps is a fibration *and* all three objects are fibrant. We can therefore talk about the left-derived pushout, and right-derived pullback, in any model category.

5.2.5 In summary

A model structure on **C** is three classes of maps C, W, F, such that W has 2-out-of-3, all three have retracts, and we have factorizations and lifts as pictured.

$$\bullet \xrightarrow{C} \bullet \xrightarrow{W \cap F} \bullet \qquad\qquad \bullet \xrightarrow{W \cap C} \bullet \xrightarrow{F} \bullet$$

A cofibrantly generated model structure arises from sets of maps I and J so that

- $C =$ retracts of I-cell complexes, $\quad W \cap C =$ retracts of J-cell complexes,
- $F = J$-fibrations, $\quad W \cap F = I$-fibrations.

This structure makes it easier to study objects in **C** up to weak equivalence, because:

- the Fundamental Theorem and Whitehead's Theorem make it feasible to count maps in the homotopy category,
- there is a built-in notion of which objects are "nice" (cofibrant and/or fibrant), and everything is weakly equivalent to a nice object, and
- Ken Brown's lemma and the generating acyclics J make it easier to check that a given functor F preserves equivalences of nice objects. (The weak equivalences break up into "chunks.")

5.3 Diagrams (presheaves) of spaces

In this section we give many, many examples of model categories. For any small topological category **I**, the category of **I**-shaped diagrams $\mathbf{Top}^{\mathbf{I}}$ has a model structure. One might also call this the category of topological presheaves on $\mathbf{I}^{\mathrm{op}}$.

In particular, we show that spectra are the same thing as diagrams on some category **I**. We use this to give the proof of the level model structure for spectra.

5.3.1 Diagrams of unbased spaces

Definition 5.3.1. A **topological category** or **category enriched in spaces** is

- a collection of objects $\mathrm{ob}\,\mathbf{I}$,
- for each pair of objects a, b, a topological space of morphisms $\mathbf{I}(a,b)$,
- continuous composition maps $\mathbf{I}(a,b) \times \mathbf{I}(b,c) \to \mathbf{I}(a,c)$, and
- for each object a a unit map $\{\mathrm{id}_a\} \to \mathbf{I}(a,a)$,

such that the composition in $\mathbf{I}$ is associative and each id_a acts as an identity element. A **functor** of topological categories $F : \mathbf{I} \to \mathbf{J}$ is just a functor such that the maps $\mathbf{I}(a,b) \to \mathbf{J}(F(a),F(b))$ are continuous.

Example 5.3.2. Every ordinary category is a topological category, by giving the mapping sets the discrete topology.

Example 5.3.3. The categories **Top** and $\mathbf{Top}_*$ are topological categories, using the usual space of continuous maps $\mathrm{Map}(X,Y)$ or the subspace of based maps $\mathrm{Map}_*(X,Y)$. Recall from Definition 2.3.12 that the category of sequential spectra, **Sp**, is also a topological category. The space of maps $\mathbf{Sp}(X,Y)$ is the product of the mapping spaces $\mathrm{Map}(X_n,Y_n)$, restricted to the subspace of those maps $\{f_n : X_n \to Y_n\}_{n\geq 0}$ that commute with the bonding maps of X and Y.

Example 5.3.4. If $\mathbf{I}$ is a topological category, its opposite category $\mathbf{I}^{\mathrm{op}}$ is also a topological category in a canonical way.

Recall that a category or topological category $\mathbf{I}$ is **small** if its objects form a set, rather than a proper class.

Definition 5.3.5. Let $\mathbf{I}$ be any small category, or more generally, any small topological category.[2] A **diagram** (of spaces) consists of

- a space $X(i)$ for each $i \in \mathrm{ob}\,\mathbf{I}$, and
- a continuous action map $X(i) \times \mathbf{I}(i,j) \to X(j)$,

such that the identity map of i acts by the identity on $X(i)$, and $g \circ f$ acts by the composite of g and f. A **map of diagrams** is a natural transformation $X \to Y$. Equivalently, it consists of continuous maps $X(i) \to Y(i)$ for all $i \in \mathrm{ob}\,\mathbf{I}$, commuting with the action of every map $f : i \to j$. We let $\mathbf{Top}^{\mathbf{I}}$ refer to the category of diagrams.

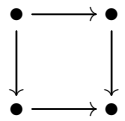
Remark 5.3.6. A diagram in the above sense is essentially the same thing as a topologically enriched functor $\mathbf{I} \to \mathbf{Top}$, in other words a functor such that each map $\mathbf{I}(i,j) \to \mathrm{Map}(X(i),X(j))$ is continuous. We see this using the homeomorphism from (1.1.17). This is one place where it is important to assume that all of our spaces are CGWH.

Example 5.3.7.

- If $\mathbf{I} = \{\bullet\}$ has a single object and only the identity morphism, an $\mathbf{I}$-diagram is just a space. A map of diagrams is a map of spaces.
- If $\mathbf{I} = \{\bullet, \bullet\}$ has two objects but only identity morphisms, an $\mathbf{I}$-diagram is a pair of spaces (X,Y), with no relationship. (In other words, we are not asking for Y to be a subspace of X.) A map of diagrams is a pair of maps of spaces.

[2] We regard every category as a topological category by giving the hom-sets the discrete topology.

- If $\mathbf{I} = \{0 \to 1\}$ has a single nontrivial morphism, an $\mathbf{I}$-diagram is an arrow $X(0) \to X(1)$. A map of diagrams is a commuting square.
- If $\mathbf{I} = \{1 \leftarrow 0 \to 2\}$ is the span category, an $\mathbf{I}$-diagram is a span $X(1) \leftarrow X(0) \to X(2)$. A map of diagrams is a map of spans. As remarked in Example 3.4.26, there is a functor $\mathbf{Top}^{\{1\leftarrow 0\to 2\}} \to \mathbf{Top}$ taking each span to its pushout.
- If $\mathbf{I}$ is the commuting square category



  then an $\mathbf{I}$-diagram is a commuting square. A map of diagrams is a commuting cube.

Definition 5.3.8. A **topological monoid** is a space G with a multiplication map $G \times G \to G$ that is continuous, associative, and has unit $1 \in G$:

$$g \cdot 1 = g = 1 \cdot g, \qquad g_1(g_2 g_3) = (g_1 g_2) g_3.$$

Note that this is the same thing as

- a topological category $\mathbf{I}$ with one object, or
- a monoid (Definition 4.1.12) in the symmetric monoidal category $(\mathbf{Top}, \times, *)$.

We say that G is a **topological group** if in addition it has inverses, and the map $g \mapsto g^{-1}$ is continuous.

A **right action** of G on a space X is a continuous map $X \times G \to G$ that is associative and unital in the sense that

$$x \cdot 1 = x, \qquad (x g_1) g_2 = x(g_1 g_2).$$

A **left action** of G on Y is a continuous map $G \times Y \to Y$ such that

$$1 \cdot y = y, \qquad g_2(g_1 y) = (g_2 g_1) y.$$

In this case we call X a **left G-space** and Y a **left G-space**.

A map G-spaces $f : Y \to Z$ is **equivariant** if it commutes with the G-action. For left actions, this means that $f(gy) = gf(y)$ for all $y \in Y$ and $g \in G$.

Example 5.3.9. If G is any topological monoid, then as noted above, it also defines a category with one object. A G-diagram is the same thing as a space with right G-action, and a map of G-diagrams is the same thing as an equivariant map. Similarly, an G^{op}-diagram is a space with a left G-action.[3]

[3] Is it a right action or a left action? It depents on which convention you use. And you are encouraged to use whichever convention feels more natural to you.

Definition 5.3.10. For each object $i \in \mathrm{ob}\,\mathbf{I}$ and space A, the **free diagram** F_iA is the $\mathbf{I}$-diagram whose value at j is the product

$$(F_iA)(j) = A \times \mathbf{I}(i,j),$$

with $\mathbf{I}$ acting by composition on the $\mathbf{I}(i,j)$ term.

Lemma 5.3.11. *This is the left adjoint of the forgetful functor* $\mathrm{ev}_i : \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}$ *sending the diagram* X *to the space* $X(i)$.

Proof. We just have to see why a map of *diagrams* $F_iA \to X$ is the same data as a map of *spaces* $A \to X(i)$. Once you know where $A \times \{\mathrm{id}_i\}$ goes in X, the rest of the maps $A \times \{f\}$ are determined by the fact that $F_iA \to X$ has to be a map of diagrams. □

Example 5.3.12.

- If $\mathbf{I} = \{\bullet\}$, a free diagram is any space (A).
- If $\mathbf{I} = \{\bullet,\ \bullet\}$, a free diagram is any pair in which one of the spaces is empty: $(A, \varnothing)$ or $(\varnothing, A)$.
- If $\mathbf{I} = \{\bullet \to \bullet\}$, a free diagram is either an arrow of the form $A = A$, or $\varnothing \to A$.
- If $\mathbf{I} = \{\bullet \leftarrow \bullet \to \bullet\}$ is the span category, a free diagram is one of the following three forms:

$$\begin{array}{ccc} A & \longrightarrow & A \\ \downarrow & & \\ A & & \end{array} \qquad \begin{array}{ccc} \varnothing & \longrightarrow & \varnothing \\ \downarrow & & \\ A & & \end{array} \qquad \begin{array}{ccc} \varnothing & \longrightarrow & A \\ \downarrow & & \\ \varnothing & & \end{array}$$

- If $\mathbf{I}$ is the commuting square category, a free diagram is one of the following four forms:

$$\begin{array}{ccc} A & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \end{array} \qquad \begin{array}{ccc} \varnothing & \longrightarrow & \varnothing \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \end{array} \qquad \begin{array}{ccc} \varnothing & \longrightarrow & A \\ \downarrow & & \downarrow \\ \varnothing & \longrightarrow & A \end{array} \qquad \begin{array}{ccc} \varnothing & \longrightarrow & \varnothing \\ \downarrow & & \downarrow \\ \varnothing & \longrightarrow & A. \end{array}$$

- If G is a topological monoid, a free G-diagram is a space of the form $A \times G$, with G acting on the right by $(a, g_0) \cdot g_1 = (a, g_0 g_1)$.

5.3.2 The projective model structure

Our next goal is to prove that $\mathbf{I}$-diagrams are a model category.

Lemma 5.3.13. *The category* $\mathbf{Top}^{\mathbf{I}}$ *has all colimits and limits. The colimit or limit of a diagram of diagrams is computed at each object* $i \in \mathbf{I}$ *separately.*

Proof. This is straightforward to check: form the diagram where we take the colimit or limit at each object $i \in \mathbf{I}$ separately, and verify it has the universal property of the colimit or the limit in the category of $\mathbf{I}$-shaped diagrams. □

Definition 5.3.14. A map of diagrams $X \to Y$ is a **level equivalence** (or **pointwise equivalence**) if $X(i) \to Y(i)$ is a weak equivalence for every $i \in \operatorname{ob}\mathbf{I}$. Similarly $X \to Y$ is a **level fibration** if $X(i) \to Y(i)$ is a Serre fibration for every i.

Example 5.3.15. For spaces with G-action, a level equivalence is an equivariant map $X \xrightarrow{\sim} Y$, that is a weak equivalence if we forget the G-action. These are often called **Borel equivalences**, or coarse equivalences, and the homotopy category $\operatorname{Ho}\mathbf{Top}^G$ formed using these equivalences is the **Borel homotopy category**. We have this special notation because there is also a more sophisticated kind of weak equivalence, that keeps track of the subspaces of H-fixed points X^H for $H \leq G$. See Theorem A.1.2.

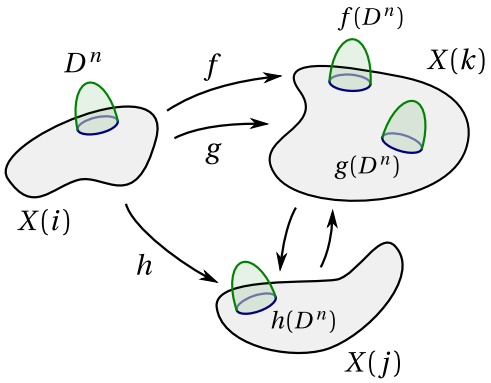
Let $I = \{S^{n-1} \to D^n\}_{n\geq 0}$ be the class of cells from Example 5.1.2. Let FI be the set of all maps obtained by applying the free diagram functors F_i of Definition 5.3.10 to the maps in I:

$$FI = \{\, F_i S^{n-1} \to F_i D^n : n \geq 0,\ i \in \operatorname{ob}\mathbf{I} \,\}.$$

A **relative cell complex of diagrams** is an FI-cell complex. Note that if $X \to Y$ is a cell complex of diagrams then $X(j) \to Y(j)$ is not necessarily a relative cell complex in the traditional sense. It is built out of the "cells" of the form $\mathbf{I}(i,j) \times (S^{n-1} \to D^n)$.

A **free cofibration** of diagrams is any retract of an FI-cell complex.

Intuitively, an FI-cell complex is obtained by starting with a diagram X, and attaching cells to the spaces $X(i)$. But, every time you attach a cell to one space $X(i)$, you also have to attach a cell to $X(j)$ for *every* map $f : i \to j$. That way, your new cell in $X(i)$ has somewhere to go under the map f, so that you still have a diagram.



As in Example 5.1.10, a square of diagrams

$$\begin{array}{ccc} F_i S^{n-1} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ F_i D^n & \longrightarrow & Y \end{array}$$

is equivalent to a square of spaces

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & X(i) \\ \downarrow & \nearrow & \downarrow{\scriptstyle p_i} \\ D^n & \longrightarrow & Y(i), \end{array}$$

using the adjunction between F_i and ev_i. Therefore, a map of diagrams $X \to Y$ is an FI-fibration iff it is a weak equivalence and Serre fibration at each level. In other words, if it is a level equivalence and a level fibration.

We define FJ similarly:

$$FJ = \{\, F_i(D^n \times \{0\}) \to F_i(D^n \times I) : n \geq 0,\ i \in \mathrm{ob}\,\mathbf{I} \,\}.$$

A map of diagrams $X \to Y$ is an FJ-fibration iff it is a level fibration.

Theorem 5.3.16. *The category* $\mathbf{Top}^{\mathbf{I}}$ *has a* ***projective model structure*** *in which*

- *the cofibrations are the free cofibrations (retracts of FI-cell complexes),*
- *the weak equivalences are the level equivalences, and*
- *the fibrations are the level fibrations.*

Proof. We check that the level equivalences, and the sets of maps FI and FJ defined just above, satisfy the conditions of Proposition 5.2.6.

1. W **is closed under 2-out-of-3 and retracts.** Can be checked at each $i \in \mathrm{ob}\,\mathbf{I}$ separately.
2. FI **satisfies the countable smallness condition.** By the free-forgetful adjunction, a map $F_i S^{n-1} \to X$ is the same as a map $S^{n-1} \to X(i)$. It is easy to see that $\mathbf{I}(i,j) \times (S^{n-1} \to D^n)$ is a closed inclusion. It follows that the skeleta of any FI-cell complex of diagrams are related by closed inclusions. So, by Exercise 1.7.31, the map from S^{n-1} factors through some finite stage.
3. FJ **satisfies the countable smallness condition.** Same as the previous point.
4. FJ**-cell complexes are in** $W \cap (FI$**-cof).** The proof is essentially the same as when we proved Theorem 5.2.5.

5 & 6. FI**-fib** $= W \cap (FJ$**-fib).** As discussed above, by the free-forgetful adjunction, a map of diagrams is an FI-fibration iff it is a level equivalence and a level fibration. This is true iff it is in $W \cap (FJ\text{-fib})$.

□

The weak equivalences and fibrations in this model category are pretty boring, but the cofibrations are interesting.

Example 5.3.17. We say a diagram X is **cellular** if the map from the empty diagram $\varnothing \to X$ is a cell complex of diagrams, and **cofibrant** if $\varnothing \to X$ is a free cofibration, in other words, a retract of a cell complex of diagrams.

- If $\mathbf{I} = \{\bullet,\ \bullet\}$, a diagram (X_0, X_1) is cellular iff both X_0 and X_1 are cell complexes.
- If $\mathbf{I} = \{\bullet \to \bullet\}$, a diagram $X_0 \to X_1$ is cellular iff X_0 is a cell complex and $X_0 \to X_1$ is a relative cell complex.
- For $\mathbf{I} = G$, a cell complex of diagrams is called a **free G-cell complex**. It is a complex built out of cells of the form $D^n \times G$, attached along the boundary $S^{n-1} \times G$.

Example 5.3.18. Applying Theorem 5.3.16 to the category $\mathbf{Top}^G$ of spaces with left G-action, we get a projective model structure or **Borel model structure** on $\mathbf{Top}^G$ in which

- the cofibrations are the retracts of free G-cell complexes,
- the weak equivalences are the Borel equivalences (weak equivalences after forgetting the G-action), and
- the fibrations are the maps that are Serre fibrations after forgetting the G-action.

5.3.3 Homotopy colimits and homotopy orbits

We give a short example to illustrate the power of this theory, by proving that the colimit has a left-derived functor. Consider the functor $\mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}$ that takes every diagram X to its colimit.

Proposition 5.3.19. *This colimit functor preserves all equivalences between cofibrant diagrams.*

Proof. We first show that each map in FJ goes to a weak equivalence. The colimit of a free diagram is the same space back:

$$\operatorname*{colim}_{\mathbf{I}} (F_i A) \cong A,$$

so each map in FJ goes to the map of spaces $D^n \times \{0\} \to D^n \times I$, which is clearly a weak equivalence.

Similarly, a coproduct of maps in FJ goes to a coproduct of maps of the form $D^n \times \{0\} \to D^n \times I$, which is a weak equivalence. A pushout of maps in FJ goes to a pushout in spaces,

because colim is a left adjoint. A pushout of maps of the form $D^n \times \{0\} \to D^n \times I$ is a collection of elementary expansions, which we have already noted is a weak equivalence. Finally, a sequential composition of such maps is a weak equivalence, because it is a colimit along a sequence of closed inclusions, using Exercise 1.7.31.

Therefore every FJ-cell complex goes to a weak equivalence. Every retract of an FJ-cell complex goes to a retract of a weak equivalence, which is therefore a weak equivalence. In conclusion, colim sends every acyclic cofibration (FJ-cofibration, or retract of an FJ-cell complex) to a weak equivalence. By Ken Brown's Lemma (Lemma 5.2.25), we conclude that all weak equivalences of cofibrant diagrams go to weak equivalences. □

Remark 5.3.20. There is a much faster proof of Proposition 5.3.19 using Quillen adjunctions – see Exercise 5.8.19. We give this version of the proof here as an illustration, before we apply the technique to diagram spectra in the next chapter.

Recall from Definition 3.4.29 that the **homotopy colimit** is the left-derived functor of colim: $\mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}$, if it exists.

Proposition 5.3.21. *Homotopy colimits in* **Top** *exist, for diagrams of any shape* **I**. *In other words, the colimit functor* $\mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}$ *always has a left-derived functor, which we call the homotopy colimit.*

Proof. By Theorem 5.3.16, we can replace each diagram X by an equivalent cofibrant diagram QX, and by Proposition 5.3.19, the colimit functor preserves weak equivalences between cofibrant diagrams. Therefore the colimit functor can be left-derived as in Definition 3.4.15. □

Remark 5.3.22. We have stated this for **I** an ordinary category (so $\mathbf{I}(i,j)$ has the discrete topology), but we have actually proven it when **I** is a topological category. When **I** has a topology, we sometimes call the left-derived functor of colim the **enriched homotopy colimit**, though we still write it as hocolim.

Remark 5.3.23. We will later give a second proof that the homotopy colimit exists using the bar construction (Definition 7.4.23 and Theorem 7.4.22). By Corollary 3.5.3, the construction we give here is equivalent to any other construction of the left-derived colimit, so it does not matter which one you use.

Example 5.3.24. Suppose $\mathbf{I} = \{\bullet \quad \bullet\}$, or more generally **I** is any category that only has identity morphisms. Then the homotopy colimit along **I** is obtained by replacing each of the spaces $X(i)$ by a cell complex $QX(i)$, then taking their coproduct.

If **I** is the span category $\{\bullet \leftarrow \bullet \rightarrow \bullet\}$, the homotopy colimit is obtained by making all the spaces in the span into cell complexes, and making the maps into inclusions of subcomplexes, then taking the pushout.

If $\mathbf{I}$ is a sequential colimit category

$$\mathbf{I} = \{ \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \},$$ [4]

the homotopy colimit is obtained by making each space X_n into a cell complex, and the maps $X_n \to X_{n+1}$ into subcomplexes, then taking the colimit.

In each of these three cases, we can see that the homotopy colimit as defined in Proposition 5.3.21 is equivalent (though not isomorphic) to the homotopy colimit as defined in Section 1.5. Of course, they're both examples of left-derived functors of the colimit, so by Corollary 3.5.3 they *must* be equivalent to each other.

Recall that for a space with G-action X, the orbit space X_G is obtained by identifying every point x to all of its images under the G-action, $x \sim xg$. This is the same thing as taking the colimit of X as a diagram over the category G. Recall also that a Borel equivalence is a map that is both G-equivariant, and a weak equivalence after forgetting the G-action.

Corollary 5.3.25. *Any Borel equivalence of free G-cell complexes $X \xrightarrow{\sim} Y$ induces a weak equivalence of orbit spaces $X_G \xrightarrow{\sim} Y_G$.*

This is not true if we drop the "free" condition. For instance, the infinite dimensional sphere S^∞ has a $\mathbb{Z}/2$-action by the antipodal map. The one-point space $*$ has a trivial action by $\mathbb{Z}/2$. The map $S^\infty \to *$ is a Borel equivalence, but on orbits, it induces the map $\mathbb{RP}^\infty \to *$, which is decidedly not a weak equivalence.

Definition 5.3.26. The **homotopy orbit space** X_{hG} is the is the left-derived functor of the orbits, in other words, the homotopy colimit of X as a diagram over G. Concretely, we replace X by an equivalent free G-cell complex, then take its G-orbits. In Exercise 5.8.22 we also show this is given by the **Borel construction**

$$X_{hG} = EG \times_G X$$

where EG is any free contractible G-CW complex. The homotopy orbits X_{hG} is a left-derived functor, so it preserves all equivalences. Therefore every Borel equivalence $X \xrightarrow{\sim} Y$ induces an equivalence of homotopy orbits $X_{hG} \to Y_{hG}$.

Example 5.3.27. The homotopy orbits of both S^∞ and $*$ under the $\mathbb{Z}/2$-action are homotopy equivalent to $\mathbb{RP}^\infty$.

The homotopy colimit generalizes to the **homotopy coend**, see Exercise 5.8.23.

[4] Here as usual we are only drawing the generating morphisms of $\mathbf{I}$. See Exercise 1.7.10 for details.

Remark 5.3.28. The constructions of the homotopy colimit and the homotopy orbits in this section are not very explicit – you use the small-object argument to replace the diagram X by a huge diagram QX with an absurd number of cells, then take its colimit. In Definition 7.4.23 we give a weakly equivalent, and much more explicit, model for the homotopy colimit.

Remark 5.3.29. One might hope that we can define homotopy limits in an analogous way, by showing that the limit functor $\mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}$ preserves equivalences of fibrant diagrams. Unfortunately, it does not! We would have to change the model structure to one in which the *cofibrations* are the level cofibrations, and the fibrations are defined by a right lifting property. This would be called the **injective model structure** on $\mathbf{Top}^{\mathbf{I}}$. Unfortunately, it does not exist!

There are a few standard ways around this: we could pass to the setting of simplicial sets where the injective model structure does exist, or we could argue abstractly that a right-derived functor of the limit exists, as in [DHKS04]. We'll follow the approach of [Shu06] and use bar constructions to derive the limit instead. See Definition 7.5.27 and Corollary 7.5.28.

5.3.4 Based diagrams

Now we consider the category $\mathbf{Top}_*^{\mathbf{I}}$ of diagrams of based spaces. This requires a few changes to the definitions, but the proofs do not change.

Definition 5.3.30. A **based topological category** or **category enriched in based spaces** is

- a collection of objects $\mathrm{ob}\,\mathbf{C}$,
- for each pair of objects a, b, a based space of morphisms $\mathbf{C}(a,b)$,
- continuous composition maps $\mathbf{C}(a,b) \wedge \mathbf{C}(b,c) \to \mathbf{C}(a,c)$, and
- for each object a a unit map $\{\mathrm{id}_a\} \to \mathbf{C}(a,a)$,

with the same conditions as in Definition 5.3.1. For a functor of based categories, we require that the maps $\mathbf{C}(a,b) \to \mathbf{D}(F(a),F(b))$ preserve the basepoint.

Remark 5.3.31. A based topological category the same thing as a topological category in which there is a "zero map" $*\colon a \to b$ for every pair of objects a and b, such that the composition of any morphism with zero is always equal to zero.

Example 5.3.32. The categories $\mathbf{Top}_*$ and $\mathbf{Sp}$ are based topological categories.

Definition 5.3.33. Suppose $\mathbf{I}$ is a based category. A **based diagram** is a continuous functor $X : \mathbf{I} \to \mathbf{Top}_*$. Equivalently, it consists of

- a based space $X(i)$ for each $i \in \operatorname{ob}\mathbf{I}$, and
- a continuous composition map $X(i) \wedge \mathbf{I}(i,j) \to X(j)$,

with the same conditions as in Definition 5.3.5. By abuse of notation, we let $\mathbf{Top}_*^{\mathbf{I}}$ refer to the category of based diagrams.

Definition 5.3.34. The **sphere category $\mathbf{S}$** is a based topological category, with one object for each nonnegative integer $n \geq 0$. The morphism spaces are spheres

$$\mathbf{S}(m,n) = \begin{cases} S^{n-m} & \text{when } n \geq m, \\ * & \text{when } n < m. \end{cases}$$

The nontrivial compositions are all homeomorphisms

$$S^{n-m} \wedge S^{p-n} \cong S^{p-m}.$$

To be specific, we use the canonical homeomorphism of Definition 2.1.5. This is the one-point compacitification of the the linear isomorphism $\mathbb{R}^{n-m} \times \mathbb{R}^{p-n} \cong \mathbb{R}^{p-m}$ that concatenates the coordinates.

Lemma 5.3.35. *There is an isomorphism of categories* $\mathbf{Top}_*^{\mathbf{S}} \cong \mathbf{Sp}$.

In other words, spectra are a special case of diagrams!

Proof. Given a spectrum X, form a diagram over $\mathbf{S}$ using the spaces X_n. The maps $X_m \wedge S^{n-m} \to X_n$ are the compositions of the bonding maps. It is easy to check this forms a diagram. Going the other way, for any $\mathbf{S}$-diagram, the action of $\mathbf{S}(n, n+1) \cong S^1$ gives maps $X_n \wedge S^1 \to X_{n+1}$, making the spaces X_n into a spectrum.

These operations are inverses, and along this identification, a map of diagrams corresponds to a map of spectra. We therefore get an isomorphism of categories. □

Definition 5.3.36. For each object $i \in \operatorname{ob}\mathbf{I}$ and based space A, the **free based diagram** $F_i A$ is the based $\mathbf{I}$-diagram whose value at j is the smash product

$$(F_i A)(j) = A \wedge \mathbf{I}(i,j),$$

with $\mathbf{I}$ acting by composition on the $\mathbf{I}(i,j)$ term.

Lemma 5.3.37. *This is the left adjoint of the forgetful functor* $\mathrm{ev}_i : \mathbf{Top}_*^{\mathbf{I}} \to \mathbf{Top}_*$ *sending the based diagram X to the based space $X(i)$.*

The proof is the same as that of Lemma 5.3.11.

Example 5.3.38. For the sphere category **S**, a free diagram $F_d A$ is precisely the same thing as a shift desuspension spectrum from Example 2.1.8. At level n, if $n \geq d$, we get the suspension $A \wedge S^{n-d} = \Sigma^{n-d} A$. When $n < d$, we get the zero space $*$, because $A \wedge * \cong *$.

Definition 5.3.39. As in Definition 5.3.14, a map of based diagrams is **level equivalence** if $X(i) \to Y(i)$ is a weak equivalence for every $i \in \mathrm{ob}\,\mathbf{I}$. Similarly $X \to Y$ is a **level fibration** if $X(i) \to Y(i)$ is a Serre fibration for every i.

As before, we define the sets

$$FI_+ = \{\ F_i S^{n-1}_+ \to F_i D^n_+ : n \geq 0,\ i \in \mathrm{ob}\,\mathbf{I}\ \},$$
$$FJ_+ = \{\ F_i(D^n \times \{0\})_+ \to F_i(D^n \times I)_+ : n \geq 0,\ i \in \mathrm{ob}\,\mathbf{I}\ \}.$$

A free cofibration in $\mathbf{Top}^{\mathbf{I}}_*$ is a retract of an FI_+-cell complex.

Theorem 5.3.40. *The category of based diagrams* $\mathbf{Top}^{\mathbf{I}}_*$ *has a* ***projective model structure*** *in which*

- *the cofibrations are the free cofibrations,*
- *the weak equivalences are the level equivalences, and*
- *the fibrations are the level fibrations.*

Proof. The proof is identical to that of Theorem 5.3.16, using the sets FI_+ and FJ_+. □

Example 5.3.41. For the sphere category **S**, this gives the level model structure on spectra from Theorem 5.2.12.

5.3.5 Based diagrams on unbased categories

A large class of interesting based diagrams comes from the following construction.

Example 5.3.42. For any unbased category **I**, we form a based category $\mathbf{I}_+$ by adding a disjoint basepoint to each of the mapping spaces. The composition becomes

$$\mathbf{I}(a,b)_+ \wedge \mathbf{I}(b,c)_+ \cong (\mathbf{I}(a,b) \times \mathbf{I}(b,c))_+ \longrightarrow \mathbf{I}(a,c)_+.$$

Lemma 5.3.43. *A based diagram* $\mathbf{I}_+ \to \mathbf{Top}_*$ *is the same thing as a functor* $\mathbf{I} \to \mathbf{Top}_*$. *It is also the same thing as an unbased diagram* $\mathbf{I} \to \mathbf{Top}$ *that contains the constant diagram* $*$ *as a retract.*

Proof. These all describe the same thing: a based space $X(i)$ for each $i \in \mathrm{ob}\,\mathbf{I}$, and a based map $X(i) \to X(j)$ for each $i \to j$ in $\mathbf{I}$. □

Example 5.3.44. Recall that a topological category with one object is given by a topological monoid G. A **based G-space** is a based space X with basepoint-preserving action by G. By Lemma 5.3.43, this could also be described as a based space with action maps of the form

$$G_+ \wedge X \longrightarrow X,$$

or as a G-space that contains the trivial G-space $*$ as a retract.

We write the category of based diagrams on $\mathbf{I}_+$ as $\mathbf{Top}^{\mathbf{I}}_*$, because by Lemma 5.3.43, this is the same as a functor from $\mathbf{I}$ to based spaces. Consider the forgetful map $\mathbf{Top}^{\mathbf{I}}_* \to \mathbf{Top}^{\mathbf{I}}$ that forgets the basepoints.

Lemma 5.3.45. *A map in $\mathbf{Top}^{\mathbf{I}}_*$ is a cofibration, weak equivalence, or fibration in the model structure of Theorem 5.3.40 iff its image in $\mathbf{Top}^{\mathbf{I}}$ is respectively a cofibration, weak equivalence, or fibration in the model structure of Theorem 5.3.16.*

Proof. For weak equivalences and fibrations, this is immediate. For cofibrations, we have to observe that attaching a cell of the form

$$S^{n-1}_+ \wedge \mathbf{I}(i,-)_+ \longrightarrow D^n_+ \wedge \mathbf{I}(i,-)_+$$

gives a homeomorphic result if we instead write the cell in the form

$$S^{n-1} \times \mathbf{I}(i,-) \longrightarrow D^n \times \mathbf{I}(i,-).$$

□

Remark 5.3.46. One might expect that you could take the colimit or homotopy colimit of a based diagram X on a based category $\mathbf{I}$. However, this colimit is always the one-point space $*$, because the zero morphisms in $\mathbf{I}$ identify everything to zero. As a result, you can't meaningfully take a "colimit of a spectrum along itself."

You *can* use Lemma 5.3.43, and consider a diagram of based spaces indexed by an unbased category $\mathbf{I}$. Then you take the colimit along $\mathbf{I}$, *not* the larger category $\mathbf{I}_+$. This works, and the corresponding **based homotopy colimit** turns out to be the quotient of unbased homotopy colimits

$$\operatorname*{hocolim}^{(b)}_{i\in\mathbf{I}} X(i) \simeq \left(\operatorname*{hocolim}^{(u)}_{i\in\mathbf{I}} X(i)\right)\Big/\left(\operatorname*{hocolim}^{(u)}_{i\in\mathbf{I}} (*)\right).$$

See Exercise 5.8.40. It turns out that it is also possible to define more general kinds of colimits (weighted colimits and left Kan extensions) for based diagrams, see Section 5.4.3.

5.4 Quillen adjunctions and equivalences

Now that we have many, many examples of model categories, we can dig more into the relationships between different model categories. The two basic kinds of relationships are Quillen adjunctions (which include Quillen equivalences as a special case), and Quillen tensors.

5.4.1 Quillen adjunctions

Suppose **C** and **D** are model categories. In particular, they are categories with weak equivalences, so we can talk about derived functors.

Recall from Definition 3.4.15 that a functor $F : \mathbf{C} \to \mathbf{D}$ is left-deformable if we can define a functor QX, equivalent to X by a map $QX \xrightarrow{\sim} X$, and landing in a full subcategory $\mathbf{A} \subseteq \mathbf{C}$ on which F preserves all weak equivalences. The left-derived functor is then $\mathbb{L}F = F \circ Q$.

If **C** is a model category, there is an obvious choice to make for **A**: the subcategory of cofibrant objects $\mathbf{C}^c \subseteq \mathbf{C}$. By Ken Brown's Lemma (Lemma 5.2.25), to check that F preserves equivalences on **A**, it is enough to show that F of any acyclic cofibration is a weak equivalence. For instance, we used this technique to show that the colimit has a left-derived functor in Proposition 5.3.19.

Definition 5.4.1. A functor $F : \mathbf{C} \to \mathbf{D}$ is **left Quillen** if it is a left adjoint, and preserves both the cofibrations and the acyclic cofibrations:

$$F(C) \subseteq C, \qquad F(W \cap C) \subseteq W \cap C.$$

Similarly, $G : \mathbf{D} \to \mathbf{C}$ is **right Quillen** if it is a right adjoint, and preserves both the fibrations and the acyclic fibrations.

A **Quillen adjunction**, or Quillen pair, is an adjoint pair $(F \dashv G)$ in which F is left Quillen and G is right Quillen.

From Definition 3.4.15, it follows that any such F has a **left-derived functor** $\mathbb{L}F(X) = F(QX)$, any such G has a **right-derived functor** $\mathbb{R}G(Y) = G(RY)$. By Lemma 3.4.18, these derived functors preserve weak equivalences, so they define maps on the homotopy category

$$\mathbb{L}F : \operatorname{Ho}\mathbf{C} \to \operatorname{Ho}\mathbf{D}, \qquad \mathbb{R}G : \operatorname{Ho}\mathbf{D} \to \operatorname{Ho}\mathbf{C}.$$

By Proposition 3.5.20, these derived functors are again adjoint to each other, $(\mathbb{L}F \dashv \mathbb{R}G)$.

To give examples, it will be helpful if we first make the definition a little easier to check.

Lemma 5.4.2. *If $(F \dashv G)$ is an adjoint pair of functors on the model categories* **C** *and* **D***, then F is left Quillen iff G is right Quillen.*

We leave the proof to Exercise 5.8.17. As a result, we only have to check that F is left Quillen, or that G is right Quillen.

This gets even easier if $\mathbf{C}$ is cofibrantly generated:

Lemma 5.4.3. *If $\mathbf{C}$ is cofibrantly generated, $F : \mathbf{C} \to \mathbf{D}$ is a left adjoint, and*

$$F(I) \subseteq C, \qquad F(J) \subseteq W \cap C,$$

then F is left Quillen (and therefore its right adjoint G is right Quillen).

Proof. Since F is a left adjoint, it preserves coproducts, pushouts, sequential compositions, and retracts – exactly the moves we use to build an I-cofibration out of the maps in I. Therefore, if $F(I) \subseteq C$, it follows that $F(C) \subseteq C$. The same argument applies to J and $W \cap C$. □

Example 5.4.4. Using Lemma 5.4.2 or Lemma 5.4.3, or both, we can see that the following pairs are Quillen adjunctions:

Left adjoint	**Right adjoint**
Disjoint basepoint $(-)_+ : \mathbf{Top} \to \mathbf{Top}_*$	Forget basepoint $U : \mathbf{Top}_* \to \mathbf{Top}$
$A \times (-) : \mathbf{Top} \to \mathbf{Top}$ (if A is a cell complex or cofibrant)	$\mathrm{Map}(A, -) : \mathbf{Top} \to \mathbf{Top}$
$P \otimes (-) : \mathbf{Ch}_{\geq 0}(\mathbb{Z}) \to \mathbf{Ch}_{\geq 0}(\mathbb{Z})$ (if P is projective)	$\mathrm{Hom}(P, -) : \mathbf{Ch}_{\geq 0}(\mathbb{Z}) \to \mathbf{Ch}_{\geq 0}(\mathbb{Z})$
$A \wedge (-) : \mathbf{Top}_* \to \mathbf{Top}_*$ (if A is cofibrant)	$\mathrm{Map}_*(A, -) : \mathbf{Top}_* \to \mathbf{Top}_*$
Tensor $A \wedge (-) : \mathbf{Sp} \to \mathbf{Sp}$ (if A is cofibrant)	Cotensor $F(A, -) : \mathbf{Sp} \to \mathbf{Sp}$
Free diagram $F_i : \mathbf{Top} \to \mathbf{Top}^{\mathbf{I}}$	Evalulation $\mathrm{ev}_i : \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}$
Unbased colimit, $\mathrm{colim}^{(u)} : \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}$	Constant diagram $\Delta : \mathbf{Top} \to \mathbf{Top}^{\mathbf{I}}$
Based colimit, $\mathrm{colim}^{(b)} : \mathbf{Top}_*^{\mathbf{I}} \to \mathbf{Top}_*$	Constant diagram $\Delta : \mathbf{Top}_* \to \mathbf{Top}_*^{\mathbf{I}}$
Free diagram $F_i : \mathbf{Top}_* \to \mathbf{Top}_*^{\mathbf{I}}$	Evalulation $\mathrm{ev}_i : \mathbf{Top}_*^{\mathbf{I}} \to \mathbf{Top}_*$
$F_i(-)_+ : \mathbf{Top} \to \mathbf{Top}_*^{\mathbf{I}}$	$U \circ \mathrm{ev}_i : \mathbf{Top}_*^{\mathbf{I}} \to \mathbf{Top}$

As a special case we also get the following:

Left adjoint	Right adjoint
Suspension $\Sigma\colon \mathbf{Top}_* \to \mathbf{Top}_*$	Loopspace $\Omega\colon \mathbf{Top}_* \to \mathbf{Top}_*$
Suspension $\Sigma\colon \mathbf{Sp} \to \mathbf{Sp}$	Loopspace $\Omega\colon \mathbf{Sp} \to \mathbf{Sp}$
Suspension spectrum $\Sigma^\infty\colon \mathbf{Top}_* \to \mathbf{Sp}$	0th space $\mathrm{ev}_0\colon \mathbf{Sp} \to \mathbf{Top}_*$
Suspension spectrum $\Sigma^\infty_+\colon \mathbf{Top} \to \mathbf{Sp}$	0th space $U \circ \mathrm{ev}_0\colon \mathbf{Sp} \to \mathbf{Top}$
Free spectrum $F_k\colon \mathbf{Top}_* \to \mathbf{Sp}$	kth space $\mathrm{ev}_k\colon \mathbf{Sp} \to \mathbf{Top}_*$

Remark 5.4.5. We have already seen that many of these functors can be derived. By Corollary 3.5.3, derived functors are unique up to equivalence, so it doesn't matter whether we use a model structure, or some other technique, to form the derived functor. The result is the same either way.

Remark 5.4.6. If F is left Quillen, the left-derived functor $\mathbb{L}F$ is sampling the behavior of F on the cofibrant objects, and extending that behavior to the rest of $\mathbf{C}$. Of course, $\mathbb{L}F$ is equivalent to F on every cofibrant object. Following Definition 3.5.10, we say that F is correct, or has the correct homotopy type, when it is evaluated on a cofibrant object.

Quillen pairs can be composed. If we have three model categories and two left Quillen functors

$$\mathbf{C}_1 \xrightarrow{F_1} \mathbf{C}_2 \xrightarrow{F_2} \mathbf{C}_3,$$

then the composite $F_2 \circ F_1$ is left Quillen. If the right adjoints are

$$\mathbf{C}_1 \xleftarrow{G_1} \mathbf{C}_2 \xleftarrow{G_2} \mathbf{C}_3,$$

then the composite $G_1 \circ G_2$ is right Quillen, and $(F_2 \circ F_1 \dashv G_1 \circ G_2)$ form a Quillen pair. By Lemma 3.5.13, we have equivalences of derived functors

$$\mathbb{L}(F_2 \circ F_1) \simeq (\mathbb{L}F_2) \circ (\mathbb{L}F_1), \qquad \mathbb{R}(G_1 \circ G_2) \simeq (\mathbb{R}G_1) \circ (\mathbb{R}G_2).$$

Example 5.4.7. Composing the disjoint basepoint and free diagram functors from Example 5.4.4 gives the free based diagram on an unbased space:

$$\mathbf{Top} \xrightarrow{(-)_+} \mathbf{Top}_* \xrightarrow{F_i(-)} \mathbf{Top}_*^{\mathbf{I}}$$

The right adjoints are all forgetful functors, which compose to give a forgetful functor:

$$\mathbf{Top} \xleftarrow{U} \mathbf{Top}_* \xleftarrow{\mathrm{ev}_i} \mathbf{Top}_*^{\mathbf{I}}$$

The composite Quillen pair is $(F_i(-)_+, U \circ \mathrm{ev}_i)$.

Example 5.4.8. Let $\alpha\colon \mathbf{I} \to \mathbf{J}$ be a functor. Then composing with α gives a functor $\mathbf{Top}^{\mathbf{J}} \to \mathbf{Top}^{\mathbf{I}}$ that we call the pullback α^*. We will see shortly in Definition 5.4.14 that this has a left adjoint $\alpha_!\colon \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}^{\mathbf{J}}$ called the "left Kan extension." Without even knowing what this is, we can prove that $\alpha_!$ must send each free diagram $F_i A\colon \mathbf{I} \to \mathbf{Top}$ to the free diagram $F_{\alpha(i)}A$, and that these extensions must compose by the rule $(\beta \circ \alpha)_! \cong \beta_! \circ \alpha_!$. See Exercises 5.8.14 and 5.8.15. By the above discussion, the left-derived functor of left Kan extension also composes in the same way.

5.4.2 Quillen equivalences

Suppose $(F \dashv G)$ is a Quillen adjunction. By Proposition 3.5.20, the functors $(\mathbb{L}F \dashv \mathbb{R}G)$ form an adjoint pair on the homotopy categories $\mathrm{Ho}\,\mathbf{C}$ and $\mathrm{Ho}\,\mathbf{D}$. However, this adjunction is much easier to describe using the fundamental theorem of model categories, Theorem 5.2.27.

It suffices to restrict attention to cofibrant $X \in \mathbf{C}$, so that $\mathbb{L}F(X) \simeq F(X)$, and fibrant $Y \in \mathbf{D}$, so that $\mathbb{R}G(Y) \simeq G(Y)$. Since F preserves cofibrations and the initial object, $F(X) \in \mathbf{D}$ is cofibrant. Since G preserves fibrations and the terminal object, $G(Y) \in \mathbf{C}$ is fibrant. By Theorem 5.2.27 we therefore get

$$\begin{aligned}\mathrm{Ho}\,\mathbf{C}(X, G(Y)) &\cong \mathbf{C}(X, G(Y))/(\text{homotopy}) \\ &\cong \mathbf{D}(F(X), Y)/(\text{homotopy}) \\ &\cong \mathrm{Ho}\,\mathbf{D}(F(X), Y),\end{aligned}$$

giving the adjunction between $\mathbb{L}F$ and $\mathbb{R}G$.[5]

Definition 5.4.9. A **Quillen equivalence** is a Quillen adjunction $(F \dashv G)$ such that $(\mathbb{L}F \dashv \mathbb{R}G)$ is an *equivalence* of homotopy categories $\mathrm{Ho}\,\mathbf{C} \simeq \mathrm{Ho}\,\mathbf{D}$.

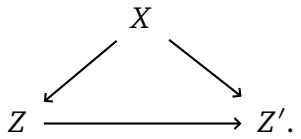
Recall that an adjunction is an equivalence if its counit and unit maps are isomorphisms. Equivalently, if isomorphisms $F(X) \to Y$ correspond to isomorphisms $X \to G(Y)$. Therefore,

Lemma 5.4.10. *A Quillen adjunction $(F \dashv G)$ is a Quillen equivalence if, for all cofibrant X in $\mathbf{C}$ and fibrant Y in $\mathbf{D}$, a map $F(X) \to Y$ is a weak equivalence in $\mathbf{D}$ iff its adjunct $X \to G(Y)$ is a weak equivalence in $\mathbf{C}$.*

Example 5.4.11. Σ and Ω form a Quillen equivalence between $\mathbf{Sp}$ and itself. The other entries in the table in Example 5.4.4 are not Quillen equivalences.

[5] We are skipping some details here about why the adjunction sends homotopies of maps $F(X) \to Y$ to homotopies of maps $X \to G(Y)$. In examples, this is usually obvious. It's also true in general, but you have to use the notions of left and right homotopy in a model category to prove it – see [Hov99, Lem 1.3.10].

Example 5.4.12. For any object X in a model category X, the **slice category** $\mathbf{C}_{X/}$, also known as the comma category $(X \downarrow \mathbf{C})$, is the category of maps $X \to Z$, with morphisms the commuting triangles

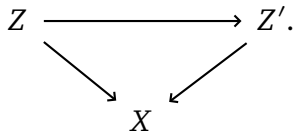


This is a model category with essentially the same cofibrations, weak equivalences, and fibrations as $\mathbf{C}$, see Exercise 5.8.8.

Any map $f : X \to Y$ induces a map of slice categories $f^* : (Y \downarrow \mathbf{C}) \to (X \downarrow \mathbf{C})$ by pre-composing with f. This operation has a left adjoint $f_! : (X \downarrow \mathbf{C}) \to (Y \downarrow \mathbf{C})$ that takes every $X \to Z$ to the pushout $Y \to Y \cup_X Z$. It is very easy to see that f^* preserves weak equivalences and fibrations, so $(f_! \dashv f^*)$ forms a Quillen pair.

Furthermore, if f is a weak equivalence and $\mathbf{C}$ is left proper (Definition 5.2.31), then $(f_! \dashv f^*)$ is a Quillen equivalence. (This is also true if f is a weak equivalence and both X and Y are cofibrant.)

Example 5.4.13. The previous example can be dualized. The slice category $\mathbf{C}_{/X}$, also known as the comma category $(\mathbf{C} \downarrow X)$, is the category of maps $Z \to X$, with morphisms the commuting triangles



This is also model category with essentially the same cofibrations, weak equivalences, and fibrations as $\mathbf{C}$, see Exercise 5.8.9.

Any map $f : X \to Y$ induces a map of slice categories $f^* : (\mathbf{C} \downarrow Y) \to (\mathbf{C} \downarrow \mathbf{X})$ by sending $Z \to Y$ to the pullback $X \times_Y Z$. This operation has a left adjoint $f_! : (\mathbf{C} \downarrow X) \to (\mathbf{C} \downarrow Y)$ that post-composes with f. Again, it is very easy to see that $f_!$ preserves weak equivalences and cofibrations, so $(f_! \dashv f^*)$ forms a Quillen pair.

Furthermore, if f is a weak equivalence and $\mathbf{C}$ is right proper (Definition 5.2.31), then $(f_! \dashv f^*)$ is a Quillen equivalence. (This is also true if f is a weak equivalence and both X and Y are fibrant.)

5.4.3 Restriction and left Kan extension

Perhaps the most important Quillen adjunction is the one between "restriction" and "left Kan extension." This can be done in a great deal of generality, but we focus here on diagrams of spaces.

Let $\mathbf{I}$ and $\mathbf{J}$ be any small topological categories, and $\alpha\colon \mathbf{I} \to \mathbf{J}$ a topological functor. First, let's consider the case that the categories are unbased.

Definition 5.4.14. The **restriction** $\alpha^*\colon \mathbf{Top}^{\mathbf{J}} \to \mathbf{Top}^{\mathbf{I}}$ composes every diagram $\mathbf{J} \to \mathbf{Top}$ with the functor F. In other words, it produces the diagram whose value at $i \in \mathbf{I}$ is the original diagram at $\alpha(i) \in \mathbf{J}$:

$$(\alpha^* X)(i) = X(\alpha(i)).$$

The **left Kan extension** $\alpha_!\colon \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}^{\mathbf{J}}$ is the left adjoint of α^*. By analogy with the theory of rings, this is also sometimes called **extension of scalars**, and written as $X \times_{\mathbf{I}} \mathbf{J}$ or even $X \otimes_{\mathbf{I}} \mathbf{J}$.

To spell it out explicitly, if the morphism spaces of $\mathbf{I}$ and $\mathbf{J}$ are discrete, then $\alpha_!$ takes the colimit of all of the terms $X(i)$ for each morphism $\alpha(i) \to j$ in $\mathbf{J}$:

$$(\alpha_! X)(j) = \operatorname*{colim}_{\alpha(i)\to j} X(i)$$

See Exercise 5.8.13 for more details. More generally, if $\mathbf{I}$ and $\mathbf{J}$ have a topology, then $\alpha_! X = X \times_{\mathbf{I}} \mathbf{J}$ is defined by taking the *spaces*

$$X(i) \times \mathbf{J}(\alpha(i), j)$$

and modding out by a relation for each morphism $\alpha\colon i_0 \to i_1$ in $\mathbf{I}$:

$$(\alpha_! X)(j) = \left(\coprod_{i\in\mathbf{I}} X(i) \times \mathbf{J}(\alpha(i), j) \right) \Big/ \big((x, p \circ \alpha(f)) \sim (\alpha(x), p) \big)$$

We could also express this as a coequalizer. In other words, it is a colimit of a diagram that just has two maps between the same pair of objects:

$$\left(\coprod_{i_0,i_1\in\mathbf{I}} X(i_0) \times \mathbf{I}(i_0, i_1) \times \mathbf{J}(\alpha(i_1), j) \right) \rightrightarrows \left(\coprod_{i\in\mathbf{I}} X(i) \times \mathbf{J}(\alpha(i), j) \right) \longrightarrow (\alpha_! X)(j). \quad (5.4.15)$$

The first map sends (x, f, p) to $(f(x), p)$, while the second takes it to $(x, p \circ \alpha(f))$. See also Exercise 5.8.23.

Lemma 5.4.16. *The functor $\alpha_!$ thus described is the left adjoint of α^*.*

Proof. A map of $\mathbf{J}$-diagrams $h\colon \alpha_! X \to Y$ consists of maps $h_{i,j}\colon X(i) \times \mathbf{J}(\alpha(i), j) \to Y(j)$ for all i and j, commuting with the coequalizer relation in i, and natural in j. Using the fact that they are natural in j, we show that they are determined by the maps $h_{i,\alpha(i)}$, restricted to points of the form $(x, \mathrm{id}_{\alpha(i)}) \in X(i) \times \mathbf{J}(\alpha(i), \alpha(i))$. Furthermore, on these spaces we get a map of diagrams $X(i) \to Y(\alpha(i))$, in other words $g\colon X \to \alpha^* Y$.

To reconstruct the map $h\colon \alpha_! X \to Y$ from $g\colon X \to \alpha^* Y$, we take for each $i \in \mathbf{I}$ and $j \in \mathbf{J}$ the composite

$$X(i) \times \mathbf{J}(\alpha(i), j) \xrightarrow{g_i \times \mathrm{id}} (\alpha^* Y)(i) \times \mathbf{J}(\alpha(i), j) = Y(\alpha(i)) \times \mathbf{J}(\alpha(i), j) \xrightarrow{\circ} Y(j).$$

These agree along the relation defining the coequalizer (5.4.15), giving for each $j \in \mathbf{J}$ a well-defined map of spaces

$$(\alpha_! X)(j) \longrightarrow Y(j).$$

This construction commutes with the action of $\mathbf{J}(j, j')$ on the right, giving a well-defined map of diagrams $h\colon \alpha_! X \to Y$, that along the identity morphisms of $\alpha(i)$ gives back our original map $g\colon X \to \alpha^* Y$.

In summary, we showed how to go from maps $\alpha_! X \to Y$ to maps $X \to \alpha^* Y$ without losing information (so the construction is injective), and then showed that every map $X \to \alpha^* Y$ came from a map $\alpha_! X \to Y$ (so the construction is surjective). We therefore get a bijection between the two notions. It is straightforward to check that this bijection for each X and Y is also natural with respect to maps of diagrams $X \to X'$ and $Y \to Y'$. □

Proposition 5.4.17. *The functors $(\alpha_! \dashv \alpha^*)$ form a Quillen adjunction.*

Proof. By Lemma 5.4.2, it suffices to check that α^* preserves level equivalences and level fibrations of diagrams. This is obviously true, so we are done! □

Of course α^* is homotopical, so its right-derived functor $\mathbb{R}\alpha^*$ is equivalent to α^* itself. The left-derived functor $\mathbb{L}\alpha_!$ is called the **homotopy left Kan extension**. It is formed by first making the diagram X cofibrant, then taking its left Kan extension.

Example 5.4.18. If $\mathbf{J} = *$ is the one-point category with only the identity map, a diagram over $\mathbf{J}$ is just a single space. There is a unique functor $\alpha\colon \mathbf{I} \to *$. The restriction α^* is the functor that takes each space to the constant diagram on that space. The left Kan extension $\alpha_!$ is the colimit. The homotopy left Kan extension $\mathbb{L}\alpha_!$ is therefore the homotopy colimit.

Example 5.4.19. If $\mathbf{I} = *$ is the one-point category, there is a functor $\alpha\colon * \to \mathbf{J}$ for every object $j \in \mathbf{J}$. The restriction α^* is the evaluation ev_j that takes the jth level of the diagram $X(j)$ and forgets everything else. The left Kan extension $\alpha_!$ is the free diagram functor $F_j(-)$ from Definition 5.3.10. This preserves equivalences, so it is also the homotopy left Kan extension.

Example 5.4.20. Every topological group G is a category with one object, and a functor $\alpha\colon H \to G$ is a group homomorphism. Suppose the homomorphism is injective. Then α^* is the functor from G-spaces to H-spaces that forgets the action of those elements of G not in the subgroup H. The left Kan extension $\alpha_!$ is the extension or balanced product

$$\alpha_! X = G \times_H X = \big(G \times X\big)\big/(g, hx) \sim (gh, x),$$

see Exercise 5.8.22. This is also sometimes denoted $G \otimes_H X$ to emphasize the analogy with extension of scalars in ring theory.

Example 5.4.21. Exercise 5.8.16 gives conditions under which $(\alpha_!, \alpha^*)$ is a Quillen equivalence. Essentially, it is enough if the categories $\mathbf{I}$ and $\mathbf{J}$ have the same mapping spaces up to equivalence.

The case of based diagrams is entirely similar. Let $\mathbf{I}$ and $\mathbf{J}$ be any small based topological categories, and $\alpha\colon \mathbf{I} \to \mathbf{J}$ a based topological functor. The restriction $\alpha^*\colon \mathbf{Top}_*^{\mathbf{J}} \to \mathbf{Top}_*^{\mathbf{I}}$ composes every based diagram $\mathbf{J} \to \mathbf{Top}_*$ with the functor F, just as before. The left Kan extension $\alpha_!\colon \mathbf{Top}_*^{\mathbf{I}} \to \mathbf{Top}_*^{\mathbf{J}}$ is the left adjoint of α^*. It is defined just as before but using smash products:

$$(\alpha_! X)(j) = \left(\bigvee_{i \in \mathbf{I}} X(i) \wedge \mathbf{J}(\alpha(i), j) \right) \Big/ \big((x, p \circ \alpha(f)) \sim (f(x), p) \big). \tag{5.4.22}$$

Example 5.4.23. Again let $\mathbf{I} = *$ be the one-point category, and $S^0 = \mathbf{I}_+$ the based category obtained by adding a disjoint basepoint. It has one object, and morphism set S^0, and the composition is the isomorphism $S^0 \wedge S^0 \cong S^0$. A based functor $\alpha\colon S^0 \to \mathbf{J}$ is the same thing as an object $j \in \mathbf{J}$. As before, the restriction α^* is the evaluation ev_j, and the left Kan extension $\alpha_!$ is the free diagram functor $F_j(-)$ from Definition 5.3.36. The homotopy left Kan extension is the same thing, except that we make the space A well-based before taking the free diagram $F_j A$.

Example 5.4.24. If instead we let $\mathbf{J}_+ = S^0$, then there isn't a unique functor $\mathbf{I} \to \mathbf{J}_+$ anymore for an arbitrary based category $\mathbf{I}$. This is consistent with the idea that you can't really take colimits of diagrams on based categories (Remark 5.3.46).

5.5 Quillen tensors

Quillen tensors are essentially Quillen adjunctions that have two inputs instead of one. So they are functors from a pair of model categories $\mathbf{C}$ and $\mathbf{D}$ to a third model category $\mathbf{E}$:

$$\otimes\colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}.$$

We develop their properties here. We will need them to finish the proof of the stable model structure for spectra.

5.5.1 Two-variable adjunctions

Definition 5.5.1. A **two-variable left adjoint** is a functor $\otimes\colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ that has a right adjoint in each slot. In other words, each of the functors $X \otimes -$ and $- \otimes Y$ has a right adjoint.

It follows from the abstract properties of adjoints (Exercise 5.8.25) that the right adjoints assemble together into functors

$$\mathrm{Hom}_{\mathbf{C}} \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{E} \to \mathbf{D}, \qquad \mathrm{Hom}_{\mathbf{D}} \colon \mathbf{D}^{\mathrm{op}} \times \mathbf{E} \to \mathbf{C},$$

and we have bijections

$$\mathbf{C}(X, \mathrm{Hom}_{\mathbf{D}}(Y,Z)) \cong \mathbf{E}(X \otimes Y, Z) \cong \mathbf{D}(Y, \mathrm{Hom}_{\mathbf{C}}(X,Z))$$

that are natural in X, Y, and Z. All together this is called a **two-variable adjunction**. (cf. Remark 3.5.25)

Example 5.5.2. The Cartesian product and unbased mapping space

$$\begin{aligned} \times &\colon \mathbf{Top} \times \mathbf{Top} \to \mathbf{Top} \\ \mathrm{Map} &\colon \mathbf{Top}^{\mathrm{op}} \times \mathbf{Top} \to \mathbf{Top} \\ \mathrm{Map} &\colon \mathbf{Top}^{\mathrm{op}} \times \mathbf{Top} \to \mathbf{Top} \end{aligned}$$

form a two-variable adjunction. So we have natural bijections

$$\mathbf{Top}(X, \mathrm{Map}(Y,Z)) \cong \mathbf{Top}(X \times Y, Z) \cong \mathbf{Top}(Y, \mathrm{Map}(X,Z))$$

for any three unbased spaces X, Y, and Z.

Example 5.5.3. The smash product and based mapping space

$$\begin{aligned} \wedge &\colon \mathbf{Top}_* \times \mathbf{Top}_* \to \mathbf{Top}_* \\ \mathrm{Map}_* &\colon \mathbf{Top}_*^{\mathrm{op}} \times \mathbf{Top}_* \to \mathbf{Top}_* \\ \mathrm{Map}_* &\colon \mathbf{Top}_*^{\mathrm{op}} \times \mathbf{Top}_* \to \mathbf{Top}_* \end{aligned}$$

form a two-variable adjunction. So we have natural bijections

$$\mathbf{Top}_*(X, \mathrm{Map}_*(Y,Z)) \cong \mathbf{Top}_*(X \wedge Y, Z) \cong \mathbf{Top}_*(Y, \mathrm{Map}_*(X,Z))$$

for any three based spaces X, Y, and Z.

Example 5.5.4. Consider the functors

$$\begin{aligned} \times &\colon \mathbf{Top} \times \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}^{\mathbf{I}} \\ \mathrm{Map} &\colon \mathbf{Top}^{\mathrm{op}} \times \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}^{\mathbf{I}} \\ \mathrm{Map}^{\mathbf{I}} &\colon (\mathbf{Top}^{\mathbf{I}})^{\mathrm{op}} \times \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}. \end{aligned}$$

The first functor takes a space X and applies $X \times -$ to every level of a diagram Y. The second functor similarly applies $\mathrm{Map}(X,-)$ to every level. The third functor takes the space

of maps of diagrams $\mathrm{Map}^{\mathbf{I}}(Y,Z)$. It is the product of mapping spaces $\mathrm{Map}(Y(i),Z(i))$, restricted to the subspace of those maps that form a natural transformation of diagrams.

These form a two-variable adjunction. So we have natural bijections

$$\mathbf{Top}(X,\mathrm{Map}^{\mathbf{I}}(Y,Z)) \cong \mathbf{Top}^{\mathbf{I}}(X\times Y,Z) \cong \mathbf{Top}^{\mathbf{I}}(Y,\mathrm{Map}(X,Z))$$

for any space X and diagrams Y and Z.

Example 5.5.5. The previous example has a variant where we use based spaces and the smash product instead. Taking $\mathbf{I}$ to be the sphere category $\mathbf{S}$ of Definition 5.3.34, we get the tensor, cotensor, and the mapping space

$$\begin{aligned}\wedge &: \mathbf{Top}_* \times \mathbf{Sp} \to \mathbf{Sp}\\ F &: \mathbf{Top}_*^{\mathrm{op}} \times \mathbf{Sp} \to \mathbf{Sp}\\ \mathrm{Map}_* &: \mathbf{Sp}^{\mathrm{op}} \times \mathbf{Sp} \to \mathbf{Top}_*\end{aligned}$$

So we have natural bijections

$$\mathbf{Top}_*(X,\mathrm{Map}_*(Y,Z)) \cong \mathbf{Sp}(X\wedge Y,Z) \cong \mathbf{Sp}(Y,F(X,Z))$$

for any space X and spectra Y and Z.

5.5.2 Pushout-products and pullback-homs

We would like to say that the product of two topological spaces

$$\times : \mathbf{Top}\times\mathbf{Top}\to\mathbf{Top}$$

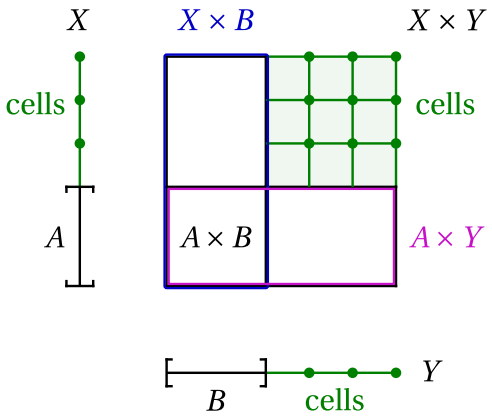
is a left Quillen functor, in some sense. The problem is that, if $A\to X$ and $B\to Y$ are both Quillen cofibrations (retracts of relative cell complexes), the product map

$$A\times B\to X\times Y$$

will not be a Quillen cofibration.

But products of spaces are nice, right? So what's going on? The problem is that we are asking for the wrong condition.

To illustrate the problem, assume further that both maps are relative cell complexes. So the complement $X \setminus A$ is composed of cells, and the complement $Y \setminus B$ is also composed of cells. The complement $(X \times Y) \setminus (A \times B)$ has all the products of these cells, but it also has products of A with all the cells in $Y \setminus B$, and also the product of B with all the cells in $X \setminus A$.



What is the correct statement? The cells in $X \times Y$ are located away from the points in which *either* the first coordinate is in A or the second coordinate is in B. So we have to consider $X \times Y$ relative to a larger subspace.

Given two maps of spaces $f : A \to X$ and $g : B \to Y$, we define the pushout-product $f \square g$ to be the inclusion

$$(A \times Y) \cup_{(A \times B)} (X \times B) \longrightarrow X \times Y.$$

More generally, suppose $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{E}$ are categories, $\mathbf{E}$ has all pushouts, and

$$\otimes : \mathbf{C} \times \mathbf{D} \to \mathbf{E}$$

is a functor that preserves colimits in each slot.

Definition 5.5.6. If $f : A \to X$ is a map in $\mathbf{C}$ and $g : B \to Y$ is a map in $\mathbf{D}$, we use $\otimes$ to form the following commuting square.

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\mathrm{id} \otimes g} & A \otimes Y \\ {\scriptstyle f \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle f \otimes \mathrm{id}} \\ X \otimes B & \xrightarrow[\mathrm{id} \otimes g]{} & X \otimes Y \end{array}$$

The **pushout-product** $f \square g$ is the map from the pushout of the first three terms to the final vertex:

$$f \square g : (X \otimes B) \cup_{A \otimes B} (A \otimes Y) \longrightarrow X \otimes Y.$$

Note that $f \square g$ is a morphism in the category $\mathbf{E}$.

In particular, if $\otimes$ is a functor $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$, the pushout-product of any two maps in $\mathbf{C}$ is another map in $\mathbf{C}$.

Example 5.5.7. Since $\otimes$ preserves colimits in each slot, we have $\varnothing \otimes X \cong \varnothing$. Therefore,

$$(\varnothing \longrightarrow X) \square (B \xrightarrow{g} Y) = (X \otimes B \xrightarrow{\mathrm{id} \otimes g} X \otimes Y)$$

$$(A \xrightarrow{f} X) \square (\varnothing \longrightarrow Y) = (A \otimes Y \xrightarrow{f \otimes \mathrm{id}} X \otimes Y)$$

In other words, pushout-product with a map of the form $(\varnothing \to X)$ is the same thing as tensoring the map with the object X.

As a further special case, we get

$$(\varnothing \longrightarrow X) \square (\varnothing \longrightarrow Y) = (\varnothing \longrightarrow X \otimes Y).$$

Example 5.5.8. In based spaces $\mathbf{Top}_*$, we have two products, $\times$ and $\wedge$. We can take pushout-product with respect to either one, to make another map of based spaces. If there is any risk of confusion, we use a subscript, $f \square_\times g$ or $f \square_\wedge g$, to denote which one we are taking.

Example 5.5.9. In **Top** with the Cartesian product $\times$, there are isomorphisms of maps

$$(S^{m-1} \to D^m) \square (S^{n-1} \to D^n) \cong (S^{m+n-1} \to D^{m+n})$$

$$(S^{m-1} \to D^m) \square (D^n \times \{0\} \to D^n \times I) \cong (D^{m+n} \times \{0\} \to D^{m+n} \times I).$$

Once we have checked this, we observe that $A \times (-)$ commutes with the pushout-product in unbased spaces, so we get

$$(A \times S^{m-1} \to A \times D^m) \square (S^{n-1} \to D^n) \cong (A \times S^{m+n-1} \to A \times D^{m+n})$$

$$(A \times S^{m-1} \to A \times D^m) \square (D^n \times \{0\} \to D^n \times I) \cong (A \times D^{m+n} \times \{0\} \to A \times D^{m+n} \times I).$$

We could prove these more elegantly by proving that $\square$ is associative (Exercise 5.8.26). Then in the above two isomorphisms, each side can be written as a three-fold pushout-product

$$(\varnothing \to A) \square (S^{m-1} \to D^m) \square (S^{n-1} \to D^n),$$
$$(\varnothing \to A) \square (S^{m-1} \to D^m) \square (D^n \times \{0\} \to D^n \times I).$$

Similarly, if we work in based spaces and use the smash product, we have isomorphisms

$$(S^{m-1}_+ \to D^m_+) \square_\wedge (S^{n-1}_+ \to D^n_+) \cong (S^{m+n-1}_+ \to D^{m+n}_+)$$

$$(S^{m-1}_+ \to D^m_+) \square_\wedge ((D^n \times \{0\})_+ \to (D^n \times I)_+) \cong ((D^{m+n} \times \{0\})_+ \to (D^{m+n} \times I)_+)$$

and similar isomorphisms that smash one of the maps with a based space A.

The pushout-product is a functor of arrow categories

$$\square \colon \mathbf{C}^{\{\bullet \to \bullet\}} \times \mathbf{D}^{\{\bullet \to \bullet\}} \longrightarrow \mathbf{E}^{\{\bullet \to \bullet\}}.$$

If $\otimes$ has a right adjoint in each slot, and $\mathbf{C}$ and $\mathbf{D}$ have all limits, then $\square$ has a right adjoint in each slot as well. The right adjoint is called the pullback-hom.

Definition 5.5.10. If $g : B \to Y$ is a map in $\mathbf{D}$ and $h\colon C \to Z$ is a map in $\mathbf{E}$, we can use $\mathrm{Hom}_{\mathbf{D}}(-,-)$ to form the following commuting square.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{D}}(Y,C) & \longrightarrow & \mathrm{Hom}_{\mathbf{D}}(B,C) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbf{D}}(Y,Z) & \longrightarrow & \mathrm{Hom}_{\mathbf{D}}(B,Z) \end{array}$$

The **pullback-hom** $\mathrm{Hom}_{\mathbf{D},\square}(g,h)$, or just $\mathrm{Hom}_{\square}(g,h)$, is the map in $\mathbf{C}$ from the first vertex of this square to the pullback of the remaining three terms:

$$\mathrm{Hom}_{\square}(g,h)\colon \mathrm{Hom}_{\mathbf{D}}(Y,C) \to \mathrm{Hom}_{\mathbf{D}}(B,C) \times_{\mathrm{Hom}_{\mathbf{D}}(B,Z)} \mathrm{Hom}_{\mathbf{D}}(Y,Z).$$

This can be visualized as the space of all choices of diagonal in the square below, mapping to the space of all choices of two horizontal maps making the square commute.

$$\begin{array}{ccc} B & \dashrightarrow & C \\ {\scriptstyle g}\downarrow & \nearrow & \downarrow{\scriptstyle h} \\ Y & \dashrightarrow & Z \end{array}$$

Example 5.5.11. Similarly to Example 5.5.7, if we plug an initial object in for B, the pullback-hom simplifies to $\mathrm{Hom}_{\mathbf{D}}(Y,-)$

$$\mathrm{Hom}_{\square}((\varnothing \longrightarrow Y),\ (C \xrightarrow{h} Z)) = (\mathrm{Hom}_{\mathbf{D}}(Y,C) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(Y,Z)),$$

and if we plug in a terminal object for Z, the pullback-hom simplifies to $\mathrm{Hom}_{\mathbf{D}}(-,C)$:

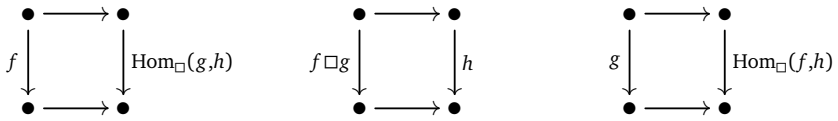
$$\mathrm{Hom}_{\square}((B \xrightarrow{g} Y),\ (C \longrightarrow *)) = (\mathrm{Hom}_{\mathbf{D}}(Y,C) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(B,C)),$$

If we do both, we get a single mapping space and its map to the terminal object:

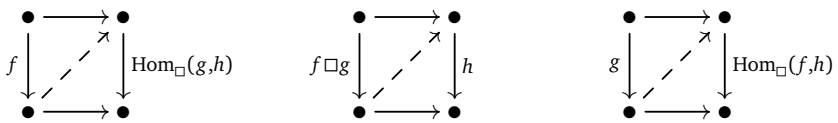
$$\mathrm{Hom}_{\square}((\varnothing \longrightarrow Y),\ (C \longrightarrow *)) = (\mathrm{Hom}_{\mathbf{D}}(Y,C) \longrightarrow *).$$

We are now ready to prove that $\mathrm{Hom}_{\square}$ is the right adjoint of $\square$ in each slot.

Proposition 5.5.12. *We have natural bijections between commuting squares of the following forms.*



Moreover, a lift in any one square gives lifts in the others:



Proof. Pick any three maps

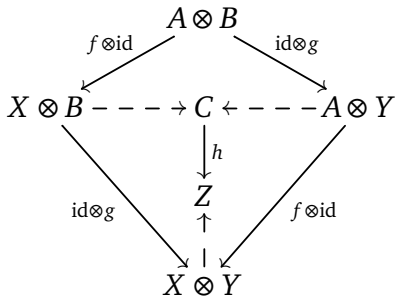
$$f : A \to X \text{ in } \mathbf{C}, \qquad g : B \to Y \text{ in } \mathbf{D}, \qquad h : C \to Z \text{ in } \mathbf{E}.$$

A choice of dotted maps making the square

$$\begin{array}{ccc} A & \dashrightarrow & \mathrm{Hom}_{\mathbf{D}}(Y, C) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{Hom}_{\square}(g,h)} \\ X & \dashrightarrow & \mathrm{Hom}_{\mathbf{D}}(B, C) \times_{\mathrm{Hom}_{\mathbf{D}}(B,Z)} \mathrm{Hom}_{\mathbf{D}}(Y, Z) \end{array}$$

commute in $\mathbf{C}$ can be expressed uniquely by the data of three maps $A \otimes Y \to C$, $X \otimes B \to C$, and $X \otimes Y \to Z$ in $\mathbf{E}$, subject to three compatibility conditions, two that say the above square commutes, and one that says that the maps to the lower-right land not just in the product but in the fiber product.

These three compatibility conditions correspond to the three regions in the diagram below, and they hold if the maps around these regions commute.



This rearranges to the statement that the square

$$\begin{array}{ccc} X \otimes B \cup_{A \otimes B} A \otimes Y & \dashrightarrow & C \\ \downarrow{\scriptstyle f \square g} & & \downarrow{\scriptstyle h} \\ X \otimes Y & \dashrightarrow & Z \end{array}$$

commutes (two of the conditions) and that the top map is in fact a map out of the pushout and not just the coproduct (the last condition).

Similarly, a choice of lift $X \to \mathrm{Hom}_{\mathbf{D}}(Y, C)$ in the first square corresponds to a map $X \otimes Y \to C$ in the last square, and in each case we need the same three conditions to guarantee that these lifts commute with the other maps in the square. □

In summary, the pushout-product $\square$ is an operation on *maps* in the categories $\mathbf{C}$ and $\mathbf{D}$. And if the tensor $\otimes : \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ has a right adjoint in each slot, then $\square$ has a right adjoint in each slot as well.

5.5.3 Definition of a Quillen tensor

If we have both a tensor product and a model structure, how should they interact? By the previous subsection, the right condition to ask for is that the *pushout-product* of any two cofibrations is a cofibration.

Suppose that **C**, **D**, and **E** are model categories, and $\otimes\colon \mathbf{C}\times\mathbf{D}\to\mathbf{E}$ a functor.

Definition 5.5.13. We say that $\otimes$ is a **Quillen tensor**, or a **left Quillen bifunctor**, if the following conditions hold.

- $\otimes$ is a left adjoint in each slot.
- For each cofibration f in **C** and cofibration g in **D**, the pushout-product $f\square g$, formed using $\otimes$, is a cofibration.
- In the previous point, if in addition one of the maps f or g is a weak equivalence, then the pushout-product $f\square g$ is also a weak equivalence.

We can summarize these conditions as follows.

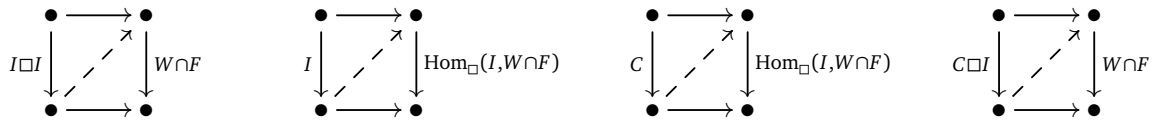
$$C\square C\subseteq C,\qquad (W\cap C)\square C\subseteq (W\cap C),\qquad C\square (W\cap C)\subseteq (W\cap C).$$

Proposition 5.5.14. *If* **C** *and* **D** *are cofibrantly generated, then it suffices to check the last two conditions of Definition 5.5.13 on the generating cofibrations I and generating acyclic cofibrations J in the categories* **C** *and* **D**. *In shorthand,*

$$I\square I\subseteq C,\qquad J\square I\subseteq (W\cap C),\qquad I\square J\subseteq (W\cap C).$$

Proof. One might think this proceeds exactly as in Lemma 5.4.3, and indeed it is possible to prove it that way, but the proof is a bit harder because you have to figure out what $-\square g$ does to pushouts and sequential compositions. It turns out to be faster to use the adjoints of $\square$ and lifting properties instead.

The argument can be given pictorially as follows.



In words, since each map in $I\square I$ is a cofibration, lifts in the first square always exist. By Proposition 5.5.12, this is equivalent to the existence of lifts in the second square. Since maps with the left-lifting property are closed under coproducts, pushouts, sequential compositions, and retracts, we can pass from I to C, giving the third square. Finally, rearranging again gives the fourth square. We conclude that $C\square I\subseteq C$.

Repeating the argument on the other side, we conclude that $C\square C\subseteq C$. The other two cases are similar. □

Example 5.5.15. The following are Quillen tensors, using Proposition 5.5.14 and Example 5.5.9.

Functor	**Notes**
$\times\colon \mathbf{Top}\times\mathbf{Top}\to\mathbf{Top}$	Cartesian product of unbased spaces.
$\wedge\colon \mathbf{Top}_*\times\mathbf{Top}_*\to\mathbf{Top}_*$	Smash product of based spaces.
$\times\colon \mathbf{Top}^{\mathbf{I}}\times\mathbf{Top}\to\mathbf{Top}^{\mathbf{I}}$	Multiply each space in the diagram $X(i)$ by a fixed space A.
$\wedge\colon \mathbf{Top}_*^{\mathbf{I}}\times\mathbf{Top}_*\to\mathbf{Top}_*^{\mathbf{I}}$	Smash each $X(i)$ with a fixed based space A.
$\wedge\colon \mathbf{Sp}\times\mathbf{Top}_*\to\mathbf{Sp}$	Smashing a spectrum with a space.[6]

There is another entry for this list called the "coend," see Exercise 5.8.23.

Example 5.5.16. The tensor product of chain complexes

$$\otimes\colon \mathbf{Ch}_{\geq 0}(\mathbb{Z})\times\mathbf{Ch}_{\geq 0}(\mathbb{Z})\longrightarrow\mathbf{Ch}_{\geq 0}(\mathbb{Z})$$

is also a Quillen tensor.

Remark 5.5.17. We can use Proposition 5.5.12 to rearrange the conditions of a Quillen tensor. For instance, if $\otimes$ has a right adjoint in each slot, then $\otimes$ is a Quillen tensor iff one of the right adjoints $\mathrm{Hom}(-,-)$ has

$$\mathrm{Hom}_{\square}(C,F)\subseteq F,\quad \mathrm{Hom}_{\square}((W\cap C),F)\subseteq(W\cap F),\quad \mathrm{Hom}_{\square}(C,(W\cap F))\subseteq(W\cap F).$$

This is often called **Quillen's SM7 axiom**. See Exercise 5.8.30.

5.5.4 Properties of Quillen tensors

What does the condition of a Quillen tensor give us? For one, it guarantees that $\otimes$ will be well-behaved on all cofibrant objects.

Lemma 5.5.18. *Suppose $\otimes$ is a Quillen tensor.*

- *If X and Y are cofibrant, then $X\otimes Y$ is cofibrant.*
- *If A is cofibrant and $B\to Y$ is an (acyclic) cofibration, then the product $A\otimes B\to A\otimes Y$ is an (acyclic) cofibration.*
- *If B is cofibrant and $A\to X$ is an (acyclic) cofibration, then the product $A\otimes B\to X\otimes B$ is an (acyclic) cofibration.*

Proof. This follows from the definition of a Quillen tensor, and the simplifications described in Example 5.5.7. □

Lemma 5.5.19. *If $\otimes$ is a Quillen tensor, then the functor $(-)\otimes(-)$ preserves all weak equivalences between pairs of cofibrant objects.*

Proof. This follows from Lemma 5.5.18 and Ken Brown's Lemma (Lemma 5.2.25). □

Corollary 5.5.20. *If $\otimes$ is a Quillen tensor and $X \in \mathbf{C}$ is cofibrant, then $X \otimes (-)$ is left Quillen functor $\mathbf{D} \to \mathbf{E}$. Similarly if $Y \in \mathbf{D}$ is cofibrant, then $(-)\otimes Y$ is a left Quillen functor $\mathbf{C} \to \mathbf{E}$.*

We can recover some of the examples of Example 5.4.4 from Example 5.5.15 using this fact. For instance, the smash product of spaces $\wedge$ is a Quillen tensor. Therefore, if X is a cofibrant space, then $X \wedge (-)$ is left Quillen.

We can dualize the above results, and say that the right adjoints $\mathrm{Hom}(-,-)$ are well-behaved whenever the source is cofibrant and the target is fibrant.

Lemma 5.5.21. *Suppose $\mathrm{Hom}(-,-)$ is one of the right adjoints of a Quillen tensor.*

- *If X is cofibrant and Y is fibrant, then $\mathrm{Hom}(X,Y)$ is fibrant.*
- *If A is cofibrant and $X \to Y$ is an (acyclic) fibration, then the induced map $\mathrm{Hom}(A,X) \to \mathrm{Hom}(A,Y)$ is an (acyclic) fibration.*
- *If Y is fibrant and $A \to B$ is an (acyclic) cofibration, then the induced map going the other way $\mathrm{Hom}(B,Y) \to \mathrm{Hom}(A,Y)$ is an (acyclic) fibration.*

Proof. This follows from Example 5.5.11 and Proposition 5.5.12.

To spell out one case in detail, if $i : \varnothing \to A$ is a cofibration and $p : X \to Y$ is a fibration, we use Example 5.5.11 to write the map $\mathrm{Hom}(A,X) \to \mathrm{Hom}(A,Y)$ as $\mathrm{Hom}_{\square}(i,p)$. We show this is a fibration by showing it has lifts along any acyclic cofibration j:

$$\begin{array}{ccc} \bullet & \longrightarrow & \mathrm{Hom}(A,X) \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle \mathrm{Hom}_{\square}(i,p)} \\ \bullet & \longrightarrow & \mathrm{Hom}(A,Y) \end{array}$$

By Proposition 5.5.12, this is equivalent to finding a lift in

$$\begin{array}{ccc} \bullet & \longrightarrow & X \\ {\scriptstyle j\square i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \bullet & \longrightarrow & Y. \end{array}$$

But p is a fibration, and $j\square i$ is an acyclic cofibration, because $\otimes$ is a Quillen tensor. Therefore the required lift exists.

The remaining cases are left to Exercise 5.8.31. □

Lemma 5.5.22. *If* $\mathrm{Hom}(-,-)$ *is one of the right adjoints of a Quillen tensor, then it preserves all weak equivalences when the first input is cofibrant and the second input is fibrant.*

Proof. This follows from Lemma 5.5.21 and Ken Brown's Lemma for fibrations, Lemma 5.2.26. □

Corollary 5.5.23. *If* $\mathrm{Hom}(-,-)$ *is one of the right adjoints of a Quillen tensor, and* $X \in \mathbf{C}$ *is cofibrant, then* $\mathrm{Hom}(X,-)$ *is right Quillen.*

Example 5.5.24. The unbased mapping space $\mathrm{Map}(X,Y)$ is the right adjoint of a Quillen tensor. Therefore it preserves equivalences if X is cofibrant and Y is fibrant. You could prove this directly using the Whitehead theorem and cellular arguments, but it would be quite tedious!

Example 5.5.25. When X and Y are spectra, the mapping space $\mathrm{Map}_*(X,Y)$ is the right adjoint of a Quillen tensor. Therefore it preserves equivalences if X is cofibrant and Y is fibrant. This gives a second proof of Lemma 3.4.32.

Example 5.5.26. When K is a based space and X is a spectrum, the cotensor $F(K,X)$ is the right adjoint of a Quillen tensor. So it preserves equivalences if K is a retract of a cell complex, and X is an Ω-spectrum.

To summarize, a Quillen tensor $\otimes$ is well-behaved on pairs of cofibrant objects, and its right adjoints $\mathrm{Hom}_{\mathbf{C}}(-,-)$ and $\mathrm{Hom}_{\mathbf{D}}(-,-)$ are well-behaved when the source is cofibrant and the target is fibrant. We therefore have a left-derived tensor and right-derived homs

$$X \otimes^{\mathbb{L}} Y := QX \otimes QY,$$

$$\mathbb{R}\mathrm{Hom}_{\mathbf{C}}(X,Z) := \mathrm{Hom}_{\mathbf{C}}(QX,RZ), \qquad \mathbb{R}\mathrm{Hom}_{\mathbf{D}}(Y,Z) := \mathrm{Hom}_{\mathbf{D}}(QY,RZ).$$

As in Remark 3.5.25, these form a two-variable adjunction on the homotopy category, so we have natural isomorphisms

$$[X, \mathbb{R}\mathrm{Hom}_{\mathbf{D}}(Y,Z))]_{\mathrm{Ho}\,\mathbf{C}} \cong [X \otimes^{\mathbb{L}} Y, Z]_{\mathrm{Ho}\,\mathbf{E}} \cong [Y, \mathbb{R}\mathrm{Hom}_{\mathbf{C}}(X,Z)]_{\mathrm{Ho}\,\mathbf{D}}.$$

Concretely, these come about by the identifications

$$\begin{aligned}\mathbf{C}(X, \mathrm{Hom}_{\mathbf{D}}(Y,Z))/(\text{homotopy}) &\cong \mathbf{E}(X \otimes Y, Z)/(\text{homotopy}) \\ &\cong \mathbf{D}(Y, \mathrm{Hom}_{\mathbf{C}}(X,Z))/(\text{homotopy})\end{aligned}$$

where we assume X and Y are cofibrant and Z is fibrant, we take the identifications of mapping sets in Definition 5.5.1, and we pass to homotopy classes of maps.

Corollary 5.5.27. *If* $\mathbf{C}$ *is both a model category and a closed symmetric monoidal category, if the unit* I *is cofibrant and* $\otimes$ *is a Quillen tensor, then the left-derived tensor* $\otimes^{\mathbb{L}}$ *and right-derived hom* $\mathbb{R}\mathrm{Hom}$ *make the homotopy category* $\mathrm{Ho}\,\mathbf{C}$ *into a closed symmetric monoidal category.*

Proof. Follows from the above discussion along with Lemma 4.1.8 and Lemma 4.1.43. □

See Section 6.3.1 for a more extensive discussion of model categories with symmetric monoidal structures of this form.

5.6 Proof of the stable model structure*

We are finally ready to establish the stable model structure on **Sp** from Theorem 5.2.12.

Definition 5.6.1. We define the generating cofibrations to be

$$I = \left\{ F_i S^{n-1}_+ \longrightarrow F_i D^n_+ \ : \ n, i \geq 0 \right\}.$$

We have already observed that an I-cell complex is the same thing as a cellular spectrum.

For each $i, j \geq 0$, let

$$\lambda_{i,j} \colon F_{i+j} S^j \longrightarrow F_i S^0$$

be the map adjoint to the identity $S^j \cong S^j$. Concretely, this is a truncation map that cuts off a few of the levels of $F_i S^0$, and includes the remaining levels back into the spectrum. It follows that $\lambda_{i,j}$ is a stable equivalence. However, it is not a cofibration, so we use a mapping cylinder to fix that.

Let $\mathrm{Cyl}_{i,j}$ denote the based mapping cylinder of $\lambda_{i,j}$, and let $k_{i,j}$ inclusion of the front end of that mapping cylinder. This is summarized in the following diagram.

$$\begin{array}{ccccccc} F_{i+j}S^j & \xleftarrow{\cong} & \{1\}_+ \wedge F_{i+j}S^j & \xrightarrow{\sim} & I_+ \wedge F_{i+j}S^j & \xleftarrow{\sim} & \{0\}_+ \wedge F_{i+j}S^j \\ \downarrow{\scriptstyle \lambda_{i,j}} & & \downarrow{\scriptstyle \lambda_{i,j}} & & \downarrow & \swarrow{\scriptstyle k_{i,j}} & \\ F_i S^0 & \xleftarrow{\cong} & \{1\}_+ \wedge F_i S^0 & \xrightarrow{\sim} & \mathrm{Cyl}_{i,j} & & \end{array} \tag{5.6.2}$$

Definition 5.6.3. We define the generating acyclic cofibrations to be

$$\begin{aligned} J = &\left\{ F_i(D^n \times \{0\})_+ \longrightarrow F_i(D^n \times I)_+ \ : \ n, i \geq 0 \right\} \\ &\cup \left\{ k_{i,j} \square \left(S^{n-1}_+ \longrightarrow D^n_+ \right) \ : \ i, j, n \geq 0 \right\}. \end{aligned}$$

Here the pushout-product $\square$ is being taken with respect to the tensoring functor

$$\wedge \colon \mathbf{Sp} \times \mathbf{Top}_* \to \mathbf{Sp}.$$

We have already established in Example 5.5.15 that this is a Quillen tensor, using the level model structure on spectra.

Lemma 5.6.4. *The map $k_{i,j}$ is a stable equivalence and an I-cofibration.*

Proof. In (5.6.2), the maps marked $\sim$ are all level equivalences. Since $\lambda_{i,j}$ is a stable equivalence, we conclude that $k_{i,j}$ is a stable equivalence. The following diagram shows that $k_{i,j}$ is a composition of two I-cell complexes of spectra, and is therefore an I-cofibration.

$$\begin{array}{ccccc} & & \{0,1\}_+ \wedge F_{i+j}S^j & \longrightarrow & I_+ \wedge F_{i+j}S^j \\ & & \downarrow & & \downarrow \\ \{0\}_+ \wedge F_{i+j}S^j & \longrightarrow & (\{0\}_+ \wedge F_{i+j}S^j) \vee (\{1\}_+ \wedge F_iS^0) & \longrightarrow & \mathrm{Cyl}_{i,j} \\ & \searrow^{k_{i,j}} & & \nearrow & \end{array}$$

□

Lemma 5.6.5. *The map $k_{i,j} \square \left(S^{n-1}_+ \to D^n_+\right)$ is a stable equivalence and an I-cofibration.*

Proof. Since $\wedge$ is a Quillen tensor using the level model structure, and this uses the same set of maps I, we know that $k_{i,j} \square \left(S^{n-1}_+ \to D^n_+\right)$ is an I-cofibration.

To show it is a stable equivalence, we consider more generally any stable equivalence of cofibrant spectra $k\colon X \to Y$, and show that $k \square \left(S^{n-1}_+ \to D^n_+\right)$ is a stable equivalence. Consider the square

$$\begin{array}{ccc} X \wedge S^{n-1}_+ & \longrightarrow & X \wedge D^n_+ \\ {\scriptstyle k\wedge \mathrm{id}}\downarrow & & \downarrow{\scriptstyle k\wedge \mathrm{id}} \\ Y \wedge S^{n-1}_+ & \longrightarrow & Y \wedge D^n_+. \end{array}$$

By Corollary 2.4.25, or alternatively Lemma 5.5.19, the vertical maps are both stable equivalences. On the other hand, the horizontal maps are cofibrations at each spectrum level, so the strict pushout

$$(Y \wedge S^{n-1}_+) \cup_{X\wedge S^{n-1}_+} (X \wedge D^n_+)$$

is equivalent to the homotopy pushout. Since $X \wedge S^{n-1}_+ \to Y \wedge S^{n-1}_+$ is an equivalence, this homotopy pushout is equivalent to $X \wedge D^n_+$. Therefore the map from the pushout to $Y \wedge D^n_+$ is an equivalence. □

Lemma 5.6.6. *Let $p\colon X \to Y$ be a map of spectra. Then p is an I-fibration if and only if each $p_i\colon X_i \to Y_i$ is an acyclic Serre fibration.*

Furthermore p is a J-fibration if and only if two conditions hold: each $p_i\colon X_i \to Y_i$ is a Serre fibration, and each square (5.6.7) *is a homotopy pullback square.*

$$\begin{array}{ccc} X_i & \longrightarrow & \Omega^j X_{i+j} \\ \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle \Omega^j p_{i+j}} \\ Y_i & \longrightarrow & \Omega^j Y_{i+j} \end{array} \tag{5.6.7}$$

Definition 5.6.8. A **stable fibration** is a map of spectra $X \to Y$ that is a J-fibration. By Lemma 5.6.6, this is the same as being a level fibration and (5.6.7) being a homotopy pullback square. Note that $X \to *$ is a stable fibration iff X is an Ω-spectrum.

Proof of Lemma 5.6.6. The first claim has already been proven in Theorem 5.3.40, using the free-forgetful adjunction (F_n, ev_n). See Theorem 5.3.16.

By the same argument, p has the right-lifting property with respect to the first set of maps in J, iff each p_i is a Serre fibration. It remains to prove that, under this assumption, p has the right-lifting property with respect to the second set of maps in J iff each square (5.6.7) is a homotopy pullback square. So we assume that p is a level fibration for the rest of the proof.

By Proposition 5.5.12, p has the right-lifting property with respect to the second set of maps in J iff $\mathrm{Hom}_\square(k_{i,j}, p)$ has the right-lifting property with respect to $S^{n-1}_+ \to D^n_+$, in other words iff $\mathrm{Hom}_\square(k_{i,j}, p)$ is an acyclic Serre fibration.

Note that in the level model structure on spectra, $k_{i,j}$ is a cofibration and p is a fibration. By Remark 5.5.17, therefore $\mathrm{Hom}_\square(k_{i,j}, p)$ is a Serre fibration. So we only have to consider whether it is a weak equivalence as well.

Recall that the map $\mathrm{Hom}_\square(k_{i,j}, p)$ is written as

$$\mathrm{Map}_*(\mathrm{Cyl}_{i,j}, X) \longrightarrow \mathrm{Map}_*(F_{i+j}S^j, X) \times_{\mathrm{Map}_*(F_{i+j}S^j, Y)} \mathrm{Map}_*(\mathrm{Cyl}_{i,j}, Y). \tag{5.6.9}$$

Since Map_* is the right adjoint of a Quillen tensor for the level model structure, Lemma 5.5.21 applies. It tells us two things.

First, $\mathrm{Map}_*(F_{i+j}S^j, X) \to \mathrm{Map}_*(F_{i+j}S^j, Y)$ is a Serre fibration. Therefore the pullback on the right-hand side of (5.6.9) is a homotopy pullback.

Second, Map_* preserves equivalences when the source is a cellular spectrum. Therefore we may replace $\mathrm{Cyl}_{i,j}$ by the level equivalent spectrum $F_i S^0$:

$$\mathrm{Map}_*(F_i S^0, X) \longrightarrow \mathrm{Map}_*(F_{i+j}S^j, X) \times^h_{\mathrm{Map}_*(F_{i+j}S^j, Y)} \mathrm{Map}_*(F_i S^0, Y). \tag{5.6.10}$$

This map is weakly equivalent to (5.6.9), because we replaced two of the terms by weakly equivalent spaces, and the pullback is a homotopy pullback.

We can simplify one more time. By composing adjoints, we get an isomorphism $\mathrm{Map}_*(F_m S^n, X) \cong \Omega^n X_m$. This allows us to rewrite (5.6.10) as the map

$$X_i \longrightarrow \Omega^j X_{i+j} \times^h_{(\Omega^j Y_{i+j})} Y_i \tag{5.6.11}$$

arising from the adjunct bonding maps of X and Y.

In summary, we have shown that p has the right-lifting property with respect to the second set of maps in J iff (5.6.11) is a weak equivalence. However, (5.6.11) is just the map

from the first vertex to the homotopy pullback of the other three vertices, in the square (5.6.7). So p has the right-lifting property with respect to the second set of maps in J iff (5.6.7) is a homotopy pullback square. □

Remark 5.6.12. To check that the squares (5.6.7) are homotopy pullbacks, it suffices to check the cases in which $j = 1$. For instance, the square for $j = 2$ can be subdivided as follows.

$$\begin{array}{ccccc} X_i & \longrightarrow & \Omega X_{i+1} & \longrightarrow & \Omega^2 X_{i+2} \\ \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle \Omega p_{i+1}} & & \downarrow{\scriptstyle \Omega^2 p_{i+2}} \\ Y_i & \longrightarrow & \Omega Y_{i+1} & \longrightarrow & \Omega^2 Y_{i+2} \end{array}$$

If the squares with $j = 1$ are all homotopy pullbacks, then the left-hand square above is a homotopy pullback, and applying Ω, we see that the right-hand square is as well. Homotopy pullback squares are preserved under pasting, hence the large rectangle is a homotopy pullback. The same argument works for larger values of j as well.

As a result, we could have been a little more efficient in defining J. We only had to take the maps $k_{i,j} \square \left(S^{n-1}_+ \to D^n_+\right)$ for $j = 1$, not for all $j \geq 0$.

Proof of Theorem 5.2.12. (The stable model structure on **Sp**)

We verify the six conditions from Proposition 5.2.6.

1. W **is closed under 2-out-of-3 and retracts.** This follows because the stable equivalences are exactly those maps that are sent to isomorphisms under π_*.

2. I **satisfies the countable smallness condition.** As in $\mathbf{Top}^I$, this follows because the domain of each map in I is a free spectrum on a compact space, and every I-cell complex $\cdots \to X^{(n)} \to \cdots \to X^{(\infty)}$ is a cell complex at each spectrum level. The factorization therefore exists by Exercise 1.7.31.

3. J **satisfies the countable smallness condition.** Every map in J is a composite of I-cell attachments. Therefore any J-cell complex is also I-cell complex. So now we have to show the domains of J factor through some finite stage in any I-cell complex.

   For the first set of maps in J, the proof is the same as the previous point. For the second set of maps, we note that we are factoring a map $K \to X^{(\infty)}$ through a finite stage $K \to X^{(n)}$. The inclusion $X^{(n)} \to X^{(\infty)}$ is a closed inclusion, at each spectrum level. Therefore, if n is large enough that the factorization $K \to X^{(n)}$ exists, the factorization must also be unique.

   If K is a finite union of spectra K_i, and each piece K_i factors through some finite stage of the cell complex, then the factorizations must agree with each other (assuming n is large enough that they are all defined). Therefore K factors through some finite

stage as well. In particular, if K can be expressed as a finite union of free spectra on compact spaces, then this factorization exists.

By (5.6.2), the spectrum $\mathrm{Cyl}_{i,j}$ is a union of three free spectra on compact spaces. It therefore factors through some finite stage in any I-cell complex. The same is true if we take the tensor with a sphere, $\mathrm{Cyl}_{i,j} \wedge S^{n-1}_+$, because smash product commutes with free spectrum,

$$(F_i A) \wedge B \cong F_i(A \wedge B).$$

Now we turn to the domain of $k_{i,j} \square \left(S^{n-1}_+ \to D^n_+\right)$. This is of the form

$$(\mathrm{Cyl}_{i,j} \wedge S^{n-1}_+) \cup_{F_{i+j}S^j \wedge S^{n-1}_+} (F_{i+j}D^n_+).$$

We now know that each of these three pieces factors through some finite stage, and therefore the entire union does as well.

4. **J-cell complexes are in $W \cap$ (I-cof).** As in the above point, every J-cell complex is also an I-cell complex, and so is an I-cofibration. By Lemma 5.6.5, each map $K_\alpha \to L_\alpha$ in J is also a stable equivalence. A coproduct of such maps $\bigvee_\alpha K_\alpha \to \bigvee_\alpha L_\alpha$ is a stable equivalence by Exercise 2.7.30. Any pushout of this coproduct map is also a stable equivalence, because the pushout map has the same homotopy cofiber (Lemma 2.4.16), and this homotopy cofiber is contractible. Therefore in a J-cell complex, the successive stages $X^{(n)} \to X^{(n+1)}$ are all stable equivalences. Finally, the sequential composition of these stable equivalences is a stable equivalence, by Exercise 2.7.28. We conclude that every J-cell complex is a stable equivalence.

5. **I-fib $\subseteq W \cap$ (J-fib).** By Lemma 5.6.6, if p is an I-fibration then it is a both a fibration and a weak equivalence at each spectrum level. Therefore it is a stable equivalence, so $p \in W$. Moreover, since p is a level equivalence, the squares (5.6.7) have both verticals weak equivalences, and therefore they are homotopy pullback squares. Combining this with the fact that p is a level fibration, by Lemma 5.6.6 again we conclude that p is a J-fibration.

6. **$W \cap$ (J-fib) $\subseteq I$-fib.** If $p \in W \cap (J\text{-fib})$, then by Lemma 5.6.6, each p_i is a Serre fibration and the squares (5.6.7) are homotopy pullback squares. We just need to prove that each p_i is also a weak equivalence, using the fact that p is a stable equivalence.

   Examine the strict fiber spectrum F of the map p, in other words the pullback of X along the basepoint $* \to Y$. Because p is a level fibration, F is equivalent to the homotopy fiber spectrum. Since p is a stable equivalence, by the long exact sequence of Lemma 2.4.10, F is stably equivalent to the zero spectrum.

   Since (5.6.7) is a homotopy pullback square, the induced map of fibers of the vertical maps is a weak equivalence:

$$F_i \xrightarrow{\sim} \Omega^j F_{i+j}.$$

Therefore F is an Ω-spectrum. The map $F \to *$ is a stable equivalence of Ω-spectra, so by Lemma 2.2.5, it is also a level equivalence. Therefore each of the spaces F_i is weakly contractible.

Since F_i is the homotopy fiber of $p_i : X_i \to Y_i$ at the basepoint of Y_i, we conclude that p_i is a weak equivalence, at least over the basepoint component of Y_i. Taking the loopspace, $\Omega p_i : \Omega X_i \to \Omega Y_i$ is an equivalence on every component. By the homotopy pullback square (5.6.7), this implies that p_{i-1} is a weak equivalence, for every value of i. Therefore p is an acyclic Serre fibration at each level. By Lemma 5.6.6 again we conclude that p is an I-fibration.

□

In summary, once we developed the theory of Quillen tensors, we were able to show that the category of spectra **Sp** has a model structure with the stable equivalences. All of the properties of model categories from Section 5.2.5 now apply: we get factorizations

$$X \xrightarrow[\text{stable equivalence}]{\text{cellular spectrum}} X' \xrightarrow{\text{stable fibration}} Y$$

$$X \xrightarrow{\text{cellular spectrum}} X' \xrightarrow[\text{stable equivalence}]{\text{stable fibration}} Y,$$

and lifts of the form

Note that since the stable and level model structures have the same cofibrations C, they also have the same acyclic fibrations $W \cap F$: a map that is both a stable equivalence and a stable fibration must be a level equivalence and level fibration.

One big upshot of this work is the ability to check that a functor preserves weak equivalences, by using Ken Brown's Lemma and checking the functor on one map of J at a time. We will use this heavily as we develop symmetric and orthogonal spectra, and their smash product. See for instance the proof in the next chapter that the smash product preserves stable equivalences (Proposition 6.3.4 and Corollary 6.3.5).

5.7 Bousfield localization of a model category

A localization of a category **C** is when we invert a class of weak equivalences, turning them into isomorphisms, as in Definition 3.1.1. In this section we analyze what localization does to a model category, and in particular what happens to the stable model structure on

Sp when we invert the E-local equivalences, giving the E-local homotopy category that we studied in Section 3.3.

Now, when $\mathbf{C}$ is a model category with a class of weak equivalences V, then it has a homotopy category $\mathrm{Ho}\,\mathbf{C} = \mathbf{C}\langle V^{-1}\rangle$, in which we have turned each map in V into an isomorphism. However, we don't always work with this homotopy category. Most of the time, we do things like cofibrant replacements or homotopy colimits in the model category $\mathbf{C}$ itself, passing to the homotopy category $\mathrm{Ho}\,\mathbf{C}$ only at the very end.

So what should "localization" mean here? It should mean that we *expand* the class of weak equivalences V to a larger class W. We don't actually invert the maps in W, we only "mark" them so that now they are weak equivalences too. On the homotopy category, this has precisely the desired effect, turning the morphisms in W into isomorphisms. In other words, the map of homotopy categories $\mathbf{C}\langle V^{-1}\rangle \to \mathbf{C}\langle W^{-1}\rangle$ is a localization in the old sense from Definition 3.1.9.

So we want to find a way to take a model structure on $\mathbf{C}$ with weak equivalences V, and find a new model structure with a larger class of weak equivalences W, in a compatible way. There is a standard framework for this called left Bousfield localization.

Definition 5.7.1. Suppose $\mathbf{C}$ is a category and (C, V, F) is a model structure on $\mathbf{C}$ with cofibrations C, weak equivalences V, and fibrations F. A **left Bousfield localization** is a model structure (C, W, LF) in which

- the cofibrations are the same,
- the weak equivalences are a larger class of maps W containing V, and
- the fibrations LF are the maps with the right-lifting property with respect to $W \cap C$.

Note that this new model structure is determined by the new weak equivalences W, because the fibrations *must* be the maps with the RLP with respect to $W \cap C$, by Lemma 5.2.22. So the Bousfield localization along W is unique, if it exists.

Example 5.7.2. The stable model structure on **Sp** is a left Bousfield localization of the level model structure. They both have the same cofibrations, but the stable model structure has more equivalences.

Example 5.7.3. A map of spectra $X \to Y$ is a **rational equivalence** if it induces isomorphisms on the rational stable homotopy groups $\pi_*(-) \otimes \mathbb{Q}$, see Example 2.5.31 and Example 3.3.2. The **rational stable model structure** on **Sp** is defined to be the left Bousfield localization of the stable model structure, using these rational equivalences.

We can directly show that this model structure exists. You take I to be the same as in Definition 5.6.1, and in J you add to Definition 5.6.3 the mapping cylinders of the maps

$$F_i S^n \longrightarrow F_i S^n$$

that apply a degree k map ϕ_k to the n-sphere, for all positive integer values of k. These new maps are all rational equivalences, as is any cell complex built out of them. On the other hand, the proof of Lemma 5.6.6 shows that having the right-lifting property with respect to these new maps is equivalent to asking that the following squares are homotopy pullback squares:

$$\begin{array}{ccc} \Omega^n X_i & \xrightarrow{-\circ\phi_k} & \Omega^n X_i \\ {\scriptstyle \Omega^n p_i}\downarrow & & \downarrow{\scriptstyle \Omega^n p_i} \\ \Omega^n Y_i & \xrightarrow[-\circ\phi_k]{} & \Omega^n Y_i \end{array}$$

This is equivalent to the fiber of $X \to Y$ (which is already an Ω-spectrum) having $k \cdot -$ act by an isomorphism on the stable homotopy groups, in other words, the fiber is rationally contractible. All together, this is everything we need to show that I and J generate a model structure with W equal to the rational equivalences.

Example 5.7.4. Let p be any prime. A map of spectra $X \to Y$ is a **p-local equivalence** if it induces isomorphisms on the p-local stable homotopy groups $\pi_*(-) \otimes \mathbb{Z}_{(p)}$, see Example 2.5.32 and Example 3.3.23. The **p-local stable model structure** on **Sp** is defined to be the left Bousfield localization of the stable model structure, using these p-local equivalences. Its construction is nearly identical to the previous example.

Left Bousfield localizations have a few important features:

- We always have $LF \subseteq F$, because $V \cap C \subseteq W \cap C$. So the model structure (C, W, LF) has fewer fibrations, and therefore fewer fibrant objects. We say that an object is **local** if it is V-equivalent to a fibrant object in the new model structure (C, W, LF). The fibrant replacement in the new model structure is called **localization**

  $$X \longrightarrow LX.$$

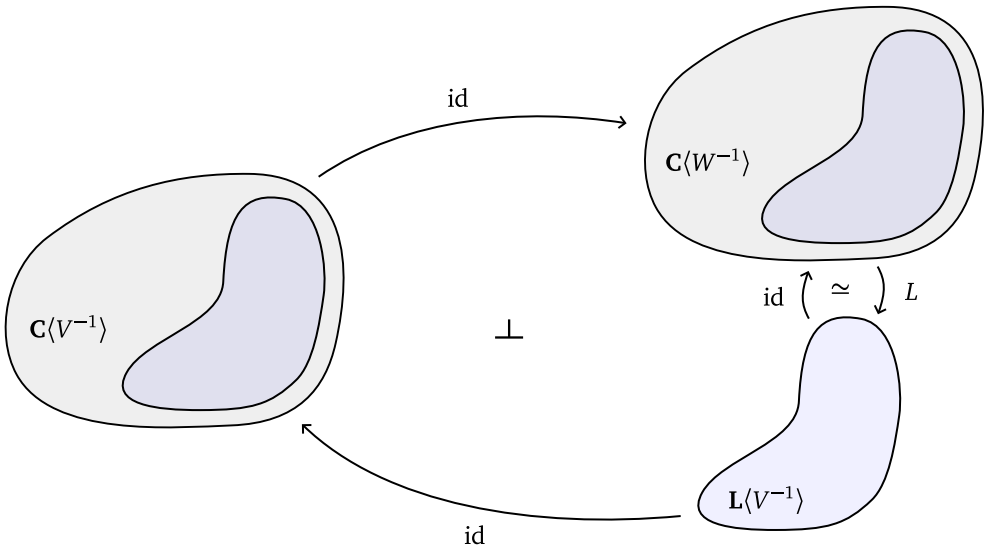
  Think of LX as the universal approximation of X by a local object, up to V-equivalence.

- We always have $W \cap LF = V \cap F$, because the cofibrations are the same. It follows that $W = V$ when restricted to the local objects. So objects up to W-equivalence are the same thing as local objects up to V-equivalence.

  We can visualize this by letting **L** be the subcategory of local objects, and seeing that when we invert W in **C**, the resulting big category $\mathbf{C}\langle W^{-1}\rangle$ is equivalent to its subcategory $\mathbf{L}\langle V^{-1}\rangle$, as shown below.

- The subcategory of local objects $\mathbf{L} \subseteq \mathbf{C}$ is a **reflective subcategory**. That means that on the homotopy category, the inclusion of the local objects has a left adjoint, which depending on your perspective is either the functor $L\colon \mathbf{C}\langle V^{-1}\rangle \to \mathbf{L}\langle V^{-1}\rangle$ or

the identity functor $\mathbf{C}\langle V^{-1}\rangle \to \mathbf{C}\langle W^{-1}\rangle$. The following picture illustrates the two ways of thinking about this adjunction:



Or in symbols:

$$\mathbf{C}\langle V^{-1}\rangle \underset{L}{\overset{\mathrm{id}}{\rightleftarrows}} \mathbf{C}\langle W^{-1}\rangle \qquad \text{or} \qquad \mathbf{C}\langle V^{-1}\rangle \underset{\mathrm{id}}{\overset{L}{\rightleftarrows}} \mathbf{L}\langle V^{-1}\rangle$$

It's a little confusing that L could be the left or the right adjoint, depending on how you describe the category on the right. But either way, the smaller category $\mathbf{C}\langle W^{-1}\rangle \simeq \mathbf{L}\langle V^{-1}\rangle$ sits inside the larger category $\mathbf{C}\langle V^{-1}\rangle$, and the left adjoint gives a universal way to approximate anything in the larger category by something in the smaller category.

The fact that the localized homotopy category sits inside the original one is very convenient. It means we can quickly prove things about the localized homotopy category from things we knew about the original homotopy category. For instance, if we localize the stable homotopy category Ho **Sp** in this way, we'll always get a category enriched in abelian groups, because it's just a subcategory of Ho **Sp**!

To illustrate this universal approximation in one example, if $W_{\mathbb{Q}}$ is the set of rational equivalences of spectra, then the local objects are the rational spectra, whose homotopy groups are rational vector spaces. The functor L is rationalization $H\mathbb{Q} \wedge -$, which applies $\mathbb{Q} \otimes -$ to the homotopy groups. So L turns every spectrum into its "closest" rational spectrum.

Let us return to the question of whether a Bousfield localization exists for a given class of weak equivalences W. An important example is the localization of spaces or spectra with respect to a homology theory.

Example 5.7.5. Let E be any spectrum. Recall from Definition 3.3.1 that a map of spectra $X \to Y$ is an E**-local equivalence** if it induces an isomorphism $E_*(X) \cong E_*(Y)$. The E**-local model structure** on **Sp** is the left Bousfield localization of the stable model structure, using the E-local equivalences. Its homotopy category is precisely the E-local stable homotopy category $\mathrm{Ho}\,\mathbf{Sp}_E$ that we studied in Section 3.3.

Example 5.7.6. When X and Y are topological spaces, we say similarly that a map $X \to Y$ is an E-local equivalence if it induces an isomorphism $E_*(X) \cong E_*(Y)$. The E-local model structure on **Top** or $\mathbf{Top}_*$ is the left Bousfield localization of the Quillen model structure on **Top** or $\mathbf{Top}_*$, using the E-local equivalences.

Theorem 5.7.7 (Bousfield)**.** *The E-local model structures on* **Sp***,* **Top***, and* $\mathbf{Top}_*$ *always exist, for any spectrum E.*

In Exercise 5.8.41 you will prove that the fibrant objects in **Top** and $\mathbf{Top}_*$ are the E**-local spaces**, i.e. spaces Z such that for every E-local equivalence of CW complexes $X \to Y$, the map of homotopy classes $[Y, Z] \to [X, Z]$ is a bijection. Similarly, the fibrant objects in **Sp** are the Ω-spectra that are also E-local in the sense of Definition 3.3.3. Therefore the fibrant replacement or localization in this model structure is Bousfield's functor $L_E X$.

Example 5.7.8. We covered many examples of E-localization for spectra in Section 3.3.3. Here are some analogous localizations for spaces:

- If $E = H\mathbb{Q}$ then we get the rational equivalences of spaces and the rational model structure on spaces. The fibrant replacement is the rationalization $X \to X_{\mathbb{Q}}$.
- If $E = H\mathbb{Z}_{(p)}$ then we get the p-local model structure on spaces. The fibrant replacement is the p-localization $X \to X_{(p)}$.
- If $E = H\mathbb{Z}/p$ then we get the p**-adic equivalences** and a corresponding model structure. A map is an p-adic equivalence if it induces an equivalence on homology groups mod p. The fibrant replacement is the p-completion $X \to X^\wedge_p$.

If X is either a spectrum or a simply-connected topological space, there are nice descriptions of the homotopy and homology groups of the localizations $X_{\mathbb{Q}}$, $X_{(p)}$, and $X^\wedge_p$ [MP12]. On the other hand, if X is a space that is not simply-connected, then its localization still exists, but it is harder to calculate anything about it.[7]

We spend the rest of the section proving Theorem 5.7.7, following the original sources [Bou75, Bou79] and the proof for spaces in [MP12, §19.3]. This also proves the existence of Bousfield's functor $L_E X$ that we described in Theorem 3.3.7.

[7] In fact, we also get computational results about $X_{\mathbb{Q}}$, $X_{(p)}$, and $X^\wedge_p$ if X is **nilpotent** in the sense of [MP12]. But if X fails to be nilpotent then there is little that we can say.

For simplicity, we will assume that the sum of all the groups $E_n(*)$ is countable. This covers essentially all examples of interest, and makes the proof easier to read. The reader interested in the general case need only replace every instance of "countable" with "cardinality $< \kappa$," where κ is some fixed regular cardinal (see Exercise 5.8.42) that is at least as big as $\aleph_1$, and the sum of all the groups $E_n(*)$.

We begin with some notation, which will apply in any of the settings **Top**, $\mathbf{Top}_*$, or **Sp**. Recall that the first two of these have a Quillen model structure, and the last has a stable model structure. When we say "cell complex," we mean a map built out of the generating cofibrations of this model structure by taking coproducts, pushouts, and a countable composition, just as in Definition 1.1.11 and Proposition 2.6.11. A "subcomplex" is built in the same way by selecting only some of the cells, such that the attaching map of each cell we select also lands in the subcomplex. This is always possible because in a cell complex, the image of the attaching map for any cell always lands in a finite subcomplex. (Think about what this means in the case of spectra!)

Definition 5.7.9. An **E-acyclic cell complex** is a cell complex B and an inclusion of a subcomplex $A \to B$ that is an E-local equivalence (Example 5.7.5).

Definition 5.7.10. Let K be the set of all E-acyclic cell complexes $A \to B$ in which B (and therefore A) has countably many cells, and the underlying set of B is a subset of $\mathbb{N} \times \mathbb{R}$.[8] This last condition ensures that K is a set, not a proper class. But K is still large enough to contain every countable E-acyclic cell complex, up to isomorphism, because we can always find an injection from the interior of each cell into one copy of $\mathbb{R}$. In particular, K contains (up to isomorphism) the usual set of generating acyclic cofibrations J, in other words the relative cell complexes $D^n \times \{0\} \to D^n \times I$ for all $n \geq 0$.

Note that every map in K has $E_*(B,A) = 0$, by the long exact sequence on E-homology. The following lemma is the heart of the proof, and it's essentially a check that the set K "generates" the rest of the E-acyclic cofibrations.

Lemma 5.7.11. *If $X \to Y$ is a K-fibration then it has the right-lifting property with respect to every E-acyclic cell complex $A \to B$.*

Proof. Let $X \to Y$ be a K-fibration. So it has lifts of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & Y. \end{array}$$

for every countable E-acyclic cell complex $A \to B$. Now suppose that $A \to B$ is another E-acyclic cell complex that is not countable. Consider subcomplexes $B' \subseteq B$ containing A,

[8] When generalizing this to larger cardinals κ, we ask for B to be a subset of $\kappa \times \mathbb{R}$.

such that $A \to B' \to B$ are both E-local equivalences, and B' has a chosen lift to X:

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ B' & \longrightarrow & Y. \end{array}$$

These form a partially ordered set in which every chain has a maximal element, so by Zorn's lemma there must be a maximal such subcomplex $B' \subseteq B$.

Now if B' isn't all of B, then take any countable nonempty collection of cells of B that are not in B', and let B_1 be the smallest subcomplex of B containing these cells, which is still countable. For each of the countably many elements $x \in E_*(B_1, A \cap B_1)$, x goes to zero in $E_*(B,A)$. But $E_*(B,A)$ is a colimit of $E_*(B_1 \cup C, A \cap (B_1 \cup C))$ where C ranges over all finite subcomplexes. Therefore there exists a finite complex C_x in B such that x goes to zero in $E_*(B_1 \cup C_x, A \cap (B_1 \cup C_x))$. Taking the union of B_1 with the countably many finite complexes C_x gives a new countable subcomplex $B_2 \supseteq B_1$ such that $E_*(B_1, A \cap B_1) \to E_*(B_2, A \cap B_2)$ is the zero homomorphism.

Unfortunately, the group $E_*(B_2, A \cap B_2)$ may be nonzero. Fortunately, we can repeat this process countably many times[9] and taking the colimit, we get a countable subcomplex $B_\infty \subseteq B$ such that $E_*(B_\infty, A \cap B_\infty) = 0$. So $A \cap B_\infty \to B_\infty$ is (isomorphic to) a map in the set K. Therefore the given lift on B' extends to the strictly larger subcomplex $B' \cup B_\infty$. But B' was supposed to be maximal, so this is a contradiction. □

You may have been reading the above proof imagining that we were working in **Top**, but in fact the same proof works in **Sp** as well, provided we interpret "cell complex" in the correct way, as in Proposition 2.6.11.[10]

Proof of Theorem 5.7.7. We begin with the case of **Top**. We check the six conditions in Proposition 5.2.6, using the usual generating cofibrations $I = \{S^{n-1}_+ \to D^n_+\}_{n \geq 0}$, and using the set K from Definition 5.7.10 as the generating acyclic cofibrations. Part (1) is easy, and (2) has already been done since the set I hasn't changed. For (3) we use the transfinite version of the small object argument from Exercise 5.8.42. (We need this even in this special case that $E_*(*)$ is countable!) Since the cell complexes $A \to B$ that appear in K have countably many cells, if we take a transfinite K-cell complex X indexed by an $\aleph_1$-filtered limit ordinal β (as in Exercise 5.8.42), any map $A \to X_\beta$ must factor through some stage $A \to X_\alpha$, $\alpha < \beta$. (Each cell of A factors through one stage by Exercise 5.8.43, and this gives countably many stages $\alpha_i < \beta$. Since β is $\aleph_1$-filtered, these stages have an upper

[9] This instance of "countable" actually does not need to be replaced when you do the general case.

[10] This proof also will also work perfectly well for "cell complexes" in symmetric or orthogonal spectra, once we define those.

bound $\alpha < \beta$.) Therefore by Theorem 5.8.1, every map $X \to Y$ factors into a transfinite K-cell complex, which in particular is an E-acyclic cell complex, followed by a K-fibration.

For (4), every map in K is an E-local equivalence and I-cofibration (retract of an I-cell complex) by definition, and these properties extend to K-cell complexes as well. For (5), if $X \to Y$ is an I-fibration then it is a weak homotopy equivalence, so it is certainly an E-local equivalence. It also has the right-lifting property for every relative cell complex, so in particular it has lifts along every map in K, and is therefore a K-fibration. For (6), suppose that $X \to Y$ that is both an E-local equivalence and a K-fibration. Then we form the square

$$\begin{array}{ccc} QX & \xrightarrow{\sim} & X \\ \downarrow & \nearrow & \downarrow \\ QY & \xrightarrow{\sim} & Y \end{array}$$

where QX is a cell complex, $QX \to QY$ is a relative cell complex, and the horizontal maps are weak homotopy equivalences. The dashed lift exists by Lemma 5.7.11 because $QX \to QY$ is a relative cell complex and an E-local equivalence. The dashed lift proves that $X \to Y$ is invertible in Ho **Top**, and is therefore a weak equivalence. This concludes the proof of the model structure for **Top**.

The model structure for $\mathbf{Top}_*$ follows immediately by Exercise 5.8.8. The model structure for **Sp** follows by exactly the same proof, where we replace "weak homotopy equivalence" by "stable equivalence." In this case K is a set of maps of *spectra* $A \to B$ that are relative cell complexes and E-local equivalences, with one representative for each isomorphism class in which both spectra have countably many cells. □

Again, by Exercise 5.8.41, the fibrant spectra in this model structure are the E-local Ω-spectra, so the fibrant replacement in this model structure is Bousfield's functor L_E. This proves the existence of the functor L_E that we claimed in Theorem 3.3.7, and therefore the E-local stable homotopy category has all of the desired properties.

In [Hir03, Thm 4.1.1] this proof is greatly generalized, giving a method for producing left Bousfield localizations that works in any cofibrantly generated model category $(\mathbf{C}, C, V, F)$ satisfying mild assumptions (Definition 10.4.66).

To describe Hirschhorn's version in brief, given any set of maps S_0 in $\mathbf{C}$, we can form a class of "S_0-local objects" and then a class of "S_0-local equivalences." Intuitively, the S_0-local equivalences are the maps generated by V and S_0 under homotopy colimits of maps, pushouts along arbitrary morphisms in $\mathbf{C}$, and 2-out-of-3. Then the left Bousfield localization with respect to the S_0-local equivalences is guaranteed to exist, and its local objects are exactly the S_0-local objects (Theorem 10.4.67). (The E-local equivalences can be generated this way out of a set of maps S_0, by Lemma 5.7.11). We will wait until Section 10.4.6 to discuss this more general formulation, because we need homotopy

function complexes to state the theorem precisely.

5.8 Exercises

5.8.1. (a) Use Lemma 5.2.22 to show that in a model category **C**, any map isomorphic to a cofibration must also be a cofibration. The same is also true for weak equivalences, and fibrations.

(b) If **C** is a model category and **D** is another category equivalent to **C**, show that **D** has a model structure in which a map is a cofibration, weak equivalence, or fibration iff its image in **C** is such. (The previous part of this exercise is needed to show that this is well-defined.)

When the model structures on **C** and **D** are related in this very strong way, we call $\mathbf{C} \simeq \mathbf{D}$ an **equivalence of model categories**.

5.8.2. Let **C** be any category. Show that there is always a **trivial model structure** on **C** in which the weak equivalences W are the isomorphisms, and both C and F consist of all of the morphisms in **C**. Show that there is also a model structure in which C is the isomorphisms, and W and F are every map in **C**.

5.8.3. Let **C** be any model category. Show that the opposite category $\mathbf{C}^{\mathrm{op}}$ is also a model category, in which we swap the cofibrations with the fibrations and keep the weak equivalences the same.

5.8.4. Suppose **C** is a category and I is any set of maps in **C**. Recall from Definition 5.1.19 that we defined an "I-cofibration" to be a retract of an I-cell complex.

Show that $X \to Y$ is an I-cofibration iff Y is a retract of a third object Z, and the composite $X \to Y \to Z$ is an I-cell complex. In other words, without loss of generality, the map $X \to Y$ is a retract of a cell complex with the same domain X.

5.8.5. Finish the proof of Proposition 5.2.6 by assuming that in point (6) we have $W \cap (I\text{-cof}) \subseteq J\text{-cof}$, and proving that $W \cap (J\text{-fib}) \subseteq I\text{-fib}$. (Hint: Dualize the proof of Proposition 5.1.23.)

5.8.6. Let g be any morphism in any category **C**.

(a) Prove that the set of maps with the left lifting property (LLP) with respect to g is closed under coproducts, pushouts, sequential compositions, and retracts. In particular, I-cofibrations in any category, and cofibrations in any model category, are closed under these operations.

(b) Dually, prove that the maps with the right lifting property (RLP) with respect to g are closed under products, pullbacks, sequential limits, and retracts. Therefore fibrations in any model category are closed under these operations.

5.8.7. Prove that both model structures on spectra are proper (Definition 5.2.31). You may find it helpful to use the facts about homotopy cofibers and fibers that we proved back in Section 2.4.

5.8.8. Recall the definition of the slice category $(X \downarrow \mathbf{C})$ and $(\mathbf{C} \downarrow X)$ from Example 5.4.12 and Example 5.4.13.

(a) If $\mathbf{C}$ is a model category and X is any object, prove that the slice category $(X \downarrow \mathbf{C})$ is a model category, in which the cofibrations, fibrations, and weak equivalences are determined by forgetting the map from X.

(b) If $\mathbf{C}$ is cofibrantly generated by I and J, show that $(X \downarrow \mathbf{C})$ is cofibrantly generated by the sets

$$X \amalg I = \{X \amalg K \xrightarrow{\mathrm{id}\amalg f} X \amalg L : f \in I\}$$

$$X \amalg J = \{X \amalg K \xrightarrow{\mathrm{id}\amalg f} X \amalg L : f \in J\}.$$

(c) Recover the model structure on $\mathbf{Top}_* = (* \downarrow \mathbf{Top})$ as a special case.

5.8.9. (a) As in Exercise 5.8.8, prove that the slice category $(\mathbf{C} \downarrow X)$ has a model structure in which the cofibrations, fibrations, and weak equivalences are determined by forgetting the map from X

(b) If $\mathbf{C}$ is cofibrantly generated by I and J, show that $(\mathbf{C} \downarrow X)$ is cofibrantly generated by the sets I/X and J/X, where I/X has one morphism for each commuting triangle

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

with $f \in I$, and similarly J/X has a morphism for each such triangle with $f \in J$.

(c) Use this to give a model structure on $(\mathbf{Top} \downarrow B)$ for an unbased space B. What do Whitehead's theorem and the fundamental theorem tell us in this setting? (This is very useful to know when studying fibrations and fiber bundles!)

(d) Combine this exercise and Exercise 5.8.8 to give a model structure on the category $\mathscr{R}(B)$ of topological spaces containing B as a retract.

5.8.10. Use Proposition 5.2.6 to establish Quillen's model structure on chain complexes $\mathbf{Ch}_{\geq 0}(R)$ from Example 5.2.11. (You might want to figure out what the I-cell complexes, I-fibrations, and J-fibrations are.)

5.8.11. Generalize Proposition 5.2.6 to give a model structure on unbounded ($\mathbb{Z}$-graded) chain complexes $\mathbf{Ch}(R)$, with weak equivalences the quasi-isomorphisms and fibrations the maps that are surjective in every degree. Use essentially the same generating maps I and J.

5.8.12. In any model category $\mathbf{C}$:

- Show that any span $X \xleftarrow{i} A \xrightarrow{j} Y$ is weakly equivalent to a span in which all three objects are cofibrant, and the map i is a cofibration.
- If in addition the map i is a weak equivalence, explain why the pushout of i, $Y \to X \cup_A Y$, is also a weak equivalence.
- If i is a cofibration, j is a weak equivalence, and all three objects are cofibrant, prove that the pushout of j, $X \to X \cup_A Y$, is also a weak equivalence. (Hint: Imitate the proof of Ken Brown's Lemma, Lemma 5.2.25.)

Therefore the left properness property of Definition 5.2.31 "almost" holds in any model category: it holds when all the objects in the diagram are cofibrant. It follows that the gluing lemma, Lemma 5.2.34, also holds when all objects are cofibrant.

5.8.13. Let $\mathbf{I}$ be a discrete category, and $F : \mathbf{I} \to \mathbf{J}$ a functor. For each $j \in \mathbf{J}$, define the slice category $(F \downarrow j)$ to have an object for each $i \in \mathbf{I}$ and morphism $p : F(i) \to j$, and a morphism for each morphism $f : i_0 \to i_1$ in $\mathbf{I}$ that forms a commuting triangle

$$\begin{array}{ccc} F(i_0) & \xrightarrow{F(f)} & F(i_1) \\ & {\scriptstyle p_0}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & j. & \end{array}$$

(a) Verify that this forms a category, with a forgetful functor $(F \downarrow j) \to \mathbf{I}$.

(b) Explain how the slice category $(\mathbf{C} \downarrow X)$ from Example 5.4.13 is a special case of this.

(c) If $X : \mathbf{I} \to \mathbf{Top}$ is a diagram of spaces on $\mathbf{I}$, then along the forgetful functor, it is a diagram of spaces on $(F \downarrow j)$. Prove that the spaces

$$(F_! X)(j) = \operatorname*{colim}_{F(i) \to j} X(i)$$

form a diagram on $\mathbf{J}$.

(d) Prove that each map of $\mathbf{J}$-diagrams $F_! X \to Y$ corresponds to a map of $\mathbf{I}$-diagrams $X \to F^* Y$. In other words, this is a formula for the left Kan extension of X from Definition 5.4.14.

(e) If the mapping spaces $\mathbf{J}(j_0, j_1)$ have a topology, explain why this construction does not give a valid $\mathbf{J}$-diagram in general. (In that case we use Lemma 5.4.16.)

5.8.14. (a) If $\alpha\colon \mathbf{I} \to \mathbf{J}$ is a functor, and $F_i A$ is a free diagram on $\mathbf{I}$ at the object $i \in \mathbf{I}$, prove that the left Kan extension $\alpha_! F_i A$ is isomorphic to the free diagram $F_{\alpha(i)} A$. (You might find it easiest to prove this by taking the right adjoints, composing them, then taking the left adjoint of the result.)

(b) Verify that the previous part also holds if the categories are based and the diagrams are diagrams of based spaces.

5.8.15. Similarly to the previous exercise, show that if F and G are composable functors

$$\mathbf{A} \xrightarrow{F} \mathbf{B} \xrightarrow{G} \mathbf{D}$$

then the left Kan extensions compose, in the sense that $(G \circ F)_!$ is naturally isomorphic to $G_! \circ F_!$. Conclude that for a commuting square of functors

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{F} & \mathbf{B} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle G} \\ \mathbf{C} & \xrightarrow{K} & \mathbf{D} \end{array}$$

we have $K_! \circ H_! \cong G_! \circ F_!$.

5.8.16. (a) Suppose $\alpha\colon \mathbf{I} \to \mathbf{J}$ is a **pointwise equivalence** of topological categories. That is, it is a bijection on objects, and each of the maps $\mathbf{I}(a,b) \to \mathbf{J}(a,b)$ is a weak equivalence. Prove that the left Kan extension and restriction functors $(\alpha_!, \alpha^*)$ from Section 5.4.3 form a Quillen equivalence between $\mathbf{Top}^{\mathbf{I}}$ and $\mathbf{Top}^{\mathbf{J}}$ with the projective model structures.

(Hint: The key step is to notice that the natural maps $X(a) \to \alpha_! X(a)$ are equivalences for each cofibrant diagram X. Prove this by working one cell of X at a time.)

(b) (*) Now prove this more generally in the case that α is a **Dwyer-Kan equivalence** in the sense of Exercise 7.7.27. (Essentially, this means that α is a weak equivalence on mapping spaces, and every object of $\mathbf{J}$ is "homotopy equivalent" to an object in the image of α.)

5.8.17. If $(F \dashv G)$ are a pair of adjoint functors between model categories, prove that F preserves cofibrations and acyclic cofibrations iff G preserves fibrations and acyclic fibrations. (Hint: by Lemma 5.2.22, a map is a fibration *if and only if* it has the right lifting property with respect to acyclic cofibrations.)

5.8.18. Verify that the functors listed in Example 5.4.4 are all Quillen adjunctions.

5.8.19. Use the characterization of Quillen pairs in Lemma 5.4.2 to give a much shorter proof that the colimit functor preserves equivalences between cofibrant diagrams (Proposition 5.3.19).

5.8.20. Dualize the proof of Ken Brown's Lemma for cofibrant objects (Lemma 5.2.25) to prove the lemma for fibrant objects (Lemma 5.2.26).

5.8.21. Though we have not proven this, in every model category, homotopy colimits and homotopy limits exist.

(a) Use Exercise 3.6.29 to show that for a Quillen adjunction $(F \dashv G)$, F preserves homotopy colimits and G preserves homotopy limits.

(b) Explain what this looks like in the special case of a left Bousfield localization, where F is the identity functor of a category $\mathbf{C}$, going from $(\mathbf{C}, C, V, F)$ to $(\mathbf{C}, C, W, LF)$.

(c) Explain why any Bousfield localization of spectra gives a homotopy category $\mathbf{Sp}\langle W^{-1}\rangle$ that is automatically an additive category, and why computational results such as Proposition 3.2.20 and Proposition 3.2.21 still hold in the localized homotopy category $\mathbf{Sp}\langle W^{-1}\rangle$.

5.8.22. This exercise is a preview of Section 7.3. Suppose G is a topological monoid, X is a right G-space, and Y is a left G-space. The **balanced product** $X \times_G Y$ is the quotient of $X \times Y$ by the equivalence relation

$$(xg, y) \sim (x, gy)$$

for all $x \in X$, $y \in Y$, and $g \in G$. In other words, it is the coequalizer of the two action maps of G,

$$X \times G \times Y \rightrightarrows X \times Y \longrightarrow X \times_G Y.$$

The two maps take (x, g, y) to either (xg, y) or (x, gy).

(a) Prove that $X \times_G * $ is the orbit space X_G.

(b) Prove that this is a Quillen tensor

$$\mathbf{Top}^G \times \mathbf{Top}^{G^{\mathrm{op}}} \to \mathbf{Top}.$$

Therefore the balanced product has a left-derived functor $X \times_G^{\mathbb{L}} Y$, and it is computed by making both X and Y into cofibrant G-spaces,

$$X \times_G^{\mathbb{L}} Y = QX \times_G QY.$$

(c) Prove that if only X is cofibrant then the functor $X \times_G (-)$ preserves weak equivalences. (Hint: WLOG X is a free G-cell complex, built by attaching cells of the form $G \times D^n$. Prove this by an induction on the skeleta of X.) Conclude that $QX \times_G Y$ is also a valid model for the left-derived functor.

(d) Conclude in the special case that $X = EG$, the cofibrant replacement of the point $*$ in the category of right G-spaces, that the left-derived functor of the orbits $* \times_G Y$ is given by the **Borel construction** $EG \times_G Y$:

$$Y_{hG} \simeq EG \times_G Y.$$

This verifies the claim we made in Definition 5.3.26.

5.8.23. This exercise is a preview of Section 7.4. Suppose $\mathbf{I}$ is a topological category, $X : \mathbf{I} \to \mathbf{Top}$ is a diagram, and $Y : \mathbf{I}^{\mathrm{op}} \to \mathbf{Top}$ is a contravariant diagram. Define the **coend** $X \times_{\mathbf{I}} Y$ to be the topological space

$$X \times_{\mathbf{C}} Y = \left(\coprod_{c \in \mathbf{C}} X(c) \times Y(c) \right) \Big/ (f(x), y) \sim (x, f(y))$$

where the relation is taken over all $f : c \to d$, $x \in X(c)$, and $y \in Y(d)$. In other words, it is the coequalizer of the two action maps of $\mathbf{I}$,

$$\coprod_{i_0, i_1 \in \mathbf{C}} X(i_0) \times \mathbf{I}(i_0, i_1) \times Y(i_1) \rightrightarrows \coprod_{i \in \mathbf{C}} X(i) \times Y(i) \longrightarrow X \times_{\mathbf{C}} Y.$$

The two maps take (x, f, y) to either $(f(x), y)$ or $(x, f^*(y))$.

Essentially, X and Y are glued along the category $\mathbf{C}$ to create a single space, in much the same way that we form the balanced product $X \times_G Y$ in the previous exercise.

(a) Prove that $X \times_{\mathbf{I}} *$ is the colimit of X.

(b) Prove that the coend defines a Quillen tensor

$$\mathbf{Top}^{\mathbf{I}} \times \mathbf{Top}^{\mathbf{I}^{\mathrm{op}}} \to \mathbf{Top}.$$

(c) Let $F : \mathbf{I} \to \mathbf{J}$ be a functor, and $j \in \mathbf{J}$. Notice that $\mathbf{J}(F(-), j)$ is a diagram on $\mathbf{I}^{\mathrm{op}}$. Prove that $X \times_{\mathbf{I}} \mathbf{J}(F(-), j)$ is the left Kan extension $(F_! X)(j)$ from Definition 5.4.14.

5.8.24. Suppose $F_1, F_2 : \mathbf{C} \rightrightarrows \mathbf{D}$ are two functors, with right adjoints $G_1, G_2 : \mathbf{D} \rightrightarrows \mathbf{C}$, respectively. Prove that any natural transformation $\phi : F_1 \to F_2$ has a "mate," a natural transformation $\phi^* : G_2 \to G_1$ going the opposite direction. It should be set up so that the following square commutes.

$$\begin{array}{ccc} \mathbf{D}(F_2(X), Y) & \xleftrightarrow{\cong} & \mathbf{C}(X, G_2(Y)) \\ {\scriptstyle -\circ\phi}\downarrow & & \downarrow{\scriptstyle \phi^* \circ -} \\ \mathbf{D}(F_1(X), Y) & \xleftrightarrow{\cong} & \mathbf{C}(X, G_1(Y)) \end{array}$$

5.8.25. Let **C**, **D**, and **E** be categories. A functor $F : \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ is a **two-variable left adjoint** if

- $\forall c \in \mathbf{C}$, the functor $F(c,-)\colon \mathbf{D} \to \mathbf{E}$ has a right adjoint $G(c,-)\colon \mathbf{E} \to \mathbf{D}$, and
- $\forall d \in \mathbf{D}$, the functor $F(-,d)\colon \mathbf{C} \to \mathbf{E}$ has a right adjoint $H(d,-)\colon \mathbf{E} \to \mathbf{C}$.

Using Exercise 5.8.24, explain how as c and d varies, the operations G and H give functors of two variables

$$G(-,-)\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{E} \to \mathbf{D},$$
$$H(-,-)\colon \mathbf{D}^{\mathrm{op}} \times \mathbf{E} \to \mathbf{C}$$

and natural isomorphisms of functors $\mathbf{C}^{\mathrm{op}} \times \mathbf{D}^{\mathrm{op}} \times \mathbf{E} \to \mathbf{Set}$,

$$\mathbf{C}(c, H(d,e)) \cong \mathbf{E}(F(c,d), e) \cong \mathbf{D}(d, G(c,e)).$$

This is called a **two-variable adjunction**.

5.8.26. Define the three-fold pushout-product $f \square g \square h$, and use it to prove that the pushout-product is associative,

$$(f \square g) \square h \cong f \square g \square h \cong f \square (g \square h).$$

5.8.27. Establish the isomorphisms in Example 5.5.9:

$$(S^{m-1} \to D^m) \square (S^{n-1} \to D^n) \cong (S^{m+n-1} \to D^{m+n})$$

$$(S^{m-1} \to D^m) \square (D^n \times \{0\} \to D^n \times I) \cong (D^{m+n} \times \{0\} \to D^{m+n} \times I).$$

5.8.28. Let us say that a map in the arrow category $\mathbf{C}^{\bullet\to\bullet}$ is a "pushout morphism" if, when we think of the map as a commuting square in **C**, it is a pushout square. Prove that $(-)\square g$ preserves pushout morphisms.

5.8.29. Prove that the tensor product $\otimes$ on chain complexes $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$ is a Quillen tensor (Example 5.5.16).

5.8.30. Prove the claim in Remark 5.5.17, that $\otimes$ is a Quillen tensor iff one of the right adjoints satisfies:

$$\mathrm{Hom}_{\square}(C, F) \subseteq F, \quad \mathrm{Hom}_{\square}((W \cap C), F) \subseteq (W \cap F), \quad \mathrm{Hom}_{\square}(C, (W \cap F)) \subseteq (W \cap F).$$

5.8.31. Spell out the remaining cases in the proof of Lemma 5.5.21.

5.8.32. Suppose $\otimes\colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ is any Quillen tensor. In this exercise we show that any tensor product of cofibrations *between cofibrant objects* is a cofibration.

(a) If $f : A \to X$ is a cofibration and Z is cofibrant, prove that $f \otimes \mathrm{id}_Y : A \otimes Z \to X \otimes Z$ is a cofibration.

(b) Using the previous part, prove that if A and B are cofibrant and $f : A \to X$ and $g : B \to Y$ are cofibrations, then $f \otimes g : A \otimes B \to X \otimes Y$ is a cofibration.

(c) Show by counterexample that the previous part fails if A or B is not cofibrant. (Hint: Try assuming that $\mathbf{C} = \mathbf{D} = \mathbf{E}$, the tensor has a unit object $I \in \mathbf{C}$, and that $\mathbf{C}$ contains a non-cofibrant object A.)

5.8.33. Finish the details of the proof that the rational stable model structure exists (Example 5.7.3).

5.8.34. If $\mathbf{I}$ and $\mathbf{J}$ are two based topological categories, define their smash product $\mathbf{I} \wedge \mathbf{J}$ to have objects $(\mathrm{ob}\,\mathbf{I}) \times (\mathrm{ob}\,\mathbf{J})$, and morphisms

$$(\mathbf{I} \wedge \mathbf{J})((a,b),(c,d)) = \mathbf{I}(a,c) \wedge \mathbf{J}(b,d),$$

with the evident composition.

Let $\mathbf{S}$ be the sphere category $\mathbf{S}(m,n) = S^{n-m}$ from Definition 5.3.34. Define a **bispectrum** to be a diagram of based spaces indexed by $\mathbf{S} \wedge \mathbf{S}$. Explain how this is a grid of based spaces $X_{p,q}$, and bonding maps $X_{p,q} \wedge S^1 \to X_{p+1,q}$ and $X_{p,q} \wedge S^1 \to X_{p,q+1}$ that form the evident commuting squares.

5.8.35. (*) Recall from the previous exercise that a bispectrum is a $(\mathbf{S} \wedge \mathbf{S})$-diagram of based spaces. This long exercise develops the properties of bispectra.

(a) For each spectrum X, define a suspension bispectrum $\Sigma^\infty X$ by defining level (p,q) to be $X_p \wedge S^q$. More generally, if X and Y are spectra, explain how the smash products $X_p \wedge Y_q$ form a bispectrum.

(b) Define free bispectra and cellular bispectra. Construct the level model structure on bispectra.

(c) Define stable homotopy groups for bispectra, as the colimit (along both directions!) of the homotopy groups of each level. Define a stable equivalence to be a map inducing isomorphisms on these homotopy groups.

(d) Verify that the proof of the stable model structure from Section 5.6 establishes a stable model structure on bispectra.

(e) Prove that the suspension bispectrum functor $\mathbf{Top}_*^{\mathbf{S}} \to \mathbf{Top}_*^{\mathbf{S}\wedge\mathbf{S}}$ is a Quillen left adjoint.

5.8.36. (*) Prove that the suspension bispectrum functor $\mathbf{Top}_*^{\mathbf{S}} \to \mathbf{Top}_*^{\mathbf{S}\wedge\mathbf{S}}$ from the previous exercise is actually a Quillen *equivalence*.

Conclude that bispectra are Quillen equivalent to spectra. Once we've stabilized once to form spectra, stabilizing a second time doesn't change the theory.

5.8.37. Let $\mathbf{C}$ be any cofibrantly generated model category, and $\mathbf{I}$ any discrete category (no topology on its morphisms). Prove that the category of diagrams $\mathbf{C}^{\mathbf{I}}$ has a projective model structure, as in Theorem 5.3.16. You might need to think a little about how to define the free diagrams $F_i A$ here. (Remember, $\mathbf{C}$ has all coproducts!) Recover the model structures on $\mathbf{Top}^{\mathbf{I}}$ and $\mathbf{Top}_*^{\mathbf{I}}$ as special cases.

5.8.38. This exercise proves a classical result of Crans [Cra95], see also [HKRS17]. Suppose $\mathbf{M}$ is a model category, $\mathbf{D}$ is any category, and we have an adjunction in which $F\colon \mathbf{M} \to \mathbf{D}$ is the left adjoint and $U\colon \mathbf{D} \to \mathbf{M}$ is the right adjoint.

We define the **right-induced model structure** on $\mathbf{D}$ by saying that a map f in $\mathbf{D}$ is a weak equivalence iff Uf is a weak equivalence in $\mathbf{M}$, and similarly f is a fibration in $\mathbf{D}$ iff Uf is a fibration in $\mathbf{M}$. The cofibrations are then defined to be maps that have the left-lifting property with respect to the acyclic fibrations.

Prove that this is a model structure, if we assume that

- $\mathbf{M}$ is cofibrantly generated by sets of maps I and J,
- $F(I)$ and $F(J)$ each satisfy the countable smallness condition in $\mathbf{D}$, and
- $F(J)$-cell complexes are weak equivalences: applying U to them gives weak equivalences in $\mathbf{M}$.

Explain how the model structure on the diagram category $\mathbf{M}^{\mathbf{I}}$ from Exercise 5.8.37 can be obtained in this way.

(The verification of the last point above is sometimes best done one $F(J)$-cell at a time. At other times, it is easier to use the following result: these $F(J)$-cell complexes are all weak equivalences if we can find a fibrant replacement functor on $\mathbf{D}$, and every object $Y \in \mathbf{D}$ has a path object PY and natural maps $Y \to PY \to Y \times Y$ that are a weak equivalence and fibration, respectively, and whose composite is the diagonal map of Y.)

5.8.39. Now let $\mathbf{C} = \mathbf{Sp}$ be the category of spectra, and let $\mathbf{I}$ be any topological category, either based or unbased. Prove that the category of diagrams $\mathbf{Sp}^{\mathbf{I}}$ has a projective model structure, as in Theorem 5.3.16 and Theorem 5.3.40.

Note that a weak equivalence in $\mathbf{Sp}^{\mathbf{I}}$ is a **pointwise stable equivalence**: a map of diagrams of spectra $X \to Y$ such that for each i, the map of spectra $X(i) \to Y(i)$ is a stable equivalence.

(There are many more categories $\mathbf{C}$ for which the category of diagrams $\mathbf{C}^{\mathbf{I}}$ has such a model structure. However, it is difficult to give good, clean, general conditions on $\mathbf{C}$ that make the proof work, specifically the smallness condition we need for the small-object argument to work, as in Exercise 5.8.38. See e.g. [MMSS01] for one example of a general framework – but it does not apply to parametrized spectra!)

5.8.40. Let $\mathbf{I}$ be any unbased topological category, such that the maps $* \to \mathbf{I}(c,c)$ and $\varnothing \to \mathbf{I}(a,b)$ are Quillen cofibrations. (When $\mathbf{I}$ is discrete, this is automatically true.) In this exercise we prove that homotopy colimits along $\mathbf{I}$ exist, both for based spaces and for spectra, and we express them in terms of homotopy colimits for unbased spaces.

(a) Prove that the colimit functors

$$\operatorname{colim}\colon \mathbf{Top}_*^{\mathbf{I}} \to \mathbf{Top}_*$$
$$\operatorname{colim}\colon \mathbf{Sp}^{\mathbf{I}} \to \mathbf{Sp}$$

are left Quillen, using the projective model structure of Theorem 5.3.40, respectively Exercise 5.8.39. (Hint: This is easy, think about the right adjoints!) Conclude that the homotopy colimit, in other words the left-derived functor of the colimit, exists for based spaces and for spectra.

(b) Suppose X is a diagram of based spaces indexed by the unbased category $\mathbf{I}$, as in Lemma 5.3.43, and let $*$ denote the constant diagram on the one-point space. Show that we can find a cellular diagram $QX \xrightarrow{\sim} X$ that contains the cellular diagram $Q* \xrightarrow{\sim} *$ as a subcomplex. Explain why the unbased homotopy colimit of X is the colimit of QX, while the based homotopy colimit of X is the colimit of $QX/Q*$. (You might find Lemma 5.3.45 helpful.)

Conclude that, as in Remark 5.3.46, the based homotopy colimit is obtained as a quotient of unbased homotopy colimits,

$$\operatorname*{hocolim}_{i\in\mathbf{I}}{}^{(b)}\, X(i) \simeq \Big(\operatorname*{hocolim}_{i\in\mathbf{I}}{}^{(u)}\, X(i)\Big)\Big/\Big(\operatorname*{hocolim}_{i\in\mathbf{I}}{}^{(u)}\, (*)\Big).$$

(c) Now suppose X is a diagram of spectra indexed by the unbased category $\mathbf{I}$. Modify the model structure of Exercise 5.8.39 by using *level* equivalences of spectra instead of stable equivalences (see Theorem 5.2.12). Let Q be the cofibrant replacement in the resulting model structure. Observe that QX is also cofibrant in the projective *stable* model structure, and $QX \to X$ is a *stable* equivalence. Therefore the colimit of QX is the homotopy colimit of X.

Finally, check that at each spectrum level $n \geq 0$, QX_n is cofibrant in the projective model structure for based spaces (Theorem 5.3.40), and $QX_n \to X_n$ is an equivalence. Conclude that the homotopy colimit of X in the category of spectra, is computed by taking the homotopy colimit in the category of based spaces, at each spectrum level.

(You can also do part (c) faster using Exercise 5.8.21(a).)

5.8.41. (a) Suppose Z is a topological space, and for every E-local equivalence of CW complexes $X \to Y$, the induced map of homotopy classes $[Y,Z] \longrightarrow [X,Z]$ is a bijection. Prove that Z is fibrant in the E-local model structure on **Top** (Theorem 5.7.7). (Hint: The homotopy extension property is useful here.)

(b) Conversely, if Z is fibrant in the E-local model structure on **Top** from Theorem 5.7.7, prove that for every E-local equivalence of CW complexes $X \to Y$, the induced map of homotopy classes $[Y,Z] \longrightarrow [X,Z]$ is a bijection.[11] (Hint: Consider lifts along $X \to Y$ and along the double mapping cylinder $Y \cup (X \times I) \cup Y \to (Y \times I)$.)

(c) Adapt these arguments to prove that a spectrum Z is fibrant in the E-local model structure on **Sp** iff Z is an Ω-spectrum and Z is E-local in the sense of Definition 3.3.3, in other words E-local equivalences $X \to Y$ give bijections $[Y,Z]_s \to [X,Z]_s$.

5.8.42. (*) [Small-object argument, general case] In this exercise we give the more general version of the small-object argument, found in e.g. [Hov99, §2.1].

Recall that an ordinal number β is a canonical representative for every isomorphism class of well-ordered sets. (See Section 10.4.1 for slightly more detailed review of ordinal numbers.) When we pick a well-ordering for the set S, or equivalently a bijection to an ordinal $\beta \cong S$, we think of this as a "coordinate system" for S. It organizes the elements of S so that we can prove things about S using a transfinite version of induction.

Every ordinal number β is a set whose elements are the ordinals $\{\alpha : \alpha < \beta\}$. We also regard it as a small category with a unique morphism $\alpha \to \gamma$ whenever $\alpha \leq \gamma$. A successor ordinal is one that has a maximum element λ, so the ordinal itself is $(\lambda+1) = \{\alpha : \alpha \leq \lambda\}$. A limit ordinal is one that does not have a maximum element.

Let **C** be a category with all colimits, and let I be any set of maps in **C**.

- Suppose β is an ordinal and there is a diagram indexed by its successor ordinal, $X : (\beta+1) \to \mathbf{C}$. Suppose that for each $\alpha \leq \beta$, if α is a limit ordinal then the diagram induces an isomorphism

$$X_\alpha \cong \operatorname*{colim}_{\gamma<\alpha} X_\gamma.$$

  Then we say that the map $X_0 \to X_\beta$ is a **transfinite composition** of the maps $X_\alpha \to X_{\alpha+1}$ for all $\alpha < \beta$. This is the precise way to make sense of an "infinite composition" of morphisms.

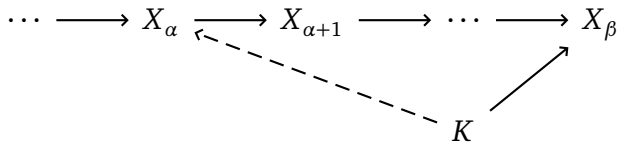
- We replace Definition 5.1.1 with the following: a **transfinite I-cell complex** is any transfinite composition in which each of the maps $X_\alpha \to X_{\alpha+1}$ is a pushout of a single map in I (not a coproduct of maps in I!)

- Recall that an infinite cardinal number κ is **regular** if it cannot be written as a union of $< \kappa$ sets, each of which has cardinality $< \kappa$. (See also Defini-

[11] Equivalently, for any E-local equivalence of spaces whatsoever, we get a bijection on the set of maps in $\mathrm{Ho}\,\mathbf{Top} = \mathbf{Top}\langle w^{-1}\rangle$.

tion 10.4.1.) So for instance, the very first infinite cardinal $\kappa = \aleph_0$ is regular, because a finite union of finite sets must be finite.

- If κ is a regular cardinal, we say that β is **κ-filtered** if every subset of β of cardinality $< \kappa$ has a least upper bound that is still $< \beta$. In other words, β is so big that you can't pick a cofinal subset of β of cardinality $< \kappa$, you can only pick one whose cardinality is $\geq \kappa$.[12] This condition is trivially true when β is a successor ordinal, so we only consider it when β is a limit ordinal.
- We replace Definition 5.1.12 with the following: I satisfies the **smallness condition** if for each map $K \to L$ in the set I, there exists a regular cardinal κ with the following property. For each transfinite I-cell complex $X : (\beta + 1) \to \mathbf{C}$ indexed by a κ-filtered limit ordinal β, every map $K \to X_\beta$ lifts to some X_α for $\alpha < \beta$:



  Furthermore, this lift is eventually unique: if we pick any two lifts $K \to X_{\alpha_1}$ and $K \to X_{\alpha_2}$, the two lifts agree once we map forward to some third space X_{α_3} with $\alpha_3 \geq \alpha_1, \alpha_2$. More succinctly, the following map of sets is a bijection.[13]

$$\operatorname*{colim}_{\alpha<\beta} \mathbf{C}(K, X_\alpha) \to \mathbf{C}(K, X_\beta).$$

  Again, this is intuitively about size. The "size" of K is bounded above by κ, in the sense that for any transfinite cell complex that indexed by a large enough ordinal, K can't stretch out to touch all of its cells.

That's all of the definitions! Now it's your turn:

(a) Assume that every set has a well-ordering, and prove that any transfinite composition of pushouts of *coproducts* of maps in I can be rewritten as a (larger) transfinite composition of pushouts of maps in I.

(b) Use this to prove that every I-cell complex in the classical sense (Definition 5.1.1) is also a transfinite I-cell complex.

(c) This is the easy part! Now that the definitions are set up, adapt the proof of Theorem 5.1.13 and prove the following.

[12] We are differing slightly from [Hov99, §2.1] by asking for sets of cardinality $< \kappa$, not $\leq \kappa$. This is not a huge difference: being $\leq \kappa$ is equivalent to being $< \kappa'$, where κ' is the successor cardinal of κ, and successor cardinals that are infinite are always regular.

[13] We don't actually need the uniqueness statement for the small-object argument to run. Uniqueness only becomes important when we want to construct such lifts for I-cell complexes, by lifting the cells individually and then arguing that the lifts glue together.

Theorem 5.8.1 (Small-object argument, general case)**.** *Suppose* **C** *is a category with all colimits, and I is a set of maps in* **C** *satisfying the smallness condition. Then every map in* **C** *can be factored into a transfinite I-cell complex followed by an I-fibration:*

$$X \xrightarrow{I\text{-cell complex}} X_\beta \xrightarrow{I\text{-fibration}} Y.$$

Furthermore, this factorization is functorial.

(You can assume that any set of cardinals is bounded above by some regular cardinal κ, and that there exists a κ-filtered ordinal β.)

5.8.43. In this exercise we show that for spaces and spectra, the classical and general settings for the small object argument coincide.

(a) When I satisfies the countable smallness condition of Definition 5.1.12, prove the converse to Exercise 5.8.42(b): every transfinite I-cell complex is also a classical I-cell complex. (The main idea is to move the attaching map for each cell to a map that lands in the initial segment ω that corresponds to the natural numbers, and show that this gives the same object of **C** up to isomorphism. To show you can make these choices consistently, try using Zorn's lemma on the set of all ordinals $\alpha \leq \beta$ and rearrangements of $X_0 \to X_\alpha$ into a classical cell complex, where the ordering relation comes from adding cells to one classical cell complex to make a larger one. Make this precise!)

(b) Use Exercises 1.7.31(a) and 2.7.30(a) to prove that for each of the model structures we have constructed on **Top**, $\mathbf{Top}_*$, and **Sp**, the generating cofibrations I and J satisfy the generalized smallness condition from Exercise 5.8.42. (You will need to use part (a) of this exercise as well. You can take $\kappa = \aleph_0$, so being κ-filtered just means that β is a limit ordinal.)

Chapter 6

Construction of the smash product

In this chapter we define the smash product of spectra $\wedge$ explicitly, and show that it has all of the properties we promised in Chapter 4. To do this, we introduce new kinds of spectra called symmetric spectra and orthogonal spectra. Together with ordinary (sequential) spectra, we refer to these as **diagram spectra**, because they can all be defined as diagrams on some based topological category $\mathbf{I}$.

Recall that we defined the smash product in Definition 2.3.23 by picking a sequence of values of p and q and defining $(X \wedge Y)_{p+q} = X_p \wedge Y_q$. As already mentioned, this depends on choices, and different choices give spectra that are not even isomorphic, only equivalent. As a result, it is impossible to prove things like $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$, and define a symmetric monoidal category of spectra.

With symmetric and orthogonal spectra, we can define a smash product that is completely well-defined, and has isomorphisms

$$(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z), \qquad X \wedge Y \cong Y \wedge X, \qquad X \wedge \mathbb{S} \cong X,$$

making spectra into a symmetric monoidal category, just like the category of abelian groups with the tensor product $\otimes$. As we saw in Chapter 4, lots of things we can do with abelian groups then carry over to spectra. For instance, we can define ring spectra as spectra that have multiplication maps $X \wedge X \to X$ that are associative and that have a unit given by a map $\mathbb{S} \to X$.

Moreover, symmetric and orthogonal spectra both form model categories, that are Quillen equivalent to the usual category of sequential spectra. In other words, they model the same homotopy theory. Therefore, we can use them to put a symmetric monoidal structure onto the homotopy theory $\mathrm{Ho}\,\mathbf{Sp}$ we have been studying all along.

Almost all of the material in this chapter comes from the landmark book project [Sch07] and the seminal paper [MMSS01], which in turn draws heavily on [HSS00].

6.1 Symmetric and orthogonal spectra

In this section we introduce new kinds of spectra called symmetric and orthogonal spectra. To differentiate them from the old notion we've been studying them up to this point, we'll sometimes call the old notion **sequential spectra** or **prespectra**, denote the category of such by $\mathbf{Sp}^{\mathbb{N}}$, and denote the free spectra by $F_i^{\mathbb{N}}A$.

6.1.1 Definition and examples

Definition 6.1.1. The symmetric group Σ_n is the group of permutations on n letters. Specifically, if we let $\underline{n} = \{1, 2, \ldots, n\}$, then Σ_n is the group of bijections $\underline{n} \to \underline{n}$, with composition written from right to left.

The orthogonal group $O(n)$ is the group of linear isometries $\mathbb{R}^n \to \mathbb{R}^n$.

Taking the one-point compactification, we get a basepoint-preserving action of $O(n)$ on S^n. The symmetric group $\Sigma_n \leq O(n)$ is the subgroup of permutation matrices, and therefore also acts on S^n by permuting coordinates.

The idea of a symmetric or orthogonal spectrum is that the nth level of the spectrum has an action by Σ_n, respectively $O(n)$, which mimics this action on the n-sphere.

Definition 6.1.2. A **symmetric spectrum** is a spectrum X, with a basepoint-preserving left action of Σ_n on X_n for all $n \geq 0$, with the following condition. Define $\xi_{m,n}$ to be the composite of n bonding maps:

$$\xi_{m,n} : X_m \wedge S^n \xrightarrow{\xi_m \wedge \mathrm{id}} X_{m+1} \wedge S^{n-1} \xrightarrow{\xi_{m+1} \wedge \mathrm{id}} \ldots \xrightarrow{\xi_{m+n-1}} X_{m+n}.$$

Then we require that $\xi_{m,n}$ is $\Sigma_m \times \Sigma_n$-equivariant. Here Σ_n acts on S^n by permuting coordinates, and $\Sigma_m \times \Sigma_n$ acts on X_{m+n} along the inclusion $\Sigma_m \times \Sigma_n \longrightarrow \Sigma_{m+n}$ given by block permutations.

Note that by Example 5.3.44, the based action of Σ_n on X_n can be described as a map out of the smash product

$$(\Sigma_n)_+ \wedge X_n \longrightarrow X_n.$$

A map of symmetric spectra is a map of spectra $X \to Y$ commuting with the Σ_n-action at each level n. We let $\mathbf{Sp}^{\Sigma}$ denote the category of symmetric spectra.

Definition 6.1.3. A **orthogonal spectrum** is a spectrum X, with a continuous basepoint-preserving left action

$$O(n)_+ \wedge X_n \longrightarrow X_n$$

for all $n \geq 0$, so that the composite maps $\xi_{m,n}$ are $O(m) \times O(n)$-equivariant.

A map of orthogonal spectra is a map of spectra $X \to Y$ commuting with the $O(n)$-action at each level n. We let $\mathbf{Sp}^O$ denote the category of orthogonal spectra.

Every orthogonal spectrum is a symmetric spectrum, and every symmetric spectrum is a (sequential) spectrum, by neglect of structure. So we have forgetful functors

$$\mathbf{Sp}^O \underset{U^O_{\mathbb{N}}}{\overset{U^O_\Sigma}{\rightleftarrows}} \mathbf{Sp}^\Sigma \xrightarrow{U^\Sigma_{\mathbb{N}}} \mathbf{Sp}^{\mathbb{N}}.$$

We will shortly define model structures on the categories $\mathbf{Sp}^\Sigma$ and $\mathbf{Sp}^O$, so that these forgetful functors are right Quillen, and give equivalences of homotopy categories

$$\mathrm{Ho}\,\mathbf{Sp}^O \simeq \mathrm{Ho}\,\mathbf{Sp}^\Sigma \simeq \mathrm{Ho}\,\mathbf{Sp}^{\mathbb{N}}.$$

In other words, symmetric and orthogonal spectra are *different* point-set models for the *same* stable homotopy category.

Example 6.1.4. The sphere spectrum $\mathbb{S}$ is an orthogonal spectrum, and therefore also a symmetric spectrum. The group $O(n)$ acts on S^n as in Definition 6.1.1, and so Σ_n acts by permuting the coordinates. In fact, this is the only way to make $\mathbb{S}$ into a symmetric or orthogonal spectrum!

More generally, the suspension spectrum $\Sigma^\infty A$ is an orthogonal spectrum. At level n, $O(n)$ acts on $A \wedge S^n$ by acting just on the sphere S^n and leaving A alone.

Example 6.1.5. We can build a model for the Eilenberg–Mac Lane spectrum HG that is an orthogonal spectrum, and therefore a symmetric spectrum. The idea is to take level n to be the space of configurations of points in S^n, with labels in G. When points collide their labels add, and when a point goes to the basepoint of S^n, it disappears. The orthogonal group acts on the sphere as in Definition 6.1.1. See Section 9.2.3 for more details.

6.1.2 Recasting as diagrams

Recall from Lemma 5.3.35 that a spectrum is the same thing as a based diagram on the category $\mathbf{S}$ from Definition 5.3.34, with one object for each nonnegative integer $n \geq 0$, and morphism spaces the spheres

$$\mathbf{S}(m,n) = \begin{cases} S^{n-m} & \text{when } n \geq m, \\ * & \text{when } n < m. \end{cases}$$

The compositions use the canonical homeomorphisms of Definition 2.1.5 that concatenate coordinates.

To generalize this construction to symmetric spectra, we have to include more morphisms in $\mathbf{S}$, to account for the actions of the symmetric groups.

Definition 6.1.6. Let $\mathbf{S}^\Sigma$ be the based topological category with an object for each integer $n \geq 0$. The morphisms are

$$\mathbf{S}^\Sigma(m,n) = \begin{cases} (\Sigma_n)_+ \wedge_{\Sigma_{n-m}} S^{n-m} & \text{when } n \geq m, \\ * & \text{when } n < m. \end{cases}$$

In particular, we get a wedge of spheres S^{n-m}, one for each element of Σ_n, but then we glue these spheres together along the relation

$$(\sigma\tau, x) \sim (\sigma, \tau x) \qquad \text{for } \sigma \in \Sigma_n,\ \tau \in \Sigma_{n-m},\ x \in S^{n-m}.$$

To define the composition in $\mathbf{S}^\Sigma$, it is easiest to rewrite the morphism space in a different way. Recall from Definition 6.1.1 that $\underline{n} = \{1, 2, \ldots, n\}$. For each subset $A \subseteq \underline{n}$, we let the A-sphere $S^A \subseteq S^n$ be the one-point compactification of $\mathbb{R}^A \subseteq \mathbb{R}^n$. This is a sphere of dimension $|A|$. Note that for disjoint subsets A and B we have

$$S^A \wedge S^B \cong S^{A \amalg B}.$$

For each injective map $\alpha\colon \underline{m} \hookrightarrow \underline{n}$, we can therefore take the sphere on the complement of the image of α,

$$S^{\underline{n} - \alpha(\underline{m})} \cong S^{n-m}.$$

We can now rewrite the morphism space as the wedge sum of these spheres over all the injective maps $\alpha\colon \underline{m} \hookrightarrow \underline{n}$:

$$\mathbf{S}^\Sigma(m,n) = \bigvee_{\alpha\colon \underline{m} \hookrightarrow \underline{n}} S^{\underline{n} - \alpha(\underline{m})} \qquad \text{when } n \geq m.$$

With this rewriting, we can finally compose the maps. For two composable injective maps $\alpha\colon \underline{m} \hookrightarrow \underline{n}$ and $\beta\colon \underline{n} \hookrightarrow \underline{p}$, the complement of the image of $\beta\alpha$ splits into the disjoint union of two sets, the complement of β and the complement of α:

$$\begin{aligned} \underline{p} - \beta\alpha(\underline{m}) &= (\underline{p} - \beta(\underline{n})) \amalg \beta(\underline{n} - \alpha(\underline{m})) \\ &\cong (\underline{p} - \beta(\underline{n})) \amalg (\underline{n} - \alpha(\underline{m})) \end{aligned}$$

Therefore we get a homeomorphism of spheres

$$S^{\underline{p} - \beta\alpha(\underline{m})} \cong S^{\underline{p} - \beta(\underline{n})} \wedge S^{\underline{n} - \alpha(\underline{m})}. \tag{6.1.7}$$

The composition in $\mathbf{S}^\Sigma$ is now defined by composing the injective maps α and β, and using the homeomorphism (6.1.7).

Lemma 6.1.8. *There is an isomorphism of categories* $\mathbf{Top}_*^{\mathbf{S}^\Sigma} \cong \mathbf{Sp}^\Sigma$.

This and the next lemma are left to Exercise 6.6.4.

Definition 6.1.9. Let $\mathbf{S}^O = \mathscr{J}$ be the based topological category with an object for each integer $n \geq 0$. The morphisms are

$$\mathbf{S}^O(m,n) = \begin{cases} O(n)_+ \wedge_{O(n-m)} S^{n-m} & \text{when } n \geq m, \\ * & \text{when } n < m. \end{cases}$$

Again we rewrite this composition in a different way. For each inclusion of inner product spaces $i\colon V \subseteq W$, let $W - i(V)$ refer to the orthogonal complement of V in W, and S^{W-V} its one-point compactification. For each pair of integers $m, n \geq 0$, let $O(m,n)$ be the space of all linear isometries $\mathbb{R}^m \to \mathbb{R}^n$. That is, all linear maps that preserve the standard inner product, and are therefore injective. Note that $O(n,n) = O(n)$.

There is a vector bundle $E(m,n) \to O(m,n)$ whose fiber over each embedding $i\colon \mathbb{R}^m \to \mathbb{R}^n$ is the orthogonal complement $\mathbb{R}^n - i(\mathbb{R}^m)$. Taking the one-point compactification of $E(m,n)$, in other words its Thom space, gives the space $\mathbf{S}^O(m,n)$.

Intuitively, this is a "wedge sum" of spheres S^{n-m}, but the spheres are indexed by the space $O(m,n)$, so it is not really a wedge sum. Instead, the spheres are continuously glued together.

For any three integers $m \leq n \leq p$, there is a product map

$$E(m,n) \times E(n,p) \longrightarrow E(m,p)$$

that composes the embeddings $i\colon \mathbb{R}^m \to \mathbb{R}^n$ and $j\colon \mathbb{R}^n \to \mathbb{R}^p$, and identifies each pair of points $y \in \mathbb{R}^p - j(\mathbb{R}^n)$ and $x \in \mathbb{R}^n - i(\mathbb{R}^m)$ with the point $y + j(x) \in \mathbb{R}^p - ji(\mathbb{R}^m)$. Passing to one-point compactifications gives the composition map

$$\mathbf{S}^O(m,n) \wedge \mathbf{S}^O(n,p) \longrightarrow \mathbf{S}^O(m,p).$$

Lemma 6.1.10. *There is an isomorphism of categories* $\mathbf{Top}_*^{\mathbf{S}^O} \cong \mathbf{Sp}^O$.

There is also a more "coordinate free" way of describing $\mathbf{S}^O$. Let $\mathscr{J}$ be the larger category with one object for each finite-dimensional inner product space V. To be definite, we'll only take those V that are subspaces of some fixed countable-dimensional inner product space $\mathscr{U}$. Then we define $O(V,W)$ to be the space of all linear isometric injective maps $i\colon V \to W$, and $E(V,W) \to O(V,W)$ the vector bundle whose fiber over i is the orthogonal complement $W - i(V)$. Finally, let $\mathscr{J}(V,W)$ be the Thom space of this vector bundle.

By construction, there is an inclusion of based topological categories $\mathbf{S}^O \to \mathscr{J}$. In fact, every object in $\mathscr{J}$ is isomorphic to something in $\mathbf{S}^O$. It follows that:

Lemma 6.1.11. *There is an equivalence of categories* $\mathbf{Top}_*^{\mathscr{J}} \simeq \mathbf{Sp}^O$.

That is, a $\mathscr{J}$-diagram of spaces always gives an orthogonal spectrum, and every orthogonal spectrum can have its structure extended to a $\mathscr{J}$-diagram. Going back and forth gives the same $\mathscr{J}$-diagram up to canonical isomorphism, but not literally the same diagram on the nose. This point of view is common in the equivariant setting, because then we take V to range over all finite-dimensional orthogonal G-representations.

6.1.3 Cellular spectra and the level model structure

By Theorem 5.3.40, there is a **level model structure** on the category of symmetric spectra $\mathbf{Sp}^\Sigma$, and also on the category of orthogonal spectra $\mathbf{Sp}^O$.

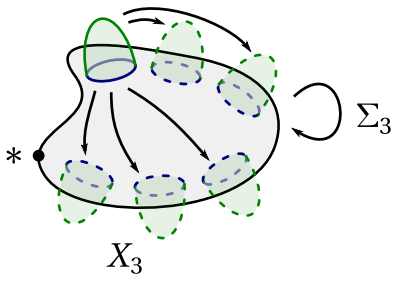
The weak equivalences in this model structure are the level equivalences, the maps of spectra $X \to Y$ that are weak equivalences on each level, $X_n \xrightarrow{\sim} Y_n$. Similarly, the fibrations are the maps of spectra $X \to Y$ that are fibrations at each level.

To describe the cofibrations, recall from Definition 5.3.36 that for each $m \geq 0$ and based space A, we can make a free symmetric spectrum on A at level m. We call this spectrum $F_m^\Sigma A$. At level n, it is the smash product

$$(F_m^\Sigma A)(n) = A \wedge \mathbf{S}^\Sigma(m,n) \simeq \bigvee_{\alpha\colon \underline{m} \hookrightarrow \underline{n}} (A \wedge S^{\underline{n}-\alpha(\underline{m})}).$$

A **cellular symmetric spectrum** is a spectrum built from $*$ by successively attaching "cells" of the form $F_m^\Sigma S_+^{n-1} \to F_m^\Sigma D_+^n$. These objects, and their retracts, are the cofibrant objects in the level model structure. The cofibrations are the retracts of the relative cell complexes.

Intuitively, this is like the notion of a cellular spectrum from Definition 2.6.2 and Proposition 2.6.11. The difference is that, when we attach a cell at level m, we not only have to attach its suspensions to the higher levels, but we *also* have to attach extra copies of the same cell so that the symmetric group actions Σ_n have somewhere to send them.



So if we attach a cell D_+^n at level 3, we have to attach 6 copies of that cell, so that Σ_3 can act by permuting them. If we only attached one, there would be no Σ_3 action on the result! Then we have to attach 24 copies of its suspension ΣD_+^n at level 4, with Σ_4 permuting them. Then 60 copies of $\Sigma^2 D_+^n$ at level 5, with Σ_5 acting by permuting them, and also flipping the two suspension coordinates on each of them. At level p, we attach as many copies of $\Sigma^{p-3} D_+^n$ as there are injective maps $\underline{3} \to \underline{p}$. The action of Σ_p permutes these cells and also their $(p-3)$ suspensions.

For orthogonal spectra $\mathbf{Sp}^O$, we get the same situation, except that the free spectra $F^O_m A$ now look like

$$(F^O_m A)(n) = A \wedge \mathbf{S}^O(m,n).$$

So when we attach a cell at level m, we also have to attach its image under all of the *orthogonal* group actions at each level. This is quite complicated to think about geometrically. We encourage the reader to think about it more formally instead.

Proposition 6.1.12. *Using the level model structure, the forgetful functors*

$$\mathbf{Sp}^{\mathbb{N}} \underset{U^O_{\mathbb{N}}}{\overset{U^\Sigma_{\mathbb{N}}}{\leftarrow}} \mathbf{Sp}^\Sigma \overset{U^O_\Sigma}{\leftarrow} \mathbf{Sp}^O$$

are right Quillen.

Proof. This is immediate – they clearly preserve both weak equivalences and fibrations. □

We call the left adjoint of each of these functors **prolongation**. Prolongation is a special case of the left Kan extension from Section 5.4.3. We denote it as

$$\mathbf{Sp}^{\mathbb{N}} \underset{P^O_{\mathbb{N}}}{\overset{P^\Sigma_{\mathbb{N}}}{\rightarrow}} \mathbf{Sp}^\Sigma \overset{P^O_\Sigma}{\rightarrow} \mathbf{Sp}^O,$$

but you could also denote it as "extension of scalars,"

$$P^\Sigma_{\mathbb{N}} X = X \wedge_{\mathbf{S}^{\mathbb{N}}} \mathbf{S}^\Sigma, \qquad P^O_\Sigma X = X \wedge_{\mathbf{S}^\Sigma} \mathbf{S}^O, \qquad P^O_{\mathbb{N}} X = X \wedge_{\mathbf{S}^{\mathbb{N}}} \mathbf{S}^O.$$

These can be written very explicitly using coequalizers. For instance, the prolongation of a sequential spectrum X to a symmetric spectrum is written as the coequalizer

$$\left(\bigvee_{m_0, m_1 \geq 0} X_{m_0} \wedge \mathbf{S}^{\mathbb{N}}(m_0, m_1) \wedge \mathbf{S}^\Sigma(m_1, n) \right) \rightrightarrows \left(\bigvee_{m \geq 0} X_m \wedge \mathbf{S}^\Sigma(m,n) \right) \longrightarrow (P^\Sigma_{\mathbb{N}} X)_n \quad (6.1.13)$$

in which the two maps multiply $\mathbf{S}^{\mathbb{N}}(m_0, m_1)$ into either X_{m_0} or $\mathbf{S}^{\mathbb{N}}(m_1, n)$, as in (5.4.15).

The formula (6.1.13) looks pretty complicated, but here is a simpler way to think about it. By composition of adjoints, the prolongation of a free spectrum is always a free spectrum:

$$P^\Sigma_{\mathbb{N}}(F^{\mathbb{N}}_n A) \cong F^\Sigma_n A, \qquad P^O_{\mathbb{N}}(F^{\mathbb{N}}_n A) \cong F^O_n A, \qquad P^O_\Sigma(F^\Sigma_n A) \cong F^O_n A.$$

Prolongation also commutes with the formation of cell complexes, and preserves weak equivalences between cell complexes, because it is left Quillen. So, if we think of "cell complexes" abstractly, then when we use prolongation to move between these different categories, the cell complexes don't change.

Therefore, if we replace all of our spectra by cellular spectra, then prolongation essentially does nothing, it just changes the exact meaning of the word "cell" as we pass from one category to another.

Example 6.1.14. The prolongation of a suspension spectrum is always a suspension spectrum on the same space. As a special case, the prolongation of $\mathbb{S}$ is $\mathbb{S}$ again.

Example 6.1.15. The prolongation of the spectrum $F_1^{\mathbb{N}}A$ to symmetric spectra is the symmetric spectrum $F_1^{\Sigma}A$. At level n, it is a wedge sum of n copies of the suspension $\Sigma^{n-1}A$, whereas the original spectrum F_1A had a single copy of $\Sigma^{n-1}A$ at each spectrum level.

For any space A and symmetric or orthogonal spectrum X, we define the tensor $A \wedge X$ just as in Definition 2.3.6, by smashing each level of X with A. This produces a symmetric or orthogonal spectrum in which Σ_n or $O(n)$ is acting on the X_n inside $A \wedge X_n$ and leaving the A alone.

Lemma 6.1.16. *This defines Quillen tensors*

$$\wedge\colon \mathbf{Top}_* \times \mathbf{Sp}^{\Sigma} \to \mathbf{Sp}^{\Sigma}$$

$$\wedge\colon \mathbf{Top}_* \times \mathbf{Sp}^{O} \to \mathbf{Sp}^{O}$$

using the level model structure on spectra.

Proof. This is just a special case of Example 5.5.15. □

The right adjoints give us the space of maps $\mathrm{Map}_*(X,Y)$ between two diagram spectra, and the cotensor $F(A,X)$. The cotensor just applies $F(A,-)$ at each spectrum level, as in Definition 2.3.8. The space of maps is the subspace of the product $\prod_{n\geq 0}\mathrm{Map}_*(X_n,Y_n)$ of those maps that commute with both the bonding maps and the symmetric or orthogonal group actions.

We can similarly define the homotopy cofiber Cf and homotopy fiber Ff of any map of symmetric or orthogonal spectra, by doing the same construction we did for sequential spectra in Example 2.3.19 and Example 2.3.22. The result has a canonical Σ_n or $O(n)$-action, so that it is a symmetric or orthogonal spectrum.

Remark 6.1.17. More generally, the colimit of any diagram of symmetric or orthogonal spectra can be defined by taking the colimit of the underlying spectra. The Σ_n and $O(n)$ actions on the individual terms define an action on the colimit as well. In particular, we can take wedge sums $X \vee Y$ and pushouts $X \cup_A Y$ of symmetric or orthogonal spectra.

Similarly, we can take any limit, or homotopy colimit or limit, by doing the same construction to the underlying spectrum. For homotopy limits, we do want to restrict attention to Ω-spectra when doing this. We'll see a little later how to replace each symmetric or orthogonal spectrum by an equivalent Ω-spectrum.

6.1.4 π_*-isomorphisms and stable equivalences

Recall that a "diagram spectrum" is either a sequential spectrum in $\mathbf{Sp}^{\mathbb{N}}$, a symmetric spectrum in $\mathbf{Sp}^{\Sigma}$, or an orthogonal spectrum in $\mathbf{Sp}^{O}$. Every diagram spectrum X has an underlying sequential spectrum, and therefore has stable homotopy groups $\pi_*(X)$, defined as in Definition 2.1.2.

Definition 6.1.18. A **π_*-isomorphism** of diagram spectra is a map $X \to Y$ inducing an isomorphism $\pi_*(X) \xrightarrow{\cong} \pi_*(Y)$.

We might be tempted to define a stable equivalence to be a π_*-isomorphism, as we did in $\mathbf{Sp}^{\mathbb{N}}$. However, if we did that in symmetric spectra $\mathbf{Sp}^{\Sigma}$, then the forgetful functor on homotopy categories

$$U^{\Sigma}_{\mathbb{N}} \colon \operatorname{Ho}\mathbf{Sp}^{\Sigma} \longrightarrow \operatorname{Ho}\mathbf{Sp}^{\mathbb{N}}$$

would *not* be an equivalence of categories. In other words, symmetric spectra would not be a valid model for the stable homotopy category.

To give a heuristic for why this fails, suppose that $U^{\Sigma}_{\mathbb{N}}$ is an equivalence of homotopy categories $\operatorname{Ho}\mathbf{Sp}^{\Sigma} \simeq \operatorname{Ho}\mathbf{Sp}^{\mathbb{N}}$. It turns out that the left adjoint $P^{\Sigma}_{\mathbb{N}}$ is left-deformable.[1] But this means that the derived left adjoint $\mathbb{L}P^{\Sigma}_{\mathbb{N}}$ must be an equivalence of categories as well. If we take the free sequential spectrum on the circle, $F^{\mathbb{N}}_1 S^1$, and move it back and forth through this equivalence, we get the free symmetric spectrum on the circle, $F^{\Sigma}_1 S^1$, regarded as a sequential spectrum.

So if this were an equivalence of categories, then the two sequential spectra $F^{\mathbb{N}}_1 S^1$ and $F^{\Sigma}_1 S^1$ would be isomorphic in $\operatorname{Ho}\mathbf{Sp}^{\mathbb{N}}$. Therefore, they would have the same homotopy groups. However, we can see directly that

$$\pi_0(F^{\Sigma}_1 S^1) \cong \bigoplus_{n\geq 0} \mathbb{Z},$$
$$\pi_0(F^{\mathbb{N}}_1 S^1) \cong \mathbb{Z},$$

and we have our contradiction. (See also Exercise 6.6.7 for a different proof.)

This is a pickle! To get around this problem, we take motivation from the idea of left Bousfield localization. We *want* $F^{\Sigma}_1 S^1$ to have the correct homotopy type, the same as $F^{\Sigma}_0 S^0 = \mathbb{S}$. There is a map of symmetric spectra $F^{\Sigma}_1 S^1 \to F^{\Sigma}_0 S^0$ that should be the equivalence. So, we just *formally* declare that this map is an equivalence, and see what class of stable equivalences this gives us in the category $\mathbf{Sp}^{\Sigma}$.

How will we characterize such maps? Well, maps of the form $F^{\Sigma}_1 S^1 \to Z$ correspond to maps of based spaces $S^1 \to Z_1$. Maps $F^{\Sigma}_0 S^0 \to Z$ correspond to based maps $S^0 \to Z_0$. If Z is an Ω-spectrum, then these are equivalent, because $Z_0 \simeq \Omega Z_1$. This gives us an idea: define the stable equivalences using maps to Ω-spectra.

[1] We skip the proof of this since it takes us too far afield and we are only trying to illustrate the problem.

Definition 6.1.19. A **stable equivalence** of diagram spectra is a map $f : X \to Y$ such that, for any diagram spectrum Z that is also an Ω-spectrum, the induced map

$$(-) \circ f : \ [Y, Z]_\ell \longrightarrow [X, Z]_\ell$$

is a bijection. Here $[-,-]_\ell$ means morphisms in the homotopy category of diagram spectra, with the level equivalences inverted.

In other words, if we replace X and Y by level-equivalent cellular spectra QX and QY, a map $f : X \to Y$ is a stable equivalence if it induces a bijection on homotopy classes of maps

$$(-) \circ Qf : \ [QY, Z]_h \xrightarrow{\cong} [QX, Z]_h.$$

Example 6.1.20. In sequential spectra $\mathbf{Sp}^{\mathbb{N}}$, a map is a stable equivalence iff it is a π_*-isomorphism. To see this, notice that $X \to Y$ is a π_*-isomorphism iff it induces a bijection on $[-, Z]_s$ for all Z. (See Exercises 3.6.4 and 3.6.7.) Without loss of generality Z is an Ω-spectrum. Applying Theorem 5.2.27 twice, we see that stable maps $X \to Z$ are the same as the homotopy classes of maps $QX \to Z$, which are the same as maps in the level homotopy category $X \to Z$.

We will also see shortly that in orthogonal spectra $\mathbf{Sp}^O$, a map is a stable equivalence iff it is a π_*-isomorphism. So symmetric spectra are to blame for the extra complexity in Definition 6.1.19. However, symmetric spectra are very useful, because for instance orthogonal spectra can't be defined in the motivic setting. So we will bite the bullet and develop the theory in a way that works for symmetric spectra too.

Example 6.1.21. As discussed before, the canonical map $F_1^\Sigma S^1 \to F_0^\Sigma S^0$ is a stable equivalence of symmetric spectra, even though it is not a π_*-isomorphism. Similarly, for any well-based space A the canonical maps

$$F_{i+j}\Sigma^i A \longrightarrow F_j A$$

are all stable equivalences, in any of the three categories of diagram spectra $\mathbf{Sp}^{\mathbb{N}}$, $\mathbf{Sp}^\Sigma$, or $\mathbf{Sp}^O$. Therefore the maps $k_{i,j}$ in Definition 5.6.3 are all stable equivalences.

Our goal in the remainder of the section is to prove that every π_*-isomorphism is a stable equivalence. We restrict attention to symmetric spectra since the argument for orthogonal spectra is identical.

Definition 6.1.22. Let X be any symmetric spectrum. We define symmetric spectra ΣX and $\mathrm{sh} X$ as

$$(\Sigma X)_n = \Sigma X_n = S^1 \wedge X_n$$
$$(\mathrm{sh} X)_n = X_{1+n}$$

with the same bonding maps as X. We write $(1+n)$ instead of $(n+1)$ in the second subscript to indicate that we take the Σ_n-action on X_{1+n} that arises from the inclusion $\Sigma_n \leq \Sigma_{1+n}$ as the permutations on the *last* n letters, not the first n letters. This is necessary to actually make $\mathrm{sh}X$ a symmetric spectrum, since the bonding maps will add more elements to the right-hand side of the set $\{1, 2, \ldots, n+1\}$, not the left.

We define a map of symmetric spectra

$$\lambda_* \colon \Sigma X \to \mathrm{sh}X$$

by sending $S^1 \wedge X_n \to X_{1+n}$ by applying the bonding map $\xi_n \colon X_n \wedge S^1 \to X_{n+1}$, then applying the action of the block permutation $\tau_{n,1} \in \Sigma_{n+1}$ that moves the first n elements past the last element. Intuitively, the permutation $\tau_{n,1}$ is "correcting" for the fact that we had to swap the S^1 past the X_n to define the bonding map.

It is an exercise that λ_* is a map of symmetric spectra, and that it would fail to be a map of symmetric spectra if we didn't include $\tau_{n,1}$ (Exercise 6.6.9). You need to use the equivariance condition in Definition 6.1.2 when checking this!

Proposition 6.1.23. *If X is an Ω-spectrum, then the adjoint map $\lambda^* \colon X \to \Omega\,\mathrm{sh}X$ is a level equivalence.*

Proof. At each level it is the composite

$$X_n \xrightarrow[\sim]{\widetilde{\xi}_n} \Omega X_{n+1} \xrightarrow[\cong]{\Omega\tau_{n,1}} \Omega X_{1+n}.$$

The first map is an equivalence because X is an Ω-spectrum. The second map is a homeomorphism because it has an inverse. Therefore their composite is a weak equivalence. □

Let LX be the based homotopy colimit (reduced mapping telescope) of iterations of the map λ^*.

Proposition 6.1.24. *There is a natural isomorphism $\pi_{k+n}((LX)_n) \cong \pi_k(X)$.*

Proof. If λ^* consisted only of the adjoint maps $\widetilde{\xi}_n$, the colimit would be the spectrum RX from Proposition 2.2.9, and so this would be true because RX is an Ω-spectrum:

$$\pi_k(X) \cong \pi_k(RX) \cong \pi_{n+k}((RX)_n).$$

The full truth is that we added some isomorphisms $\tau_{n,1}$ into the colimit system. However, this does not change the colimit up to isomorphism, so the colimit group $\pi_{n+k}((LX)_n)$ is still isomorphic to $\pi_k(X)$. Interestingly, LX may fail to be an Ω-spectrum, even though $\pi_{k+n}((LX)_n)$ is abstractly isomorphic to $\pi_{k+n+1}((LX)_{n+1})$. □

Proposition 6.1.25. *Every π_*-isomorphism $X \to Y$ is a stable equivalence.*

Proof. Since homotopy colimits preserve equivalences of based spaces, L preserves level equivalences, so it induces a map on the level homotopy category $[X,Z]_\ell \to [LX,LZ]_\ell$. If we let $\iota : X \to LX$ denote the natural map, then for every map of spectra $f : X \to Z$ where Z is an Ω-spectrum, we have a commuting square

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\iota} \\ LX & \xrightarrow[Lf]{} & LZ, \end{array}$$

and therefore a commuting diagram

$$\begin{array}{ccc} [LX,Z]_\ell & & [X,Z]_\ell \\ {\scriptstyle\cong}\downarrow{\scriptstyle\iota\circ -} & \swarrow{\scriptstyle L} & {\scriptstyle\cong}\downarrow{\scriptstyle\iota\circ -} \\ [LX,LZ]_\ell & \xrightarrow[-\circ\iota]{} & [X,LZ]_\ell. \end{array}$$

The isomorphisms are because Z is an Ω-spectrum, and so by applying Proposition 6.1.23 and taking the colimit, the map $\iota : Z \to LZ$ is a level equivalence.

We conclude that the set $[X,Z]_\ell$ is naturally a retract of the set $[LX,Z]_\ell$. Now suppose $f : X \to Y$ is a π_*-isomorphism. By Proposition 6.1.24, Lf is a level equivalence, so it induces an isomorphism $(-\circ Lf) : [LY,Z]_\ell \xrightarrow{\cong} [LX,Z]_\ell$. Since $(-\circ f) : [Y,Z]_\ell \to [X,Z]_\ell$ is a retract of this isomorphism, it is also an isomorphism. Therefore f is a stable equivalence. □

We round out this section with the observation:

Lemma 6.1.26. *Between Ω-spectra, every stable equivalence is a level equivalence.*

Proof. This is a simple application of the Yoneda lemma, to the category of Ω-spectra with level equivalences inverted. □

To summarize, we have implications

$$\text{level equivalence} \;\Rightarrow\; \pi_*\text{-isomorphism} \;\Rightarrow\; \text{stable equivalence}$$

and for Ω-spectra, all three are equivalent.

6.1.5 Stability theorems

Since our definition of stable equivalence has changed, we need to re-prove some of the theorems from the first few chapters of the book, for this new notion of stable equivalence.

It will be useful to prove some of these theorems for maps of spectra that have the homotopy extension property. There are three closely related versions of this:

Definition 6.1.27. Let $f : X \to Y$ be a map of diagram spectra. By smashing these spectra with spaces, we can form another map

$$(X \wedge I_+) \cup_{(X \wedge \{0\}_+)} (Y \wedge \{0\}_+) \longrightarrow Y \wedge I_+. \tag{6.1.28}$$

We can also do this at each spectrum level separately:

$$(X_n \wedge I_+) \cup_{(X_n \wedge \{0\}_+)} (Y_n \wedge \{0\}_+) \longrightarrow Y_n \wedge I_+. \tag{6.1.29}$$

Or, we could use Cartesian products at each spectrum level, giving the simpler map

$$(X_n \times I) \cup_{(X_n \times \{0\})} (Y_n \times \{0\}) \longrightarrow Y_n \times I. \tag{6.1.30}$$

We say that f is an **h-cofibration of spectra** if the first of these maps (6.1.28) has a retract in the category of spectra. So there is a map of spectra from $Y \wedge I_+$ back to the domain of this map that induces the identity on the domain. Put another way, any map of spectra $Y \to Z$ and a homotopy of that map on X, extends to a homotopy of spectra on all of Y.

Similarly we say that f is a **levelwise based h-cofibration** if the second map (6.1.29) has a retract, and a **levelwise unbased h-cofibration** if the third map (6.1.30) has a retract. In other words, each of the maps $X_n \to Y_n$ has the homotopy extension property, either for based maps or for unbased maps.

By Exercise 6.6.15, we have the implications

$$\begin{array}{ccc} \text{free (stable) cofibration} & \Longrightarrow & \text{levelwise unbased } h\text{-cofibration} \\ \Downarrow & & \Downarrow \\ h\text{-cofibration} & \Longrightarrow & \text{levelwise based } h\text{-cofibration,} \end{array}$$

so the levelwise based h-cofibrations are the most general of the four notions.

We are now ready to start proving the stability theorems.

Theorem 6.1.31. *1. The map f is a stable equivalence iff Σf is a stable equivalence.*

2. For stable equivalences $X_\alpha \to Y_\alpha$, the wedge sum $\bigvee_\alpha X_\alpha \to \bigvee_\alpha Y_\alpha$ is a stable equivalence.

3. *A map $f : X \to Y$ is a stable equivalence iff the homotopy cofiber Cf is stably equivalent to $*$.*
4. *If $A \to X$ is a stable equivalence and either $A \to X$ or $A \to Y$ is a levelwise based h-cofibration, the pushout map $Y \to X \cup_A Y$ is also a stable equivalence.*
5. *If we have a sequence of maps $X^{(n)} \to X^{(n+1)}$ that are stable equivalences and levelwise based h-cofibrations, their sequential composition is also a stable equivalence.*
6. *A retract of a stable equivalence is a stable equivalence.*

It follows that (4) and (5) are also true for any of the above four notions of "cofibration."

We also get several more corollaries, since we already know how π_* behaves on the underlying sequential spectra. For example:

Corollary 6.1.32. *1. Finite wedges are stably equivalent to finite products (because they are π_*-isomorphic).*
2. *The map f is a stable equivalence iff Σf is a stable equivalence (because $X \to \Omega\Sigma X$ is a π_*-isomorphism).*
3. *The map f is a stable equivalence iff Ωf is a stable equivalence (because $\Sigma\Omega X \to X$ is a π_*-isomorphism).*
4. *There is a stable equivalence $\Sigma F f \xrightarrow{\sim} Cf$ (because it is a π_*-isomorphism).*
5. *The map f is a stable equivalence iff Ff is stably equivalent to $*$.*

Proof of Theorem 6.1.31. The key idea is to think about the properties of the functor $[-, Z]_\ell$ for any Ω-spectrum Z. Since Z is an Ω-spectrum, the construction of Example 3.2.9 makes $[X, Z]_\ell$ into an abelian group in a natural way. This will be important for the steps below that form exact sequences out of these abelian groups.

1. There are natural isomorphisms
$$[\Sigma X, Z]_\ell \cong [X, \Omega Z]_\ell,$$
$$[X, Z]_\ell \cong [X, \Omega \operatorname{sh} Z]_\ell \cong [\Sigma X, \operatorname{sh} Z]_\ell$$
using Proposition 6.1.23. Also, the operations Ω and sh preserve Ω-spectra. It follows that f induces an isomorphism on $[-, Z]_\ell$ iff it induces an isomorphism on $[\Sigma(-), Z]_\ell$ for all Ω-spectra Z.
2. The proof of Proposition 3.2.2 gives a natural isomorphism
$$\left[\bigvee_\alpha X_\alpha, Z\right]_\ell \cong \prod_\alpha [X, Z]_\ell,$$
so a wedge sum of stable equivalences is a stable equivalence.

3. We define a long exact sequence of abelian groups

$$[\Sigma Cf, Z]_\ell \longrightarrow [\Sigma Y, Z]_\ell \xrightarrow{-\circ\Sigma f} [\Sigma X, Z]_\ell \longrightarrow [Cf, Z]_\ell \longrightarrow [Y, Z]_\ell \xrightarrow{-\circ f} [X, Z]_\ell$$

just as in Proposition 3.2.20. If f is a stable equivalence then f and Σf induce bijections, so $[Cf, Z]_\ell = 0$, so Cf is stably equivalent to zero. Conversely, if Cf is stably equivalent to zero then so is ΣCf. By the exact sequence, Σf is a stable equivalence, and therefore f is a stable equivalence.

4. The cofibration assumption implies that the pushout $X \cup_A Y$ is level equivalent to the homotopy pushout $X \cup^h_A Y$, in other words the based double mapping cylinder $X \cup_A (A \wedge I_+) \cup_A Y$ at each spectrum level. (The based version of the homotopy extension property allows us to define a homotopy inverse from $X \cup_A Y$ at each spectrum level separately.)

   As in Lemma 2.4.16, the homotopy cofiber of $f : A \to X$ is level equivalent to the homotopy cofiber of $\bar{f} : Y \to X \cup_A Y$, because each one is level equivalent to its corresponding strict cofiber, and the strict cofibers are homeomorphic. So if f is a stable equivalence, Cf is stably equivalent to zero, hence $C\bar{f}$ is stably equivalent to zero, so $\bar{f}$ is a stable equivalence.

5. As in Proposition 3.2.21, we get a $\lim^1$ exact sequence of abelian groups

$$0 \longrightarrow \lim{}^1 [\Sigma X^{(n)}, Z]_\ell \longrightarrow [X^{(\infty)}, Z]_\ell \longrightarrow \lim [X^{(n)}, Z]_\ell \longrightarrow 0.$$

   If all of the maps $[X^{(n)}, Z]_\ell \to [X^{(0)}, Z]_\ell$ are bijections, the $\lim^1$ term vanishes and the map $[X^{(\infty)}, Z]_\ell \to [X^{(0)}, Z]_\ell$ is therefore a bijection.

6. A retract of an isomorphism on $[-, Z]_\ell$ is an isomorphism, so a retract of a stable equivalence is a stable equivalence.

□

Remark 6.1.33. The reader might be disturbed that we used such a weak notion of "cofibration" in the proof of part (4). In general, if a map of spaces $f : A \to X$ has the based homotopy extension property, then its cofiber X/A and its mapping cone Cf are homotopy equivalent, but they have the "wrong" homotopy type (unless we also ask for A and X to be well-based). What's going on in spectra is that the mapping cone has the "wrong" homotopy type at each level, but it still manages to have the "right" homotopy type stably, because the proof of Lemma 2.4.11 never uses any assumption about being well-based. This is a well-known phenomenon, see e.g. [MMSS01, Thm 7.4]. The reader is warned, however, that this does not work for every flavor of spectra that we know about. For instance, it doesn't work for parametrized spectra [MS06, Mal23].

The above theorems are all we need to get the stable model structure, but it's good to know that the other stability theorems carry over as well. For instance:

Proposition 6.1.34. *A square of diagram spectra is a homotopy pushout iff it is a homotopy pullback.*

Proof. We follow the earlier proof from Chapter 2. To start, it follows from Theorem 6.1.31(3) that a square of diagram spectra is homotopy pushout iff the map of homotopy cofibers of the horizontal maps, $Cf \to Cg$, is a stable equivalence. (In case you didn't do Exercise 2.7.21, the trick is to show that the homotopy cofiber of the map $Cf \to Cg$ and the homotopy cofiber of the map $B \cup^h_A C \to D$ are isomorphic as spectra.) Similarly it follows from Corollary 6.1.32(5) that a square is homotopy pullback iff the map of homotopy fibers $Ff \to Fg$ is a stable equivalence.

By Corollary 6.1.32(4), $Cf \to Cg$ is a stable equivalence iff $Ff \to Fg$ is a stable equivalence, and we are done. □

We also get the variant of Corollary 2.4.25 for symmetric and orthogonal spectra. This will be useful for proving the "monoid axiom" later in the chapter.

Corollary 6.1.35. *If $f : X \to Y$ is a stable equivalence of diagram spectra, with no conditions on basepoints, and K is a based CW complex, then* $\mathrm{id} \wedge f : K \wedge X \to K \wedge Y$ *is also a stable equivalence.*

Proof. As in the proof of Corollary 2.4.25, we let $K^{(n)}$ be the nth level of the skeletal filtration of K, and we show by induction on n that

$$\mathrm{id} \wedge f : K^{(n)} \wedge X \to K^{(n)} \wedge Y$$

is a stable equivalence. For $n = 0$, this follows from Theorem 6.1.31(2). For larger n, smash the cofiber sequence of spaces

$$K^{(n-1)} \xrightarrow{i} K^{(n)} \longrightarrow Ci \cong \bigvee_\alpha S^n$$

with the spectra X and Y to get a map of cofiber sequences of spectra

$$\begin{array}{ccccc} K^{(n-1)} \wedge X & \xrightarrow{i\wedge\mathrm{id}} & K^{(n)} \wedge X & \longrightarrow & \bigvee_\alpha \Sigma^n X \\ \sim\downarrow{\scriptstyle \mathrm{id}\wedge f} & & \downarrow{\scriptstyle \mathrm{id}\wedge f} & & \sim\downarrow{\scriptstyle \Sigma^n f} \\ K^{(n-1)} \wedge Y & \xrightarrow{i\wedge\mathrm{id}} & K^{(n)} \wedge Y & \longrightarrow & \bigvee_\alpha \Sigma^n Y. \end{array}$$

Applying the functor $[-, Z]_\ell$ for any Ω-spectrum Z gives long exact sequences of abelian groups. The vertical maps on the left and right give isomorphisms of groups, so therefore by

the 5-lemma we get an isomorphism in the middle as well, showing that $K^{(n)} \wedge X \to K^{(n)} \wedge Y$ is a stable equivalence.

Now that we know that $K^{(n)} \wedge X \to K^{(n)} \wedge Y$ is a stable equivalence for all $n \geq 0$, we use Theorem 6.1.31(3) to say that each of the spectra $K^{(n)} \wedge Cf$ is stably contractible. Taking the colimit over n gives $K \wedge Cf$, which is stably contractible by Theorem 6.1.31(5). Using Theorem 6.1.31(3) one more time, we conclude that $K \wedge X \to K \wedge Y$ is a stable equivalence. □

6.1.6 The stable model structure

Now that we have proven the stability theorems, we have all the ingredients we need to give symmetric and orthogonal spectra a model structure with the stable equivalences.

Theorem 6.1.36 (Hovey–Shipley–Smith)**.** *The category of symmetric spectra* $\mathbf{Sp}^\Sigma$ *has a* ***stable model structure*** *in which*

- *the cofibrations are the retracts of the relative cellular spectra,*
- *the weak equivalences are the stable equivalences, and*
- *the fibrations are the stable fibrations, i.e. the maps that are Serre fibrations at each spectrum level and such that the squares*

$$\begin{array}{ccc} X_i & \longrightarrow & \Omega^j X_{i+j} \\ \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle \Omega^j p_{i+j}} \\ Y_i & \longrightarrow & \Omega^j Y_{i+j} \end{array}$$

are homotopy pullback squares.

Theorem 6.1.37 (Mandell–May–Schwede–Shipley)**.** *The category of orthogonal spectra* $\mathbf{Sp}^O$ *has a stable model structure in which the cofibrations, weak equivalences, and fibrations have the same description as Theorem 6.1.36.*

Proof. All of the arguments in Section 5.6 now apply, using Lemma 6.1.16, Theorem 6.1.31, and Corollary 6.1.32 instead of the earlier results on sequential spectra. □

Remark 6.1.38. This model structure is proper in the sense of Definition 5.2.31. It is left proper by part (4) of Theorem 6.1.31. It is right proper by the dual of the proof of (4), taking homotopy fibers in a pullback square and using part (4) of Corollary 6.1.32.

Now that we have proven the model structure, we can prove that stable equivalences and π_*-isomorphisms coincide for orthogonal spectra. (This is not true for symmetric spectra.)

Lemma 6.1.39. *If A is a well-based space, the canonical map from the free sequential spectrum on A to the free orthogonal spectrum on A,*

$$F_i^{\mathbb{N}} A \longrightarrow F_i^O A,$$

is a π_*-isomorphism.

Proof. At spectrum level n, this is the map

$$A \wedge S^{n-i} \longrightarrow A \wedge \mathbf{S}^O(i,n).$$

It suffices to check that the map $S^{n-i} \to \mathbf{S}^O(i,n)$, that includes the compactification of a single fiber into the entire Thom space, becomes more highly connected as $n \to \infty$. This follows from the geometric fact that the space of embeddings of $\mathbb{R}^i$ into $\mathbb{R}^n$ becomes more highly connected as $n \to \infty$. □

Proposition 6.1.40. *For orthogonal spectra, every stable equivalence is a π_*-isomorphism.*

Proof. Since level equivalences are π_*-isomorphisms, without loss of generality X and Y are cofibrant. We want to show that every stable equivalence $X \to Y$ is a π_*-isomorphism.

Let J be the generating acyclic cofibrations for orthogonal spectra. The maps of J are built from free orthogonal spectra, each of which has the same homotopy groups as the corresponding free sequential spectrum by Lemma 6.1.39. It is now straightforward to argue that each J-cofibration is a π_*-isomorphism. By Ken Brown's Lemma (Lemma 5.2.25), therefore every stable equivalence of cofibrant orthogonal spectra $X \to Y$ is a π_*-isomorphism. □

Remark 6.1.41. For symmetric spectra, we have shown that stable equivalences and π_*-isomorphisms coincide when we restrict attention to Ω-spectra. There is a somewhat larger class of **semistable** symmetric spectra, on which the stable equivalences also agree with the π_*-isomorphisms.

A spectrum is semistable if the symmetric group actions Σ_n induce no interesting action on the stable homotopy groups $\pi_*(X)$, only the action that multplies by the sign of the permutation. Equivalently, if the map λ^* in Proposition 6.1.23 is a π_*-isomorphism. Every Ω-spectrum is semistable, as is the underlying symmetric spectrum of any orthogonal spectrum. See [Sch07] for a much more extensive discussion.

Proposition 6.1.42. *The prolongation and forgetful functors form Quillen equivalences* $\mathbf{Sp}^{\mathbb{N}} \simeq \mathbf{Sp}^{\Sigma} \simeq \mathbf{Sp}^O$.

Proof. It is easy to see that the forgetful functors preserve fibrations. The acyclic fibrations are the same thing as the I-fibrations, which in every case are the level acyclic fibrations, so they are also preserved. Therefore all of the forgetful functors are right Quillen.

We next check that $(P^O_{\mathbb{N}}, U^O_{\mathbb{N}})$ is a Quillen equivalence. We abbreviate the forgetful and prolongation functors to P and U. The unit of the adjunction $(\mathbb{L}P \dashv \mathbb{R}U)$ can be described on any cofibrant sequential spectrum X as the composite

$$X \longrightarrow UPX \longrightarrow URPX$$

that applies the unit of $(P \dashv U)$ and then a fibrant replacement in the middle. But this is equivalent to the plain old unit map $X \to UPX$, in other words the canonical map from each sequential spectrum to the free orthogonal spectrum on it.

If X is a free spectrum $F^{\mathbb{N}}_i A$ on a cell complex A, this canonical map is π_*-isomorphism by Lemma 6.1.39. We observe using (6.1.13) that prolongation P commutes with homotopy cofibers. So by Lemma 2.4.11 and the 5-lemma, if the unit map is an equivalence for any two spectra in a cofiber sequence $X \to Y \to Cf$, it is also an equivalence on the third. Using this, we can show by induction that the unit map is an equivalence for each of the skeleta $X^{(n)}$ of a cellular spectrum X. Since P preserves sequential compositions of cofibrations, we conclude also that the unit map is an equivalence on X itself.

The forgetful functor $\mathbb{R}U \simeq U$ reflects weak equivalences, in the sense that $UX \to UY$ is a weak equivalence iff $X \to Y$ is a weak equivalence. It follows formally that the derived counit map $PQUY \to PUY \to Y$ must also be an equivalence. (See Exercise 6.6.16.) Therefore we have a Quillen equivalence between sequential spectra and orthogonal spectra.

Next, we apply the same argument to the pair (P^O_Σ, U^O_Σ). We need to check that:

- U^O_Σ preserves weak equivalences. This is true because weak equivalences of orthogonal spectra are π_*-isomorphisms by Proposition 6.1.40, which are weak equivalences of symmetric spectra by Proposition 6.1.25.
- U^O_Σ reflects weak equivalences. This is true because up to π_*-isomorphism, we may as well assume the orthogonal spectra are Ω-spectra, and if a map of orthogonal Ω-spectra $X \to Y$ is a stable equivalence as a map of symmetric spectra, then by Lemma 6.1.26 it's a level equivalence, and therefore it's a stable equivalence as a map of orthogonal spectra too.
- The unit map $X \to U^O_\Sigma P^O_\Sigma X$ is a stable equivalence (not a π_*-isomorphism!) when X is a free spectrum $F^\Sigma_n A$. Using the five-lemma and Theorem 6.1.31, we reduce this to the case where $A = S^n$. Then we compare to $F^\Sigma_0 S^0$:

$$\begin{array}{ccc} F^\Sigma_n S^n & \xrightarrow{\sim} & F^\Sigma_0 S^0 \\ {\scriptstyle\eta}\downarrow & & \uparrow{\scriptstyle\eta} \\ U^O_\Sigma F^O_n S^n & \xrightarrow{\sim} & U^O_\Sigma F^O_0 S^0 \end{array}$$

The horizontal maps are stable equivalences of symmetric spectra by Example 6.1.21, and the unit map on the right-hand side is an isomorphism $\mathbb{S} \cong \mathbb{S}$, so the unit map on the left-hand side must be a stable equivalence of symmetric spectra as well.

Finally, we give a gasp of dismay: this argument does not apply to $(P^\Sigma_{\mathbb{N}}, U^\Sigma_{\mathbb{N}})$, because of the mismatch between π_*-isomorphisms and stable equivalences in $\mathbf{Sp}^\Sigma$. But it doesn't matter! We know that $\mathbb{L}P^O_\Sigma$ is an equivalence of categories, and the composite

$$\mathbb{L}P^O_{\mathbb{N}} \simeq (\mathbb{L}P^O_\Sigma) \circ (\mathbb{L}P^\Sigma_{\mathbb{N}})$$

is an equivalence of categories too. It follows that $\mathbb{L}P^\Sigma_{\mathbb{N}}$ must also be an equivalence of categories. This proves that $(P^\Sigma_{\mathbb{N}}, U^\Sigma_{\mathbb{N}})$ is a Quillen equivalence and finishes the proof. □

Since the stable equivalences between Ω-spectra are π_*-isomorphisms, we can use the fibrant replacement functor R to right-derive the homotopy groups of a symmetric spectrum.

Definition 6.1.43. In symmetric spectra, the right-derived functor of the homotopy groups

$$\mathbb{R}\pi_*(X) := \pi_*(RX)$$

is called the **true homotopy groups**.

A map is a stable equivalence iff it induces isomorphisms on the true homotopy groups. Along the Quillen equivalences of Proposition 6.1.42, the true homotopy groups of a symmetric spectrum X correspond to the homotopy groups of its associated orthogonal spectrum $\mathbb{L}P^O_\Sigma X$ or sequential spectrum $\mathbb{R}U^\Sigma_{\mathbb{N}} X$. We therefore get all of the same theorems and calculations for the true homotopy groups that we did back in sequential spectra – for instance, cofiber sequences $X \to Y \to C(X \to Y)$ induce long exact sequences on the true homotopy groups, and for a symmetric spectrum X we have

$$[\mathbb{S}^k, X]_s \cong \mathbb{R}\pi_*(X)$$

where $\mathbb{S}^k = F^\Sigma_0 S^k$ when $k \geq 0$ and $F^\Sigma_{|k|} S^0$ when $k < 0$.

For this reason, we often drop the $\mathbb{R}$ and think of these as the real homotopy groups, and the ones we defined before as the "naïve homotopy groups" of symmetric spectra.

Having the true homotopy groups available makes some things much easier, for instance the following relative version of Theorem 6.1.31(4) and (5):

Corollary 6.1.44. *1. (Gluing Lemma) Given a map of pushout diagrams of diagram spectra*

$$\begin{array}{ccccc} X & \longleftarrow & A & \longrightarrow & Y \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ X' & \longleftarrow & A' & \longrightarrow & Y', \end{array}$$

in which each map marked $\sim$ is a stable equivalence, if each pushout diagram has one leg that is a levelwise based h-cofibration, the map of pushouts

$$X \cup_A Y \xrightarrow{\sim} X' \cup_{A'} Y'$$

is also a stable equivalence.

2. *(Colimit lemma) Given a stable equivalence of sequential colimit diagrams of diagram spectra*

$$\begin{array}{ccccccc} X^{(0)} & \longrightarrow & X^{(1)} & \longrightarrow & X^{(2)} & \longrightarrow & \dots \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim & & \\ Y^{(0)} & \longrightarrow & Y^{(1)} & \longrightarrow & Y^{(2)} & \longrightarrow & \dots, \end{array}$$

if each map in each colimit diagram is a levelwise based h-cofibration, the map of colimits

$$\operatorname*{colim}_{n\to\infty} X^{(n)} \xrightarrow{\sim} \operatorname*{colim}_{n\to\infty} Y^{(n)}$$

is also a stable equivalence.

Proof. For (1), use the level model structure to make the spectra A and A' cofibrant, and the maps $A \to X$ and $A \to X'$ cofibrations. This changes both pushouts by a level equivalence, and so the new map $X \cup_A Y \longrightarrow X' \cup_{A'} Y'$ is still equivalent to the old one. As in the proof of Theorem 6.1.31(4), since $A \to A'$ and $X \to X'$ are stable equivalences, the map of homotopy cofibers $C(A \to X) \to C(A' \to X')$ is an equivalence, using the long exact sequence for the true homotopy groups of $A \to X \to C(A \to X)$. This map of homotopy cofibers is equivalent to the map of homotopy cofibers $C(Y \to X \cup_A Y) \to C(Y' \to X' \cup_{A'} Y')$, which is therefore an equivalence as well. Again using the long exact sequence on true homotopy groups, since $Y \to Y'$ is an equivalence, so is $X \cup_A Y \longrightarrow X' \cup_{A'} Y'$.

For (2), we similarly replace all the spectra by cofibrant ones up to level equivalence, and then examine the map of $\lim^1$ exact sequences of abelian groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \lim{}^1[\Sigma X^{(n)}, Z]_\ell & \longrightarrow & [X^{(\infty)}, Z]_\ell & \longrightarrow & \lim[X^{(n)}, Z]_\ell & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \\ 0 & \longrightarrow & \lim{}^1[\Sigma Y^{(n)}, Z]_\ell & \longrightarrow & [Y^{(\infty)}, Z]_\ell & \longrightarrow & \lim[Y^{(n)}, Z]_\ell & \longrightarrow & 0 \end{array}$$

We get isomorphisms on the left and right and therefore by the 5-lemma an isomorphism in the middle as well. □

6.2 The smash product

In this section we define the smash product $X \wedge Y$ of symmetric or orthogonal spectra. We show it is equivalent to the handicrafted smash product of Definition 2.3.23, and that

it is associative, commutative, and unital up to isomorphism:

$$(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z), \qquad X \wedge Y \cong Y \wedge X, \qquad \mathbb{S} \wedge X \cong X.$$

We also show that it has right adjoints in each variable, and plays well with the stable equivalences, so that it defines a Quillen tensor, and passes to a smash product on the homotopy category Ho **Sp**. The construction of this smash product was a major development in the subject in the 1990s.

6.2.1 Explicit definition

In the handicrafted smash product, $(X \wedge Y)_n$ is defined to be $X_p \wedge Y_q$ for *some* values of p and q that add to n. However, it's much better to take *all* values of p and q that add to n, and use the structure of a symmetric spectrum to glue them together.

Definition 6.2.1. Let X and Y be symmetric spectra. The **smash product** $X \wedge Y$ is the symmetric spectrum that at level n is the quotient

$$(X \wedge Y)_n = \bigvee_{p+q=n} (\Sigma_{p+q})_+ \wedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q)/\sim$$

that identifies the images of the following maps for all p, q, and r adding to n:

$$\begin{array}{ccc} (\Sigma_{p+q+r})_+ \wedge X_p \wedge Y_q \wedge S^r & \xrightarrow{\mathrm{id}\wedge\mathrm{id}\wedge\upsilon_{q,r}} & (\Sigma_{p+q+r})_+ \wedge X_p \wedge Y_{q+r} \\ \downarrow{\scriptstyle (-\circ(1\times\tau_{r,q}))\wedge\mathrm{id}\wedge\gamma} & & \\ (\Sigma_{p+q+r})_+ \wedge X_p \wedge S^r \wedge Y_q & \xrightarrow{\mathrm{id}\wedge\xi_{p,r}\wedge\mathrm{id}} & (\Sigma_{p+q+r})_+ \wedge X_{p+r} \wedge Y_q \end{array}$$

Here $G \wedge_H X$ denotes the quotient of $G \wedge X$ by the relation $(gh, x) \sim (g, hx)$, $\xi_{p,r}$ and $\upsilon_{q,r}$ are the iterated bonding maps from Definition 6.1.2, γ is the symmetry isomorphism $Y_q \wedge S^r \cong S^r \wedge Y_q$, and $- \circ (1 \times \tau_{r,q})$ is the operation that pre-composes the permutation $\sigma \in \Sigma_{p+q+r}$ with the block permutation $1 \times \tau_{r,q}$ that leaves the first p letters alone, and switches the next r letters past the last q letters.

The symmetric group Σ_n acts on $(X \wedge Y)_n$ by multiplying into Σ_{p+q} on the left, and the bonding map $(X \wedge Y)_n \wedge S^1 \to (X \wedge Y)_{n+1}$ is obtained from the bonding map of Y. (However, because of the quotient we applied, we get the same result if we use the bonding map of X, along with some extra permutations.)

Intuitively, the smash product is the disjoint union of all of the terms along one diagonal in the grid in Definition 2.3.23, modulo relations that glue these terms together along bonding maps coming from the terms below that diagonal. We also add in the symmetric

group $(\Sigma_{p+q})_+$ so that the full symmetric group will act on the resulting space. For instance, at spectrum level 2, we get a single space built out of the following six spaces:

$$\begin{array}{ccccc} X_0 \wedge Y_2 & & & & \\ | & & & & \\ X_0 \wedge Y_1 & \text{——} & X_1 \wedge Y_1 & & \\ | & & | & & \\ X_0 \wedge Y_0 & \text{——} & X_1 \wedge Y_0 & \text{——} & X_2 \wedge Y_0 \end{array}$$

This identification comes with an extra factor of $\tau_{r,q}$, which you should think of as the Koszul sign rule for spectra.

You can also think of $\tau_{r,q}$ as a correcting factor that undoes the permutation we do when we switch the S^r past Y_q. That switch feels like a block permutation $\tau_{q,r}$ that wasn't there before, so we add in the inverse $\tau_{r,q}$ to "undo" it. See also Remark 6.2.5.

If X and Y are orthogonal spectra, the smash product is defined the same way, but with orthogonal groups in the place of the symmetric groups:

$$(X \wedge Y)_n = \bigvee_{p+q=n} O(p+q)_+ \wedge_{O(p)\times O(q)} (X_p \wedge Y_q)/\sim$$

Example 6.2.2. The smash product $\mathbb{S} \wedge \mathbb{S}$ is isomorphic to $\mathbb{S}$ again. To see this, we check that all of the terms $X_p \wedge Y_q$ are n-spheres S^n, and all of the relations give the identity map of S^n. The permutations $\tau_{r,q}$ are absolutely necessary for this! If we didn't work them in, we would get the orbit space $(S^n)_{\Sigma_n}$ at level n, which is contractible!

Example 6.2.3. The argument of the previous example also shows that $\Sigma^\infty A \wedge \Sigma^\infty B \cong \Sigma^\infty(A \wedge B)$. So a smash product of suspension spectra is a suspension spectrum again.

Example 6.2.4. Generalizing in a different direction, if X is any symmetric spectrum then $X \wedge \mathbb{S} \cong X$. This is also not hard to show: all the terms $X_p \wedge S^q$ get identified to $X_n \wedge S^0$, and we're left with just X_n at spectrum level n. More generally, we get $X \wedge \Sigma^\infty K \cong X \wedge K$, as required by our black box in Chapter 4.

Remark 6.2.5. You should imagine a symmetric spectrum as a spectrum in which X_n has "secret sphere coordinates," and Σ_n permutes those coordinates. Of course, there actually is a map $\Sigma^n X_0 \to X_n$ that commutes with the Σ_n-action, so on the image of that map, the points really do have sphere coordinates that Σ_n is permuting around.

With this intuition, the smash product identifies all copies of $X_p \wedge Y_q$ together along bonding maps, but when we use the bonding maps to move some spheres from X to Y, the sphere coordinates get shuffled out of order. The Koszul sign rule (in other words the permutation $\tau_{r,q}$) is there to shuffle everything back into the correct order.

Proposition 6.2.6. *There are natural isomorphisms*

$$X \wedge \mathbb{S} \cong X, \qquad X \wedge Y \cong Y \wedge X, \qquad (X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z).$$

Proof sketch. For the first isomorphism, we use the equivalence relation to identify all the points in $X_p \wedge S^q$ to X_{p+q} using the bonding map of X.

For the middle isomorphism, we switch X_p past Y_q, but when we do so we also multiply the element of Σ_{p+q} on the right by the block permutation $\tau_{q,p}$ that switches the first q letters past the last p letters. This is another instance of the spectral Koszul sign rule: the permutation $\tau_{q,p}$ "corrects" the switch by "undoing" its effect on the sphere coordinates. So in the special case that $X_p = S^p$ and $Y_q = S^q$, the net effect of both of these operations is the identity map of S^{p+q}.

For the last isomorphism, we define a three-fold variant of the smash product $X \wedge Y \wedge Z$, and then identify both $(X \wedge Y) \wedge Z$ and $X \wedge (Y \wedge Z)$ with this three-fold smash product. □

Remark 6.2.7. It's important to emphasize how strong this is: $\mathbb{S} \wedge X$ is *isomorphic* to X. Not just stably equivalent, or something wishy-washy like that. They are isomorphic.

One can proceed further and carefully check that the isomorphisms in Proposition 6.2.6 are coherent in the sense outlined above Theorem 4.1.5. We conclude:

Theorem 6.2.8. *Symmetric spectra* $\mathbf{Sp}^\Sigma$ *and orthogonal spectra* $\mathbf{Sp}^O$ *are both symmetric monoidal categories under the smash product* $\wedge$.

There is also a somewhat easier way to do this step, see Theorem 6.2.22 below.

6.2.2 Explicit function spectra

To define function spectra $F(X, Y)$, it's convenient to first capture the universal property of the smash product of symmetric spectra, in a way that's similar to the universal property of the tensor product $\otimes$ of abelian groups.

Definition 6.2.9. Let X, Y, and Z be symmetric spectra, with bonding maps labeled ξ, υ, and ζ, respectively. A **bimorphism** $(X, Y) \longrightarrow Z$ is a collection of $\Sigma_p \times \Sigma_q$-equivariant maps of pointed spaces

$$b_{p,q} : X_p \wedge Y_q \longrightarrow Z_{p+q}$$

such that the following diagram commutes.

$$\begin{array}{ccccc}
X_p \wedge Y_q \wedge S^1 & = & X_p \wedge Y_q \wedge S^1 & \xrightarrow{\mathrm{id}\wedge\tau} & X_p \wedge S^1 \wedge Y_q \\
\downarrow{\scriptstyle b_{p,q}\wedge\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}\wedge\upsilon_q} & & \downarrow{\scriptstyle \xi_p\wedge\mathrm{id}} \\
Z_{p+q} \wedge S^1 & & X_p \wedge Y_{q+1} & & X_{p+1} \wedge Y_q \\
\downarrow{\scriptstyle \zeta_{p+q}} & & \downarrow{\scriptstyle b_{p,q+1}} & & \downarrow{\scriptstyle b_{p+1,q}} \\
Z_{p+q+1} & = & Z_{p+q+1} & \xrightarrow[1\times\tau_{q,1}]{} & Z_{p+1+q}
\end{array} \tag{6.2.10}$$

The following is a straightforward check of the definitions.

Lemma 6.2.11. *The smash product $X \wedge Y$ is the universal symmetric spectrum that receives a bimorphism from (X, Y).*

Definition 6.2.12. We define the **function spectrum** $F(Y, Z)$ as follows. At level p, it is the subspace of the product

$$F(Y,Z)_p \subseteq \prod_{q\geq 0} \mathrm{Map}_*(Y_q, Z_{p+q})$$

consisting of all tuples of Σ_q-equviariant maps $f_q \colon Y_q \to Z_{p+q}$ that commute with the bonding maps of Y and Z:

$$\begin{array}{ccc} Y_q \wedge S^1 & \xrightarrow{\upsilon_q} & Y_{q+1} \\ \downarrow{\scriptstyle f_q \wedge \mathrm{id}} & & \downarrow{\scriptstyle f_{q+1}} \\ Z_{p+q} \wedge S^1 & \xrightarrow{\zeta_{p+q}} & Z_{p+q+1} \end{array}$$

We define a Σ_p-action on this space by composing the maps f_q with the action of $\Sigma_p \times 1 \leq \Sigma_{p+q}$ on the space Z_{p+q}. We define the bonding map

$$F(Y,Z)_p \to \Omega F(Y,Z)_{p+1} \cong F(Y, \Omega Z)_{p+1}$$

by taking each collection of maps $f_q \colon Y_q \to Z_{p+q}$ to the collection of maps

$$Y_q \xrightarrow{f_q} Z_{p+q} \xrightarrow{\widetilde{\zeta}_{p+q}} \Omega Z_{p+q+1} \xrightarrow{(1\times\tau_{q,1})} \Omega Z_{p+1+q}.$$

It is straightforward to check this satisfies the definition of a symmetric spectrum (Definition 6.1.2).

Now, if we have a map of symmetric spectra $X \to F(Y,Z)$, the individual maps $X_p \to F(Y,Z)_p$ rearrange to maps $b_{p,q} \colon X_p \wedge Y_q \to Z_{p+q}$, making the left-hand region in (6.2.10) commute. Furthermore, requiring that $X_p \to F(Y,Z)_p$ is Σ_p-equivariant and that each f_q is Σ_q-equivariant is equivalent to asking that each $b_{p,q}$ is $\Sigma_p \times \Sigma_q$-equivariant. Finally, the maps $X_p \to F(Y,Z)_p$ must commute with the bonding maps:

$$\begin{array}{ccc} X_p & \xrightarrow{\widetilde{\xi}_p} & \Omega X_{p+1} \\ \downarrow & & \downarrow \\ F(Y,Z)_p & \longrightarrow & F(Y, \Omega Z)_{p+1}. \end{array}$$

This rearranges to the commutativity of the outside rectangle in (6.2.10). All together, this proves:

Lemma 6.2.13. *Maps of symmetric spectra $X \to F(Y,Z)$ correspond to maps of symmetric spectra $X \wedge Y \to Z$. In other words, $F(Y,-)$ is the right adjoint of $(-)\wedge Y$.*

It is clear that we can form the same constructions using orthogonal spectra, and the same result holds, with the same proof. In conclusion:

Theorem 6.2.14. *The symmetric monoidal structures on $\mathbf{Sp}^\Sigma$ and $\mathbf{Sp}^O$ from Theorem 6.2.14 are both closed.*

As mentioned in Example 4.1.45, this means that we also have isomorphisms of spectra

$$F(X\wedge Y,Z)\cong F(X,F(Y,Z)),\qquad F(\mathbb{S},X)\cong X,$$

and that a function spectrum out of a suspension spectrum $F(\Sigma^\infty K,X)$ is isomorphic to the cotensor spectrum $F(K,X)$ where we just apply $\mathrm{Map}_*(K,-)$ to each level of X separately. See also Exercise 6.6.10.

6.2.3 Abstract definition

In this section we give a second definition of the smash product. It is isomorphic to the first definition, but it is described in a more abstract and global way, which will make it easier to prove some of its properties.

Recall from Lemma 6.1.8 that symmetric spectra are the same thing as diagrams of based spaces on the category $\mathbf{S}^\Sigma$, whose mapping spaces are

$$\mathbf{S}^\Sigma(m,n)=\bigvee_{\alpha\colon \underline{m}\hookrightarrow\underline{n}} S^{\underline{n}-\alpha(\underline{m})}\qquad \text{when } n\geq m.$$

If X and Y are two symmetric spectra, we can consider all of the smash products $X_p\wedge Y_q$ for $p,q\geq 0$. These form a diagram on the smash product category $\mathbf{S}^\Sigma\wedge\mathbf{S}^\Sigma$ defined in Exercise 5.8.34. The objects of this category are pairs of integers, and the maps are the smash product of the mapping spaces:

$$\begin{aligned}(\mathbf{S}^\Sigma\wedge\mathbf{S}^\Sigma)((p_0,q_0),(p_1,q_1)) &= \mathbf{S}^\Sigma(p_0,p_1)\wedge\mathbf{S}^\Sigma(q_0,q_1)\\ &= \left(\bigvee_{\alpha\colon \underline{p_0}\hookrightarrow\underline{p_1}} S^{\underline{p_1}-\alpha(\underline{p_0})}\right)\wedge\left(\bigvee_{\beta\colon \underline{q_0}\hookrightarrow\underline{q_1}} S^{\underline{q_1}-\beta(\underline{q_0})}\right).\end{aligned}$$

Definition 6.2.15. We define a functor $\amalg\colon \mathbf{S}^\Sigma\wedge\mathbf{S}^\Sigma\to\mathbf{S}^\Sigma$ as follows. On objects, we take each pair (p,q) to the sum $p+q$. On morphisms, we take each pair of maps $\alpha\colon \underline{p_0}\hookrightarrow\underline{p_1}$ and $\beta\colon \underline{q_0}\hookrightarrow\underline{q_1}$ to the disjoint union

$$(\alpha\amalg\beta)\colon \underline{p_0+q_0}\hookrightarrow\underline{p_1+q_1}.$$

We have bijections of sets

$$\underline{p_1+q_1}-(\alpha \amalg \beta)(\underline{p_0+q_0}) \cong (\underline{p_1}-\alpha(\underline{p_0})) \amalg (\underline{q_1}-\beta(\underline{q_0}))$$

which give homeomorphisms of spheres

$$S^{\underline{p_1+q_1}-(\alpha\amalg\beta)(\underline{p_0+q_0})} \cong S^{\underline{p_1}-\alpha(\underline{p_0})} \wedge S^{\underline{q_1}-\beta(\underline{q_0})}, \tag{6.2.16}$$

and together these define the desired maps

$$\begin{aligned} \left(\bigvee_{\alpha\colon \underline{p_0}\hookrightarrow\underline{p_1}} S^{\underline{p_1}-\alpha(\underline{p_0})}\right) \wedge \left(\bigvee_{\beta\colon \underline{q_0}\hookrightarrow\underline{q_1}} S^{\underline{q_1}-\beta(\underline{q_0})}\right) &\longrightarrow \bigvee_{\gamma\colon \underline{n_0}\hookrightarrow\underline{n_1}} S^{\underline{n_1}-\gamma(\underline{n_0})} \\ \mathbf{S}^\Sigma(p_0,p_1)\wedge\mathbf{S}^\Sigma(q_0,q_1) &\longrightarrow \mathbf{S}^\Sigma(p_0+q_0,p_1+q_1) \end{aligned}$$

finishing the definition of the functor $\mathbf{S}^\Sigma \wedge \mathbf{S}^\Sigma \to \mathbf{S}^\Sigma$. It is straightforward to check this respects composition (Exercise 6.6.17). We call this functor $\amalg$, because in essence it is taking the disjoint union of finite sets, and concatenating sphere coordinates.

Definition 6.2.17. The **smash product** $X \wedge Y$ is the $\mathbf{S}^\Sigma$-diagram obtained from the smash products $(X_p \wedge Y_q)_{p,q\geq 0}$ by left Kan extension along $\amalg$ (Definition 5.4.14):

$$X \wedge Y = (\amalg)_!(X_- \wedge Y_-).$$

We can also define an analogous functor $\oplus\colon \mathbf{S}^O \wedge \mathbf{S}^O \to \mathbf{S}^O$ for orthogonal spectra, see Exercise 6.6.18.

Proposition 6.2.18. *This abstract smash product is isomorphic to the explicit smash product from Definition 6.2.1.*

Proof. The expression for the left Kan extension from (5.4.15), adapted to based spaces (so that the Cartesian products $\times$ becomes smash products $\wedge$), gives in this case the coequalizer of the two maps

$$\left(\bigvee_{(p_0,q_0),(p_1,q_1)} X_{p_0}\wedge Y_{q_0}\wedge \mathbf{S}^\Sigma(p_0,p_1)\wedge\mathbf{S}^\Sigma(q_0,q_1)\wedge\mathbf{S}^\Sigma(q_0+q_1,n)\right) \rightrightarrows \left(\bigvee_{(p,q)} X_p\wedge Y_q\wedge\mathbf{S}^\Sigma(p+q,n)\right)$$

that either have $\mathbf{S}^\Sigma(p_0,p_1)\wedge\mathbf{S}^\Sigma(q_0,q_1)$ act on $X_{p_0}\wedge Y_{q_0}$, or apply $\amalg$ to $\mathbf{S}^\Sigma(p_0,p_1)\wedge\mathbf{S}^\Sigma(q_0,q_1)$ and then compose the result with $\mathbf{S}^\Sigma(q_0+q_1,n)$.

Along these relations, the terms in which $p+q<n$ get identified to terms in which $p+q=n$, so that we get a quotient of the sum

$$\bigvee_{p+q=n} X_p\wedge Y_q\wedge\mathbf{S}^\Sigma(p+q,n) \cong \bigvee_{p+q=n} X_p\wedge Y_q\wedge(\Sigma_{p+q})_+.$$

The remaining relations turn out to give the relations we imposed in Definition 6.2.1. □

The advantage of this definition of the smash product is that we can prove the following by a formal argument:

Lemma 6.2.19. *The smash product of two free spectra is free:*

$$F_d A \wedge F_e B \cong F_{d+e}(A \wedge B).$$

This holds if the free spectra and smash product are both taken in the category of symmetric spectra, or if they are both taken in the category of orthogonal spectra.

Proof. The smash products of the levels of $F_d A$ and $F_e B$ become

$$(F_d A)_p \wedge (F_e B)_q \cong A \wedge \mathbf{S}^{\Sigma}(d,p) \wedge B \wedge \mathbf{S}^{\Sigma}(e,q).$$

This is the free diagram on the category $\mathbf{S}^{\Sigma} \wedge \mathbf{S}^{\Sigma}$, at the object (d,e), using the space $(A \wedge B)$. The left Kan extension preserves free diagrams (Exercise 5.8.14) and therefore we get $F_{d+e}(A \wedge B)$ as the result. □

Recall that $\Sigma^{\infty} A \wedge \Sigma^{\infty} B \cong \Sigma^{\infty}(A \wedge B)$ is a special case of this result. As in Exercise 4.4.21, we can also conclude that the smash product preserves cell complex spectra:

Corollary 6.2.20. $\wedge$ *preserves cellular spectra, and* $X \wedge Y$ *has a stable* $(m+n)$*-cell for every stable* m*-cell of* X *and stable* n*-cell of* Y*.*

This is illustrated at the beginning of Chapter 4.

It is likewise easier to prove that the smash products for symmetric and orthogonal spectra agree strictly in the following sense.

Proposition 6.2.21. *The prolongation from symmetric to orthogonal spectra commutes with the smash product,*

$$(P^O_{\Sigma} X) \wedge^{(O)} (P^O_{\Sigma} Y) \cong P^O_{\Sigma}(X \wedge^{(\Sigma)} Y).$$

Proof. Let $i \colon \mathbf{S}^{\Sigma} \to \mathbf{S}^O$ be the obvious inclusion. Using the fact that the smash product preserves colimits in each slot, the smash product of the levels of $P^O_{\Sigma} X$ and $P^O_{\Sigma} Y$ can be rewritten as the left Kan extension of $X_- \wedge Y_-$ along $i \wedge i \colon \mathbf{S}^{\Sigma} \wedge \mathbf{S}^{\Sigma} \to \mathbf{S}^O \wedge \mathbf{S}^O$. To finish taking the smash product in orthogonal spectra, we left Kan extend to $\mathbf{S}^O$ along the sum functor $\oplus$.

On the other hand, if we smashed X and Y in symmetric spectra first, we'd take the left Kan extension of $X_- \wedge Y_-$ along the sum functor II to get a symmetric spectrum, then left

Kan extend along i to get an orthogonal spectrum. So, each side of the desired equation is a left Kan extension along one of the two routes in the following commuting square:

$$\begin{array}{ccc} \mathbf{S}^{\Sigma} \wedge \mathbf{S}^{\Sigma} & \xrightarrow{\amalg} & \mathbf{S}^{\Sigma} \\ {\scriptstyle i\wedge i}\downarrow & & \downarrow{\scriptstyle i} \\ \mathbf{S}^{O} \wedge \mathbf{S}^{O} & \xrightarrow[\oplus]{} & \mathbf{S}^{O}. \end{array}$$

The results agree up to isomorphism because left Kan extensions are closed under composition (Exercise 5.8.15). □

6.2.4 Point-set rigidity of the smash product

The smash product has a convenient property that keeps us from getting errors when we pass between different definitions, or when we check that the associativity and commutativity isomorphisms are coherent.

Think of the smash product as a functor $\wedge\colon \mathbf{Sp}^{\Sigma} \times \mathbf{Sp}^{\Sigma} \to \mathbf{Sp}^{\Sigma}$, and consider natural transformations from this functor to itself. These are maps

$$\eta_{X,Y}\colon X \wedge Y \to X \wedge Y$$

that commute with maps in each variable.

Theorem 6.2.22 (Point-set rigidity of $\wedge$)**.** *The only such natural transformations are the identity transformation, and the zero transformation sending everything to the basepoint. Moreover, the same is also true for:*

- the n-fold smash product

  $$(X_1, X_2 \ldots, X_n) \mapsto (((X_1 \wedge X_2) \wedge \ldots) \wedge X_n)$$

  for every $n \geq 0$,
- the n-fold smash product of orthogonal spectra for every $n \geq 0$,
- the suspension spectrum functor Σ^{∞}, to either symmetric or orthogonal spectra,
- the n-fold smash product of spaces, and also the suspension spectrum of the n-fold smash product of spaces.

This is a useful fact because it allows us to lock down the definitions of various isomorphisms between different models of the smash product.

Corollary 6.2.23. *For any two functors isomorphic to the n-fold smash product, there is a unique natural isomorphism between them.*

Proof. Any two isomorphisms α, β can be composed to give a natural transformation $\beta^{-1}\alpha$ from the n-fold smash product to itself, that is not zero. This transformation must be the identity by Theorem 6.2.22, and therefore $\beta = \alpha$. □

In particular, the two definitions of the smash product we gave in Definition 6.2.1 and Definition 6.2.17 are isomorphic by a unique isomorphism.

Here are a few more applications. The first is another proof that symmetric spectra and orthogonal spectra form symmetric monoidal categories.

Second proof of Theorem 6.2.8. We explained how to define the associativity, commutativity, and unit isomorphisms in Proposition 6.2.6. Now we have to check the coherences listed in Theorem 4.1.5. But in each of these diagrams, the maps are all natural transformations. Going around the diagram gives a natural transformation from the n-fold smash product to itself. This must be the identity by Theorem 6.2.22. Therefore the diagram commutes. □

Lemma 6.2.24. Σ^∞ *is a strong symmetric monoidal functor* $\mathbf{Top}_* \to \mathbf{Sp}^\Sigma$ *or* $\mathbf{Sp}^O$.

Proof. The isomorphisms $\Sigma^\infty A \wedge \Sigma^\infty B \cong \Sigma^\infty(A \wedge B)$ were constructed in Lemma 6.2.19, and $\Sigma^\infty S^0 \cong \mathbb{S}$ is obvious. As in the previous proof, all of the coherences for a symmetric monoidal functor are automatically satisfied, because the functor $\Sigma^\infty X_1 \wedge \ldots \wedge X_n$ has no nontrivial automorphisms. □

Proof of Theorem 6.2.22. We handle the n-fold smash product of symmetric spectra; the other cases are analogous. Let η be a natural transformation from the n-fold smash product functor to itself.

We first examine what η does to n-tuples of free spectra $(F^\Sigma_{d_1}A_1, \ldots, F^\Sigma_{d_n}A_n)$. By Lemma 6.2.19, the smash product of these n spectra is equivalent to the free spectrum $F^\Sigma_{d_1+\ldots+d_n}(A_1 \wedge \ldots \wedge A_n)$.

The simplest case is the n-tuple consisting entirely of sphere spectra, $(F^\Sigma_0 S^0, \ldots, F^\Sigma_0 S^0)$, whose smash product is $F^\Sigma_0 S^0$. Here η gives a map of spectra

$$F^\Sigma_0 S^0 \longrightarrow F^\Sigma_0 S^0.$$

But $F^\Sigma_0 S^0$ is a free spectrum, so this is determined a choice of point in S^0 at level 0. There are only two possibilities here: η is the identity map, or the zero map.

Let's examine the case where η is the identity on $(F_0^\Sigma S^0, \ldots, F_0^\Sigma S^0)$. Moving next to $(F_{d_1}^\Sigma S^0, \ldots, F_{d_n}^\Sigma S^0)$, η gives a map

$$F^\Sigma_{d_1+\ldots+d_n} S^0 \longrightarrow F^\Sigma_{d_1+\ldots+d_n} S^0.$$

Again this is determined by what it does at spectrum level $(d_1 + \ldots + d_n)$, where it gives a point in the space $(\Sigma_{d_1+\ldots+d_n})_+$. In other words, a permutation σ, or the basepoint.

We claim that this point has to be the identity permutation $\sigma = \mathrm{id}$. To see this, consider any point $(t_1, \ldots, t_n) \in S^{d_1} \wedge \ldots \wedge S^{d_n}$. We can choose maps of spectra $F^\Sigma_{d_i} S^0 \to F^\Sigma_0 S^0$ which at level d_i send the non-basepoint of S^0 to the point $t_i \in S^{d_i}$. Since η is a natural transformation, this square commutes for all choices of $(t_1, \ldots, t_n)$:

$$\begin{array}{ccc} (\Sigma_{d_1+\ldots+d_n})_+ & \xrightarrow{\cdot\sigma} & (\Sigma_{d_1+\ldots+d_n})_+ \\ {\scriptstyle (-)(t_1,\ldots,t_n)}\downarrow & & \downarrow{\scriptstyle (-)(t_1,\ldots,t_n)} \\ S^{d_1+\ldots+d_n} & \xrightarrow{\mathrm{id}} & S^{d_1+\ldots+d_n} \end{array}$$

Since $\Sigma_{d_1+\ldots+d_n}$ acts faithfully on the sphere $S^{d_1+\ldots+d_n}$, we must have $\sigma = \mathrm{id}$. In summary, η acts as the identity on the n-tuple $(F^\Sigma_{d_1} S^0, \ldots, F^\Sigma_{d_n} S^0)$.

Next we examine an arbitrary n-tuple of free spectra $(F^\Sigma_{d_1} A_1, \ldots, F^\Sigma_{d_n} A_n)$. Each collection of choices of point $a_i \in A_i$ gives a sequence of maps $F^\Sigma_{d_i} S^0 \to F^\Sigma_{d_i} A_i$, and applying η to this sequence of maps gives a commuting square

$$\begin{array}{ccc} F^\Sigma_{d_1+\ldots+d_n} S^0 \wedge \ldots \wedge S^0 & \xrightarrow{\mathrm{id}} & F^\Sigma_{d_1+\ldots+d_n} S^0 \wedge \ldots \wedge S^0 \\ \downarrow & & \downarrow \\ F^\Sigma_{d_1+\ldots+d_n} A_1 \wedge \ldots \wedge A_n & \xrightarrow{\eta} & F^\Sigma_{d_1+\ldots+d_n} A_1 \wedge \ldots \wedge A_n \end{array}$$

The commutativity of the square implies that the bottom map must be the identity on any point of the form

$$(a_1, \ldots, a_n, f) \in A_1 \wedge \ldots \wedge A_n \wedge \mathbf{S}^\Sigma(d_1 + \ldots + d_n, m).$$

But by varying $(a_1, \ldots, a_n)$ this gives every point, and so η is the identity on every n-tuple of free spectra.

It is left as an exercise to conclude that η is therefore the identity on all n-tuples of spectra (Exercise 6.6.24). In the remaining case, we assume that $\eta = 0$ on the n-tuple $(F^\Sigma_0 S^0, \ldots, F^\Sigma_0 S^0)$, and follow the same steps to conclude that $\eta = 0$ on all n-tuples of spectra. □

We can prove the same for prolongation, which also helps us check it is a symmetric monoidal functor. We combine the two results into one:

Theorem 6.2.25. *Prolongation P^O_Σ is a strong symmetric monoidal functor.*

Proof. As in the previous proofs, it suffices to show that the functor that takes an n-tuple of symmetric spectra, takes their smash product, and prolongs the result to orthogonal spectra, has no nontrivial automorphisms. The proof is the same as the proof of Theorem 6.2.22. For the part deferred to Exercise 6.6.24, we observe that if $X \to Y$ is a levelwise surjection of symmetric spectra, the map of prolongations $P^O_\Sigma X \to P^O_\Sigma Y$ is also levelwise surjective by examination of the formula (5.4.22). □

6.2.5 Symmetric and orthogonal ring spectra

Recall from Definition 4.1.18 that a ring spectrum is a monoid for the smash product. Now that we have an honest definition of the smash product, we get an honest definition of ring spectra:

Definition 6.2.26. A **symmetric ring spectrum** is a monoid in $\mathbf{Sp}^\Sigma$. So it consists of a symmetric spectrum R with maps

$$\mu\colon R \wedge R \longrightarrow R, \qquad \eta\colon \mathbb{S} \longrightarrow R,$$

such that the diagrams in Definition 4.1.12 commute. A **commutative symmetric ring spectrum** is a commutative monoid object in $\mathbf{Sp}^\Sigma$, and an **R-module spectrum** is a module object over R, with action map

$$\alpha\colon R \wedge M \longrightarrow M,$$

such that the diagrams in Definition 4.1.12 commute. An **orthogonal ring spectrum** and modules over it are defined in the same way in $\mathbf{Sp}^O$.

Remark 6.2.27. Using the definition of a bimorphism from Definition 6.2.9, we can see that the ring structure is determined by a collection of maps

$$R_p \wedge R_q \longrightarrow R_{p+q}$$

that interact in the right way with the bonding maps of R. Similarly an R-module is determined by maps

$$R_p \wedge M_q \longrightarrow M_{p+q}.$$

Lemma 6.2.28. *The prolongation P^O_Σ of a symmetric ring spectrum is an orthogonal ring spectrum, and the underlying symmetric spectrum U^O_Σ of an orthogonal ring spectrum is a symmetric ring spectrum. These functors likewise preserve modules over ring spectra, and commutative ring spectra.*

Proof. Follows immediately from Proposition 4.1.27 and Exercise 4.4.11. □

Example 6.2.29. The sphere spectrum $\mathbb{S}$ is an orthogonal ring spectrum in a canonical way. It is therefore also a symmetric ring spectrum.

Example 6.2.30. If G is a topological group or monoid, the **spherical group ring**

$$\mathbb{S}[G] = \mathbb{S} \wedge G_+ = \Sigma^\infty_+ G$$

is the (symmetric or orthogonal) ring spectrum given by the suspension spectrum of G, with multiplication coming from that of G. This is a ring spectrum by Proposition 4.1.27 and the fact that Σ^∞_+ is a strong symmetric monoidal functor (Lemma 6.2.24). See Example 4.1.31 for a discussion of why we need the fact that Σ^∞_+ is a strong symmetric monoidal functor.

Remark 6.2.31. We will prove in Proposition 9.2.27 that for any ordinary ring R, the Eilenberg–Mac Lane spectrum HR can also be made into an orthogonal ring spectrum.

Definition 6.2.32. We define the category of symmetric ring spectra $\mathrm{Alg}(\mathbf{Sp}^\Sigma)$ by saying that an object is a symmetric ring spectrum R, and a morphism is a map of spectra $f : R \to S$ that commutes with the multiplication and unit maps:

$$\begin{array}{ccc} R \wedge R & \xrightarrow{f \wedge f} & S \wedge S \\ {\scriptstyle \mu_R}\downarrow & & \downarrow{\scriptstyle \mu_S} \\ R & \xrightarrow[f]{} & S \end{array} \qquad \begin{array}{ccc} \mathbb{S} & = & \mathbb{S} \\ {\scriptstyle \eta_R}\downarrow & & \downarrow{\scriptstyle \eta_S} \\ R & \xrightarrow[f]{} & S. \end{array}$$

We also sometimes call such a map a **ring homomorphism**. The category of commutative ring spectra $\mathrm{CAlg}(\mathbf{Sp}^\Sigma)$ is the full subcategory on the rings that are commutative. (In other words, the morphisms are the same.)

For a fixed symmetric ring spectrum R, the category of module spectra $\mathrm{Mod}(R)$ has objects the R-modules, and morphisms the maps of spectra $f : M \to N$ that commute with the R-action:

$$\begin{array}{ccc} R \wedge M & \xrightarrow{\mathrm{id} \wedge f} & R \wedge N \\ {\scriptstyle \alpha_M}\downarrow & & \downarrow{\scriptstyle \alpha_N} \\ M & \xrightarrow[f]{} & N. \end{array}$$

We also sometimes call this an **R-linear map**. We can also do the same definitions with orthogonal spectra, or really in any symmetric monoidal category (Definition 6.5.1).

Example 6.2.33. By Exercise 6.6.25, the category of $\mathbb{S}$-modules in symmetric spectra, $\mathrm{Mod}(\mathbb{S}^{(\Sigma)})$, is equivalent to the category of symmetric spectra $\mathbf{Sp}^\Sigma$. Likewise, the category of $\mathbb{S}$-modules in orthogonal spectra, $\mathrm{Mod}(\mathbb{S}^{(O)})$, is equivalent to $\mathbf{Sp}^O$. In other words, every spectrum is a module over the sphere spectrum in a canonical way.

Example 6.2.34. We will later see that $H\mathbb{Z}$-module spectra up to stable equivalence correspond to unbounded chain complexes over $\mathbb{Z}$.

Lemma 6.2.28 tells us that we get functors on ring spectra

$$P_\Sigma^O \colon \mathrm{Alg}(\mathbf{Sp}^\Sigma) \leftrightarrow \mathrm{Alg}(\mathbf{Sp}^O) \colon U_\Sigma^O.$$

As with ordinary spectra, we really want to think about ring spectra up to stable equivalence.

Definition 6.2.35. A **stable equivalence of ring spectra** is a homomorphism of ring spectra $R \to S$ that is a stable equivalence on the underlying spectra. A stable equivalence of modules, or of commutative rings, is defined similarly.

We said after Definition 2.1.10 that stable homotopy theory is the study of spectra up to stable equivalence. By analogy with this:

Higher algebra is the study of ring spectra and their modules up to stable equivalence.

To do this properly, we need to define a model category on ring spectra with these stable equivalences. This will be done in Section 6.5. For now, we state the result:

Theorem 6.2.36. *For each of the following categories, there is a stable model structure, in which the weak equivalences and fibrations are the morphisms that are stable equivalences, respectively stable fibrations, on the underlying spectra.*

- $\mathrm{Mod}(R)$, $R \in \mathrm{Alg}(\mathbf{Sp}^\Sigma)$
- $\mathrm{Alg}(\mathbf{Sp}^\Sigma)$
- $\mathrm{CAlg}(\mathbf{Sp}_+^\Sigma)$
- $\mathrm{Mod}(R)$, $R \in \mathrm{Alg}(\mathbf{Sp}^O)$
- $\mathrm{Alg}(\mathbf{Sp}^O)$
- $\mathrm{CAlg}(\mathbf{Sp}_+^O)$

The "+" in the commutative case indicates that we have to change the model structure on spectra a little bit before the theorem becomes true: we have to use the "positive" stable model structure, which is like the stable model structure except that all of the generators start at spectrum level 1, instead of spectrum level 0. See Theorem 6.4.2.

We also have the following compatibility with our earlier model structure, proven in Exercise 6.6.32.

Lemma 6.2.37. *A cofibrant ring spectrum always has a cofibrant underlying spectrum. A cofibrant module over a cofibrant ring spectrum always has a cofibrant underlying spectrum.*

Remark 6.2.38. The theory of commutative ring spectra is more subtle – cofibrant commutative rings are actually not cofibrant as spectra, but something weaker called "flat cofibrant." See Example 6.5.34.

These results are enough to start working with rings and modules up to stable equivalence. For instance, suppose we want a left-derived version of prolongation of ring spectra,

$$P_\Sigma^O : \mathrm{Alg}(\mathbf{Sp}^\Sigma) \to \mathrm{Alg}(\mathbf{Sp}^O).$$

We might try to take a symmetric ring spectrum X and cofibrantly replace it. But we have to do this in the model category of symmetric *ring* spectra, not the model category of symmetric spectra. A cofibrant replacement in symmetric spectra won't have a multiplication anymore!

The cofibrant replacement in symmetric ring spectra, $QX \to X$, is a ring homomorphism by definition. Then the prolongation $P_\Sigma^O QX$ is an orthogonal ring spectrum. By Lemma 6.2.37, QX is also cofibrant as a spectrum, so $P_\Sigma^O QX$ preserves weak equivalences, as desired.

This shows that we can effectively pass between symmetric and orthogonal ring spectra without losing information. The theory of *ring spectra up to equivalence* is the same in both cases.

6.3 How the smash product interacts with the model structure

Now that we understand how the smash product works *up to isomorphism*, let us consider how it works *up to weak equivalence*. The big thing to show is that the smash product preserves weak equivalences on cofibrant spectra. To do that, we prove that it plays nicely with the stable model structure, making spectra into something called a symmetric monoidal model category.

6.3.1 Symmetric monoidal model categories

Suppose that $\mathbf{C}$ is both a model category and a closed symmetric monoidal category. For instance, $\mathbf{C}$ might be:

- topological spaces $(\mathbf{Top}, \times, *)$ with the Quillen model structure (Theorem 5.2.5),
- based spaces $(\mathbf{Top}_*, \wedge, S^0)$ with the model structure of Example 5.2.9,
- bounded chain complexes $(\mathbf{Ch}_{\geq 0}(k), \otimes_k, k[0])$, where k is a commutative ring, with the model structure of Example 5.2.11,
- unbounded chain complexes $(\mathbf{Ch}(k), \otimes_k, k[0])$ with the model structure of Exercise 5.8.11,

- symmetric spectra $(\mathbf{Sp}^\Sigma, \wedge, \mathbb{S})$ with the stable model structure (Theorem 6.1.36), or
- orthogonal spectra $(\mathbf{Sp}^O, \wedge, \mathbb{S})$ with the stable model structure (Theorem 6.1.37).

At the moment, we haven't asked for the model structure and the symmetric monoidal structure on $\mathbf{C}$ to be compatible at all. But it is reasonable to ask for something.

Recall that the **pushout-product** of two maps $f : A \to X$ and $g : B \to Y$ in a symmetric monoidal category $\mathbf{C}$ is the map

$$f \square g : (X \otimes B) \cup_{A \otimes B} (A \otimes Y) \longrightarrow X \otimes Y.$$

We say that $\otimes$ is a **Quillen tensor** if $X \otimes (-)$ always has a right adjoint $\mathrm{Hom}(X, -)$, and if the pushout-product preserves cofibrations and acyclic cofibrations:

$$C \square C \subseteq C, \qquad C \square (W \cap C) \subseteq (W \cap C).$$

In other words, if f and g are cofibrations then $f \square g$ is a cofibration, and if additionally one of them is a weak equivalence then $f \square g$ is a weak equivalence.

(We first saw pushout-products in Definition 5.5.6, and Quillen tensors in Definition 5.5.13. One of the three conditions in Definition 5.5.13 has become redundant because of the symmetry of $\otimes$.)

Definition 6.3.1. A **symmetric monoidal model category** is a closed symmetric monoidal category $(\mathbf{C}, \otimes, I)$, and the structure of a model category on $\mathbf{C}$, such that

- $\otimes$ is a Quillen tensor, and
- either the unit I is cofibrant, or the map $QI \otimes QX \to I \otimes QX$ is a weak equivalence for all $X \in \mathbf{C}$.

Remark 6.3.2. By Proposition 5.5.14, for a cofibrantly generated model category $\mathbf{C}$, it suffices to check the pushout-product condition on the generating cofibrations I and generating acyclic cofibrations J. We also recalled in Remark 5.5.17 a common rearrangement of the pushout-product axiom called **Quillen's SM7 axiom**.

Example 6.3.3. All of the examples in the above list are closed symmetric monoidal model categories. For the last two, we use the following proposition.

Proposition 6.3.4. *The smash product $\wedge$ is a Quillen tensor, both in symmetric spectra $\mathbf{Sp}^\Sigma$ and orthogonal spectra $\mathbf{Sp}^O$.*

Proof. We write $F_d A$ as generic notation for a free spectrum, since our proof will apply equally well to symmetric or orthogonal spectra.

We need to calculate the pushout-product of two maps using the smash product. We use the fact that smashing with a suspension spectrum $\Sigma^\infty K$ is the same as smashing with the space K, and that the free spectrum $F_d A$ is the smash product of the spectrum $F_d S^0$ with the space A.

Using the simplifications of the pushout-product from Example 5.5.7 and Example 5.5.9, the pushout-product of two maps in I from Definition 5.6.1 is rewritten as

$$\begin{aligned} F_d(S^{k-1} \to D^k)_+ \square F_e(S^{\ell-1} \to D^\ell)_+ &\cong (* \to F_d S^0) \square (S^{k-1} \to D^k)_+ \square (* \to F_e S^0) \square (S^{\ell-1} \to D^\ell)_+ \\ &\cong (* \to F_d S^0) \square (* \to F_e S^0) \square (S^{k-1} \to D^k)_+ \square (S^{\ell-1} \to D^\ell)_+ \\ &\cong (* \to F_{d+e} S^0) \square (S^{k+\ell-1} \to D^{k+\ell})_+ \\ &\cong F_{d+e}(S^{k+\ell-1} \to D^{k+\ell})_+. \end{aligned}$$

This is a map in I, so we've shown that $I \square I \subseteq C$. (Recall from Exercise 5.8.1 that any map isomorphic to a cofibration is again a cofibration.)

Next we take pushout-product of maps in I with maps in J. For the first set of maps in J from Definition 5.6.3, the proof is largely the same:

$$\begin{aligned} F_d(S^{k-1} \to D^k)_+ \square F_e(D^\ell \to D^\ell \times I)_+ &\cong (* \to F_d S^0) \square (S^{k-1} \to D^k)_+ \square (* \to F_e S^0) \square (D^\ell \to D^\ell \times I)_+ \\ &\cong (* \to F_d S^0) \square (* \to F_e S^0) \square (S^{k-1} \to D^k)_+ \square (D^\ell \to D^\ell \times I)_+ \\ &\cong (* \to F_{d+e} S^0) \square (D^{k+\ell} \to D^{k+\ell} \times I)_+ \\ &\cong F_{d+e}(D^{k+\ell} \to D^{k+\ell} \times I)_+, \end{aligned}$$

which is another map in J. For the second set of maps from Definition 5.6.3,

$$\begin{aligned} F_d(S^{k-1} \to D^k)_+ \square k_{i,j} \square (S^{\ell-1} \to D^\ell)_+ &\cong (* \to F_d S^0) \square (S^{k-1} \to D^k)_+ \square k_{i,j} \square (S^{\ell-1} \to D^\ell)_+ \\ &\cong (* \to F_d S^0) \square k_{i,j} \square (S^{k-1} \to D^k)_+ \square (S^{\ell-1} \to D^\ell)_+ \\ &\cong k_{d+i,j} \square (S^{k+\ell-1} \to D^{k+\ell})_+, \end{aligned}$$

which is another map in J. The last manipulation uses the fact that smashing a free spectrum $F_d S^0$ into the definition of $k_{i,j}$ increases the indices on all of the free spectra in the definition by d, giving the map $k_{d+i,j}$. In conclusion, we have shown that $I \square J \subseteq (W \cap C)$, so the smash product is a Quillen tensor. □

By far, the biggest reason why we care about this is the following corollary.

Corollary 6.3.5. *The smash product $X \wedge Y$ preserves stable equivalences when X and Y are cofibrant. The function spectrum $F(X, Y)$ preserves stable equivalences when X is cofibrant and Y is fibrant.*

Proof. This follows from Lemma 5.5.19 and Lemma 5.5.22, and the just-established fact that $\wedge$ is a Quillen tensor. □

Remark 6.3.6. In fact, more is true: the smash product $X \wedge Y$ preserves stable equivalences when X is cofibrant and Y is *arbitrary*. See Exercise 6.6.12. Also, the function spectrum $F(X,Y)$ preserves stable equivalences if X is a finite cell spectrum and Y is arbitrary. See Exercise 6.6.13.

We therefore have a smash product of spectra that is well-behaved on the point-set level, and preserves stable equivalences when we need it to!

Recall from Section 5.5.4 that in any symmetric monoidal model category $\mathbf{C}$, the tensor $X \otimes Y$ preserves equivalences when X and Y are cofibrant, and $\mathrm{Hom}(X,Y)$ preserves equivalences when X is cofibrant and Y is fibrant. We can therefore left-derive the tensor and right-derive the hom:

$$X \otimes^{\mathbb{L}} Y := QX \otimes QY, \qquad \mathbb{R}\mathrm{Hom}(X,Y) := \mathrm{Hom}(QX, RY).$$

These functors preserve all equivalences, so they define functors on the homotopy category as well. We proved in Corollary 5.5.27 that these make the homotopy category $\mathrm{Ho}\,\mathbf{C}$ into a closed symmetric monoidal category.

Since we have equivalences of homotopy categories

$$\mathrm{Ho}\,\mathbf{Sp}^O \simeq \mathrm{Ho}\,\mathbf{Sp}^\Sigma \simeq \mathrm{Ho}\,\mathbf{Sp}^{\mathbb{N}},$$

by Exercise 4.4.13, a symmetric monoidal structure on any one of these categories becomes a symmetric monoidal structure on all of them. We have therefore, finally, achieved the goal of placing a closed symmetric monoidal structure on the stable homotopy category $\mathrm{Ho}\,\mathbf{Sp}$.[2]

Corollary 6.3.7. *The stable homotopy category* $\mathrm{Ho}\,\mathbf{Sp}$ *has a closed symmetric monoidal structure using* $\wedge^{\mathbb{L}}$ *and* $\mathbb{R}F(-,-)$*, and satisfying all of the properties from Example 4.1.10 and Example 4.1.45.*

There were more promised properties in Example 4.1.31, but those have to do with functors between this an other categories, and Example 4.2.11, but that had to do with duality. See Exercise 6.6.3.

6.3.2 Symmetric monoidal Quillen adjunctions

We next turn to functors between symmetric monoidal model categories.

Definition 6.3.8. Suppose $\mathbf{C}$ and $\mathbf{D}$ are symmetric monoidal model categories. A **strong symmetric monoidal Quillen adjunction** is

[2]The following section checks the important fact that $\mathbf{Sp}^\Sigma$ and $\mathbf{Sp}^O$ give the *same* symmetric monoidal structure, up to isomorphism.

- a Quillen adjunction $(F \dashv G)$ in which
- the left Quillen functor $F : \mathbf{C} \to \mathbf{D}$ is strong symmetric monoidal,[3] and
- the map of units $F(QI_{\mathbf{C}}) \to F(I_{\mathbf{C}}) \cong I_{\mathbf{D}}$ is a weak equivalence.

A **strong symmetric monoidal Quillen equivalence** is the same thing, except that $(\mathbb{L}F \dashv \mathbb{R}G)$ is a Quillen equivalence, i.e. the derived functors also give an equivalence on the homotopy category. See also Exercise 6.6.27 for a weaker notion.

To check the first two conditions, we need to check that F is a left adjoint, that F preserves cofibrations and acyclic cofibrations, and that F preserves tensors and the units up to coherent isomorphism:

$$F(C) \subseteq C, \quad F(W \cap C) \subseteq W \cap C, \quad F(X) \otimes_{\mathbf{D}} F(Y) \cong F(X \otimes_{\mathbf{C}} Y), \quad F(I_{\mathbf{C}}) \cong I_{\mathbf{D}}.$$

Also, the third condition on the map $F(QI_{\mathbf{C}}) \to F(I_{\mathbf{C}}) \cong I_{\mathbf{D}}$ is automatically true if $I_{\mathbf{C}}$ is cofibrant.

Example 6.3.9. The following adjoint pairs are strong symmetric monoidal Quillen adjunctions:

- The disjoint basepoint $(-)_+ : \mathbf{Top} \to \mathbf{Top}_*$ and its right adjoint the forgetful functor $U : \mathbf{Top} \to \mathbf{Top}_*$.
- Suspension spectrum $\Sigma^\infty : \mathbf{Top}_* \to \mathbf{Sp}^\Sigma$ and its right adjoint the evaluation functor $\mathrm{ev}_0 : \mathbf{Sp}^\Sigma \to \mathbf{Top}_*$.
- Prolongation $P^O_\Sigma : \mathbf{Sp}^\Sigma \to \mathbf{Sp}^O$ and its right adjoint the forgetful functor $U^O_\Sigma : \mathbf{Sp}^O \to \mathbf{Sp}^\Sigma$.

These are closed under composition. So for instance, suspension with a disjoint basepoint Σ^∞_+ is also a symmetric monoidal left Quillen functor. We summarize with the following diagram of strong symmetric monoidal left Quillen functors:

$$\mathbf{Top} \xrightarrow{(-)_+} \mathbf{Top}_* \xrightarrow{\Sigma^\infty} \mathbf{Sp}^\Sigma \xrightarrow[\sim]{P^O_\Sigma} \mathbf{Sp}^O. \tag{6.3.10}$$

Remark 6.3.11. As in Remark 4.1.32, any time $F : \mathbf{C} \to \mathbf{D}$ is a strong symmetric monoidal functor, we can think of $\mathbf{D}$ as a "module" or "algebra" over $\mathbf{C}$. So unbased spaces act on the category of based spaces by

$$X \cdot Y := X_+ \wedge Y, \qquad X \in \mathbf{Top},\ Y \in \mathbf{Top}_*,$$

[3] We state it this way for simplicity, but you can get away with a little less. It's enough if F is oplax symmetric monoidal and the map $F(X \otimes_{\mathbf{D}} Y) \to F(X) \otimes_{\mathbf{C}} F(Y)$ is an isomorphism when X and Y are cofibrant.

based spaces act on symmetric or orthogonal spectra by

$$X \cdot Y := X \wedge Y \cong (\Sigma^\infty X) \wedge Y, \qquad X \in \mathbf{Top}_*,\ Y \in \mathbf{Sp},$$

and so on. In the diagram (6.3.10), every category "acts" in this way on every category to its right.

If $(F \dashv G)$ is a strong symmetric monoidal Quillen adjunction, it follows from Exercise 4.4.11 that the right Quillen functor $G\colon \mathbf{D} \to \mathbf{C}$ is lax symmetric monoidal. We also get:

Lemma 6.3.12. *The left-derived functor $\mathbb{L}F\colon \operatorname{Ho}\mathbf{C} \to \operatorname{Ho}\mathbf{D}$ is strong monoidal, and the right-derived functor $\mathbb{R}G\colon \operatorname{Ho}\mathbf{D} \to \operatorname{Ho}\mathbf{C}$ is lax monoidal.*

The proof is left to Exercise 6.6.26. As a consequence of Proposition 4.1.27, we also get

Corollary 6.3.13. *F and G both preserve monoid objects, and $\mathbb{L}F$ and $\mathbb{R}G$ preserve monoid objects in the homotopy category.*

We already observed this in Lemma 6.2.28, where we concluded that the forgetful and prolongation functors between $\mathbf{Sp}^\Sigma$ and $\mathbf{Sp}^O$ preserve ring spectra.

6.3.3 The monoid axiom

We next prove an important property about the smash product called the "monoid axiom." We will use this later in Section 6.5 to build the model category whose objects are ring spectra.

Suppose that $\mathbf{C}$ is a symmetric monoidal model category, and let $A = W \cap C$ denote the acyclic cofibrations. Let $A \otimes \mathbf{C}$ denote the class of maps of the form

$$X \otimes Z \xrightarrow{f \otimes \mathrm{id}} Y \otimes Z,$$

with $f : X \to Y$ an acyclic cofibration. As in Definition 5.1.1, a $(A \otimes \mathbf{C})$-cell complex is any countable composition of coproducts of maps in $A \otimes \mathbf{C}$.

Definition 6.3.14. We say that $\mathbf{C}$ satisfies the **monoid axiom** if every $(A \otimes \mathbf{C})$-cell complex is a weak equivalence in $\mathbf{C}$.

Example 6.3.15. Consider the symmetric monoidal model categories from Section 6.3.1:

- (CGWH) topological spaces $(\mathbf{Top}, \times, *)$,
- based spaces $(\mathbf{Top}_*, \wedge, S^0)$,

- bounded chain complexes $(\mathbf{Ch}_{\geq 0}(k), \otimes_k, k[0])$,
- unbounded chain complexes $(\mathbf{Ch}(k), \otimes_k, k[0])$,
- symmetric spectra $(\mathbf{Sp}^\Sigma, \wedge, \mathbb{S})$, or
- orthogonal spectra $(\mathbf{Sp}^O, \wedge, \mathbb{S})$.

All of these categories satisfy the monoid axiom. In most cases it is straightforward to check this, but for spectra we use the following proposition.

Proposition 6.3.16. *The categories* $\mathbf{Sp}^\Sigma$ *and* $\mathbf{Sp}^O$ *both satisfy the monoid axiom.*

Proof. Let $J \otimes \mathbf{C}$ be the smaller collection of maps of the form $f \wedge \mathrm{id}_Z$ in which f is in the generating acyclic cofibrations J. The domain and codomain of f is cofibrant, so by Exercise 6.6.12, every such map in $J \otimes \mathbf{C}$ is a stable equivalence. We would like to take pushouts and compositions of these, but we have a problem: they are not usually cofibrations, because Z is not cofibrant.

To fix this, recall from Definition 6.1.27 that an h**-cofibration** is a map of spectra $X \to Y$ such that the inclusion

$$(X \wedge I_+) \cup_{(X \wedge \{0\}_+)} (Y \wedge \{0\}_+) \longrightarrow Y \wedge I_+$$

has a retract. Notice that this property is preserved by taking a smash product with any spectrum $- \wedge Z$, even if Z is not cofibrant, because the smash product commutes with the definition of the mapping cylinder. It is clear that the suspension spectrum of $(S^{k-1} \to D^k)_+$ is an h-cofibration. Smashing this with $F_d S^0$, we conclude that each of the generating cofibrations in I is an h-cofibration. Then we observe that h-cofibrations are preserved by coproducts, pushouts, transfinite compositions, and retracts, and we conclude that every cofibration in the stable model structure is an h-cofibration.

In particular, every map in J is an h-cofibration. Again since these are preserved by smashing with any spectrum, every map in $J \otimes \mathbf{C}$ is an h-cofibration, in addition to being a stable equivalence.

Now we weaken the condition a little bit. Every h-cofibration is a levelwise based h-cofibration (Definition 6.1.27), so it has the based homotopy extension property at every spectrum level. By Theorem 6.1.31, the class of maps that are both stable equivalences and levelwise based h-cofibrations is closed under coproducts, pushouts, compositions, and retracts.

Each map in A is a retract of a J-cell complex, and therefore each map in $A \otimes \mathbf{C}$ is a retract of a $(J \otimes \mathbf{C})$-cell complex. Therefore each map in $A \otimes \mathbf{C}$ is a stable equivalence and a levelwise h-cofibration. Finally, we form any $(A \otimes \mathbf{C})$-cell complex, and again these properties are preserved, so the $(A \otimes \mathbf{C})$-cell complex is a stable equivalence. □

6.4 Other symmetric monoidal models of spectra

We spent this chapter focusing on symmetric and orthogonal spectra, but there are other important models of spectra that also have a smash product. We develop a few of them in this section.

6.4.1 The positive stable model structure

Our first example is not really a new example, because it's just symmetric spectra again. But the model structure is a little different.

Definition 6.4.1. A **positive cofibration** in $\mathbf{Sp}^\Sigma$ is a retract of a relative cell complex, in which all of the cells $F_i^\Sigma(S^{n-1} \to D^n)_+$ that appear have $i > 0$. There is no restriction on n.

In other words, we can attach cells of any stable dimension, but we can only attach them at spectrum level 1 and above. A positive cofibration in $\mathbf{Sp}^O$ is defined the same way.

Theorem 6.4.2. *The category of symmetric spectra $\mathbf{Sp}^\Sigma$ has a **positive stable model structure** in which*

- *the cofibrations are the positive cofibrations,*
- *the weak equivalences are the stable equivalences, and*
- *the fibrations are the maps $X \to Y$ such that $X_i \to Y_i$ is a Serre fibration, and the squares*

$$\begin{array}{ccc} X_i & \longrightarrow & \Omega^j X_{i+j} \\ \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle \Omega^j p_{i+j}} \\ Y_i & \longrightarrow & \Omega^j Y_{i+j} \end{array}$$

  are homotopy pullbacks, for all positive values of i.

The category of orthogonal spectra $\mathbf{Sp}^O$ also has a positive stable model structure with the same description.

The proof is the same as the proof of Theorem 6.1.36 and its earlier variant Theorem 5.2.12, only we change the sets of maps I and J from Definition 5.6.1 and Definition 5.6.3 to only use positive values of i:

$$\begin{aligned} I^+ &= \{\ F_i S^{n-1}_+ \longrightarrow F_i D^n_+ \ :\ n \geq 0,\ i > 0\ \}, \\ J^+ &= \{\ F_i(D^n \times \{0\})_+ \longrightarrow F_i(D^n \times I)_+ \ :\ n \geq 0,\ i > 0\ \} \\ &\quad \cup \{\ k_{i,j} \,\square\, \left(S^{n-1}_+ \longrightarrow D^n_+\right) \ :\ j, n \geq 0,\ i > 0\ \}. \end{aligned}$$

A fibrant object in this model structure is a **positive Ω-spectrum**: it is a spectrum X such that the maps $X_n \to \Omega X_{n+1}$ are all weak equivalences when $n \geq 1$, not necessarily when $n = 0$.

Lemma 6.4.3. *The identity functor of* $\mathbf{Sp}^\Sigma$ *gives a Quillen equivalence between the positive stable model structure and the stable model structure.*

Proof. This can be checked directly from the definitions of the two model structures in Theorems 6.1.36 and 6.4.2. The identity is left Quillen when going from the positive one to the ordinary one, and right Quillen when going from the ordinary one to the positive one. □

We let $\mathbf{Sp}^\Sigma_+$ denote the category of symmetric spectra with the positive stable model structure. So it is the same category as $\mathbf{Sp}^\Sigma$, but the model structure is different.

Lemma 6.4.4. $\mathbf{Sp}^\Sigma_+$ *is a symmetric monoidal model category. The identity functor* $\mathbf{Sp}^\Sigma_+ \to \mathbf{Sp}^\Sigma$ *is a strong symmetric monoidal left Quillen adjoint.*

Proof. This is proven just as in Proposition 6.3.4, only the variables d, e, and i in that proof all have to be positive. □

To summarize, the diagram (6.3.10) can be expanded to the following diagram of strong symmetric monoidal Quillen left adjoints:

$$\begin{array}{ccccccc} \mathbf{Top} & \xrightarrow{(-)_+} & \mathbf{Top}_* & \xrightarrow{\Sigma^\infty} & \mathbf{Sp}^\Sigma & \xrightarrow[\sim]{P^O_\Sigma} & \mathbf{Sp}^O \\ & & & & \uparrow{\scriptstyle \mathrm{id}}\ \uparrow{\scriptstyle \sim} & & \uparrow{\scriptstyle \mathrm{id}}\ \uparrow{\scriptstyle \sim} \\ & & & & \mathbf{Sp}^\Sigma_+ & \xrightarrow[\sim]{P^O_\Sigma} & \mathbf{Sp}^O_+. \end{array} \tag{6.4.5}$$

Remark 6.4.6. The suspension spectrum functor $\Sigma^\infty \colon \mathbf{Top}_* \to \mathbf{Sp}^\Sigma$ is no longer left Quillen when we give $\mathbf{Sp}$ the positive stable model structure!

6.4.2 EKMM spectra

In the early 1990s, Hovey, Shipley, and Smith developed the theory of symmetric spectra, providing a solution to the problem of giving the category of spectra a well-behaved smash product. At about the same time, a second group of authors developed in parallel an *entirely different* category of spectra that has a smash product. We call it the category of **EKMM spectra** after its creators.[4]

[4] Michael Cole also provided significant simplifications to the theory, see [EKMM97] for more details.

Theorem 6.4.7 (Elmendorf–Kriz–Mandell–May, [EKMM97, MM02])**.** *There is a category $\mathscr{M}$, whose objects we call EKMM spectra, with the following properties.*

- $\mathscr{M}$ is a closed symmetric monoidal category, whose product we call the smash product $\wedge$ and whose unit we call the sphere spectrum $\mathbb{S}$.
- $\mathscr{M}$ is also a model category, whose weak equivalences we call the stable equivalences.
- Every object in $\mathscr{M}$ is fibrant. The sphere spectrum $\mathbb{S}$ fails to be cofibrant.
- The smash product $\wedge$ is a Quillen tensor, and the map $Q\mathbb{S} \wedge QX \to \mathbb{S} \wedge QX$ is always a stable equivalence. Therefore, $\mathscr{M}$ is a symmetric monoidal model category.
- There is a strong symmetric monoidal Quillen equivalence $(\mathbb{N}, \mathbb{N}^{\#})$ between positive orthogonal spectra $\mathbf{Sp}^O_+$ and $\mathscr{M}$.
- The left adjoint $\mathbb{N}\colon \mathbf{Sp}^O_+ \to \mathscr{M}$ preserves all stable equivalences between cofibrant orthogonal spectra, not just the ones that are positive cofibrant. However, if there are any cells at spectrum level 0, they don't go to cofibrant objects of $\mathscr{M}$.

We don't give the definition of EKMM spectra in this book. It is more complicated than the definition of symmetric or orthogonal spectra, so if you want to get your hands dirty and work with a definition directly, diagram spectra are usually the way to go. The main advantage of EKMM spectra is that every object is fibrant. This can be very useful.

You can work effectively with EKMM spectra by using Theorem 6.4.7 as a black box. For instance, since $(\mathbb{N}, \mathbb{N}^{\#})$ is a Quillen equivalence, the homotopy category $\mathrm{Ho}\,\mathscr{M}$ is equivalent to the stable homotopy category $\mathrm{Ho}\,\mathbf{Sp}$. Since $\mathscr{M}$ is a symmetric monoidal category, we can define EKMM ring spectra just as before. And since the Quillen equivalence $(\mathbb{N}, \mathbb{N}^{\#})$ is strong symmetric monoidal, EKMM ring spectra up to equivalence are the same thing as symmetric or orthogonal ring spectra up to equivalence.

We define the EKMM suspension spectrum of a based space A to be $\mathbb{N}\Sigma^\infty A$. This is usually not cofibrant, even if A is a cell complex. You might find it odd that the sphere is not cofibrant, but this is actually impossible:

Theorem 6.4.8. *There does not exist a symmetric monoidal model category of spectra* $\mathbf{Sp}$ *with the following properties:*

- The homotopy category $\mathrm{Ho}\,\mathbf{Sp}$ is equivalent to the classical stable homotopy category, as a symmetric monoidal category.
- The unit object of $\mathbf{Sp}$ is both cofibrant and fibrant.

Proof. If there were such a category, the space of maps from the unit object to itself would be equivalent to $QS^0 = \Omega^\infty \mathbb{S}$, and would also be a commutative monoid. Any grouplike commutative monoid in $(\mathbf{Top}_*, \times, *)$ is equivalent to a product of Eilenberg–Mac Lane spaces (see Lemma 9.1.14, Corollary 9.1.25, and Lemma 9.2.18 for details), but QS^0 is not equivalent to such a product by Exercise 9.6.14, so we have a contradiction. □

Remark 6.4.9. This is a weaker form of a theorem of Gaunce Lewis. Lewis's result does not need to assume that Ho **Sp** is equivalent to the classical stable homotopy category: it is enough to ask for a strong symmetric monoidal adjunction $(\Sigma^\infty, \Omega^\infty)$ between $\mathbf{Top}_*$ and **Sp**, such that $\Omega^\infty\Sigma^\infty X$ has the homotopy type we expect. See [Lew91].

Remark 6.4.10. In the category of symmetric spectra, the sphere spectrum $\mathbb{S}$ is cofibrant but not fibrant. In the category of EKMM spectra, the sphere spectrum $\mathbb{S}$ is fibrant but not cofibrant. By Theorem 6.4.8, it is impossible for $\mathbb{S}$ to be both.

To summarize, the diagram (6.4.5) can be further expanded to the following diagram of strong symmetric monoidal Quillen left adjoints:

$$\begin{array}{ccccccc} \mathbf{Top} & \xrightarrow{(-)_+} & \mathbf{Top}_* & \xrightarrow{\Sigma^\infty} & \mathbf{Sp}^\Sigma & \xrightarrow[\sim]{P^O_\Sigma} & \mathbf{Sp}^O \\ & & & & \uparrow{\scriptstyle \mathrm{id}} \sim & & \uparrow{\scriptstyle \mathrm{id}} \sim \\ & & & & \mathbf{Sp}^\Sigma_+ & \xrightarrow[\sim]{P^O_\Sigma} & \mathbf{Sp}^O_+ \xrightarrow[\sim]{\mathbb{N}} \mathscr{M}. \end{array} \tag{6.4.11}$$

6.4.3 Reduced excisive functors*

Another important perspective is that spectra are the same thing as functors from spaces to spaces that have certain properties. Recall that $F : \mathbf{Top}_* \to \mathbf{Top}_*$ is a **homotopy functor** if it sends every weak equivalence $A \xrightarrow{\sim} B$ to a weak equivalence $F(A) \xrightarrow{\sim} F(B)$.

Definition 6.4.12. For any homotopy functor $F : \mathbf{Top}_* \to \mathbf{Top}_*$, we say that F is

- **finitary** if for any CW complex K and its finite subcomplexes $K_\alpha \subseteq K$ the canonical map (see (8.6.2))
$$\operatorname*{hocolim}_\alpha F(K_\alpha) \longrightarrow F(K)$$
is an equivalence,
- **reduced** if $F(*)$ is contractible, and
- **excisive** if for each homotopy pushout square
$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D, \end{array}$$

applying F gives a homotopy pullback square

$$\begin{array}{ccc} F(A) & \xrightarrow{F(g)} & F(B) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F(h)} \\ F(C) & \xrightarrow[F(k)]{} & F(D). \end{array}$$

In this section we show that up to equivalence, a spectrum is the same thing as a finitary, reduced, excisive homotopy functor $F : \mathbf{Top}_* \to \mathbf{Top}_*$. We also show how to define the smash product from this point of view.

To make our life easier, we first restrict attention to homotopy functors

$$F : \mathbf{FinCW}_* = \mathscr{W}_0 \to \mathbf{Top}_*,$$

where $\mathbf{FinCW}_* = \mathscr{W}_0$ is the category of finite based CW complexes.[5] This is not actually a restriction because finitary homotopy functors $F : \mathbf{Top}_* \to \mathbf{Top}_*$ are the same thing as homotopy functors $\mathscr{W}_0 \to \mathbf{Top}_*$, up to weak equivalence. (This is proven in Exercise 8.7.27, after we develop more of the properties of homotopy colimits.)

Let $\mathrm{Fun}(\mathscr{W}_0, \mathbf{Top}_*)$ be the category of all such functors, and let us use the superscripts "red" and "exc" for reduced and excisive, respectively. So $\mathrm{Fun}^{\mathrm{red\ exc}}(\mathscr{W}_0, \mathbf{Top}_*)$ is the category of reduced excisive functors.

Example 6.4.13. If $E \in \mathbf{Sp}^{\mathbb{N}}$ is any sequential spectrum, we define a functor on $\mathscr{W}_0$ by sending $A \in \mathscr{W}_0$ to the based space

$$F(A) = \Omega^\infty(A \wedge E).$$

It is left to Exercise 6.6.28 to show that F is a reduced excisive functor.

In fact, every reduced excisive functor is of this form. To give a simple description of the inverse operation, given any reduced excisive functor F, we define an Ω-spectrum E by

$$E_n = F(S^n).$$

We have the homotopy pullback square

$$\begin{array}{ccc} F(S^n) & \longrightarrow & F(D^{n+1}) \\ \downarrow & & \downarrow \\ F(D^{n+1}) & \longrightarrow & F(S^{n+1}), \end{array}$$

[5] We fix a model of this category in which the collection of objects forms a set, not a proper class. One way to do this is to only consider finite CW complexes whose underlying set is a subset of $\mathbb{R}$. Another way to do it is to consider finite CW complexes that are continuously embedded into $\mathbb{R}^\infty$.

where $F(D^{n+1})$ is contractible. Therefore the homotopy pullback is equivalent to the based loopspace ΩE_{n+1}. Since this is a homotopy pullback square, we therefore get an equivalence $E_n \xrightarrow{\sim} \Omega E_{n+1}$, though it is only well-defined up to homotopy. Choosing a representative of the homotopy class for each value of n, we get bonding maps that make E into an Ω-spectrum.

Theorem 6.4.14. *These two operations give an equivalence of homotopy categories*

$$\mathrm{Ho}\,\mathrm{Fun}^{\text{red exc}}(\mathscr{W}_0, \mathbf{Top}_*) \simeq \mathrm{Ho}\,\mathbf{Sp}^{\mathbb{N}}.$$

In other words, reduced excisive functors are equivalent to spectra.

We describe the proof in the remainder of this section. As you might imagine, to make the proof run, we first need to give a more canonical construction of the spectrum E associated to a reduced excisive functor F. To do this, we change the category of reduced excisive functors to something that is equivalent, but has more structure.

Let us say that the homotopy functor F is **strictly reduced** if $F(*) \cong *$. In other words, $F(*)$ is literally a one-point space, not just contractible.

Lemma 6.4.15. *The inclusion of strictly reduced functors into reduced functors gives an equivalence of homotopy categories*

$$\mathrm{Ho}\,\mathrm{Fun}^{\text{str red}}(\mathscr{W}_0, \mathbf{Top}_*) \simeq \mathrm{Ho}\,\mathrm{Fun}^{\text{red}}(\mathscr{W}_0, \mathbf{Top}_*).$$

The proof of this proposition is relatively easy but uses simplicial sets, so it is deferred to Exercise 8.7.28. Next we enrich the functors.

Definition 6.4.16. A **$\mathscr{W}$-spectrum** is a strictly reduced functor $F : \mathbf{FinCW}_* \to \mathbf{Top}_*$ that is enriched in based spaces. In other words, if we let $\mathscr{W}$ denote the version of $\mathbf{FinCW}_*$ in which the mapping spaces have a topology, it is a diagram of based spaces $\mathscr{W} \to \mathbf{Top}_*$ such that $* \mapsto *$. Let $\mathbf{Sp}^{\mathscr{W}}$ denote the category of $\mathscr{W}$-spectra.

So a $\mathscr{W}$-spectrum is a functor from finite CW complexes to based spaces, but the maps

$$\mathrm{Map}_*(A, B) \xrightarrow{F} \mathrm{Map}_*(F(A), F(B))$$

induced by F are continuous.[6] A map of enriched functors is the same as a map of underlying functors. See Exercise 6.6.29.

Every enriched functor has **assembly maps**

$$F(A) \wedge B \longrightarrow F(A \wedge B) \tag{6.4.17}$$

[6]Notice that this is a condition on F, not extra data, though in general giving a functor an enrichment requires extra data. We get away with murder here because the forgetful functor $\mathbf{Top} \to \mathbf{Set}$ is faithful.

for all based CW complexes A and B, defined by sending each $b \in B$ to the map $F(A) \to F(A \wedge B)$ that is F applied to the inclusion $A \times \{b\} \to A \wedge B$. If F is enriched then these assembly maps are continuous, and satisfy an associativity and unitality property – this is left to Exercise 8.7.36.

As a result, we have an **underlying orthogonal spectrum** $F[S^0]$ of the $\mathscr{W}$-spectrum F, whose levels are the spaces $F(S^n)$, whose bonding maps are the assembly maps $F(S^n) \wedge S^1 \to F(S^{n+1})$, and whose $O(n)$-actions come from the canonical $O(n)$-action on $S^n = \mathbb{R}^n \cup \{\infty\}$. Similarly for each $A \in \mathscr{W}_0$ we can define an orthogonal spectrum $F[A]$ whose nth level is $F(A \wedge S^n)$.

So enriched functors give us a more structured way of turning functors into spectra.

Lemma 6.4.18. *If $F : \mathscr{W} \to \mathbf{Top}_*$ is enriched, then when considered as an ordinary functor $\mathscr{W}_0 \to \mathbf{Top}_*$, it is a strictly reduced homotopy functor.*

Proof. The key observation is a homotopy of maps $A \to B$ is the same thing as a map $I_+ \to \mathrm{Map}_*(A, B)$. Since F is enriched, this becomes a homotopy of maps $F(A) \to F(B)$. Therefore F sends homotopy equivalences to homotopy equivalences. Since the domain is restricted to CW complexes, this means that F is a homotopy functor. □

Proposition 6.4.19. *The operation that forgets the enrichment has an inverse up to weak equivalence: every strictly reduced homotopy functor $\mathscr{W}_0 \to \mathbf{Top}_*$ can be turned into an enriched functor $\mathscr{W} \to \mathbf{Top}_*$, and going back and forth between the two notions gives the same functor up to weak equivalence. In other words, we get an equivalence of homotopy categories*

$$\mathrm{Ho}\,\mathrm{Fun}(\mathscr{W}, \mathbf{Top}_*) \simeq \mathrm{Ho}\,\mathrm{Fun}^{hsr}(\mathscr{W}_0, \mathbf{Top}_*).$$

The proof of this proposition makes heavy use of simplicial sets, so it is deferred to Exercise 8.7.29. See also [Blo25, Sec 3].

At this point we have shown that finitary, reduced homotopy functors are the same thing as enriched functors, in other words diagrams $\mathscr{W} \to \mathbf{Top}_*$. So to finish the proof of Theorem 6.4.14 we only need to explain how the excisive functors fit into this, and why the excisive enriched functors are the same thing as spectra.

Following [MMSS01], we do this by putting a stable model structure on $\mathrm{Fun}(\mathscr{W}, \mathbf{Top}_*)$ such that the fibrant objects are the excisive functors, and the weak equivalences between fibrant objects are just the pointwise equivalences. Then the proof is finished by showing that this model category is Quillen equivalent to orthogonal spectra.

Definition 6.4.20. Given two diagrams $X, Y : \mathscr{W} \to \mathbf{Top}_*$, a map $X \to Y$ is a *stable equivalence* if it is a stable equivalence on the underlying spectra $X[S^0] \to Y[S^0]$. It is an **absolute stable equivalence** if it induces a stable equivalence $X[A] \to Y[A]$ for all finite CW complexes A.

Lemma 6.4.21. *[BF78, 4.1], [MMSS01] A map of functors $X \to Y$ is a stable equivalence iff it is an absolute stable equivalence.*

Proof. The assembly maps give a natural map of spectra $A \wedge X[S^0] \to X[A]$. It suffices to prove that this is always a stable equivalence of orthogonal spectra. We first prove this when X is a free diagram of the form $F_B(K)$, whose value at C is $K \wedge \mathscr{W}(B,C)$. In this case $X[S^0]$ is the cotensor spectrum $F(B,\mathbb{S})$, $X[A]$ is the cotensor spectrum $F(B,A\wedge\mathbb{S})$, and the natural map between them is the interchange

$$A \wedge F(B,\mathbb{S}) \to F(B,A\wedge\mathbb{S}).$$

We see that this is a stable equivalence for all B by induction on the cells of B, since adding one new cell creates a map of cofiber/fiber sequences, and then we can use the five lemma to do the inductive step.

Now that we know that the natural map $A \wedge X[S^0] \to X[A]$ is an equivalence when X is a free diagram, we take wedges and cofibers of the result to show that it is an equivalence when X is any cofibrant diagram in the projective model structure. But since $A \wedge (-)$ preserves all level equivalences, this map is also a stable equivalence for all X. □

Definition 6.4.22. We say that a functor $X : \mathscr{W} \to \mathbf{Top}_*$ is an **absolute Ω-spectrum** if every adjoint map $X(A) \to \Omega X(A \wedge S^1)$ is an equivalence.

Lemma 6.4.23. *The functor $X : \mathscr{W} \to \mathbf{Top}_*$ is an absolute Ω-spectrum iff it is excisive.*

Proof. If X is excisive then each of the squares

$$\begin{array}{ccc} X(S^n) & \longrightarrow & X(D^n) \\ \downarrow & & \downarrow \\ X(*) & \longrightarrow & X(S^{n+1}) \end{array}$$

is a homotopy pullback, whence the map from the first term to loops of the last term is a weak equivalence. (You should also convince yourself that the map to the homotopy pullback agrees with the bonding map $X(S^n) \to \Omega X(S^{n+1})$.)

Conversely, if X is an absolute Ω-spectrum then we get a natural equivalence $X(A) \simeq \Omega^\infty(A \wedge X[S^0])$ by the proof of Lemma 6.4.21. So for any two maps of CW complexes $B \leftarrow A \to C$ with pushout $B \cup_A C$, the square below on the left is weakly equivalent to the one on the right. This is the infinite loopspace of a homotopy pushout/pullback square of spectra, and so it is a homotopy pullback square of spaces. Therefore X is excisive.

$$\begin{array}{ccc} X(A) & \longrightarrow & X(C) \\ \downarrow & & \downarrow \\ X(B) & \longrightarrow & X(B \cup_A C) \end{array} \qquad \begin{array}{ccc} \Omega^\infty(A \wedge X[S^0]) & \longrightarrow & \Omega^\infty(C \wedge X[S^0]) \\ \downarrow & & \downarrow \\ \Omega^\infty(B \wedge X[S^0]) & \longrightarrow & \Omega^\infty((B \cup_A C) \wedge X[S^0]) \end{array}$$

□

Theorem 6.4.24. *The category* $\mathbf{Sp}^{\mathscr{W}}$ *has an* ***absolute stable model structure*** *in which*

- *the cofibrations are the same as in the projective model structure,*
- *the weak equivalences are the (absolute) stable equivalences, and*
- *the fibrations are the maps* $X \to Y$ *such that* $X(A) \to Y(A)$ *is a Serre fibration, and the squares*

$$\begin{array}{ccc} X(A) & \longrightarrow & \Omega X(\Sigma A) \\ \downarrow{\scriptstyle p_A} & & \downarrow{\scriptstyle \Omega p_{\Sigma A}} \\ Y(A) & \longrightarrow & \Omega Y(\Sigma A) \end{array}$$

are homotopy pullbacks for every $A \in \mathscr{W}$.

In particular, the fibrant objects are the excisive functors, by Lemma 6.4.3.

Proof. As in Theorem 6.1.36, the proof follows the same outline as Section 5.6, using the analogues of Lemma 6.1.16, Theorem 6.1.31, and Corollary 6.1.32, which follow quickly because a stable equivalence of $\mathscr{W}$-spectra is the same thing as a stable equivalence on the underlying orthogonal spectrum. □

Proposition 6.4.25. *The underlying spectrum functor* $U_O^{\mathscr{W}} \colon \mathbf{Sp}^{\mathscr{W}} \to \mathbf{Sp}^O$ *and its left adjoint* $P_O^{\mathscr{W}}$ *given a Quillen equivalence between this model category and the stable model structure on* $\mathbf{Sp}^O$.

Proof. The adjunction $(P_O^{\mathscr{W}}, U_O^{\mathscr{W}})$ is just another example of restriction and left Kan extension. As in the proof of Proposition 6.1.42, we only have to check that for each well-based space K, the canonical map of orthogonal spectra from $F_i^O K$ to the free $\mathscr{W}$-spectrum $F_i^{\mathscr{W}} K = F_{S^i} K$ is a π_*-isomorphism. But this latter spectrum is just the cotensor $F(S^i, \mathbb{S}) = \Omega^i \mathbb{S}$, which allows us to check that this map is an equivalence. □

This finishes the proof of Theorem 6.4.14, that reduced excisive finitary homotopy functors are equivalent to orthogonal spectra. Indeed, by the above theorem, orthogonal spectra are equivalent to $\mathscr{W}$-spectra up to stable equivalence, or fibrant $\mathscr{W}$-spectra up to level equivalence, which are excisive enriched functors. Then the previous results show that "enriched" is equivalent to "reduced finitary homotopy," so that we end up with reduced excisive finitary homotopy functors.[7]

We close out the section by discussing the smash product of $\mathscr{W}$-spectra. It is defined in exactly the same way as in Definition 6.2.17, only using $\mathscr{W}$. To be more specific, given

[7] We have followed the proof of this theorem from [MMSS01]; see [BCR07] for a different approach.

two $\mathscr{W}$-spectra X and Y, we form the diagram on the category $\mathscr{W} \wedge \mathscr{W}$ that sends (A, B) to $X(A) \wedge Y(B)$, and then take its left Kan extension along the smash product functor $\mathscr{W} \wedge \mathscr{W} \to \mathscr{W}$ to define a single $\mathscr{W}$-spectrum that we call $X \wedge Y$.

Proposition 6.4.26. *This makes* $\mathbf{Sp}^{\mathscr{W}}$ *into a symmetric monoidal model category, and the prolongation functor* $P_O^{\mathscr{W}}$ *is strong symmetric monoidal.*

The proof is the same as in Theorem 6.2.25. As a result, we get a strong symmetric monoidal Quillen equivalence between orthogonal spectra and $\mathscr{W}$-spectra, so the two notions of ring spectra are equivalent as well. We have now extended the diagram (6.4.11) to the following diagram of strong symmetric monoidal Quillen left adjoints:

$$\begin{array}{ccccccccc} \mathbf{Top} & \xrightarrow{(-)_+} & \mathbf{Top}_* & \xrightarrow{\Sigma^\infty} & \mathbf{Sp}^\Sigma & \xrightarrow[\sim]{P_\Sigma^O} & \mathbf{Sp}^O & \xrightarrow[\sim]{P_O^{\mathscr{W}}} & \mathbf{Sp}^{\mathscr{W}} \\ & & & & \uparrow{\scriptstyle \mathrm{id}} \; \sim & & \uparrow{\scriptstyle \mathrm{id}} \; \sim & & \\ & & & & \mathbf{Sp}^\Sigma_+ & \xrightarrow[\sim]{P_\Sigma^O} & \mathbf{Sp}^O_+ & \xrightarrow[\sim]{\mathbb{N}} & \mathscr{M}. \end{array} \tag{6.4.27}$$

Monoids in $\mathscr{W}$-spectra are called **functors with smash product (FSP)**. Their multiplicative structure is determined by maps

$$S^n \longrightarrow F(S^n), \qquad F(A) \wedge F(B) \longrightarrow F(A \wedge B)$$

that are associative and unital in the appropriate sense (see e.g. [MMSS01, 22.5]).

Remark 6.4.28. Reduced excisive functors on $\mathbf{FinCW}_* = \mathscr{W}_0$, or equivalently finitary reduced excisive functors on $\mathbf{Top}_*$, are sometimes called "homology theories." Conceptually, this is because they satisfy axioms like the axioms for an extraordinary homology theory. Practically, this is because they are equivalent to spectra, which classify extraordinary homology theories. These two perspectives give the same answer, in the sense that for a spectrum E, the associated functor $\mathbf{Top}_* \to \mathbf{Top}_*$ takes the space K to $\Omega^\infty(K \wedge E)$, whose homotopy groups are the extraordinary homology groups $E_*(K)$, at least in nonnegative degrees.

Remark 6.4.29. When we define symmetric and orthogonal spectra, we might wonder what is so special about the groups Σ_n and $O(n)$ that make the construction work. Well, it turns out they aren't special. We could have used a larger subgroup or even submonoid of $\mathrm{Map}_*(S^n, S^n)$ to act on spectrum level n. In $\mathscr{W}$-spectra we make the largest possible choice, namely the entire monoid $\mathrm{Map}_*(S^n, S^n)$, while in symmetric spectra we make the smallest possible choice, namely the symmetric groups Σ_n. The orthogonal groups $O(n)$ are a convenient middle ground, because they are large enough to make the stable equivalences into π_*-isomorphisms, while still being much smaller than all of $\mathrm{Map}_*(S^n, S^n)$.

6.5 Model categories of ring spectra and module spectra*

In this section we prove Theorem 6.2.36. In other words, we define a model category Alg(**Sp**) whose objects are (symmetric or orthogonal) ring spectra, and whose weak equivalences are ring homomorphisms that are stable equivalences on the underlying spectra. We also do the same for modules Mod(R), and for commutative ring spectra CAlg(**Sp**). This means that we can replace rings or modules up to equivalence with better-behaved versions (cofibrant or fibrant, or both), in order to get various constructions to work. For instance, we will use this when we define the bar construction for ring spectra (Corollary 8.3.20).

6.5.1 The formal part

We begin by defining the categories of algebras and modules, and spelling out what we need for the proof in each case. Let **C** be any symmetric monoidal category (Definition 4.1.1), such as $\mathbf{C} = \mathbf{Sp}^\Sigma$ or $\mathbf{Sp}^O$. Recall that an **algebra** in **C** is an object R together with a multiplication map and a unit map

$$\mu : R \otimes R \longrightarrow R, \qquad \eta : I \longrightarrow R,$$

such that the diagrams in Definition 4.1.12 commute. In essence, these diagrams say the multiplication μ is associative and has η as a unit. An **commutative algebra** is one in which the diagram in Definition 4.1.15 also commutes; in other words the multiplication μ commutes with swapping its two inputs. Finally, a **module** over an algebra R is an object M together with an action map

$$\alpha : R \otimes M \longrightarrow M,$$

making the diagrams in Definition 4.1.13 commute.

In this section, we will put more emphasis on the fact that there is a category of all such algebras, and a category of all such modules.

Definition 6.5.1. If **C** is a symmetric monoidal category, the **category of algebras** Alg(**C**), also called the category of monoids Mon(**C**), has an object for each algebra R in **C**. A morphism $\phi : R \to S$ is a morphism in **C** that commutes with the multiplication and unit:

$$\begin{array}{ccc} R \otimes R & \xrightarrow{\phi\otimes\phi} & S \otimes S \\ {\scriptstyle \mu_R}\downarrow & & \downarrow{\scriptstyle \mu_S} \\ R & \xrightarrow[\phi]{} & S \end{array} \qquad \begin{array}{ccc} I & = & I \\ {\scriptstyle \eta_R}\downarrow & & \downarrow{\scriptstyle \eta_S} \\ R & \xrightarrow[\phi]{} & S. \end{array}$$

The **category of commutative algebras** CAlg(**C**) or CMon(**C**) is the full subcategory consisting of those algebras that are commutative. In other words, a map of commutative algebras is the same thing as a map of the underlying algebras.

Similarly for a fixed algebra R, the **category of left R-modules** Mod(R) has an object for each R-module M and a morphism for each map $f : M \to N$ in **C** that is "R-linear" in the sense that it commutes with the R-action:

$$\begin{array}{ccc} R \otimes M & \xrightarrow{\mathrm{id} \otimes f} & R \otimes N \\ {\scriptstyle \alpha_M}\downarrow & & \downarrow{\scriptstyle \alpha_S} \\ M & \xrightarrow[f]{} & N. \end{array}$$

We already saw these in the special case of ring and module spectra in Definition 6.2.32.

We construct model structures on all three of these categories using the same formal recipe. Letting **D** denote any one of the above three categories associated to **C**, the procedure is as follows:

- Show that there is a forgetful functor $U : \mathbf{D} \to \mathbf{C}$ and a left adjoint $F : \mathbf{C} \to \mathbf{D}$ that we call the free functor.
- Prove that **D** has all (small) colimits and limits.
- Assume that **C** is cofibrantly generated, and let I and J are the generating cofibrations and generating acyclic cofibrations, respectively.
- Define a **free cofibration** in **D** to be a retract of a relative $F(I)$-cell complex, where $F(I)$ is the set of maps in **D** obtained by applying F to every map in I.
- Define a **underlying weak equivalence** or an **underlying fibration** in **D** to be a map that is a weak equivalence or fibration in **C**, respectively, after applying the forgetful functor U.
- Show that $F(I)$ and $F(J)$ each satisfy the countable smallness condition in **D**.
- Prove that every $F(J)$-cell complex is an (underlying) weak equivalence.

Theorem 6.5.2. *If these steps are completed, then* **D** *is a cofibrantly generated model category with the free cofibrations, the underlying weak equivalences, and the underlying fibrations. The cofibrations and acyclic cofibrations are generated by $F(I)$ and $F(J)$, respectively.*

Proof. The proof is immediate using either Proposition 5.2.6 or Exercise 5.8.38. In the literature this is often called a **right-induced model structure** on **D**. □

Remark 6.5.3. One might like to give a cleaner statement of this theorem that doesn't directly ask for a smallness condition, and that holds for, say, any symmetric monoidal model category with the monoid axiom. However, as we mentioned in Exercise 5.8.39, it is difficult to give good, clean, general conditions under which the smallness condition holds, that actually holds in all of the examples of interest and not just some of them. We will have to be content with just checking these assumptions in each example.

Well, that concludes the formal part. It remains to verify these hypotheses for modules, algebras, and commutative algebras in spectra.

6.5.2 The model structure on R-modules

Let $\mathbf{C}$ be a symmetric monoidal category and let R be an algebra in $\mathbf{C}$. Let

$$U_R \colon \mathrm{Mod}(R) \to \mathbf{C}$$

be the functor that forgets the R-action. Going the other way, for each object X of $\mathbf{C}$, we can form the **free R-module** by taking the abelian group

$$T_R(X) := R \otimes X$$

and giving it the R-action

$$R \otimes R \otimes X \xrightarrow{\mu \otimes \mathrm{id}} R \otimes X.$$

We let $F_R(X)$ denote the resulting R-module, whose underlying object of $\mathbf{C}$ is $T_R(X)$.

Lemma 6.5.4. *The functor F_R is the left adjoint of U_R.*

Proof. Left as an exercise. For spectra, see Exercise 6.6.25. □

Lemma 6.5.5. *Suppose the symmetric monoidal category $\mathbf{C}$ has all colimits and limits, and that $\otimes$ preserves colimits in each slot. Then the category* Mod(R) *has all colimits and limits as well.*

Proof. Let $M \colon \mathbf{I} \to \mathrm{Mod}(R)$ be a diagram of modules, sending i to the module M_i. We form the limit in Mod(R) by taking the limit in $\mathbf{C}$, which we denote $\lim_{(\mathbf{C})} M_i$. To give it an R-action, we form the diagram

$$\begin{array}{ccc} R \otimes \left(\lim\limits_{(\mathbf{C})} M_i \right) & \dashrightarrow & \lim\limits_{(\mathbf{C})} M_i \\ \downarrow & & \downarrow \\ R \otimes M_i & \xrightarrow{\alpha_{M_i}} & M_i \end{array}$$

in which the vertical maps project from the limit to one of the terms of the diagram. We check that the resulting maps $R \otimes \lim M_i \to M_i$ commute with the maps of the diagram $M_i \to M_j$, and therefore lift to a map to the limit, as indicated by the dashed arrow. A straightforward diagram-chase shows that the diagrams in Definition 4.1.13 commute for this limit module, so that $\lim M_i$ is indeed an R-module. The map from this back to each of the M_i is an R-module map by construction.

Given any module N and compatible R-module maps $f_i : N \to M_i$, giving a map of objects $f : N \to \lim M_i$ in $\mathbf{C}$, we get the larger diagram

$$\begin{array}{ccc} R \otimes N & \xrightarrow{\alpha_N} & N \\ \downarrow{\scriptstyle \mathrm{id}\otimes f} & & \downarrow{\scriptstyle f} \\ R \otimes \left(\underset{(\mathbf{C})}{\lim} M_i\right) & \dashrightarrow & \underset{(\mathbf{C})}{\lim} M_i \\ \downarrow & & \downarrow \\ R \otimes M_i & \xrightarrow{\alpha_{M_i}} & M_i \end{array}$$

in which map f exists and is unique by the universal property of the limit in $\mathbf{C}$. Since the map from N to each M_i is a module map, the big rectangle commutes for each i. Therefore the upper square commutes after composing with $\lim M_i \to M_i$ for each i. Therefore the upper square commutes. This proves that the unique lift $f : N \to \lim M_i$ is in fact a map of R-modules, and finishes the proof that $\mathrm{Mod}(R)$ has limits.

Next we do colimits. Since $\otimes$ preserves colimits in each slot, we get a canonical isomorphism

$$R \otimes \left(\underset{(\mathbf{C})}{\operatorname{colim}} M_i\right) \cong \underset{(\mathbf{C})}{\operatorname{colim}} (R \otimes M_i)$$

determined by the property that it commutes with the map from $R \otimes M_i$ to each side, for each value of i. Using this isomorphism, the actions $R \otimes M_i \to M_i$ give an action of R on $\operatorname{colim} M_i$, as indicated in the top row of the following diagram. It is now straightforward to check the diagrams in Definition 4.1.13 for this colimit module. Given any module N and compatible R-module maps $f_i : M_i \to N$, giving a map of objects $f : \operatorname{colim} M_i \to N$ in $\mathbf{C}$, the diagram below shows that f is an R-module map, and is therefore the unique such map extending the f_i, which proves that $\mathrm{Mod}(R)$ has colimits.

$$\begin{array}{ccccc} R \otimes \left(\underset{(\mathbf{C})}{\operatorname{colim}} M_i\right) & \xleftarrow{\cong} & \underset{(\mathbf{C})}{\operatorname{colim}} (R \otimes M_i) & \xrightarrow{\operatorname{colim} \alpha_{M_i}} & \underset{(\mathbf{C})}{\operatorname{colim}} M_i \\ \downarrow{\scriptstyle \mathrm{id}\otimes f} & & \downarrow{\scriptstyle \operatorname{colim}(\mathrm{id}\otimes f_i)} & & \downarrow{\scriptstyle f} \\ R \otimes N & \longrightarrow & R \otimes N & \xrightarrow{\alpha_N} & N. \end{array}$$

□

Continuing along our formal procedure, the next step is to assume that $\mathbf{C}$ is a cofibrantly generated model category, and to define a **free cofibration** in $\mathrm{Mod}(R)$ to be a retract of a relative $F_R(I)$-cell complex. In other words, it is retract of a "cell complex of R-modules," where a "cell" of R-modules is obtained by applying $R\otimes -$ to a "cell" in the original category $\mathbf{C}$.

Theorem 6.5.6. *Suppose* $\mathbf{C} = \mathbf{Top}$, $\mathbf{Top}_*$, $\mathbf{Ch}_{\geq 0}(k)$, $\mathbf{Ch}(k)$, $\mathbf{Sp}^\Sigma$, *or* $\mathbf{Sp}^O$.

Then the assumptions of Theorem 6.5.2 hold for $\mathbf{D} = \mathrm{Mod}(R)$, *for any algebra R in* $\mathbf{C}$.

Therefore in any of these settings, $\mathrm{Mod}(R)$ *is a model category, with*

- *the cofibrations are the free cofibrations (retracts of $F_R(I)$-cell complexes), and*
- *the weak equivalences and fibrations are the underlying ones.*

The cofibrations and acyclic cofibrations are generated by

$$R\otimes I = \{\ \mathrm{id}\otimes f : R\otimes X \to R\otimes Y \mid f \in I\ \},$$
$$R\otimes J = \{\ \mathrm{id}\otimes f : R\otimes X \to R\otimes Y \mid f \in J\ \}.$$

Proof. We have already completed most of the steps for Theorem 6.5.2 to apply. We know that $F_R(J)$-cell complexes are weak equivalences because the monoid axiom applies in all of these examples, see Proposition 6.3.16.

Only the smallness condition remains. In the topological examples, the smallness condition is satisfied because a map of R-modules out of $F_R(X)$ is a map in the underlying category $\mathbf{C}$ out of X, and then in every case X is compact and is mapping into a countable colimit along closed inclusions, so a factorization exists by Exercise 1.7.31. In the chain complex examples, the smallness condition is even easier because we are analyzing a map out of a free rank-one k-module, which is the same thing as an element in the target module. □

Example 6.5.7. In symmetric or orthogonal spectra, this gives a model structure on R-modules for any fixed ring spectrum R. The cofibrations and acyclic cofibrations are generated by smashing R with the maps from Definition 5.6.1 and Definition 5.6.3:

$$R\wedge I = \{\ R\wedge f\ :\ f\in I\ \} = \{\ R\wedge F_i S^{n-1}_+ \longrightarrow R\wedge F_i D^n_+\ :\ n,i\geq 0\ \},$$
$$R\wedge J = \{\ R\wedge f\ :\ f\in J\ \} = \{\ R\wedge F_i(D^n\times\{0\})_+ \longrightarrow R\wedge F_i(D^n\times I)_+\ :\ n,i\geq 0\ \}$$
$$\cup\{\ R\wedge k_{i,j}\ \square\ \left(S^{n-1}_+ \longrightarrow D^n_+\right)\ :\ i,j,n\geq 0\ \}.$$

What happens when we change the ring? Returning to the setting of any symmetric monoidal category $\mathbf{C}$, let $\phi : R\to S$ be any homomorphism of algebras. We define functors

$$\mathrm{Mod}(R) \underset{\phi^*}{\overset{S\otimes_R -}{\rightleftarrows}} \mathrm{Mod}(S).$$

Definition 6.5.8. The **restriction of scalars** functor ϕ^* takes each S-module N and makes it into an R-module with action map

$$R \otimes N \xrightarrow{\phi \otimes \mathrm{id}} S \otimes N \xrightarrow{\alpha} N.$$

The **extension of scalars** functor $S \otimes_R -$ takes each R-module M to the coequalizer

$$S \otimes R \otimes M \underset{\mathrm{id}\otimes\alpha}{\overset{\mu_S(\mathrm{id}\otimes\phi)\otimes\mathrm{id}}{\rightrightarrows}} S \otimes M \longrightarrow S \otimes_R M.$$

Note that we can extend this to a functor of two variables $- \otimes_R -$, which takes a right R-module in the first slot and a left R-module in the second slot. In the case of spectra, we usually call this operation $- \wedge_R -$.

Lemma 6.5.9. *The functor $S \otimes_R -$ is the left adjoint of ϕ^*.*

Proof. Let $M \in \mathrm{Mod}(R)$ and $N \in \mathrm{Mod}(S)$. We form the bijection

$$S \otimes_R M \to N \qquad \leftrightarrow \qquad M \to \phi^*(N)$$

by sending each S-linear map $f : S \otimes_R M \to N$ to its restriction to $I \otimes M$, which is an R-linear map $M \to \phi^*(N)$, and extending each R-linear map $g : M \to \phi^*(N)$ to the S-linear map

$$S \otimes M \xrightarrow{\mathrm{id}\otimes g} S \otimes N \xrightarrow{\alpha} N.$$

A straightforward diagram-chase shows that these are inverses. □

Proposition 6.5.10. *Whenever the model structure of Theorem 6.5.6 exists, the extension and restriction of scalars define a Quillen adjunction. If in addition $\phi : R \to S$ is a weak equivalence of algebras, and for each cofibrant R-module M, the functor $(-) \otimes_R M$ preserves weak equivalences of right R-modules, then this is a Quillen equivalence.*

Proof. The right adjoint commutes with the forgetful functor U to $\mathbf{C}$, so it preserves both weak equivalences and fibrations in the categories of modules. Therefore it is a Quillen right adjoint.

If ϕ is a weak equivalence and the given assumption holds, then since $R \to S$ is a weak equivalence of right R-modules, the map $M \cong R \otimes_R M \to S \otimes_R M$ is a weak equivalence for every cofibrant R-module M. Therefore if N is a fibrant S-module, a map $S \otimes_R M \to N$ is a weak equivalence of S-modules iff its restriction to $M \to N$ is a weak equivalence of R-modules. Therefore we have a Quillen equivalence by Lemma 5.4.10. □

Corollary 6.5.11. *In any of the categories mentioned in Theorem 6.5.6, the extension and restriction of scalars form a Quillen adjunction. If ϕ is a weak equivalence of algebras, then they form a Quillen equivalence.*

Proof. In each case we show that $(-) \otimes_R M$ preserves equivalences of right R-modules by an induction on the cells or skeleta of M. We do the case of spectra in detail.

Let $P \xrightarrow{\sim} P'$ be any map of right R-module spectra that is a stable equivalence, and let M be any cofibrant R-module spectrum. We wish to show that the map

$$P \wedge_R M \longrightarrow P' \wedge_R M$$

is a stable equivalence. Without loss of generality $* \to M$ is a $F_R(I)$-cell complex, so it is built by successive pushouts of the form

$$\begin{array}{ccc} \bigvee_\alpha R \wedge F_{i_\alpha} S^{n_\alpha - 1}_+ & \longrightarrow & \bigvee_\alpha R \wedge F_{i_\alpha} D^{n_\alpha}_+ \\ \downarrow & & \downarrow \\ M^{(n-1)} & \longrightarrow & M^{(n)}, \end{array}$$

with $M = \operatorname*{colim}_{n\to\infty} M^{(n)}$. Applying $P \wedge_R (-)$ and simplifying gives

$$\begin{array}{ccc} \bigvee_\alpha P \wedge F_{i_\alpha} S^{n_\alpha - 1}_+ & \longrightarrow & \bigvee_\alpha P \wedge F_{i_\alpha} D^{n_\alpha}_+ \\ \downarrow & & \downarrow \\ P \wedge_R M^{(n-1)} & \longrightarrow & P \wedge_R M^{(n)}. \end{array}$$

The map $P \to P'$ induces a map of two such squares, that is an equivalence on the top two terms by Exercise 6.6.12. By inductive assumption it is an equivalence on the lower-left term $P \wedge_R M^{(n-1)}$ as well. The top horizontal map in each square is a levelwise h-cofibration by the proof of Proposition 6.3.16, and therefore each of these squares is a homotopy pushout square of spectra. Therefore the induced map on the lower-right term $P \wedge_R M^{(n)}$ is an equivalence as well. Taking the colimit as $n \to \infty$ and using Corollary 6.1.44(2), the induced map $P \wedge_R M \to P' \wedge_R M$ is an equivalence as desired. □

6.5.3 The model structure on algebras

Now we start our formal procedure again from the beginning, this time considering the category of algebras $\mathrm{Alg}(\mathbf{C})$ in a symmetric monoidal category $\mathbf{C}$, from Definition 6.5.1. When $\mathbf{C} = \mathbf{Sp}^\Sigma$ or $\mathbf{Sp}^O$, this is the category of ring spectra.

There is a functor

$$U_{\mathrm{Alg}} \colon \mathrm{Alg}(\mathbf{C}) \to \mathbf{C}$$

that forgets the algebra structure and only remembers the underlying object of $\mathbf{C}$. Going the other way, given an object X of $\mathbf{C}$, we form the **free algebra** by taking

$$T_{\mathrm{Alg}}(X) := \coprod_{n\geq 0} (X^{\otimes n}) = (I) \amalg (X) \amalg (X \otimes X) \amalg (X \otimes X \otimes X) \amalg \cdots .$$

Since $\otimes$ preserves colimits in each slot, the tensor product $T_{\text{Alg}}(X) \otimes T_{\text{Alg}}(X)$ is written as a coproduct over all pairs $m, n \geq 0$ of the tensor products $(X^{\otimes m}) \otimes (X^{\otimes n})$. We map this to $T_{\text{Alg}}(X)$ by making the obvious identification of each of these terms with $X^{\otimes(m+n)}$. (No shuffling or rearranging allowed!) We also map $I \to T_{\text{Alg}}(X)$ by including the $n = 0$ term. It is straightforward to check that this gives $T_{\text{Alg}}(X)$ the structure of an algebra in $\mathbf{C}$.

We let $F_{\text{Alg}}(X)$ denote the resulting algebra, whose underlying object of $\mathbf{C}$ is $T_{\text{Alg}}(X)$.

Lemma 6.5.12. *The functor F_{Alg} is the left adjoint of U_{Alg}.*

Proof. Let $X \in \mathbf{C}$ and $S \in \text{Alg}(\mathbf{C})$. We form the bijection

$$f : \coprod_{n \geq 0} X^{\otimes n} \to S \qquad \longleftrightarrow \qquad g : X \to U_{\text{Alg}}(S)$$

by restricting each map f to the $n = 1$ term to get g, and extending each map g to the remaining terms by applying $g^{\otimes n}$ and then applying the multiplication of S (or the unit of S when $n = 0$). A straightforward diagram-chase shows that every map f must be of this form, so these are inverses. □

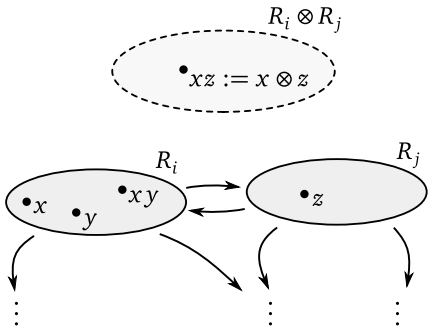
Lemma 6.5.13. *Suppose the symmetric monoidal category $\mathbf{C}$ has all colimits and limits, and that $\otimes$ preserves colimits in each slot. Then the category $\text{Alg}(\mathbf{C})$ has all colimits and limits as well.*

Proof. Let $R \colon \mathbf{I} \to \text{Alg}(\mathbf{C})$ be a diagram of algebras, sending i to the algebra R_i. As in Lemma 6.5.5, we form the limit in $\text{Alg}(\mathbf{C})$ by taking the limit in $\mathbf{C}$, which we denote $\lim_{(\mathbf{C})} R_i$. Its multiplication arises from the universal property of the limit and the following diagram, which commutes for all i.

$$\begin{array}{ccc} \left(\lim_{(\mathbf{C})} R_i\right) \otimes \left(\lim_{(\mathbf{C})} R_i\right) & \dashrightarrow & \lim_{(\mathbf{C})} R_i \\ \downarrow & & \downarrow \\ R_i \otimes R_i & \xrightarrow{\mu_{R_i}} & R_i \end{array}$$

The unit map $I \to \lim R_i$ arises in the same way. The proof that this defines an algebra and that it has the universal property of the limit in $\text{Alg}(\mathbf{C})$ is a straightforward adaptation of the proof in Lemma 6.5.5, so we leave it to Exercise 6.6.30.

Colimits are more complicated. The colimit of the algebras R_i will *not* be the colimit in the underlying category **C**, as there is no way to define a multiplication on that. This is illustrated on the right: the colimit contains R_i and R_j, and since R_i is an algebra, we can multiply elements of R_i with each other. But to multiply an element of R_i with an element of R_j, we have to add a place for the product to live. The universal way to do this is to add a copy of $R_i \otimes R_j$.



To handle all possible products, we therefore take all possible tensor products

$$\coprod_{n\geq 0} \coprod_{i_1,\dots,i_n} R_{i_1} \otimes \cdots \otimes R_{i_n}, \tag{6.5.14}$$

and we impose the following three relations.

- For each arrow $f : i \to j$ in **I**, each expression involving R_i is identified with the corresponding expression with R_j in its place, along the map f. So for instance $R_i \otimes R_k$ is identified with $R_j \otimes R_k$ along the map $f \otimes \mathrm{id}$.
- Each expression involving two adjacent copies of the same algebra, $R_i \otimes R_i$, is identified with the corresponding expression in which they are multiplied together. So for instance $R_j \otimes R_i \otimes R_i$ is identified with $R_j \otimes R_i$ along $\mathrm{id} \otimes \mu_{R_i}$.
- Each expression is identified with each corresponding expression obtained by inserting a copy of the unit I anywhere, and then applying the unit map to land in any algebra R_i. So for instance $R_j \otimes R_k$ is identified with $R_j \otimes R_i \otimes R_k$ along $\mathrm{id} \otimes \eta_{R_i} \otimes \mathrm{id}$.

We call the resulting quotient $\operatorname*{colim}_{\mathrm{Alg}(\mathbf{C})} R_i$.

We are in an arbitrary category **C**, not necessarily a category whose objects are sets with extra structure, so we have to think about what we mean when we say "quotient by these relations." Each of the above relations is encoded by a pair of morphisms going into (6.5.14). For instance, the first relation gives two maps

$$R_i \otimes R_k \rightrightarrows \coprod_{n\geq 0} \coprod_{i_1,\dots,i_n} R_{i_1} \otimes \cdots \otimes R_{i_n},$$

one of which includes the term $R_i \otimes R_k$ in the coproduct, the other of which applies $f \otimes \mathrm{id}$ and then includes the result into the term $R_j \otimes R_k$ in the coproduct. The quotient is defined to be the initial object of **C** with a morphism

$$\coprod_{n\geq 0} \coprod_{i_1,\dots,i_n} R_{i_1} \otimes \cdots \otimes R_{i_n} \longrightarrow \operatorname*{colim}_{\mathrm{Alg}(\mathbf{C})} R_i$$

that respects all of the relations. In other words, for each relation, the two resulting composite maps into $\operatorname{colim}_{\mathrm{Alg}(\mathbf{C})} R_i$ agree with each other. So if there is a morphism in $\mathbf{C}$ to any other object Y that respects the relations, there must be a unique dashed arrow in $\mathbf{C}$ that makes the following diagram commute.

$$\begin{array}{ccc} \coprod_{n\geq 0}\coprod_{i_1,\dots,i_n} R_{i_1}\otimes\cdots\otimes R_{i_n} & \longrightarrow & \operatorname*{colim}_{\mathrm{Alg}(\mathbf{C})} R_i \\ & \searrow & \downarrow \exists! \\ & & Y \end{array}$$

This universal object is guaranteed to exist, by repeated application of the fact that $\mathbf{C}$ has all colimits. And in practical situations where $\mathbf{C}$ is a category of sets with extra structure, this category-theoretic formalism agrees with our intuition that we "quotient out by the relations."

At any rate, we define the multiplication and unit map on this colimit algebra by defining them on the individual terms of (6.5.14). The multiplication concatenates terms, just like in the definition of F_{Alg}, and the unit includes the unique term where $n = 0$. We then check that this multiplication respects each of the relations, and therefore descends to a multiplication on the colimits. (This is where we use the fact that $\otimes$ preserves colimits in each slot.) This defines the algebra structure on $\operatorname{colim}_{\mathrm{Alg}(\mathbf{C})} R_i$.

For each i we get a map $R_i \to \operatorname{colim}_{\mathrm{Alg}(\mathbf{C})} R_j$ by including the term R_i in the coproduct (6.5.14). This is a map of algebras by construction. Furthermore, given any algebra S and maps of algebras $f_i : R_i \to S$ that commute with the homomorphisms $R_i \to R_j$, we define a map from the coproduct (6.5.14) into S by applying the map $f_{i_1}\otimes\cdots\otimes f_{i_n}$, followed by iterated application of μ_S until we get down to a single copy of S. (When $n = 0$, we instead send I into S by the unit map η_S.) We check once more that this respects the relations, giving a map $f : \operatorname{colim}_{\mathrm{Alg}(\mathbf{C})} R_i \to S$, and that f is a map of algebras.

Finally, f is the only map of algebras that extends the f_i, because the f_i determine f on all of the terms with $n = 1$, and the requirement that f be a map of algebras extends the definition in a unique way to all of the terms with $n \neq 1$. Therefore our colimit algebra satisfies the universal property of the colimit in $\mathrm{Alg}(\mathbf{C})$. □

Example 6.5.15. This construction of the colimit in $\mathrm{Alg}(\mathbf{C})$ may seem complicated, but you probably saw it when you learned the van Kampen theorem. When $\mathbf{C} = \mathbf{Set}$, an algebra is the same thing as a monoid (a set with a multiplication that is associative and unital). The coproduct of the monoids G_1, …, G_n constructed above is the **free product** $G_1 * \cdots * G_n$. Its elements are equivalence classes of words, in which each letter is in one of the monoids G_i. The relations allow us to insert or remove identity elements, and to multiply adjacent letters in the same monoid G_i together.

If the monoids G_i happen to groups, then $G_1 * \cdots * G_n$ is a group as well, and this is the free product of groups! In other words, the same construction gives the coproduct in category of groups as well. So for any group H, a homomorphism $G_1 * \cdots * G_n \to H$ is the same data as a homomorphism $G_i \to H$ for each value of i. (This happens very generally – see Exercise 8.7.46.)

Continuing along our formal procedure, the next step is to assume that **C** is a cofibrantly generated model category, and to define a **free cofibration** in Alg(**C**) to be a retract of a relative $F_{\mathrm{Alg}}(I)$-cell complex. In other words, it is retract of a "cell complex of algebras," where a cell of algebras is a free tensor algebra on a single cell. This doesn't sound that bad, but remember that when we attach a cell of algebras $K \to L$ to an existing algebra R, we take the pushout *in the category of algebras*. This is quite a bit more complicated than taking a pushout in the original category **C**.

We will need to understand this procedure better, in order to prove that Alg(**C**) has a model structure. So assume that $\otimes$ preserves colimits in each slot, as in Lemma 6.5.13, and take any pushout diagram of algebras of the form

$$\begin{array}{ccc} F_{\mathrm{Alg}}(K) & \xrightarrow{F_{\mathrm{Alg}}(f)} & F_{\mathrm{Alg}}(L) \\ \downarrow{\scriptstyle \widetilde{g}} & & \downarrow \\ X & \longrightarrow & P, \end{array}$$

where $f : K \to L$ is a map in the underlying category **C**, and the map $\widetilde{g}$ is the unique extension of some map $g : K \to X$ to a map of algebras $F_{\mathrm{Alg}}(K) \to X$.

Lemma 6.5.16. *The map $X \to P$, considered as a map in the underlying category* **C***, is a countable composition of pushouts of maps of the form*

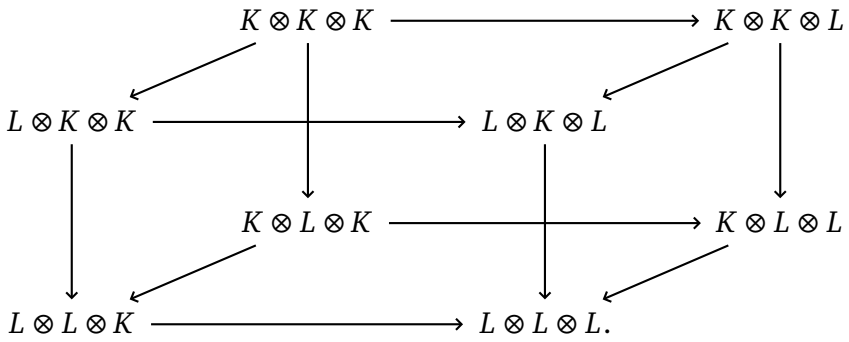
$$f^{\square n} \otimes \mathrm{id}_X^{\otimes(n+1)} = (K \to L)^{\square n} \square (\varnothing \to X)^{\square(n+1)}.$$

In other words, there is a filtration in **C**

$$X = P_0 \to P_1 \to P_2 \to \cdots \to \operatorname*{colim}_{n\to\infty} P_n = P.$$

Then we go from P_{n-1} to P_n as follows. Take n copies of the map $i : K \to L$ and tensor them together to make a cube-shaped diagram, as we did when we defined the pushout-product

in Definition 5.5.6. For instance, when $n = 3$ we get the cube



Then we let $f^{\square n}$ be the map from the colimit of the punctured cube (so the cube minus the last vertex $L^{\otimes n}$) to the term $L^{\otimes n}$. Finally, we take the tensor product of this with the identity of $X^{\otimes(n+1)}$, or equivalently, we tensor this entire cube by $X^{\otimes(n+1)}$ before taking the map from the colimit to the last vertex. Either way, we get the map $f^{\square n} \otimes \mathrm{id}_X^{\otimes(n+1)}$, and the map $P_{n-1} \to P_n$ is a pushout of this.

Proof. Our proof follows [SS00, Lemma 6.2] but is a little shorter because we can lean more heavily on the explicit description of the colimit that we gave in Lemma 6.5.13. Recall that P is defined as the quotient of

$$\coprod_{k \geq 0} \coprod_{i_1, \ldots, i_k} X_{i_1} \otimes \cdots \otimes X_{i_k}, \tag{6.5.17}$$

where each X_i is either X, $F_{\mathrm{Alg}}(K)$, or $F_{\mathrm{Alg}}(L)$, and the relations apply the maps $F_{\mathrm{Alg}}(f)$ and $\widetilde{g}$, multiply adjacent terms, and insert units. The terms in $F_{\mathrm{Alg}}(K)$ and $F_{\mathrm{Alg}}(L)$ break apart into tensors of terms in K and L, and the multiplication maps simply concatenate these terms. Therefore the definition of P rearranges to the following: it is the quotient of (6.5.17), where each X_i is either X, K, or L, up to relations that apply f or g to any term in K, that multiply adjacent terms that are in X (but not those in K or L), and that insert copies of I and then apply the unit map $I \to X$. For instance, we have identifications along the maps

$$X \otimes L \otimes L \xleftarrow{\mathrm{id} \otimes f \otimes \mathrm{id}} X \otimes K \otimes L \xrightarrow{\mathrm{id} \otimes g \otimes \mathrm{id}} X \otimes X \otimes L \xrightarrow{\mu \otimes \mathrm{id}} X \otimes L$$

$$K \otimes K \xrightarrow{\cong} K \otimes I \otimes K \xrightarrow{\mathrm{id} \otimes \eta \otimes \mathrm{id}} K \otimes X \otimes K \xleftarrow{\mathrm{id} \otimes \mu \otimes \mathrm{id}} K \otimes X \otimes X \otimes K.$$

We define P_n by taking the smaller coproduct in which L can only appear up to n times in the expression $X_{i_1} \otimes \cdots \otimes X_{i_k}$, and then we impose all the same relations as before, except that we don't allow ourselves to change a K to an L if L already appears n times. By comparing universal properties, we see there are maps $P_{n-1} \to P_n$, and that P is the colimit of the P_n over all values of n.

When $n = 0$, there are no copies of L, and then by rearranging the universal properties we see that P_0 is isomorphic to X. For larger values of n, we define expressions of the form

$$W(S) = X \otimes C_1 \otimes X \otimes C_2 \otimes X \otimes \cdots \otimes X \otimes C_n \otimes X$$

for each n and each subset $S \subseteq \{1, \ldots, n\}$. The term C_i is equal to K when $i \notin S$, and L when $i \in S$. As S varies over all subsets of $\{1, \ldots, n\}$, these define the vertices of a cube-shaped diagram, essentially the one we described above but with $(n+1)$ copies of X multiplied in. The arrows of this cube correspond to inclusions of subsets $S \subseteq T$, and each one applies the map i for $i \in T \setminus S$ to transform the K terms into L terms.

We define maps $W(S) \to P_n(X)$ by including each expression as the corresponding term in the coproduct (6.5.17). When S is a proper subset, this map factors through $P_{n-1}(X)$. All together this gives a commuting square in $\mathbf{C}$ of the form

$$\begin{array}{ccc} \operatorname*{colim}_{S \subsetneq \{1,\ldots,n\}} W(S) & \longrightarrow & W(\{1, \ldots, n\}) \\ \downarrow & & \downarrow \\ P_{n-1}X & \longrightarrow & P_nX, \end{array} \tag{6.5.18}$$

where the top-left corner is the colimit over the punctured cube, in other words it does not include the last vertex where $S = \{1, \ldots, n\}$. Note that the map along the top of this square is isomorphic to $f^{\square n} \otimes \mathrm{id}_X^{\otimes(n+1)}$, the isomorphism given by moving all of the X terms to the right-hand side. (We can't rearrange the terms in this way when mapping into P_nX. We have to preserve their ordering for the map to be well-defined.)

The proof is finished once we show that (6.5.18) is a pushout square in $\mathbf{C}$. To do this, we take any $Y \in \mathbf{C}$ and compare the universal properties for a map to Y. A map $P_nX \to Y$ consists of maps

$$X_{i_1} \otimes \cdots \otimes X_{i_k} \to Y$$

for each possible tensor product of the terms X, K, or L, with at most n copies of L, respecting the unit and multiplication relations for X and the relations that apply $f : K \to L$ and $g : K \to X$. A map from the other three terms of (6.5.18) to Y consists of the same data and relations but with at most $(n-1)$ copies of L, which gives maps $W(S) \to Y$ for all proper subsets S, and then a choice of how to extend this to $W(\{1, \ldots, n\})$.

Given a map from the pushout of these three terms to Y, we construct a map from all of the remaining tensors $X_{i_1} \otimes \cdots \otimes X_{i_k}$ in P_nX to Y as follows. We apply g to all copies of K that appear, so that the expression only has n copies of L, and the remaining terms are copies of X. We then use μ and η to condense the copies of X down until we have one copy of X between every pair of copies of L, and one copy of X on the left-hand side and one copy of X on the right-hand side as well. This gives an expression of the form $X \otimes L \otimes X \otimes \cdots \otimes L \otimes X$. We then apply the given map $W(\{1, \ldots, n\}) \to Y$, and this gives

a rule for how to map our original tensor $X_{i_1} \otimes \cdots \otimes X_{i_k}$ to Y. We check that this new rule respects all the required relations, and so gives a map $P_n X \to Y$. Furthermore, since we only used the relations of $P_n X$ to manipulate the tensor $X_{i_1} \otimes \cdots \otimes X_{i_k}$ to one of the form $W(\{1, \ldots, n\})$, this is the only possible extension, so the map $P_n X \to Y$ is unique. This shows the universal properties agree, so the proof is complete. □

Let us use C and $A = W \cap C$ to refer to the cofibrations and acyclic cofibrations in a model category $\mathbf{C}$.

Corollary 6.5.19. *In any symmetric monoidal model category* $\mathbf{C}$*, for any pushout square of algebras of the form*

$$\begin{array}{ccc} F_{\mathrm{Alg}}(K) & \xrightarrow{F_{\mathrm{Alg}}(f)} & F_{\mathrm{Alg}}(L) \\ \downarrow{\scriptstyle \tilde{g}} & & \downarrow \\ X & \longrightarrow & P, \end{array}$$

- *If $f \in C$, then $X \to P$ is a $(C \otimes \mathbf{C})$-cell complex, as a map in $\mathbf{C}$.*
- *If $f \in A$, then $X \to P$ is an $(A \otimes \mathbf{C})$-cell complex, as a map in $\mathbf{C}$.*
- *If $f \in C$ and X is cofibrant in $\mathbf{C}$, then $X \to P$ is a cofibration in $\mathbf{C}$.*

Proof. By Lemma 6.5.16, the map $X \to P$ is a countable composition of maps of the form $(K \to L)^{\square n} \otimes \mathrm{id}_X^{\otimes(n+1)}$, with $n \geq 1$. Since $\otimes$ is a Quillen tensor, the pushout-product $(K \to L)^{\square n}$ is an (acyclic) cofibration whenever $f : K \to L$ is an (acyclic) cofibration. Tensoring it with the identity of $X^{\otimes(n+1)}$ and then composing the results therefore gives a cell complex of the desired form. If in addition X is cofibrant, then we also get a cofibration, again since $\otimes$ is a Quillen tensor. □

Theorem 6.5.20. *Suppose* $\mathbf{C} = \mathbf{Top}$, $\mathbf{Top}_*$, $\mathbf{Ch}_{\geq 0}(k)$, $\mathbf{Ch}(k)$, $\mathbf{Sp}^\Sigma$, *or* $\mathbf{Sp}^O$.

Then the assumptions of Theorem 6.5.2 hold for $\mathbf{D} = \mathrm{Alg}(\mathbf{C})$.

Therefore in any of these settings, $\mathrm{Alg}(\mathbf{C})$ *is a model category, with*

- *the cofibrations are the free cofibrations of algebras,*
- *the weak equivalences and fibrations are the underlying ones.*

Moreover, any cofibration between cofibrant objects in $\mathrm{Alg}(\mathbf{C})$ *is also a cofibration in* $\mathbf{C}$.

Proof. We have already completed most of the steps for Theorem 6.5.2 to apply. We still need to check that a $F_{\mathrm{Alg}}(J)$-cell complex must be a weak equivalence. By Corollary 6.5.19, it is enough to check that any composition of $(A \otimes \mathbf{C})$-cell complexes is a weak equivalence

in each of these examples. (This is a composition in the category $\mathrm{Alg}(\mathbf{C})$, but it's also a composition in the underlying category $\mathbf{C}$ by Exercise 6.6.31.) For spectra, we recall from the proof of Proposition 6.3.16 that each $(A \otimes \mathbf{C})$-cell complex is not only a stable equivalence but also a levelwise h-cofibration, and therefore even after taking further composition of these, we still get a stable equivalence.[8] For the non-spectra examples, the proof is easier and is left as an exercise.

Only the smallness condition remains. As in the proof of Theorem 6.5.20, the only difficulty occurs in the topological examples, and then in that case the proof reduces to checking that pushouts along maps in $F_{\mathrm{Alg}}(C)$ are closed inclusions (at each spectrum level, in the spectrum case). This also follows from Corollary 6.5.19 and the fact that the maps in $C \otimes \mathbf{C}$ are h-cofibrations and are therefore closed inclusions (at each spectrum level).

The final claim about cofibrations in $\mathrm{Alg}(\mathbf{C})$ is left to Exercise 6.6.32. □

Example 6.5.21. This gives a model structure on symmetric or orthogonal ring spectra. If R is a cofibrant ring spectrum then it is also cofibrant as a spectrum, proving Lemma 6.2.37. This achieves (finally!) our goal of making our ring spectra cofibrant as spectra.

Remark 6.5.22. We also get a model structure for algebras in $\mathrm{Mod}(R)$ for any commutative ring spectrum R. This is because the algebras in $\mathrm{Mod}(R)$ are the same thing as algebras in $\mathbf{C}$ with an algebra map from R. In other words, they are just a comma category in $\mathrm{Alg}(\mathbf{C})$. See Exercise 6.6.33.

6.5.4 Commutative algebras

We are ready to follow our formal procedure one more time. Let $\mathbf{C}$ be any symmetric monoidal category with all colimits, and let $\mathrm{CAlg}(\mathbf{C})$ be the category of commutative algebras in the symmetric monoidal category $\mathbf{C}$. The functor

$$U_{\mathrm{CAlg}} \colon \mathrm{CAlg}(\mathbf{C}) \to \mathbf{C}$$

forgets the algebra structure. Given an object X of $\mathbf{C}$, we form the **free commutative algebra** by taking

$$T_{\mathrm{CAlg}}(X) := \coprod_{n \geq 0} (X^{\otimes n})/\Sigma_n = (I) \amalg (X) \amalg (X \otimes X)/\Sigma_2 \amalg (X \otimes X \otimes X)/\Sigma_3 \amalg \cdots,$$

where the $(-)/\Sigma_n$ means taking the colimit in $\mathbf{C}$ along the diagram that permutes the copies of X using the symmetric monoidal structure. We make this into a commutative

[8] This is where allowing more general transfinite compositions as in Exercise 5.8.42, and not just countable ones, is helpful. It makes the $(A \otimes \mathbf{C})$-cell complexes closed under composition.

algebra as before, with product that concatenates the strings of Xs, and check that it respects the relations, giving a map $T_{\mathrm{CAlg}}(X) \otimes T_{\mathrm{CAlg}}(X) \to T_{\mathrm{CAlg}}(X)$. Note that the quotient by the Σ_n-action is necessary to make this commutative!

As before, let $F_{\mathrm{CAlg}}(X)$ denote the resulting algebra, whose underlying object of $\mathbf{C}$ is $T_{\mathrm{CAlg}}(X)$.

Lemma 6.5.23. *The functor F_{CAlg} is the left adjoint of U_{CAlg}.*

Proof. As in the proof of Lemma 6.5.12, for $X \in \mathbf{C}$ and $S \in \mathrm{CAlg}(C)$, we restrict each map

$$f : \coprod_{n \geq 0} (X^{\otimes n})/\Sigma_n \to S$$

to the $n = 1$ term to get a map $g : X \to S$ in $\mathbf{C}$, and then we see we can recover f uniquely from this map if we assume that f is a map of commutative algebras. □

Lemma 6.5.24. *Suppose the symmetric monoidal category $\mathbf{C}$ has all colimits and limits, and that $\otimes$ preserves colimits in each slot. Then the category $\mathrm{CAlg}(\mathbf{C})$ has all colimits and limits as well.*

Proof. Let $R : \mathbf{I} \to \mathrm{CAlg}(\mathbf{C})$ be a diagram of commutative algebras, sending i to the algebra R_i. We form the limit in $\mathrm{CAlg}(\mathbf{C})$ by taking the limit in $\mathbf{C}$ and giving it the same multiplication that we gave in Lemma 6.5.13. In other words, we take the limit in the category of algebras and observe that it is a commutative algebra as well. See Exercise 6.6.30.

We form the colimit in $\mathrm{CAlg}(\mathbf{C})$ by taking the quotient in $\mathbf{C}$ of the coproduct

$$\coprod_{n \geq 0} \coprod_{i_1, \ldots, i_n} R_{i_1} \otimes \cdots \otimes R_{i_n} \tag{6.5.25}$$

by the three relations we gave below (6.5.14), and an additional fourth relation that swaps any expression of the form $R_i \otimes R_j$ with $R_j \otimes R_i$. In particular, when a term R_i is repeated multiple times, we pass to the colimit along the Σ_n-action that permutes the repeated terms. The proof that this quotient in $\mathbf{C}$ gives the colimit in $\mathrm{CAlg}(\mathbf{C})$ is almost identical to the proof in Lemma 6.5.13. □

Example 6.5.26. When $\mathbf{C} = \mathbf{Set}$, the coproduct of the abelian monoids (or abelian groups) $G_1, \ldots, G_n$ constructed above is the Cartesian product $G_1 \times \cdots \times G_n$. More generally:

Lemma 6.5.27. *In any symmetric monoidal category $\mathbf{C}$, the category $\mathrm{CAlg}(\mathbf{C})$ has all finite coproducts, and they are given by the tensor product in $\mathbf{C}$.*

To be more specific, given commutative algebras $R_1, \ldots, R_n$, the tensor product $R_1 \otimes \cdots \otimes R_n$ is a commutative algebra by shuffling:

$$(R_1 \otimes \cdots \otimes R_n) \otimes (R_1 \otimes \cdots \otimes R_n) \cong (R_1 \otimes R_1) \otimes \cdots \otimes (R_n \otimes R_n) \xrightarrow{\mu_1 \otimes \cdots \otimes \mu_n} R_1 \otimes \cdots \otimes R_n.$$

Proof. We could use the description of the colimit from Lemma 6.5.24 and simplify it down in the case of the coproduct, but it is faster and more general to just directly check that for commutative algebras $R_1, \ldots, R_n$, the tensor product $R_1 \otimes \cdots \otimes R_n$ has the required universal property. We include each of the terms R_i into the tensor product by tensoring with the unit maps $I \to R_j$ for $j \neq i$. Given a commutative algebra S and algebra maps $f_i : R_i \to S$, we define the map $R_1 \otimes \cdots \otimes R_n \to S$ by taking $f_1 \otimes \cdots \otimes f_n$ and then applying the maps μ_S until we get down to a single copy of S. This is a commutative algebra homomorphism, it extends the homomorphisms f_i, and it is the only possible extension, hence $R_1 \otimes \cdots \otimes R_n$ has the required universal property. □

Example 6.5.28. Suppose we have an abelian group A, and a non-abelian group G acts on A through group homomorphisms. We can think of this as a functor $G \to \mathrm{CAlg}(\mathbf{Set})$. The limit of this functor is given by taking the limit in sets, in other words the G-fixed subgroup $A^G \leq A$. The colimit as constructed in Lemma 6.5.24 is the **coinvariants** of A. It's built by considering all finite tuples of elements of A, letting different elements of g act on the terms, and then adding them together. In other words, the equivalence relation is generated by the relations

$$a_1 + \cdots + a_n \sim g_1 a_1 + \cdots + g_n a_n.$$

The coinvariants are not the same thing as the set of orbits of A under the G-action. That would be the colimit in the category **Set**, and it is different because it only allows relations of the form

$$a_1 + \cdots + a_n \sim g(a_1 + \cdots + a_n).$$

For instance, if $A = \mathbb{Z}$ and $G = \mathbb{Z}/2$ acts by negation, then the orbits are the set $\mathbb{Z}_{\geq 0}$, which is no longer an abelian group, but the coinvariants are $\mathbb{Z}/2$, which is still an abelian group.

Continuing along our formal procedure, the next step is to assume that $\mathbf{C}$ is a cofibrantly generated model category, and to define a **free cofibration** in $\mathrm{CAlg}(\mathbf{C})$ to be a retract of a relative $F_{\mathrm{CAlg}}(I)$-cell complex. As in the non-commutative case, we examine a pushout of the form

$$\begin{array}{ccc} F_{\mathrm{CAlg}}(K) & \xrightarrow{F_{\mathrm{Alg}}(f)} & F_{\mathrm{CAlg}}(L) \\ \downarrow{\scriptstyle \widetilde{g}} & & \downarrow \\ X & \longrightarrow & P, \end{array}$$

where $f : K \to L$ is a map in the underlying category $\mathbf{C}$, and the map $\widetilde{g}$ is given by the data of a map $g : K \to X$.

Lemma 6.5.29. *The map $X \to P$, considered as a map in the underlying category $\mathbf{C}$, is a countable composition of pushouts of maps of the form*

$$f^{\square n}/\Sigma_n \otimes \mathrm{id}_X = ((K \to L)^{\square n}/\Sigma_n) \square (\emptyset \to X).$$

This is almost the same as Lemma 6.5.16, except that we have quotiented by symmetric group actions, giving a $/\Sigma_n$ on the left, and only one copy of X on the right.

Proof. Our proof follows [Whi17, Prop B.2]. As in the proof of Lemma 6.5.16, because two of the algebras in this diagram are free commutative algebras, the definition of the colimit P simplifies: we get the quotient in $\mathbf{C}$ of

$$\coprod_{k\geq 0} \coprod_{i_1,\dots,i_k} X_{i_1} \otimes \cdots \otimes X_{i_k}, \tag{6.5.30}$$

where each X_i is either X, K, or L, up to relations that apply f or g to any term in K, that multiply adjacent terms that are in X (but not those in K or L), and that insert units and then include the result into a new term in X, or that rearrange the order of the terms.

We define P_n by taking the smaller coproduct in which L can only appear up to n times in the expression $X_{i_1} \otimes \cdots \otimes X_{i_k}$, with the same relations. As before, we get $P_0 \cong X$, there are maps $P_{n-1} \to P_n$, and P is the colimit of the P_n over all values of n. For each $n \geq 1$ and each subset $S \subseteq \{1,\dots,n\}$, we define

$$W(S) = C_1 \otimes C_2 \otimes \cdots \otimes C_n \otimes X.$$

The term C_i is equal to K when $i \notin S$, and L when $i \in S$. As before, these define a cube-shaped diagram, and we get a commuting square in $\mathbf{C}$ of the form

$$\begin{array}{ccc} \left(\operatorname*{colim}_{S\subsetneq\{1,\dots,n\}} W(S)\right)/\Sigma_n & \longrightarrow & W(\{1,\dots,n\})/\Sigma_n \\ \downarrow & & \downarrow \\ P_{n-1}X & \longrightarrow & P_nX. \end{array} \tag{6.5.31}$$

Since $\otimes$ preserves colimits in each slot, the map along the top is isomorphic to $f^{\square n}/\Sigma_n \otimes \mathrm{id}_X$. So the proof is finished once we show that this is a pushout square in $\mathbf{C}$. For any $Y \in \mathbf{C}$, a map $P_nX \to Y$ consists of maps

$$X_{i_1} \otimes \cdots \otimes X_{i_k} \to Y$$

for each possible tensor product of the terms X, K, or L, with at most n copies of L, respecting the unit and multiplication relations for X and the relations that apply $f : K \to L$ and $g : K \to X$, and the relations that swap the order of the terms in the tensor product. A map from the other three terms of (6.5.31) to Y consists of the same data and relations but with at most $(n-1)$ copies of L, which gives maps $W(S) \to Y$ for all proper subsets S, and then a choice of how to extend this to $W(\{1,\dots,n\})$. Finally, these maps all respect the Σ_n-action on $\coprod_S W(S)$ that rearranges the terms C_1, …, C_n.

Given a map from the pushout of these three terms in (6.5.31) to Y, we construct a map from all of the remaining tensors $X_{i_1} \otimes \cdots \otimes X_{i_k}$ in $P_n X$ to Y as follows. We apply g to all copies of K that appear, so that the expression only has n copies of L, and every other term is a copy of X. We then use the reordering relation to put all the copies of X on the right, and then use μ and η to condense this down to a single copy of X. This gives an expression of the form $L \otimes L \otimes \cdots \otimes L \otimes X$. We then apply the given map $W(\{1,\dots,n\}) \to Y$, and this gives a rule for how to map our original tensor $X_{i_1} \otimes \cdots \otimes X_{i_k}$ to Y. We check that this new rule respects all the required relations, and so gives a map $P_n X \to Y$. Furthermore, since we only used the relations of $P_n X$ to manipulate the tensor to one of the form $W(\{1,\dots,n\})$, this is the only possible extension, so the map $P_n X \to Y$ is unique. This shows the universal properties agree, so the proof is complete. □

We state the next corollary in a weaker form because it is not always the case that the assignment $f \mapsto f^{\square n}/\Sigma_n$ preserves cofibrations and acyclic cofibrations. Suppose instead that this functor sends cofibrations to some larger class of maps C', and acyclic cofibrations some larger class A'.

Corollary 6.5.32. *In any symmetric monoidal model category* **C***, for any pushout square of commutative algebras of the form*

$$\begin{array}{ccc} F_{\mathrm{Alg}}(K) & \xrightarrow{F_{\mathrm{Alg}}(f)} & F_{\mathrm{Alg}}(L) \\ \downarrow{\scriptstyle \widetilde{g}} & & \downarrow \\ X & \longrightarrow & P, \end{array}$$

- *If $f \in C$, then $X \to P$ is a $(C' \otimes \mathbf{C})$-cell complex, as a map in* **C**.
- *If $f \in A$, then $X \to P$ is an $(A' \otimes \mathbf{C})$-cell complex, as a map in* **C**.

The proof is the same as in Corollary 6.5.19. In Section A.1 we will see that for spaces, the functor $f \mapsto f^{\square n}/\Sigma_n$ does indeed preserve cofibrations and acyclic cofibrations (Corollary A.1.24). However for spectra, these functors send the positive cofibrations in $\mathbf{Sp}^{\Sigma}_{+}$ or $\mathbf{Sp}^{O}_{+}$ to a larger class of maps called the "positive flat cofibrations," and similarly the acyclic cofibrations are sent to "positive flat acyclic cofibrations." However this is still good enough because these flat cofibrations are all h-cofibrations, so we can prove that $F_{\mathrm{CAlg}}(J)$-cell complexes are stable equivalences just as in Theorem 6.5.20.

Theorem 6.5.33. *Suppose* $\mathbf{C} = \mathbf{Top}$ *or* $\mathbf{Top}_*$, $\mathbf{Ch}_{\geq 0}(k)$ *or* $\mathbf{Ch}(k)$ *where k is a commutative $\mathbb{Q}$-algebra, or* $\mathbf{Sp}^{\Sigma}_{+}$*, or* $\mathbf{Sp}^{O}_{+}$*, with the positive stable model structure.*

Then the assumptions of Theorem 6.5.2 hold for $\mathbf{D} = \mathrm{CAlg}(\mathbf{C})$.

Therefore in any of these settings, $\mathrm{CAlg}(\mathbf{C})$ *is a model category, with*

- *the cofibrations are the free cofibrations of commutative algebras,*
- *the weak equivalences and fibrations are the underlying ones.*

Proof. For the chain complex examples see [Whi17, 5.1]. For the topological examples we have to check that maps in $F_{\mathrm{CAlg}}(C)$ are closed inclusions (at each spectrum level, in the spectrum case), and that $F_{\mathrm{CAlg}}(J)$-cell complexes are weak equivalences. For spaces this follows from Corollary A.1.24, or more easily from the trick that the deformation retraction of $D^n \times I$ onto D^n passes to a deformation retraction of $F_{\mathrm{CAlg}}(D^n \times I)$ onto $F_{\mathrm{CAlg}}(D^n)$, as in Proposition 9.1.3. For spectra, this follows from Corollary A.1.45. □

Example 6.5.34. This gives a model structure on symmetric or orthogonal commutative ring spectra. It is not true that the underlying spectrum of a cofibrant commutative ring is cofibrant, essentially because the quotient by Σ_n destroys the cofibrancy. However the underlying spectrum is still flat cofibrant, in the sense given in Definition A.1.29.

6.6 Exercises

6.6.1. Given a symmetric spectrum E, define the E-homology and E-cohomology of another symmetric spectrum X by:

$$E_n(X) = [\Sigma^n \mathbb{S}, X \wedge^{\mathbb{L}} E]_s = \pi_n(X \wedge^{\mathbb{L}} E)$$

$$E^n(X) = [\Sigma^{-n} X, E]_s = \pi_{-n}(\mathbb{R}F(X, E)).$$

Explain what the Eilenberg-Steenrod axioms should be in this setting, and verify that they hold for these theories.

Note that the $\mathbb{L}$ decoration on the smash product can be removed if E and X are cofibrant, and the $\mathbb{R}$ decoration on F can be removed if X is cofibrant and E is fibrant.

By Exercise 6.6.22, these definitions of homology and cohomology agree with the earlier ones from Definition 2.6.37, Definition 3.2.18, and Example 3.2.22.

6.6.2. Suppose that E is a symmetric ring spectrum. Show that the cohomology of any *space* X inherits a "cup product" by taking any two maps in the stable homotopy category

$$\Sigma^{-m} X \xrightarrow{f} E, \Sigma^{-n} X \xrightarrow{g} E$$

to the composite

$$\Sigma^{-(m+n)} X \xrightarrow{\Delta} \Sigma^{-(m+n)}(X \wedge X) \xrightarrow{f \wedge g} E \wedge E \xrightarrow{\mu} E.$$

6.6.3. Prove that a spectrum X is dualizable in the stable homotopy category $\mathrm{Ho}\,\mathbf{Sp}^O$ iff it is finite, in other words it is a free spectrum on a finite CW complex. This proves that the Spanier-Whitehead category contains all of the dualizable spectra.

(Hint: If X is dualizable, make both X and DX cellular, and show that the coevaluation map $\mathbb{S} \to X \wedge DX$ must factor through $X' \wedge DX$ for a finite cellular subspectrum $X' \subseteq X$. Argue that X is a retract in $\mathrm{Ho}\,\mathbf{Sp}$ of X', and conclude that condition (3) in Exercise 2.7.43 holds.)

6.6.4. Prove Lemma 6.1.8 and Lemma 6.1.10. In other words, show that symmetric and orthogonal spectra are indeed described as diagrams over a category of based topological spaces.

6.6.5. In this exercise we give the inverse to the shift functor $\mathrm{sh}^1\colon \mathbf{Sp}^\Sigma \to \mathbf{Sp}^\Sigma$ from Definition 6.1.22. Recall that $(\mathrm{sh}^1 X)_n = X_{1+n}$, with Σ_n acting as the subgroup $1 \times \Sigma_n \leq \Sigma_{n+1}$ that permutes only the last n letters.

(a) Construct a symmetric spectrum by the rule

$$(\mathrm{sh}^{-1} X)_n = (\Sigma_n)_+ \wedge_{\Sigma_{n-1}} X_{n-1},$$

with bonding maps coming from those of X.

(b) Prove that this defines the left adjoint to sh^1.

(c) Argue that both of these functors preserve level equivalences, and that we get an adjunction in the level homotopy category

$$[\mathrm{sh}^{-1} X, Z]_\ell \cong [X, \mathrm{sh}^1 Z]_\ell.$$

(d) Conclude that sh^{-1} preserves stable equivalences.

6.6.6. Generalize Exercise 6.6.5 to orthogonal spectra. To prove that

$$O(n)_+ \wedge_{O(n-1)} A$$

preserves weak equivalences of spaces, you might find it helpful to recall that $O(n)$ is built out of cells of the form $O(n-1) \times D^m$ for varying m.

6.6.7. This exercise gives another proof that the π_*-isomorphisms in symmetric spectra give the wrong homotopy category. Suppose we form the homotopy category $\mathrm{Ho}\,\mathbf{Sp}^\Sigma$ using the π_*-isomorphisms (Definition 6.1.18). The forgetful functor to sequential spectra $U^\Sigma_{\mathbb{N}}$ clearly preserves these weak equivalences, so we get a map of homotopy categories

$$U^\Sigma_{\mathbb{N}}\colon \mathrm{Ho}\,\mathbf{Sp}^\Sigma \longrightarrow \mathrm{Ho}\,\mathbf{Sp}^{\mathbb{N}}.$$

(a) Recall from Exercise 6.6.5 the shift functor sh^1, and that the inverse of sh^1 on sequential spectra is $(\mathrm{sh}^{-1}_{\mathbb{N}} X)_n = X_{n-1}$. Explain why this does not define a symmetric spectrum.

(b) Show that the left adjoint $\mathrm{sh}^{-1}_{\Sigma} \colon \mathbf{Sp}^{\Sigma} \to \mathbf{Sp}^{\Sigma}$ to sh^1 from Exercise 6.6.5 is *not* π_*-isomorphic to $\mathrm{sh}^{-1}_{\mathbb{N}}$.

(c) Observe that sh^1 commutes with the forgetful functor, and conclude using Lemma 3.5.13 that we have a square of functors of homotopy categories

$$\begin{array}{ccc} \mathrm{Ho}\,\mathbf{Sp}^{\Sigma} & \xrightarrow{\mathrm{sh}^1_{\Sigma}} & \mathrm{Ho}\,\mathbf{Sp}^{\Sigma} \\ {\scriptstyle U^{\Sigma}_{\mathbb{N}}}\downarrow & & \downarrow{\scriptstyle U^{\Sigma}_{\mathbb{N}}} \\ \mathrm{Ho}\,\mathbf{Sp}^{\mathbb{N}} & \xrightarrow[\mathrm{sh}^1_{\mathbb{N}}]{} & \mathrm{Ho}\,\mathbf{Sp}^{\mathbb{N}} \end{array}$$

that commutes up to isomorphism.

Now, *assume* that $U^{\Sigma}_{\mathbb{N}}$ gives an equivalence of homotopy categories. This implies that the left adjoints agree up to isomorphism as well:

$$\begin{array}{ccc} \mathrm{Ho}\,\mathbf{Sp}^{\Sigma} & \xleftarrow{\mathrm{sh}^{-1}_{\Sigma}} & \mathrm{Ho}\,\mathbf{Sp}^{\Sigma} \\ {\scriptstyle U^{\Sigma}_{\mathbb{N}}}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle U^{\Sigma}_{\mathbb{N}}} \\ \mathrm{Ho}\,\mathbf{Sp}^{\mathbb{N}} & \xleftarrow[\mathrm{sh}^{-1}_{\mathbb{N}}]{} & \mathrm{Ho}\,\mathbf{Sp}^{\mathbb{N}} \end{array}$$

But this contradicts part (b), so $U^{\Sigma}_{\mathbb{N}}$ cannot be an equivalence of homotopy categories $\mathrm{Ho}\,\mathbf{Sp}^{\Sigma} \simeq \mathrm{Ho}\,\mathbf{Sp}^{\mathbb{N}}$. This proves that the π_*-isomorphisms are the "wrong" class of weak equivalences to take in $\mathbf{Sp}^{\Sigma}$.

6.6.8. What happens to the argument in Exercise 6.6.7 in the setting of orthogonal spectra? Which parts change, and how do we not get a contradiction?

6.6.9. Prove that the map λ_* in Definition 6.1.22 would fail to be a map of symmetric spectra if we did not include the permutation $\tau_{n,1}$ in its definition.

6.6.10. Check directly that $F(\mathbb{S}, X) \cong X$, where F is the function spectrum from Definition 6.2.12.

6.6.11. Suppose we work in either symmetric or orthogonal spectra.

(a) Generalizing Exercise 6.6.10, check that $F(F_1 S^0, X) \cong \mathrm{sh}^1 X$, where the shift functor sh^1 is defined in Exercise 6.6.5.

(b) Use this to conclude that $F(F_p S^0, X) \cong \mathrm{sh}^p X$, where sh^n is defined by iterating sh^1. More concretely, $(\mathrm{sh}^p X)_n = X_{p+n}$, with Σ_n acting through the permutations on $(p+n)$ letters that only permute the last n letters.

(c) Deduce formally from this that for any based space K,

$$F(F_pK, X) \cong F(K, \mathrm{sh}^p X) \cong \mathrm{sh}^p F(K, X)$$

where $F(K,-)$ denotes cotensor spectrum, and also

$$F_pK \wedge X \cong K \wedge \mathrm{sh}^{-p} X$$

where $\mathrm{sh}^{-p} = (\mathrm{sh}^{-1})^{\circ p}$ is the p-fold iterate of the functor sh^{-1} from Exercise 6.6.5.

(d) Using Corollary 6.1.35 and Exercises 6.6.5 and 6.6.6, conclude that if K is a cell complex, smashing with the free spectrum $F_pK \wedge (-)$ preserves all stable equivalences of spectra.

6.6.12. Suppose we work in either symmetric or orthogonal spectra. Prove that the smash product $X \wedge Y$ preserves stable equivalences when X is cofibrant and Y is *arbitrary*, generalizing Corollary 6.3.5.

(You'll need to do Exercises 6.6.5, 6.6.6, and 6.6.11 first, which show that $X \wedge (-)$ preserves equivalences if X is a free spectrum on a based cell complex K. Then you can use Theorem 6.1.31 to work your way up from this to any cofibrant spectrum X. Finally, you can use Corollary 6.3.5 to show that equivalences in the X variable are preserved as well.)

6.6.13. Use the argument of Exercise 6.6.12 to show that the function spectrum $F(X,Y)$ preserves equivalences if X is a *finite* cell spectrum and Y is arbitrary. (That is, if X is cofibrant but only has finitely many cells.)

What goes wrong if X has infinitely many cells? (You might want to recall Exercise 2.7.26.)

6.6.14. Define "unitary spectra." Show that every orthogonal spectrum gives a unitary spectrum using the inclusion of groups $U(n) \to O(2n)$.

6.6.15. Prove the four implications given in Definition 6.1.27. You might want to first prove:

- all of these notions of "cofibration" are preserved by coproducts, pushouts, sequential compositions, and retracts,
- by Lemma 1.2.7, the pushout-product of two unbased h-cofibrations of topological spaces is another unbased h-cofibration, and that
- if we take the pushout-product of two unbased h-cofibrations using the *smash product* instead, we still get an unbased h-cofibration!

6.6.16. Suppose $(F \dashv G)$ is a Quillen adjunction, or more generally an adjunction of categories with weak equivalences in which F can be left-derived and G can be right-derived. Suppose the derived unit map

$$\eta : X \to (\mathbb{R}G)(\mathbb{L}F)X$$

is a weak equivalence, and that $\mathbb{R}G$ **reflects weak equivalences**, meaning that $X \to Y$ is a weak equivalence iff $\mathbb{R}GX \to \mathbb{R}GY$ is a weak equivalence. Prove that the derived counit map

$$\epsilon : (\mathbb{L}F)(\mathbb{R}G)Y \to Y$$

is also a weak equivalence, and therefore $\mathbb{L}F$ and $\mathbb{R}G$ induce an equivalence of homotopy categories.

(Hint: This is easy, it takes longer to state than it does to prove! Start with the triangle identity that states that the composite

$$(\mathbb{R}G)Y \xrightarrow{\eta\circ\mathrm{id}} (\mathbb{R}G)(\mathbb{L}F)(\mathbb{R}G)Y \xrightarrow{\mathrm{id}\circ\epsilon} (\mathbb{R}G)Y$$

is equal to the identity in the homotopy category.)

6.6.17. Check that the functor $\amalg : \mathbf{S}^\Sigma \wedge \mathbf{S}^\Sigma \to \mathbf{S}^\Sigma$ defined in Definition 6.2.15 respects composition and identity maps, so that it is in fact a functor.

6.6.18. Generalize $\amalg$ to a functor $\oplus : \mathbf{S}^O \wedge \mathbf{S}^O \to \mathbf{S}^O$, where $\mathbf{S}^O = \mathscr{J}$ is the category from Definition 6.1.9. You should be taking direct sum of vector spaces and the direct sum of their orthogonal complements. The isomorphisms in Exercise 2.7.5 will probably be helpful.

6.6.19. Show that the functors $\amalg$ and $\oplus$ from Exercises 6.6.17 and 6.6.18 make $\mathbf{S}^\Sigma$ and $\mathbf{S}^O$, respectively, into symmetric monoidal categories.

6.6.20. Suppose $(\mathbf{I}, +, 0)$ and $(\mathbf{C}, \otimes, I)$ are symmetric monoidal categories, $\mathbf{I}$ is small, $\mathbf{C}$ has all colimits, and $\otimes$ preserves colimits in each slot. The **Day convolution** is a tensor product defined on the category of diagrams

$$\boxtimes : \mathbf{C}^\mathbf{I} \times \mathbf{C}^\mathbf{I} \to \mathbf{C}^\mathbf{I}$$

in the following way. Given two diagrams $X, Y : \mathbf{I} \to \mathbf{C}$, the tensor products $X(i) \otimes Y(j)$ form a diagram

$$X \otimes Y : \mathbf{I} \times \mathbf{I} \to \mathbf{C}.$$

We define the Day convolution to be the left Kan extension of this diagram along the sum functor $+$:

$$X \boxtimes Y = (+)_!(X \otimes Y) : \mathbf{I} \to \mathbf{C}$$

The smash product of diagram spectra from Definition 6.2.17 is a variant of this construction where $\mathbf{I}$ is enriched in based spaces and $\mathbf{C}$ has a tensor product with based spaces, but for the purposes of this exercise, let $\mathbf{I}$ and $\mathbf{C}$ be ordinary categories, enriched only in sets.

- Prove that there is an isomorphism $(X \boxtimes Y) \boxtimes Z \cong X \boxtimes (Y \boxtimes Z)$. You might find Exercise 5.8.15 helpful.
- Prove that there is an isomorphism $X \boxtimes Y \cong Y \boxtimes X$.
- Identify the unit diagram U, and prove that there is an isomorphism $X \boxtimes U \cong X$.

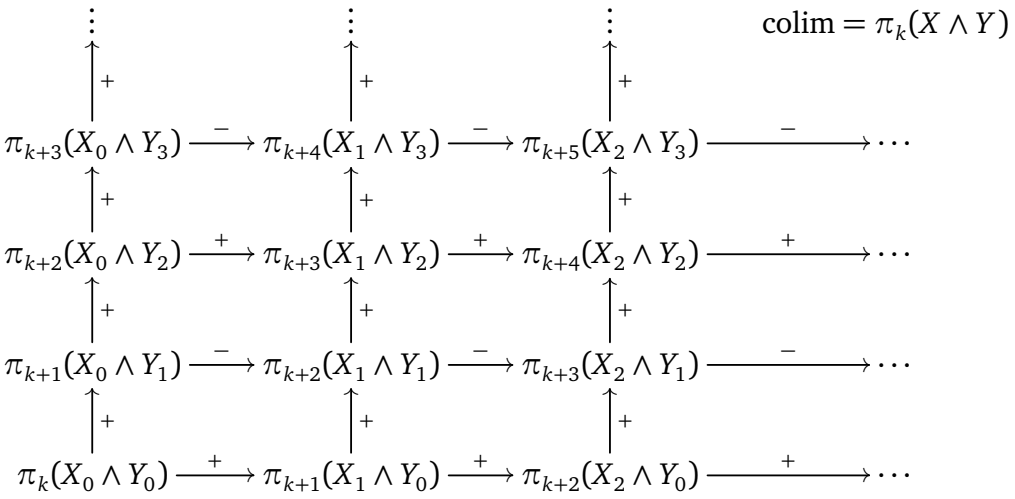
Together with the coherences between these isomorphisms, this makes $\mathbf{C}^{\mathbf{I}}$ into a symmetric monoidal category. This gives a more formal way of proving Theorem 6.2.8.

6.6.21. Recall from Exercise 5.8.35 that a bispectrum is a spectrum that has two directions. Formally, it is a diagram over the smash product category $\mathbf{S} \wedge \mathbf{S}$, where $\mathbf{S}$ is the sphere category $\mathbf{S}(m,n) = S^{n-m}$ from Definition 5.3.34.

(a) Define a symmetric bispectrum to be a diagram on $\mathbf{S}^{\Sigma} \wedge \mathbf{S}^{\Sigma}$. Explain how these arise in the definition of the smash product in Definition 6.2.17.

(b) Taking on faith that bispectra have a stable model structure in which the cofibrations are the relative cellular bispectra, and that suspension bispectrum $\mathbf{Top}_*^{\mathbf{S}^{\Sigma}} \to \mathbf{Top}_*^{\mathbf{S}^{\Sigma} \wedge \mathbf{S}^{\Sigma}}$ is a Quillen equivalence, prove that the left Kan extension $(\amalg)_! : \mathbf{Top}_*^{\mathbf{S}^{\Sigma} \wedge \mathbf{S}^{\Sigma}} \to \mathbf{Top}_*^{\mathbf{S}^{\Sigma}}$ is also a Quillen equivalence.

6.6.22. Using Exercise 6.6.21, show that the homotopy groups of the smash product of orthogonal spectra $X \wedge Y$ agrees with the homotopy groups of the bispectrum $X \overline{\wedge} Y$, which is the colimit of the grid of groups from Definition 2.3.23:



Of course, the same conclusion follows in symmetric spectra as well if we use the true homotopy groups from Definition 6.1.43. Finally, use this to show that the handicrafted smash product, and the underlying sequential spectrum of the orthogonal smash product, are stably equivalent.

6.6.23. Let $\mathbf{S}$ be the sphere category $\mathbf{S}(m,n) = S^{n-m}$ from Definition 5.3.34. Prove that the most "obvious" definition of a functor $\mathbf{S} \wedge \mathbf{S} \to \mathbf{S}$, does not in fact define a functor. This is one way of explaining why we couldn't define a good smash product on sequential spectra.

6.6.24. Finish the proof of the point-set rigidity theorem (Theorem 6.2.22) using the following ideas. We say that a map of spectra $X \to Y$ is levelwise surjective if each map $X_n \to Y_n$ is surjective.

(a) Show that if $X \to Y$ and $W \to Z$ are levelwise surjective then so is their smash product $X \wedge W \to Y \wedge Z$.

(b) If $f : X \to Y$ is levelwise surjective and we have a commuting square of the form

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\eta]{} & Y, \end{array}$$

explain why we must have $\eta = \mathrm{id}$.

(c) Prove that for any spectrum X, the canonical map

$$\bigvee_{n\geq 0} F_n X_n \longrightarrow X$$

is levelwise surjective.

(d) Use these ideas to prove that the natural transformation η in the proof of Theorem 6.2.22 must be the identity on any n-tuple of spectra.

6.6.25. Fix a symmetric or orthogonal ring spectrum R and let $\mathrm{Mod}(R)$ be the category of R-module spectra from Definition 6.2.32.

(a) Show that $\mathrm{Mod}(\mathbb{S}) \cong \mathbf{Sp}$, in other words $\mathbb{S}$-modules are the same thing as spectra.

(b) Show that the forgetful functor $\mathrm{Mod}(R) \to \mathbf{Sp}$ has as its left adjoint the functor $R \wedge -$ that takes the smash product with R, and has R act by

$$R \wedge R \wedge X \xrightarrow{\mu\wedge\mathrm{id}} R \wedge X.$$

6.6.26. Prove Lemma 6.3.12, that a strong monoidal Quillen left adjoint F induces monoidal functors $\mathbb{L}F$ and $\mathbb{R}G$ on the homotopy category. You might want to use ideas from the proof of Lemma 4.1.8.

6.6.27. Define a **lax symmetric monoidal Quillen adjunction** to be

- a Quillen adjunction $(F \dashv G)$ in which
- the left Quillen functor $F\colon \mathbf{C} \to \mathbf{D}$ is *oplax* symmetric monoidal, meaning the maps go in the direction

$$F(X \otimes_{\mathbf{C}} Y) \longrightarrow F(X) \otimes_{\mathbf{D}} F(Y), \quad I_{\mathbf{D}} \longrightarrow F(I_{\mathbf{C}}),$$

- the map of units $F(QI_{\mathbf{C}}) \to F(I_{\mathbf{C}}) \to I_{\mathbf{D}}$ is a weak equivalence, and
- for *cofibrant* X and Y, the map

$$F(X \otimes_{\mathbf{C}} Y) \longrightarrow F(X) \otimes_{\mathbf{D}} F(Y)$$

  is a weak equivalence.

Prove Lemma 6.3.12 under these weaker assumptions. In other words, the left-derived functor $\mathbb{L}F\colon \operatorname{Ho}\mathbf{C} \to \operatorname{Ho}\mathbf{D}$ is strong monoidal, and the right-derived functor $\mathbb{R}G\colon \operatorname{Ho}\mathbf{D} \to \operatorname{Ho}\mathbf{C}$ is lax monoidal.

6.6.28. Prove that the functor

$$F(A) = \Omega^\infty(A \wedge E)$$

of Example 6.4.13 is excisive. Here E is a sequential spectrum and A is a based finite CW complex.

6.6.29. Suppose that F and G are enriched functors and $\eta\colon F \to G$ is a natural transformation on the underlying functors. Explain why this gives a commuting square of topological spaces, not just sets,

$$\begin{array}{ccc} \operatorname{Map}_*(A,B) & \xrightarrow{F} & \operatorname{Map}_*(F(A),F(B)) \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle \eta\circ -} \\ \operatorname{Map}_*(G(A),G(B)) & \xrightarrow{-\circ\eta} & \operatorname{Map}_*(F(A),G(B)). \end{array}$$

This is called the **enriched naturality condition**. If this weren't automatically true, we would have to assume it as a condition on η in order to get a good theory of enriched functors.

6.6.30. Suppose $\mathbf{C}$ is a symmetric monoidal category that has all colimits and limits, and that $\otimes$ preserves colimits in each slot.

(a) For any diagram of algebras $R\colon \mathbf{I} \to \mathrm{Alg}(\mathbf{C})$, prove that the algebra $\lim R_i$ constructed in Lemma 6.5.13 is indeed the limit of the algebras R_i in the category $\mathrm{Alg}(\mathbf{C})$.

(b) If the diagram lies in the subcategory of commutative algebras, prove that the algebra structure constructed in this way is commutative as well.

6.6.31. In this exercise we show that *filtered* colimits in $\mathrm{Alg}(\mathbf{C})$ agree with the corresponding filtered colimit in $\mathbf{C}$. A category $\mathbf{I}$ is **filtered** if

- For every pair of objects a and b, there exists a third object c that receives morphisms $a \to c$ and $b \to c$.
- For any pair of parallel morphisms $d \rightrightarrows e$, there is a morphism $e \to f$ such that the two composites $d \rightrightarrows f$ agree.

Equivalently, for any diagram in $\mathbf{I}$ with finitely many objects and morphisms, there is a co-cone, a single object c receiving maps from all the objects of the diagram, commuting with all the morphisms of the diagram.

(a) For any symmetric monoidal category $\mathbf{C}$ with tensor $\otimes$ preserving colimits in each variable, and any diagram of algebras $R\colon \mathbf{I} \to \mathrm{Alg}(\mathbf{C})$, Let $\operatorname{colim}_{\mathbf{C}} R_i$ be the colimit in $\mathbf{C}$ and let $\operatorname{colim}_{\mathrm{Alg}(\mathbf{C})} R_i$ be the colimit in $\mathrm{Alg}(\mathbf{C})$ that we constructed in Lemma 6.5.13. Prove that there is a canonical map

$$\operatorname*{colim}_{\mathbf{C}} R_i \to \operatorname*{colim}_{\mathrm{Alg}(\mathbf{C})} R_i$$

that includes all those tensors $R_{i_1} \otimes \cdots \otimes R_{i_n}$ in which there is only one term.

(b) If $\mathbf{I}$ is filtered, prove that this canonical map is an isomorphism. In other words, the forgetful functor $\mathrm{Alg}(\mathbf{C}) \to \mathbf{C}$ preserves filtered colimits.

(c) Check that the same statement holds in $\mathrm{CAlg}(\mathbf{C})$ as well.

6.6.32. Suppose $\mathbf{C}$ is a symmetric monoidal model category.

(a) Let R be an algebra in $\mathbf{C}$, and suppose that the model structure on $\mathrm{Mod}(R)$ from Theorem 6.5.6 exists. Prove that if R cofibrant in $\mathbf{C}$, then any cofibration $M \to N$ in $\mathrm{Mod}(R)$ is also a cofibration in $\mathbf{C}$.

(b) Suppose that the model structure on $\mathrm{Alg}(\mathbf{C})$ from Theorem 6.5.20 exists. Prove that if R is an algebra that is cofibrant in $\mathbf{C}$, then any cofibration $R \to S$ in $\mathrm{Alg}(\mathbf{C})$ is a cofibration in $\mathbf{C}$ as well. (You might find Corollary 6.5.19 helpful.) In particular, if the unit I is cofibrant, then every cofibrant algebra in $\mathrm{Alg}(\mathbf{C})$ is cofibrant in $\mathbf{C}$ as well.

6.6.33. Suppose that $\mathbf{C}$ is a symmetric monoidal model category and that R is a commutative algebra in $\mathbf{C}$. An **R-algebra** is an algebra A in $\mathbf{C}$, not necessarily commutative, and an algebra homomorphism $R \to A$. Let $\mathrm{Alg}_R(\mathbf{C})$ be the category of R-algebras.

(a) If $\mathrm{Alg}(\mathbf{C})$ has a right-induced model structure, in other words a model structure in which the weak equivalences and fibrations are the underlying ones in $\mathbf{C}$, then prove that $\mathrm{Alg}_R(\mathbf{C})$ has a right-induced model structure as well. (You might find Exercise 5.8.8 helpful.)

(b) Explain how we can identify left R-modules and right R-modules when R is commutative. Then prove that the operation $\otimes_R$ from Definition 6.5.8 makes the category $\mathrm{Mod}(R)$ into a symmetric monoidal category.

(c) Prove that an R-algebra is the same thing as an algebra in the symmetric monoidal category $\mathrm{Mod}(R)$. Therefore $\mathrm{Alg}(\mathrm{Mod}(R))$ has a right-induced model structure as well.

The above claims apply to commutative algebras as well, by the same proof.

6.6.34. Continuing the previous exercise, suppose that $\mathbf{C}$ is a symmetric monoidal model category, and $\phi : R \to S$ is a map of commutative algebras in $\mathbf{C}$. We already know that under the assumptions in Proposition 6.5.10, the restriction $\phi^* : \mathrm{Mod}(S) \to \mathrm{Mod}(R)$ and extension of scalars $S \otimes_R (-) : \mathrm{Mod}(R) \to \mathrm{Mod}(S)$ induce a Quillen adjunction between $\mathrm{Mod}(R)$ and $\mathrm{Mod}(S)$.

(a) Prove that ϕ^* is lax symmetric monoidal and $S \otimes_R (-)$ is strong symmetric monoidal. Therefore they induce functors on the category of S-algebras and R-algebras, respectively.

(b) Prove that the resulting maps between R-algebras and S-algebras are adjoint, and Quillen adjoint if the right-induced model structure exists on $\mathrm{Alg}_R(\mathbf{C})$ and $\mathrm{Alg}_S(\mathbf{C})$.

(c) If $\phi : R \to S$ is a weak equivalence, prove that this is a Quillen equivalence under the assumptions in Proposition 6.5.10, and therefore in all of the examples in Corollary 6.5.11.

Chapter 7

Simplicial sets, simplicial spaces, and the bar construction

The bar construction is one of the most important ideas in homotopy theory. Its raw power is comparable to that of the small-object argument (and is therefore considerable). In this book, among other things, the bar construction will allow us to define homotopy colimits for diagrams of any shape, and to define the Eilenberg–Mac Lane spectrum $H\mathbb{Z}$ as a ring spectrum.

The bar construction can be visualized as follows. Suppose G is a group, X is a set with a right action by G, and Y is a set with a left action by G. The bar construction $B(X, G, Y)$ is a topological space. It has one point for every configuration of finitely many points on the interval as shown:

The intermediate points are labeled by elements $g_i \in G$, and the points on the end are labeled by $x \in X$ and $y \in Y$. The space is topologized so that the points labeled by g_i can move along the interval, and furthermore the value of each label g_i can vary in G, and also the value of x in X and y in Y. The point g_i can also collide with g_{i+1} and become a single point labeled by the product $g_i g_{i+1}$. If g_1 collides with the x, it changes the x to xg_1, and similarly if g_n collides with y then it changes to $g_n y$. Finally, any point labeled by the identity $1 \in G$ may be deleted.

In order to develop the bar construction, we will use the language of simplicial sets. We have written this book so that the first six chapters can be read without knowledge of simplicial sets, but from this point forward we will begin to use them liberally. They are a necessary ingredient for the remaining chapters on homotopy colimits, operads, higher categories, and higher algebra.

We do not assume the reader has seen simplicial sets before, so we begin by introducing

them, and their generalization to simplicial spaces. We then use simplicial spaces to define the bar construction $B(X,G,Y)$ for any topological group G, and a more general bar construction where G is replaced by any topological category $\mathbf{I}$. This allows us to define homotopy colimits in **Top** for diagrams of any shape! We also describe the dual theory of cosimplicial spaces and the cobar construction, which gives an explicit formula for homotopy limits. In particular, we finally prove that homotopy limits in **Top** exist, which we were not able to do earlier using the model structure on **Top**.

In the next chapter, we formalize all of this and define the bar construction in any category $\mathbf{C}$ with a notion of "n-simplex." Specializing to the case where $\mathbf{C}$ is chain complexes, based spaces, or spectra gives many important examples, including the bar complex from homological algebra, and a corresponding bar complex for module spectra over ring spectra.

Much of this chapter can be read independently of the rest of the book. If you skip the comments about model categories and spectra, there is a good deal of information in here purely about the bar construction, that can be understood using only the preliminaries from Chapter 1.

7.1 Simplicial sets

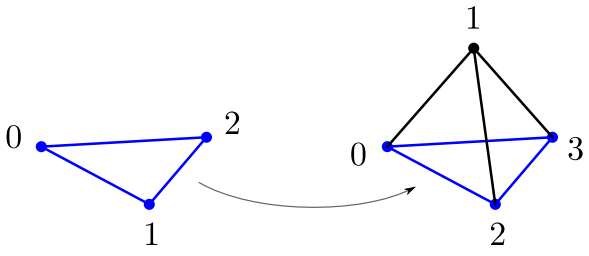
7.1.1 The simplex category Δ

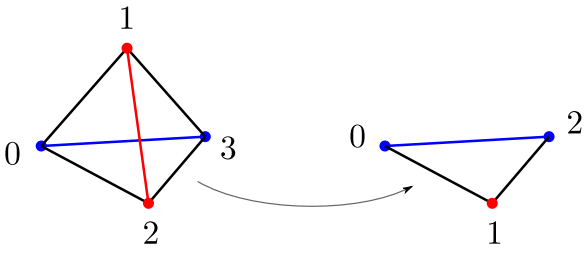
Recall that for each $n \geq 0$, the n-simplex is defined to be the convex hull of the standard basis vectors in $\mathbb{R}^{n+1}$:

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0 \ \forall i \ , \ \sum_i t_i = 1 \right\}$$

This is homeomorphic to the convex hull of any $(n+1)$ points in general position in Euclidean space of dimension at least n.

We can relate simplices of varying dimensions together. For each $0 \leq i \leq n$, the ith face of Δ^n is the subset of all points in which $t_i = 0$. This is identified with Δ^{n-1} in the obvious way, preserving the order of the remaining vertices. The resulting map $\Delta^{n-1} \to \Delta^n$ we call the ith **coface map** d^i.



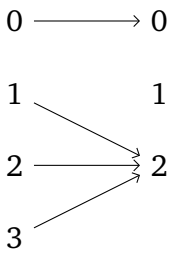


For each $0 \le i \le n$, we can also define a surjective map $\Delta^{n+1} \to \Delta^n$ that folds vertices i and $i+1$ together to a single point, and linearly interpolates to the rest of the simplex. In other words, the coordinates t_i and t_{i+1} are added together. We call this the ith **codegeneracy map** s^i.

The previous two constructions generalize. Let $[n]$ be the set with $(n+1)$ elements,

$$[n] = \{0, 1, \ldots, n\}.$$

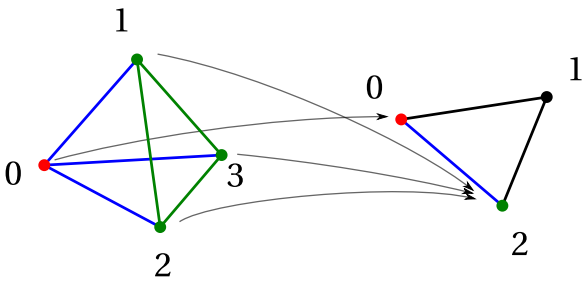
It has a total ordering given by $0 \le 1 \le \cdots \le n$. We say that a function $\alpha\colon [m] \to [n]$ is **totally ordered** or **nondecreasing** if $i \le j$ implies $\alpha(i) \le \alpha(j)$. For instance, the following is a totally ordered map $[3] \to [2]$.



Definition 7.1.1. The **simplex category** $\mathbf{\Delta}$ has objects $[n] = \{0, 1, \ldots, n\}$ for $n \ge 0$, and morphisms the totally ordered maps.

Each map of totally ordered sets $\alpha\colon [m] \to [n]$ gives an obvious map from the vertices of Δ^m to the vertices of Δ^n. This extends uniquely to a linear map $\mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ that sends Δ^m into Δ^n. By abuse of notation, we also call this map $\alpha\colon \Delta^m \to \Delta^n$.

In more detail, each coordinate of the output is the sum of the coordinates of its preimages under α, $t_j = \sum_{i \in \alpha^{-1}(j)} t_i$. The action of the map $\alpha\colon [3] \to [2]$ given above is illustrated to the right.



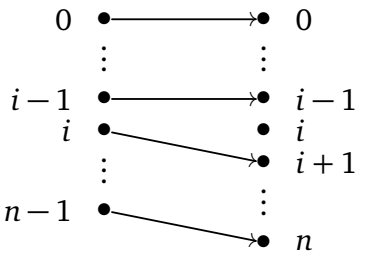
The procedure we just described makes the n-simplices Δ^n into a diagram over the category $\mathbf{\Delta}$, in other words a functor

$$\Delta^{(-)}\colon \mathbf{\Delta} \to \mathbf{Top}.$$

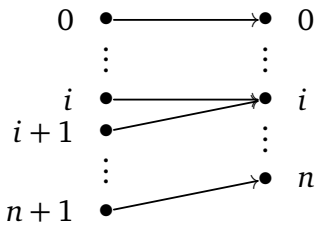
The coface and codegeneracy maps show up in $\mathbf{\Delta}$ as the injective maps $[n-1] \to [n]$, and the surjective maps $[n+1] \to [n]$. See Figure 7.1.2.

$$d^i : [n-1] \longrightarrow [n] \qquad s^i : [n+1] \longrightarrow [n]$$

$$d^i(j) = \begin{cases} j & \text{if} \quad j < i \\ j+1 & \text{if} \quad j \geq i \end{cases} \qquad s^i(j) = \begin{cases} j & \text{if} \quad j \leq i \\ j-1 & \text{if} \quad j > i \end{cases}$$



(a) The coface map d^i.



(b) The codegeneracy map s^i.

Figure 7.1.2: Generators for Δ

Lemma 7.1.3. *The morphisms of the category* $\mathbf{\Delta}$ *are generated by the maps* d^i *and* s^i *for all* n *and* i*, subject to the following relations:*

1. $d^i d^j = d^{j+1} d^i$ for $i \leq j$,
2. $s^j s^i = s^i s^{j+1}$ for $i \leq j$,
3. $s^j d^i = d^i s^{j-1}$ for $i < j$,
4. $s^j d^i = 1$ for $i = j, j+1$, and
5. $s^j d^i = d^{i-1} s^j$ for $i > j+1$.

Proof. It is not hard to see that any map $\alpha\colon [m] \to [n]$ can be decomposed into codegeneracies (fold together any points that have the same image under α), followed by cofaces (include any points into $[n]$ that are not in the image of α). The relations above clearly hold.

To see that these relations suffice, take any composite of cofaces and codegeneracies that gives $\alpha\colon [m] \to [n]$. We use the last three relations to rearrange the composite so that all the codegeneracies happen first (on the right), and all the face maps happen second (on the left), for instance

$$d^1 d^3 d^1 s^2 s^1 s^4.$$

We then use the first relation to rearrange the cofaces so that they happen in increasing order, and the second relation to rearrange the codegeneracies so they happen in increasing order. This puts the composite into a standard form that is determined by the underlying map of sets $\alpha\colon [m] \to [n]$. □

7.1.2 Simplicial sets

Definition 7.1.4. A **simplicial set** is a contravariant diagram of sets on $\mathbf{\Delta}$. In other words, a functor

$$X_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}.$$

Concretely, this is a sequence of sets $\{X_n\}_{n\geq 0}$, and for each map $\alpha\colon [m] \to [n]$, we get a map going the other way $\alpha^*\colon X_n \to X_m$. Furthermore, this respects composition and identity maps in the category $\mathbf{\Delta}$.

You should think of each X_n as a "collection of n-simplices in some space." The diagram goes in the opposite direction as $\mathbf{\Delta}$, because each map of simplices $\alpha\colon \Delta^m \to \Delta^n$ gives a function from the set of n-simplices to the set of m-simplices. (Namely, for each n-simplex in a space, we can compose with α to get an m-simplex in the same space.)

In particular, the ith coface map $d^i\colon \Delta^{n-1} \to \Delta^n$ gives us a way to associate each n-simplex to an $(n-1)$-simplex, by restricting to its ith face. We call the corresponding map in $X_\bullet$ the ith **face map**

$$d_i \colon X_n \to X_{n-1}.$$

Similarly, the ith codegeneracy map $s^i\colon \Delta^{n+1} \to \Delta^n$ gives us a way to associate each n-simplex to an $(n+1)$-simplex, the denenerate one given by composing with s^i. We call the corresponding map in $X_\bullet$ the ith **degeneracy map**

$$s_i \colon X_n \to X_{n+1}.$$

Lemma 7.1.5. *A simplicial set is given by a sequence of sets $\{X_n\}_{n\geq 0}$, together with face maps $d_i\colon X_n \to X_{n-1}$ and degeneracy maps $s_i\colon X_n \to X_{n+1}$ for each $n \geq 0$ and $0 \leq i \leq n$, subject to the following relations.*

1. $d_j d_i = d_i d_{j+1}$ for $i \leq j$,
2. $s_i s_j = s_{j+1} s_i$ for $i \leq j$,
3. $d_i s_j = s_{j-1} d_i$ for $i < j$,
4. $d_i s_j = 1$ for $i = j, j+1$, and
5. $d_i s_j = s_j d_{i-1}$ for $i > j+1$.

Proof. This follows from Lemma 7.1.3. Since $\mathbf{\Delta}$ is generated by the maps d^i and s^i, it suffices to give the action of those maps on the sets X_n. □

Example 7.1.6. If X is any topological space, the **singular simplices** of X are a simplicial set $S_\bullet(X)$ where level n is the set of all continuous maps $\Delta^n \to X$,

$$S_n(X) := \{\ \Delta^n \to X\ \}.$$

Each map $\alpha\colon [m] \to [n]$ gives a map of simplices $\alpha\colon \Delta^m \to \Delta^n$. Composing with α gives a rule that assigns to each n-simplex in X an m-simplex in X:

$$S_n(X) \to S_m(X).$$

Concretely, the face map $d_i\colon S_n(X) \to S_{n-1}(X)$ restricts each simplex to its ith face, and the degeneracy map $s_i\colon S_n(X) \to S_{n+1}(X)$ creates out of each n-simplex a degenerate $(n+1)$-simplex.

Example 7.1.7. Define a simplicial set $X_\bullet = \Delta[1]_\bullet$ by

$$X_0 = \{0,1\}, \quad X_1 = \{00,01,11\}, \quad X_2 = \{000,001,011,111\}, \quad \dots$$

where X_n consists of all non-decreasing sequences in the terms $\{0,1\}$. We call the first term in the sequence the "0th term", the next term the "1st term," and so on. With this convention in place, the ith face map deletes the ith term in the sequence, and the ith degeneracy map duplicates the ith term. We call this the **standard 1-simplex** $\Delta[1]_\bullet$.

More generally, we define the **standard k-simplex** $\Delta[k]_\bullet$ by allowing the sequences to take values in $\{0,1,\cdots,k\}$. So for $\Delta[2]$ we have

$$X_0 = \{0,1,2\}, \quad X_1 = \{00,01,02,11,12,22\}, \quad \dots$$

It is left to Exercise 7.7.1 to check that $\Delta[k]_\bullet$ is the functor on $\mathbf{\Delta}^{\mathrm{op}}$ represented by the object $[k]$. In other words, $\Delta[k]_n = \mathbf{\Delta}([n],[k])$.

Remark 7.1.8. We often refer to elements $x \in X_n$ as "n-simplices in X." There are many good reasons to use this language. For one, a map of simplicial sets $\Delta[n]_\bullet \to X_\bullet$ is the same thing as a choice of element $x \in X_n$.

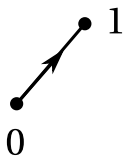
Example 7.1.9. Recall from [Hat02] that a **Δ-complex** is a CW complex X in which the n-cells are described as simplices $\Delta^n \to X$, such that for each i, the ith face $\Delta^{n-1} \to \Delta^n \to X$ is one of the $(n-1)$-cells, with the same ordering on its vertices. It follows that X is expressed as the disjoint union of the simplices Δ^n, modulo a relation that glues each face to a lower-dimensional simplex.

If X is a Δ-complex, we associate to it a simplicial set $X_\bullet$. It will be a subset of the singular simplices $S_\bullet(X)$, where we only take those maps $\Delta^n \to X$ of the form

$$\Delta^n \longrightarrow \Delta^k \longrightarrow X,$$

where the first map comes from a map of totally ordered sets $[n] \to [k]$, and the second map is one of the k-simplices in the Δ-complex structure on X. (We consider two such composites to be the same if they give the same map $\Delta^n \to X$.) It is easy to check this is preserved under taking faces and degeneracies, so this gives a simplicial set $X_\bullet$.

For instance, the 1-simplex Δ^1 is a Δ-complex in an obvious way. The associated simplicial set has an n-simplex for every map of totally ordered sets $[n] \to [1]$. We can write these as non-decreasing sequences in the terms $\{0,1\}$, so



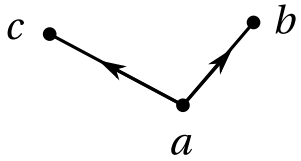
$$X_0 = \{0,1\}, \quad X_1 = \{00,01,11\}, \quad X_2 = \{000,001,011,111\}, \quad \dots$$

This is exactly the standard 1-simplex $\Delta[1]_\bullet$ from Example 7.1.7.

Similarly, the k-simplex Δ^k is a Δ-complex. Its associated simplicial set is the standard k-simplex $\Delta[k]_\bullet$. This has an n-simplex for every map of totally ordered sets $[n] \to [k]$.

The boundary $\partial\Delta^k \subseteq \Delta^k$ is the subcomplex which consists of all the faces. The associated simplicial set $\partial\Delta[k]_\bullet \subseteq \Delta[k]_\bullet$ is at level n the subset of all maps $[n] \to [k]$ that are not surjective, i.e. they miss at least one point of $[k]$. Missing a point corresponds to $\Delta^n \to \Delta^k$ landing in one of the faces of Δ^k.

To give another example, the Δ-complex pictured has two 1-simplices glued end-to-end. In the associated simplicial set, we have



$$X_0 = \{a,b,c\}, \quad X_1 = \{aa,ab,bb,ac,cc\},$$
$$X_2 = \{aaa,aab,abb,bbb,aac,acc,ccc\}, \quad \dots$$

So X_0 has three points, while X_1 has five points: one for each of the pictured 1-simplices, and three degenerate ones that collapse onto the points a, b, and c. In general, X_n has $(2n+3)$ points.

Example 7.1.10. Let $\mathbf{C}$ be a category. The **nerve** $N_\bullet\mathbf{C}$ is the simplicial set with one n-simplex for every n-tuple of composable maps in $\mathbf{C}$:

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} c_n.$$

When $n = 0$, we interpret this to mean a choice of object $c_0 \in \operatorname{ob}\mathbf{C}$.

The ith face map deletes c_i, composing the maps f_i and f_{i+1} if $0 < i < n$. The ith degeneracy map duplicates c_i and inserts an identity map between the two copies.

This also generalizes Example 7.1.7. Consider the category $\{0 \to 1 \to \dots \to k\}$, which has a unique morphism $i \to j$ if $i \le j$, and no other morphisms. An n-tuple of composable arrows in this category is the same thing as a totally ordered map $[n] \to [k]$. Therefore the nerve of this category is the standard k-simplex,

$$N_\bullet(\{0 \to 1 \to \dots \to k\}) \cong \Delta[k]_\bullet.$$

7.1.3 Geometric realization

Definition 7.1.11. For a simplicial set $X_\bullet$, the **geometric realization** $|X_\bullet|$ is a topological space defined as the quotient

$$|X_\bullet| = \left(\coprod_{n\geq 0} \Delta^n \times X_n \right) / \, (\alpha(t), x) = (t, \alpha^*(x)) \tag{7.1.12}$$

where the relation varies over all pairs $m, n \geq 0$, all points $t \in \Delta^m$, $x \in X_n$, and all maps $\alpha\colon [m] \to [n]$.

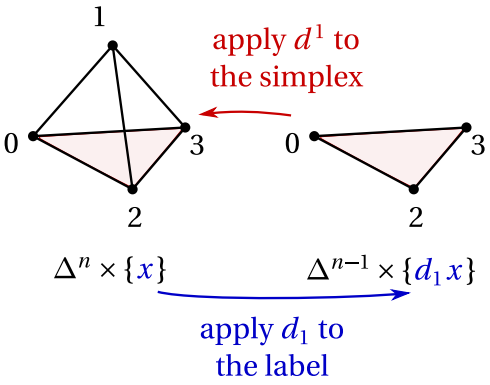
Lemma 7.1.13. *The geometric realization can be described using only face and degeneracy maps:*

$$|X_\bullet| = \left(\coprod_{n\geq 0} \Delta^n \times X_n \right) / \, (d^i t, x) = (t, d_i x),\ (s^i t, x) = (t, s_i x). \tag{7.1.14}$$

Proof. This follows because each map α is a composite of maps d^i and s^i (Lemma 7.1.3). □

Intuitively, the realization creates an actual n-simplex for each point $x \in X_n$. Then it glues them together using the face and degeneracy maps.

For example, the ith face of the simplex given by $x \in X_n$ ought to be glued to the $(n-1)$ simplex given by $(d_i x) \in X_{n-1}$, as illustrated on the right. The relation $(d^i t, x) = (t, d_i x)$ accomplishes this. It identifies the copy of Δ^{n-1} associated to $d_i x$ to the ith face of the Δ^n associated to x.



The degeneracy relations do essentially the same thing, collapsing down all of the degenerate simplices labeled by $s_i x$ onto lower-dimensional simplices.

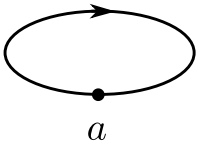
Proposition 7.1.15. *If $X_\bullet$ is the simplicial set associated to a Δ-complex X (Example 7.1.9), its realization is homeomorphic to X.*

Proof. Each n-simplex in the Δ-complex corresponds to a point of X_n, namely the composite $\Delta^n = \Delta^n \to X$ where the first map is the identity. This gives a rule that maps $X \to |X_\bullet|$, by taking this n-simplex of X to the corresponding n-simplex in $\Delta^n \times X_n$. It is well-defined because the relations that glue the simplices of X together are contained in the relations that define $|X_\bullet|$.

It is similarly straightforward to define a map $|X_\bullet| \to X$, to check that the composite $X \to |X_\bullet| \to X$ is the identity, and to check that $X \to |X_\bullet|$ is surjective. All together this shows that the two spaces are homeomorphic. □

Corollary 7.1.16. *The geometric realization of the standard k-simplex $|\Delta[k]_\bullet|$ is homeomorphic to the topological k-simplex Δ^k.*

Example 7.1.17. The circle S^1 is a Δ-complex with one 1-simplex, whose endpoints are glued together. We call the associated simplicial set the **simplicial circle** $S^1_\bullet$. It can be described as the quotient $\Delta[1]_\bullet/\partial\Delta[1]_\bullet$. At level n, it consists of all the maps in $\mathbf{\Delta}$ from $[n]$ to $[1]$, but with the two constant maps identified to each other. By the above, the realization of this simplicial set gives the circle S^1 again.



Remark 7.1.18. Not every simplicial set arises from a Δ-complex. For instance, we can take

$$\Delta[2]_\bullet/\partial\Delta[2]_\bullet,$$

the standard 2-simplex with all of the simplices in the boundary $\partial\Delta[2]$ identified together. The realization of this simplicial set is easy to understand: it is the two-sphere

$$\Delta^2/\partial\Delta^2 \cong S^2.$$

However, this simplicial set did not come from a Δ-complex, because in a Δ-complex the edges on the boundary of Δ^2 cannot be collapsed down to a point.

Therefore, we think of simplicial sets as "like Δ-complexes, except that we are allowed to collapse the face of a simplex down to a simplex of smaller dimension."

The geometric realization can be a little hard to picture when $X_\bullet$ isn't defined from a Δ-complex. One way to picture it is to throw out the points $x \in X_n$ that are degenerate, then build a CW complex out of the remaining points.

Definition 7.1.19. If $X_\bullet$ is a simplicial set, we say $x \in X_n$ is **degenerate** if $x = s_i(y)$ for some $0 \le i \le n-1$ and $y \in X_{n-1}$. We say x is **nondegenerate** otherwise.

Theorem 7.1.20. *When $X_\bullet$ is a simplicial set, the realization $|X_\bullet|$ is a CW complex with one n-cell for every nondegenerate point of X_n.*

We defer the proof to the next chapter, where it is an immediate consequence of Lemma 8.2.36. You can sketch a proof by applying the relations $(s^i t, x) = (t, s_i x)$ from (7.1.14) to collapse away all of the simplices labeled by degenerate x. The remaining simplices are then glued along the face relations $(d^i t, x) = (t, d_i x)$. But the trick to making this precise is to handle the possibility of strange combinations of the relations that further reduce the result to something smaller!

Example 7.1.21. Consider the standard 1-simplex of Example 7.1.7. Since the degeneracy maps duplicate the terms of a sequence, a sequence is degenerate precisely when it has a repeated term. So there are three nondegenerate sequences in the entire simplicial set, highlighted in blue. The remaining sequences, highlighted in red, are degenerate.

$$X_0 = \{0, 1\}, \quad X_1 = \{00, 01, 11\}, \quad X_2 = \{000, 001, 011, 111\}, \quad \dots$$

We conclude that the realization has two 0-cells and one 1-cell. Of course, the realization is the 1-simplex Δ^1, so this is what we expected.

Example 7.1.22. The simplicial circle is similar to the last example, except that the constant sequence $11\dots1$ is identified to the constant sequence $00\dots0$:

$$X_0 = \{0\}, \quad X_1 = \{00, 01\}, \quad X_2 = \{000, 001, 011\}, \quad \dots$$

We conclude that the realization has one 0-cell and one 1-cell. This gives the circle S^1 with its standard cell complex structure.

7.1.4 The category of simplicial sets

Let **sSet** be the category whose objects are simplicial sets, in other words diagrams

$$X_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}.$$

The morphisms in this category $f : X_\bullet \to Y_\bullet$ are natural transformations, in other words diagrams

$$\mathbf{\Delta}^{\mathrm{op}} \times \{\bullet \to \bullet\} \to \mathbf{Set}.$$

Concretely, this means we have maps $f_n \colon X_n \to Y_n$ for $n \geq 0$, commuting with each of the face and degeneracy maps.

Example 7.1.23. If $\Delta[k]_\bullet$ is the standard k-simplex from Example 7.1.7, a map of simplicial sets

$$\Delta[k]_\bullet \longrightarrow Y_\bullet$$

is the same thing as a point $y \in Y_k$. This is because $\Delta[k]_\bullet$ is a free diagram on the category $\mathbf{\Delta}^{\mathrm{op}}$ at the object k. Or, to put it another way, it is a representable functor, and this is a consequence of the Yoneda lemma.

Example 7.1.24. If $S^1_\bullet$ is the simplicial circle from Example 7.1.17, there are only two maps of simplicial sets

$$S^1_\bullet \longrightarrow S^1_\bullet,$$

namely the identity map, and the zero map that sends everything to the basepoint. This is in contrast to the topological space S^1, which can have degree n maps for any $n \in \mathbb{Z}$. The combinatorial model is much more rigid than the topological space.

Example 7.1.25. Singular simplices gives a functor **Top** → **sSet**. In other words, every continuous map of topological spaces $X \to Y$ induces a map of simplicial sets $S_\bullet X \to S_\bullet Y$.

For the circle, this means there are many interesting maps of simplicial sets $S_\bullet(S^1) \to S_\bullet(S^1)$. Of course, $S_\bullet(S^1)$ is a much larger simplicial set than the simplicial circle $S^1_\bullet$.

Example 7.1.26. The nerve of a category gives a functor **Cat** → **sSet**. In other words, every functor of categories $\mathbf{C} \to \mathbf{D}$ induces a map of simplicial sets $N_\bullet\mathbf{C} \to N_\bullet\mathbf{D}$.

Example 7.1.27. Geometric realization gives a functor **sSet** → **Top**. So every map of simplicial sets $X_\bullet \to Y_\bullet$ induces a map on the realizations $|X_\bullet| \to |Y_\bullet|$.

It is an exercise to check that the realization functor is the left adjoint of singular simplices. In other words, if $X_\bullet$ is a simplicial set and Y is a topological space, then a map of topological spaces

$$|X_\bullet| \longrightarrow Y$$

is the same thing as a map of simplicial sets $X_\bullet \to S_\bullet(Y)$. See Exercise 7.7.4.

7.1.5 Colimits and limits

We can take a product of two simplicial sets $X_\bullet \times Y_\bullet$ by taking the product at each level, $X_n \times Y_n$, and having $\mathbf{\Delta}^{\mathrm{op}}$ act on both factors. We can similarly take a limit of any diagram of simplicial sets $\lim_{i\in\mathbf{I}} X_\bullet(i)$ in the same way: the set of n-simplices will be the limit of the sets $X_n(i)$ over all values of i:

$$\left(\lim_{i\in\mathbf{I}} X_\bullet(i)\right)_n := \lim_{i\in\mathbf{I}} X_n(i).$$

It is easy to check that this really is the limit in the category of simplicial sets **sSet**: a map of simplicial sets

$$Z_\bullet \longrightarrow \lim_{i\in\mathbf{I}} X_\bullet(i)$$

is the same thing as maps $Z_\bullet \to X_\bullet(i)$ for all i, commuting with the maps of the diagram.

Example 7.1.28. Given two maps of simplicial sets $X_\bullet \to B_\bullet \leftarrow Y_\bullet$, we form the pullback $X_\bullet \times_{B_\bullet} Y_\bullet$ by taking the pullbacks $X_n \times_{B_n} Y_n$ at each simplicial level.

Dually, we can take a coproduct $X_\bullet \amalg Y_\bullet$ by taking the disjoint union at each level, $X_n \amalg Y_n$. More generally, we can take a colimit of any diagram of simplicial sets $\operatorname{colim}_{i\in\mathbf{I}} X_\bullet(i)$ by letting the n-simplices by the colimit of the sets $X_n(i)$ over all values of i:

$$\left(\operatorname*{colim}_{i\in\mathbf{I}} X(i)\right)_n := \operatorname*{colim}_{i\in\mathbf{I}} X_n(i).$$

So for instance we can take pushouts $X_\bullet \cup_{A_\bullet} Y_\bullet$.

What happens to the realizations when we do this? For colimits, the answer is simple. Geometric realization $|-|$ is a left adjoint, by Exercise 7.7.4. By Theorem 1.6.9, it therefore preserves colimits:

Lemma 7.1.29. *Geometric realization commutes with all colimits,*

$$\left|\operatorname*{colim}_{i\in\mathbf{I}} X_\bullet(i)\right| \cong \operatorname*{colim}_{i\in\mathbf{I}} |X_\bullet(i)|.$$

In particular, it commutes with coproducts and pushouts:

$$|X_\bullet \amalg Y_\bullet| \cong |X_\bullet| \amalg |Y_\bullet|, \qquad |X_\bullet \cup_{A_\bullet} Y_\bullet| \cong |X_\bullet| \cup_{|A_\bullet|} |Y_\bullet|.$$

This is not very hard to see in concrete examples anyway.

It would be strange if geometric realization commuted with products. That would suggest that if X and Y are Δ-complexes, you could build $X \times Y$ by taking one n-simplex for each choice of an n-simplex in X and an n-simplex in Y. And this seems false, because you expect a copy of $\Delta^n \times \Delta^n$ in the product for each such pair, not just a single copy of Δ^n.

However, a miracle occurs: geometric realization actually does preserve products. The reason for this is the degeneracy maps: you can cut all of the products of simplices $\Delta^n \times \Delta^n$ into simplices, in such a way that each one comes from a pair of simplices in Δ^n, *which may be degenerate*, even if the simplex in $\Delta^n \times \Delta^n$ is not degenerate.

To be more precise, let $X_\bullet \times Y_\bullet$ be a product of two simplicial sets. We have projection maps to $X_\bullet$ and $Y_\bullet$, giving maps on the realizations

$$|X_\bullet \times Y_\bullet| \longrightarrow |X_\bullet|, \qquad |X_\bullet \times Y_\bullet| \longrightarrow |Y_\bullet|.$$

This induces a map to the product of the realizations

$$|X_\bullet \times Y_\bullet| \xrightarrow{\cong} |X_\bullet| \times |Y_\bullet|. \tag{7.1.30}$$

Theorem 7.1.31 (Milnor)**.** *The map is* (7.1.30) *is an isomorphism. More generally, the map*

$$\left|\lim_{i\in\mathbf{I}} X_\bullet(i)\right| \xrightarrow{\cong} \lim_{i\in\mathbf{I}} |X_\bullet(i)|$$

is an isomorphism if $\mathbf{I}$ *has finitely many objects and finitely many morphisms.*

We defer the proof to the exercises (Exercises 7.7.7, 7.7.8, and 7.7.9), and just illustrate a couple of examples here. See also [May72, 11.6], [GZ67].

Example 7.1.32. Let us illustrate this theorem for the product of two 1-simplices. Let $X_\bullet = Y_\bullet = \Delta[1]_\bullet$. Then $X_\bullet \times Y_\bullet$ is the following simplicial set:

$X_0 \times Y_0 = \{(0,0),(0,1),(1,0),(1,1)\}$

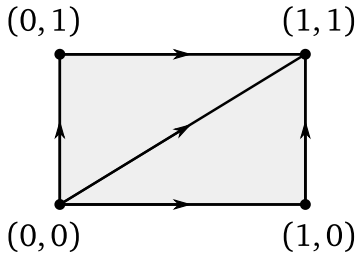
$X_1 \times Y_1 = \{(00,00),(00,01),(00,11),(01,00),(01,01),(01,11),(11,00),(11,01),(11,11)\}$

$X_2 \times Y_2 = \{\ \ldots, (001,011),(011,001), \ldots\}$

The face maps delete an element from the ith slot of every sequence, and the degeneracy maps duplicate the element in the ith slot of every sequence. All of the nondegenerate simplices are pictured. Focusing on the nondegenerate simplices, we can write them differently as sequences of ordered pairs:

$$X_0 \times Y_0 = \{(0,0),(0,1),(1,0),(1,1)\}$$
$$X_1 \times Y_1 = \{\ \ldots, (0,0)(0,1),(0,0)(1,0),(0,0)(1,1),(0,1)(1,1),(1,0)(1,1), \ldots\}$$
$$X_2 \times Y_2 = \{\ \ldots, (0,0)(0,1)(1,1),(0,0)(1,0)(1,1), \ldots\}$$

These describe the following Δ-complex structure on the square $\Delta^1 \times \Delta^1$. This verifies $|X_\bullet \times Y_\bullet| \cong |X_\bullet| \times |Y_\bullet|$ in one special case.



Example 7.1.33. More generally, let $X_\bullet = \Delta[k]_\bullet$ and $Y_\bullet = \Delta[\ell]_\bullet$ be two standard simplices. The product simplicial set at level n is $X_n \times Y_n$, the set of sequences of the form

$$(a_0, b_0),(a_1, b_1), \ldots (a_n, b_n),$$
$$0 \le a_0 \le a_1 \le \ldots \le a_n \le k,$$
$$0 \le b_0 \le b_1 \le \ldots \le b_n \le \ell.$$

The sequence is nondegenerate if the pairs are all distinct, in other words either a_i or b_i increases at each step. Given a sequence with this property, removing one of the terms gives another sequence with this property. In other words, for any nondegenerate point in $X_n \times Y_n$, every one of its faces is another nondegenerate point. It follows that the realization $|X_\bullet \times Y_\bullet|$ is a Δ-complex.

This Δ-complex has one n-simplex for every nondegenerate sequence described above. We map this n-simplex into the space $\Delta^k \times \Delta^\ell$ by sending the ith vertex to the product of vertices (a_i, b_i), and linearly interpolating for the rest. These maps induce a homeomorphism between the realization $|X_\bullet \times Y_\bullet|$ and the product $|X_\bullet| \times |Y_\bullet|$ (Exercise 7.7.7), verifying Theorem 7.1.31 in this special case.

7.1.6 Replacing topological spaces with simplicial sets

The "fundamental theorem of simplicial sets" is that simplicial sets are not just a convenient model for *some* topological spaces, but for *all* topological spaces.

Theorem 7.1.34. *For any topological space X, there is a weak homotopy equivalence*

$$|S_\bullet(X)| \xrightarrow{\sim} X.$$

The map is the canonical one that takes each n-simplex in the realization $|S_\bullet(X)|$, corresponding to a point in the set $S_n(X)$, and applies the corresponding map $\Delta^n \to X$. We omit the proof that this map is a weak homotopy equivalence. It is easy to show that it is surjective on homotopy groups, but to prove injectivity requires simplicial approximation, a technique that we don't develop in this book.

So topological spaces up to weak equivalence are the same thing as simplicial sets, up to maps that give weak equivalences on the realization. In other words, simplicial sets are a way to model topological spaces.

We can give this theorem a lot more structure by making **sSet** into a model category and proving that the result is Quillen equivalent to **Top**.

Definition 7.1.35. For any $n \geq 1$ and any $0 \leq i \leq n$, consider the Δ-complex Λ^n_i, which is the union of all of the faces of Δ^n except for the ith face. We call its associated simplicial set the ith **horn** $\Lambda_i[n]_\bullet$.

Concretely, $\Lambda_i[n]_k$ consists of all totally-ordered maps $[k] \to [n]$ whose image misses some point that is not i. So it's a map that's not surjective, even if we add i to its image.

Note that Λ^n_i is homeomorphic to a disc D^{n-1}, and the inclusion $\Lambda^n_i \subseteq \Delta^n$ is homeomorphic to the inclusion $D^{n-1} \times \{0\} \to D^{n-1} \times I$. Therefore, we can use the maps of simplicial sets

$$\Lambda_i[n]_\bullet \longrightarrow \Delta[n]_\bullet \tag{7.1.36}$$

as combinatorial analogs of $D^{n-1} \times \{0\} \to D^{n-1} \times I$.

Definition 7.1.37. A map of simplicial sets $X_\bullet \to Y_\bullet$ is a **cofibration** if each $X_n \to Y_n$ is injective.

It is a **weak equivalence** if the map of realizations $|X_\bullet| \to |Y_\bullet|$ is a weak homotopy equivalence.

It is a **(Kan) fibration** if each square of the following form has a lift, where the left-hand map is the inclusion from (7.1.36).

$$\begin{array}{ccc} \Lambda_i[n]_\bullet & \longrightarrow & X_\bullet \\ \downarrow & \nearrow & \downarrow \\ \Delta[n]_\bullet & \longrightarrow & Y_\bullet \end{array}$$

A **Kan complex** is a simplicial set $X_\bullet$ such that the map to the constant simplicial set on the one-point space $*$ is a Kan fibration. In other words, every horn $\Lambda_i[n]_\bullet \to X_\bullet$ extends to an n-simplex $\Delta[n]_\bullet \to X_\bullet$.

Theorem 7.1.38. *The geometric realization of any Kan fibration is a Serre fibration.*

The proof is given in [GJ99, I, Thm 10.10]. Note that this is not an if and only if: the unique map $\Lambda_1[2]_\bullet \to \Delta[0]_\bullet$ is not a Kan fibration, but its realization is homeomorphic to $I \to *$, which is a Serre fibration.

Theorem 7.1.39 (Kan model structure)**.** *The category of simplicial sets* **sSet** *has a cofibrantly-generated model structure with the above cofibrations, weak equivalences, and fibrations,*

- $I = \{\ \partial\Delta[n]_\bullet \to \Delta[n]_\bullet\ \}_{n\geq 0}$, *and*
- $J = \{\ \Lambda_i[n]_\bullet \to \Delta[n]_\bullet\ \}_{n\geq 1,\ 0\leq i\leq n}$.

Every object is cofibrant, and the fibrant objects are the Kan complexes. Furthermore, geometric realization is the left adjoint of a Quillen equivalence $\mathbf{sSet} \simeq \mathbf{Top}$.

The proof is given in [GJ99, I, Thm 11.3]. The reader is invited to contemplate what goes into proving this: the first four conditions of Proposition 5.2.6 are easy to check, and it remains to show that the I-fibrations are the same as the acyclic J-fibrations. This is the tricky part: it amounts to showing that we have all lifts of the form

$$\begin{array}{ccc} \partial\Delta_i[n]_\bullet & \longrightarrow & X_\bullet \\ \downarrow & \nearrow & \downarrow \\ \Delta[n]_\bullet & \longrightarrow & Y_\bullet \end{array}$$

iff $X_\bullet \to Y_\bullet$ is a Kan fibration and the map of realizations $|X_\bullet| \to |Y_\bullet|$ is a weak equivalence. Once this is done, it is straightforward to check that realization sends the maps in I and J to cofibrations and acyclic cofibrations of spaces, respectively.

Lemma 7.1.40. *The Kan model structure is proper (Definition 5.2.31).*

Proof. This follows from the same fact for the Quillen model structure on **Top**, and the fact that the geometric realization preserves both cofibrations and fibrations. □

In Exercises 7.7.15 and 7.7.16 we check that this is also a symmetric monoidal model category, under the Cartesian product $X_\bullet \times Y_\bullet$. The right adjoint is a simplicial set of maps $\mathrm{Map}_\bullet(Y, Z)$, whose n-simplices are maps of simplicial sets $\Delta[n]_\bullet \times Y_\bullet \to Z_\bullet$.

Theorem 7.1.39 plays a very important role in homotopy theory. It shows that if we are only interested in studying spaces up to weak equivalence, we never even have to think

about topological spaces. We could live our entire lives in the combinatorial world of simplicial sets. (One might worry that we defined weak equivalences of simplicial sets using the geometric realization functor to topological spaces. But it is possible to get around this, and define weak equivalences in an entirely combinatorial way. See e.g. [GJ99, I, Section 1.7].)

Many homotopy theorists take this point of view, and never think about topological spaces at all. When such a person says "a space," they always mean a simplicial set. Therefore a spectrum is a sequence of simplicial sets related by bonding maps; see Section 7.6 for more details.

Of course, if you are interested in anything geometric, like a manifold or a vector bundle, then you still need to know about topological spaces. However, even then, simplicial sets are a useful tool.

Example 7.1.41. Consider the diffeomorphism group $\mathrm{Diff}(M)$ of a smooth manifold M. It is possible to define this as a topological space, by defining a family of seminorms, which measure distance between two diffeomorphisms by the difference between the values of all of their higher partial derivatives. See e.g. [Hir94].

However, it is far easier to describe this as a simplicial set. Let $\mathrm{Diff}_\bullet(M)$ be the simplicial set whose k-simplices are maps

$$\Delta^k \times M \longrightarrow \Delta^k \times M$$

of the form $(t,x) \mapsto (t, f(t,x))$, where f is smooth and $f(t,-)$ is a diffeomorphism for all t. It is easy to check that this is a simplicial set, and it is a relatively well-known fact that its realization is weakly equivalent to the diffeomorphism group $\mathrm{Diff}(M)$.

In essence, the topology is sometimes easier to describe if you only need to know all *k-simplices* in the space, and not all the open sets.

7.2 Simplicial spaces

7.2.1 Definition and examples

Recall that $\mathbf{\Delta}$ is the category whose objects are $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$, and whose morphisms are the totally ordered maps. We defined a simplicial set to be a contravariant functor to sets, in other words a functor $X_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$. One way to generalize this definition is to allow topological spaces instead of sets.

Definition 7.2.1. A **simplicial space** is a contravariant diagram of topological spaces on $\mathbf{\Delta}$. In other words, a functor

$$X_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Top}.$$

As in Lemma 7.1.5, this is determined by a sequence of topological spaces $\{X_n\}_{n\geq 0}$ with face maps $d_i : X_n \to X_{n-1}$ and degeneracy maps $s_i : X_n \to X_{n+1}$. The only difference is that now X_n has a topology, and the face and degeneracy maps are continuous.

The **geometric realization** of $X_\bullet$ is defined as the quotient of the topological space $\coprod_{n\geq 0} \Delta^n \times X_n$ by the same relations as before, see (7.1.12) and (7.1.14). Note that since X_n has a topology, the space $\Delta^n \times X_n$ is not just a disjoint union of simplices, but a product space.

Example 7.2.2. Every simplicial set $X_\bullet$ is a simplicial space, by giving the sets X_n the discrete topology. Thinking of it as a simplicial set or as a simplicial space does not matter, the realization is the same.

Example 7.2.3. For any topological space A, the **constant simplicial space** at A has every level $X_n = A$, and every face and degeneracy the identity map of A. Its geometric realization is canonically isomorphic to A again.

Example 7.2.4. If $X_\bullet$ is a simplicial space and A is a topological space, we define the product $A \times X_\bullet$ to be the simplicial space whose nth level is $A \times X_n$. The face and degeneracy maps act only on X_n, leaving A alone. Since quotients commute with products, the realization of this space becomes $A \times |X_\bullet|$.

Example 7.2.5. A **bisimplicial set** is a functor $X : \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$, in other words, a $\mathbf{\Delta}^{\mathrm{op}}$-diagram of simplicial sets. If someone takes "space" to mean "simplicial set," then this is what they mean when they say "simplicial space."

If $X_{\bullet,\bullet}$ is a bisimplicial set, then by taking realization in one direction, we get a simplicial space $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Top}$. By taking realization again, we get a topological space $|X_{\bullet,\bullet}|$. One can check that it doesn't matter which order we took the realizations in: we get a product $\Delta^p \times \Delta^q$ for each level $X_{p,q}$ either way.

What is more surprising is that we could have taken the diagonal simplicial set $X_n := X_{n,n}$, and the realization of this diagonal is *homeomorphic* to the double realization $|X_{\bullet,\bullet}|$! This is essentially a consequence of Milnor's theorem (Theorem 7.1.31), see Exercise 7.7.10. As a result, the term "realization of a bisimplicial set" is completely unambiguous!

Example 7.2.6. Suppose we have a pushout diagram of topological spaces

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g} & & \\ C. & & \end{array}$$

We define a simplicial space that resembles the standard 1-simplex Δ^1, except that each point has been replaced by a copy of A, B, or C:

$$X_0 = B_0 \amalg C_1, \quad X_1 = B_{00} \amalg A_{01} \amalg C_{11}, \quad X_2 = B_{000} \amalg A_{001} \amalg A_{011} \amalg C_{111}, \quad \dots$$

The face maps delete one of the terms in the sequence of 0s and 1s, and apply f or g if necessary to go from A to B or C. The degeneracy maps duplicate one of the 0s or 1s.

It is straightforward to see this is a simplicial space. When computing its realization, as for $\Delta[1]$, the terms highlighted in red collapse away, and we get the three spaces

$$\Delta^0 \times B, \qquad \Delta^1 \times A, \qquad \Delta^0 \times C$$

glued along the faces of Δ^1. In other words, we get the double mapping cylinder

$$B \cup_A (A \times I) \cup_A C.$$

This is our first hint that simplicial spaces are a useful way to build homotopy colimits.

This last example feels somewhat nonrigorous – how do we compute these realizations and make sure that we get the correct answer? One way to do this is to use the fact that Theorems 8.2.8 and 7.1.31 apply equally well to simplicial spaces.

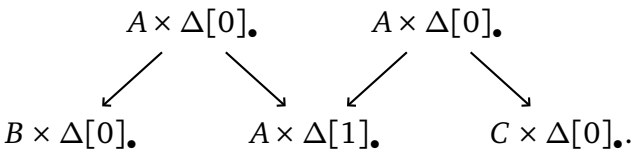
Theorem 7.2.7. *Geometric realization commutes with all colimits*

$$\left|\operatorname*{colim}_{i\in\mathbf{I}} X_\bullet(i)\right| \cong \operatorname*{colim}_{i\in\mathbf{I}} |X_\bullet(i)|,$$

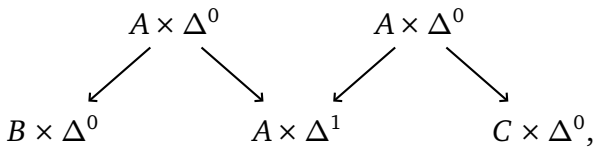
and all finite limits

$$\left|\lim_{i\in\mathbf{I}} X_\bullet(i)\right| \xrightarrow{\cong} \lim_{i\in\mathbf{I}} |X_\bullet(i)|.$$

Example 7.2.8. We can think of the simplicial space of Example 7.2.6 as the colimit of the following five simplicial spaces:



Since realization commutes with both pushouts and products, the realization becomes the colimit of the diagram



which is exactly the double mapping cylinder.

(Another way to make Example 7.2.6 rigorous is to use Proposition 8.2.31.)

7.2.2 Latching maps and Reedy cofibrancy

Intuitively, the realization $|X_\bullet|$ of a simplicial space $X_\bullet$ is built out of the nondegenerate points of X_n for each $n \geq 0$. This is more complicated now because X_n has a topology, and we have to keep track of how the nondegenerate points are connected to the degenerate points.

Recall from Definition 7.1.19 that a point $x \in X_n$ is **degenerate** if it is in the image of a degeneracy map $s_i : X_{n-1} \to X_n$.

Definition 7.2.9. The nth **latching space** $L_nX \subseteq X_n$ is the subspace of degenerate points. The inclusion of this subspace into X_n is called the nth **latching map**.

Example 7.2.10. In the simplicial circle $S^1_\bullet$, the latching spaces are

$$\begin{aligned} L_0S^1 = \varnothing \subseteq \{0\} = S^1_0, \\ L_1S^1 = \{00\} \subseteq \{00, 01\} = S^1_1, \\ L_2S^1 = \{000, 001, 011\} = \{000, 001, 011\} = S^1_2, \end{aligned}$$

and $L_nS^1 = S^1_n$ for all $n \geq 2$.

In general, the nth latching space is a union of n subspaces of X_n, each of which is homeomorphic to X_{n-1}. (They are the images of X_{n-1} under each of the degeneracy maps.) The two-fold intersections of these subspaces are homeomorphic to X_{n-2}, the three-fold intersections are homeomorphic to X_{n-3}, and so on. We strongly recommend that the reader try out Exercise 7.7.11 to build this geometric intuition for latching spaces.

If this is what the degenerate points look like, then at each level X_n, the remaining nondegenerate points are being attached to this subspace L_nX. As we might expect from Chapter 1, this is only well-behaved if X_n is obtained from the subspace L_nX by cell attachments, or at the very least if it has the homotopy extension property.

Definition 7.2.11. A simplicial space $X_\bullet$ is **Reedy cofibrant** if the latching maps $L_nX \to X_n$ are all cofibrations (have the homotopy extension property).

Example 7.2.12. If $X_\bullet$ is a simplicial set, then we can think of it as a discrete simplicial space, and it is always Reedy cofibrant.

Example 7.2.13. Let $\mathbb{N}^*$ be the one-point compactification of the natural numbers, as in Example 1.4.14, with basepoint the point at infinity. Let $X_\bullet$ be the simplicial space in which $X_0 = *$ and $X_n = \mathbb{N}^*$ for $n > 0$. The face and degeneracy maps are all identity maps of $\mathbb{N}^*$, save for the ones that involve X_0, which have to be the zero map that sends everything to the basepoint. The first latching map $L_1X \to X_1$ is the inclusion $* \to \mathbb{N}^*$, which does not have the homotopy extension property. So $X_\bullet$ is not Reedy cofibrant.

A map of simplicial spaces $X_\bullet \to Y_\bullet$ is a **weak equivalence** if each $X_n \to Y_n$ is a weak homotopy equivalence. This agrees with the notion of "level equivalence" from Definition 5.3.14 or "pointwise equivalence" from Definition 3.4.29, for diagrams on the category $\mathbf{\Delta}^{\mathrm{op}}$.

Proposition 7.2.14. *If $X_\bullet \to Y_\bullet$ is a weak equivalence of Reedy cofibrant simplicial spaces, it induces a weak equivalence of realizations $|X_\bullet| \to |Y_\bullet|$.*

The proof of this appears in the next chapter, as a special case of Theorem 8.2.27. Intuitively, you should believe this because at each simplicial level the new space X_n is related to the union of the old spaces by a cofibration, so we have control over what the homotopy type of $|X_\bullet|$ looks like.

Example 7.2.15. Suppose $X_\bullet$ is a Reedy cofibrant simplicial space, and each space X_n is a union of contractible components. Then taking π_0 at each level gives a simplicial set $Y_\bullet$ and a quotient map of simplicial sets $X_\bullet \to Y_\bullet$. By Proposition 7.2.14, the induced map on realizations $|X_\bullet| \to |Y_\bullet|$ is a weak equivalence.

Example 7.2.16. Let $X_\bullet$ be the non-Reedy cofibrant simplicial space from Example 7.2.13. Its realization is homeomorphic to $\Sigma\mathbb{N}^*$, which in turn is homeomorphic to the infinite shrinking wedge of circles in $\mathbb{R}^2$. On the other hand, if we let $\mathbb{N}_+$ be the natural numbers with a disjoint basepoint, then we have a weak equivalence $\mathbb{N}_+ \to \mathbb{N}^*$. We can build a second simplicial space $X'_\bullet$ in the same way we built $X_\bullet$, but with $\mathbb{N}_+$ instead of $\mathbb{N}^*$. We get a weak equivalence of simplicial spaces $X'_\bullet \to X_\bullet$, that on realizations is the map from a countable wedge of circles to the infinite shrinking wedge of circles in $\mathbb{R}^2$. Although this map is a bijection, it fails to be a weak equivalence, since π_1 of the infinite shrinking wedge is much larger than the free product of the summands $\pi_1(S^1) \cong \mathbb{Z}$.

This counterexample demonstrates that the Reedy cofibrancy assumption in Proposition 7.2.14 cannot be dropped.

As the notation suggests, there is a Reedy model structure on **sTop** whose cofibrant objects have the above description, except that we use Quillen cofibrations instead of the homotopy extension property. Furthermore the realization functor **sTop** $\to$ **Top** is left Quillen. As a result, it is always possible to replace a simplicial space by a weakly equivalent one that is Reedy cofibrant (in the stronger, Quillen sense). We won't need this for a while, so we postpone its development to Exercise 8.7.37. (We could also use the projective model structure to accomplish this purpose. See Exercise 7.7.5.)

Remark 7.2.17. At this point in the book, it not important to develop a model structure for **sTop**, because we don't need to use simplicial spaces as a substitute for topological spaces. We only use simplicial spaces as a tool for building new topological spaces, such as the bar construction of the next section.

7.3 The bar construction for spaces with G-action

Our primary motivation for introducing simplicial objects is to define the bar construction. As we mentioned the introduction of this chapter, this is one of the most pervasive and useful constructions in homotopy theory.

7.3.1 Definition

We begin with the simplest case. Recall from Definition 5.3.8 that a **topological monoid** is a space G with a multiplication map $G \times G \to G$ that is continuous, associative, and has unit $e \in G$. A **topological group** is a topological monoid with inverses, such that the map $g \mapsto g^{-1}$ is continuous.

A **right action** of G on a space X is a map $X \times G \to X$ that is associative and unital in the sense that

$$x \cdot e = x, \qquad (xg_1)g_2 = x(g_1g_2).$$

A **left action** of G on Y is a map $G \times Y \to Y$ such that

$$e \cdot y = y, \qquad g_2(g_1y) = (g_2g_1)y.$$

The difference between a left action and a right action lies in how the elements compose: if we apply g_1 first, then g_2, we get either g_1g_2 or g_2g_1, depending on whether we have a right action or a left action.

The bar construction will be a thickened or derived version of the following construction.

Definition 7.3.1. Suppose G is a topological monoid, X has a right G-action, and Y has a left G-action. The **balanced product** $X \times_G Y$ is the quotient of $X \times Y$ by the equivalence relation

$$(xg, y) \sim (x, gy)$$

for all $x \in X$, $y \in Y$, and $g \in G$. In other words, it is the coequalizer of the two action maps of G,

$$X \times G \times Y \rightrightarrows X \times Y \longrightarrow X \times_G Y. \tag{7.3.2}$$

The two maps take (x, g, y) to either (xg, y) or (x, gy).

Example 7.3.3. The balanced product $X \times_G *$ is just the orbit space X_G, in other words X modulo the relation $x \sim xg$ for all $x \in X$ and $g \in G$. Similarly the product $* \times_G Y$ is the orbit space $Y_G = Y/(y \sim gy)$.

Example 7.3.4. If G is a group and $H \leq G$, the balanced product $G \times_H * = G/H$ is the left cosets of H in G. The balanced product $* \times_H G = H \setminus G$ is the right cosets of H in G.

Example 7.3.5. If G is a group, $H \leq G$, and Y is any space with left H-action, the balanced product $G \times_H Y$ is a space with left G-action as follows:

$$g_1(g_2, y) := (g_1 g_2, y).$$

This is the universal way of extending the H-action on Y to a G-action. In other words, there is an H-equivariant map $Y \to G \times_H Y$ by $y \mapsto (e, y)$, and this is inital among all H-equivariant maps from Y to a space with G-action.

Put yet another way, the functor $G \times_H (-)$ is the left adjoint to the forgetful functor from G-spaces to H-spaces (the one that forgets the action of elements of G that aren't in H).

Example 7.3.6. As a special case of the previous example, we have an isomorphism

$$G \times_G Y \cong Y$$

as spaces with left G-action. (In other words the map is **equivariant**; it commutes with the action of each $g \in G$.) Similarly $X \times_G G$ is isomorphic to X as spaces with right G-action.

Definition 7.3.7. Suppose G is a topological monoid, X is a right G-space, and Y is a left G-space. The **two-sided nerve** $N_\bullet(X, G, Y)$ is a simplicial space whose nth level is the product

$$N_n(X, G, Y) = X \times G^n \times Y.$$

The face maps d_i multiply adjacent terms together, while the degeneracy maps s_i insert copies of the identity element $e \in G$: (underlines added for emphasis)

$$\begin{aligned}
d_0(x, g_1, \ldots, g_n, y) &= (\underline{x g_1}, g_2, \ldots, g_n, y), \\
d_i(x, g_1, \ldots, g_n, y) &= (x, g_1, \ldots, \underline{g_i g_{i+1}}, \ldots, g_n, y) \text{ when } 0 < i < n, \\
d_n(x, g_1, \ldots, g_n, y) &= (x, g_1, \ldots, g_{n-1}, \underline{g_n y}), \text{ and} \\
s_i(x, g_1, \ldots, g_n, y) &= (x, g_1, \ldots, g_i, \underline{e}, g_{i+1}, \ldots, g_n, y) \text{ when } 0 \leq i \leq n.
\end{aligned}$$

The **two-sided bar construction**, or just the **bar construction**, is the topological space we obtain by taking the realization:

$$B(X, G, Y) = |N_\bullet(X, G, Y)|.$$

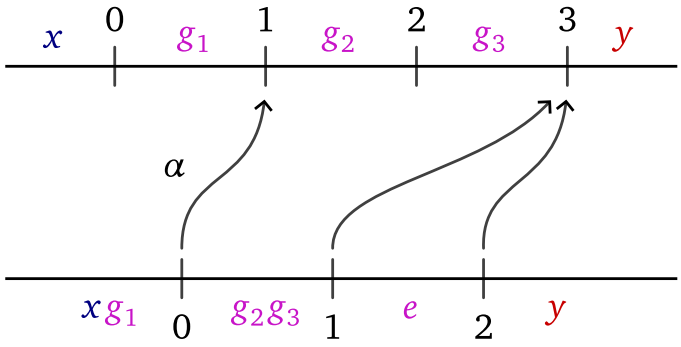
As a special case, when X and Y are the one-point space $*$ we just call this BG:

$$BG = B(*, G, *) = |N_\bullet(*, G, *)|.$$

Now, in order to verify that $N_\bullet(X, G, Y)$ forms a simplicial space, the identities from Lemma 7.1.5 can be checked directly, but it is much more fun to prove this with a picture.

Draw intervals between the points of $[n]$, and label the intervals by elements of G. The interval to the left of 0 is labeled by x, and to the right of n is labeled by y.

Any totally ordered map $\alpha\colon [m] \to [n]$ can be drawn from bottom-to-top as shown. The gaps between the arrows of α give channels along which each interval along the top is associated to a unique interval along the bottom. We move the symbols $x, g_1, \ldots, g_n, y$ down along these channels, and multiply them together if they end in the same interval, as shown.



In the special case of a face map d_i or a degeneracy map s_i, this agrees with the formula in Definition 7.3.7. Stacking two of these figures on top of each other gives a "picture proof" that the rule respects composition. Therefore $N_\bullet(X, G, Y)$ is a simplicial space.

Example 7.3.8. If we take G to be $\mathbb{Z}$, the abelian group of the integers under addition, then $B\mathbb{Z} \simeq S^1$. In fact if we take $G = \mathbb{N}$ then we also get $B\mathbb{N} \simeq S^1$. The inclusion of the 1-simplex corresponding to $1 \in \mathbb{N}$ gives the equivalence $S^1 \simeq B\mathbb{N}$, and the homomorphism $\mathbb{N} \to \mathbb{Z}$ induces the other equivalence $B\mathbb{N} \simeq B\mathbb{Z}$.

In the case of $B\mathbb{Z}$, this follows since $B\mathbb{Z}$ is the quotient of $E\mathbb{Z} = B(*, \mathbb{Z}, \mathbb{Z})$ by a free $\mathbb{Z}$-action. The bar lemma below (Lemma 7.3.11) shows that $E\mathbb{Z}$ is contractible, and therefore $B\mathbb{Z}$ must be equivalent to the circle. The same proof works for $B\mathbb{N}$ as well, except that we write it as the quotient of the classifying space of the category $\mathbf{I}$ that has one object for every $n \in \mathbb{Z}$ and a unique morphism $m \to n$ whenever $m \leq n$. See Lemma 9.1.24.

Remark 7.3.9. Informally, the bar construction is a thickened version of the balanced product. To see this, observe that for every $g \in G$, the 1-cell corresponding to (x, g, y) connects the two 0-cells corresponding to (xg, y) and (x, gy). So these two points are connected by a path, instead of being literally equal to each other.

It's also helpful to draw out all the levels of the simplicial space $N_\bullet(X, G, Y)$ and the maps

between them, giving a very natural-looking extension of the diagram (7.3.2):

$$\cdots \;\; \overleftarrow{\overleftarrow{\overleftarrow{\overleftarrow{\rightarrow}}}} \; X \times G \times G \times Y \;\; \overleftarrow{\overleftarrow{\overleftarrow{\rightarrow}}} \; X \times G \times Y \;\; \rightleftarrows \;\; X \times Y \longrightarrow B(X,G,Y)$$

We'll make the above observation more precise by saying that the balanced product is a functor $\mathbf{Top}^G \times {}^G\mathbf{Top} \to \mathbf{Top}$, and the bar construction is the left-derived functor of this product, at least when G is nice enough that the map $\{e\} \to G$ has the homotopy extension property. See Theorem 7.3.14.

7.3.2 Homotopy invariance and the bar lemma

Our first theorem about the bar construction is the following. Recall that a space is **well-based** if the inclusion of its basepoint is a Hurewicz cofibration, i.e. has the homotopy extension property. For a topological monoid G, we consider this basepoint to be the unit element $e \in G$. In other words, G is well-based if $\{e\} \to G$ is a cofibration.

Theorem 7.3.10 (Homotopy invariance of the bar construction)**.** *If G is well-based then any equivalences of G-spaces $X \to X'$, $Y \to Y'$ induce an equivalence of bar constructions*

$$B(X,G,Y) \xrightarrow{\sim} B(X',G,Y').$$

Proof. The nth latching map of $N_\bullet(X,G,Y)$ is the subspace of $X \times G^{\times n} \times Y$ consisting of all points where at least one of the G-coordinates is $e \in G$. This is obtained as an n-fold pushout product

$$X \times (\{e\} \to G)^{\square n} \times Y,$$

or as the $(n+2)$-fold pushout-product

$$(\varnothing \to X) \square (\{e\} \to G)^{\square n} \square (\varnothing \to Y).$$

If $\{e\} \to G$ is a cofibration, then by Lemma 1.2.7, this pushout-product is a cofibration as well. Therefore the simplicial space $N_\bullet(X,G,Y)$ is Reedy cofibrant.

The equivalences $X \to X'$ and $Y \to Y'$ induce a map of simplicial spaces that at each level is an equivalence

$$X \times G^n \times Y \to X' \times G^n \times Y'.$$

Since both simplicial spaces are Reedy cofibrant, by Proposition 7.2.14 this gives a weak equivalence on the realizations. □

More generally, the same argument works if we change G, X, and Y at the same time, so long as the actions are all compatible so that we actually get a map of bar constructions.

If all three change by a weak equivalence, then the map of bar constructions is a weak equivalence

$$B(X,G,Y) \xrightarrow{\sim} B(X',G',Y').$$

See Exercise 7.7.26.

The next result allows us to "contract" the bar construction when an argument is repeated. It is analogous to the fact in algebra that $M \otimes_R R \cong M$, when R is a ring and M is a right R-module.

Lemma 7.3.11 (Bar Lemma)**.** *For any group or monoid G and any space X with right G-action, there is a homotopy equivalence*

$$m_X : B(X,G,G) \xrightarrow{\sim} X.$$

This homotopy equivalence respects the G-action, but its inverse does not. We also get a similar homotopy equivalence $B(G,G,Y) \xrightarrow{\sim} Y$.

Proof. The map $m\colon B(X,G,G) \to X$ is defined as the realization of the map of simplicial spaces from $N_\bullet(X,G,G)$ to the constant simplicial space at X, with formula

$$\begin{aligned} X \times G^{n+1} &\to X \\ (x, g_1, \dots, g_n, g) &\mapsto x g_1 \cdots g_n g. \end{aligned}$$

It is easy to check this commutes with faces and degeneracies, so it defines a map of simplicial spaces. It respects the right G-action because

$$(x, g_1, \dots, g_n, gh) \mapsto x g_1 \cdots g_n g h = (x g_1 \cdots g_n g)h.$$

The inverse map $i\colon X \to B(X,G,G)$ is defined by

$$x \mapsto (x, e, \dots, e, e).$$

The composite $m \circ i$ is the identity. The composite $i \circ m$ comes from the map of simplicial spaces

$$\begin{aligned} X \times G^{n+1} &\to X \times G^{n+1} \\ (x, g_1, \dots, g_n, g) &\mapsto (x g_1 \cdots g_n g, e, \dots, e, e). \end{aligned}$$

We define a map $\Delta[1]_\bullet \times N_\bullet(X,G,G) \to N_\bullet(X,G,G)$ that at one end of $\Delta[1]$ is $i \circ m$, and at the other end is the identity of $N_\bullet(X,G,G)$. At level n, this homotopy takes sends each $\alpha\colon [n] \to [1]$ to the map

$$\begin{aligned} X \times G^{n+1} &\to X \times G^{n+1} \\ (x, g_1, \dots, g_n, g) &\mapsto (x, g_1, \cdots, g_i, g_{i+1} \cdots g_n g, e, \dots, e, e), \end{aligned}$$

where i is the value such that $\alpha(i) = 0$ but $\alpha(i+1) = 1$. When $i = n$ this gives the identity map, and when $i = 0$ it gives the map that squashes all the terms to the left. It is straightforward to check that this rule $\Delta[1]_n \times N_n(X,G,G) \to N_n(X,G,G)$ defines a map of simplicial sets.

Taking realizations and using Milnor's theorem Theorem 7.1.31, this gives a homotopy

$$\Delta^1 \times B(X,G,G) \to B(X,G,G)$$

from the composite $i \circ m$ to the identity, showing that m and i give a homotopy equivalence. □

Example 7.3.12. If G is a well-based topological group, this imples that the space $EG = B(*,G,G)$ is contractible. Using Proposition 7.3.17 we will see that this is the universal space with free G-action, and that its quotient $BG = EG/G \cong B(*,G,*)$ is the classifying space of G.

7.3.3 The bar construction is a left-derived functor

In this section we relate the bar construction $B(X,G,Y)$ back to the balanced product $X \times_G Y$. As a warm-up, suppose that in addition to having a left G-action, Y has a right action by another topological monoid H, and furthermore this right H-action commutes with the left G-action. We summarize this by saying that Y is a (G,H)-space. In addition, suppose that Z is another left H-space.

Lemma 7.3.13. *In this setting we get a homeomorphism*

$$B(X,G,Y) \times_H Z \cong B(X,G,Y \times_H Z).$$

Proof. Since realization commutes with colimits (Proposition 8.2.8), both of these are realizations of the same simplicial space that at level n is $X \times G^n \times Y \times_H Z$. □

As a special case, the bar construction commutes with orbits,

$$B(X,G,Y)_H \cong B(X,G,Y_H).$$

We also get the more general identity

$$B(X,G,G) \times_G Y \cong B(X,G,Y) \cong X \times_G B(G,G,Y).$$

Theorem 7.3.14. *Assume G is well-based. Then the bar construction is the left-derived functor of the balanced product:*

$$B(X,G,Y) \cong (Q_1X) \times_G (Q_2Y), \qquad Q_1X \xrightarrow{\sim} X, \qquad Q_2Y \xrightarrow{\sim} Y.$$

In fact, it can be constructed by only deforming X,

$$B(X,G,Y) \cong (QX) \times_G Y, \qquad QX \xrightarrow{\sim} X,$$

or by only deforming Y.

Therefore the bar construction $B(X,G,Y)$ is the left-derived functor of $X \times_G Y$, even if we hold Y constant and only consider it as a functor of X! It's also the left-derived functor if we hold X constant and only consider it as a functor of Y. We only have to replace one of the inputs to get the correct homotopy type.

Proof. This proof is inspired by [Shu06, Thm 13.7]. Consider the collection of all right G-spaces of the form $B(X,G,G)$. The maps between these are G-equivariant maps $f : B(X,G,G) \to B(X',G,G)$, not necessarily arising from a map $X \to X'$. However, if such a map f is a weak equivalence, then by the bar lemma (Lemma 7.3.11), it fits into a commuting square of G-equivariant weak equivalences

$$\begin{array}{ccc} B(X,G,G) & \xrightarrow{f} & B(X',G,G) \\ {\scriptstyle m_{B(X,G,G)}}\uparrow & & \uparrow{\scriptstyle m_{B(X',G,G)}} \\ B(B(X,G,G),G,G) & \xrightarrow{B(f,\mathrm{id},\mathrm{id})} & B(B(X',G,G),G,G). \end{array}$$

Applying $- \times_G Y$ to the terms of this square and using Lemma 7.3.13, we get a commuting square

$$\begin{array}{ccc} B(X,G,Y) & \xrightarrow{f \times_G \mathrm{id}} & B(X',G,Y) \\ {\scriptstyle m_{B(X,G,G)} \times_G \mathrm{id}}\uparrow & & \uparrow{\scriptstyle m_{B(X,G,G)} \times_G \mathrm{id}} \\ B(B(X,G,G),G,Y) & \xrightarrow{B(f,\mathrm{id},\mathrm{id})} & B(B(X',G,G),G,Y). \end{array}$$

The bottom horizontal map is an equivalence by Theorem 7.3.10. To show that the vertical maps are equivalences, we observe that $B(B(X,G,G),G,Y)$ is the realization of a bisimplicial space, and that the two realizations can be taken in either order, as in Example 7.2.5. Taking the realizations in the opposite order gives the maps

$$\begin{array}{ccc} B(X,G,Y) & & B(X',G,Y) \\ {\scriptstyle B(\mathrm{id},\mathrm{id},m_Y)}\uparrow & & \uparrow{\scriptstyle B(\mathrm{id},\mathrm{id},m_Y)} \\ B(X,G,B(G,G,Y)) & & B(X',G,B(G,G,Y)). \end{array}$$

These maps are equivalences by Theorem 7.3.10. By 2-out-of-3, we conclude that the top horizontal map $f \times_G \mathrm{id}$ is a weak equivalence as well.

This proves that $- \times_G -$ preserves equivalences on pairs of spaces where the first space is of the form $B(X, G, G)$. By the bar lemma again, there is a left deformation of the identity to a functor Q taking X to $B(X, G, G)$. We are now done because

$$(QX) \times_G Y := B(X, G, G) \times_G Y \cong B(X, G, Y).$$

The case where we only deform Y is done in a symmetric way. □

Example 7.3.15. If G is well-based and Y has a left G-action, the **homotopy orbit space** is defined to be

$$Y_{hG} = EG \times_G Y = B(*, G, G) \times_G Y \cong B(*, G, Y),$$

see the discussion after Lemma 7.3.13. Similarly, if X has a right G-action then X_{hG} is defined to be the bar construction $B(X, G, *)$. By Theorem 7.3.10, weak equivalences of spaces with G-action $X \to X'$ induce weak equivalences on the homotopy orbits,

$$X_{hG} \xrightarrow{\sim} X'_{hG}.$$

By Theorem 7.3.14, this definition of homotopy orbits is the left-derived functor of the strict orbits Y_G or X_G. So this construction of the homotopy orbits is equivalent (not homeomorphic!) to the one that we described in Definition 5.3.26. (You may have already proven this in Exercise 5.8.22.)

7.3.4 Principal bundles and classifying spaces

If G is a well-based topological group, let us define

$$EG = B(*, G, G), \qquad BG = B(*, G, *).$$

Note that EG has a free right G-action, and by Lemma 7.3.13, $EG/G \cong BG$. In this section we show that $EG \to BG$ is a principal G-bundle whose total space is contractible, and that BG is the classifying space of G. Let's first recall what this means.

Definition 7.3.16. Let G be a topological group. A **principal G-bundle** is a space E with free right G-action $E \times G \to E$, and a map $p : E \to B$ such that near any point in B, there is an open set U and a homeomorphism $p^{-1}(U) \cong U \times G$ along which the map p is the projection to U and the action of G is by $(u, h)g = (u, hg)$. In other words, $E \to B$ is locally homeomorphic to the product projection $U \times G \to U$.

A **classifying space** for G is defined to be any space of the form E/G, where E is weakly contractible (i.e. all homotopy groups vanish) and $E \to E/G$ is a principal G-bundle.

Proposition 7.3.17. *Suppose G is a topological group (not just a monoid) and G is well-based. Then $B(X, G, G) \to B(X, G, *)$ is a principal G-bundle.*

We omit the proof of Proposition 7.3.17, referring the reader to [May75, Thm 8.2]. To sketch the main idea, observe that each point in the realization $B(X,G,*)$ must lie in the interior of a k-simplex $\Delta^k \subseteq B(X,G,*)$, corresponding to the tuple $(x, g_1, \ldots, g_k, *)$. The set of all such points for a fixed value of k can be described as

$$(\Delta^k - \partial\Delta^k) \times X \times G^k \subseteq B(X,G,*).$$

On this set, the preimage in $B(X,G,G)$ is exactly

$$(\Delta^k - \partial\Delta^k) \times X \times G^{k+1} \cong ((\Delta^k - \partial\Delta^k) \times X \times G^k) \times G,$$

commuting with the right G-action on the last copy of G. So on this set, the map is indeed a projection of the form $U \times G \to U$. The remainder of the argument is a technical exercise in patching these local models together using the homotopy extension property.

Since $B(*,G,G)$ is contractible by the bar lemma (Lemma 7.3.11) and its projection to $B(*,G,*)$ is a principal G-bundle by Proposition 7.3.17, we conclude

Corollary 7.3.18. *For any well-based topological group G, the space $BG = B(*,G,*)$ is a classifying space for G.*

Since classifying spaces are unique up to weak homotopy equivalence (Exercise 7.7.19), the space $B(*,G,*)$ is weakly equivalent to any other classifying space one might come up with. So we often just say that BG is *the* classifying space of G.

Example 7.3.19. We can form a classifying space for $\mathbb{Z}$ by taking the real line with $\mathbb{Z}$ acting by translation, $n \cdot (x) = n + x$. The quotient of $\mathbb{R}$ by this action gives the circle S^1. Therefore S^1 is a classifying space for $\mathbb{Z}$. The bar construction $B\mathbb{Z} = B(*,\mathbb{Z},*)$ gives another classifying space. These must be homotopy equivalent,

$$S^1 \simeq B\mathbb{Z},$$

even though they are not homeomorphic.

Example 7.3.20. We can form a classifying space for $\mathbb{Z}/2$ by taking its action on the infinite sphere S^∞ by the antipodal map. The quotient is $\mathbb{RP}^\infty$, and therefore $B\mathbb{Z}/2 = B(*,\mathbb{Z}/2,*)$ is homotopy equivalent to $\mathbb{RP}^\infty$. In fact, they have the same cells, and the attaching maps have the same degree.

Example 7.3.21. We can form a classifying space for the orthogonal group $O(n)$ by taking the Stiefel manifold $V_n^O(\mathbb{R}^\infty)$ of all n-tuples of orthonormal vectors in $\mathbb{R}^\infty$, topologized as a subspace of $(\mathbb{R}^\infty)^{\times n}$. This is weakly contractible, and the orbit space is the Grassmannian $G_n(\mathbb{R}^\infty)$ of all n-dimensional subspaces of $\mathbb{R}^\infty$. Therefore $BO(n) = B(*,O(n),*)$ is homotopy equivalent to $G_n(\mathbb{R}^\infty)$.

Example 7.3.22. We can form a classifying space for the general linear group $GL_n(\mathbb{R})$ by taking the Stiefel manifold $V_n(\mathbb{R}^\infty)$ of all n-tuples of linearly independent vectors in $\mathbb{R}^\infty$. Again this is weakly contractible, and the orbit space is the Grassmannian $G_n(\mathbb{R}^\infty)$ of all n-dimensional subspaces of $\mathbb{R}^\infty$. This is exactly the same space we obtained above, so $BO(n) \simeq BGL_n(\mathbb{R})$.

We could also obtain this result from Theorem 7.3.10. Any weak equivalence of well-based topological monoids $G \to G'$ gives a weak equivalence of classifying spaces $BG \to BG'$. For example, the inclusion $O(n) \to GL_n(\mathbb{R})$ is a weak equivalence, because the Gram-Schmidt orthogonalization process provides a homotopy inverse. Therefore this inclusion induces a weak equivalence

$$BO(n) \xrightarrow{\sim} BGL_n(\mathbb{R}).$$

Example 7.3.23. For two monoids G and H we have

$$B(G \times H) \cong BG \times BH,$$

because the product of the simplicial spaces on the right gives the simplicial space on the left. So for instance $B(\mathbb{Z}/2 \times \mathbb{Z}/2) \simeq \mathbb{RP}^\infty \times \mathbb{RP}^\infty$.

7.4 The bar construction for diagrams on a category

The bar construction generalizes easily from groups to categories. This will allow us to give a new definition of homotopy colimits!

7.4.1 The coend and the bar construction

Suppose $\mathbf{I}$ is a small category, in which the spaces $\mathbf{I}(a,b)$ are allowed to have a topology, as in Definition 5.3.1. If there is no topology to put on the sets $\mathbf{I}(a,b)$, we say that $\mathbf{I}$ is discrete and give each set $\mathbf{I}(a,b)$ the discrete topology.

Suppose X is a topological diagram $\mathbf{I} \to \mathbf{Top}$, and Y is a contravariant topological diagram $\mathbf{I}^{\mathrm{op}} \to \mathbf{Top}$. In other words, the action maps

$$X(a) \times \mathbf{I}(a,b) \to X(b) \qquad \mathbf{I}(a,b) \times Y(b) \to Y(a)$$

are continuous.[1] The categorical bar construction will be a thickened or derived version of the following construction.

[1] The reader who prefers to write composition as $\mathbf{I}(a,b) \times X(a) \to X(b)$ is welcome to do so. This is a matter of conventions and does not affect any of the mathematics.

Definition 7.4.1. The **coend** of X and Y is the topological space

$$X \times_{\mathbf{I}} Y = \left(\coprod_{c \in \mathbf{I}} X(c) \times Y(c) \right) \Big/ (f(x), y) \sim (x, f(y))$$

where the relation is taken over all $f : c \to d$, $x \in X(c)$, and $y \in Y(d)$. In other words, it is the coequalizer of the two action maps of $\mathbf{I}$,

$$\coprod_{c_0, c_1 \in \mathrm{ob}\,\mathbf{I}} X(c_0) \times \mathbf{I}(c_0, c_1) \times Y(c_1) \rightrightarrows \coprod_{c \in \mathrm{ob}\,\mathbf{I}} X(c) \times Y(c) \longrightarrow X \times_{\mathbf{I}} Y.$$

The two maps take (x, f, y) to either $(f(x), y)$ or $(x, f^*(y))$.

Essentially, X and Y are glued along the category $\mathbf{I}$ to create a single space, in much the same way that we form the balanced product $X \times_G Y$ in Definition 7.3.1.

Example 7.4.2. If $\mathbf{I}$ has one object and morphism set G, then X is a right G-space, Y is a left G-space, and the coend $X \times_{\mathbf{I}} Y$ is just the balanced product $X \times_G Y$.

Example 7.4.3. If $\mathbf{I} = \mathbf{\Delta}$ is the simplex category, we can take the coend of the diagram of simplices $\Delta^{(-)} \colon \mathbf{\Delta} \to \mathbf{Top}$ and any simplicial space $X_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Top}$. This gives the geometric realization of $X_\bullet$,

$$|X_\bullet| \cong \Delta^{(-)} \times_{\mathbf{\Delta}} X_\bullet.$$

Example 7.4.4. For fixed $c \in \mathbf{I}$, the sets $\mathbf{I}(-, c)$ form a diagram on $\mathbf{I}^{\mathrm{op}}$. The coend $X \times_{\mathbf{I}} \mathbf{I}(-, c)$ is isomorphic to $X(c)$ (Exercise 7.7.21). Similarly, the coend $\mathbf{I}(c, -) \times_{\mathbf{I}} Y$ is isomorphic to $Y(c)$. This is the generalization of Example 7.3.6 to categories.

Example 7.4.5. In Example 7.3.5 considered a subgroup $H \leq G$ and showed how the balanced product $G \times_H X$ turns every H-space into a G-space in a universal way. When we generalize to categories, we consider any functor $\alpha \colon \mathbf{I} \to \mathbf{J}$ and turn every $\mathbf{I}$-diagram into a $\mathbf{J}$-diagram in a universal way. The idea is to form a diagram called

$$\alpha_! X = X \times_{\mathbf{I}} \mathbf{J}.$$

This is the **left Kan extension** or **prolongation** of X from $\mathbf{I}$ to $\mathbf{J}$. You might have already seen this in Section 5.4.3 and Exercise 5.8.23, but it doesn't hurt to describe it again. It is a diagram on $\mathbf{J}$, and its value at each object $j \in \mathbf{J}$ is given by the coend of X and the diagram $\mathbf{J}(\alpha(-), j)$. Notice that X is an $\mathbf{I}$-diagram, while $\mathbf{J}(\alpha(-), j)$ is an $\mathbf{I}^{\mathrm{op}}$-diagram, so we can indeed take their coend and get a space for each $j \in \mathbf{J}$.

If $\mathbf{J}$ has no topology, then for each $j \in \mathbf{J}$, the result simplifies to a single colimit,

$$(X \times_{\mathbf{I}} \mathbf{J})(j) = X \times_{\mathbf{I}} \mathbf{J}(\alpha(-), j) \cong \operatorname*{colim}_{\alpha(i) \to j} X(j),$$

where the colimit is taken over the **comma category** $(\alpha \downarrow d)$ from Exercise 7.7.33.

Example 7.4.6. As a special case of the preceding example, the coend of X and a constant diagram $*$ is exactly the colimit of X:

$$\operatorname*{colim}_{i\in\mathbf{I}} X(i) = \operatorname*{colim}_{\mathbf{I}} X \cong X \times_{\mathbf{I}} *.$$

Now that we have explored the coend a bit, we are ready to thicken it up to the bar construction.

Definition 7.4.7. The categorical version of the **two-sided nerve** is the simplicial space whose nth level is the disjoint union of products

$$N_n(X,\mathbf{I},Y) = \coprod_{c_0,c_1,\cdots,c_n\in \operatorname{ob}\mathbf{I}} X(c_0)\times \mathbf{I}(c_0,c_1)\times \mathbf{I}(c_1,c_2)\times\cdots\times \mathbf{I}(c_{n-1},c_n)\times Y(c_n).$$

When $\mathbf{I}$ is discrete, the categorical nerve is equivalently described as

$$N_n(X,\mathbf{I},Y) = \coprod_{c_0\to c_1\to\cdots\to c_n} X(c_0)\times Y(c_n).$$

The **categorical bar construction** $B(X,\mathbf{I},Y)$ is the realization of this,

$$B(X,\mathbf{I},Y) = |N_\bullet(X,\mathbf{I},Y)|.$$

The maps are the same as in Definition 7.3.7, except that we use the composition in the category $\mathbf{I}$, and its action on the diagrams X and Y: (underlines added for emphasis)

$$\begin{aligned}
d_0(x,f_1,\dots,f_n,y) &= (\underline{f_1(x)},f_2,\dots,f_n,y),\\
d_i(x,f_1,\dots,f_n,y) &= (x,f_1,\dots,\underline{f_{i+1}\circ f_i},\dots,f_n,y) \text{ when } 0<i<n,\\
d_n(x,f_1,\dots,f_n,y) &= (x,f_1,\dots,f_{n-1},\underline{f_n^*(y)}), \text{ and}\\
s_i(x,f_1,\dots,f_n,y) &= (x,f_1,\dots,f_i,\underline{\mathrm{id}_{c_i}},f_{i+1},\dots,f_n,y) \text{ when } 0\le i\le n.
\end{aligned}$$

In particular, at level 0 it is the disjoint union of the spaces

$$N_0(X,\mathbf{I},Y) = \coprod_{c_0\in\operatorname{ob}\mathbf{I}} X(c_0)\times Y(c_0),$$

and at level 1 it is the disjoint union

$$N_1(X,\mathbf{I},Y) = \coprod_{c_0,c_1\in\operatorname{ob}\mathbf{I}} X(c_0)\times \mathbf{I}(c_0,c_1)\times Y(c_1).$$

The face maps between these levels imitate the definition of the coend from Definition 7.4.1. We then get a larger simplicial object that continues this pattern:

$$\cdots \quad \Longdownarrow \quad \coprod_{c_0,c_1,c_2} X(c_0)\times \mathbf{I}(c_0,c_1)\times \mathbf{I}(c_1,c_2)\times Y(c_2) \quad \Longdownarrow \quad \coprod_{c_0,c_1} X(c_0)\times \mathbf{I}(c_0,c_1)\times Y(c_1) \quad \Longdownarrow \quad \coprod_{c_0} X(c_0)\times Y(c_0) \quad \downarrow \quad B(X,\mathbf{I},Y)$$

Example 7.4.8. If $\mathbf{I}$ has one object and morphism space G, the two diagrams become a right G-space and a left G-space, and this becomes the earlier bar construction in Definition 7.3.7.

Example 7.4.9. In the special case that X and Y are constant diagrams on the one-point space $*$, this becomes the definition of the nerve of a category from Example 7.1.10:

$$N_n\mathbf{I} = N_n(*,\mathbf{I},*) = \left\{ \; c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} c_n. \; \right\}.$$

We define the **classifying space** of the category $\mathbf{I}$ to be the realization:[2]

$$B\mathbf{I} = B(*,\mathbf{I},*) = |N_\bullet \mathbf{I}|.$$

By Exercise 7.7.12, any functor gives a map of classifying spaces, and any natural transformation gives a homotopy.

7.4.2 A useful simplification and some examples

The bar construction for a category $B(X,\mathbf{I},Y)$ looks more complicated than the bar construction for a group $B(X,G,Y)$. But in many cases, it's actually simpler:

[2] Strictly speaking, this is only the classifying space if the spaces $\mathbf{I}(c,c)$ are well-based at the basepoint id_c. See Definition 7.4.16.

Example 7.4.10. Suppose $\mathbf{I}$ is discrete, and that the non-identity morphisms of $\mathbf{I}$ are closed under composition. (For instance, this is true when $\mathbf{I}$ is the category associated to a partially ordered set.) Then we have the following simplified formula for $B(X,\mathbf{I},Y)$.

Consider all $n \geq 0$, and all strings of composable, non-identity morphisms $c_0 \to \cdots \to c_n$. We allow $n = 0$, in which case the string is just an object c_0 and no morphisms. We take the disjoint union over all of these strings of the spaces $\Delta^n \times X(c_0) \times Y(c_n)$:

$$\coprod_{n\geq 0}\left(\coprod_{c_0\to c_1\to\cdots\to c_n} \Delta^n \times X(c_0) \times Y(c_n)\right).$$

Then we apply only the face relations,

$$\begin{aligned}
&(d^0 t, x, f_1, \ldots, f_n, y) \sim (t, f_1(x), f_2, \ldots, f_n, y),\\
&(d^i t, x, f_1, \ldots, f_n, y) \sim (t, x, f_1, \ldots, f_{i+1} \circ f_i, \ldots, f_n, y) \text{ when } 0 < i < n, \text{ and}\\
&(d^n t, x, f_1, \ldots, f_n, y) \sim (t, x, f_1, \ldots, f_{n-1}, f_n^*(y)).
\end{aligned}$$

In other words, we identify the ith face of $\Delta^n \times X(c_0) \times Y(c_n)$ with the corresponding term $\Delta^{n-1} \times X(c_0) \times Y(c_n)$, when $i = 0$ we also apply f_1 to the coordinate in $X(c_0)$, and when $i = n$ we also apply f_n to the coordinate in $Y(c_n)$.

Proposition 7.4.11. *The quotient space described just above is homeomorphic to $B(X,\mathbf{I},Y)$.*

This is proven in Proposition A.2.15. The homeomorphism is the canonical map from this space to $B(X,\mathbf{I},Y)$, which exists because $B(X,\mathbf{I},Y)$ is defined using a larger disjoint union, modulo more relations.

What makes this new model so much simpler is that in many cases there are only finitely many terms, and a human can just explicitly write them all out.

Example 7.4.12. Suppose $\mathbf{I}$ is the category with two objects and only identity maps:

$$\mathbf{I} = \{\ 0 \quad 1\ \}.$$

There are only two strings of non-identity morphisms: the empty string at 0, and the empty string at 1. Therefore by Proposition 7.4.11, the bar construction $B(X,\mathbf{I},*)$ is homeomorphic to the disjoint union of the spaces $X(0)$ and $X(1)$.

Example 7.4.13. Suppose $\mathbf{I}$ is the category with two objects and one nontrivial map between them:

$$\mathbf{I} = \{\ 0 \to 1\ \}.$$

There are only three strings of non-identity morphisms: the empty string at 0, the empty string at 1, and the string $0 \to 1$. Therefore by Proposition 7.4.11, the bar construction $B(X,\mathbf{I},*)$ only has three pieces:

$$\Delta^0 \times X(0), \qquad \Delta^1 \times X(0), \qquad \Delta^0 \times X(1).$$

The face relations identify one end of the cylinder $\Delta^1 \times X(0)$ with $\Delta^0 \times X(0)$, while the other end is glued to $\Delta^0 \times X(1)$ along the given map $X(0) \to X(1)$. In total, $B(X, \mathbf{I}, *)$ is homeomorphic to the mapping cylinder of $X(0) \to X(1)$.

Example 7.4.14. Suppose $\mathbf{I}$ is the category that indexes pushouts:

$$\mathbf{I} = \{\ 1 \leftarrow 0 \rightarrow 2\ \}.$$

Now there are five strings of non-identity morphisms, so by Proposition 7.4.11, the bar construction $B(X, \mathbf{I}, *)$ is the disjoint union of the pieces

$$\Delta^0 \times X(1), \qquad \Delta^1 \times X(0), \qquad \Delta^0 \times X(0), \qquad \Delta^1 \times X(0), \qquad \Delta^0 \times X(2),$$

modulo relations that glue these together to form the double mapping cylinder

$$X(1) \cup_{X(0)} (X(0) \times I) \cup_{X(0)} (X(0) \times I) \cup_{X(0)} X(2).$$

Note that $B(X, \mathbf{I}, *)$ does not give the same simplicial space as in Example 7.2.6, even though it has the same geometric realization.

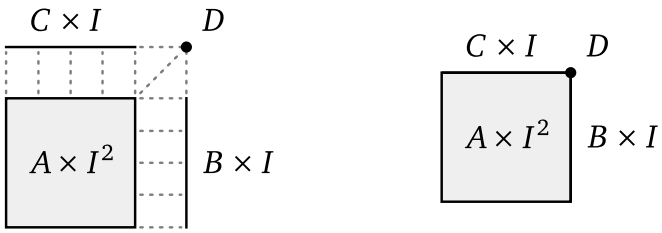
Example 7.4.15. Suppose $\mathbf{I}$ is the commuting square category. A diagram $X : \mathbf{I} \to \mathbf{Top}$ is a commuting square

$$\begin{array}{ccc} C & \longrightarrow & D \\ \uparrow & & \uparrow \\ A & \longrightarrow & B. \end{array}$$

It is an exercise (Exercise 7.7.37) to check that the bar construction $B(X, \mathbf{I}, *)$ is formed out of the pieces

$$A \times I \times I, \qquad B \times I, \qquad C \times I, \qquad D$$

by taking the square $A \times I \times I$, gluing its right-hand edge $A \times \{1\} \times I$ to $B \times I$, gluing the top edge $A \times I \times \{1\}$ to $C \times I$, and gluing the top-right corner $A \times \{1\} \times \{1\}$ to D, as shown:



7.4.3 Homotopy invariance and the bar lemma

We will prove the analogue of Theorem 7.3.10 for the categorical bar construction. The assumption we need here is the following:

Definition 7.4.16. We say $\mathbf{I}$ is **well-based** if for each object $c \in \mathbf{I}$, the inclusion of the identity map $\{\mathrm{id}_c\} \subseteq \mathbf{I}(c,c)$ is a Hurewicz cofibration. In other words, it has the homotopy extension property. When $\mathbf{I}$ is discrete, this is automatically true!

Theorem 7.4.17 (Homotopy invariance of the bar construction)**.** *Suppose that* $\mathbf{I}$ *is well-based. Then any two maps of diagrams over* $\mathbf{I}$*,* $X \to X'$ *and* $Y \to Y'$*, that are weak equivalences at each* $c \in \mathbf{I}$*, induce a weak equivalence*

$$B(X,\mathbf{I},Y) \xrightarrow{\sim} B(X',\mathbf{I},Y').$$

As in Theorem 7.3.10, we can also have maps of categories $\mathbf{I} \to \mathbf{I}'$ and compatible maps of diagrams $X \to X'$ and $Y \to Y'$, and these induce a map

$$B(X,\mathbf{I},Y) \longrightarrow B(X',\mathbf{I},Y'). \tag{7.4.18}$$

See Exercise 7.7.26. The map (7.4.18) is an equivalence if $X \to X'$ and $Y \to Y'$ are pointwise equivalences of diagrams (Definition 3.4.29 and Definition 5.3.14), and if $\mathbf{I} \to \mathbf{I}'$ is a **pointwise equivalence of topological categories**, meaning it is a bijection on objects and a weak equivalence on mapping spaces,

$$\mathbf{I}(a,b) \xrightarrow{\sim} \mathbf{I}'(a,b).$$

Remark 7.4.19. This last hypotheses can be weakened even more. The map of categories $\mathbf{I} \to \mathbf{I}'$ only has to be a **Dwyer-Kan equivalence** of topological categories, meaning it is a weak equivalence on mapping spaces, and a bijection on "homotopy equivalence" classes of objects. See Exercise 7.7.27.

The analogue of the bar lemma (Lemma 7.3.11) for categories is as follows. For each object $c \in \mathbf{I}$, the spaces of morphisms $\mathbf{I}(-,c)$ form a contravariant diagram $\mathbf{I}^{\mathrm{op}} \to \mathbf{Top}$. So, we can form the bar construction

$$B(X,\mathbf{I},\mathbf{I}(-,c)).$$

As c varies, these spaces form a diagram over $\mathbf{I}$, that we call "$B(X,\mathbf{I},\mathbf{I})$."

Lemma 7.4.20 (Bar Lemma for categories)**.** *There are homotopy equivalences*

$$B(X,\mathbf{I},\mathbf{I}(-,c)) \xrightarrow{\sim} X(c)$$

that together form a pointwise equivalence of diagrams $B(X,\mathbf{I},\mathbf{I}) \xrightarrow{\sim} X$.

The proof is the same as Lemma 7.3.11, see Exercise 7.7.24. As before, the bar lemma does not require $\mathbf{I}$ to be well-based. Also, by the same argument, if Y is a contravariant diagram then the bar constructions $B(\mathbf{I}(c,-),\mathbf{I},Y)$ form a contravariant diagram that we call "$B(\mathbf{I},\mathbf{I},Y)$," and we also have equivalences

$$B(\mathbf{I}(c,-),\mathbf{I},Y) \xrightarrow{\sim} Y(c)$$

giving a pointwise equivalence of diagrams $B(\mathbf{I},\mathbf{I},Y) \to Y$.

7.4.4 The bar construction is a left-derived functor

To prove this, we first need the category analogue of the isomorphism $B(X,G,G)\times_G Y \cong B(X,G,Y)$ from Lemma 7.3.13.

Lemma 7.4.21. *We have isomorphisms* $B(X,\mathbf{I},\mathbf{I})\times_{\mathbf{I}} Y \cong B(X,\mathbf{I},Y)\cong X\times_{\mathbf{I}} B(\mathbf{I},\mathbf{I},Y)$.

Proof. The space $B(X,\mathbf{I},\mathbf{I})\times_{\mathbf{I}} Y$ is defined as the coequalizer of the two maps

$$\coprod_{c,d\in \mathrm{ob}\,\mathbf{I}} B(X,\mathbf{I},\mathbf{I}(-,c))\times \mathbf{I}(c,d)\times Y(d) \rightrightarrows \coprod_{c\in\mathrm{ob}\,\mathbf{I}} B(X,\mathbf{I},\mathbf{I}(-,c))\times Y(c),$$

in which the $\mathbf{I}(c,d)$ is multiplied either into the $B(X,\mathbf{I},\mathbf{I}(-,c))$ or the $Y(d)$. Since geometric realization commutes with all colimits and products, this is the realization of the simplicial space that at level n is given by the coequalizer of the maps

$$\begin{array}{c}\displaystyle\coprod_{c_0,\dots,c_n,c,d\in\mathrm{ob}\,\mathbf{I}} X(c_0)\times\mathbf{I}(c_0,c_1)\times\cdots\times\mathbf{I}(c_{n-1},c_n)\times\mathbf{I}(c_n,c)\times\mathbf{I}(c,d)\times Y(d)\\ \downarrow\downarrow\\ \displaystyle\coprod_{c_0,\dots,c_n,c\in\mathrm{ob}\,\mathbf{I}} X(c_0)\times\mathbf{I}(c_0,c_1)\times\cdots\times\mathbf{I}(c_{n-1},c_n)\times\mathbf{I}(c_n,c)\times Y(c)\end{array}$$

that multiply the $\mathbf{I}(c,d)$ into either the $\mathbf{I}(c_n,c)$ or the $Y(d)$. These maps only involve the last three (or two) terms of the product, so we can rewrite this as $\coprod_{c_0,\dots,c_n,c\in\mathrm{ob}\,\mathbf{I}} X(c_0)\times \mathbf{I}(c_0,c_1)\times\cdots\times\mathbf{I}(c_{n-1},c_n)$ times the coequalizer of just the two maps

$$\coprod_{c,d\in\mathrm{ob}\,\mathbf{I}} \mathbf{I}(c_n,c)\times\mathbf{I}(c,d)\times Y(d) \rightrightarrows \coprod_{c\in\mathrm{ob}\,\mathbf{I}} \mathbf{I}(c_n,c)\times Y(c).$$

But this is the definition of $\mathbf{I}(c_n,-)\times_{\mathbf{I}} Y$, which by Example 7.4.4 is identified with $Y(c_n)$. Making this substitution, we get the simplicial space that at level n is

$$\coprod_{c_0,\dots,c_n\in\mathrm{ob}\,\mathbf{I}} X(c_0)\times\mathbf{I}(c_0,c_1)\times\cdots\times\mathbf{I}(c_{n-1},c_n)\times Y(c_n).$$

We check that it also has the same face and degeneracy maps as $N_\bullet(X,\mathbf{I},Y)$, and therefore its realization is $B(X,\mathbf{I},Y)$ as desired. The other isomorphism to $X\times_{\mathbf{I}} B(\mathbf{I},\mathbf{I},Y)$ is proven by a symmetric argument. □

Theorem 7.4.22. *When* $\mathbf{I}$ *is well-based, the bar construction* $B(X,\mathbf{I},Y)$ *is the left-derived functor of the coend* $X\times_{\mathbf{I}} Y$*:*

$$B(X,\mathbf{I},Y)\cong (Q_1X)\times_{\mathbf{I}}(Q_2Y),\qquad Q_1X\xrightarrow{\sim} X,\qquad Q_2Y\xrightarrow{\sim} Y.$$

In fact, it can be constructed by only deforming X,

$$B(X, \mathbf{I}, Y) \cong (QX) \times_{\mathbf{I}} Y, \qquad QX \xrightarrow{\sim} X,$$

or by only deforming Y.

The proof is the same as Theorem 7.3.14, and the deformation of X is $QX = B(X, \mathbf{I}, \mathbf{I})$. See Exercise 7.7.25.

7.4.5 The Bousfield–Kan homotopy colimit

One of the main applications of the categorical bar construction (Definition 7.4.7) is to give an explicit construction of the homotopy colimit.

Definition 7.4.23. If $\mathbf{I}$ is well-based, the **Bousfield–Kan homotopy colimit** of the diagram $X : \mathbf{I} \to \mathbf{Top}$ is the defined to be the categorical bar construction $B(X, \mathbf{I}, *)$.[3]

Now, we've already defined the homotopy colimit as the left-derived functor of the colimit (Definition 3.4.29), and we've proven that at least one left-derived functor exists using the small-object argument (Proposition 5.3.21). But by Theorem 7.4.22, this new construction is *another* left-derived functor of the colimit:

Corollary 7.4.24. *When $\mathbf{I}$ is well-based, the Bousfield–Kan homotopy colimit $B(X, \mathbf{I}, *)$ is a left-derived functor of the colimit $X \times_{\mathbf{I}} *$.*

Since left-derived functors are unique (Corollary 3.5.3) this new homotopy colimit *must be* equivalent to every other homotopy colimit we have already seen, including the one we constructed in Proposition 5.3.21 using the small-object argument, and the basic homotopy colimits we defined in Section 1.5. (At least, this is true when $\mathbf{I}$ is well-based. But when $\mathbf{I}$ is not well-based, we only have to modify the construction slightly. See Exercise 7.7.31.)

Remark 7.4.25. When $\mathbf{I}$ has a topology, this is the left-derived functor of the colimit on the category of *continuous* diagrams. As we mentioned in Remark 5.3.22, we sometimes call this the "enriched homotopy colimit" to emphasize that.

Although we have already proven the Bousfield–Kan homotopy colimit agrees with our earlier definitions, it is helpful to write it out anyway in the special case of the coproduct, the pushout, and the sequential colimit, so we can see what's going on.

[3] The idea that homotopy colimits can be built by a bar construction was seemingly in the air in the early 1970s. We call this construction the Bousfield–Kan homotopy colimit after its appearance in [BK87], but a similar construction was also considered by Boardman and Vogt in [Vog73].

Example 7.4.26. Suppose $\mathbf{I}$ is the category with two objects and only identity maps,

$$\bullet \qquad \bullet.$$

A diagram on $\mathbf{I}$ is a pair of spaces A and B. The colimit is the disjoint union $A \amalg B$. By Example 7.4.12, the Bousfield–Kan homotopy colimit $B(X, \mathbf{I}, *)$ is also the disjoint union $A \amalg B$. So the homotopy colimit agrees with the colimit in this case.

Example 7.4.27. Suppose $\mathbf{I}$ is the category that indexes pushouts,

$$\bullet \longleftarrow \bullet \longrightarrow \bullet.$$

A diagram on $\mathbf{I}$ is a pushout diagram $B \leftarrow A \rightarrow C$, and its colimit is the pushout $B \cup_A C$. By Example 7.4.14, the Bousfield–Kan homotopy colimit $B(X, \mathbf{I}, *)$ is the space

$$B \cup_A (A \times I) \cup_A (A \times I) \cup_A C,$$

which is homeomorphic to the homotopy pushout $B \cup_A (A \times I) \cup_A C$.

Using Proposition 7.4.11, we can also show that the thickened diagram $B(X, \mathbf{I}, \mathbf{I})$ is the diagram of mapping cylinders

$$B \cup_A (A \times I) \longleftarrow A \longrightarrow (A \times I) \cup_A C,$$

where the maps include A into the front of each mapping cylinder, as in Example 3.4.26. The colimit of this diagram is the double mapping cylinder. This is a special case of the isomorphism $B(X, \mathbf{I}, \mathbf{I}) \times_{\mathbf{I}} * \cong B(X, \mathbf{I}, *)$ from Lemma 7.4.21.

Example 7.4.28. Suppose $\mathbf{I}$ is the category that indexes sequential colimits,

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

A diagram on $\mathbf{I}$ is a sequential diagram $A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots$. By Proposition 7.4.11, the Bousfield–Kan homotopy colimit $B(X, \mathbf{I}, *)$ has a term $\Delta^n \times A_{i_0}$ for every tuple of integers $i_0 < i_1 < \cdots < i_n$, modulo face relations. This is not quite the mapping telescope – it is a larger space that is homotopy equivalent to the mapping telescope.

In detail, the mapping telescope has a cylinder $A_0 \times I$ connecting A_0 to A_1, and a cylinder $A_1 \times I$ connecting A_1 to A_2. This space also has a third cylinder $A_0 \times I$ connecting A_0 to A_2, and a 2-simplex $A_0 \times \Delta^2$ connecting these three cylinders together. However, the extra cylinder and 2-simplex admit a deformation retract onto the original two cylinders.

This pattern continues to the higher simplices, showing that this bar construction $B(X, \mathbf{I}, *)$ contains the mapping telescope as a deformation retract. This is good. They must be weakly equivalent to each other, because they are both "the homotopy colimit" of the diagram. See Exercise 7.7.36.

Example 7.4.29. We could replace the sequential category of the previous example with any totally ordered set **I**, for instance any ordinal number β. Then it becomes more difficult to define the mapping telescope, but there is no extra difficulty in defining the Bousfield–Kan homotopy colimit. It is still built out of one piece $\Delta^n \times A_{i_0}$ for every *finite* chain of elements $i_0 < i_1 < \cdots < i_n$ in the totally-ordered set, and then we glue these pieces along face and degeneracy maps as before. We still get the isomorphisms

$$H_k\left(\operatorname*{hocolim}_{c\in\mathbf{I}} X_c\right) \cong \operatorname*{colim}_{c\in\mathbf{I}} H_k(X_c), \qquad \pi_k\left(\operatorname*{hocolim}_{c\in\mathbf{I}} X_c\right) \cong \operatorname*{colim}_{c\in\mathbf{I}} \pi_k(X_c)$$

when **I** is the category associated to any totally ordered set. See also Exercise 7.7.38 for the more general case of a filtered homotopy colimit.

Example 7.4.30. If we take **I** to have one object and morphism set G, then a diagram over **I** is the same thing as a space X with a G-action (left or right depending on how we set up the conventions). The homotopy colimit of this diagram is the homotopy orbit space $X_{hG} = B(X, G, *)$. So, as we have seen before, homotopy orbits are a special case of homotopy colimits.

Finally, we return to the more general case of left Kan extensions. If $X : \mathbf{I} \to \mathbf{Top}$ is any diagram and $F : \mathbf{I} \to \mathbf{J}$ is any functor, we have seen how the coend $X \times_{\mathbf{I}} \mathbf{J}$ agrees with the left Kan extension $F_! X : \mathbf{J} \to \mathbf{Top}$. By Theorem 7.4.22, or really a slight variant in which we allow several objects, if we fix F and think of this as a functor of X, the left-derived functor is also given by a bar construction:

Corollary 7.4.31. *When* **I** *is well-based, the homotopy left Kan extension of X along F, in other words the left-derived functor of $F_!$ on X, is given by $B(X, \mathbf{I}, \mathbf{J})$.*

If the categories are discrete, then for each $j \in \mathbf{J}$, the resulting functor's value at j is simply the homotopy colimit of X over the comma category $(\mathbf{I} \downarrow j)$. See Exercise 7.7.33.

7.4.6 Homotopy colimits and bundles

The categorical bar construction is closely related to fibrations and fiber bundles, when we evaluate it on diagrams that are almost-constant.

Definition 7.4.32. If **I** is a topological category, a diagram $X : \mathbf{I} \to \mathbf{Top}$ is **almost-constant** if every map of the diagram $X(a) \to X(b)$ is a weak equivalence.

Theorem 7.4.33 (Quillen's Lemma)**.** *[Qui73, p.98] If* **I** *is well-based and $X : \mathbf{I} \to \mathbf{Top}$ is almost-constant, then the projection map*

$$B(X, \mathbf{I}, *) \longrightarrow B(*, \mathbf{I}, *)$$

*is a **quasifibration**. That is, for each point $b \in B(*, \mathbf{I}, *)$, the map from the fiber to the homotopy fiber at this point is an equivalence.*[4]

So if we replaced the map by an actual fibration, the fibers of that fibration would all be equivalent to the spaces $X(a)$. In other words, the map $B(X, \mathbf{I}, *) \to B(*, \mathbf{I}, *)$ already looks and behaves somewhat like a fibration.

This is a very useful result that will get deployed heavily in Chapter 9. We explain the proof of this and a more general version called "descent for homotopy colimits" in Section A.2.1.

Example 7.4.34. Suppose $\mathbf{I}$ has one object, and the monoid of morphisms from this object to itself is the natural numbers $\mathbb{N}$, in other words it is generated by a single nontrivial morphim. Then a diagram on $\mathbf{I}$ consists of a space X and one self-map $f : X \to X$. The Bousfield–Kan homotopy colimit is the **mapping torus**

$$T_f = (X \times I)/((x, 1) \sim (f(x), 0)).$$

This maps down to $B\mathbf{I} \cong S^1$ by forgetting the X coordinate. Quillen's lemma says that if f is a weak equivalence, then $T_f \to S^1$ is a quasifibration. So if we replaced it by a fibration, the fibers would still be equivalent to X. In fact, fibrations over X are nothing more than a space and one self-map, and Quillen's lemma shows us how to go explicitly from the space with a self-map to the fibration over S^1.

To indicate a much larger class of examples, let us say that a topological monoid M is **grouplike** if $\pi_0(M)$ is a group. In other words, each element $g \in M$ has an inverse-up-to-homotopy $h \in M$, so that gh and hg are in the same path component as the identity. (See Exercise 9.6.1.) Of course, every topological group is grouplike.

Proposition 7.4.35. *If M is a well-based grouplike monoid acting on the space X, then $B(X, M, *) \to B(*, M, *)$ is a quasifibration, and therefore we have a fiber sequence of the form*

$$X \longrightarrow X_{hM} = B(X, M, *) \longrightarrow BM.$$

Proof. For each $g \in M$, the map $(-)\cdot g : X \to X$ has a homotopy inverse, given by $(-)\cdot h$ for some inverse-up-to-homotopy $h \in M$. Therefore the M-action on X is an almost-constant diagram, so we can apply Theorem 7.4.33. Another proof can be found in [May75, Thm 7.6]. □

In the special case that $M = G$ is a topological group, we could also prove this from Proposition 7.3.17, because if $EG \to BG$ is a fiber bundle then with fiber G, then $EG \times_G X \to BG$, is a fiber bundle with fiber X, by working one trivialization at a time. Therefore $X_{hG} \to BG$ is a fiber bundle with fiber X.

[4] "Quillen's lemma" is not a standard name for this result. We call it this because it comes from Quillen, and it is the main step in the proof of his Theorem B (Corollary 8.6.21).

Example 7.4.36. For any topological space F, the **space of homotopy automorphisms** $\mathrm{hAut}(F)$ is the subspace of $\mathrm{Map}(F,F)$ consisting of all maps from F to itself that are homotopy equivalences. This is just a union of path components of $\mathrm{Map}(F,F)$, since anything homotopic to a homotopy equivalence is a homotopy equivalence. We generally assume that F is homotopy equivalent to a CW complex so that homotopy equivalences are the same thing as the weak equivalences.

Then $\mathrm{hAut}(F)$ is a topological monoid, and it is grouplike because every homotopy equivalence has an inverse up to homotopy. We can similarly make a based version $\mathrm{hAut}_*(F)$ when F is a based space.

We form the **universal fibration** with fiber F by taking the map of bar constructions

$$B(*, |S_\bullet \mathrm{hAut}(F)|, F) \longrightarrow B(*, |S_\bullet \mathrm{hAut}(F)|, *)$$

and then replacing it with an equivalent fibration (see Section 1.2.3). (The use of geometric realization and singular simplices is just a convenient way of making $\mathrm{hAut}(F)$ well-based.)

This constructs a fibration whose fiber is equivalent to F. It turns out that every fibration $E \to B$ with CW base B and with fiber equivalent to F is equivalent to a pullback of this universal fibration, by a map $B \to B(*, |S_\bullet \mathrm{hAut}(F)|, *)$ that is unique up to homotopy (see e.g. [May75]). So this gives a classifying space for fibrations up to weak equivalence, in much the same way that $BO(n)$ gives a classifying space for n-dimensional real vector bundles up to isomorphism.

7.5 Cosimplicial spaces and the cobar construction

In this section we introduce cosimplicial spaces, which are like simplicial spaces except that all the maps are reversed. (Cosimplicial *sets* don't occur very often – you usually want to have spaces to get something interesting.) These objects come with their own dual notion of realization, called "totalization." This allows us to give explicit models for homotopy limits.

7.5.1 Cosimplicial spaces

Definition 7.5.1. A **cosimplicial space** $X^\bullet$ is a functor $\mathbf{\Delta} \to \mathbf{Top}$. So it is a sequence of spaces X^n with coface maps $d^i : X^{n-1} \to X^n$ and codegeneracy maps $s^i : X^{n+1} \to X^n$ satisfying the relations in Lemma 7.1.3.

For instance, the diagram of simplices $\Delta^{(-)}$ is a cosimplicial space. If we take a simplicial space $X_\bullet$ and apply any contravariant functor to it, such as $\mathrm{Map}(-, Y)$, the result is a cosimplicial space.

Note that we put the index on $X^\bullet$ as a *superscript* instead of a subscript. This helps us remember that we have a diagram on $\mathbf{\Delta}$ instead of $\mathbf{\Delta}^{\mathrm{op}}$.

Definition 7.5.2. The **totalization** $\mathrm{Tot}(X^\bullet)$ of a cosimplicial space $X^\bullet$ is defined to be the subspace of the product

$$\mathrm{Tot}(X^\bullet) \subseteq \prod_{n\geq 0} \mathrm{Map}(\Delta^n, X^n)$$

consisting of those systems of maps $\Delta^n \to X^n$ that form a natural transformation, or map of diagrams, over the category $\mathbf{\Delta}$. In other words, every coface and codegeneracy map forms a commuting square as shown.

$$\begin{array}{ccc} \Delta^{n-1} & \xrightarrow{d^i} & \Delta^n \\ \downarrow & & \downarrow \\ X^{n-1} & \xrightarrow{d^i} & X^n \end{array} \qquad\qquad \begin{array}{ccc} \Delta^{n+1} & \xrightarrow{s^i} & \Delta^n \\ \downarrow & & \downarrow \\ X^{n+1} & \xrightarrow{s^i} & X^n. \end{array}$$

Example 7.5.3. If $X_\bullet$ is a simplicial space, then applying $\mathrm{Map}(-, Y)$ to each level gives a cosimplicial space $\mathrm{Map}(X_\bullet, Y)$. Its totalization is homeomorphic to the mapping space $\mathrm{Map}(|X_\bullet|, Y)$ (Exercise 7.7.22).

Example 7.5.4. As a special case of the previous exercise, let A be a topological space, and consider the cosimplicial space $\mathrm{Map}(\Delta[1]_\bullet, A)$. At each level this is a product of copies of A, labeled as follows:

$$X^0 = A_0 \times A_1, \quad X^1 = A_{00} \times A_{01} \times A_{11}, \quad X^2 = A_{000} \times A_{001} \times A_{011} \times A_{111}, \quad \dots$$

The maps are all duals of the maps in the simplicial set $\Delta[1]_\bullet$. So the coface maps map from a product to a product; on each factor of the target product space, we have a sequence, we delete one of the terms of that sequence, and that tells us which factor in the source space to take. So for example the 0th coface map $A \times A \to A \times A \times A$ sends $(a, b) \mapsto (a, b, b)$, while the 1st coface sends $(a, b) \mapsto (a, a, b)$. The codegeneracy maps similarly take each factor in the target product, duplicate a term in its sequence, and use that to tell which factor in the source to use.

The homeomorphism

$$\mathrm{Tot}(\mathrm{Map}(\Delta[1]_\bullet, A)) \cong \mathrm{Map}(|\Delta[1]_\bullet|, A) \cong A^I$$

takes each map of diagrams $\Delta^\bullet \to \mathrm{Map}(\Delta[1]_\bullet, A)$, extracts just the map from Δ^1 into the factor A_{01}, and forgets everything else.

Example 7.5.5. This example is dual to Example 7.2.6. Suppose we have a pullback diagram of topological spaces

$$\begin{array}{ccc} & & B \\ & & \downarrow k \\ C & \xrightarrow{h} & D. \end{array}$$

We define a cosimplicial space

$$X^0 = B_0 \times C_1, \quad X^1 = B_{00} \times D_{01} \times C_{11}, \quad X^2 = B_{000} \times D_{001} \times D_{011} \times C_{111}, \quad \dots$$

whose coface and codegeneracy maps are as in the previous example, except that we apply the maps h and k when appropriate. So for instance the 0th coface map $X^0 \to X^1$ maps $B \times C \to B \times D \times C$ by the identity maps of B and C and the map $h\colon C \to D$, while the 1st coface map uses $k\colon B \to D$ instead.

It follows from Exercise 7.7.23 that the totalization is homeomorphic to the homotopy pullback $B \times^h_D C$. In other words, a map of diagrams from $\Delta^{(-)}$ into this diagram is determined by the map from Δ^1 into the factor D_{01}, and lifts of the two endpoints to two maps $\Delta^0 \to B_0$ and $\Delta^1 \to C_1$. This agrees with the definition of the homotopy pullback from Definition 1.5.18.

7.5.2 The end and the cobar construction

We dualize the coend from Definition 7.4.1 in the following way. Suppose $X, Y : \mathbf{I} \to \mathbf{Top}$ are two diagrams on the category $\mathbf{I}$.

Definition 7.5.6. The **end** of X and Y is the topological space

$$\begin{aligned} \mathrm{Map}^{\mathbf{I}}(X,Y) &= \left\{ f_c : X(c) \to Y(c) \ : \ \forall f : c \to d,\ Y(f) \circ f_c = f_d \circ X(f) \right\} \\ &\subseteq \prod_{c \in \mathrm{ob}\, I} \mathrm{Map}(X(c), Y(c)). \end{aligned}$$

Informally, it is the space of all natural transformations $X \Rightarrow Y$, topologized as a subspace of the product of the spaces $\mathrm{Map}(X(c), Y(c))$.

Alternatively, we can think of the end as an equalizer, the limit of a diagram that just has two parallel maps:

$$\mathrm{Map}^{\mathbf{I}}(X,Y) \longrightarrow \prod_{c \in \mathrm{ob}\,\mathbf{I}} \mathrm{Map}(X(c), Y(c)) \rightrightarrows \prod_{c_0, c_1 \in \mathrm{ob}\,\mathbf{I}} \mathrm{Map}(X(c_0) \times \mathbf{I}(c_0, c_1), Y(c_1)).$$

The two maps take the set of functions $\{\phi_c : X(c) \to Y(c)\}_{c \in \mathrm{ob}\,\mathbf{I}}$ to either the set

$$\left\{(x, f) \mapsto \phi_{c_1}(f(x))\right\}_{c_0, c_1 \in \mathrm{ob}\,\mathbf{I}}$$

or to the set

$$\left\{(x, f) \mapsto f(\phi_{c_0}(x))\right\}_{c_0, c_1 \in \mathrm{ob}\,\mathbf{I}}.$$

In other words, we either pre-compose with the composition maps $X(c_0) \times \mathbf{I}(c_0, c_1) \to X(c_1)$, or we post-compose with the composition maps $Y(c_0) \times \mathbf{I}(c_0, c_1) \to Y(c_1)$.

Example 7.5.7. The totalization of a cosimplicial space $X^\bullet$ is isomorphic to the end of the diagram of simplices $\Delta^{(-)}$ and $X^\bullet$ itself:

$$\mathrm{Tot}(X^\bullet) = \mathrm{Map}^{\mathbf{\Delta}}(\Delta^{(-)}, X^\bullet).$$

Example 7.5.8. If $\mathbf{I}$ has one object and morphism set G, then X and Y are both right G-spaces. The end becomes $\mathrm{Map}^G(X,Y)$, the space of G-equivariant maps from X to Y.

Example 7.5.9. For fixed $c \in \mathbf{I}$, the sets $\mathbf{I}(c,-)$ form a diagram on $\mathbf{I}$. Then $\mathrm{Map}^{\mathbf{I}}(\mathbf{I}(c,-),X)$ is isomorphic to $X(c)$ (Exercise 7.7.21), and this is natural in c. In other words, we have an isomorphism of $\mathbf{I}$-diagrams

$$\mathrm{Map}^{\mathbf{I}}(\mathbf{I}, X) \cong X.$$

As a special case, the space of G-equivariant maps from G itself to X is just X,

$$\mathrm{Map}^G(G,X) \cong X.$$

Here $\mathrm{Map}^G(G,X)$ is formed using the action of G on itself on one side (say the right-hand side). It then receives another right action by pre-composing $\phi : G \to X$ with the left action of G on itself. The isomorphism $\mathrm{Map}^G(G,X) \cong X$ commutes with this G-action.

We now give the thickened or derived version of the end.

Definition 7.5.10. If $X, Y : \mathbf{I} \to \mathbf{Top}$ are two diagrams on the category $\mathbf{I}$, the **categorical cobar construction** $F(X,\mathbf{I},Y)$ is the totalization of the cosimplicial space whose nth level is the product of mapping spaces,

$$\begin{aligned} F^n(X,\mathbf{I},Y) &= \prod_{c_0,c_1,\cdots,c_n \in \mathrm{ob}\,\mathbf{I}} \mathrm{Map}\big(X(c_0)\times \mathbf{I}(c_0,c_1)\times\cdots\times\mathbf{I}(c_{n-1},c_n), Y(c_n)\big) \\ &\cong \prod_{c_n\in\mathrm{ob}\,\mathbf{I}} \mathrm{Map}\left(\coprod_{c_0,c_1,\cdots,c_{n-1}\in\mathrm{ob}\,\mathbf{I}} X(c_0)\times \mathbf{I}(c_0,c_1)\times\cdots\times\mathbf{I}(c_{n-1},c_n), Y(c_n)\right). \end{aligned}$$

The coface maps $d^i : F^{n-1}(X,\mathbf{I},Y) \to F^n(X,\mathbf{I},Y)$ and codegeneracy maps $s^i : F^{n+1}(X,\mathbf{I},Y) \to F^n(X,\mathbf{I},Y)$ are the duals of those in Definition 7.4.7. To write these explicitly in a way that is not completely terrible, we think of a point in the above space as a single function

$$\phi : \coprod_{c_0,c_1,\cdots,c_n} X(c_0)\times\mathbf{I}(c_0,c_1)\times\cdots\times\mathbf{I}(c_{n-1},c_n) \to \coprod_{c_n} Y(c_n),$$

where each piece of the domain has to go to the piece of the codomain with the same

value of c_n. Then the coface and codegeneracy maps are as follows:

$$\begin{aligned}
&\text{If } \phi = \phi(x, f_1, \dots, f_{n-1}),\\
(d^0\phi)(x, f_1, \dots, f_n) &= \phi(\underline{f_1(x)}, f_2, \dots, f_n),\\
(d^i\phi)(x, f_1, \dots, f_n) &= \phi(x, f_1, \dots, \underline{f_{i+1} \circ f_i}, \dots, f_n) \text{ when } 0 < i < n,\\
(d^n\phi)(x, f_1, \dots, f_n) &= \underline{f_n \circ \phi}(x, f_1, \dots, f_{n-1}).
\end{aligned}$$

$$\begin{aligned}
&\text{If } \phi = \phi(x, f_1, \dots, f_{n+1}),\\
(s^i\phi)(x, f_1, \dots, f_n) &= \phi(x, f_1, \dots, f_{i-1}, \underline{\mathrm{id}_{c_i}}, f_i, \dots, f_n) \text{ when } 0 \le i \le n.
\end{aligned}$$

Note that the function ϕ is the information that d^i and s^i acts on, turning it into another function with one more input or one fewer input, given by the formula above. The value of ϕ depends on $x, f_1, \dots, f_n$ and therefore also on the objects $c_0, \dots, c_n$, even though we did not explicitly include them in the formula. When $n = 0$, ϕ depends on $x \in X(c_0)$ and therefore implicitly depends on both x and c_0. In this case the two coface maps are

$$\begin{aligned}
(d^0\phi)(x, f_1) &= \phi(f_1(x)),\\
(d^1\phi)(x, f_1) &= f_1 \circ \phi(x),
\end{aligned}$$

as in the definition of the end (Definition 7.5.6).

7.5.3 Two rearrangements of the cobar construction

Proposition 7.5.11. *The cobar construction is homeomorphic to the end of $B(X, \mathbf{I}, \mathbf{I})$ and Y:*

$$F(X, \mathbf{I}, Y) \cong \mathrm{Map}^{\mathbf{I}}(B(X, \mathbf{I}, \mathbf{I}), Y).$$

This is dual to the fact that $B(X, \mathbf{I}, \mathbf{I}) \times_{\mathbf{I}} Y \cong B(X, \mathbf{I}, Y)$ from Lemma 7.4.21. Conceptually, it means that the cobar construction is the result of thickening up the diagram X to the diagram $B(X, \mathbf{I}, \mathbf{I})$, and then taking the space of maps to Y.

Proof. The bar construction $B(X, \mathbf{I}, \mathbf{I})(c)$ is a disjoint union of spaces

$$\Delta^n \times X(c_0) \times \mathbf{I}(c_0, c_1) \times \cdots \times \mathbf{I}(c_{n-1}, c_n) \times \mathbf{I}(c_n, c),$$

modulo the relations that come from the face and degeneracy maps,

$$\begin{aligned}
(d^0 t, x, f_1, \dots, f_n, f) &= (t, \underline{f_1(x)}, f_2, \dots, f_n, f),\\
(d^i t, x, f_1, \dots, f_n, f) &= (t, x, f_1, \dots, \underline{f_{i+1} \circ f_i}, \dots, f_n, f) \text{ when } 0 < i < n,\\
(d^n t, x, f_1, \dots, f_n, f) &= (t, x, f_1, \dots, f_{n-1}, \underline{f \circ f_n}), \text{ and}\\
(s^i t, x, f_1, \dots, f_n, f) &= (t, x, f_1, \dots, f_i, \underline{\mathrm{id}_{c_i}}, f_{i+1}, \dots, f_n, f) \text{ when } 0 \le i \le n.
\end{aligned}$$

Here $t \in \Delta^{n+1}$ in the first three lines, while $t \in \Delta^{n-1}$ in the last line, so that $d^i t$ or $s^i t$ is always in Δ^n. By Exercise 1.7.27 (d), we deduce that the space

$$\mathrm{Map}(B(X, \mathbf{I}, \mathbf{I})(-, c), Y(c))$$

is subspace of the mapping space

$$\prod_{c_n} \mathrm{Map}\left(\coprod_{c_0,\ldots,c_{n-1}} \Delta^n \times X(c_0) \times \mathbf{I}(c_0, c_1) \times \cdots \times \mathbf{I}(c_{n-1}, c_n) \times \mathbf{I}(c_n, c), Y(c) \right)$$

consisting of those maps ϕ such that

$$\begin{aligned}
\phi(d^0 t, x, f_1, \ldots, f_n, f) &= \phi(t, \underline{f_1(x)}, f_2, \ldots, f_n, f), \\
\phi(d^i t, x, f_1, \ldots, f_n, f) &= \phi(t, x, f_1, \ldots, \underline{f_{i+1} \circ f_i}, \ldots, f_n, f) \text{ when } 0 < i < n, \\
\phi(d^n t, x, f_1, \ldots, f_n, f) &= \phi(t, x, f_1, \ldots, f_{n-1}, \underline{f \circ f_n}), \text{ and} \\
\phi(s^i t, x, f_1, \ldots, f_n, f) &= \phi(t, x, f_1, \ldots, f_i, \underline{\mathrm{id}_{c_i}}, f_{i+1}, \ldots, f_n, f) \text{ when } 0 \le i \le n.
\end{aligned}$$

Here we are again writing a point in the product of mapping spaces as a single function ϕ, to make the formulas less cluttered. To form $\mathrm{Map}^{\mathbf{I}}(B(X, \mathbf{I}, \mathbf{I}), Y)$, we take the product of these over all c, and then restrict to the subspace of those maps such that $\phi(x, f_1, \ldots, f_n, f) = f \circ \phi(x, f_1, \ldots, f_n, \mathrm{id}_{c_n})$. Up to homeomorphism, we get the same space if we restrict ϕ to those tuples of the form $(x, f_1, \ldots, f_n, \mathrm{id}_{c_n})$. These now have the conditions

$$\begin{aligned}
\phi(d^0 t, x, f_1, \ldots, f_n, \mathrm{id}_{c_n}) &= \phi(t, \underline{f_1(x)}, f_2, \ldots, f_n, \mathrm{id}_{c_n}), \\
\phi(d^i t, x, f_1, \ldots, f_n, \mathrm{id}_{c_n}) &= \phi(t, x, f_1, \ldots, \underline{f_{i+1} \circ f_i}, \ldots, f_n, \mathrm{id}_{c_n}) \text{ when } 0 < i < n, \\
\phi(d^n t, x, f_1, \ldots, f_n, \mathrm{id}_{c_n}) &= \phi(t, x, f_1, \ldots, f_{n-1}, \underline{\mathrm{id}_{c_n} \circ f_n}), \text{ and} \\
\phi(s^i t, x, f_1, \ldots, f_n, \mathrm{id}_{c_n}) &= \phi(t, x, f_1, \ldots, f_i, \underline{\mathrm{id}_{c_i}}, f_{i+1}, \ldots, f_n, \mathrm{id}_{c_n}) \text{ when } 0 \le i \le n,
\end{aligned}$$

except that we have to rewrite the third line in terms of $\phi(t, x, f_1, \ldots, f_{n-1}, \mathrm{id}_{c_{n-1}})$, which gives $f_n \circ \phi(t, x, f_1, \ldots, f_{n-1}, \mathrm{id}_{c_{n-1}})$. Rewriting $\phi(t, x, \ldots)$ as $\phi_t(x, \ldots)$ to emphasize that the functions depend on the choice of point $t \in \Delta^n$, the above relations become $\phi_{d^i t} = d^i \phi_t$ and $\phi_{s^i t} = s^i \phi_t$, which is exactly the formula for the totalization of $F^\bullet(X, \mathbf{I}, Y)$. □

We will need another rearrangement of the cobar construction. Note that for each object $c \in \mathbf{I}$, the spaces $\mathbf{I}(c, -)$ form a covariant diagram on $\mathbf{I}$, so that we may form the cobar construction $F(\mathbf{I}(c, -), \mathbf{I}, Y)$. As c varies, this forms a covariant diagram on $\mathbf{I}$ that we call $F(\mathbf{I}, \mathbf{I}, Y)$.[5]

[5]This is because $F(-, \mathbf{I}, -)$ is *contravariant* in the first slot, so it flips the contravariance back around and makes it covariance.

Proposition 7.5.12. *The cobar construction is homeomorphic to the end of X and $F(\mathbf{I},\mathbf{I},Y)$:*

$$F(X,\mathbf{I},Y) \cong \mathrm{Map}^{\mathbf{I}}(X, F(\mathbf{I},\mathbf{I},Y)).$$

Proof. This proof is similar to Proposition 7.5.11, but easier. The right-hand side is a subspace of the product of mapping spaces

$$\prod_c \mathrm{Map}\left(X(c), \mathrm{Map}\left(\coprod_{c_0,\dots,c_n} \Delta^n \times \mathbf{I}(c,c_0) \times \mathbf{I}(c_0,c_1) \times \cdots \times \mathbf{I}(c_{n-1},c_n), Y(c_n)\right)\right).$$

Before we describe the subspace, we first rearrange it to

$$\prod_{c_n} \mathrm{Map}\left(\coprod_{c,c_0,\dots,c_{n-1}} \Delta^n \times X(c) \times \mathbf{I}(c,c_0) \times \mathbf{I}(c_0,c_1) \times \cdots \times \mathbf{I}(c_{n-1},c_n), Y(c_n)\right).$$

Then the subspace consists of all maps ϕ such that

$$\begin{aligned}
\phi(t,x,f,f_1,\dots,f_n) &= \phi(t,\underline{f(x),\mathrm{id}_{c_0}},f_1,f_2,\dots,f_n),\\
\phi(d^0t,x,f,f_1,\dots,f_n) &= \phi(t,x,\underline{f_1\circ f},f_2,\dots,f_n),\\
\phi(d^it,x,f,f_1,\dots,f_n) &= \phi(t,x,f,f_1,\dots,\underline{f_{i+1}\circ f_i},\dots,f_n) \text{ when } 0<i<n,\\
\phi(d^nt,x,f,f_1,\dots,f_n) &= \underline{f_n\circ\phi}(t,x,f,f_1,\dots,f_{n-1}), \text{ and}\\
\phi(s^it,x,f,f_1,\dots,f_n) &= \phi(t,x,f,f_1,\dots,f_i,\underline{\mathrm{id}_{c_i}},f_{i+1},\dots,f_n) \text{ when } 0\le i\le n.
\end{aligned}$$

The first condition comes from the end $\mathrm{Map}^{\mathbf{I}}(-,-)$, while the others come from the definition of $F(\mathbf{I},\mathbf{I},Y)$. As before, we get the same space by restricting the function to inputs of the form $(t,x,\mathrm{id}_{c_0},f_1,\dots,f_n)$. This eliminates the first condition above, while the second condition becomes

$$\phi(d^0t,x,\mathrm{id}_{c_0},f_1,\dots,f_n) = \phi(t,x,\underline{f_1\circ\mathrm{id}_{c_0}},f_2,\dots,f_n).$$

The right-hand side isn't in standard form since the function is not id_{c_0}, so we rewrite it as $\phi(t,f_1(x),\mathrm{id}_{c_0},f_2,\dots,f_n)$. This leaves us with the conditions that define the totalization of $F^\bullet(X,\mathbf{I},Y)$. □

7.5.4 A useful simplification and some examples

In this section we dualize Section 7.4.2, by giving a simplified formula for $F(X,\mathbf{I},Y)$ when $\mathbf{I}$ is particularly nice.

Example 7.5.13. Suppose $\mathbf{I}$ is discrete, and that the non-identity morphisms of $\mathbf{I}$ are closed under composition. As in Example 7.4.10, consider all strings of composable, non-identity morphisms $c_0 \to \cdots \to c_n$. We allow $n = 0$, in which case the string is just an object c_0 and no morphisms. We take the product over all of these strings of the spaces $\mathrm{Map}(\Delta^n \times X(c_0), Y(c_n))$:

$$\prod_{n\geq 0}\left(\prod_{c_0\to c_1\to\cdots\to c_n} \mathrm{Map}(\Delta^n \times X(c_0), Y(c_n))\right).$$

Then we restrict to those maps that satisfy just the face relations coming from Definition 7.5.10:

$$\begin{aligned}
&\phi(d^0 t, x, f_1, \ldots, f_n) = \phi(t, x, f_1(x), f_2, \ldots, f_n),\\
&\phi(d^i t, x, f_1, \ldots, f_n) = \phi(t, x, f_1, \ldots, f_{i+1} \circ f_i, \ldots, f_n) \text{ when } 0 < i < n, \text{ and}\\
&\phi(d^n t, x, f_1, \ldots, f_n) = f_n \circ \phi(t, x, f_1, \ldots, f_{n-1}).
\end{aligned}$$

Proposition 7.5.14. *The subspace described just above is homeomorphic to $F(X, \mathbf{I}, Y)$.*

Proof. There is a forgetful map from $F(X, \mathbf{I}, Y)$ to this space, where we only remember the value of ϕ on the strings of non-identity morphisms. Along the isomorphism of Proposition 7.5.11, this forgetful map is obtained by applying $\mathrm{Map}^{\mathbf{I}}(-, Y)$ to the homeomorphism of Proposition 7.4.11. The forgetful map is therefore also a homeomorphism. □

Example 7.5.15. Suppose $\mathbf{I}$ is the category with two objects and only identity maps:

$$\mathbf{I} = \{\ 0 \quad 1\ \}.$$

There are only two strings of non-identity morphisms: the empty string at 0, and the empty string at 1. Therefore by Proposition 7.5.14, the cobar construction $F(*, \mathbf{I}, Y)$ is homeomorphic to the product of the spaces $Y(0)$ and $Y(1)$.

Example 7.5.16. This example is dual to Example 7.4.13. Suppose $\mathbf{I}$ is the category with two objects and one nontrivial map between them:

$$\mathbf{I} = \{\ 0 \to 1\ \}.$$

There are only three strings of non-identity morphisms: the empty string at 0, the empty string at 1, and the string $0 \to 1$. Therefore by Proposition 7.5.14, the cobar construction $F(*, \mathbf{I}, Y)$ only has three pieces:

$$\mathrm{Map}(\Delta^0, Y(0)) \cong Y(0), \qquad \mathrm{Map}(\Delta^1, Y(1)) \cong Y(1)^I, \qquad \mathrm{Map}(\Delta^0, Y(1)) \cong Y(1).$$

The face relations tell us that the map $\Delta^1 \to Y(1)$ has to restrict at one end to the point $\Delta^0 \to Y(1)$, and at the other end gives the image in $Y(1)$ of the point $\Delta^0 \to Y(0)$. In other words, $F(*, \mathbf{I}, Y)$ is homeomorphic to the mapping co-cylinder $Y(0) \times_{Y(1)} Y(1)^I$.

Example 7.5.17. Suppose $\mathbf{I}$ is the category that indexes pullbacks:

$$\mathbf{I} = \{\ 1 \to 3 \leftarrow 2\ \}.$$

As in Example 7.4.14, there are five strings of non-identity morphisms, so by Proposition 7.5.14, the bar construction $F(*, \mathbf{I}, Y)$ is the product of the pieces

$$Y(1), \qquad Y(3)^I, \qquad Y(3), \qquad Y(3)^I, \qquad Y(2),$$

restricted to the subspace where the maps $I \to Y(3)$ agree on one end with the chosen point in $Y(3)$, and on the other end with the point in $Y(1)$ or $Y(2)$. So we get the space

$$Y(1) \times_{Y(3)} Y(3)^I \times_{Y(3)} Y(3)^I \times_{Y(3)} Y(2).$$

The two paths $I \to Y(3)$ link together to form a single path $I \to Y(3)$, and this gives a homeomorphism to the space

$$Y(1) \times_{Y(3)} Y(3)^I \times_{Y(3)} Y(2),$$

which is exactly the homotopy pullback. Note the similarity to Example 7.5.5, although this cosimplicial space is not isomorphic to that one, it only has a homeomorphic totalization.

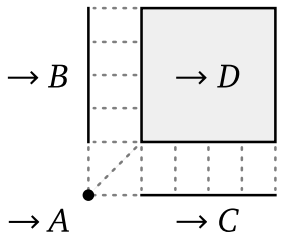
Example 7.5.18. Suppose $\mathbf{I}$ is the commuting square category, as in Example 7.4.15. A diagram $Y : \mathbf{I} \to \mathbf{Top}$ is a commuting square

$$\begin{array}{ccc} C & \longrightarrow & D \\ \uparrow & & \uparrow \\ A & \longrightarrow & B. \end{array}$$

It is an exercise (Exercise 7.7.37) to check that the cobar construction $F(*, \mathbf{I}, Y)$ is formed out of the pieces

$$A, \qquad B^I, \qquad C^I, \qquad D^{I \times I}$$

by asking that the maps $I \to B$ and $I \to C$ agree on the left endpoint $\{0\}$ with the point in A, and that the map $I \times I \to D$ agrees along the bottom edge $I \times \{0\}$ with the map $I \to B$, along the left edge $\{0\} \times I$ with the map $I \to C$, and along the bottom-left corner $\{0\} \times \{0\}$ with the point $* \to A$. This is summarized in the following picture.



7.5.5 Homotopy invariance and the cobar lemma

We begin with a lemma about the homotopy invariance of the end:

Lemma 7.5.19. *The end* $\mathrm{Map}^{\mathbf{I}}(X,Y)$ *preserves equivalences when* X *is free cofibrant (Theorem 5.3.16) and* Y *is arbitrary.*

Proof. To establish this we use the theory of Quillen tensors from Definition 5.5.13. We recall from Example 5.5.3 that the end $\mathrm{Map}^{\mathbf{I}}(-,-)$ is one of the right adjoints of an adjunction of two variables. The two-variable left adjoint is the functor

$$\times : \mathbf{Top} \times \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}^{\mathbf{I}}$$

that takes the space K and the diagram $X(-)$ and produces the diagram $K \times X(-)$.

We checked in Example 5.5.15 that this is a Quillen tensor, using the Quillen model structure for spaces (Theorem 5.2.5) and the projective model structure for diagrams (Theorem 5.3.16). To briefly summarize the proof, the pushout-product of the map of spaces $(S^{m-1} \to D^m)$ and the map of diagrams $F_c(S^{n-1} \to D^n)$ is the map of diagrams $F_c(S^{m+n-1} \to D^{m+n})$, which verifies the condition for pairs of cofibrations. The verification for the acyclic cofibrations is similar.

Therefore by Lemma 5.5.22, the functor $\mathrm{Map}^{\mathbf{I}}(X,Y)$ preserves all equivalences when X is free cofibrant and Y is level fibrant. We are in **Top**, where every space is fibrant, so this condition on Y is always true. □

Now for homotopy invariance of the cobar construction, it won't do if $\mathbf{I}$ is only well-based. We need a slightly stronger assumption:

Definition 7.5.20. We say that $\mathbf{I}$ is **pointwise cofibrant** if for each object $c \in \mathbf{I}$, the inclusion of the identity map $\{\mathrm{id}_c\} \subseteq \mathbf{I}(c,c)$ is a Quillen cofibration, in other words a retract of a cell complex. Furthermore, we require that every mapping space $\mathbf{I}(a,b)$ is Quillen cofibrant as well. Note that when $\mathbf{I}$ is discrete, this condition is automatically true!

We also say that the diagram $X : \mathbf{I} \to \mathbf{Top}$ is **pointwise cofibrant** if each space $X(c)$ is Quillen cofibrant.

Proposition 7.5.21 (Homotopy invariance of the cobar construction)**.** *Suppose that* $\mathbf{I}$ *is pointwise cofibrant. If* $X \to X'$ *and* $Y \to Y'$*, are two pointwise equivalences of* $\mathbf{I}$*-diagrams, and if* X *is also pointwise cofibrant, then the maps induce a weak equivalence*

$$F(X', \mathbf{I}, Y) \xrightarrow{\sim} F(X, \mathbf{I}, Y').$$

Proof. By Lemma 7.5.19, the end $\mathrm{Map}^{\mathbf{I}}(X,Y)$ preserves all equivalences when X is free cofibrant. By Exercise 7.7.32, if X is pointwise cofibrant, then $B(X,\mathbf{I},\mathbf{I})$ is free cofibrant.

Therefore an equivalence of pointwise cofibrant diagrams $X \to X'$ becomes an equivalence of free cofibrant diagrams $B(X, \mathbf{I}, \mathbf{I}) \to B(X', \mathbf{I}, \mathbf{I})$. Feeding this into the end gives the equivalence in the statement of the theorem. □

The analogue of the bar lemma for categories (Lemma 7.4.20) is as follows.

Lemma 7.5.22 (Cobar Lemma)**.** *If* $\mathbf{I}$ *is pointwise cofibrant, then there are weak equivalences*

$$Y(c) \xrightarrow{\sim} F(\mathbf{I}(c,-), \mathbf{I}, Y)$$

that together form an equivalence of diagrams $Y \xrightarrow{\sim} F(\mathbf{I}, \mathbf{I}, Y)$.

Proof. The bar lemma (Lemma 7.4.20) gives an equivalence of spaces

$$B(\mathbf{I}(c,-), \mathbf{I}, \mathbf{I}(-,d)) \xrightarrow{\sim} \mathbf{I}(c,d) \tag{7.5.23}$$

that commutes with the action of $\mathbf{I}$ on both the left and the right. Fixing c and feeding this into the first slot of the end gives a map

$$\mathrm{Map}^{\mathbf{I}}(\mathbf{I}(c,-), Y) \longrightarrow \mathrm{Map}^{\mathbf{I}}(B(\mathbf{I}(c,-), \mathbf{I}, \mathbf{I}), Y). \tag{7.5.24}$$

This is an equivalence because $B(\mathbf{I}(c,-), \mathbf{I}, \mathbf{I}) \to \mathbf{I}(c,-)$ is an equivalence of free cofibrant diagrams and $\mathrm{Map}^{\mathbf{I}}(-, Y)$ preserves such equivalences by Lemma 7.5.19.

By Example 7.5.9 and Proposition 7.5.11, the equivalence (7.5.24) is homeomorphic to a map of the desired form, $Y(c) \xrightarrow{\sim} F(\mathbf{I}(c,-), \mathbf{I}, Y)$. It is natural in c since the original equivalence (7.5.23) was natural in c. □

7.5.6 The cobar construction is a right-derived functor

Theorem 7.5.25. *When* $\mathbf{I}$ *and* X *are pointwise cofibrant, the cobar construction* $F(X, \mathbf{I}, Y)$ *is the right-derived functor of the end* $\mathrm{Map}^{\mathbf{I}}(X, Y)$*:*

$$F(X, \mathbf{I}, Y) \cong \mathrm{Map}^{\mathbf{I}}(QX, RY), \qquad QX \xrightarrow{\sim} X, \qquad Y \xrightarrow{\sim} RY.$$

In fact, it can be constructed by only deforming X*,*

$$F(X, \mathbf{I}, Y) \cong \mathrm{Map}^{\mathbf{I}}(QX, Y), \qquad QX \xrightarrow{\sim} X,$$

or by only deforming Y*,*

$$F(X, \mathbf{I}, Y) \cong \mathrm{Map}^{\mathbf{I}}(X, RY), \qquad Y \xrightarrow{\sim} RY.$$

Proof. There are now two statements to prove. The first is easier: by Lemma 7.5.19, $\mathrm{Map}^{\mathbf{I}}(-,-)$ preserves all equivalences when the first diagram is free cofibrant, and $QX = B(X,\mathbf{I},\mathbf{I})$ is free cofibrant if X is pointwise cofibrant. This shows that we can right-derive $\mathrm{Map}^{\mathbf{I}}(-,-)$ by only replacing X. By Proposition 7.5.11, the resulting right-derived functor is $\mathrm{Map}^{\mathbf{I}}(B(X,\mathbf{I},\mathbf{I}),Y) \cong F(X,\mathbf{I},Y)$.

For the second statement, we hold X fixed and replace Y by $RY = B(\mathbf{I},\mathbf{I},Y)$. We will prove that $\mathrm{Map}^{\mathbf{I}}$ preserves all equivalences if the first input is pointwise cofibrant and the second input is of the form $B(\mathbf{I},\mathbf{I},Y)$.

From here the proof is dual to that of Theorem 7.3.14. Consider the collection of all $\mathbf{I}$-diagrams of the form $F(\mathbf{I},\mathbf{I},Y)$. The maps between these are all maps of diagrams $f : F(\mathbf{I},\mathbf{I},Y) \to F(\mathbf{I},\mathbf{I},Y')$, not necessarily arising from a map $Y \to Y'$. However, if such a map f is a weak equivalence, then we use Proposition 7.5.12 to fit it into the following commuting diagram:

$$\begin{array}{ccc} \mathrm{Map}^{\mathbf{I}}(X,F(\mathbf{I},\mathbf{I},Y)) & \xrightarrow{\mathrm{Map}^{\mathbf{I}}(\mathrm{id},f)} & \mathrm{Map}^{\mathbf{I}}(X,F(\mathbf{I},\mathbf{I},Y')) \\ \downarrow{\scriptstyle \mathrm{Map}^{\mathbf{I}}(m_X,\mathrm{id})} & & \downarrow{\scriptstyle \mathrm{Map}^{\mathbf{I}}(m_X,\mathrm{id})} \\ \mathrm{Map}^{\mathbf{I}}(B(X,\mathbf{I},\mathbf{I}),F(\mathbf{I},\mathbf{I},Y)) & \xrightarrow{\mathrm{Map}^{\mathbf{I}}(\mathrm{id},f)} & \mathrm{Map}^{\mathbf{I}}(B(X,\mathbf{I},\mathbf{I}),F(\mathbf{I},\mathbf{I},Y')). \end{array} \tag{7.5.26}$$

It suffices to show that this is a square of weak equivalences. For the bottom horizontal map this follows from Lemma 7.5.19. For the vertical maps we use Proposition 7.5.12 and Proposition 7.5.11 to change them into

$$\begin{array}{ccc} \mathrm{Map}^{\mathbf{I}}(B(X,\mathbf{I},\mathbf{I}),Y) & & \mathrm{Map}^{\mathbf{I}}(B(X,\mathbf{I},\mathbf{I}),Y) \\ \downarrow{\scriptstyle \mathrm{Map}^{\mathbf{I}}(B(m_X,\mathrm{id},\mathrm{id}),\mathrm{id})} & & \downarrow{\scriptstyle \mathrm{Map}^{\mathbf{I}}(B(m_X,\mathrm{id},\mathrm{id}),\mathrm{id})} \\ \mathrm{Map}^{\mathbf{I}}(B(B(X,\mathbf{I},\mathbf{I}),\mathbf{I},\mathbf{I}),Y) & & \mathrm{Map}^{\mathbf{I}}(B(B(X,\mathbf{I},\mathbf{I}),\mathbf{I},\mathbf{I}),Y'). \end{array}$$

The maps $B(m_X,\mathrm{id},\mathrm{id}) \colon B(X,\mathbf{I},\mathbf{I}) \to B(B(X,\mathbf{I},\mathbf{I}),\mathbf{I},\mathbf{I})$ are equivalences of free cofibrant diagrams, so feeding them into $\mathrm{Map}^{\mathbf{I}}$ produces an equivalence, as desired. We can now conclude that the top horizontal map of (7.5.26) is an equivalence.

Now that we've proven that $\mathrm{Map}^{\mathbf{I}}$ preserves equivalences when the second diagram is of the form $F(\mathbf{I},\mathbf{I},Y)$, it follows from Proposition 7.5.12 that the resulting right-derived functor is $\mathrm{Map}^{\mathbf{I}}(X,F(\mathbf{I},\mathbf{I},Y)) \cong F(X,\mathbf{I},Y)$. □

7.5.7 The Bousfield–Kan homotopy limit

One of the main applications of the cobar construction is to give an explicit construction of the homotopy limit.

Definition 7.5.27. If $\mathbf{I}$ is pointwise cofibrant (Definition 7.5.20), the **Bousfield–Kan homotopy limit** of the diagram $Y : \mathbf{I} \to \mathbf{Top}$ is the defined to be the categorical cobar construction $F(*, \mathbf{I}, Y)$.

We've already defined the homotopy limit abstractly as the right-derived functor of the limit (Definition 3.4.37), but we were *unable* to prove that this exists using model categories. So this is our first successful definition of the homotopy limit for topological spaces. And indeed it works, because as a result of Theorem 7.5.25, we get:

Corollary 7.5.28. *When* $\mathbf{I}$ *is pointwise cofibrant, the Bousfield–Kan homotopy limit* $F(*, \mathbf{I}, Y)$ *is the right-derived functor of the limit* $\mathrm{Map}^{\mathbf{I}}(*, Y)$.

We have finally proven that homotopy limits of topological spaces exist, for diagrams of any shape. However we already knew of their existence in a few special cases, and it's illuminating to compare that to this new model in the special case of the product, the pullback, and the sequential limit.

Example 7.5.29. Suppose $\mathbf{I}$ is the category with two objects and only identity maps,

$$\bullet \qquad \bullet.$$

A diagram on $\mathbf{I}$ is a pair of spaces A and B. The limit is the product $A \times B$. By Example 7.5.15, the Bousfield–Kan homotopy limit $F(*, \mathbf{I}, Y)$ is also the product $A \times B$. So the homotopy limit agrees with the limit in this case.

Example 7.5.30. Suppose $\mathbf{I}$ is the category that indexes pullbacks,

$$\bullet \longrightarrow \bullet \longleftarrow \bullet.$$

A diagram on $\mathbf{I}$ is a pullback diagram $B \to D \leftarrow C$, and its limit is the pullback $B \times_D C$. By Example 7.5.17, the Bousfield–Kan homotopy limit $F(*, \mathbf{I}, Y)$ is the space

$$B \times_D D^I \times_D D^I \times_D C,$$

which is homeomorphic to the homotopy pullback $B \times_D D^I \times_D C$.

We can also compute this in two other ways. First, by Proposition 7.5.11, we can compute $F(X, \mathbf{I}, Y)$ as the space $\mathrm{Map}^{\mathbf{I}}(B(*, \mathbf{I}, \mathbf{I}), Y)$. Using Proposition 7.4.11, the thickened diagram $B(*, \mathbf{I}, \mathbf{I})$ is the diagram that includes the two ends of the interval,

$$\{0\} \longrightarrow I \longleftarrow \{1\}.$$

Then $\mathrm{Map}^{\mathbf{I}}(B(*, \mathbf{I}, \mathbf{I}), Y)$ is the space of maps from the above diagram into $B \to D \leftarrow C$. This is exactly the homotopy pullback $B \times_D D^I \times_D C$.

The other way to compute this is to use Proposition 7.5.11, which expresses the homotopy limit $F(*, \mathbf{I}, Y)$ as the space $\mathrm{Map}^{\mathbf{I}}(*, F(\mathbf{I}, \mathbf{I}, Y))$, in other words the limit of the thickened diagram $F(\mathbf{I}, \mathbf{I}, Y)$. By Proposition 7.5.14, the thickened diagram $F(\mathbf{I}, \mathbf{I}, Y)$ is of the form

$$B \times_D D^I \longrightarrow D \longleftarrow D^I \times_D C,$$

where the maps evaluate D^I at the end of I other than the one used for B or C. The limit of this diagram is again the homotopy pullback $B \times_D D^I \times_D C$.

Example 7.5.31. Suppose $\mathbf{I}$ is the category that indexes sequential limits,

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet.$$

A diagram on $\mathbf{I}$ is a sequential diagram $\cdots \to A_2 \to A_1 \to A_0$. The Bousfield–Kan homotopy limit $F(*, \mathbf{I}, Y)$ has a term $\mathrm{Map}(\Delta^n, A_{i_n})$ for every tuple of integers $i_0 > i_1 > \cdots > i_n$, modulo face relations. This is not quite the mapping microscope

$$\cdots \times_{A_2} A_2^I \times_{A_1} A_1^I \times_{A_0} A_0^I$$

from Definition 1.5.21, but it is a larger space that is homotopy equivalent to it. The homotopy equivalence forgets almost all of the data and remembers just the maps $\Delta^1 \to A_i$ for the pairs of integers $i + 1 > i$. Again, these two models of the homotopy limit must be weakly equivalent to each other, because they are both "the homotopy limit" of the diagram.

Example 7.5.32. As in Example 7.4.29, we could replace the category in the previous example with any totally ordered set $\mathbf{I}$. The Bousfield–Kan homotopy limit is still built out of one piece $\mathrm{Map}(\Delta^n, A_{i_n})$ for every *finite* chain of elements $i_0 < i_1 < \cdots < i_n$ in the totally-ordered set, and then we glue these pieces along face and degeneracy maps as before. The effect of this on homotopy is more difficult to describe; even in the case of a sequential homotopy limit, there was a $\lim^1$ term!

Example 7.5.33. If we take $\mathbf{I}$ to have one object and morphism set G, then a diagram over $\mathbf{I}$ is the same thing as a space X with a G-action (left or right depending on how we set up the conventions). The homotopy limit of this diagram is the homotopy fixed points $X^{hG} = F(*, G, X) = \mathrm{Map}^G(B(*, G, G), X) = \mathrm{Map}^G(EG, X)$. So homotopy fixed points are a special case of homotopy limits.

Example 7.5.34. Suppose $\alpha \colon \mathbf{I} \to \mathbf{J}$ is a functor. Then for fixed $d \in \mathbf{J}$, the sets $\mathbf{J}(d, \alpha(-))$ form a diagram on $\mathbf{I}$. The end with this diagram becomes the limit

$$\mathrm{Map}^{\mathbf{I}}(\mathbf{J}(d, \alpha(-)), X) \cong \lim_{d \to \alpha(c)} X(c)$$

where the limit is taken over the comma category $(d \downarrow \alpha)$, see Exercise 7.7.34 for more details.

By varying d, this defines a diagram on $\mathbf{J}$, called the **right Kan extension** of X along $\mathbf{I} \to \mathbf{J}$, denoted $\mathrm{Map}^{\mathbf{I}}(\mathbf{J}, X)$ or $\alpha_* X$. As a special case, if $\mathbf{J} = *$ is the one-object category with only the identity map, this example becomes the limit:

$$\mathrm{Map}^{\mathbf{I}}(*, X) \cong \lim_{c \in \mathbf{I}} X(c).$$

We can define the **homotopy right Kan extension** to be the right-derived functor of this, which is $F(\mathbf{J}, \mathbf{I}, X)$.

We have at long last shown that the pullback functor $\alpha^* : \mathbf{Top}^{\mathbf{J}} \to \mathbf{Top}^{\mathbf{I}}$ not only has a left adjoint $\alpha_!$ and a right adjoint α_*, but that each of the adjunctions passes to an adjunction of derived functors on the homotopy category,

$$(\mathbb{L}\alpha_! \dashv \alpha^*) \quad \text{and} \quad (\alpha^* \dashv \mathbb{R}\alpha_*).$$

It wasn't possible to do all of this using the small object argument. The bar and cobar constructions give you another way to derive functors, and in this case they are the more powerful technique!

7.5.8 The cobar construction for coalgebras

Finally, we briefly mention that there is another construction $\Omega(Y, X, Z)$ that is also called the "cobar construction," but that exists under very different hypotheses than in Definition 7.5.10. Instead of dualizing Definition 7.3.7 by taking mapping spaces, we dualize it by reversing the direction of all of the multiplications.

Definition 7.5.35. Let X be a topological space. Then its diagonal map

$$\delta : X \to X \times X, \qquad \delta(x) = (x, x)$$

makes X into something called a "topological comonoid," which is like a topological monoid but where all of the maps are reversed. We won't give the definition here, because it turns out, this diagonal map δ is the *only* way to give X such a structure.

A **right coaction** of X on a space Y is a map $Y \to Y \times X$ of the form $y \mapsto (y, \rho(y))$ for some continuous function $\rho : Y \to X$. This is "counital" in the sense that $Y \to Y \times X \to Y$ is the identity of Y, and "coassociative" in the sense that the two ways of forming a map $Y \to Y \times X \times X$ agree. Again, this is the only way to give Y such a structure. A **left coaction** is the same except that the map is of the form $Z \to X \times Z$, $z \mapsto (\sigma(z), z)$.

Definition 7.5.36. Suppose X is a space and we have two maps $\rho : Y \to X$ and $\sigma : Z \to X$. The **coalgebra cobar construction** $\Omega(Y, X, Z)$ is the totalization of the cosimplicial space $\Omega^\bullet(Y, X, Z)$, whose nth level is the product

$$\Omega_n(Y, X, Z) = Y \times X^n \times Z.$$

The coface maps d^i apply ρ, δ, or σ to individual terms ("comultiplications"), while the codegeneracy maps delete copies of X ("counits"):

$$\begin{aligned}
d^0(y,x_1,\dots,x_{n-1},z) &= (y,\underline{\rho(y)},x_1,\dots,x_{n-1},z),\\
d^i(y,x_1,\dots,x_{n-1},z) &= (y,x_1,\dots,\underline{x_i,x_i},\dots,x_{n-1},z) \text{ when } 0<i<n,\\
d^n(y,x_1,\dots,x_{n-1},z) &= (y,x_1,\dots,x_{n-1},\underline{\sigma(z)},z), \text{ and}\\
s^i(y,x_1,\dots,x_{n+1},z) &= (y,x_1,\dots,x_i,\underline{*},x_{i+2},\dots,x_{n+1},z) \text{ when } 0\le i\le n.
\end{aligned}$$

The identities from Lemma 7.1.5 can be checked directly, so that $\Omega^\bullet(Y,X,Z)$ forms a cosimplicial space. In fact, up to changing the names of the spaces, this is exactly the cosimplicial space we considered in Example 7.5.5. Therefore:

Lemma 7.5.37. *The cobar construction is just the homotopy pullback,*

$$\Omega(Y,X,Z) \cong Y \times_X X^I \times_X Z.$$

In particular, if X is a based space, $Y = Z = *$, and we take the coaction of X on $*$ to be the inclusion of the basepoint, then the cobar construction is a model for the loopspace,

$$\Omega(*,X,*) \cong \Omega X.$$

This explains the use of the notation Ω. Although this construction does not do much in the category of topological spaces, the cobar construction in chain complexes has many more applications.

7.6 The simplicial version of based spaces and spectra

In this section we develop some of the properties of based simplicial sets $\mathbf{sSet}_*$ and spectra of simplicial sets $\mathbf{Sp}_{\mathbf{sSet}}$. The objects of $\mathbf{Sp}_{\mathbf{sSet}}$ are spectra whose *levels* are simplicial sets instead of topological spaces. This is not to be confused with *simplicial objects* in the category of spectra, which we denote **sSp** instead.

Our ultimate goal is to extend the bar construction and the Bousfield–Kan homotopy colimit to based spaces, spectra, and these simplicial versions of based spaces and spectra. However, since the number of possible settings is quickly multiplying, we will wait until the next chapter to do this. There we will use the language of simplicial model categories and do everything in a uniform way.

7.6.1 Based simplicial sets

Definition 7.6.1. Let $\mathbf{sSet}_*$ be the category of **based simplicial sets**. A based simplicial set is either a simplicial set with a map from the constant simplicial set $* \to X_\bullet$, or

equivalently it is a simplicial set in which each set X_n has a basepoint $* \in X_n$ and each face and degeneracy map preserves the basepoint.

Similarly, we let $\mathbf{sTop}_*$ be the category of **based simplicial spaces**. This has the same two descriptions, except that the X_n are topological spaces and the face and degeneracy maps are continuous maps of based spaces.

The geometric realization of any based simplicial space $X_\bullet \in \mathbf{sTop}_*$ is defined using the same formula as in (7.1.14):

$$|X_\bullet| = \left(\coprod_{n\geq 0} \Delta^n \times X_n \right) / \ (d^i t, x) = (t, d_i x),\ (s^i t, x) = (t, s_i x). \tag{7.6.2}$$

This produces a space with a canonical basepoint, given by the realization of the constant simplicial space $|*| \cong *$ mapping to $|X_\bullet|$. One might imagine that it's more natural to define the geometric realization using the smash products and wedge sums instead:

$$|X_\bullet| = \left(\bigvee_{n\geq 0} \Delta^n_+ \wedge X_n \right) / \ (d^i t, x) = (t, d_i x),\ (s^i t, x) = (t, s_i x). \tag{7.6.3}$$

However, this always gives the same result.

Lemma 7.6.4. *If $X_\bullet$ is a based simplicial space, the above two definitions for $|X_\bullet|$ are canonically homeomorphic.*

Proof. There is a map from (7.6.2) to (7.6.3) that applies all of the additional relations needed to form the smash product and the wedge sum from the Cartesian product and the disjoint union. Writing these out, we see that the extra relations have the effect of quotienting all points of the form $(t, *)$ down to a single point. However, these points were already identified in (7.6.2): because the degeneracy maps are required to preserve the basepoint, every point of the form $(t, *)$ can be simplified through the degeneracy relations to the point $(1, *)$ where $1 \in \Delta^0$ is the unique point. Therefore the additional relations that bring us from (7.6.2) to (7.6.3) have no effect, and the canonical map is a homeomorphism. □

Next we develop the model structure on $\mathbf{sSet}_*$. As in Remark 7.2.17, we are not going to prioritize putting a model structure on $\mathbf{sTop}_*$ as well.

Given any unbased simplicial set X, we can make a based simplicial set X_+ which at each level is $(X_n)_+$, the set X_n with a disjoint basepoint. This forms the left adjoint of the forgetful functor $\mathbf{sSet}_* \to \mathbf{sSet}$.

Proposition 7.6.5. *The category of based simplicial sets $\mathbf{sSet}_*$ has a model structure with the same cofibrations, weak equivalences, and fibrations as in Definition 7.1.37. It is proper and is cofibrantly generated by*

- $I = \{ (\partial \Delta[n])_+ \to \Delta[n]_+ \}_{n \geq 0}$, *and*
- $J = \{ \Lambda_i[n]_+ \to \Delta[n]_+ \}_{n \geq 0,\ 0 \leq i \leq n}$.

Geometric realization and singular simplices define a Quillen equivalence $\mathbf{sSet}_* \simeq \mathbf{Top}_*$.

Proof. This follows immediately from the model structure on **sSet** from Theorem 7.1.39, and the model structure on a slice category from Exercise 5.8.8. □

In particular, the cofibrations are the injective maps at each simplicial level, the weak equivalences are the maps that are weak homotopy equivalences after realization, and the fibrations are the Kan fibrations, which have lifts along all of the maps in J.

Next we develop the symmetric monoidal structure on $\mathbf{sSet}_*$. The category has all colimits and limits already, which are computed at each level by taking the colimit or limit in the category of pointed sets $\mathbf{Set}_*$. So the coproduct is the wedge sum, and the product is the Cartesian product:

$$(X_\bullet \vee Y_\bullet)_n \cong X_n \vee Y_n, \qquad (X \times Y)_n \cong X_n \times Y_n.$$

Since the geometric realization is a left adjoint, it preserves all colimits, and in particular

$$|X_\bullet \vee Y_\bullet| \cong |X_\bullet| \vee |Y_\bullet|.$$

Since the geometric realization and the limit both commute with forgetting the basepoint, the realization also commutes with all finite limits. In particular,

$$|X_\bullet \times Y_\bullet| \cong |X_\bullet| \times |Y_\bullet|.$$

So we get two symmetric monoidal structures (using $\vee$ or $\times$) that make $\mathbf{sSet}_*$ into a symmetric monoidal category, such that $|-|$ is a strong symmetric monoidal functor.

However, the most interesting symmetric monoidal structure is the one given by the smash product.

Definition 7.6.6. The **smash product of based simplicial sets** $A_\bullet \wedge B_\bullet$ is just the smash product of pointed sets at each level:

$$(A_\bullet \wedge B_\bullet)_n := A_n \wedge B_n = (A_n \times B_n)/(A_n \vee B_n).$$

Since the face and degeneracy maps of A and B preserve the basepoints, they induce well-defined maps on these quotients, making $A_\bullet \wedge B_\bullet$ into a based simplicial set.

Remark 7.6.7. As with the Cartesian product, we could also form the smash product by taking the bisimplicial set $\{A_m \wedge B_n : m, n \geq 0\}$, and then restricting to the diagonal. This description is sometimes helpful when proving things about the smash product, see e.g. Lemma 9.4.17.

As for $\mathbf{Top}_*$, there are isomorphisms

$$(A_\bullet \wedge B_\bullet) \wedge C_\bullet \cong A_\bullet \wedge (B_\bullet \wedge C_\bullet), \qquad A_\bullet \wedge B_\bullet \cong B_\bullet \wedge A_\bullet, \qquad S^0_\bullet \wedge A_\bullet \cong A_\bullet$$

that make $\mathbf{sSet}_*$ into a symmetric monoidal category. Here $S^0_\bullet$ is the constant simplicial set on the 0-sphere S^0.

Lemma 7.6.8. *The realization of simplicial based spaces commutes with the smash product,*

$$|X_\bullet \wedge Y_\bullet| \cong |X_\bullet| \wedge |Y_\bullet|.$$

Moreover this isomorphism may be chosen so that it makes $|-|$ into a strong symmetric monoidal functor $\mathbf{sSet}_* \to \mathbf{Top}_*$.

Proof. The isomorphism is constructed from the fact that realization preserves the product, the wedge, and all pushouts. (The smash product is the pushout $(A \times B) \cup_{(A \vee B)} (*)$.) The resulting isomorphism is characterized by the property that the following square commutes:

$$\begin{array}{ccc} |X_\bullet \times Y_\bullet| & \xleftarrow{\cong} & |X_\bullet| \times |Y_\bullet| \\ \downarrow & & \downarrow \\ |X_\bullet \wedge Y_\bullet| & \xleftarrow{\cong} & |X_\bullet| \wedge |Y_\bullet| \end{array}$$

The coherence of these isomorphisms then follows immediately from the coherence of the same isomorphisms for the Cartesian product. □

By a formal argument (Exercise 4.4.11), the right adjoint of the geometric realization is also a lax monoidal functor. This means that there is a natural map

$$S_\bullet X \wedge S_\bullet Y \longrightarrow S_\bullet(X \wedge Y). \tag{7.6.9}$$

Concretely, the map (7.6.9) takes two continuous maps $f : \Delta^n \to X$ and $g : \Delta^n \to Y$ to the map

$$\Delta^n \xrightarrow{(f,g)} X \times Y \longrightarrow\!\!\!\!\!\rightarrow X \wedge Y.$$

Lemma 7.6.10. *The map in* (7.6.9) *is a weak equivalence.*

Proof. Form the following commuting diagram and note that the realization of each row is a cofiber sequence. Therefore, since the vertical maps in the middle and on the left are weak equivalences, the vertical map on the right is a weak equivalence as well.

$$\begin{array}{ccccc} S_\bullet X \vee S_\bullet Y & \longrightarrow & S_\bullet X \times S_\bullet Y & \longrightarrow & S_\bullet X \wedge S_\bullet Y \\ \downarrow \simeq & & \downarrow \cong & & \downarrow \\ S_\bullet(X \vee Y) & \longrightarrow & S_\bullet(X \times Y) & \longrightarrow & S_\bullet(X \wedge Y) \end{array}$$

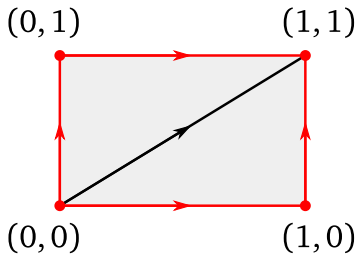
□

Example 7.6.11. Recall from Example 7.1.17 that the **simplicial circle** is $S^1_\bullet = \Delta[1]/\partial\Delta[1]$. We define the **simplicial n-sphere** to be the n-fold smash product

$$S^n_\bullet := S^1_\bullet \wedge \cdots \wedge S^1_\bullet.$$

By Lemma 7.6.8, the realization of $S^n_\bullet$ is homeomorphic to the n-sphere $S^n \cong S^1 \wedge \cdots \wedge S^1$.

In particular, $S^2_\bullet$ is obtained from the simplicial set illustrated in Example 7.1.32 by quotienting all of the simplices in the boundary of the square to the basepoint. So all of the edges highlighted in red are the basepoint:



We are left with a single basepoint 0-simplex, one nondegenerate 1-simplex, and two nondegenerate 2-simplices; everything else is degenerate. For $S^3_\bullet$ we get a similar description as a cube cut into $3! = 6$ tetrahedra, with all of the simplices in the boundary of the cube identified to a point.

In Exercise 7.7.15 we check that the smash product has a right adjoint and plays well with the model structure. This proves:

Lemma 7.6.12. *The smash product makes* $\mathbf{sSet}_*$ *into a symmetric monoidal model category.*

The right adjoint in each slot is the simplicial set of based maps

$$(\mathrm{Map}_*)_n(Y, Z) = \left\{(\Delta[n]_+)_\bullet \wedge Y_\bullet \to Z_\bullet\right\}.$$

We will need to understand its homotopical behavior a little better in order to set up the model category of spectra further below. We have a natural map

$$|\mathrm{Map}_*(Y, Z)| \longrightarrow \mathrm{Map}_*(|Y|, |Z|) \tag{7.6.13}$$

defined as the adjoint of the map of based spaces

$$|Y| \wedge |\mathrm{Map}_*(Y, Z)| \xleftarrow{\cong} |Y \wedge \mathrm{Map}_*(Y, Z)| \xrightarrow{|ev|} |Z|.$$

Don't worry too much about tracing through the exact definition of the map (7.6.13). We'll only need to use that it is natural and that we know what it does when $Y = S^0_\bullet$, where it becomes the identity map of $|Z|$.

Lemma 7.6.14. *If $Z_\bullet$ is a Kan complex, then the map* (7.6.13) *is an equivalence.*

In particular, this means that when $Z_\bullet$ is a Kan complex, $\mathrm{Map}_*(S^1_\bullet, Z_\bullet)$ is a simplicial set whose realization is equivalent to the based loopspace $\Omega|X_\bullet|$. So $\mathrm{Map}_*(S^1_\bullet, Z_\bullet)$ is the simplicial version of the loopspace of $Z_\bullet$.

Proof. Since $Z_\bullet$ is a Kan complex, the natural map $Z_\bullet \to S_\bullet|Z|$ is a weak equivalence of simplicial sets. (This follows from the fact that $|-|$ and $S_\bullet$ form a Quillen equivalence.) From this we deduce that for any set A, even an infinite one, the map of indexed products

$$\prod_{\alpha\in A} Z_\bullet \longrightarrow \prod_{\alpha\in A} S_\bullet|Z| \cong S_\bullet \prod_{\alpha\in A} |Z|$$

is an equivalence. Taking realizations gives an equivalence

$$\left|\prod_{\alpha\in A} Z_\bullet\right| \longrightarrow \left|S_\bullet \prod_{\alpha\in A} |Z|\right| \longrightarrow \prod_{\alpha\in A} |Z|,$$

from which we deduce that realization commutes with infinite products *of Kan complexes*. (This is true even if all the factors in the product are distinct.)

Now we express Y by starting with $*$ and attaching a coproduct of copies of the standard simplex $\Delta[n]$ along the boundary $\partial\Delta[n]$, one value of n at a time. This is possible by Lemma 8.2.36. The pushout square

$$\begin{array}{ccc} \bigvee \partial\Delta[n]_+ & \longrightarrow & \bigvee \Delta[n]_+ \\ \downarrow & & \downarrow \\ Y^{(n-1)} & \longrightarrow & Y^{(n)} \end{array}$$

becomes a map of pullback squares

$$\begin{array}{ccccc} \prod |\mathrm{Map}_*(\partial\Delta[n]_+, Z)| & \longleftarrow & \prod |\mathrm{Map}_*(\Delta[n]_+, Z)| & & \\ \uparrow & & \uparrow & & \\ |\mathrm{Map}_*(Y^{(n-1)}, Z)| & \longleftarrow & |\mathrm{Map}_*(Y^{(n)}, Z)| & & \\ \downarrow & & \downarrow & & \\ \prod \mathrm{Map}_*(\partial\Delta^n_+, |Z|) & \longleftarrow & \prod \mathrm{Map}_*(\Delta^n_+, |Z|) & & \\ \uparrow & & \uparrow & & \\ \mathrm{Map}_*(|Y^{(n-1)}|, |Z|) & \longleftarrow & \mathrm{Map}_*(|Y^{(n)}|, |Z|) & & \end{array}$$

using the fact that geometric realization preserves pullbacks, that maps out of a pushout square become a pullback square, and the fact we just proved that realization commutes

with infinite products of Kan complexes. Using the fact that $\mathrm{Map}_*(-,-)$ is a Quillen tensor and that realization preserves fibrations, the two horizontal maps marked with a double arrow head are Serre fibrations. Therefore we have a map of homotopy pullback squares.

We now work inductively on the dimension of Y; for $n \geq 0$, if the theorem has already been proven for $(n-1)$-dimensional complexes Y, then the two vertical maps on the left are equivalences, and the vertical map on the far right is an equivalence because it simplifies to the identity map of $|Z|$. Therefore the last vertical map for $Y^{(n)}$ is also an equivalence, completing the inductive step.

When Y is infinite-dimensional, we then take the limit of the results as $n \to \infty$. This is a limit along Serre fibrations, so if each of the maps $|\mathrm{Map}_*(Y^{(n)}, Z)| \to \mathrm{Map}_*(|Y^{(n)}|, |Z|)$ is an equivalence, so is the induced map of limits $|\mathrm{Map}_*(Y, Z)| \to \mathrm{Map}_*(|Y|, |Z|)$, by Lemma 1.5.22 and Theorem 1.5.24. □

7.6.2 Spectra of simplicial sets

We next define categories of spectra whose levels are simplicial sets.

Definition 7.6.15. A **sequential spectrum of simplicial sets** X is a sequence of based simplicial sets $\{X(n)_\bullet\}_{n\geq 0}$ and bonding maps

$$\xi(n)\colon X(n)_\bullet \wedge S^1_\bullet \to X(n+1)_\bullet$$

where $S^1_\bullet$ is the simplicial circle from Example 7.6.11.

A map of sequential spectra $f : X \to Y$ is a sequence of maps of based simplicial sets $f(n)\colon X(n)_\bullet \to Y(n)_\bullet$ that commutes with the bonding maps as in Definition 2.1.1. The category of sequential spectra is denoted $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{sSet}}$.

Definition 7.6.16. A **symmetric spectrum of simplicial sets** has in addition an action of Σ_n on $X(n)_\bullet$ through maps of based simplicial sets, for all $n \geq 0$, such that the composite

$$\xi(m,n)\colon X(m)_\bullet \wedge S^n_\bullet \xrightarrow{\xi(m)\wedge \mathrm{id}} X(m+1)_\bullet \wedge S^{n-1}_\bullet \xrightarrow{\xi(m+1)\wedge \mathrm{id}} \dots \xrightarrow{\xi(m+n-1)} X(m+n)_\bullet.$$

is $\Sigma_m \times \Sigma_n$-equivariant, as in Definition 6.1.2. Here Σ_n acts on $S^n_\bullet$ by permuting the factors of $S^1_\bullet$.

A map of symmetric spectra is a map of sequential spectra such that each $f(n)$ is Σ_n-equivariant. The category of symmetric spectra is denoted $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}}$.

Remark 7.6.17. Orthogonal spectra are conspicuously absent here. They require the topological space $O(n)$ to work, and it's $O(n)$ up to homeomorphism, not just up to equivalence, so taking singular simplices of $O(n)$ isn't good enough. As a result, orthogonal spectra don't appear in settings where topological spaces are not naturally present.

Example 7.6.18. The **simplicial sphere spectrum**, which we denote $\mathbb{S}$ by abuse of notation, has level n equal to $S^n_\bullet$, and the bonding maps are the canonical ismorphisms of simplicial sets $S^n_\bullet \wedge S^1_\bullet \cong S^{n+1}_\bullet$.

More generally, if $A_\bullet$ is any based simplicial set, the **simplicial suspension spectrum** $\Sigma^\infty A$ has level n equal to $A_\bullet \wedge S^n_\bullet$, and the bonding maps are also canonical isomorphisms of simplicial sets.

Example 7.6.19. If $X \in \mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}}$ is a spectrum of topological spaces, the singular simplices $S_\bullet X_n$ define a spectrum of simplicial sets. The bonding maps

$$S_\bullet X_n \wedge S^1_\bullet \longrightarrow S_\bullet X_{n+1}$$

send each continuous map $f : \Delta^k \to X_n$ and morphism $\alpha \in \mathbf{\Delta}([k],[1])$, which corresponds to a continuous map $\alpha\colon \Delta^k \to \Delta^1$, to the corresponding composite

$$\Delta^k \xrightarrow{(f,\alpha)} X_n \times \Delta^1 \longrightarrow\!\!\!\!\rightarrow X_n \wedge S^1 \xrightarrow{\xi_n} X_{n+1},$$

as in (7.6.9). If X is a symmetric spectrum then $S_\bullet X$ is a symmetric spectrum of simplicial sets.

This operation has a left adjoint, the geometric realization. As you might expect, the geometric realization of a spectrum of simplicial sets takes the realization of each level $|X_\bullet(n)|$, and defines the bonding maps as the composites

$$|X_\bullet(n)| \wedge S^1 \xleftarrow{\cong} |X_\bullet(n)| \wedge |S^1_\bullet| \xleftarrow{\cong} |X_\bullet(n) \wedge S^1_\bullet| \xrightarrow{|\xi(n)|} |X_\bullet(n+1)|.$$

Again, if $X_\bullet(-)$ is a symmetric spectrum then its realization is also a symmetric spectrum.

Lemma 7.6.20. *These operations define adjunctions:*

$$\mathbf{Sp}^{\mathbb{N}}_{\mathbf{sSet}} \underset{S_\bullet}{\overset{|-|}{\rightleftarrows}} \mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}} \qquad \mathbf{Sp}^{\Sigma}_{\mathbf{sSet}} \underset{S_\bullet}{\overset{|-|}{\rightleftarrows}} \mathbf{Sp}^{\Sigma}_{\mathbf{Top}}.$$

Proof. If X is a spectrum of simplicial sets and Y is a spectrum of topological spaces, a map $|X_\bullet| \to Y$ is given by maps $|X_\bullet(n)| \to Y_n$, which are adjoint to maps $X_\bullet(n) \to S_\bullet Y_n$. Along this bijection, the maps that commute with the bonding maps in one sense become the maps that commute with the bonding maps in the other sense. □

Definition 7.6.21. A **stable equivalence** of spectra of simplicial sets is a map $X \to Y$ that becomes a stable equivalence on the realizations $|X| \to |Y|$, in the sense of Definition 6.1.19. A **level equivalence** or a π_***-isomorphism** is similarly defined as a map that is a level equivalence or π_*-isomorphism, respectively, after geometric realization.

For sequential spectra, stable equivalences are the same thing as π_*-isomorphisms. For symmetric spectra, every π_*-isomorphism is a stable equivalence but not the other way around (Proposition 6.1.25).

Definition 7.6.22. A **level fibration** of spectra of simplicial sets is a map $p : X \to Y$ that is a Kan fibration at each spectrum level. A **stable fibration** is a level fibration such that each square of the following form is a homotopy pullback:

$$\begin{array}{ccc} X_\bullet(i) & \longrightarrow & \mathrm{Map}_*(S^j_\bullet, X_\bullet(i+j)) \\ \downarrow{\scriptstyle p(i)} & & \downarrow{\scriptstyle \mathrm{Map}_*(\mathrm{id}, p(i+j))} \\ Y_\bullet(i) & \longrightarrow & \mathrm{Map}_*(S^j_\bullet, Y_\bullet(i+j)). \end{array} \tag{7.6.23}$$

An **Ω-spectrum** is a spectrum X such that $X \to *$ is a stable fibration.

Lemma 7.6.24. *When $X \to Y$ is a level fibration, the homotopy pullback condition* (7.6.23) *is equivalent to the corresponding condition on the realizations:*

$$\begin{array}{ccc} |X_\bullet(i)| & \longrightarrow & \Omega^j |X_\bullet(i+j)| \\ \downarrow{\scriptstyle |p(i)|} & & \downarrow{\scriptstyle \Omega^j |p(i+j)|} \\ |Y_\bullet(i)| & \longrightarrow & \Omega^j |Y_\bullet(i+j)| \end{array} \tag{7.6.25}$$

Proof. Let $F_\bullet(i)$ be the fiber of $X_\bullet(i) \to Y_\bullet(i)$. Then $F_\bullet(i)$ is a Kan complex, and the square (7.6.23) is a homotopy pullback iff the map $F_\bullet(i) \to \mathrm{Map}_*(S^j_\bullet, F_\bullet(i+j))$ is an equivalence. By Lemma 7.6.14, this is equivalent to $|F_\bullet(i)| \to \Omega^j |F_\bullet(i+j))|$ being an equivalence, which then rearranges to (7.6.25) being a homotopy pullback. □

In particular, a spectrum of simplicial sets $X_\bullet$ is an Ω-spectrum when each spectrum level $X_\bullet(n)$ is a Kan complex, *and* the realization is an Ω-spectrum of topological spaces.

To describe the cofibrations of spectra, as before, we recast spectra as diagrams.

Definition 7.6.26. The **sphere category $\mathbf{S}_\bullet$** is a category with one object for each non-negative integer $n \geq 0$, and whose morphisms are given by the simplicial sets

$$\mathbf{S}_\bullet(m,n) = \begin{cases} S^{n-m}_\bullet & \text{when } n \geq m, \\ * & \text{when } n < m. \end{cases}$$

We compose these simplicial sets by the canonical isomorphisms

$$S^{n-m}_\bullet \wedge S^{p-n}_\bullet \cong S^{p-m}_\bullet$$

as in Definition 5.3.34.

The **symmetric sphere category** $\mathbf{S}^{\Sigma}_{\bullet}$ has the morphisms

$$\mathbf{S}^{\Sigma}_{\bullet}(m,n) = \begin{cases} (\Sigma_n)_+ \wedge_{\Sigma_{n-m}} S^{n-m}_{\bullet} & \text{when } n \geq m, \\ * & \text{when } n < m. \end{cases}$$

or equivalently, a wedge sum of spheres over all the injective maps $\alpha\colon \underline{m} \hookrightarrow \underline{n}$:

$$\mathbf{S}^{\Sigma}_{\bullet}(m,n) = \bigvee_{\alpha\colon \underline{m}\hookrightarrow\underline{n}} S^{\underline{n}-\alpha(\underline{m})}_{\bullet} \qquad \text{when } n \geq m.$$

The composition composes the maps α and identifies their complements, as in Definition 6.1.6.

These are categories enriched in based simplicial sets. This is the same notion as in Definition 5.3.30, except that each mapping object $\mathbf{I}_{\bullet}(a,b)$ is a based simplicial set, instead of a based topological space. In particular, the composition is of the form $\mathbf{I}_{\bullet}(a,b) \wedge \mathbf{I}_{\bullet}(b,c) \to \mathbf{I}_{\bullet}(a,c)$, and the identity of each object a is given by a map $S^0_{\bullet} \to \mathbf{I}_{\bullet}(a,a)$.

We can therefore consider based diagrams $X : \mathbf{I} \to \mathbf{sSet}_*$ as in Definition 5.3.33. These take each object a to a based simplicial set $X_{\bullet}(a)$, and the action of the category is through maps of based simplicial sets $X_{\bullet}(a) \wedge \mathbf{I}_{\bullet}(a,b) \to X_{\bullet}(b)$.

As in Lemma 6.1.8, sequential spectra are the same thing as based diagrams $\mathbf{S}_{\bullet} \to \mathbf{sSet}_*$, and symmetric spectra are the same thing as based diagrams $\mathbf{S}^{\Sigma}_{\bullet} \to \mathbf{sSet}_*$. We can therefore define free spectra as free diagrams in the sense of Definition 5.3.36:

Definition 7.6.27. For any $A_{\bullet} \in \mathbf{sSet}_*$ and $d \geq 0$, the **free spectrum** $F_d A_{\bullet}$ is the free diagram $A_{\bullet} \wedge \mathbf{S}_{\bullet}(d,-)$, or $A_{\bullet} \wedge \mathbf{S}^{\Sigma}_{\bullet}(d,-)$ in the symmetric spectrum case.

A **free cofibration** is an I-cofibration for the class of maps

$$I = \left\{ F_d(\partial \Delta[n]_{\bullet})_+ \to F_d(\Delta[n]_{\bullet})_+ \right\}_{d,n\geq 0}.$$

In particular, it is a retract of a relative cell complex built out of these maps.

Theorem 7.6.28. *Each of the categories* $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{sSet}}$ *and* $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}}$ *has a* ***level model structure*** *in which*

- *the cofibrations are the free cofibrations,*
- *the weak equivalences are the level equivalences, and*
- *the fibrations are the level fibrations.*

Each one also has a ***stable model structure*** *in which*

- *the cofibrations are again the free cofibrations,*
- *the weak equivalences are the stable equivalences, and*
- *the fibrations are the stable fibrations.*

Proof. The proof of the level model structure is identical to the topological case from Theorem 5.3.16, using the Kan model structure on $\mathbf{sSet}_*$ in the place of the Quillen model structure on $\mathbf{Top}_*$. The proof of the stable model structure follows Theorem 6.1.36, in other words it follows the same format as Section 5.6. The analogues of the stability theorems Lemma 6.1.16, Theorem 6.1.31, and Corollary 6.1.32 all hold in the simplicial setting, by taking realizations and using the fact that realization preserves all cofibrations and weak equivalences. □

Remark 7.6.29. This model structure is proper in the sense of Definition 5.2.31, as we see by taking geometric realization (which preserves the cofibrations, fibrations, and weak equivalences) and then recalling that the model structures for $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}}$ and $\mathbf{Sp}^{\Sigma}_{\mathbf{Top}}$ were proper.

Note that the geometric realization of $\mathbf{S}_\bullet$ and $\mathbf{S}^{\Sigma}_\bullet$ gives the topological sphere categories from Definition 5.3.34 and Definition 6.1.6, respectively. From this it follows that the realization of a free cofibration in $\mathbf{Sp}_{\mathbf{sSet}}$ is a free cofibration in $\mathbf{Sp}_{\mathbf{Top}}$. We can also define prolongation from sequential spectra to symmetric spectra, $P^{\Sigma}_{\mathbb{N}}$, as we did in (6.1.13), following the general formula for a left Kan extension from Section 5.4.3.

Proposition 7.6.30. *Geometric realization and prolongation form a commuting diagram (up to canonical isomorphism) of left Quillen equivalences:*

$$\begin{array}{ccc} \mathbf{Sp}^{\mathbb{N}}_{\mathbf{sSet}} & \xrightarrow{P^{\Sigma}_{\mathbb{N}}} & \mathbf{Sp}^{\Sigma}_{\mathbf{sSet}} \\ \downarrow{\scriptstyle |-|} & & \downarrow{\scriptstyle |-|} \\ \mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}} & \xrightarrow{P^{\Sigma}_{\mathbb{N}}} & \mathbf{Sp}^{\Sigma}_{\mathbf{Top}} \end{array}$$

Proof. It is easy to check that each one fits into a Quillen adjunction, and that the realization functors are Quillen equivalences. We have already proven that the prolongation functor along the bottom is a Quillen equivalence in Proposition 6.1.42, so the one along the top must be as well. □

Next we define the smash product following either the explicit definition in Definition 6.2.1 or the more abstract definition in Definition 6.2.17. This has the same properties as before, and the geometric realization gives the topological smash product back, $|X \wedge Y| \cong |X| \wedge |Y|$.

Proposition 7.6.31. *The category* $\mathbf{Sp}^\Sigma_{\mathbf{sSet}}$ *is a symmetric monoidal model category under the smash product. Furthermore the suspension spectrum functor* $\Sigma^\infty : \mathbf{sSet}_* \to \mathbf{Sp}^\Sigma_{\mathbf{sSet}}$ *and the geometric realization functor* $|-|: \mathbf{Sp}^\Sigma_{\mathbf{sSet}} \to \mathbf{Sp}^\Sigma_{\mathbf{Top}}$ *are strong symmetric monoidal Quillen left adjoints.*

Proof. This is mostly a straightforward exercise. The fact that the smash product has the pushout-product property follows the proof of Proposition 6.3.4 but uses Exercise 7.7.16 to calculate the pushout-products of the generating cofibrations in **sSet**. □

Finally, we get model structures on modules, rings, and commutative rings just as in Theorem 6.2.36.

Theorem 7.6.32. *The category of ring spectra* $\mathrm{Alg}(\mathbf{Sp}^\Sigma_{\mathbf{sSet}})$ *and the category of module spectra* $\mathrm{Mod}(R)$ *for any fixed ring* R *have right-induced model structures coming from* $\mathbf{Sp}^\Sigma_{\mathbf{sSet}}$.

In other words, a map of rings or modules is an equivalence or fibration if it is an equivalence or fibration in $\mathbf{Sp}^\Sigma_{\mathbf{sSet}}$ after forgetting the ring or module structure. The same is true for commutative ring spectra if we instead use the positive model structure $\mathbf{Sp}^\Sigma_{\mathbf{sSet},+}$, defined as in Theorem 6.4.2.

Proof. The proof is the same as in Section 6.5. The monoid axiom (Definition 6.3.14) follows easily from the topological case, and the smallness condition is also easier to verify because a lift can be found one nondegenerate simplex at a time. □

Remark 7.6.33. One might wonder if we can define spectrum and symmetric spectrum objects in any symmetric monoidal category $\mathbf{C}$, following the above procedure for $\mathbf{C} = \mathbf{sSet}_*$ in a more formal way. We would have to pick an object $S^1 \in \mathbf{C}$ to play the role of the circle, and define a spectrum to be a sequence of objects X_n of $\mathbf{C}$ with bonding maps $X_n \otimes S^1 \to X_{n+1}$. A symmetric spectrum is defined by adding in Σ_n-actions and asking for the same equivariance condition as in Definition 6.1.2.

In [Hov01], Hovey proves that if $\mathbf{C}$ is left proper and "cellular" (Definition 10.4.66), in addition to being a symmetric monoidal model category, then the category of such spectra $\mathbf{Sp}^\mathbb{N}_\mathbf{C}$ has a stable model structure [Hov01, Def 3.3]. If furthermore the domains of the generating cofibrations of $\mathbf{C}$ are cofibrant, then the category of symmetric spectra $\mathbf{Sp}^\Sigma_\mathbf{C}$ has both a stable model structure and a smash product, making it into a symmetric monoidal model category [Hov01, Thm 8.11].

7.7 Exercises

7.7.1. Recall from Example 7.1.7 that the simplicial set $\Delta[k]_\bullet$ has at level n all non-decreasing sequences in with values in $\{0, 1, \cdots, k\}$. Explain why this agrees with

the functor on $\mathbf{\Delta}^{\mathrm{op}}$ represented by the object $[k]$:

$$\Delta[k]_n = \mathbf{\Delta}([n],[k]).$$

7.7.2. Recall that a **simplicial complex** is a set of vertices V, together with a collection of finite subsets $\mathscr{S} \subseteq P(V)$ that contains all the singletons and is closed under taking subsets. Suppose we fix a total ordering on V. Show that we get a Δ-complex by taking one n-simplex for each set in $\mathscr{S}$ of size $(n+1)$. Conclude that we get a simplicial set with one n-simplex for every finite subset $A \in \mathscr{S}$ and surjective map of totally ordered sets $[n] \to A$.

7.7.3. Prove that a Δ-complex is the same thing as a diagram $\mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \to \mathbf{Set}$, where $\mathbf{\Delta}_{\mathrm{inj}} \subseteq \mathbf{\Delta}$ is the subcategory of injective totally-ordered maps. (These diagrams are sometimes called **semi-simplicial sets**.)

7.7.4. (a) Prove that geometric realization $|-|\colon \mathbf{sSet} \to \mathbf{Top}$ is the left adjoint of singular simplices $S_\bullet(-)\colon \mathbf{Top} \to \mathbf{sSet}$. In other words, for a simplicial set $X_\bullet$ and space Y, continuous maps $|X_\bullet| \to Y$ are the same thing as maps of simplicial sets $X_\bullet \to S_\bullet(Y)$.

(b) Prove that geometric realization of simplicial *spaces*, $|-|\colon \mathbf{sTop} \to \mathbf{Top}$, is also a left adjoint. What is its right adjoint?

7.7.5. Explain why the category of simplicial spaces $\mathbf{sTop} = \mathbf{Top}^{\mathbf{\Delta}^{\mathrm{op}}}$ has a projective model structure as in Theorem 5.3.16, whose cofibrations we call the free cofibrations.

(a) Prove that the geometric realization functor to $\mathbf{Top}$ is left Quillen, so it preserves equivalences of cofibrant objects.

(b) Check that every free cofibration is a Reedy cofibration, but not conversely. So it is harder to prove that realization preserves equivalences of Reedy cofibrant simplicial spaces.

7.7.6. We say that a simplicial set $X_\bullet$ is **discrete** if the face and degeneracy maps are all isomorphisms.

(a) Prove that $X_\bullet$ is discrete iff it is isomorphic to a constant simplicial set, in which every face and degeneracy map is an identity map.

(b) Prove that $X_\bullet$ is discrete iff its geometric realization is a discrete topological space.

7.7.7. Suppose as in Example 7.1.33 we have a sequence

$$(a_0,b_0),(a_1,b_1),\ldots(a_n,b_n),$$

$$0 \le a_0 \le a_1 \le \ldots \le a_n \le k,$$

$$0 \leq b_0 \leq b_1 \leq \ldots \leq b_n \leq \ell.$$

This gives a map $\Delta^n \to \Delta^k \times \Delta^\ell$ by sending the ith vertex to the product of vertices (a_i, b_i), and linearly interpolating for the rest. So in barycentric coordinates,

$$(t_0, t_1, \ldots, t_n) \mapsto \left(\left(\sum_{a_i = j} t_i \right)_{j \in [k]}, \left(\sum_{b_i = j} t_i \right)_{j \in [\ell]} \right).$$

(a) Verify that this is indeed a map $\Delta^n \to \Delta^k \times \Delta^\ell$, taking the ith vertex to the vertex (a_i, b_i). (The vertices are when one of the $t_i = 1$ and the other coordinates are zero.)

(b) Prove that every point in $((u_0, \ldots, u_j), (v_0, \ldots, v_\ell)) \in \Delta^k \times \Delta^\ell$ is in the image of such a map. (Hint: arrange all the numbers $u_0, u_0 + u_1, \ldots, v_0, v_0 + v_1, \ldots$ in order from smallest to largest, and let the t_i be the difference between successive values.)

(c) Prove furthermore that the preimage of $((u_0, \ldots, u_j), (v_0, \ldots, v_\ell))$ is unique, if we stipulate that the sequence is nondegenerate (no repeated tuples) and the point is in the interior of Δ^n (every $t_i > 0$). Use this to show that these maps define a Δ-complex structure on the product space $\Delta^k \times \Delta^\ell$, finishing the proof of Theorem 7.1.31 in this special case.

7.7.8. Prove Theorem 7.1.31 for any product by the following method. Suppose $X_\bullet$ is a simplicial space. Prove that $X_\bullet$ is a coequalizer in simplicial sets

$$\coprod_{k,\ell \geq 0} \Delta[k]_\bullet \times \mathbf{\Delta}([k],[\ell]) \times X_\ell \rightrightarrows \coprod_{k \geq 0} \Delta[k]_\bullet \times X_k \to X_\bullet$$

where the two maps have $\mathbf{\Delta}([k],[\ell])$ act on either $\Delta[k]_\bullet$ or X_ℓ. Multiply by $\Delta[m]_\bullet$ to get the coequalizer

$$\coprod_{k,\ell \geq 0} \Delta[k]_\bullet \times \mathbf{\Delta}([k],[\ell]) \times X_\ell \times \Delta[m]_\bullet \rightrightarrows \coprod_{k \geq 0} \Delta[k]_\bullet \times X_k \times \Delta[m]_\bullet \to X_\bullet \times \Delta[m]_\bullet.$$

Use Exercise 7.7.7 and Proposition 8.2.8 to get a coequalizer of spaces

$$\coprod_{k,\ell \geq 0} \Delta^k \times \mathbf{\Delta}([k],[\ell]) \times X_\ell \times \Delta^m \rightrightarrows \coprod_{k \geq 0} \Delta^k \times X_k \times \Delta^m \to |X_\bullet \times \Delta[m]_\bullet|.$$

Argue that the coequalizer computes $|X_\bullet| \times \Delta^m$. Therefore the product commutes with realization, if one of the simplicial spaces is $\Delta[m]_\bullet$. Using this fact, repeat the above argument with $\Delta[m]_\bullet$ replaced by an arbitrary simplicial space $Y_\bullet$, and conclude that the product always commutes with realization.

7.7.9. (*) Prove Theorem 7.1.31 for any finite limit by the following method.

(a) Adapt Exercise 7.7.8 to show that realization commutes with pullbacks,

$$|X_\bullet \times_{B_\bullet} Y_\bullet| \cong |X_\bullet| \times_{|B_\bullet|} |Y_\bullet|.$$

(b) Show that any functor F that preserves pullbacks and the terminal object must also preserve all finite limits.

7.7.10. Suppose we have a bisimplicial set $X : \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$, as in Example 7.2.5. Prove that the diagonal maps $\Delta^n \to \Delta^n \times \Delta^n$, $t \mapsto (t,t)$, induce a map from the realization of the diagonal $X_n := X_{n,n}$ to the product realization $|X_{\bullet,\bullet}|$, and that this map is a homeomorphism. (You might want to use Milnor's theorem, Theorem 7.1.31.)

7.7.11. Suppose $X_\bullet$ is a simplicial space.

(a) Prove that the degeneracy map $s_0 : X_0 \to X_1$ identifies X_0 with a closed subspace of X_1. (You will need to use the fact that in CGWH spaces, if A is a retract of X then A is identified with a closed subspace of X.)

(b) Prove that the two degeneracy maps $s_0, s_1 : X_1 \to X_2$ give two subspaces of X_2 homeomorphic to X_1, whose intersection is homeomorphic to X_0.

You might now have a good guess as to what should happen in general. We will prove this in Proposition 8.2.18.

7.7.12. Recall from Example 7.1.10 that the nerve of a small category $\mathbf{C}$ is the simplicial set $N_\bullet\mathbf{C} = N_\bullet(*, \mathbf{C}, *)$. Recall from Example 7.4.9 that the classifying space $B\mathbf{C}$ is defined to be the realization of the nerve.

(a) Suppose $\mathbf{C}$ and $\mathbf{D}$ are categories. Prove that $N_\bullet(\mathbf{C} \times \mathbf{D}) \cong (N_\bullet\mathbf{C}) \times (N_\bullet\mathbf{D})$.

(b) Prove that any functor $F : \mathbf{C} \to \mathbf{D}$ gives a map of nerves $N_\bullet\mathbf{C} \to N_\bullet\mathbf{D}$.

(c) Prove that any natural transformation $F \Rightarrow G$ induces a homotopy

$$\Delta[1]_\bullet \times N_\bullet\mathbf{C} \longrightarrow N_\bullet\mathbf{D}.$$

(Hint: The nerve of the category $\{0 \to 1\}$ is isomorphic to $\Delta[1]_\bullet$.)

Conclude that any adjunction between $\mathbf{C}$ and $\mathbf{D}$ gives a homotopy equivalence of classifying spaces

$$B\mathbf{C} = |N_\bullet\mathbf{C}| \simeq |N_\bullet\mathbf{D}| = B\mathbf{D}.$$

7.7.13. Using the previous exercise, show that if $\mathbf{C}$ has an initial or terminal object then $B\mathbf{C}$ is contractible.

7.7.14. For any simplicial space $X_\bullet$, define the **simplicial path space** $PX_\bullet$ by the rule $PX_n := X_{1+n}$, with ith face map given by $d_{1+i} : X_{1+n} \to X_n$ and ith degeneracy map $s_{1+i} : X_{2+n} \to X_{1+n}$. (So the maps d_0 and s_0 are never used.)

(a) Show that $PX_\bullet$ is also the composite of the functor $X_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Top}$ with the functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}$ that "adds 1," sending $[n] \mapsto [1+n]$ and sending each morphism $\alpha\colon [m] \to [n]$ to the corresponding morphism $0 \amalg \alpha\colon [1+m] \to [1+n]$ that sends $0 \mapsto 0$ and acts on the remaining elements by a shifted copy of the function α, in other words $i \mapsto \alpha(i-1)+1$.

(b) Prove that the unique degeneracy maps $X_0 \to X_{1+n}$ give a map from the constant simplicial space on X_0 to $PX_\bullet$, while the composite of face maps $d_1 d_2 \cdots d_n\colon X_{1+n} \to X_0$ give a map from $PX_\bullet$ back to the constant simplicial space on X_0. The composite of these is the identity of X_0.

(c) For each map $\alpha\colon [n] \to [1]$, corresponding to the subset $S = \alpha^{-1}(0) \subseteq \{0,\dots,n\}$, let $0 \amalg S \subseteq \{0,1,\dots,1+n\}$ be the corresponding shifted set that contains 0 and also $i+1$ for each $i \in S$. Let $f_\alpha\colon [1+n] \to [1+n]$ be the map that sends all of $0 \amalg S$ to 0, and is the identity on the rest of $[1+n]$. Prove that the rule $(\alpha, x) \mapsto f_\alpha^*(x)$ defines a map of simplicial sets

$$\Delta[1]_\bullet \times PX_\bullet \to PX_\bullet.$$

(Hint: For each $\beta\colon [m] \to [n]$, you only have to check that $(0 \amalg \beta) \circ f_{\alpha\circ\beta} = f_\alpha \circ (0 \amalg \beta)$, as functions $[1+m] \to [1+n]$.)

(d) Check that the map from part (c) is a simplicial homotopy from the composite $PX_\bullet \to X_0 \to PX_\bullet$ to the identity map of $PX_\bullet$. Conclude that there is a homotopy equivalence $X_0 \simeq |PX_\bullet|$.

(e) Check that $N_\bullet(G,G,X) = PN_\bullet(*,G,X)$, so that part (d) gives a new proof of the bar lemma (Lemma 7.3.11). In fact it is essentially the same proof that we have already given, just written in a different way.

7.7.15. The category **sSet** is a symmetric monoidal category under the Cartesian product $\times$. In this exercise we prove that it is also closed and we describe the internal hom.

(a) For each pair of simplicial sets $Y_\bullet$ and $Z_\bullet$ and each $n \geq 0$, let $\mathrm{Map}_n(Y,Z)$ be the set of maps of simplicial sets

$$\Delta[n]_\bullet \times Y_\bullet \to Z_\bullet.$$

Prove that these assemble into a simplicial set $\mathrm{Map}_\bullet(Y,Z)$, and that there is a bijection between maps of simplicial sets

$$X_\bullet \to \mathrm{Map}_\bullet(Y,Z) \qquad \longleftrightarrow \qquad X_\bullet \times Y_\bullet \to Z_\bullet.$$

Therefore **sSet** is a closed symmetric monoidal category.

(b) Prove that the category of based simplicial sets $\mathbf{sSet}_*$ is also a symmetric monoidal model category, using the smash product of Definition 7.6.6 and the simplicial set of based maps

$$(\mathrm{Map}_*)_n(Y,Z) = \left\{(\Delta[n]_+)_\bullet \wedge Y_\bullet \to Z_\bullet\right\}.$$

7.7.16. Consider the symmetric monoidal category **sSet** with the Cartesian product.

(a) Give a formula for the pushout-product

$$(\partial\Delta[m] \to \Delta[m]) \square (\partial\Delta[n] \to \Delta[n])$$

and prove that the map is a cofibration in the Kan model structure of Theorem 7.1.39. In other words, it is injective at each simplicial level.

(b) Do the same for the pushout-product

$$(\partial\Delta[m] \to \Delta[m]) \square (\Lambda_i[n] \to \Delta[n])$$

and verify it is both a cofibration and a weak equivalence.

(c) Conclude that **sSet** is a symmetric monoidal model category in the sense of Definition 6.3.1. Then prove the same for $\mathbf{sSet}_*$, the category of based simplicial sets.

7.7.17. If $\mathbf{I}$ is a category, prove that the category $\mathbf{sSet}^{\mathbf{I}}$ of $\mathbf{I}$-diagrams in simplicial sets has a projective model structure similar to that of Theorem 5.3.16. (These are sometimes called **simplicial presheaves** on $\mathbf{I}^{\mathrm{op}}$.) Prove that this model structure is Quillen equivalent to the one in Theorem 5.3.16, using Theorem 7.1.39. Then generalize to the case that the mapping objects $\mathbf{I}(i,j)$ are *simplicial sets*, not just sets.

7.7.18. Prove that the underlying set of $B(X,G,Y)$ has a point for each picture as shown, in which the left-hand endpoint is labeled by a point $x \in X$, the right-hand endpoint is labeled by $y \in Y$, and the finitely many points in the middle are labeled by elements $g_i \in G$. (Hint: a point in Δ^n can be specified by numbers $0 \le t_1 \le t_2 \le \cdots \le t_n \le 1$.)

Then generalize this to the categorical bar construction $B(X,\mathbf{I},Y)$.

7.7.19. (Uniqueness of classifying spaces) If G is a topological group, a classifying space for G is a space obtained as a quotient E/G, where E is weakly contractible (i.e. all homotopy groups vanish) and $E \to E/G$ is a principal G-bundle.

(a) Explain why $B(*,G,G)$ can be obtained as a G-CW complex, that is, as a space where we successively attach cells of the form $D^n \times G$, with G-action by $(x,h)g = (x,hg)$.

(b) If $E \to E/G$ is any principal bundle with E contractible, construct a G-equivariant map $B(*,G,G) \to E$, working one cell $D^n \times G$ at a time.

(c) If G is well-based, explain why any such map $B(*,G,G) \to E$ induces a weak equivalence $B(*,G,*) \to E/G$.

This proves that when G is well-based, every classifying space E/G is weakly equivalent to $B(*, G, *)$.

7.7.20. (Equivariant spaces vs parametrized spaces) Let G be a topological group whose underlying space is Quillen cofibrant (a retract of a cell complex). Let $\mathbf{Top}^G$ be the category of spaces with left G-action, and let $\mathbf{Top}_{BG}$ be the comma category of spaces with a map to BG.

(a) Use previous results to explain why $\mathbf{Top}^G$ has a projective model structure, where the weak equivalences and fibrations are determined by forgetting the G-action.

(b) Use previous results to explain why $\mathbf{Top}_{BG}$ also has a model structure where the weak equivalences and fibrations are determined by forgetting the map to BG.

(c) Define a functor $\mathbf{Top}^G \to \mathbf{Top}_{BG}$ by applying $EG \times_G (-)$. Show that this is left Quillen. What is its right adjoint?

(d) Prove that in fact $EG \times_G (-)$ is a Quillen equivalence. You might find Proposition 7.3.17 helpful.

We will see later that every connected space X is equivalent to BG for some well-based topological group G. This shows that "parametrized" spaces are the same thing as "equivariant" spaces, at least when everything is done up to weak equivalence.

7.7.21. (a) Verify the claim of Example 7.4.4 that the coend $X \times_{\mathbf{I}} \mathbf{I}(-, c)$ is isomorphic to $X(c)$.

(b) Verify the claim of Example 7.5.9 that the end $\mathrm{Map}^{\mathbf{I}}(\mathbf{I}(c, -), X)$ is isomorphic to $X(c)$.

7.7.22. If $X_\bullet$ is a simplicial space, then applying $\mathrm{Map}(-, Y)$ to each level gives a cosimplicial space $\mathrm{Map}(X_\bullet, Y)$. Verify the claim in Example 7.5.3 by proving that the totalization $\mathrm{Tot}(\mathrm{Map}(X_\bullet, Y))$ is homeomorphic to the mapping space $\mathrm{Map}(|X_\bullet|, Y)$. You may find Exercise 1.7.27 (a) and (d) helpful.

7.7.23. Prove that we can form limits of cosimplicial spaces by taking the limit at each level, and that totalization commutes with limits. (Hint: express totalization using only right adjoints!) In particular, it commutes with products and pullbacks,

$$\mathrm{Tot}(X^\bullet \times Y^\bullet) \cong \mathrm{Tot}(X^\bullet) \times \mathrm{Tot}(Y^\bullet), \qquad \mathrm{Tot}(X^\bullet \times_{B^\bullet} Y^\bullet) \cong \mathrm{Tot}(X^\bullet) \times_{\mathrm{Tot}(B^\bullet)} \mathrm{Tot}(Y^\bullet).$$

7.7.24. Prove the bar lemma for categories (Lemma 7.4.20) by adapting the proof of the bar lemma for groups (Lemma 7.3.11).

7.7.25. Adapt the proof of Theorem 7.3.14 to prove that the coend $- \times_{\mathbf{I}} -$ preserves equivalences when the diagram on the left is of the form $B(X, \mathbf{I}, \mathbf{I})$ or the diagram on the right is of the form $B(\mathbf{I}, \mathbf{I}, Y)$. This is the main part of the proof of Theorem 7.4.22.

7.7.26. In this exercise we generalize Theorem 7.3.10 by allowing the group G to change as well.

(a) Check that any group homomorphism $G \to G'$ induces a map of bar constructions $B(*, G, *) \to B(*, G', *)$.

(b) Define the notion of "homomorphism of triples" $(X, G, Y) \to (X', G', Y')$. Check that each homomorphism of triples induces a map $B(X, G, Y) \to B(X', G', Y')$.

(c) Verify that if each of the three maps $X \xrightarrow{\sim} X'$, $G \xrightarrow{\sim} G'$, and $Y \xrightarrow{\sim} Y'$ are weak equivalences and both G and G' are well-based, then the induced map

$$B(X, G, Y) \xrightarrow{\sim} B(X', G', Y')$$

is also a weak equivalence.

(d) Generalize to the case that $F : \mathbf{I} \xrightarrow{\sim} \mathbf{I}'$ is a pointwise equivalence of well-based categories, and $X \xrightarrow{\sim} X'$ and $Y \xrightarrow{\sim} Y'$ are equivalences of diagrams over this map of categories, as in Theorem 7.4.17. Prove that the resulting map

$$B(X, \mathbf{I}, Y) \xrightarrow{\sim} B(X', \mathbf{I}', Y')$$

is a weak equivalence.

7.7.27. (*) Suppose that $\mathbf{I}$ and $\mathbf{J}$ are well-based topological categories and $F : \mathbf{I} \to \mathbf{J}$ is a **Dwyer-Kan equivalence**: it is a weak equivalence on mapping spaces

$$\mathbf{I}(i, i') \xrightarrow{\sim} \mathbf{J}(F(i), F(i'))$$

and furthermore each object $j \in \mathbf{J}$ is homotopy equivalent to an object of the form $F(i)$. In other words, there are morphisms $f \in \mathbf{J}(F(i), j)$ and $g \in \mathbf{J}(j, F(i))$ such that $g \circ f$ is homotopic to the identity in the space $\mathbf{J}(F(i), F(i))$, and similarly $f \circ g$ is homotopic to the identity in the space $\mathbf{J}(j, j)$.

(a) Use the bar lemma and the above homotopy equivalence to prove that for every $j \in \mathbf{J}$, the bar construction $B(*, \mathbf{I}, \mathbf{J}(F(-), j))$ is contractible.

(b) More generally, if $X' : \mathbf{J} \to \mathbf{Top}$ is a diagram on $\mathbf{J}$, and $X \xrightarrow{\sim} F^* X'$ is any diagram on $\mathbf{I}$ that is equivalent to the pullback of X', prove that there is an equivalence of $\mathbf{J}$-diagrams

$$B(X, \mathbf{I}, \mathbf{J}) \xrightarrow{\sim} B(X', \mathbf{J}, \mathbf{J}) \xrightarrow{\sim} X'.$$

(c) Now if $Y' \colon \mathbf{J}^{\mathrm{op}} \to \mathbf{Top}$ is a contravariant diagram on $\mathbf{J}$, and $Y \xrightarrow{\sim} F^*Y'$, give a string of weak equivalences

$$B(X,\mathbf{I},Y) \simeq B(X,\mathbf{I},B(\mathbf{J},\mathbf{J},Y')) \cong B(B(X,\mathbf{I},\mathbf{J}),\mathbf{J},Y') \simeq B(X',\mathbf{J},Y').$$

(d) Use the universal property of left-derived functors (Proposition 3.5.2) to argue that this equivalence must agree up to homotopy with the canonical map $B(X,\mathbf{I},Y) \to B(X',\mathbf{J},Y')$, since it lies over the canonical map $X \times_{\mathbf{I}} Y \to X' \times_{\mathbf{J}} Y'$.

This extends the homotopy invariance of the bar construction (Theorem 7.4.17) to Dwyer-Kan equivalences.

7.7.28. Suppose that $\mathbf{I}$ is a category enriched in *based* spaces. This means that the morphism spaces $\mathbf{I}(a,b)$ all have basepoints $*$, and composing anything with $*$ gives $*$. In other words, the composition maps are of the form

$$\mathbf{I}(a,b) \wedge \mathbf{I}(b,c) \to \mathbf{I}(a,c).$$

(a) Check that we can form a version of the categorical bar construction in which we use the smash product, provided the action of $\mathbf{I}$ on the diagrams X and Y also respects basepoints. Call this version $B^{\wedge}(X,\mathbf{I},Y)$.

(b) Explain why there is no such thing as the based nerve $N_\bullet\mathbf{I}$ or the based homotopy colimit $B^{\wedge}(X,\mathbf{I},*)$. Or, if we do define such a thing, how it has to always be the one-point space.

This explains why, for homotopy colimits, we only ever consider actions of *unbased* categories on *based* spaces.

7.7.29. Let $\mathbf{I}$ be an (unbased) topological category. An $(\mathbf{I},\mathbf{I})$-**bimodule** X consists of spaces $X(a,b)$ for every pair of objects $i,j \in \mathbf{I}$, and action maps

$$\mathbf{I}(a,b) \times X(b,c) \to X(a,c), \qquad X(a,b) \times \mathbf{I}(b,c) \to X(a,c)$$

that are each associative and unital on their own, and in addition the left action commutes with the right action. We define the **cyclic bar construction** $B^{\mathrm{cyc}}(\mathbf{I},X)$ to be the realization of the **cyclic nerve**, the simplicial space whose nth level is the product

$$N_n(\mathbf{I},X) = \coprod_{c_0,c_1,\cdots,c_n \in \mathrm{ob}\,\mathbf{I}} X(c_n,c_0) \times \mathbf{I}(c_0,c_1) \times \mathbf{I}(c_1,c_2) \times \cdots \times \mathbf{I}(c_{n-1},c_n),$$

with the same face and degeneracy maps as in Definition 7.3.7, except that the last face map takes the final morphism $f_n \in \mathbf{I}(c_{n-1},c_n)$ and pulls it around to the left to act on $X(c_n,c_0)$, giving a point in $X(c_{n-1},c_0)$.

(a) Explain how the two-sided bar construction $B(X, \mathbf{I}, Y)$ is a special case. The cyclic bar construction is *actually the more general one*.

(b) Prove that $\mathbf{I}$ gives a bimodule over itself in a canonical way. If we use this bimodule then we call the cyclic bar construction $B^{\mathrm{cyc}}(\mathbf{I}) = B^{\mathrm{cyc}}(\mathbf{I}, \mathbf{I})$.

(c) Prove the homotopy invariance theorem (Theorem 7.4.17) for the cyclic bar construction. If you're feeling brave, prove it using Dwyer-Kan equivalences, as in Exercise 7.7.27.

(d) Show by example that the cyclic nerve $B^{\mathrm{cyc}}(\mathbf{I})$ behaves differently from the nerve, in that it does not send adjunctions of categories to equivalences.

7.7.30. Suppose X is a cell complex in the usual sense in **Top**, and we have a diagram $\mathbf{I} \to \mathbf{Top}$ in which each space X_i is a subcomplex of X, and each map $X_i \to X_j$ is an inclusion of subcomplexes. For simplicity, let us also assume that the union of all of the subcomplexes in the diagram is X itself.

For each cell $D_\alpha \to X$, let $\mathbf{I}_\alpha \subseteq \mathbf{I}$ be the subcategory of those complexes that contain the cell D_α.

(a) If $B\mathbf{I}_\alpha$ is connected for all α, prove that the canonical map $\operatorname{colim}_{i \in \mathbf{I}} X_i \to X$ is a homeomorphism.

(b) (*) If $B\mathbf{I}_\alpha$ is contractible for all α, prove that the canonical map $\operatorname{hocolim}_{i \in \mathbf{I}} X_i \to X$ is a weak homotopy equivalence.

You may find it helpful to use transfinite induction as in Exercise 5.8.42. Assume that X is obtained as a transfinite composition of attaching maps, and for each stage α in that process, let $X_i^{(\alpha)}$ be the corresponding subcomplex of X_i. How does the map $\operatorname{hocolim}_{i \in \mathbf{I}} X_i^{(\alpha)} \to X^{(\alpha)}$ change as we increment α?

(c) Prove that $B\mathbf{I}_\alpha$ is contractible if the intersection of any two of the subcomplexes $X_i \cap X_j$ is also in the diagram, or alternatively, if the union of any two of the subcomplexes $X_i \cup X_j$ is in the diagram. Therefore in either of these two cases, X is homeomorphic to the colimit and weakly equivalent to the homotopy colimit of the subcomplexes X_i in the diagram.

(d) Show by example that these theorems fail when the assumptions on $B\mathbf{I}_\alpha$ are dropped.

7.7.31. If $\mathbf{I}$ fails to be well-based, prove that there is a well-based category $\mathbf{I}'$ and a pointwise equivalence $\mathbf{I}' \xrightarrow{\sim} \mathbf{I}$. Adapt the proof of Theorem 7.3.14 to show that, in this case, $B(X, \mathbf{I}', *)$ is the homotopy colimit (left-derived colimit) of X. (In the case where $\mathbf{I} = G$ and $\mathbf{I}' = G'$, you just need to use G' everywhere instead of G in the proof.)

7.7.32. Let $\mathbf{I}$ be a topological category and $X : \mathbf{I} \to \mathbf{Top}$. Suppose that every space $\mathbf{I}(i,j)$ and every space $X(i)$ is a retract of a cell complex, in other words they are all cofibrant in the Quillen model structure on topological spaces (Theorem 5.2.5).

(a) Prove that the diagram $B(X,\mathbf{I},\mathbf{I})$ is cofibrant in the projective model structure from Theorem 5.3.16.

(b) Conclude that the Bousfield–Kan homotopy colimit $B(X,\mathbf{I},*) = \operatorname{colim} B(X,\mathbf{I},\mathbf{I})$ is the left-derived functor of the colimit. This gives a slightly weaker proof of Theorem 7.4.22 that only holds when all the spaces involved are cofibrant.

(c) Show furthermore that the coend $\mathbf{Top}^{\mathbf{I}} \times \mathbf{Top}^{\mathbf{I}^{\mathrm{op}}} \to \mathbf{Top}$ is a Quillen tensor. What are its right adjoints?

(d) Use this to prove that the somewhat thicker bar construction $B(X,\mathbf{I},B(\mathbf{I},\mathbf{I},Y))$ is the left-derived functor of the coend $X \times_{\mathbf{I}} Y$ when X and Y are both pointwise cofibrant. At this point the conclusion is quite a bit weaker than what we were able to get in Theorem 7.4.22.

7.7.33. (*) Let $\mathbf{I}$ and $\mathbf{J}$ be discrete categories, and $\alpha : \mathbf{I} \to \mathbf{J}$ a functor. For each $d \in \mathbf{J}$, define the comma category $(\alpha \downarrow d)$ to have an object for each $c \in \mathbf{I}$ and morphism $p : \alpha(c) \to d$, and a morphism for each morphism $f : c_0 \to c_1$ in $\mathbf{I}$ that forms a commuting triangle

$$\begin{array}{ccc} \alpha(c_0) & \xrightarrow{\alpha(f)} & \alpha(c_1) \\ & {\scriptstyle p_0}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & d. & \end{array}$$

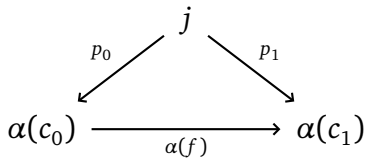
From Exercise 5.8.13, this is a category with a forgetful functor $(\alpha \downarrow d) \to \mathbf{I}$, so that every diagram X on $\mathbf{I}$ becomes a diagram on $(\alpha \downarrow d)$. Furthermore, the colimit of this diagram along $(\alpha \downarrow d)$ is the left Kan extension $\alpha_! X = X \times_{\mathbf{I}} \mathbf{J}$ evaluated at $d \in \mathbf{J}$.

(a) Recall how the coend $X \times_{\mathbf{I}} \mathbf{J}(\alpha(-), j)$ from Example 7.4.5 is isomorphic to the colimit over the comma category $\operatorname*{colim}_{\alpha(c)\to d} X(c)$.

(b) Prove that the homotopy colimit of X over $(\alpha \downarrow d)$ is isomorphic to the bar construction $B(X,\mathbf{I},\mathbf{J}(\alpha(-),d))$. In other words, we can explicitly compute any homotopy left Kan extension using only homotopy colimits.

7.7.34. (*) As in Exercise 7.7.33, assume that $\mathbf{I}$ and $\mathbf{J}$ are discrete categories. Define the comma category $(j \downarrow \alpha)$ to have an object for each $i \in \mathbf{I}$ and morphism $p : j \to \alpha(c)$,

and a morphism for each morphism $f : c_0 \to c_1$ in $\mathbf{I}$ that forms a commuting triangle



(a) Show that the end $\mathrm{Map}^{\mathbf{I}}(\mathbf{J}(j, \alpha(-)), X)$ from Example 7.5.34 is isomorphic to the limit over the comma category $\lim_{j \to \alpha(c)} X(c)$.

(b) Prove that the homotopy limit of X over $(j \downarrow \alpha)$ is isomorphic to the cobar construction $F(\mathbf{J}(j, \alpha(-)), \mathbf{I}, X)$. In other words, we can explicitly compute any homotopy right Kan extension using only homotopy limits.

7.7.35. A **partially ordered set** or **poset** is a set C with a relation $\leq$ on C that is

- reflexive ($a \leq a$ always),
- antisymmetric (if $a \leq b$ and $b \leq a$ then $a = b$), and
- transitive (if $a \leq b$ and $b \leq c$ then $a \leq c$).

(a) Explain how this is equivalent to making C into a category $\mathbf{C}$, such that each object $a \in \mathbf{C}$ has only the identity map from a to a, and for any two distinct objects $a \neq b$, there is at most one map between a and b (either from a to b, or from b to a, but not both).

(b) If $\mathbf{C}$ is a poset, prove that in the nerve $N_\bullet \mathbf{C}$, any face of a nondegenerate simplex is nondegenerate. Explain how that makes the classifying space $B\mathbf{C} = |N_\bullet \mathbf{C}|$ into a Δ-complex.

7.7.36. Let us make the following nonstandard definition. A **simple directed tree** is a category $\mathbf{I}$ that

- comes from a poset, so there is a unique morphism $a \to b$ when $a \leq b$ and no morphism otherwise,
- is directed, so for any two objects a, b there is a third object c such that $a \leq c$ and $b \leq c$
- for any two objects $a \leq b$ there are only finitely many x such that $a \leq x \leq b$, and they form a totally ordered set:

$$a \leq x_1 \leq \cdots \leq x_n \leq b.$$

We say that a and b are **adjacent** if there are no such objects x.

Assume that $\mathbf{I}$ is a simple directed tree. For any diagram $X : \mathbf{I} \to \mathbf{Top}$, define the **simple homotopy colimit** $S(X, \mathbf{I}, *)$ to be the disjoint union

$$\left(\coprod_{a \in \mathbf{I}} X(a)\right) \amalg \left(\coprod_{a<b \text{ adjacent}} X(a) \times I\right)$$

modulo the relation that glues one end of the cylinder $X(a) \times I$ to $X(a)$, and the other end to $X(b)$ using the morphism $a \to b$.

(a) For each object $c \in \mathbf{I}$, show that the subcategory $\mathbf{I}_{\leq c} = \mathbf{I}/c = (\mathbf{I} \downarrow c)$ is also a simple directed tree, and that $S(X, \mathbf{I}_{\leq c}, *)$ deformation retracts onto $X(c)$.

(b) Define a diagram $S(X, \mathbf{I}, \mathbf{I})$ whose value at $c \in \mathbf{I}$ is the simple homotopy colimit $S(X, \mathbf{I}_{\leq c}, *)$. Show that this diagram is equivalent to the original diagram X, and that its colimit is $S(X, \mathbf{I}, *)$.

(c) Show that equivalences of diagrams $X \to X'$ induce equivalences $S(X, \mathbf{I}, *) \to S(X', \mathbf{I}, *)$. (Hint: Express $S(X, \mathbf{I}, *)$ as a homotopy pushout.)

(d) If every space $X(a)$ is Quillen cofibrant, show that $S(X, \mathbf{I}, \mathbf{I})$ is cofibrant in the projective model structure. (Try starting with the free diagrams $F_a X(a)$ for all a, and then attach $F_a(X(a) \times D^1)$ along $F_a(X(a) \times S^0)$.)

(e) Conclude that $S(X, \mathbf{I}, *)$ is the homotopy colimit of X. Show by example that this fails in the more general case that $\mathbf{I}$ is any discrete category.

In particular, if $\mathbf{I}$ comes from the natural numbers $(\mathbf{N}, \leq)$, this shows that the mapping telescope construction is the homotopy colimit, and is therefore equivalent to the much larger Bousfield–Kan homotopy colimit $B(X, \mathbf{I}, *)$.

7.7.37. Prove that the bar construction on the commuting square category has the form described in Example 7.4.15, and that the cobar construction has the form described in Example 7.5.18. Can you generalize this to n-dimensional cubes?

7.7.38. Recall from Exercise 6.6.31 that a category $\mathbf{I}$ is filtered if for any diagram in $\mathbf{I}$ with finitely many objects and morphisms, there is a single object c receiving maps from all the objects of the diagram, commuting with all the morphisms of the diagram. We call a homotopy colimit along such a category a **filtered homotopy colimit**. Prove that the formulas from Example 7.4.29

$$H_k\left(\operatorname*{hocolim}_{c \in \mathbf{I}} X_c\right) \cong \operatorname*{colim}_{c \in \mathbf{I}} H_k(X_c), \qquad \pi_k\left(\operatorname*{hocolim}_{c \in \mathbf{I}} X_c\right) \cong \operatorname*{colim}_{c \in \mathbf{I}} \pi_k(X_c)$$

hold for any filtered homotopy colimit of spaces.

7.7.39. (Introduction to Group Homology.) Let G be a discrete group. Define the **group homology** $H_*(G; \mathbb{Z})$ to be the homology of the classifying space $BG = B(*, G, *)$. Explain why this agrees with the definition from homological algebra as $\operatorname{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z})$.

7.7.40. Let G be a discrete group and A an abelian group with a G-action through homomorphisms, in other words a $\mathbb{Z}[G]$-module. Define the **group homology with coefficients** $H_*(G;A)$ to be the homology of the space $BG = B(*,G,*)$ with local coefficients given by the bundle of abelian groups $B(*,G,A) \to BG$. Explain why this agrees with $\mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z},A)$.

7.7.41. Compute the group homology of:

(a) The integers $\mathbb{Z}$.

(b) The cyclic group $C_n \cong \mathbb{Z}/n$. (You might find it helpful to use the Serre spectral sequence fiber sequence $B\mathbb{Z} \to BC_n \to B^2\mathbb{Z}$, in other words $S^1 \to BC_n \to \mathbb{CP}^\infty$. Move this one to chapter 12.)

(c) The rationals $\mathbb{Q}$. (You might find Exercise 2.7.41 helpful.)

7.7.42. Explain why group homology satisfies a Künneth theorem for products of groups:

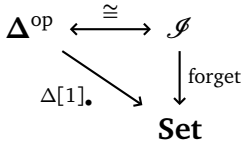
$$0 \longrightarrow \bigoplus_{i+j=n} H_i(G) \otimes H_j(H) \xrightarrow{\times} H_n(G \times H) \longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(G), H_j(H)) \longrightarrow 0.$$

7.7.43. Use Exercise 7.7.12 to show that if two homomorphisms $G \rightrightarrows H$ differ by conjugation by a single element $h \in H$, then the resulting maps $BG \rightrightarrows BH$ are homotopic as maps of unbased spaces. (They can't be homotopic as maps of based spaces, because that would force them to act the same way on π_1.)

7.7.44. Let $\mathscr{I}$ be the category with objects $[n]$ for $n \geq 0$, but where $[n]$ now denotes the totally ordered set $\{*_L, 1, 2, \ldots, n, *_R\}$, where the two endpoints are "basepoints." The maps are maps of totally ordered sets that preserve both basepoints. There is a forgetful functor $\mathscr{I} \to \mathbf{Set}$ that forgets the basepoints and the ordering condition.

(a) Prove that $\mathbf{I} \cong \mathbf{\Delta}^{\mathrm{op}}$, by taking each map of totally-ordered sets $\{0,\ldots,m\} \to \{0,\ldots,n\}$ to the backwards map it induces on the *gaps* between the points of the totally-ordered sets.

(b) Re-interpret this isomorphism as the standard 1-simplex $\Delta[1]_\bullet$ of Example 7.1.7, by showing that the following diagram of functors commutes up to a natural isomorphism.



(c) Re-interpret the bar construction using the category $\mathscr{I}$ instead of $\mathbf{\Delta}^{\mathrm{op}}$.

Chapter 8

Simplicial model categories and homotopy colimits of spectra

Now that we have introduced simplicial sets, we can begin to study categories that have the structure of a simplicial set baked into them. There are two versions of this that we will cover, **simplicial model categories** in this chapter, and **∞-categories** in chapter Chapter 10.

These "categories with extra simplicial structure" are a natural setting for all kinds of questions about homotopy coherence. **Homotopy coherence** is the philosophy that, instead of making a definition by naming elements and naming equations that they must satisfy, we only ask for *homotopies*, and then homotopies between them, and so on.

For example, the homotopy-coherent version of the colimit is the homotopy colimit, at least the one built by the bar construction in Section 7.4.5. Indeed, if X is a diagram of spaces on the category $\mathbf{I}$, a map from the homotopy colimit $B(X, \mathbf{I}, *)$ to another space Y is given first by maps $X(i) \to Y$ for each object i. But then for each morphism $f : i \to j$, the two resulting maps $X(i) \rightrightarrows Y$ are related by a *homotopy* $X(i) \times \Delta^1 \to Y$, rather than being equal. The 1-simplices of the bar construction provide this homotopy. Then for each pair of composable maps beginning at $X(i)$, we get a map $X(i) \times \Delta^2 \to Y$, which is a homotopy between some of the homotopies that we already chose. And so on.

As this example indicates, simplicial sets are an essential tool for understanding homotopy coherence. They give a natural framework for handling infinite systems of coherent homotopies, by working one level of the simplicial set at a time. This is why our categories need some kind of extra simplicial structure if we want to study homotopy coherence seriously.

In this chapter, we will focus on the framework of a simplicial model category. This is a model category $\mathbf{C}$ with a Quillen tensor $\otimes : \mathbf{sSet} \times \mathbf{C} \to \mathbf{C}$ that makes $\mathbf{C}$ into a "module" over simplicial sets. In particular, it has

- a way to multiply simplicial sets $K_\bullet$ with objects $X \in \mathbf{C}$,
- a simplicial set of maps between any pair of objects X and Y in $\mathbf{C}$, and
- a way to form a mapping object from a simplicial set $K_\bullet$ to an object $Y \in \mathbf{C}$.

Every model of spaces and spectra that we have encountered so far is a simplicial model category. And chain complexes are almost a simplicial model category.

The first bullet point above allows us to take products $\Delta^n \otimes X(i)$ for any object $X(i) \in \mathbf{C}$, and this is exactly what we need to define the bar construction $B(X, \mathbf{I}, *)$ for any diagram $X : \mathbf{I} \to \mathbf{C}$. We therefore get an explicit construction of the homotopy colimit in every simplicial model category, similar to the construction we gave for unbased spaces in Chapter 7.

This is a tremendous labor-saving device. Rather than construct the homotopy colimit in each model of spaces or spectra separately (simplicial or topological, symmetric or orthogonal), we give one construction that will always work. We can jump back and forth between the different categories without having to re-prove everything from scratch.

The homotopy colimit is one of the simplest examples of a homotopy-coherent construction, but it is not the only one. In the next chapter we introduce two more, **operads** and **homotopical group completions**, and use them to produce new examples of spectra out of categorical data.

8.1 Simplicial model categories

8.1.1 Definition and examples

Definition 8.1.1. A **simplicial model category** is a model category $\mathbf{C}$ (Definition 5.2.4) with a Quillen tensor (Definition 5.5.13) of the form

$$\otimes : \mathbf{sSet} \times \mathbf{C} \to \mathbf{C},$$

where **sSet** has the Kan model structure from Theorem 7.1.39. We require that

- $* \otimes X \cong X$, where $*$ denotes the constant simplicial set that is $*$ at every level, and
- $(A \times B) \otimes X \cong A \otimes (B \otimes X)$ for any simplical sets A and B.

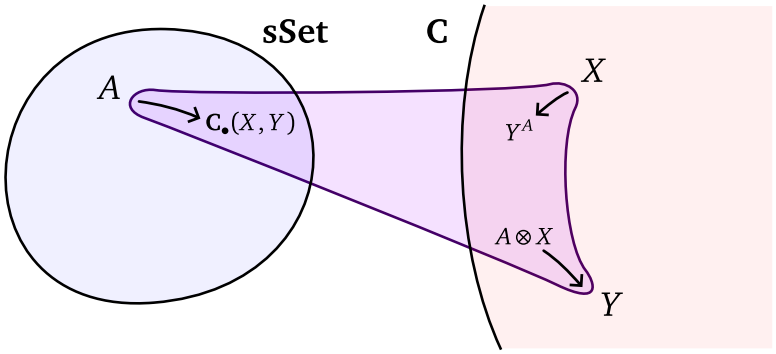
Moreover, we require that the above isomorphisms are coherent: if we apply them several times to go from $(A_1 \times \cdots \times A_n) \otimes X$ back to itself, the isomorphism thus produced is always the identity map.[1]

[1] We are beginning to drop the convention of always writing simplicial sets with a bullet $A_\bullet$, and sometimes just writing A instead.

We call the right adjoints the **cotensor** and **simplicial hom**

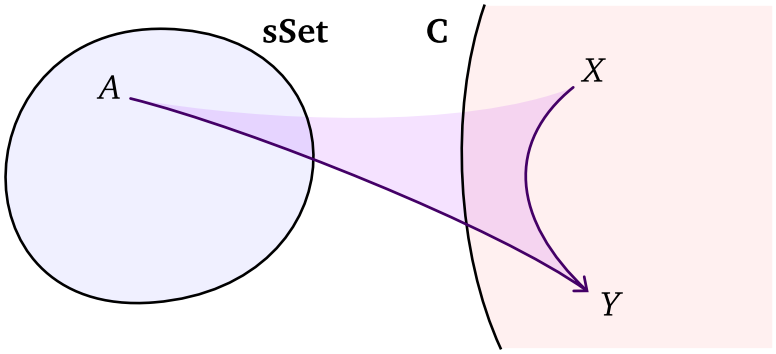
$$(-)^{(-)} \colon \mathbf{sSet}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C}$$
$$\mathbf{C}_\bullet(-,-) \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{sSet}.$$

(They are both functors, because of the magic of the Yoneda Lemma!) So for any $A \in \mathbf{sSet}$ and $X \in \mathbf{C}$ there is a cotensor $X^A \in \mathbf{C}$, and for any two objects $X, Y \in \mathbf{C}$ there is a simplicial set of maps $\mathbf{C}_\bullet(X,Y) \in \mathbf{sSet}$. The adjunctions mean that the following three things are all given by the same data:



- a morphism $A \otimes X \to Y$ in $\mathbf{C}$,
- a morphism $X \to Y^A$ in $\mathbf{C}$, or
- a map of simplicial sets $A_\bullet \to \mathbf{C}_\bullet(X,Y)$.

Informally, you can think of this as giving three equivalent definitions for a "bimorphism" from (A,X) to Y as shown.



The definition of a simplicial model category is usually easy to check, except for the coherence. The following lemma makes that part easier. Recall from Definition 6.3.1 that a symmetric monoidal model category $\mathbf{C}$ is a model category that is also (closed)

symmetric monoidal, such that the tensor product $\boxtimes$ is a Quillen tensor, and furthermore $QI \otimes QX \to I \otimes QX$ is always a weak equivalence.

Lemma 8.1.2. *If* $\mathbf{C}$ *is a symmetric monoidal model category with tensor* $\otimes$*, and* $F : \mathbf{sSet} \to \mathbf{C}$ *is a strong symmetric monoidal Quillen left adjoint, then* $\mathbf{C}$ *is a simplicial model category with tensor*

$$(K, X) \mapsto F(K) \otimes X.$$

The definition of a strong symmetric monoidal Quillen left adjoint is in Definition 6.3.8. When the domain category is **sSet**, this is just a functor that is both a Quillen left adjoint, and strong symmetric monoidal.

Proof. Since F preserves cofibrations and acyclic cofibrations, and $\otimes$ is a Quillen tensor, it follows immediately that the new tensor $F(-) \otimes -$ is also Quillen. The coherence of the isomorphisms follows from the coherence for the product $\otimes$ and for the symmetric monoidal functor F. □

Definition 8.1.3. A **symmetric monoidal simplicial model category**, or "SM simplicial model category" for short, is any category $\mathbf{C}$ satisfying the hypotheses of Lemma 8.1.2. So it is both a symmetric monoidal category and a simplicial model category in a compatible way.

We are now ready to describe a whole bunch of examples.

Example 8.1.4. **sSet** is a simplicial model category using the product:

$$K_\bullet \otimes X_\bullet := K_\bullet \times X_\bullet.$$

The cotensor and simplicial hom are both given by the functor $\mathrm{Map}_\bullet(-,-)$ from Exercise 7.7.15. This is because the category **sSet** is a symmetric monoidal model category by Exercise 7.7.16, and the identity functor **sSet** $\to$ **sSet** is (trivially) strong symmetric monoidal and a Quillen left adjoint.

Example 8.1.5. **Top** is a simplicial model category with tensor

$$K_\bullet \otimes X := |K_\bullet| \times X.$$

The cotensor is $\mathrm{Map}(|K_\bullet|, Y)$, and the simplicial hom is $S_\bullet \mathrm{Map}(X, Y)$, where $S_\bullet$ denotes singular simplices. This is because the geometric realization functor $|-|$: **sSet** $\to$ **Top** is strong symmetric monoidal, using the Cartesian product for both categories, and is a Quillen left adjoint.

Example 8.1.6. $\mathbf{Top}_*$ is a simplicial model category with tensor

$$K_\bullet \otimes X := |K_\bullet|_+ \wedge X.$$

The cotensor is $\mathrm{Map}_*(|K_\bullet|_+, Y) \cong \mathrm{Map}(|K_\bullet|, Y)$, and the simplicial hom is $S_\bullet \mathrm{Map}_*(X, Y)$. This is because the disjoint basepoint functor $\mathbf{Top} \to \mathbf{Top}_*$ is strong symmetric monoidal, $(A_+) \wedge (B_+) \cong (A \times B)_+$.

Example 8.1.7. Similarly $\mathbf{sSet}_*$ is a simplicial model category with tensor $(K_\bullet)_+ \wedge X_\bullet$.

Example 8.1.8. The categories of diagram spectra $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}}$, $\mathbf{Sp}^{\Sigma}_{\mathbf{Top}}$, and $\mathbf{Sp}^{O}_{\mathbf{Top}}$ are all simplicial model categories. The tensoring is

$$K_\bullet \otimes X := (|K_\bullet|_+) \wedge X,$$

which smashes each level of X with the based space $|K_\bullet|_+$. In $\mathbf{Sp}^\Sigma$ or $\mathbf{Sp}^O$, we could also write this as $(\Sigma^\infty_+ |K_\bullet|) \wedge X$.

For symmetric and orthogonal spectra, this follows from Lemma 8.1.2. For sequential spectra we need a different argument, because we don't have a symmetric monoidal category. We define the tensor directly and check that it is a Quillen tensor, and then observe that the structure isomorphisms such as $A \otimes (B \otimes X) \cong (A \times B) \otimes X$ are given at each spectrum level by the same isomorphisms in $\mathbf{Top}_*$. Since they were coherent in $\mathbf{Top}_*$, they are therefore coherent in $\mathbf{Sp}^{\mathbb{N}}$ as well.

Example 8.1.9. The categories of spectra of simplicial sets, $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{sSet}}$ and $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}}$, from Definition 7.6.15 and Definition 7.6.16 are simplicial model categories. The tensoring takes the smash product (Definition 7.6.6) of $(K_\bullet)_+$ with each level of the spectrum.

Example 8.1.10. For all five of the above categories of diagram spectra, if we take the *positive* stable model structure as in Theorem 6.4.2, we still get a simplicial model category, with the same tensor as before. So for instance $\mathbf{Sp}^O_+$ is a simplicial model category. This doesn't follow from Lemma 8.1.2, because the functor $(\Sigma^\infty_+ |K_\bullet|)$ is no longer left Quillen, but we can check directly that $(\Sigma^\infty_+ |K_\bullet|) \wedge X$ is still a Quillen tensor.

Example 8.1.11. The category of EKMM spectra $\mathscr{M}$ from Theorem 6.4.7 is a simplicial model category [EKMM97, VII.2.8]. The tensoring takes the smash product with the EKMM spectrum $\mathbb{N}(\Sigma^\infty_+ |K_\bullet|)$. (As above, this works even though $\mathbb{N}\Sigma^\infty_+(-)$ is not left Quillen!)

Example 8.1.12. If $\mathbf{I}$ is a topological category, category of diagrams $\mathbf{Top}^{\mathbf{I}}$ is a simplicial model category, using the tensor at each term of the diagram

$$(K_\bullet \otimes X)(c) := |K_\bullet| \times X(c).$$

As in the example of spectra, we prove this by leveraging the structure we've already proven on $\mathbf{Top}$. We get a similar simplicial model category structure on diagrams of based spaces $\mathbf{Top}^{\mathbf{I}}_*$ when $\mathbf{I}$ is enriched in based spaces.

Example 8.1.13. If $\mathbf{I}$ is an ordinary category, or a category enriched in simplicial sets (Proposition 8.1.21), we can form the the category of diagrams $\mathbf{sSet}^{\mathbf{I}}$ as in Definition 5.3.5. This has a projective model structure by Exercise 7.7.17. It is a simplicial model category, using the tensor at each term of the diagram

$$(K_\bullet \otimes X)(c) := K_\bullet \times X(c).$$

As for spaces, there is also a based version where the diagrams can take values in $\mathbf{sSet}_*$. We could also have $\mathbf{I}$ be enriched in based simplicial sets $\mathbf{sSet}_*$ instead, as in Definition 5.3.33.

Example 8.1.14. If $\mathbf{C}$ is any symmetric monoidal simplicial model category, and if its category of algebras $\mathrm{Alg}(\mathbf{C})$ has the right-induced model structure (with the underlying fibrations and weak equivalences), then this model structure makes $\mathrm{Alg}(\mathbf{C})$ into a simplicial model category as well. For algebras R and S, the simplicial hom $\mathrm{Alg}(\mathbf{C})_\bullet(R,S)$ is the subset of $\mathbf{C}_\bullet(R,S)$ consisting of maps that are algebra homomorphisms.

More explicitly, a map $f : \Delta[n] \otimes R \to S$ is an "n-simplex of algebra homomorphisms" if the following diagrams commute.

$$\begin{array}{ccc}
\Delta[n]\otimes R\otimes R & \xrightarrow{\mathrm{id}\otimes\mu_R} & \Delta[n]\otimes R \\
\downarrow{\scriptstyle \Delta\otimes\mathrm{id}\otimes\mathrm{id}} & & \downarrow{\scriptstyle f} \\
\Delta[n]\otimes\Delta[n]\otimes R\otimes R & & S \\
\updownarrow{\scriptstyle \cong} & & \uparrow{\scriptstyle f} \\
\Delta[n]\otimes R\otimes\Delta[n]\otimes R & \xrightarrow{f\otimes f} & S\otimes S
\end{array}
\qquad
\begin{array}{ccc}
\Delta[n]\otimes I & \xrightarrow{\mathrm{id}\otimes\eta_R} & \Delta[n]\otimes R \\
\downarrow{\scriptstyle r\otimes\mathrm{id}} & & \\
\Delta[0]\otimes I & & \downarrow{\scriptstyle f} \\
\updownarrow{\scriptstyle \cong} & & \\
I & \xrightarrow{\eta_S} & S.
\end{array}
\tag{8.1.15}$$

The tensor is fun to construct, see Exercise 8.7.44. The categories of commutative algebras $\mathrm{CAlg}(\mathbf{C})$, and modules over a fixed algebra $\mathrm{Mod}(R)$, are also simplicial model categories in a similar way.

In particular, the category of orthogonal ring spectra $\mathrm{Alg}(\mathbf{Sp}^O)$ is a simplicial model category. The category of commutative ring spectra and the positive stable model structure $\mathrm{CAlg}(\mathbf{Sp}^O_+)$ is also a simplicial model category.

Example 8.1.16. The category of chain complexes $\mathbf{Ch}(\mathbb{Z})$, and more generally the category $\mathbf{Ch}(R)$ of chain complexes over any ring R, has a Quillen tensor $\mathbf{sSet} \times \mathbf{Ch}(R) \to \mathbf{Ch}(R)$ given by

$$K_\bullet \otimes C := NK_* \otimes_{\mathbb{Z}} C,$$

where NK_* is a chain complex with one generator in level n for each *nondegenerate* point of K_n, and face maps following the usual formula for simplicial chains, $\partial = \sum_{i=0}^n d_i$. For instance,

$$N_*(\Delta[1]) = \left(\begin{array}{ccc} \mathbb{Z} & \xrightarrow{(-1,1)} & \mathbb{Z}\oplus\mathbb{Z} \end{array} \right),$$

with the $\mathbb{Z} \oplus \mathbb{Z}$ in degree zero. Tensoring this with a chain complex C gives the complex $\Delta[1] \otimes C$ with boundary maps as follows:

$$\begin{array}{ccccccc}
(\Delta[1]\otimes C)_{n+1} & = & C_{n+1} & \oplus & C_n & \oplus & C_{n+1} \\
\downarrow{\scriptstyle \partial_{\Delta[1]\otimes C}} & & {\scriptstyle \partial_C}\downarrow & \swarrow{\scriptstyle -\mathrm{id}} & \downarrow{\scriptstyle -\partial_C} & \searrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle \partial_C} \\
(\Delta[1]\otimes C)_{n} & = & C_{n} & \oplus & C_{n-1} & \oplus & C_{n} \\
\downarrow{\scriptstyle \partial_{\Delta[1]\otimes C}} & & {\scriptstyle \partial_C}\downarrow & \swarrow{\scriptstyle -\mathrm{id}} & \downarrow{\scriptstyle -\partial_C} & \searrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle \partial_C} \\
(\Delta[1]\otimes C)_{n-1} & = & C_{n-1} & \oplus & C_{n-2} & \oplus & C_{n-1}.
\end{array}$$

Intuitively, this is what it should mean to multiply a chain complex by an interval. In fact, if C is the cellular chain complex of a CW complex X, then $\Delta[1] \otimes C$ is exactly the cellular chain complex of the product $I \times X$.

This Quillen tensor does not make $\mathbf{Ch}(R)$ into a simplicial model category, because we don't have isomorphisms $(K_\bullet \times L_\bullet) \otimes C \cong K_\bullet \otimes (L_\bullet \otimes C)$, only weak equivalences. However, most of the results in this chapter, including the construction of homotopy colimits and limits, still work in this example, because they don't use these isomorphisms. We will do this example in more detail in Section 9.2.1.

Remark 8.1.17. It might seem strange to think of **Top** and $\mathbf{Sp}^{\mathbf{N}}_{\mathbf{Top}}$ as simplicial model categories. Isn't it more natural to say that they're **topological model categories**? In other words, each one has a Quillen tensor $\otimes\colon \mathbf{Top} \times \mathbf{C} \to \mathbf{C}$ with the same coherence properties?

Yes, that's true. However, any time you have a topological model category, you can always compose with the geometric realization functor $|-|$ and get a simplicial model category. On the other hand, you can't go back: not every simplicial model category comes from a topological one. See Exercise 8.7.2. In Exercise 8.7.3 you will be asked to generalize this to the notion of a $\mathbf{V}$-model category for any symmetric monoidal model category $\mathbf{V}$.

The point is that simplicial model categories are the better choice, if our goal is to develop a language that covers as many examples as possible.

8.1.2 Properties of the tensor, cotensor, and simplicial hom

Suppose $\mathbf{C}$ is any simplicial model category. Then the coherence part of Definition 8.1.1 allows us to speak unambiguously about tensor products

$$A_1 \otimes \cdots \otimes A_n \otimes X \cong (A_1 \times \cdots \times A_n) \otimes X$$

where the A_i are simplicial sets and X is an object of $\mathbf{C}$. We can similarly unambiguously talk about the cotensor $X^{A_1 \times \cdots \times A_n}$. More importantly, the coherence implies that $\mathbf{C}$ is enriched in simplicial sets:

Definition 8.1.18. If $\mathbf{C}$ is a simplicial model category, or even just a category with a tensor $\otimes$ that has the same coherences as above, then as in Definition 4.1.35, for any two objects X and Y of $\mathbf{C}$ we define an evaluation map in $\mathbf{C}$

$$\mathbf{C}_\bullet(X,Y)\otimes X \xrightarrow{\text{ev}} Y \tag{8.1.19}$$

as the adjunct of the identity map of simplicial sets $\mathbf{C}_\bullet(X,Y) = \mathbf{C}_\bullet(X,Y)$. For any three objects X, Y, and Z of $\mathbf{C}$, we define the composition map of simplicial sets

$$\mathbf{C}_\bullet(X,Y)\times\mathbf{C}_\bullet(Y,Z) \xrightarrow{\circ} \mathbf{C}_\bullet(X,Z) \tag{8.1.20}$$

as the adjunct of the composite

$$\mathbf{C}_\bullet(Y,Z)\otimes(\mathbf{C}_\bullet(X,Y)\otimes X) \xrightarrow{\text{ev}} \mathbf{C}_\bullet(Y,Z)\otimes Y \xrightarrow{\text{ev}} Z.$$

Finally, for each object X we define the map of simplicial sets $\iota : * \to \mathbf{C}_\bullet(X,X)$ as the adjunct of the canonical isomorphism $*\otimes X \cong X$.

Proposition 8.1.21. *These operations make* $\mathbf{C}$ ***enriched over simplicial sets****: the composition* (8.1.20) *is associative and has* ι *as an identity, and along the bijections*

$$\mathbf{sSet}(*, \mathbf{C}_\bullet(X,Y)) \cong \mathbf{C}(X,Y),$$

these composition and identity maps on the simplicial sets $\mathbf{C}_\bullet(X,Y)$ *give the original composition and identity maps on the category* $\mathbf{C}$*.*

The proof is essentially the same as the corresponding fact for closed symmetric monoidal categories from Proposition 4.1.38. The last statement tells us that simplicial level 0, $\mathbf{C}_0(X,Y)$, is just the set of morphisms in $\mathbf{C}$ from X to Y. So the enrichment in simplicial sets extends the ordinary set of morphisms to some larger *simplicial* set of morphisms.

In fact, we can be much more explicit: the set of n-simplices in the simplicial hom $\mathbf{C}_n(X,Y)$ is canonically identified with the set of morphisms in $\mathbf{C}$

$$f : \Delta[n]\otimes X \to Y,$$

where $\Delta[n]$ is the standard n-simplex from Example 7.1.7. Along this identification, the composition map (8.1.20) at simplicial level n is the map of sets

$$\mathbf{C}_n(X,Y)\times\mathbf{C}_n(Y,Z) \longrightarrow \mathbf{C}_n(X,Z)$$

that takes each pair of maps $f : \Delta[n]\otimes X \to Y$ and $g : \Delta[n]\otimes Y \to Z$ to the composite

$$\Delta[n]\otimes X \xrightarrow{\delta\otimes\text{id}} (\Delta[n]\times\Delta[n])\otimes X \xleftarrow{\cong} \Delta[n]\otimes(\Delta[n]\otimes X) \xrightarrow{\text{id}\otimes f} \Delta[n]\otimes Y \xrightarrow{g} Z, \tag{8.1.22}$$

where δ is the diagonal map that is the identity of $\Delta[n]$ in each factor. This makes it clear why the composition at simplicial level 0 is exactly the composition in $\mathbf{C}$.

The adjunction between the tensor, cotensor, and simplicial hom respects this larger enrichment. If $K_\bullet \in \mathbf{sSet}$ and $X, Y \in \mathbf{C}$, the bijections of morphisms

$$K_\bullet \otimes X \to Y \quad \leftrightarrow \quad K_\bullet \to \mathbf{C}_\bullet(X,Y) \quad \leftrightarrow \quad X \to Y^{K_\bullet}$$

are actually simplicial level 0 of isomorphims of simplicial sets

$$\mathbf{C}_\bullet(K_\bullet \otimes X, Y) \cong \mathrm{Map}_\bullet(K_\bullet, \mathbf{C}_\bullet(X,Y)) \cong \mathbf{C}_\bullet(X, Y^{K_\bullet}). \tag{8.1.23}$$

See Exercise 8.7.35. We summarize this by saying that $\mathbf{C}$ is **enriched, tensored and cotensored** over **sSet**. See [Kel05] for an extensive treatment of this and related notions.

Remark 8.1.24. Sometimes the category $\mathbf{C}$ comes to us with an enrichment $\mathbf{C}_\bullet(X,Y)$, and we know that the left adjoints $K \otimes X$ exist for each K separately. In this case we get coherent isomorphisms $(A \times B) \otimes X \cong A \otimes (B \otimes X)$, *if* we have an isomorphism of simplicial sets (8.1.23) and not just an isomorphism on simplicial level 0. If we only have a bijection at simplicial level 0, that's not enough data to tell that the tensor behaves in this way.[2]

In summary, every simplicial model category $\mathbf{C}$ is enriched in simplicial sets. So its morphisms become level 0 of an entire simplicial set of morphisms, $\mathbf{C}(X,Y) = \mathbf{C}_0(X,Y)$. We think of the rest of the simplicial set $\mathbf{C}_\bullet(X,Y)$ as providing "homotopies" between morphisms, homotopies between the homotopies, and so on. Exactly what we need to study homotopy coherence!

For the reader's convenience, we also recall the homotopical properties of the tensor, cotensor, and simplicial hom, which arise because $\otimes$ is a Quillen tensor:

Lemma 8.1.25. *Suppose* $\mathbf{C}$ *is any simplicial model category. Suppose* X *and* Y *are objects of* $\mathbf{C}$*, and* K *is a simplicial set. Then the following properties hold:*

- The tensor $K \otimes X$ preserves both cofibrations and equivalences when X is cofibrant.
- The cotensor Y^K preserves equivalences when Y is fibrant, and sends both cofibrations $K \to K'$ and fibrations $Y \to Y'$ to fibrations in $\mathbf{C}$
- The simplicial homs $\mathbf{C}_\bullet(X,Y)$ preserve equivalences when X is cofibrant and Y is fibrant. In that case it also sends both cofibrations $X \to X'$ and fibrations $Y \to Y'$ to Kan fibrations of simplicial sets.

These follow from Lemma 5.5.18 and Lemma 5.5.21, and the fact that every simplicial set is cofibrant.

[2] In Exercise 1.7.26 we used the Yoneda Lemma to upgrade a bijection of mapping sets to a homeomorphism of mapping spaces, but this relies critically on the fact that the Cartesian product is associative!

8.1.3 Homotopies and homotopy equivalences

In the last section we saw that in any simplicial model category $\mathbf{C}$, the simplicial set $\mathbf{C}_\bullet(X,Y)$ is at level 0 just the set of morphisms $X \to Y$ in $\mathbf{C}$. At level 1, we think of it as the set of "homotopies." To be more specific, for any two maps $f, g \in \mathbf{C}_0(X,Y)$, a **homotopy** is an element $h \in \mathbf{C}_1(X,Y)$ such that $d_0 h = g$ and $d_1 h = f$. In other words, h is a 1-simplex of morphisms, whose two faces are f and g.

Spelling this out, h is a map

$$\Delta[1] \otimes X \xrightarrow{h} Y$$

where $\Delta[1]$ is the standard 1-simplex of Example 7.1.7, and f and g are obtained from h by composing with the two maps

$$X \cong \Delta[0] \otimes X \underset{d^0 \otimes \mathrm{id}}{\overset{d^1 \otimes \mathrm{id}}{\rightrightarrows}} \Delta[1] \otimes X.$$

Intuitively, $\Delta[1]$ is an interval and we are including $\Delta[0]$ along its two ends. In practice, this lines up exactly with what you would think of as a "homotopy."

Definition 8.1.26. We say that f and g are **homotopic** if there is a finite string of homotopies $f = f_0 \sim f_1 \sim \cdots \sim f_n = g$, each of which can point in either direction. (So the 0th face can be f_{i-1} and the 1st face f_i, or vice-versa, for each stage separately.)

The set of homotopy classes of morphisms is called $\pi_0(\mathbf{C}_\bullet(X,Y))$, since it is the same as the connected components of the realization $|\mathbf{C}_\bullet(X,Y)|$.

A **homotopy equivalence** between X and Y is two maps $f \in \mathbf{C}_0(X,Y)$ and $f' \in \mathbf{C}_0(Y,X)$ together with a *single* homotopy $\mathrm{id}_X \sim f' \circ f$ and a single homotopy $f \circ f' \sim \mathrm{id}_Y$. Note that this is not symmetric, and is stronger than asking for $f' \circ f$ and $f \circ f'$ to be homotopic to the respective identity maps. However, it is usually equivalent to this weaker definition and is therefore symmetric.

Lemma 8.1.27. *If $\mathbf{C}$ is a simplicial model category, X is cofibrant and Y is fibrant, then $f, g : X \rightrightarrows Y$ are homotopic iff there is a single homotopy from f to g.*

So in this favorable case, homotopic maps are always related by a single 1-simplex of morphisms, pointing in whichever direction we wish.

Proof. By Lemma 8.1.25, the simplicial set $\mathbf{C}_\bullet(X,Y)$ is a Kan complex. Therefore any two homotopies h_1 and h_2 that agree along one of their faces, define two 1-simplices joined at an edge. This is exactly a horn $\Lambda_1[2]$ in the sense of Definition 7.1.37. Since $\mathbf{C}_\bullet(X,Y)$ is a Kan complex, that horn can be filled by a 2-simplex in $\mathbf{C}_2(X,Y)$, and the remaining face of that 2-simplex defines a single homotopy that accomplishes the job of h_1 and h_2.

In fact, we can order the vertices of this 2-simplex so that the remaining edge points in either direction!

By induction, a string of n homotopies can also be reduced to a single homotopy, pointing in whichever direction we wish. □

Example 8.1.28.

- In the simplicial model category **Top**, a homotopy in this sense is just a homotopy in the usual sense:
$$I \times X \xrightarrow{h} Y.$$
- In the simplicial model category $\mathbf{Top}_*$, a homotopy in this sense is a based homotopy:
$$I_+ \wedge X \xrightarrow{h} Y.$$
- In **sSet**, a homotopy in this sense is called a **simplicial homotopy**,
$$\Delta[1]_\bullet \times X_\bullet \xrightarrow{h} Y_\bullet.$$
The realization of any simplicial homotopy gives a homotopy of maps of spaces $I \times |X_\bullet| \to |Y_\bullet|$, as a result of Milnor's theorem. We saw an example of this in the proof of the bar lemma, Lemma 7.3.11.

  This gives the notion of a **simplicial homotopy equivalence** $X_\bullet \simeq Y_\bullet$. This implies that the realizations are homotopy equivalent, $|X_\bullet| \simeq |Y_\bullet|$, but is much stronger than that.
- In any of the categories of diagram spectra $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}}$, $\mathbf{Sp}^{\Sigma}_{\mathbf{Top}}$, and $\mathbf{Sp}^{O}_{\mathbf{Top}}$, a homotopy in this sense is a homotopy in the sense of Definition 2.3.11: a based homotopy at each spectrum level
$$I_+ \wedge X_n \xrightarrow{h} Y_n$$
that commutes with the bonding maps. A homotopy equivalence in this sense is a homotopy equivalence of spectra from Definition 2.1.10, which is stronger than a level equivalence. Similarly, in the categories $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{sSet}}$ and $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}}$, a homotopy consists of a simplicial homotopy at each spectrum level n
$$\Delta[1]_+ \wedge X_\bullet(n) \xrightarrow{h} Y_\bullet(n),$$
commuting with the bonding maps.
- In $\mathbf{Ch}(R)$, a homotopy in this sense is a **chain homotopy**: two maps of chain complexes $f, g : C \rightrightarrows D$ and maps $h_n : C_{n-1} \to D_n$ such that $dh = -hd + g - f$. A homotopy equivalence in this sense is a chain homotopy equivalence.

Lemma 8.1.29. *If* $\mathbf{C}$ *is a simplicial model category, every homotopy equivalence is a weak equivalence.*

Proof. We only prove this in the case that the objects are both cofibrant; the general case follows from Proposition 10.2.9 and Proposition 5.2.30.

If X is cofibrant, then the projection map $\Delta[1] \otimes X \to * \otimes X$ is a weak equivalence by Lemma 8.1.25. Therefore it is an isomorphism in the homotopy category $\mathrm{Ho}\,\mathbf{C}$. It follows from the argument in Lemma 3.1.15 that any two homotopic maps between cofibrant objects are equal $\mathrm{Ho}\,\mathbf{C}$. Therefore each homotopy equivalence of cofibrant objects is an isomorphism in $\mathrm{Ho}\,\mathbf{C}$, and by Proposition 5.2.30 it is a weak equivalences as well. □

Proposition 8.1.30. *If* $\mathbf{C}$ *is a simplicial model category, the homotopy category* $\mathrm{Ho}\,\mathbf{C}$ *is equivalent to the subcategory of cofibrant and fibrant objects* $\mathbf{C}^{\mathrm{cf}}$*, with homotopy classes of maps as defined above.*

Proof. This follows from the fundamental theorem of model categories (Theorem 5.2.27), provided we can show that when X is cofibrant and fibrant, the maps

$$X \amalg X \cong (\partial\Delta[1]) \otimes X \longrightarrow \Delta[1] \otimes X \longrightarrow (*) \otimes X \cong X$$

are a cofibration and a weak equivalence, respectively. These follow from Lemma 8.1.25, since $\partial\Delta[1] \to \Delta[1]$ is a cofibration of simplicial sets (i.e. injective at every level), and $\Delta[1] \to *$ is a weak equivalence of simplicial sets. □

Corollary 8.1.31 (Whitehead's Theorem)**.** *In any simplicial model category* $\mathbf{C}$*, if* X *and* Y *are bifibrant, then a map* $f : X \to Y$ *is a weak equivalence iff it is a homotopy equivalence in the sense described above.*

We have seen this Whitehead theorem in action for spaces and spectra already, but it is instructive to see what it says about simplicial sets and chain complexes as well.

Example 8.1.32.

- In **Top**, a weak equivalence between retracts of cell complexes is a homotopy equivalence.
- In **sSet**, a weak equivalence between Kan complexes is a simplicial homotopy equivalence.
- In any of the categories of diagram spectra $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}}$, $\mathbf{Sp}^{\Sigma}_{\mathbf{Top}}$, and $\mathbf{Sp}^{O}_{\mathbf{Top}}$, a weak equivalence between cellular Ω-spectra is a homotopy equivalence.
- In the categories of diagram spectra $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{sSet}}$ and $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}}$, a weak equivalence between cellular Ω-spectra is a simplicial homotopy equivalence of spectra. (Recall that to be an Ω-spectrum, every level of the spectrum must be a Kan complex as well.)

- In $\mathbf{Ch}_{\geq 0}(R)$, a quasi-isomorphism between a complexes that are projective at each level must be a chain homotopy equivalence.

Remark 8.1.33. We have already pointed out that $\mathbf{Ch}_{\geq 0}(R)$, at least with the tensor we have given it, is not a simplicial model category. However, it has a Quillen tensor $\mathbf{sSet} \times \mathbf{C} \to \mathbf{C}$, without the coherent isomorphisms. But that is enough to get all of the results in this subsection, because they don't use the coherent isomorphisms.

8.1.4 Simplicial functors and adjunctions

When moving between simplicial model categories, we ask for functors that respect the extra structure.

Definition 8.1.34. If $\mathbf{C}$ and $\mathbf{D}$ are simplicial model categories, a **simplicial functor** is a functor $F\colon \mathbf{C} \to \mathbf{D}$ and an extension of the map $F\colon \mathbf{C}_0(X,Y) \to \mathbf{D}_0(F(X),F(Y))$ to a map of simplicial sets

$$\mathbf{C}_\bullet(X,Y) \longrightarrow \mathbf{D}_\bullet(F(X),F(Y)),$$

such that F respects composition on the entire simplicial set, not just at simplicial level 0:

$$\begin{array}{ccc} \mathbf{C}_\bullet(X,Y) \times \mathbf{C}_\bullet(Y,Z) & \xrightarrow{F} & \mathbf{D}_\bullet(F(X),F(Y)) \times \mathbf{D}_\bullet(F(Y),F(Z)) \\ \circ\downarrow & & \downarrow\circ \\ \mathbf{C}_\bullet(X,Z) & \xrightarrow{\qquad F \qquad} & \mathbf{D}_\bullet(F(X),F(Z)) \end{array}$$

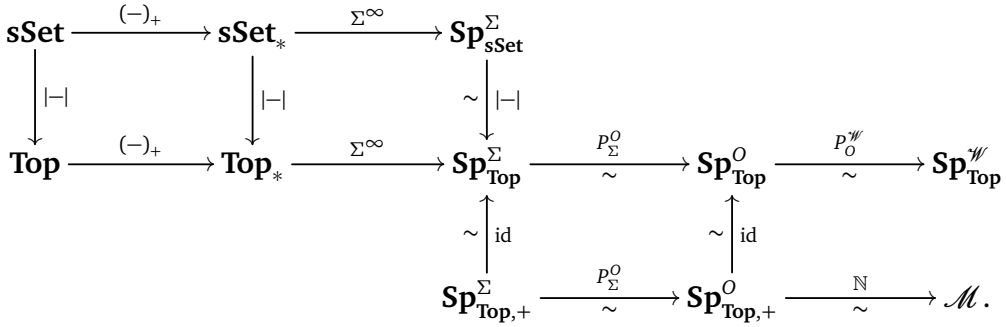
For instance, any functor that is lax monoidal with respect to $\otimes$, in other words it has coherent maps of the form $m\colon K \otimes F(X) \to F(K \otimes X)$, will be a simplicial functor. We send each n-simplex of morphisms $f \in \mathbf{C}_n(X,Y)$ to an element of $\mathbf{D}_n(F(X),F(Y))$ by the rule

$$\Delta[n] \otimes F(X) \xrightarrow{m} F(\Delta[n] \otimes X) \xrightarrow{F(f)} F(Y).$$

It is a straightforward diagram-chase to see that this respects composition. See also Exercise 8.7.36 for a more extensive treatment of this and related ideas.

Example 8.1.35. If $\mathbf{C}$ and $\mathbf{D}$ are symmetric monoidal simplicial model categories, every lax monoidal functor F is a simplicial functor. So in the following diagram extending

(6.4.27), every functor, and all of their right adjoints as well, are simplicial functors.



(8.1.36)

There are even more examples: the free spectrum $F_d(-)$ and the evaluation $\mathrm{ev}_d(-)$, even for sequential spectra, are simplicial functors. Even more generally, the free diagram $F_d(-)$ and the evaluation $\mathrm{ev}_d(-)$ are simplicial functors between **Top** and $\mathbf{Top}^{\mathbf{I}}$ for any topological category **I**.

When F is a simplicial functor we will sometimes write F_0 to denote the underlying functor, which only acts on level 0 of the simplicial sets $\mathbf{C}_\bullet(X,Y)$

Definition 8.1.37. If **C** and **D** are simplicial model categories, a **simplicial adjunction** is two simplicial functors $F\colon \mathbf{C}\to\mathbf{D}$ and $G\colon \mathbf{D}\to\mathbf{C}$, an adjunction of the underlying functors $(F_0 \dashv G_0)$ whose unit and counit maps induce isomorphisms of simplicial sets

$$\mathbf{C}_\bullet(X, G(Y)) \cong \mathbf{D}_\bullet(F(X), Y).$$

A **simplicial Quillen adjunction** is a simplicial adjunction such that F_0 is left Quillen.

Note that the *data* of an enriched adjunction is the same as the data of an adjunction, but the *condition* is stronger. And it is not automatic! Giving a bijection on the 0th level of the simplicial set $\mathbf{C}_0(X,Y)$ does not imply that we have an isomorphism of simplicial sets!

Example 8.1.38. Again, in the case that everything comes from a symmetric monoidal category, every symmetric monoidal Quillen adjunction is also a simplicial Quillen adjunction. We see this by writing out the definition of the map $\mathbf{C}_\bullet(X,G(Y)) \to \mathbf{D}_\bullet(F(X),Y)$ as taking each map $f : \Delta[n]\otimes X \to G(Y)$ to the composite

$$\Delta[n]\otimes F(X) \xrightarrow{\ m\ } F(\Delta[n]\otimes X) \xrightarrow{F(f)} F(G(Y)) \xrightarrow{\ \epsilon\ } Y.$$

The last two arrows define the adjunct of f in the underlying category, from which f is uniquely recovered, and the remaining map m is an isomorphism. It follows that this whole process is invertible, so that we get a bijection at every simplicial level $\mathbf{C}_n(X,G(Y)) \cong \mathbf{D}_n(F(X),Y)$ as desired.

In particular, every adjunction that we discussed in Example 8.1.35 is a simplicial Quillen adjunction.

8.2 Simplicial objects and their realization

In this section we prove some fundamental properties about the realization of simplicial objects in any simplicial model category **C**. In the next section we use these properties to build homotopy colimits in **C** using the bar construction.

8.2.1 Defining the realization in a general category

Recall from Definition 7.1.1 that $\mathbf{\Delta}$ is the category whose objects are $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$, and whose morphisms are the totally ordered maps. The following definition generalizes simplicial spaces to any category **C**.

Definition 8.2.1. If **C** is any category, a **simplicial object in C** is a functor $X_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}$. In other words, it is a sequence of objects X_0, X_1, X_2, ... in **C**, together with face and degeneracy morphisms in **C** that satisfy the relations from Lemma 7.1.5.

If **C** has a tensor product functor $\otimes \colon \mathbf{sSet} \times \mathbf{C} \to \mathbf{C}$ and $X_\bullet$ is a simplicial object of **C**, the **realization** $|X_\bullet|$ is defined as the coequalizer of the following two maps:

$$\coprod_{k,\ell \geq 0} \coprod_{\alpha \in \mathbf{\Delta}([k],[\ell])} \Delta[k] \otimes X_\ell \rightrightarrows \coprod_{k \geq 0} \Delta[k] \otimes X_k \longrightarrow |X_\bullet|. \tag{8.2.2}$$

The first map applies the morphism $\alpha \colon [k] \to [\ell]$ to $\Delta[k]$ to map forward to $\Delta[\ell] \times X_\ell$, while the second map applies α to X_ℓ to map back to $\Delta[k] \times X_k$.

This is just the usual definition of the realization from Definition 7.1.11, except that instead of applying an equivalence relation (which doesn't make sense in a general category), we have to use a coend as in Definition 7.4.1.

As in Lemma 7.1.13, we can make this coequalizer diagram smaller by only taking those $\alpha \colon [k] \to [\ell]$ that are coface or codegeneracy maps, and we still get the same result for the coequalizer $|X_\bullet|$, up to canonical isomorphism. This is because every α is a composite of face and degeneracy maps, so a morphism from $\coprod_{k \geq 0} \Delta[k]_\bullet \otimes X_k$ to another object of **C** that agrees with the face and degeneracy relations will automatically agree with all of the relations.

Example 8.2.3. When $\mathbf{C} = \mathbf{Top}$, this is exactly the geometric realization that we saw in Definition 7.1.11 and Definition 7.2.1.

This formal definition of realization doesn't make sense in the category $\mathbf{C} = \mathbf{Set}$, because if S is a set then there's no good definition for $\Delta[n] \otimes S$ as a set. Rather, we have to think of our sets as discrete topological spaces, and then apply this definition of geometric realization in the category **Top**.

Example 8.2.4. When $\mathbf{C} = \mathbf{Top}_*$, this gives a formula for the realization from (7.6.3), that uses smash products and wedge sums instead of Cartesian products and disjoint unions. However, by Lemma 7.6.4, this agrees with the realization in **Top**.

Example 8.2.5. Suppose $\mathbf{C} = \mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}}$, $\mathbf{Sp}^{\Sigma}_{\mathbf{Top}}$, or $\mathbf{Sp}^{O}_{\mathbf{Top}}$. Then we get the notion of a **simplicial spectrum**: a sequence of spectra X_k with face and degeneracy maps of spectra between them. Let $X(n)_k$ denote spectrum level n of the spectrum at simplicial level k. So we have face and degeneracy maps that change k, bonding maps that change n, and these operations commute with each other. The realization is just the geometric realization from (7.6.2) or (7.6.3) at each spectrum level:

$$|X(n)_\bullet| = \left(\bigvee_{k\geq 0} \Delta^k_+ \wedge X(n)_k \right) / \ (d^i t, x) = (t, d_i x),\ (s^i t, x) = (t, s_i x). \tag{8.2.6}$$

The bonding maps are

$$\Sigma |X(n_\bullet)| \cong |\Sigma X(n)_\bullet| \to |X(n+1)_\bullet|.$$

The presence of two indices tends to make formulas confusing, so whenever possible we try to think in terms of the whole spectrum, and just say "the simplicial spectrum $X_\bullet$."[3]

To give one example, if $A_\bullet$ is a simplicial based space, then the suspension spectra $\Sigma^\infty A_n$ define a simplicial spectrum $\Sigma^\infty A_\bullet$. Its realization is the suspension spectrum of $|A_\bullet|$:

$$|\Sigma^\infty A_\bullet| \cong \Sigma^\infty |A_\bullet|.$$

To give another example, if R is a ring spectrum then we can make a simplicial spectrum $N^{\mathrm{cyc}}_\bullet(R)$ whose nth level is the $(n+1)$-fold smash product $R^{\wedge(n+1)}$, and whose face maps multiply adjacent copies of R, "arranged in a circle." See Exercise 8.7.5 for more details. The realization of this is called **topological Hochschild homology** $\mathrm{THH}(R)$.

Example 8.2.7. Recall from Example 7.2.5 that a bisimplicial set is a functor $X : \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$. This is exactly a simplicial object in **sSet**. Its realization in the sense above is the diagonal simplicial set $X_n := X_{n,n}$.

There are a couple of things we can prove about realizations right away.

Proposition 8.2.8. *If $\otimes$ preserves colimits in each slot, then the realization commutes with all colimits:*

$$\left| \operatorname*{colim}_{i\in\mathbf{I}} X_\bullet(i) \right| \cong \operatorname*{colim}_{i\in\mathbf{I}} |X_\bullet(i)|.$$

[3] Note that simplicial spectra are not the same as objects in $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{sSet}}$ or $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}}$, where the bonding maps involve the simplicial structure in a more nontrivial way. One can consider simplicial objects in $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{sSet}}$ or $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}}$, which now have *three* indices. This tends to be very confusing and we will avoid this whenever possible.

Proof. This follows by comparing universal properties. □

Proposition 8.2.9. *If* **C** *is a symmetric monoidal simplicial model category, then realization also commutes with the tensor product, making* $|-|\colon \mathbf{sC} \to \mathbf{C}$ *into a strong symmetric monoidal functor:*

$$|X_\bullet \otimes Y_\bullet| \cong |X_\bullet| \otimes |Y_\bullet|.$$

To be more specific, if $X_\bullet$ and $Y_\bullet$ are two simplicial objects in **C**, then we form the levelwise tensor product, a simplicial object $X_\bullet \otimes Y_\bullet$ whose nth level is $X_n \otimes Y_n$, and whose face and degeneracy maps are applied to both X_n and Y_n at the same time. The proposition says that the realization of this levelwise tensor product is isomorphic to $|X_\bullet| \otimes |Y_\bullet|$.

Proof. We follow the formal outline of the proof of Milnor's theorem from Exercises 7.7.8 and 7.7.9.

The key idea is to write $S \otimes X = \coprod_S X$ when S is any set, and then to form a simplicial object in **C** of the form $\mathbf{\Delta}(\bullet,[k]) \otimes I$. This is not $\Delta[k] \otimes I$, which is just an object of **C**. Rather, it is a simplicial object in which every level is a coproduct of copies of the unit, $\mathbf{\Delta}([n],[k]) \otimes I \cong \coprod_{\mathbf{\Delta}([n],[k])} I$. However, if we take the realization of $\mathbf{\Delta}(\bullet,[k]) \otimes I$, we get the object $\Delta[k] \otimes I$. Multiplying this result by an arbitrary set and taking a colimit, we conclude that if we perform this process with any simplicial set $S_\bullet$, forming a simplicial object whose nth level is $S_n \otimes I$, then the realization is $S \otimes I$.

Now we take the simplicial object $X_\bullet$ and express it as a coequalizer

$$\coprod_{k,\ell \geq 0} \coprod_{\alpha \in \mathbf{\Delta}([k],[\ell])} \mathbf{\Delta}(\bullet,[k]) \otimes X_\ell \rightrightarrows \coprod_{k \geq 0} \mathbf{\Delta}(\bullet,[k]) \otimes X_k \longrightarrow X_\bullet. \tag{8.2.10}$$

If we take the levelwise tensor product with the sets $\mathbf{\Delta}(\bullet,[m])$, we get the coequalizer

$$\begin{array}{c} \displaystyle\coprod_{k,\ell \geq 0} \coprod_{\alpha \in \mathbf{\Delta}([k],[\ell])} (\mathbf{\Delta}(\bullet,[k]) \times \mathbf{\Delta}(\bullet,[m])) \otimes X_\ell \\ \downarrow\downarrow \\ \displaystyle\coprod_{k \geq 0} (\mathbf{\Delta}(\bullet,[k]) \times \mathbf{\Delta}(\bullet,[m])) \otimes X_k \longrightarrow \mathbf{\Delta}(\bullet,[m]) \otimes X_\bullet. \end{array} \tag{8.2.11}$$

By the previous discussion, taking realizations gives the coequalizer

$$\begin{array}{c} \displaystyle\coprod_{k,\ell \geq 0} \coprod_{\alpha \in \mathbf{\Delta}([k],[\ell])} \Delta[k] \otimes \Delta[m] \otimes X_\ell \\ \downarrow\downarrow \\ \displaystyle\coprod_{k \geq 0} \Delta[k] \otimes \Delta[m] \otimes X_k \longrightarrow |\mathbf{\Delta}(\bullet,[m]) \otimes X_\bullet|. \end{array} \tag{8.2.12}$$

But the coequalizer on the left gives $\Delta[m] \otimes |X_\bullet|$, so we conclude

$$|\mathbf{\Delta}(\bullet, [m]) \otimes X_\bullet| \cong \Delta[m] \otimes |X_\bullet|.$$

This is natural in m, so we tensor the result with Y_n for varying $n \geq 0$, and take one more coequalizer to get the desired isomorphism, shown on the right-hand side of the following diagram.

$$\begin{array}{ccccc}
\coprod_{m,n\geq 0} \coprod_{\alpha\in\mathbf{\Delta}([m],[n])} |\mathbf{\Delta}(\bullet,[m]) \otimes Y_n \otimes X_\bullet| & \rightrightarrows & \coprod_{n\geq 0} |\mathbf{\Delta}(\bullet,[n]) \otimes Y_n \otimes X_\bullet| & \longrightarrow & |Y_\bullet \otimes X_\bullet| \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
\coprod_{m,n\geq 0} \coprod_{\alpha\in\mathbf{\Delta}([m],[n])} \Delta[m] \otimes Y_n \otimes |X_\bullet| & \rightrightarrows & \coprod_{n\geq 0} \Delta[n] \otimes Y_n \otimes |X_\bullet| & \longrightarrow & |Y_\bullet| \otimes |X_\bullet|.
\end{array}$$

□

Corollary 8.2.13. *The realization of simplicial spectra commutes with the smash product,*

$$|X_\bullet \wedge Y_\bullet| \cong |X_\bullet| \wedge |Y_\bullet|.$$

This is the analog of Milnor's theorem for spectra, which we saw earlier for unbased spaces in Theorem 7.1.31 and for based spaces in Lemma 7.6.8.

Remark 8.2.14. The proof of Proposition 8.2.9 is very formal. However, the full proof that $|X_\bullet \wedge Y_\bullet| \cong |X_\bullet| \wedge |Y_\bullet|$ for spectra is not completely formal. We still needed to know that spectra were a simplicial model category. We proved this using the fact that $\Sigma^\infty_+|-|$ is a strong symmetric monoidal functor, and this ultimately relied on the geometric proof of Milnor's theorem in **Top**.

8.2.2 Latching maps and Reedy cofibrant simplicial objects

We can define the realization of any simplicial object $X_\bullet$ in any simplicial model category $\mathbf{C}$. But how do we understand what $|X_\bullet|$ is, up to weak equivalence? In other words, can we control its homotopy type? It turns out we can, but only if $X_\bullet$ is Reedy cofibrant in the following sense.

Definition 8.2.15. Suppose $\mathbf{C}$ is any category with all colimits, and $X_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}$ is a simplicial object in $\mathbf{C}$. For each $n \geq 0$, we form a diagram of degeneracy maps with one copy of X_i for each $i < n$ and each surjective map of totally ordered sets $[n] \to [i]$. The diagram has a map $X_i \to X_j$ for each surjective map $[j] \to [i]$ commuting with the maps in from $[n]$.

The colimit of this diagram is the n**th latching object** $L_nX \in \mathbf{C}$, and the n**th latching map**

$$L_nX \to X_n$$

is induced by the given maps $[n] \to [i]$. The 0th latching map is $\varnothing \to X_0$, where $\varnothing$ is the initial object of $\mathbf{C}$.

If we fix any notion of "cofibration" in $\mathbf{C}$, then $X_\bullet$ is **Reedy cofibrant** if the latching map $L_nX \to X_n$ is a cofibration for all $n \geq 0$.

Example 8.2.16. When $\mathbf{C} = \mathbf{Top}$, we saw this notion already in Definition 7.2.11 using the Hurewicz cofibrations or h-cofibrations. (We prove that the two definitions agree in Lemma 8.2.20.) We could also take the Quillen cofibrations, or q-cofibrations. This leads to *two different versions* of Reedy cofibrancy. We say that $X_\bullet$ is Reedy h-cofibrant in the first case, and Reedy q-cofibrant in the second case.

Example 8.2.17. When $\mathbf{C} = \mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}}$, we get a notion of a Reedy cofibrant simplicial spectrum. As in **Top**, this also depends on which notion of "cofibration" we want to use. It's common to use the free cofibrations that come from the stable model structure, or to use the h-cofibrations in the sense of Definition 6.1.27.

Returning to any category $\mathbf{C}$ with all colimits, we can describe the latching map much more explicitly using a cube-shaped diagram in $\mathbf{C}$. This is a diagram on the category $\{0 \to 1\}^{\times n}$, or equivalently the category with one object for each subset $S \subseteq \{1, \ldots, n\}$ and one morphism $S \to T$ for each subset inclusion $S \subseteq T$.

Proposition 8.2.18. *The diagram of degeneracy maps from Definition 8.2.15 is in fact a cube-shaped diagram, with X_n as the final vertex, and L_nX as the colimit of all the vertices except the final vertex, as illustrated in Figure 8.2.19.*

Proof. To see that the diagram of degeneracy maps is a cube, we observe that a surjective map $[n] \to [i]$ is described uniquely by giving the smallest element of the preimage of each point of $[i]$. These choices form a subset of $[n] = \{0, 1, \ldots, n\}$ that always contains 0, so it is of the form $\{0\} \cup S$ for a unique set $S \subseteq \{1, \ldots, n\}$, that has exactly i elements. The original surjective map $[n] \to [i]$ is then given by "rounding down" each element of $[n]$ to the nearest element of $\{0\} \cup S$, and then applying the unique order-preserving bijection $\{0\} \cup S \cong [i]$.

The subsets of $\{1, \ldots, n\}$ are the vertices of an n-dimensional cube, which means the diagram is shaped like a cube!

To see what the morphisms are, we observe that a surjective map $[n] \to [i]$ corresponding to the set S factors through the surjective map $[n] \to [j]$ corresponding to a set T precisely when $S \subseteq T$. You first round down to the closest element of $\{0\} \cup T$, and then round

n	Cube-shaped diagram	L_nX is the colimit of:	X_n
0	X_0	$\varnothing$	X_0
1	$X_0 \longrightarrow X_1$	X_0	X_1
2	$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ & \searrow & \searrow \\ & X_1 & \longrightarrow X_2 \end{array}$	$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ & \searrow & \\ & X_1 & \end{array}$	X_2
3	$\begin{array}{ccccc} X_0 & \longrightarrow & X_1 & & \\ & \searrow & \downarrow & \searrow & \\ & X_1 & \longrightarrow & X_2 & \\ & \downarrow & & \downarrow & \\ & X_1 & \longrightarrow & X_2 & \\ & & \searrow & \downarrow & \\ & & & X_3 & \end{array}$	$\begin{array}{ccccc} X_0 & \longrightarrow & X_1 & & \\ & \searrow & \downarrow & \searrow & \\ & X_1 & \longrightarrow & X_2 & \\ & \downarrow & & & \\ & X_1 & & & \\ & & \searrow & & \\ & & & X_2 & \end{array}$	X_3

Figure 8.2.19

down to the closest element of $\{0\} \cup S$. Of course, the inclusions of subsets are exactly the morphisms of a cube-shaped diagram, so this describes the morphisms as well.

Concretely, each set S is assigned to a copy of X_i that we call X_S, and each inclusion $S \subseteq T$ is assigned to the map $X_S \to X_T$ that is induced by the map $[j] \to [i]$ that rounds down. This completely describes the diagram of degeneracy maps, so removing the last vertex and taking the colimit gives the latching object L_nX. □

In light of the above proof, we may label the diagram of degeneracies using the subsets S, rather than just remembering the size of the subsets. So for instance this gives the diagram on the left rather than the diagram on the right.

$$\begin{array}{ccccc} X_\varnothing & \longrightarrow & X_{\{1\}} & & \\ & \searrow & \downarrow & \searrow & \\ & X_{\{2\}} & \longrightarrow & X_{\{1,2\}} & \\ \downarrow & & \downarrow & & \\ X_{\{3\}} & \longrightarrow & X_{\{1,3\}} & & \\ & \searrow & \downarrow & & \\ & & X_{\{2,3\}} & & \end{array} \qquad \begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ & \searrow & \downarrow \searrow \\ & X_1 & \longrightarrow X_2 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_2 \\ & \searrow & \downarrow \\ & & X_2 \end{array}$$

The following lemma shows that our definition of the latching map in Definition 8.2.15 agrees with our earlier definition from Definition 7.2.9. Suppose **C** is a category with a faithful forgetful functor to **Set**, that preserves colimits along any punctured cube-shaped

diagram in which each morphism has a retract. For instance, this is true when **C** is sets, topological spaces, simplicial sets, or even spectra. (You can send every spectrum to the disjoint union of its levels, $\coprod_{n\geq 0} X_n$.)

Lemma 8.2.20. *Under this assumption, for each simplicial object X, the latching object L_nX is just the union of the images of the degeneracy maps $X_{n-1} \to X_n$, and the latching map is the inclusion of this subset into X_n.*

Corollary 8.2.21. *For any simplicial object X in such a category* **C***, the latching map is injective.*

Proof of Lemma 8.2.20. The idea is to show that each vertex of the cube is a subset of X_n and the maps are inclusions of subsets. This shows that the colimit maps surjectively to the union of these subsets. To show it is injective, it is enough to check the intersection condition

$$X_S \cap X_T = X_{S\cap T}.$$

For each set $S \subseteq \{1,\dots,n\}$, the associated term of the diagram $X_S \cong X_i$ is already equipped with a map to X_n, given by applying the rounding-down map

$$s^S : [n] \to \{0\} \cup S \cong [i].$$

Since s^S is a morphism in the category $\mathbf{\Delta}$, it induces a map $s_S : X_S \to X_n$.

We define another map $d^S : [i] \to [n]$ that simply includes the set $\{0\} \cup S$ back into $[n]$, which gives a map $d_S : X_n \to X_S$. The composite $s^S \circ d^S$ is the identity of $[i]$, and therefore the composite $d_S \circ s_S : X_S \to X_S$ is the identity. Therefore $s_S : X_S \to X_n$ is injective, and we may identify X_S with its image in X_n.

This image is the same as the image of the idempotent map $s_S \circ d_S : X_n \to X_n$, which we call e_S for short. The map e_S is the identity on this image, so we can also characterize $X_S \subseteq X_n$ as the subset of points that are fixed by e_S. Note that e_S is induced by the map $e^S : [n] \to [n]$ that rounds points down to the nearest point of $\{0\} \cup S$, then includes them back into S.

It remains to show that $X_{S\cap T} = X_S \cap X_T$ for every pair of sets $S, T \subseteq \{1,\dots,n\}$. Let $A_i \subseteq \{1,\dots,n\}$ be the set that is missing only i, so that $S = \bigcap_{i\notin S} A_i$. It suffices to prove that $X_S = \bigcap_{i\notin S} X_{A_i}$.

We check that when i is not in S, $e^S e^{A_i} = e^S$, and therefore $e_{A_i} e_S = e_S$. This shows that anything in X_S is fixed by e_{A_i}, so it is in X_{A_i}. Conversely, e^S can be written as a composite of all the e^{A_i} for $i \notin S$, and therefore e_S is the composite of the e_{A_i}. Therefore any point in the intersection $\bigcap_{i\notin S} X_{A_i}$ is fixed every e_{A_i}, so it is fixed by their composite e_S, and is therefore in X_S. This finishes the proof. □

Remark 8.2.22. If the latching object is just the subset of degenerate points, why did we bother to give such an elaborate definition in Definition 8.2.15? Because the more elaborate definition is better for *proving theorems*.

Example 8.2.23. If $X_{\bullet\bullet}$ is any bisimplicial set, regarded as a simplicial object in **sSet**, then its latching map $L_n X_\bullet \to X_{n\bullet}$ is a map of simplicial sets. Applying Corollary 8.2.21 to each simplicial level, we see that this map of simplicial sets is injective at every level. Therefore $L_n X_\bullet \to X_{n\bullet}$ is automatically a cofibration of simplicial sets. In other words, every bisimplicial set is Reedy cofibrant.

Example 8.2.24. Taking the geometric realization of the previous example, we conclude that any simplicial space $X_\bullet$ that arises as the realization of a bisimplicial set, is automatically Reedy q-cofibrant.

8.2.3 The realization of Reedy cofibrant objects

We are ready for the first theorem of the chapter, which exerts control over the homotopy type of the realization $|X_\bullet|$. Suppose **C** is a simplicial model category, and let us take "cofibration" to mean a cofibration in the model structure on **C**. So, a simplicial object $X_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}$ is Reedy cofibrant if each latching map $L_n X \to X_n$ is a cofibration in the model structure on **C**.

Theorem 8.2.25 ([GJ99, Ch VII],[Hir03, Ch. 15])**.** *If $X_\bullet$ is Reedy cofibrant then the realization $|X_\bullet|$ is cofibrant. Furthermore, if $X_\bullet \to Y_\bullet$ is any map of Reedy cofibrant objects such that each $X_n \to Y_n$ is a weak equivalence in* **C***, then the map of realizations $|X_\bullet| \to |Y_\bullet|$ is a weak equivalence in* **C***.*

We will use this theorem later in the chapter to show that the homotopy colimit preserves all equivalences, in any simplicial model category **C**.

On the other hand, the assumptions of Theorem 8.2.25 are sometimes too restrictive in the examples we care about.

Example 8.2.26. For a simplicial space $X_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Top}$, Theorem 8.2.25 says that if the latching maps $L_n X \to X_n$ are Quillen cofibrations (retracts of relative cell complexes), then the realization $|X_\bullet|$ preserves equivalences. However, in Proposition 7.2.14 we claimed that it is enough for the latching maps $L_n X \to X_n$ to be Hurewicz cofibrations (having the homotopy extension property).

We will therefore weaken the assumptions of Theorem 8.2.25 before proving it. Suppose **C** is any category with all colimits, $\otimes \colon \mathbf{sSet} \times \mathbf{C} \to \mathbf{C}$ is a functor that preserves colimits in each slot, and **C** is given any class of "cofibrations," not necessarily fitting into a model structure, satisfying the following assumptions. (When we say X is "cofibrant," we mean that $\varnothing \to X$ is in our chosen class of "cofibrations.")

- Every isomorphism is a cofibration.
- Every pushout of a cofibration is a cofibration.
- Every countably infinite composition of cofibrations is a cofibration. (Therefore every finite composition of cofibrations is a cofibration as well.)
- (Gluing Lemma) If we consider spans of cofibrant objects in which one leg is a cofibration,
$$X \longleftarrow A \longrightarrow Y,$$
then any weak equivalence of two such spans gives a weak equivalence on the pushouts $X \cup_A Y$.
- (Colimit Lemma) If we consider sequences of cofibrations of cofibrant objects,
$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots,$$
then any weak equivalence of two such diagrams gives a weak equivalence on the the colimits X_∞.
- (Pushout-Product Lemma) If i is the map of simplicial sets $\partial\Delta[n] \to \Delta[n]$ and $f : A \to X$ is any cofibration, then the pushout-product $i \square f$ is a cofibration. Furthermore $\partial\Delta[n] \otimes -$ and $\Delta[n] \otimes -$ preserve weak equivalences of cofibrant objects.

We then declare that an object $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}$ is Reedy cofibrant if each latching map $L_n X \to X_n$ is a "cofibration" in the sense we have chosen.

Theorem 8.2.27. *Under these more general assumptions, if $X_\bullet$ is Reedy cofibrant then $|X_\bullet|$ is cofibrant. Furthermore, if $X_\bullet \to Y_\bullet$ is any map of Reedy cofibrant objects such that each $X_n \to Y_n$ is a weak equivalence in $\mathbf{C}$, then the map of realizations $|X_\bullet| \to |Y_\bullet|$ is a weak equivalence in $\mathbf{C}$.*

Example 8.2.28. We give some examples where the assumptions of Theorem 8.2.27 hold.

- In any simplicial model category $\mathbf{C}$, these assumptions hold for the cofibrations coming from the model structure in $\mathbf{C}$. Most of this is easy. The gluing lemma is done in Exercise 5.8.12, and the colimit lemma is similar.
- In **Top**, these assumptions hold for either the Quillen cofibrations or the unbased h-cofibrations, in other words the maps with the homotopy extension property. So Theorem 8.2.27 proves the claim we made in Proposition 7.2.14, that Reedy h-cofibrant simplicial spaces have well-behaved realization.
- In $\mathbf{Top}_*$, these assumptions also hold for the Quillen cofibrations and the unbased h-cofibrations. That makes sense because the realization is the same as in **Top**. However, a simplicial based space can only be Reedy h-cofibrant if each space X_n is well-based.

- In $\mathbf{Top}_*$, these assumptions do *not* hold for the based h-cofibrations, in other words the maps that have the homotopy extension property for basepoint-preserving maps. The gluing lemma and colimit lemma fail, and the weak equivalence part of the pushout-product lemma also fails. The assumptions do become true again if we restrict our attention to the subcategory of well-based spaces, $\mathbf{Top}_*^{wb}$. On this subcategory, the based and unbased h-cofibrations coincide, so this follows from the previous point.
- In $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}}$, $\mathbf{Sp}^{\Sigma}_{\mathbf{Top}}$, or $\mathbf{Sp}^{O}_{\mathbf{Top}}$, these assumptions hold for either the h-cofibrations or the levelwise unbased h-cofibrations from Definition 6.1.27. The gluing lemma and colimit lemma come from Corollary 6.1.44, while the pushout-product lemma follows from a formal pairing result by Schwänzl and Vogt. See Exercise 8.7.42.

The proof of Theorem 8.2.27 occupies the rest of this section. The idea is to express the realization using a step-by-step process that adds the n-simplices for one value of n at a time.

Definition 8.2.29. For any simplicial object $X_\bullet$ in $\mathbf{C}$ and any $n \geq 0$, define the **n-skeleton** $\mathrm{Sk}^n|X_\bullet| \in \mathbf{C}$ by taking the definition of the realization from (8.2.2) and restricting to those terms that are $\leq n$:[4]

$$\coprod_{0\leq k,\ell\leq n} \coprod_{\alpha\in\mathbf{\Delta}([k],[\ell])} \Delta[k]\otimes X_\ell \rightrightarrows \coprod_{0\leq k\leq n} \Delta[k]\otimes X_k \longrightarrow \mathrm{Sk}^n|X_\bullet|. \tag{8.2.30}$$

By comparing the universal properties, we get canonical maps $\mathrm{Sk}^{n-1}|X_\bullet| \to \mathrm{Sk}^n|X_\bullet|$, and the full realization $|X_\bullet|$ is the colimit of these maps in $\mathbf{C}$. Intuitively, we are building up the realization one dimension at a time.

Proposition 8.2.31. *For any simplicial object $X_\bullet$ in $\mathbf{C}$, the natural maps form a pushout square in $\mathbf{C}$*

$$\begin{array}{ccc} (\Delta[n]\otimes L_nX) \cup_{\partial\Delta[n]\otimes L_nX} (\partial\Delta[n]\otimes X_n) & \longrightarrow & \Delta[n]\otimes X_n \\ \downarrow & & \downarrow \\ \mathrm{Sk}^{n-1}|X_\bullet| & \longrightarrow & \mathrm{Sk}^n|X_\bullet|. \end{array} \tag{8.2.32}$$

The map on the top is the pushout-product of $\partial\Delta[n] \to \Delta[n]$ and the latching map $L_nX \to X_n$, using the tensor $\otimes\colon \mathbf{sSet}\times\mathbf{C}\to\mathbf{C}$.

[4] We are defining the n-skeleton as an object of $\mathbf{C}$ that maps to the realization $|X_\bullet|$. Note that some authors instead define the n-skeleton as a simplicial object that maps to $X_\bullet$. The realization of their n-skeleton is our n-skeleton, so they are not the same, but they are closely related.

Proof. We compare the universal property of $|\mathrm{Sk}^n X_\bullet|$ to the universal property of the pushout of the other three terms. In each case, we get that maps to some other object Y are given by maps $\Delta^\ell \otimes X_\ell \to Y$ for all $0 \leq \ell \leq n$, that agree along every face and degeneracy map. □

Lemma 8.2.33. *For any simplicial object $X_\bullet$ in $\mathbf{C}$ and any $n \geq 0$, the map $\varnothing \to L_n X$ is a finite composition of pushouts of latching maps $L_k X \to X_k$, for varying values of $k \in \{0, 1, \ldots, n-1\}$.*

Proof. The idea is to examine the cube-shaped diagram from Proposition 8.2.18. Make a list of sets of vertices, starting with the empty set, and at each stage adding one vertex to the set. Make sure not to add the vertex for $S \subseteq \{1, \ldots, n\}$ until all of the subsets of S have been added. At each step in the process, take the colimit of the diagram along all of the vertices that we have selected. This gives a sequence of maps in $\mathbf{C}$

$$\varnothing = C_0 \to C_1 \to C_2 \to \cdots \to C_{2^n-1} = L_n X.$$

Each of the maps $C_{j-1} \to C_j$ is obtained by adding a single new vertex to the colimit. By comparing universal properties, we show that this is a pushout of a latching map

$$\begin{array}{ccc} L_k X & \longrightarrow & X_k \\ \downarrow & & \downarrow \\ C_{j-1} & \longrightarrow & C_j, \end{array}$$

and the conclusion follows. □

Proposition 8.2.34. *Under the assumptions of Theorem 8.2.27, if $X_\bullet$ is Reedy cofibrant then each latching object $L_n X$ and each skeleton $\mathrm{Sk}^n |X_\bullet|$ is cofibrant, and each of the maps $\mathrm{Sk}^{n-1}|X_\bullet| \to \mathrm{Sk}^n |X_\bullet|$ is a cofibration.*

Proof. This is by induction on n. When $n = 0$, $L_0 X = \varnothing$ is tautologically cofibrant and $\mathrm{Sk}^0 |X_\bullet| = X_0$ is cofibrant since $\varnothing \to X_0$ is assumed to be a cofibration.

When moving from $(n-1)$ to n, we use Lemma 8.2.33 to see that $L_n X$ is cofibrant, since we have assumed that each map $L_k X \to X_k$ is a cofibration, and cofibrations are preserved by pushouts and compositions. We then examine the square (8.2.32). The top horizontal map is a pushout-product of cofibrations and is therefore a cofibration by assumption. Therefore its pushout $\mathrm{Sk}^{n-1}|X_\bullet| \to \mathrm{Sk}^n |X_\bullet|$ is a cofibration. By inductive hypothesis $\mathrm{Sk}^{n-1}|X_\bullet|$ is cofibrant, so therefore $\mathrm{Sk}^n |X_\bullet|$ is cofibrant as well. □

Taking the colimit over n shows that $|X_\bullet|$ is cofibrant, which finishes the proof of the first half of Theorem 8.2.27.

Proposition 8.2.35. *Under the assumptions of Theorem 8.2.27, if $X_\bullet$ and $Y_\bullet$ are Reedy cofibrant and $X_\bullet \to Y_\bullet$ is a map of simplicial objects that is an equivalence at each level, then each map of latching objects $L_nX \to L_nY$ and each map of skeleta $\mathrm{Sk}^n|X_\bullet| \to \mathrm{Sk}^n|Y_\bullet|$ is a weak equivalence.*

Proof. Again we do this by induction on n. When $n = 0$, the theorem is already trivially true. For larger values of n, we observe that the map $L_nX \to L_nY$ is the last of a sequence of maps $C_j(X) \to C_j(Y)$, with C_j being the jth term of the filtration in Lemma 8.2.33. The map $C_j(X) \to C_j(Y)$ is the pushout of a map of spans of the form

$$\begin{array}{ccccc} C_{j-1}(X) & \longleftarrow & L_kX & \longrightarrow & X_k \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ C_{j-1}(Y) & \longleftarrow & L_kY & \longrightarrow & Y_k. \end{array}$$

The maps marked $\sim$ are weak equivalences in $\mathbf{C}$ by inductive hypothesis on j and n. (Note that the only values of k that appear are $< n$.) By the gluing lemma, therefore the map of pushouts $C_j(X) \to C_j(Y)$ is a weak equivalence as well. By induction on j, this shows that $L_nX \to L_nY$ is a weak equivalence.

For the skeleta, the inductive step is similar. By Proposition 8.2.31, the map $\mathrm{Sk}^n|X_\bullet| \to \mathrm{Sk}^n|Y_\bullet|$ is the pushout of a map of spans of the form

$$\begin{array}{ccccc} \mathrm{Sk}^{n-1}|X_\bullet| & \longleftarrow & (\Delta[n]\otimes L_nX)\cup_{\partial\Delta[n]\otimes L_nX}(\partial\Delta[n]\otimes X_n) & \longrightarrow & \Delta[n]\otimes X_n \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ \mathrm{Sk}^{n-1}|Y_\bullet| & \longleftarrow & (\Delta[n]\otimes L_nY)\cup_{\partial\Delta[n]\otimes L_nY}(\partial\Delta[n]\otimes Y_n) & \longrightarrow & \Delta[n]\otimes Y_n. \end{array}$$

The map on the left marked $\sim$ is a weak equivalence by inductive hypothesis. The one on the right is a weak equivalence by the assumption that $\Delta[n]\otimes -$ preserves weak equivalences. The one in the middle is a weak equivalence by combining the already-proven fact that $L_nX \to L_nY$ is a weak equivalence, the pushout-product lemma, and the gluing lemma. Then each row of the diagram is a pushout along a cofibration by the pushout-product lemma. So by the gluing lemma once again we get a weak equivalence of pushouts $\mathrm{Sk}^n|X_\bullet| \to \mathrm{Sk}^n|Y_\bullet|$. □

Taking the the colimit of the weak equivalences $\mathrm{Sk}^n|X_\bullet| \to \mathrm{Sk}^n|Y_\bullet|$ along the cofibrations $\mathrm{Sk}^{n-1}|X_\bullet| \to \mathrm{Sk}^n|X_\bullet|$ and applying the colimit lemma one more time, we get a weak equivalence $|X_\bullet| \to |Y_\bullet|$, which finishes the proof of Theorem 8.2.27.

We are done with the main theorem, but we should also prove the following variant of Proposition 8.2.31. Although we didn't need it for the proof of Theorem 8.2.27, it is helpful in many other places. Suppose that $X_\bullet$ is any simplicial object with the property

that each latching map $L_nX \to X_n$ is the inclusion of a summand. In other words, suppose that

$$X_n \cong (L_nX) \amalg X_n^{nd}.$$

We think of X_n^{nd} as an object of $\mathbf{C}$ that contains "only the nondegenerate points."

Lemma 8.2.36. *Under this assumption, the pushout square* (8.2.32) *simplifies to*

$$\begin{array}{ccc} \partial\Delta[n] \otimes X_n^{nd} & \longrightarrow & \Delta[n] \otimes X_n^{nd} \\ \downarrow & & \downarrow \\ \mathrm{Sk}^{n-1}|X_\bullet| & \longrightarrow & \mathrm{Sk}^n|X_\bullet|. \end{array} \tag{8.2.37}$$

Proof. This is again by comparing universal properties. □

This assumption always holds for simplicial spaces that come from simplicial sets. In that case X_n^{nd} is literally the set of nondegenerate points, and the above lemma shows that $|X_\bullet|$ is a CW complex with one n-cell for every point of X_n^{nd}, as we claimed in Theorem 7.1.20.

8.3 The bar construction in any simplicial model category

Now we're ready to form the bar construction $B(X, G, Y)$ and $B(X, \mathbf{I}, Y)$ in any simplicial model category $\mathbf{C}$. This will be very general, but we encourage the reader to focus on the special case where $\mathbf{C}$ is the category of symmetric or orthogonal spectra.

8.3.1 The definition

First we're going to give a symbolic definition. Then we'll discuss what we have to assume for the symbols in this definition to actually make sense.

Definition 8.3.1. For any category $\mathbf{I}$ and any diagrams $X : \mathbf{I} \to \mathbf{C}$ and $Y : \mathbf{I}^{\mathrm{op}} \to \mathbf{C}$, the two-sided nerve $N_n^{\otimes}(X, \mathbf{I}, Y)$ is the simplicial object in $\mathbf{C}$ with the formula

$$N_n^{\otimes}(X, \mathbf{I}, Y) := \coprod_{c_0, c_1, \cdots, c_n \in \mathrm{ob}\,\mathbf{I}} X(c_0) \otimes \mathbf{I}(c_0, c_1) \otimes \cdots \otimes \mathbf{I}(c_{n-1}, c_n) \otimes Y(c_n),$$

with face and degeneracy maps given by the same formulas in Definition 7.4.7. The **bar construction** is its realization, which is an object of $\mathbf{C}$:

$$B^{\otimes}(X, \mathbf{I}, Y) = |N_\bullet^{\otimes}(X, \mathbf{I}, Y)|.$$

When $\mathbf{I}$ has one object, this can all be written more simply as

$$N_n^{\otimes}(X, G, Y) := X \otimes G^{\otimes n} \otimes Y,$$

whose realization is $B^{\otimes}(X, G, Y)$.

Now, when does this make sense? For starters, to take the realization, $\mathbf{C}$ should be a simplicial model category, or at least a category with a tensor product functor $\mathbf{sSet} \times \mathbf{C} \to \mathbf{C}$. But we have another problem: we have to be able to form these tensors $\otimes$. There are a few different situations in which this can be done:

(1) The simplest setting is when $(\mathbf{C}, \otimes, I)$ is symmetric monoidal and there is a strong symmetric monoidal functor $\mathbf{sSet} \to \mathbf{C}$. Then $\mathbf{I}$ is a diagram enriched in $\mathbf{C}$, and X and Y are enriched functors to $\mathbf{C}$. So there are multiplication maps

$$\begin{aligned} \mathbf{I}(a,b) \otimes \mathbf{I}(b,c) &\longrightarrow \mathbf{I}(a,c), & I &\longrightarrow \mathbf{I}(a,a), \\ X(a) \otimes \mathbf{I}(a,b) &\longrightarrow X(b), & \mathbf{I}(a,b) \otimes Y(b) &\longrightarrow Y(a), \end{aligned}$$

all of which are morphisms in $\mathbf{C}$.

For instance, $\mathbf{C}$ *is the category of symmetric spectra,* $\mathbf{I}$ *has one object, so it is a ring spectrum* R*, and* X *and* Y *are* R*-modules.*

(2) $\mathbf{C}$ might fail to have a tensor product, but there could be some smaller symmetric monoidal category $(\mathbf{V}, \otimes, I)$ with a strong symmetric monoidal functor $\mathbf{sSet} \to \mathbf{V}$, and a tensor product functor $\boxtimes \colon \mathbf{V} \times \mathbf{C} \to \mathbf{C}$.[5] Then $\mathbf{I}$ is a diagram enriched in $\mathbf{V}$, not $\mathbf{C}$, and only one of the diagrams X or Y is $\mathbf{C}$, while the other is in $\mathbf{V}$. So for instance if $X \colon \mathbf{I} \to \mathbf{V}$ and $Y \colon \mathbf{I}^{\mathrm{op}} \to \mathbf{C}$, then we have multiplication maps of the form

$$\begin{aligned} \mathbf{I}(a,b) \otimes \mathbf{I}(b,c) &\longrightarrow \mathbf{I}(a,c), & I &\longrightarrow \mathbf{I}(a,a), \\ X(a) \otimes \mathbf{I}(a,b) &\longrightarrow X(b), & \mathbf{I}(a,b) \boxtimes Y(b) &\longrightarrow Y(a). \end{aligned}$$

The first three of these are in $\mathbf{V}$, and only the last one is in $\mathbf{C}$. However, when we multiply everything together at the end to form the two-sided nerve, we still get a simplicial object of $\mathbf{C}$!

For instance, $\mathbf{C}$ *is the category of sequential spectra,* $\mathbf{V}$ *is unbased spaces,* X *is a topological diagram of spectra, and perhaps* $Y = *$ *is a constant diagram on the one-point space.*

[5] It won't be necessary for this tensor to have coherences as in Definition 8.1.1! But if it doesn't, we have to make sure to take all the tensor products in $\mathbf{V}$ first, and take the tensor $\boxtimes$ at the very end.

(3) The category **I** might just be discrete. Any discrete category can be included into a symmetric monoidal category **V** or **C** by sending the set S to the object $S \otimes I = \coprod_{s \in S} I$. So for instance we could form the bar construction where X takes values in **C**, but **I** and Y are discrete.

For instance, **C** *is a simplicial model category that doesn't have a tensor product, X is an ordinary diagram in* **C***, and again $Y = *$ is a constant diagram on the one-point set.*

We care about all of these settings, but (1) and (3) are special cases of (2), so we'll prove our results in the setting of (2), assuming in addition that **V** and **C** are model categories compatible with all of the tensor products. Now let's give lots of examples.

Example 8.3.2. If $\mathbf{V} = \mathbf{C} = \mathbf{Top}$, the bar construction of Definition 8.3.1 is the categorical bar construction we gave in Definition 7.4.7:

$$N_n^{\times}(X, \mathbf{I}, Y) = \coprod_{c_0, c_1, \cdots, c_n \in \mathrm{ob}\,\mathbf{I}} X(c_0) \times \mathbf{I}(c_0, c_1) \times \cdots \times \mathbf{I}(c_{n-1}, c_n) \times Y(c_n).$$

When $Y = *$ is the constant diagram and **I** is well-based, this gives the Bousfield–Kan homotopy colimit for unbased spaces, with formula $B^{\times}(X, \mathbf{I}, *)$. And when **I** has one object, its morphism monoid G is well-based, and one of the two spaces X or Y is $*$, this is the **homotopy orbits** of the remaining space, $B(*, G, Y)$ or $B(X, G, *)$.

Example 8.3.3. If $\mathbf{V} = \mathbf{Top}$ and $\mathbf{C} = \mathbf{Top}_*$, this is a new variant of the bar construction where the category is unbased but one of the diagrams is based. So for instance we could let X be a diagram of based spaces on **I**, and $Y = *$ is the contant diagram of unbased spaces.

$$\begin{aligned} N_n^{\wedge}(X, \mathbf{I}_+, S^0) &= \bigvee_{c_0, c_1, \cdots, c_n \in \mathrm{ob}\,\mathbf{I}} X(c_0) \wedge \big(\mathbf{I}(c_0, c_1) \times \cdots \times \mathbf{I}(c_{n-1}, c_n) \times *\big)_+ \\ &\cong \bigvee_{c_0, c_1, \cdots, c_n \in \mathrm{ob}\,\mathbf{I}} X(c_0)_+ \wedge \mathbf{I}(c_0, c_1)_+ \wedge \cdots \wedge \mathbf{I}(c_{n-1}, c_n)_+ \wedge *_+. \end{aligned}$$

If the spaces $X(c)$ and $\mathbf{I}(c, c)$ are well-based, the realization of this is called the **Bousfield–Kan homotopy colimit for based spaces**. It is straightforward to check that this is a quotient of unbased homotopy colimits

$$B^{\wedge}(X, \mathbf{I}_+, S^0) \cong B^{\times}(X, \mathbf{I}, *)/B^{\times}(*, \mathbf{I}, *).$$

In other words, using (u) and (b) to denote unbased and based homotopy colimits,

$$\operatorname*{hocolim}^{(b)}_{c \in \mathbf{I}} X(c) \simeq \frac{\operatorname*{hocolim}^{(u)}_{c \in \mathbf{I}} X(c)}{\operatorname*{hocolim}^{(u)}_{c \in \mathbf{I}} (*)}.$$

We could also make a version of this for two diagrams of based spaces, simply by taking the category to be $\mathbf{I}_+$ and doing everything in the category of based spaces, $\mathbf{V} = \mathbf{C} = \mathbf{Top}_*$. Again this gives the quotient of unbased bar constructions

$$B^\wedge(X, \mathbf{I}_+, Y) \cong B^\times(X, \mathbf{I}, Y)/B^\times(*, \mathbf{I}, *).$$

See Exercise 8.7.4.

Example 8.3.4. As a special case of the previous example, let $\mathbf{V} = \mathbf{Top}$, $\mathbf{C} = \mathbf{Top}_*$, and let $\mathbf{I}$ have one object, whose morphisms are a (well-based) topological group or monoid G. If X is the one-point unbased space $*$ and Y is a based space with action by G, given by $G_+ \wedge Y \to Y$, then the bar construction as we've just defined it is

$$N_n^\wedge(S^0, G_+, Y) = S^0 \wedge (G^{\times n})_+ \wedge Y.$$

The realization $B^\wedge(S^0, G_+, Y)$ is called the **based homotopy orbits** of Y. As before, this gives a quotient of unbased bar constructions

$$B^\wedge(S^0, G_+, Y) \cong B^\times(*, G, Y)/B^\times(*, G, *).$$

In other words, using Y_{uhG} and Y_{bhG} to distinguish the unbased and based versions of homotopy orbits, we have

$$Y_{bhG} \simeq (Y_{uhG})/(*_{uhG}).$$

Warning! Most authors write the based homotopy orbits as Y_{hG}, even though it is not the same as the unbased homotopy orbits! Pay attention to the context!

Example 8.3.5. If we give S^0 the trivial G-action, its unbased homotopy orbits are the space $BG \amalg BG$. Its based homotopy orbits are $BG \amalg * = BG_+$.

Remark 8.3.6. You might wonder why we aren't as interested in the case where $\mathbf{V} = \mathbf{C} = \mathbf{Top}_*$. It's mainly because there are no colimits of diagrams along based topological categories, as we discussed in Remark 5.3.46 and Exercise 7.7.28. On the other hand, we will still work in this case from time to time, because the bar construction $B^\wedge(X, \mathbf{I}, \mathbf{J})$ defines the homotopy left Kan extension along a functor $\alpha\colon \mathbf{I} \to \mathbf{J}$ of categories enriched in based spaces.

Example 8.3.7. Let $\mathbf{V} = \mathbf{C}$ be any symmetric monoidal category of spectra, for instance, symmetric spectra $\mathbf{Sp}^\Sigma_{\mathbf{Top}}$. Then $\mathbf{I}$ is a **spectral category**: the objects $\mathbf{I}(a, b)$ are spectra, and the composition maps use the smash product of spectra, $\mathbf{I}(a, b) \wedge \mathbf{I}(b, c) \to \mathbf{I}(a, c)$. The bar construction is all smash products of spectra:

$$N_n^\wedge(X, \mathbf{I}, Y) = \bigvee_{c_0, c_1, \cdots, c_n \in \mathrm{ob}\,\mathbf{I}} X(c_0) \wedge \mathbf{I}(c_0, c_1) \wedge \cdots \wedge \mathbf{I}(c_{n-1}, c_n) \wedge Y(c_n).$$

Again, there is no version of the homotopy colimit for these because $B^\wedge(X,\mathbf{I},*)\cong *$. Still, this is a very useful construction in higher algebra.

As a special case, if $\mathbf{I}$ has one object then it gives a ring spectrum R, and X and Y are right and left R-module spectra, respectively. Then the two-sided nerve is a simplicial spectrum that at simplicial level n is the smash product

$$N_n^\wedge(X,R,Y) = X\wedge R^{\wedge n}\wedge Y.$$

Its realization $B^\wedge(X,R,Y)$ turns out to be the correct version of $\mathrm{Tor}_R(X,Y)$, or the derived tensor product of X and Y, provided the spectra R, X, and Y are all cofibrant as spectra. See Corollary 8.3.20.

Example 8.3.8. For our final example, let $\mathbf{V}$ be the category of "unbased spaces" (which could be **Top** or **sSet**) and let $\mathbf{C}$ be the category of "spectra." (Any category of spectra that has a tensor product $\mathbf{V}\times\mathbf{C}\to\mathbf{C}$ will do. So for instance if $\mathbf{V}=\mathbf{Top}$, then $\mathbf{C}$ could be sequential spectra $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}}$. Up to weak equivalence, the choice of model doesn't matter.)

Then we have a category $\mathbf{I}$ enriched in unbased spaces, and a diagram X of spectra with action maps of the form

$$X(a)\wedge\mathbf{I}(a,b)_+\to X(b),$$

and a diagram Y of unbased spaces with maps of the form

$$\mathbf{I}(a,b)\times Y(b)\to Y(a).$$

Then the two sided nerve has the formula

$$\begin{aligned} N_n(X,\mathbf{I},Y) &= \bigvee_{c_0,c_1,\cdots,c_n\in\mathrm{ob}\,\mathbf{I}} X(c_0)\wedge\big(\mathbf{I}(c_0,c_1)\times\cdots\times\mathbf{I}(c_{n-1},c_n)\times Y(c_n)\big)_+ \\ \cong N_n^\wedge(X,\mathbf{I}_+,Y_+) &\cong \bigvee_{c_0,c_1,\cdots,c_n\in\mathrm{ob}\,\mathbf{I}} X(c_0)_+\wedge\mathbf{I}(c_0,c_1)_+\wedge\cdots\wedge\mathbf{I}(c_{n-1},c_n)_+\wedge Y(c_n)_+, \end{aligned}$$

and its realization is the categorical bar construction for spectra. If Y is the constant diagram of unbased spaces at $*$, this gives the **Bousfield–Kan homotopy colimit for spectra** $B(X,\mathbf{I},*)=B^\wedge(X,\mathbf{I}_+,S^0)$.[6] Notice that it really is just the homotopy colimit of based spaces at each spectrum level, as we promised earlier.

As a special case, if we have a monoid or group G in spaces, a space X with left G-action, and a spectrum Y with right G-action. The bar construction has the form

$$\begin{aligned} N_n(X,G,Y) &= (X\times G^{\times n})_+\wedge Y \\ \cong N_n^\wedge(X_+,G_+,Y) &\cong X_+\wedge G_+^{\wedge n}\wedge Y, \end{aligned}$$

[6] As before, we will also need to assume that either each space $\mathbf{I}(a,b)$ is Quillen cofibrant, or in **Top** that each space $\mathbf{I}(c,c)$ is well-based and each level of each spectrum $X(c)$ is well-based. When $\mathbf{I}$ is discrete, all of these cofibrancy assumptions on $\mathbf{I}$ are automatically true.

Taking $X = *$, this is the definition of the **homotopy orbit spectrum** Y_{hG}. It is just the based homotopy orbits at each spectrum level.[7]

Remark 8.3.9. It is not too hard to check that any strong symmetric monoidal functor $\mathbf{C} \to \mathbf{C}'$ that preserves colimits, will also preserve the bar construction. In the setting of two categories $\mathbf{V}$ and $\mathbf{C}$ a similar statement is true – think about what that statement should be!

This means that we can pass between any of the different definitions of "spaces" and "spectra," and we get the same bar construction. This will be true up to weak equivalence as well, so long as all the inputs are cofibrant, so that the bar construction is cofibrant as well. (We will prove this in Theorem 8.3.12.)

8.3.2 Homotopy invariance and the bar lemma

Now we begin proving that the bar construction in an arbitrary simplicial model category $\mathbf{C}$ has the same properties we proved for $\mathbf{C} = \mathbf{Top}$ in Section 7.4.5. We begin by making our assumptions completely precise.

Assumption 8.3.10. We assume throughout this section that

- $(\mathbf{V}, \otimes, I)$ is a symmetric monoidal simplicial model category, in other words a symmetric monoidal model category with a strong symmetric monoidal Quillen left adjoint $\mathbf{sSet} \to \mathbf{V}$,
- $\mathbf{C}$ is a model category and $\boxtimes\colon \mathbf{V} \times \mathbf{C} \to \mathbf{C}$ is a Quillen tensor,
- $I \boxtimes X \cong X$ when $I \in \mathbf{V}$ is the unit and $X \in \mathbf{C}$ is any object,[8]
- $\mathbf{I}$ is a small category enriched in $\mathbf{V}$, and
- X is a covariant diagram on $\mathbf{I}$ and Y is a contravariant diagram on $\mathbf{I}$, one of which takes values in $\mathbf{V}$ and the other of which takes values in $\mathbf{C}$.

For simplicity, the reader is encouraged to focus on the case where $\mathbf{C}$ is the category of spectra, and $\mathbf{V}$ is either unbased spaces or spectra.

Definition 8.3.11. As in Definition 7.5.20, we say that $\mathbf{I}$ is **pointwise cofibrant** if the unit maps $I \to \mathbf{I}(c, c)$ and canonical maps $\varnothing \to \mathbf{I}(a, b)$ are all cofibrations in the model category $\mathbf{V}$. We also say that X is **pointwise cofibrant** if each object $X(c)$ is cofibrant in $\mathbf{V}$ or $\mathbf{C}$.

[7] Again, in order for this to have the right homotopy type, we need to assume that G is Quillen cofibrant, or that G is well-based and the levels of Y are also well-based.

[8] This assumption isn't really necessary, but it pretty much always holds, and without it some of the theorem statements get messier.

Similarly, a **pointwise equivalence** of categories $\mathbf{I} \to \mathbf{I}'$ is one that is a weak equivalence in $\mathbf{V}$ on each mapping spaces $\mathbf{I}(a,b) \xrightarrow{\sim} \mathbf{I}'(a,b)$, and a **pointwise equivalence** of diagrams is one that is a weak equivalence in $\mathbf{V}$ or $\mathbf{C}$ at each term.

Theorem 8.3.12 (Homotopy invariance of the bar construction)**.** *If all categories and diagrams are pointwise cofibrant, then the bar construction $B(X,\mathbf{I},Y)$ is cofibrant, and any pointwise equivalence of categories $\mathbf{I} \to \mathbf{I}'$ and compatible pointwise equivalences of diagrams $X \to X'$ and $Y \to Y'$ induce a weak equivalence*

$$B(X,\mathbf{I},Y) \xrightarrow{\sim} B(X',\mathbf{I}',Y').$$

The "compatibility" between the map of categories $g\colon \mathbf{I} \to \mathbf{I}'$ and the maps of diagrams $f\colon X \to X'$ and $h\colon Y \to Y'$ is that the following diagrams commute. (You may have worked this out already in Exercise 7.7.26.)

$$\begin{array}{ccc} X(b) & \longleftarrow & X(a) \boxtimes \mathbf{I}(a,b) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\boxtimes g} \\ X'(b) & \longleftarrow & X'(a) \boxtimes \mathbf{I}'(a,b) \end{array} \qquad \begin{array}{ccc} \mathbf{I}(a,b) \otimes Y(b) & \longrightarrow & Y(a) \\ {\scriptstyle g\otimes h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathbf{I}'(a,b) \otimes Y'(b) & \longrightarrow & Y'(a) \end{array}$$

Proof. Without loss of generality X is the diagram in the larger category $\mathbf{C}$ and Y is the diagram in the smaller category $\mathbf{V}$. By Theorem 8.2.25, it suffices to show that the simplicial object $N_\bullet(X,\mathbf{I},Y)$ is Reedy cofibrant, using the cofibrations in the model structure on $\mathbf{C}$.

Notice that the degeneracy maps of the bar construction only hit those tuples in which some pair of adjacent objects repeat, $c_0,\dots,c_i,c_i,\dots,c_n$. Using this, we see that the cube in Proposition 8.2.18 can be split into a coproduct of cubes, one for each tuple of objects $c_0,c_1,\cdots,c_n \in \operatorname{ob}\mathbf{I}$, and cubes have nontrivial X_i for $i<n$ only when we have repeats in the tuple. Using this observation, we rewrite the latching map as the disjoint union of pushout-products

$$(\varnothing \to X(c_0))\square(\varnothing \to \mathbf{I}(c_0,c_1))\square\cdots\square(I \to \mathbf{I}(c_i,c_i))\square\cdots\square(\varnothing \to Y(c_n)),$$

with one nontrivial map $I \to \mathbf{I}(c_i,c_i)$ for each repeated index, and trivial maps $\varnothing \to \mathbf{I}(c_{i-1},c_i)$ for the rest. In particular, if the tuple is constant $c_0 = c_1 = \cdots = c_n$, we get n nontrivial maps

$$\begin{aligned}&(\varnothing \to X(c_0))\square(I \to \mathbf{I}(c_0,c_1))\square\cdots\square(I \to \mathbf{I}(c_{n-1},c_n))\square(\varnothing \to Y(c_n)) \\ &\quad = X(c_0) \boxtimes (I \to \mathbf{I}(c_0,c_1))\square\cdots\square(I \to \mathbf{I}(c_{n-1},c_n)) \otimes Y(c_n).\end{aligned}$$

At any rate, since we assumed that $\mathbf{I}$, X and Y are pointwise cofibrant, each of the maps in the pushout-product is a cofibration, in $\mathbf{V}$ or in $\mathbf{C}$. Since $\mathbf{V}$ is a symmetric monoidal model category and $\boxtimes\colon \mathbf{V}\times\mathbf{C} \to \mathbf{C}$ is a Quillen tensor, the big pushout-product is therefore a cofibration in $\mathbf{C}$. Therefore $N_\bullet(X,\mathbf{I},Y)$ is Reedy cofibrant and we are done. □

Remark 8.3.13. As soon as $\mathbf{I}$, X, and Y are nice enough that the bar construction $B(X,\mathbf{I},Y)$ preserves pointwise equivalences, it will also preserve Dwyer-Kan equivalences in the $\mathbf{I}$ variable. The proof is the same as in Exercise 7.7.27.

Example 8.3.14. Taking $\mathbf{V}$ and $\mathbf{C}$ to be any symmetric monoidal model category of spectra, we conclude that for any ring spectrum R and any right and left R-modules M and N, respectively, the bar construction $B^{\wedge}(M,R,N)$ preserves equivalences whenever the maps $* \to M$, $\mathbb{S} \to R$, and $* \to N$ are cofibrations of spectra.

The analogue of the bar lemma (Lemma 7.4.20) is as follows. As before, if $X : \mathbf{I} \to \mathbf{C}$, then the bar constructions $B(X,\mathbf{I},\mathbf{I}(-,c))$ also form a diagram over $\mathbf{I}$, that we call "$B(X,\mathbf{I},\mathbf{I})$." The same is true on the other side: if $Y : \mathbf{I}^{\mathrm{op}} \to \mathbf{C}$ is a contravariant diagram then the bar constructions $B(\mathbf{I}(c,-),\mathbf{I},Y)$ form a contravariant diagram that we call "$B(\mathbf{I},\mathbf{I},Y)$."

Lemma 8.3.15 (Bar Lemma)**.** *For any diagram $X : \mathbf{I} \to \mathbf{C}$, we have weak equivalences in $\mathbf{C}$*

$$B(X,\mathbf{I},\mathbf{I}(-,c)) \xrightarrow{\sim} X(c)$$

that together form a weak equivalence of diagrams $B(X,\mathbf{I},\mathbf{I}) \xrightarrow{\sim} X$.

Similarly if for any diagram $Y : \mathbf{I}^{\mathrm{op}} \to \mathbf{C}$, we have weak equivalences in $\mathbf{C}$

$$B(\mathbf{I}(c,-),\mathbf{I},Y) \xrightarrow{\sim} Y(c)$$

that together form a weak equivalence of diagrams $B(\mathbf{I},\mathbf{I},Y) \xrightarrow{\sim} Y$.

Proof. The proof in Lemma 7.3.11, as generalized in Exercise 7.7.24, shows that the map $B(X,\mathbf{I},\mathbf{I}(-,c)) \xrightarrow{\sim} X(c)$ is a simplicial homotopy equivalence in $\mathbf{C}$, in the sense of Section 8.1.3. Then by Lemma 8.1.29, this homotopy equivalence is also a weak equivalence. The proof for Y is the same. □

Note that this bar lemma also applies if X or Y takes values in $\mathbf{V}$ instead, giving a weak equivalence of diagrams in $\mathbf{V}$. We just run the same proof and take $\mathbf{C} = \mathbf{V}$.

8.3.3 The bar construction is a left-derived functor

To make sense of this, we have to say what the bar construction is a left-derived functor of! We define the coend of two diagrams, $X \otimes_{\mathbf{I}} Y$, by the same formula as in Definition 7.4.1:

$$\coprod_{c_0,c_1 \in \mathbf{I}} X(c_0) \otimes \mathbf{I}(c_0,c_1) \otimes Y(c_1) \rightrightarrows \coprod_{c \in \mathbf{I}} X(c) \otimes Y(c) \longrightarrow X \otimes_{\mathbf{I}} Y. \tag{8.3.16}$$

Again, this makes sense even if one of the diagrams is in $\mathbf{V}$ and the other is in $\mathbf{C}$; the coend will be in $\mathbf{C}$. The following is an exercise in commuting the tensor product with colimits:

Lemma 8.3.17. *If* $\mathbf{V} = \mathbf{sSet}$ *or* $\mathbf{Top}$ *and* $\mathbf{I}$ *is a discrete category, then if* X *is an* $\mathbf{I}$*-diagram in* $\mathbf{C}$ *and* $Y = *$ *is the constant diagram on the one-point space, then the coend is just the colimit,*

$$(X \otimes_{\mathbf{I}} *) \cong \operatorname*{colim}_{c \in \mathbf{I}} X(c).$$

When $\mathbf{I}$ is not discrete, it is sometimes possible to take define $\mathrm{const}(I)$, the constant diagram $\mathbf{I}^{\mathrm{op}} \to \mathbf{V}$ on the unit object $I \in \mathbf{V}$. When this exists, then $X \otimes_{\mathbf{I}} \mathrm{const}(I)$ is the correct notion of an **enriched colimit**, or colimit along a category enriched in $\mathbf{V}$. See Exercise 8.7.40.

Lemma 8.3.18. *We have isomorphisms* $B(X, \mathbf{I}, \mathbf{I}) \times_{\mathbf{I}} Y \cong B(X, \mathbf{I}, Y) \cong X \times_{\mathbf{I}} B(\mathbf{I}, \mathbf{I}, Y)$.

Proof. Simply phrase each step of the proof of Lemma 7.4.21 in category-theoretic terms, rather than a statement about sets that have a topology. (We believe it is useful to build intuition in topological spaces before carrying the argument over to any category.) The one fact we need is that the geometric realization commutes with colimits and tensor products, which we proved in Proposition 8.2.8 and Proposition 8.2.9. (We also need the variant of Proposition 8.2.9 where the tensor is $\boxtimes$, but the proof of this variant is identical to the proof of Proposition 8.2.9.) □

Theorem 8.3.19. *When* $\mathbf{I}$*,* X*, and* Y *are pointwise cofibrant, the bar construction* $B(X, \mathbf{I}, Y)$ *is the left-derived functor of the coend* $X \otimes_{\mathbf{I}} Y$*:*

$$B(X, \mathbf{I}, Y) \simeq (Q_1 X) \otimes_{\mathbf{I}} (Q_2 Y), \qquad Q_1 X \xrightarrow{\sim} X, \qquad Q_2 Y \xrightarrow{\sim} Y.$$

In fact, it can be constructed by only deforming X*,*

$$B(X, \mathbf{I}, Y) \cong (QX) \otimes_{\mathbf{I}} Y, \qquad QX \xrightarrow{\sim} X,$$

or by only deforming Y*.*

The proof is the same as in Theorem 7.3.14 and Exercise 7.7.25. For this reason, under these assumptions the bar construction $B(X, \mathbf{I}, Y)$ is also called the **homotopy coend**.

By the extension of this proof in Exercise 7.7.31, this also holds with no assumptions on X, $\mathbf{I}$, and Y, but then we need to take the deformation $B(X', \mathbf{I}', \mathbf{I}') \to X$, where $X' \to X$ is a pointwise cofibrant replacement of X and $\mathbf{I}' \to \mathbf{I}$ is a pointwise cofibrant replacement of $\mathbf{I}$. The subcategory of cofibrant objects is the collection of all pairs of diagrams of the form $(B(X', \mathbf{I}', \mathbf{I}'), Y')$, where X' and Y' are pointwise cofibrant and $\mathbf{I}'$ is a pointwise cofibrant diagram with a map to $\mathbf{I}$. (We do need to assume that these cofibrant replacements exist. See Theorem A.2.20 below.) So any deformation that lands in this subcategory gives the left-derived functor of $- \otimes_{\mathbf{I}} -$.

Corollary 8.3.20. *In any of our symmetric monoidal categories of spectra, the bar construction $B(QM, QR, QN)$ is the left-derived functor of the relative smash product $M \wedge_R N$.*

Here Q denotes cofibrant replacement in the category of R-modules or the category of ring spectra from Theorem 6.2.36. In fact, it's enough for $* \to M$, $\mathbb{S} \to R$, and $* \to N$ to be cofibrations of spectra; they don't have to be cofibrations of modules or ring spectra.

8.3.4 The Bousfield–Kan homotopy colimit

We've accomplished one major goal, showing that the bar construction in spectra preserves equivalences when the inputs are cofibrant. Our other major goal is to understand the homotopy colimit of a diagram of spectra. The following is a corollary of Theorem 8.3.19:

Corollary 8.3.21. *Assume $\mathbf{I}$ is pointwise cofibrant, the category $\mathbf{C}^{\mathbf{I}}$ has a cofibrant replacement functor Q, the unit $I \in \mathbf{V}$ is cofibrant, and the constant diagram $\mathrm{const}(I)$ exists. Then $B(QX, \mathbf{I}, \mathrm{const}(I))$ is a left-derived functor of the enriched colimit $(X \otimes_{\mathbf{I}} \mathrm{const}(I))$.*

As in Definition 7.4.23, we call $B(QX, \mathbf{I}, \mathrm{const}(I))$ the **Bousfield–Kan homotopy colimit** of X. Recall from Definition 3.4.29 that for any category $\mathbf{C}$ with weak equivalences, the homotopy colimit is defined to be the left-derived functor of the colimit, if it exists.

Corollary 8.3.22. *If $\mathbf{C}$ is any cofibrantly generated simplicial model category, and $\mathbf{I}$ is any discrete or simplicially enriched small category, then homotopy colimits of diagrams $X : \mathbf{I} \to \mathbf{C}$ always exist and are given by the Bousfield–Kan formula:*

$$B(QX, \mathbf{I}, *) \simeq \operatorname*{hocolim}_{c \in \mathbf{I}} X(c).$$

As always, the left-derived functor of the colimit is unique up to equivalence (Corollary 3.5.3), so this is equivalent to any other model of the homotopy colimit that exists.

Example 8.3.23. Let's focus on the case where $\mathbf{I}$ is discrete. Corollary 8.3.21 gives us a formula for how to build a homotopy colimit of a diagram X in any simplicial model category $\mathbf{C}$. The recipe is:

1. First check that each of the terms of the diagram $X(a)$ is cofibrant in $\mathbf{C}$, or else take a cofibrant replacement of all of them to get $QX(a)$.
2. Form the homotopy colimit using the bar construction. Concretely,
   - take a coproduct of the objects $X(a)$,
   - add a homotopy $\Delta[1] \otimes X(a)$ for each morphism $X(a) \to X(b)$,
   - add a homotopy $\Delta[2] \otimes X(a)$ for each pair of morphisms $X(a) \to X(b) \to X(c)$,

and so on.

This accomplishes the goal we stated at the beginning of the chapter, of giving a nice, uniform way to define the homotopy colimit in many different settings at the same time.

Let's apply this recipe to orthogonal spectra, $\mathbf{Sp}^O_{\mathbf{Top}}$. Given a diagram of orthogonal spectra $X : \mathbf{I} \to \mathbf{Sp}^O_{\mathbf{Top}}$, we form the homotopy colimit by making all the spectra cofibrant, and then

- taking a wedge sum of the spectra $X(a)$,
- adding a homotopy $I_+ \wedge X(a)$ for each morphism $X(a) \to X(b)$,
- adding a homotopy $\Delta^2_+ \otimes X(a)$ for each pair of morphisms $X(a) \to X(b) \to X(c)$,

and so on. The pieces are all glued together along the face and degeneracy maps of the bar construction. Since $\Delta^n_+ \wedge -$ is an operation that is applied to the spectra levelwise, we conclude that the result is really the based homotopy colimit, or the quotient of unbased homotopy colimits

$$B^\wedge(X, \mathbf{I}_+, S^0) = B(X, \mathbf{I}, *)/B(*, \mathbf{I}, *)$$

at each spectrum level separately. See Exercise 8.7.4. This is also true in all of our models of spectra:

Corollary 8.3.24. *The homotopy colimit of a diagram of spectra is computed as the based homotopy colimit at each spectrum level separately.*

This confirms that the definition of the homotopy colimit that we gave all the way back in Definition 2.3.17 is correct. (This claim can also be proven using model categories, see Exercise 5.8.40.)

In fact, we prove the following more general statement in Theorem A.2.25:

Proposition 8.3.25. *When* $\mathbf{I}$ *is discrete and* $\mathbf{C}$ *is a category of diagram spectra, the bar construction* $B(X, \mathbf{I}, *)$ *always gives the homotopy colimit, even if we don't make the spectra cofibrant first.*

So step 1 in the above recipe can be skipped!

8.3.5 Examples of homotopy colimits of spectra

Now let's write out a few examples of the Bousfield–Kan homotopy colimit $B(X, \mathbf{I}, *)$ when $\mathbf{C}$ is a category of spectra.

Example 8.3.26. Suppose **I** is the category with two objects and only identity maps:

$$\mathbf{I} = \{\ 0 \quad 1\ \}.$$

A diagram on **I** is a pair of spectra A and B, and no maps between them. As in Example 7.4.12, the only strings of non-identity morphisms in **I** are the empty string at 0, and the empty string at 1, so the homotopy colimit $B(X, \mathbf{I}, *)$ is isomorphic to the wedge sum of spectra $A \vee B$.

Example 8.3.27. Suppose **I** is the category that indexes pushouts:

$$\mathbf{I} = \{\ 1 \leftarrow 0 \rightarrow 2\ \}.$$

A diagram on **I** is a pushout diagram of spectra $B \leftarrow A \rightarrow C$. As in Example 7.4.14, there are five strings of non-identity morphisms, so the bar construction $B(X, \mathbf{I}, *)$ is generated by the five spectra

$$\Delta^0_+ \wedge B, \qquad \Delta^1_+ \wedge A \qquad \Delta^0_+ \wedge A, \qquad \Delta^1_+ \wedge A, \qquad \Delta^0_+ \wedge C,$$

modulo relations that glue these together to form the double mapping cylinder

$$B \cup_A (A \wedge I_+) \cup_A (A \wedge I_+) \cup_A C,$$

which simplifies down to

$$B \cup_A (A \wedge I_+) \cup_A C.$$

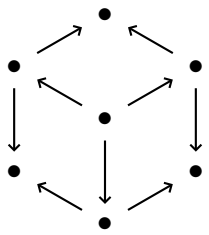
This preserves all equivalences in the A, B, and C variables by Proposition 8.3.25. We've seen this double mapping cylinder before: for instance, a commuting square of spectra

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

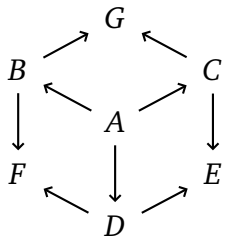
was defined to be a homotopy pushout square iff $B \cup_A (A \wedge I_+) \cup_A C \rightarrow D$ is a stable equivalence.

As a further special case, when $C = *$, the homotopy colimit is just the homotopy cofiber $C(A \rightarrow B)$. We already know this preserves all equivalences even when A and B are not cofibrant, because its homotopy groups fit into a long exact sequence (Lemma 2.4.11).

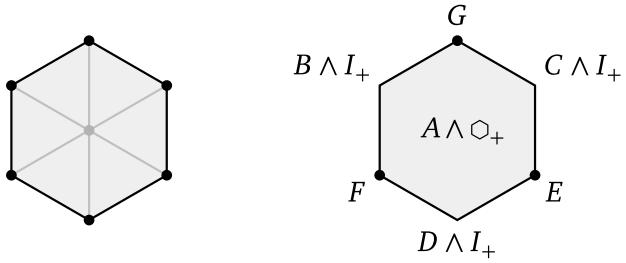
Example 8.3.28. Let's do a crazier example. Suppose **I** is a diagram of the following shape, where the three rhombus-shaped regions all commute.



An **I**-diagram of spectra consists of seven spectra and nine maps of spectra that form three commuting squares:



The homotopy colimit of this diagram, as given by the bar construction $B(X, \mathbf{I}, *)$, consists of six copies of $A \wedge \Delta^2_+$ that glue together to make A times a hexagon, $A \wedge \hexagon_+$. Then we take two copies of $B \wedge \Delta^1_+$ that glue together to make $B \wedge I_+$, which we glue to one-third of the perimeter of the hexagon as shown. We do the same to C and D, making three cylinders that cover the perimeter of the hexagon. Finally, we glue each of the spectra E, F, and G to two of the endpoints of the intervals $B \wedge I_+$, $C \wedge I_+$, and $D \wedge I_+$, as shown.



Again, by Proposition 8.3.25 this construction preserves *all* stable equivalences. We don't have to make the spectra cofibrant first before forming this homotopy colimit.

Example 8.3.29. We can again consider the category **I** that indexes sequential colimits:

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

As in Example 7.4.28, the Bousfield–Kan homotopy colimit is not the based mapping telescope, but a larger spectrum that is stably equivalent to it. As in Example 7.4.29, we could also define colimits along any totally ordered set. We deduce the isomorphisms for sequences of spectra

$$H_k\left(\operatorname*{hocolim}_{c\in\mathbf{I}} X_c\right) \cong \operatorname*{colim}_{c\in\mathbf{I}} H_k(X_c), \qquad \pi_k\left(\operatorname*{hocolim}_{c\in\mathbf{I}} X_c\right) \cong \operatorname*{colim}_{c\in\mathbf{I}} \pi_k(X_c)$$

from the same formulas for spaces from Example 7.4.29. (See also Exercise 8.7.7 for the more general case of a filtered homotopy colimit.)

Example 8.3.30. As discussed in Example 8.3.8, if X is a spectrum with an action of a topological group G, we can form the **homotopy orbit spectrum**

$$X_{hG} = B^\wedge(S^0, G_+, X),$$

or as $B^\wedge(X, G_+, S^0)$ if G acts on the right. For this to have the correct homotopy type, we need G to be Quillen cofibrant (and X is anything), or else we need G and every level X_n to be well-based. By Corollary 8.3.21 and Proposition 8.3.25, this preserves all equivalences of spectra, and is the left-derived functor of the strict orbits X_G.

It is fun to rewrite the homotopy orbit spectrum in two other equivalent ways. For one, we can think of it as the smash product over G

$$X_{hG} \cong (EG_+) \wedge_G X.$$

At each spectrum level this applies the based homotopy orbits

$$(EG_+) \wedge_G X_n = (EG \times_G X_n)/(EG \times_G *).$$

The other way to rewrite this is as the bar construction

$$X_{hG} \cong B(\mathbb{S}, \mathbb{S}[G], X)$$

where $\mathbb{S}[G] = \Sigma^\infty_+ G$ is the group ring of G, acting on $\mathbb{S}$ by collapsing G to a point, and acting on X by a map of the form

$$(\Sigma^\infty_+ G) \wedge X \cong G_+ \wedge X \to X.$$

Example 8.3.31. We can take a left Kan extension of any diagram of spectra $X : \mathbf{I} \to \mathbf{Sp}$ along any functor $F : \mathbf{I} \to \mathbf{J}$, when $\mathbf{I}$ and $\mathbf{J}$ are enriched in unbased spaces. It's given by the coend $X \otimes_{\mathbf{I}} \mathbf{J}$, which we might also write $X \wedge_{\mathbf{I}_+} \mathbf{J}_+$, because we can compute it by adding disjoint basepoints to all the categories and then smashing everything together. By a small variant of the proof of Theorem 8.3.19, the left derived functor of this, called the **homotopy left Kan extension**, is given by the bar construction $B^\wedge(X, \mathbf{I}_+, \mathbf{J}_+)$, at least when $\mathbf{I}$ and $\mathbf{J}$ are pointwise cofibrant.

We can also do then when $\mathbf{I}$ and $\mathbf{J}$ are enriched in based spaces. In that case the homotopy left Kan extension is the bar construction $B^\wedge(X, \mathbf{I}, \mathbf{J})$.

8.4 Cosimplicial objects and totalization

Next we will dualize the last two sections and discuss cobar constructions and homotopy limits in any simplicial model category $\mathbf{C}$. Again, the case we're most interested in is when $\mathbf{C}$ is the category of spectra, where we'll be able to (finally!) define homotopy limits for diagrams of any shape $\mathbf{I}$. We will be a bit more brief, since most of the arguments are dualizations of ones we have already seen.

8.4.1 Defining the totalization in a general category

As in Definition 8.2.1, we define a **cosimplicial object** in any category $\mathbf{C}$ to be a functor $X^\bullet\colon \mathbf{\Delta} \to \mathbf{C}$. In other words, it is a sequence of objects $X^0, X^1, X^2, \ldots$ in $\mathbf{C}$, together with coface and codegeneracy morphisms in $\mathbf{C}$ that satisfy the relations from Lemma 7.1.5.

Definition 8.4.1. If $\mathbf{C}$ is a simplicial model category and $X^\bullet$ is a cosimplicial object of $\mathbf{C}$, the totalization $\mathrm{Tot}(X^\bullet)$ is defined just as in Definition 7.5.2, except that instead of restricting to a subspace, we form an end as in Definition 7.5.6:

$$\mathrm{Tot}(X^\bullet) \longrightarrow \prod_{n\geq 0} (X^n)^{\Delta[n]} \rightrightarrows \prod_{m,n\geq 0} \prod_{\alpha\in\mathbf{\Delta}([m],[n])} (X^n)^{\Delta[m]}. \tag{8.4.2}$$

The first map is calculated for each coordinate of the product by applying $\alpha\colon [m]\to[n]$ to $\Delta[m]$ to get a map $(X^n)^{\Delta[n]} \to (X^n)^{\Delta[m]}$, while the second map applies α to X^n to get a map $(X^m)^{\Delta[m]} \to (X^n)^{\Delta[m]}$.

As before, we can also make this definition smaller by only taking those $\alpha\colon [m]\to[n]$ that are coface or codegeneracy maps, and we still get the same result for the equalizer $\mathrm{Tot}(X^\bullet)$.

Example 8.4.3. When $\mathbf{C} = \mathbf{Top}$, this is exactly the totalization that we saw in Definition 7.5.2. When $\mathbf{C} = \mathbf{Top}_*$, we get the same formula, and the resulting space has a canonical basepoint. We don't get two different formulas when we move between the unbased and based settings, because the product and cotensor don't change.

Example 8.4.4. When $\mathbf{C}$ is the category of spectra, this is just the totalization carried out on each level of the spectrum. The bonding maps arise as

$$\mathrm{Tot}(X(n)^\bullet) \to \mathrm{Tot}(\Omega X(n+1)^\bullet) \cong \Omega\mathrm{Tot}(X(n+1)^\bullet).$$

Example 8.4.5. When $\mathbf{C} = \mathbf{sSet}$, the object $X^\bullet$ is a **cosimplicial simplicial set**. It has two indices, a cosimplicial one and a simplicial one. Taking the totalization in the above sense produces a simplicial set.

8.4.2 Reedy fibrant cosimplicial objects

Now suppose that the category $\mathbf{C}$ has all limits. The definition of a latching object dualizes nicely:

Definition 8.4.6. If X is a cosimplicial object in $\mathbf{C}$, for each $n \geq 0$, we form a "diagram of codegeneracy maps" with one copy of X^i for each $i < n$ and each surjective map of totally ordered sets $[n] \to [i]$. The diagram has a map $X^i \to X^j$ for each surjective map $[j] \to [i]$

n	Cube-shaped diagram	X_n	M_nX is the limit of:
0	X^0	X^0	$*$
1	$X^1 \longrightarrow X^0$	X^1	X^0
2	$\begin{array}{ccc} X^2 & \longrightarrow & X^1 \\ & \searrow & \downarrow \\ & & X^1 \longrightarrow X^0 \end{array}$	X^2	$\begin{array}{ccc} X^1 & & \\ & \searrow & \\ & & X^0 \end{array}$
3	$\begin{array}{ccccc} X^3 & \longrightarrow & X^2 & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ X^2 & \longrightarrow & X^1 & \longrightarrow & X^0 \\ \downarrow & \searrow & \downarrow & \searrow & \\ X^2 & \longrightarrow & X^1 & \longrightarrow & X^0 \end{array}$	X^3	$\begin{array}{ccccc} X^2 & & & & \\ & \searrow & & & \\ X^2 & \longrightarrow & X^1 & & \\ & \searrow & & \searrow & \\ X^2 & \longrightarrow & X^1 & \longrightarrow & X^0 \end{array}$

Figure 8.4.10

commuting with the maps in from $[n]$. The n**th matching object** $M_nX \in \mathbf{C}$ is the limit of the resulting diagram, and the n**th matching map**

$$X^n \to M_nX$$

is induced by the given maps $[n] \to [i]$. The 0th matching map is $X^0 \to *$, where $*$ is the terminal object of $\mathbf{C}$.

If $\mathbf{C}$ is a simplicial model category, we say that X is **Reedy fibrant** if each of the matching maps $X^n \to M_nX$ is a fibration in the model structure on $\mathbf{C}$.

Remark 8.4.7. In contrast to Definition 8.2.15, we aren't interested in considering fibrations other than the ones in the model structure on $\mathbf{C}$. This is for practical reasons. It doesn't give us any interesting examples!

Remark 8.4.8. There is also a definition of matching objects for *simplicial* objects of $\mathbf{C}$, which behaves differently from the definition above. It is actually quite a bit simpler: it is essentially the space of maps from $\partial\Delta[n]$ into the simplicial object $X_\bullet$. See Exercise 8.7.37.

Proposition 8.4.9. *The diagram of codegeneracy maps from Definition 8.4.6 is in fact a cube-shaped diagram, with X^n as the initial vertex, and M_nX as the limit of all the vertices except the initial vertex, as illustrated in Figure 8.4.10.*

Proof. The proof is dual to the proof of Proposition 8.2.18. In other words, it is the same proof, except that when we pass from S to X^S the direction of all of the arrows is reversed. □

The main theorem of this section is that Reedy fibrant objects in any simplicial model category have well-behaved totalization.

Theorem 8.4.11. *In any simplicial model category* **C***, if* $X^\bullet$ *is Reedy fibrant then* $\mathrm{Tot}(X^\bullet)$ *is fibrant. Furthermore, if* $X^\bullet \to Y^\bullet$ *is any map of Reedy fibrant objects such that each* $X^n \to Y^n$ *is a weak equivalence in* **C***, then the map of totalizations* $\mathrm{Tot}(X^\bullet) \to \mathrm{Tot}(Y^\bullet)$ *is a weak equivalence in* **C***.*

The proof follows the same pattern as Theorem 8.2.27, so instead of giving the full proof, we just give the relevant definitions and some of the intermediate results.

Lemma 8.4.12. *In any simplicial model category* **C***, the fibrations satisfy the following dual versions of the axioms for Theorem 8.2.27:*

- Every isomorphism is a fibration.
- Every pullback of a fibration is a fibration.
- Every countably infinite inverse composition of fibrations is a fibration. (Therefore every finite composition of fibrations is a fibration as well.)
- (Co-gluing Lemma) If we consider spans of fibrant objects in which one leg is a fibration,
$$X \longrightarrow B \longleftarrow Y,$$
then any weak equivalence of two such spans gives a weak equivalence on the pullbacks $X \times_B Y$.
- (Limit Lemma) If we consider sequences of fibrations of fibrant objects,
$$\cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0,$$
then any weak equivalence of two such diagrams gives a weak equivalence on the the limits X_∞.
- (Pullback-Hom Lemma) If i is the map of simplicial sets $\partial\Delta[n] \to \Delta[n]$ and $f : X \to B$ is any cofibration, then $\mathrm{Hom}_\square(i, f)$ is a fibration. Furthermore $(-)^{\partial\Delta[n]}$ and $(-)^{\Delta[n]}$ preserve equivalences of fibrant objects.

Proof. This follows quickly from the fact that $\otimes$ is a Quillen tensor, and the dual of the argument in Exercise 5.8.12. □

Definition 8.4.13. For any cosimplicial object $X^\bullet$ in $\mathbf{C}$ and any $n \geq 0$, define the n-**coskeleton** by taking the definition of the totalization from (8.4.2) and restricting to those terms that are $\leq n$:

$$\mathrm{coSk}^n\mathrm{Tot}(X^\bullet) \longrightarrow \prod_{0\leq \ell \leq n} (X^\ell)^{\Delta[\ell]} \rightrightarrows \prod_{0\leq m,\ell\leq n} \prod_{\alpha\in \mathbf{\Delta}([m],[\ell])} (X^\ell)^{\Delta[m]}. \tag{8.4.14}$$

By comparing the universal properties, we get canonical maps

$$\mathrm{coSk}^n\mathrm{Tot}(X^\bullet) \to \mathrm{coSk}^{n-1}\mathrm{Tot}(X^\bullet),$$

and the full totalization $\mathrm{Tot}(X^\bullet)$ is the limit of these maps in $\mathbf{C}$.

Proposition 8.4.15. *The natural maps form a pullback square in* $\mathbf{C}$

$$\begin{array}{ccc} \mathrm{coSk}^n\mathrm{Tot}(X^\bullet) & \longrightarrow & \mathrm{coSk}^{n-1}\mathrm{Tot}(X^\bullet) \\ \downarrow & & \downarrow \\ (X^n)^{\Delta[n]} & \longrightarrow & (M_nX)^{\Delta[n]} \times_{(M_nX)^{\partial\Delta[n]}} (X^n)^{\partial\Delta[n]}. \end{array} \tag{8.4.16}$$

The map along the bottom is the pullback-hom of $\partial\Delta[n] \to \Delta[n]$ and the matching map $X^n \to M_nX$. The rest of the proof of Theorem 8.4.11 is dual to the proof of Theorem 8.2.27.

We also have the following dual version of Lemma 8.2.36. Suppose $X^\bullet$ is any cosimplicial object such that each matching map $X^n \to M_nX$ is the projection to a factor:

$$X^n \cong X^n_{nd} \times (M_nX).$$

So X^n_{nd} is a space of "co-nondegenerate points".

Lemma 8.4.17. *Under this assumption, the pullback square* (8.4.16) *simplifies to*

$$\begin{array}{ccc} \mathrm{coSk}^n\mathrm{Tot}(X^\bullet) & \longrightarrow & \mathrm{coSk}^{n-1}\mathrm{Tot}(X^\bullet) \\ \downarrow & & \downarrow \\ (X^n_{nd})^{\Delta[n]} & \longrightarrow & (X^n_{nd})^{\partial\Delta[n]}. \end{array} \tag{8.4.18}$$

Proof. This is again by comparing universal properties. □

8.5 The cobar construction in any simplicial model category

We are now ready to form the cobar construction $F(X, G, Y)$ and $F(X, \mathbf{I}, Y)$ in any simplicial model category $\mathbf{C}$, by dualizing Section 8.5.

8.5.1 The definition

We will again work in the setting of Assumption 8.3.10, except with two changes. First, the two diagrams X and Y should both be covariant now. Second, to make the technical part easier, we will also assume that the categories of diagrams $\mathbf{V}^{\mathbf{I}}$ and $\mathbf{C}^{\mathbf{I}}$ have the projective model structure. (This is automatically true when $\mathbf{V}$ and $\mathbf{C}$ are **sSet**, $\mathbf{sSet}_*$, **Top**, $\mathbf{Top}_*$, or any of our categories of spectra. It's also true if $\mathbf{V} = \mathbf{sSet}$ and $\mathbf{C}$ is any cofibrantly generated simplicial model category, by Exercise 8.7.12.)

Let's state the assumptions again with these changes.

Assumption 8.5.1. We assume throughout this section that

- $(\mathbf{V}, \otimes, I)$ is a symmetric monoidal simplicial model category,
- $\mathbf{C}$ is a model category and $\boxtimes\colon \mathbf{V} \times \mathbf{C} \to \mathbf{C}$ is a Quillen tensor,
- $I \boxtimes X \cong X$ when $I \in \mathbf{V}$ is the unit and $X \in \mathbf{C}$ is any object,
- $\mathbf{I}$ is a small category enriched in $\mathbf{V}$,
- the categories $\mathbf{V}^{\mathbf{I}}$ and $\mathbf{C}^{\mathbf{I}}$ of $\mathbf{V}$-enriched diagrams have the projective model structure, and
- X is a covariant diagram on $\mathbf{I}$ taking values in either $\mathbf{V}$ or $\mathbf{C}$, and Y is a covariant diagram on $\mathbf{I}$ taking values in $\mathbf{C}$.

Definition 8.5.2. Define $F^{\otimes}_n(X, \mathbf{I}, Y)$ to be the cosimplicial object with the formula

$$F^n_{\otimes}(X, \mathbf{I}, Y) := \prod_{c_0, c_1, \cdots, c_n \in \operatorname{ob}\mathbf{I}} Y(c_n)^{X(c_0) \otimes \mathbf{I}(c_0, c_1) \otimes \cdots \otimes \mathbf{I}(c_{n-1}, c_n)},$$

with coface and codegeneracy maps given by the same formulas as Definition 7.5.10. The **cobar construction** $F_{\otimes}(X, \mathbf{I}, Y)$ is its totalization.

If X takes values in $\mathbf{V}$, then the exponential in this formula refers to the right adjoint of the tensor of the form $\mathbf{V}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C}$. In this case the cobar construction is an object of $\mathbf{C}$. On the other hand, if X takes values in $\mathbf{C}$, then we take the other right adjoint $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{V}$, and the cobar construction is an object of $\mathbf{V}$. Think of these two cases as the "derived cotensor" and the "derived mapping space," respectively.

Example 8.5.3. When $\mathbf{V} = \mathbf{C} = \mathbf{Top}$ with the Cartesian product, this gives the cobar construction from Definition 7.5.10:

$$F^n(X, \mathbf{I}, Y) = \prod_{c_0, c_1, \cdots, c_n \in \operatorname{ob}\mathbf{I}} \operatorname{Map}\big(X(c_0) \times \mathbf{I}(c_0, c_1) \times \cdots \times \mathbf{I}(c_{n-1}, c_n), Y(c_n)\big).$$

When $\mathbf{I} = G$ is a Quillen cofibrant topological group and $X = *$, this is the **homotopy fixed points** $\operatorname{Map}^G(EG, Y)$. For a general category $\mathbf{I}$ and $X = *$, this is the **Bousfield–Kan homotopy limit** $F(*, \mathbf{I}, Y)$.

Example 8.5.4. When $\mathbf{V} = \mathbf{C} = \mathbf{Top}_*$ with the smash product, this gives a based version of the cobar construction

$$F^n_\wedge(X, \mathbf{I}, Y) = \prod_{c_0, c_1, \cdots, c_n \in \mathrm{ob}\,\mathbf{I}} \mathrm{Map}\big(X(c_0) \wedge \mathbf{I}(c_0, c_1) \wedge \cdots \wedge \mathbf{I}(c_{n-1}, c_n), Y(c_n)\big).$$

In the special case that the category is of the form $\mathbf{I}_+$, where $\mathbf{I}$ is a category of unbased spaces, and X comes from a diagram of unbased spaces, this is homeomorphic to the unbased version above! So the "based homotopy fixed points" are the same as the unbased homotopy fixed points.

Example 8.5.5. When $\mathbf{V} = \mathbf{C} = \mathbf{Sp}^\Sigma$, or any other symmetric monoidal model category of spectra, and $\mathbf{I}$ has one object, then we have a ring spectrum R and two right R-modules M and N. The cobar construction produces a spectrum $F_\wedge(M, R, N)$ that is the totalization of

$$F^n_\wedge(M, R, N) = F(M \wedge R^{\wedge n}, N),$$

where $F(-,-)$ denotes the function spectrum. This turns out to be the correct version of $\mathrm{Ext}_R(M, N)$, or the derived mapping space from M to N, provided the underlying spectra of M and R are cofibrant, and N is fibrant. See Corollary 8.5.15.

Example 8.5.6. Suppose $\mathbf{V}$ is unbased spaces and $\mathbf{C}$ is spectra. Then we have a category $\mathbf{I}$ enriched in unbased spaces and a diagram $Y : \mathbf{I} \to \mathbf{Sp}$. There are two possibilities for the diagram X: it could be a diagram of spaces, or a diagram of spectra.

If X is a diagram of spaces, then the cobar construction produces a spectrum with the formula

$$\begin{aligned} F^n(X, \mathbf{I}, Y) &= \prod_{c_0, c_1, \cdots, c_n \in \mathrm{ob}\,\mathbf{I}} F\Big(\big(X(c_0) \times \mathbf{I}(c_0, c_1) \times \cdots \times \mathbf{I}(c_{n-1}, c_n)\big)_+, Y(c_n)\Big) \\ \cong F^n_\wedge(X_+, \mathbf{I}_+, Y) &\cong \prod_{c_0, c_1, \cdots, c_n \in \mathrm{ob}\,\mathbf{I}} F\big(X(c_0)_+ \wedge \mathbf{I}(c_0, c_1)_+ \wedge \cdots \wedge \mathbf{I}(c_{n-1}, c_n)_+, Y(c_n)\big) \end{aligned}$$

where now $F(-,-)$ denotes the cotensor of a space with a spectrum from Definition 2.3.8. Think of this as a big derived cotensor of the diagram Y by X.

If X is the constant diagram of unbased spaces at $*$, this gives the **Bousfield–Kan homotopy limit for spectra** $F(*, \mathbf{I}, Y) = F_\wedge(S^0, \mathbf{I}_+, Y)$, at least if $\mathbf{I}$ is pointwise Quillen cofibrant and each $Y(c)$ is an Ω-spectrum. Notice that it really is just formed by making the spectra $Y(c)$ into Ω-spectra and then taking the homotopy limit at each spectrum level, as we promised earlier.

In the special case where $\mathbf{I} = G$ is a category with one object, and taking $X = *$, this is the definition of the **homotopy fixed point spectrum** Y^{hG}. Again, it is formed by making Y an Ω-spectrum, and then taking the homotopy fixed points at each spectrum level, $F^G(EG_+, Y_n)$.

In the remaining case that X is a diagram of spectra, the cobar construction produces a space, not a spectrum, with the formula

$$F^n_\wedge(X, \mathbf{I}_+, Y) \cong \prod_{c_0, c_1, \cdots, c_n \in \mathrm{ob}\,\mathbf{I}} \mathrm{Map}_*\big(X(c_0) \wedge \mathbf{I}(c_0, c_1)_+ \wedge \cdots \wedge \mathbf{I}(c_{n-1}, c_n)_+, Y(c_n)\big)$$

where now $\mathrm{Map}_*(-,-)$ denotes the mapping space between two spectra from Definition 2.3.12. Think of this as a big derived mapping space from the diagram X to the diagram Y. It is sometimes called the **homotopy end** of the diagrams X and Y, at least if $\mathbf{I}$ and X are pointwise cofibrant, and Y is pointwise fibrant.

Remark 8.5.7. We can see that the cobar construction is preserved if we move between different models of spectra by applying the right adjoints, in other words the forgetful functors between these models. Each of those forgetful functors commutes with limits and with the cotensor $F(K,-)$ up to isomorphism, and therefore gives an isomorphism on the cobar constructions. We can similarly move between our different notions of "spaces," this time using the left adjoints, and this also preserves the cobar construction up to isomorphism. Again, this will be true up to weak equivalence as well, using Theorem 8.5.12 below.

8.5.2 Two rearrangements of the cobar construction

We can re-express the cobar construction in terms of the bar construction and the end. To define the end of two diagrams, $\mathrm{Map}^{\mathbf{I}}(X,Y)$, we use the same formula as in Definition 7.5.6:

$$\mathrm{Map}^{\mathbf{I}}(X,Y) \longrightarrow \prod_{c \in \mathrm{ob}\,\mathbf{I}} Y(c)^{X(c)} \rightrightarrows \prod_{c_0, c_1 \in \mathrm{ob}\,\mathbf{I}} Y(c_1)^{X(c_0)\times \mathbf{I}(c_0,c_1)}. \tag{8.5.8}$$

As in Definition 8.5.2, there are actually two versions of this definition. In the first, X is a diagram in $\mathbf{V}$, and we use the right adjoints $\mathbf{V}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C}$ in the above definition, which makes the end $\mathrm{Map}^{\mathbf{I}}(X,Y)$ an object of $\mathbf{C}$. In the second version, X is a diagram in $\mathbf{C}$, we use the right adjoint $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{V}$ instead, which makes $\mathrm{Map}^{\mathbf{I}}(X,Y)$ into an object of $\mathbf{V}$. We use the same notation for both cases, to make the proofs that follow more uniform.

The following is an exercise in commuting the cotensor with colimits in the first slot and limits in the second slot:

Lemma 8.5.9. *If* $\mathbf{V} = \mathbf{sSet}$ *or* $\mathbf{Top}$ *and* $\mathbf{I}$ *is a discrete category, then if* $X = *$ *is the constant diagram on the one-point space and* Y *is an* $\mathbf{I}$*-diagram in* $\mathbf{C}$*, then*

$$\mathrm{Map}^{\mathbf{I}}(*, Y) \cong \lim_{c \in \mathbf{I}} Y(c).$$

When $\mathbf{I}$ is not discrete, we instead take $X = \mathrm{const}(I)$ to be the constant diagram on the unit object $I \in \mathbf{I}$, if it exists. Then $\mathrm{Map}^{\mathbf{I}}(\mathrm{const}(I), Y)$ is the correct definition of the **enriched limit**, or limit along a category enriched in $\mathbf{V}$. See Exercise 8.7.40.

Lemma 8.5.10. *We have* $\mathrm{Map}^{\mathbf{I}}(B(X,\mathbf{I},\mathbf{I}), Y) \cong F(X,\mathbf{I},Y) \cong \mathrm{Map}^{\mathbf{I}}(X, F(\mathbf{I},\mathbf{I},Y))$.

Proof. The proof is the same as in Proposition 7.5.11 and Proposition 7.5.12, except that all of the bijections are replaced by isomorphisms in $\mathbf{V}$ or $\mathbf{C}$, that we form by comparing universal properties. □

Lemma 8.5.11 (Cobar Lemma)**.** *If* $\mathbf{I}$ *is pointwise cofibrant, and* Y *is pointwise fibrant, then there are weak equivalences in* $\mathbf{C}$

$$Y(c) \xrightarrow{\sim} F(\mathbf{I}(c,-),\mathbf{I},Y)$$

that together form an equivalence of diagrams $Y \xrightarrow{\sim} F(\mathbf{I},\mathbf{I},Y)$.

Proof. Here we're only using the version of the cobar construction where the first diagram X takes values in $\mathbf{V}$. We check that the end $\mathrm{Map}^{\mathbf{I}}(-,-)$ in this case is the right adjoint of the Quillen tensor $\mathbf{V}^{\mathbf{I}} \times \mathbf{C} \to \mathbf{C}^{\mathbf{I}}$ that tensors a fixed object $Z \in \mathbf{C}$ into every term of a diagram $X(c) \in \mathbf{V}$ to form a new diagram in $\mathbf{C}$. It follows that $\mathrm{Map}^{\mathbf{I}}(-,-)$ preserves equivalences when the source diagram is free cofibrant and the target diagram is pointwise fibrant. Now we plug the weak equivalence of diagrams $B(\mathbf{I},\mathbf{I},\mathbf{I}) \to \mathbf{I}$ into the first slot of $\mathrm{Map}^{\mathbf{I}}(-,Y)$ to get the desired weak equivalence of diagrams $Y \to F(\mathbf{I},\mathbf{I},Y)$, just as in Lemma 7.5.22. □

8.5.3 Homotopy invariance and the cobar lemma

Now we can prove that the cobar construction in any nice category $\mathbf{C}$ has the same properties we proved for $\mathbf{C} = \mathbf{Top}$ in Section 7.5.7. We continue to work under Assumption 8.5.1, so that $\mathbf{C}$ is a model category, and $\mathbf{I}$ is a category enriched in some smaller symmetric monoidal model category $\mathbf{V}$ that acts on $\mathbf{C}$. Again, the reader is encouraged to focus on the case where $\mathbf{C}$ is the category of spectra, and $\mathbf{V}$ is either unbased spaces or spectra.

As in Definition 8.3.11, we say that the diagram Y is **pointwise fibrant** if each object $Y(c)$ is fibrant in $\mathbf{C}$.

Theorem 8.5.12 (Homotopy invariance of the cobar construction)**.** *If* X *and* $\mathbf{I}$ *are pointwise cofibrant and* Y *is pointwise fibrant, then the cobar construction* $F(X,\mathbf{I},Y)$ *is fibrant, and any pointwise equivalence of categories* $\mathbf{I} \to \mathbf{I}'$ *and compatible pointwise equivalences of diagrams* $X \to X'$ *and* $Y \to Y'$ *induce a weak equivalence*

$$F(X,\mathbf{I},Y) \xrightarrow{\sim} F(X',\mathbf{I}',Y').$$

The compatibility condition is the same as the one we gave after Theorem 8.3.12, except that now Y is a covariant diagram, so $\mathbf{I}$ is acting on the other side.

Proof. By Theorem 8.4.11, it suffices to show that the simplicial object $N_\bullet(X, \mathbf{I}, Y)$ is Reedy fibrant. Dualizing the proof of Theorem 8.3.12, we see that the matching object is a product of pullback-homs of maps of the form

$$\mathrm{Hom}_\square\big((\varnothing \to X(c_0))\square(\varnothing \text{ or } I \to \mathbf{I}(c_0, c_1))\square\cdots\square(\varnothing \text{ or } I \to \mathbf{I}(c_{n-1}, c_n), (Y(c_n) \to *)\big).$$

As before, we get the map $I \to \mathbf{I}(c_i, c_i)$ whenever an index repeats, and a map $\varnothing \to \mathbf{I}(c_{i-1}, c_i)$ when the index does not repeat. Since $\mathbf{V}$ is a symmetric monoidal model category, the map in the source of the $\mathrm{Hom}_\square$ is a cofibration. By assumption the map in the target is a fibration. Since $\boxtimes\colon \mathbf{V}\times\mathbf{C}\to\mathbf{C}$ is a Quillen tensor by Remark 5.5.17 this pullback-hom is therefore a fibration in $\mathbf{C}$ or $\mathbf{V}$ (depending on which version of the cobar construction we are taking). □

Example 8.5.13. Taking $\mathbf{V}$ and $\mathbf{C}$ to be the (symmetric monoidal) category of spectra, we conclude that the cobar construction $F(M, S, N)$ preserves equivalences whenever the maps $* \to M$, $\mathbb{S} \to S$ are cofibrations of spectra and N is fibrant as a spectrum.

8.5.4 The cobar construction is a right-derived functor

Theorem 8.5.14. *When $\mathbf{I}$ and X are pointwise cofibrant and Y is pointwise fibrant, the cobar construction $F(X, \mathbf{I}, Y)$ is the right-derived functor of the end $\mathrm{Map}^{\mathbf{I}}(X, Y)$:*

$$F(X, \mathbf{I}, Y) \cong \mathrm{Map}^{\mathbf{I}}(QX, RY), \qquad QX \xrightarrow{\sim} X, \qquad Y \xrightarrow{\sim} RY.$$

In fact, it can be constructed by only deforming X,

$$F(X, \mathbf{I}, Y) \cong \mathrm{Map}^{\mathbf{I}}(QX, Y), \qquad QX \xrightarrow{\sim} X,$$

or by only deforming Y,

$$F(X, \mathbf{I}, Y) \cong \mathrm{Map}^{\mathbf{I}}(X, RY), \qquad Y \xrightarrow{\sim} RY.$$

Proof. For the first statement, the left deformation is $QX = B(X, \mathbf{I}, \mathbf{I})$, and so this follows because $\mathrm{Map}^{\mathbf{I}}(-,-)$ preserves all equivalences when the first diagram is free cofibrant, for instance $B(X, \mathbf{I}, \mathbf{I})$ when X is pointwise cofibrant, and when the second diagram is pointwise fibrant. (This works whether X takes values in $\mathbf{V}$ or in $\mathbf{C}$; in the first case the Quillen tensor is of the form $\mathbf{V}^{\mathbf{I}}\times\mathbf{C}\to\mathbf{C}^{\mathbf{I}}$, while in the second case it is of the form $\mathbf{C}^{\mathbf{I}}\times\mathbf{V}\to\mathbf{C}^{\mathbf{I}}$.)

For the second statement, the right deformation is $RY = F(\mathbf{I}, \mathbf{I}, Y)$. We follow the same argument as in Theorem 7.5.25, to show that $\mathrm{Map}^{\mathbf{I}}(-,-)$ preserves all equivalences when the first diagram is pointwise cofibrant and the second diagram is of the form $F(\mathbf{I}, \mathbf{I}, Y)$ with Y pointwise fibrant, and that finishes the proof. □

For this reason, under these assumptions the cobar construction $F(X, \mathbf{I}, Y)$ is also called the **homotopy end**.

By the extension of this proof in Exercise 7.7.31, this also holds under no assumptions on X, $\mathbf{I}$, and Y, but then we need to take the deformation $B(X', \mathbf{I}', \mathbf{I}') \to X$, where $X' \to X$ is a pointwise cofibrant replacement of X and $\mathbf{I}' \to \mathbf{I}$ is a pointwise cofibrant replacement of $\mathbf{I}$, and $Y \to Y'$ where Y' is a pointwise fibrant replacement of Y. The subcategory of fibrant objects is the collection of all pairs of diagrams of the form $(B(X', \mathbf{I}', \mathbf{I}'), Y')$, or alternatively $(X', F(\mathbf{I}', \mathbf{I}', Y))$, where X' is pointwise cofibrant, $\mathbf{I}'$ is a pointwise cofibrant diagram with a map to $\mathbf{I}$, and Y' is pointwise fibrant. (We do need to assume that these replacements exist.) So any deformation that lands in this subcategory gives the right-derived functor of $\mathrm{Map}^{\mathbf{I}}(-,-)$.

Corollary 8.5.15. *In any of our symmetric monoidal categories of spectra, the cobar construction $F(QM, QS, RN)$ is the right-derived functor of the spectrum of S-linear maps $F^S(M, N)$.*

Here the cofibrant and fibrant replacements of ring spectra and module spectra come from the model structures in Theorem 6.2.36.

8.5.5 The Bousfield–Kan homotopy limit

Now that we've proven the homotopy invariance properties of the cobar construction, we can use the cotensor version of the cobar construction to define homotopy limits. The following is a corollary of Theorem 8.5.14:

Corollary 8.5.16. *Assume $\mathbf{I}$ is pointwise cofibrant, the unit $I \in \mathbf{V}$ is cofibrant, and the constant diagram $\mathrm{const}(I)$ exists. Then $F(\mathrm{const}(I), \mathbf{I}, RY)$ is a right-derived functor of the enriched limit $\mathrm{Map}^{\mathbf{I}}(\mathrm{const}(I), Y)$.*

To be consistent with Definition 7.5.27, we call $F(\mathrm{const}(I), \mathbf{I}, RY)$ the **Bousfield–Kan homotopy limit** of X.

Recall from Definition 3.4.37 that for any category $\mathbf{C}$ with weak equivalences, the homotopy limit is defined to be the right-derived functor of the limit, if it exists. Specializing to the case that $\mathbf{V} = \mathbf{sSet}$ and X is the constant diagram $*$ in $\mathbf{V}$, we get:

Corollary 8.5.17. *If $\mathbf{C}$ is any cofibrantly generated simplicial model category, and $\mathbf{I}$ is any discrete or simplicially enriched small category, then homotopy limits of diagrams $X : \mathbf{I} \to \mathbf{C}$ always exist and are given by the Bousfield–Kan formula:*

$$F(*, \mathbf{I}, RY) \simeq \operatorname*{holim}_{c \in \mathbf{I}} X(c).$$

As always, the right-derived functor of the limit is unique up to equivalence (Corollary 3.5.3), so this is equivalent to any other model of the homotopy limit that exists.

Example 8.5.18. Let's parse this formula explicitly when $\mathbf{I}$ is discrete. For any diagram Y in a simplicial model category $\mathbf{C}$, the recipe is for the homotopy limit is:

1. First check that each of the terms of the diagram $Y(a)$ is fibrant in $\mathbf{C}$, or else take a fibrant replacement of all of them to get $RY(a)$.
2. Form the homotopy limit using the cobar construction. Concretely,
   - take a product of the objects $Y(a)$,
   - take the fiber product with a path space $Y(a)^{\Delta[1]}$ for each morphism $Y(a) \to Y(b)$,
   - take the fiber product with $Y(a)^{\Delta[2]}$ for each pair of morphisms $Y(a) \to Y(b) \to Y(c)$,

   and so on.

More concretely, given a diagram of orthogonal spectra $Y : \mathbf{I} \to \mathbf{Sp}^O_{\mathbf{Top}}$, we form the homotopy limit by making all the spectra into Ω-spectra, and then following the recipe for the homotopy limit of based or unbased spaces (they are homeomorphic!) at each spectrum level.

Corollary 8.5.19. *The homotopy limit of a diagram of spectra is computed by replacing by Ω-spectra and then taking the homotopy limit at each spectrum level separately.*

Again, this confirms a claim we made all the way back in Definition 2.3.20.

In fact, we prove the following more general statement in Theorem A.2.30. We say that $\mathbf{I}$ is a **finite cell category** if each mapping space $\mathbf{I}(a,b)$ is a finite cell complex and the realization $B\mathbf{I} = |B(*,\mathbf{I},*)|$ is also a finite cell complex.

Proposition 8.5.20. *When $\mathbf{I}$ is a finite cell category and $\mathbf{C}$ is a category of diagram spectra, the cobar construction $F(*,\mathbf{I},Y)$ always gives the homotopy limit, even if we don't make the spectra fibrant first.*

8.5.6 Examples of homotopy limits of spectra

We now take $\mathbf{C}$ to be the category of spectra, and write out a few examples of the Bousfield–Kan homotopy limit $F(*,\mathbf{I},Y)$. Note that by Proposition 8.5.20, when $\mathbf{I}$ is a finite cell category, it isn't necessary to make Y fibrant first.

Example 8.5.21. Suppose $\mathbf{I}$ is the category with two objects and only identity maps:

$$\mathbf{I} = \{\ 0 \quad 1\ \}.$$

A diagram on $\mathbf{I}$ is a pair of spectra A and B, and no maps between them. The homotopy limit $F(*, \mathbf{I}, Y)$ is isomorphic to the product spectrum $A \times B$. We know that we don't have to make A and B into Ω-spectra for this to have the right homotopy type.

On the other hand, if $\mathbf{I}$ has an infinite set of objects and only identity maps, then we get an infinite product, and in that case we have to make all of the factors into Ω-spectra first.

Example 8.5.22. Suppose $\mathbf{I}$ is the category that indexes pullbacks,

$$\bullet \longrightarrow \bullet \longleftarrow \bullet.$$

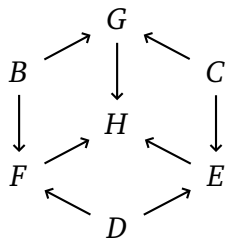
A diagram on $\mathbf{I}$ is a pullback diagram of spectra $B \to D \leftarrow C$. The homotopy limit $F(*, \mathbf{I}, Y)$ is the spectrum $B \times^h_D C = B \times_D F(I_+, D) \times_D C$, as in Example 7.5.30. This preserves all stable equivalences, as we have seen before: a commuting square of spectra

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

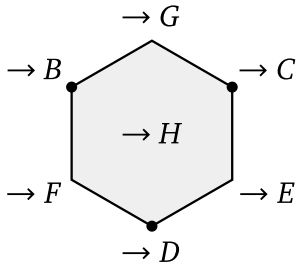
was defined to be a homotopy pullback square iff $A \to B \times_D F(I_+, D) \times_D C$ is a stable equivalence.

As a further special case, when $C = *$, the homotopy limit is just the homotopy fiber $F(B \to D)$. We already know this preserves all equivalences even when B and D are not fibrant, because its homotopy groups fit into a long exact sequence (Lemma 2.4.10).

Example 8.5.23. Dualizing Example 8.3.28, the homotopy limit of a diagram of spectra of the form



consists of the cotensor of a hexagon with H, the cotensor of three intervals with E, F, and G, and a single copy of B, C, and D, all multiplied together. Then we restrict to the subspectrum where the map from the hexagon into H restricts along three intervals of its boundary to the image in H of the given maps to E, F, and G, and these maps in turn lift to the given points in B, C, and D at the endpoints of these intervals.



Since **I** is a finite cell category (its realization is the hexagon), this construction preserves *all* stable equivalences, without needing to make the spectra fibrant first.

Example 8.5.24. Suppose **I** is the category that indexes sequential limits,

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet.$$

A diagram on **I** is a sequential diagram of spectra $\cdots \to A_2 \to A_1 \to A_0$. The Bousfield–Kan homotopy limit is formed by making all of these Ω-spectra and then taking the homotopy limit from Example 7.5.31 at each spectrum level. Equivalently, we can take the mapping microscope from Definition 1.5.21 at each spectrum level.

Example 8.5.25. As discussed in Example 8.5.3, if X is a spectrum with an action of a Quillen cofibrant topological group G, then its G-homotopy limit is the **homotopy fixed point spectrum**

$$X^{hG} = F(*, G, RX) = F^G(EG_+, RX) = \mathrm{Hom}^{\mathbb{S}[G]}(\mathbb{S}, RX).$$

Proposition 8.5.20 doesn't help us here, so we always have to make X into the Ω-spectrum RX first.[9]

Example 8.5.26. We can take a right Kan extension of any diagram of spectra $X : \mathbf{I} \to \mathbf{Sp}$ along any functor $F : \mathbf{I} \to \mathbf{J}$, by taking the end $\mathrm{Map}^{\mathbf{I}}(\mathbf{J}, X)$, or $\mathrm{Map}^{\mathbf{I}_+}(\mathbf{J}_+, X)$ if we think of **I** and **J** as categories of based spaces by adding a disjoint basepoint. By a small variant of the proof of Theorem 8.5.14, the left derived functor of this, called the **homotopy right Kan extension**, is given by the cobar construction $F_\wedge(\mathbf{J}_+, \mathbf{I}_+, X)$.

8.6 Properties of homotopy colimits and homotopy limits

Now that we have a concrete model for the homotopy colimit and the homotopy limit in any simplicial model category **C**, we will prove a few important facts about them.

[9] It is still true that if BG is *equivalent* to a finite CW complex B, and we can take the corresponding G-bundle $E \to B$, then we don't have to fibrantly replace X, so $F^G(E_+, X)$ is the correct formula. So for instance the $\mathbb{Z}$-homotopy fixed points can be computed as $F^{\mathbb{Z}}(\mathbb{R}_+, X)$.

For simplicity, let's assume that $\mathbf{I}$ is a discrete category, and that we are taking homotopy colimits of diagrams $X : \mathbf{I} \to \mathbf{C}$. The interested reader is welcome to generalize these results to the case where $\mathbf{I}$ is enriched in some other category $\mathbf{V}$.

What assumptions should we make on the category $\mathbf{C}$? There are two choices. The maximalist route is to assume that $\mathbf{C}$ is cofibrantly generated, so that $\mathbf{C}^{\mathbf{I}}$ has a projective model structure and each diagram $X : \mathbf{I} \to \mathbf{C}$ has a cofibrant replacement $QX : \mathbf{I} \to \mathbf{C}$, and also to assume that $\mathbf{C}$ is a simplicial model category, so that the bar construction $B(QX, \mathbf{I}, *)$ is the homotopy colimit. This allows us to use the bar construction in the proofs.

On the other hand, many of these results hold even when the bar construction breaks down. The minimalist route is to assume that $\mathbf{C}$ is a category with weak equivalences (Definition 3.4.13), and the left derived functor of the colimit $\mathbf{C}^{\mathbf{I}} \to \mathbf{C}$ exists. It's good to also assume that the left-derived functor of the left Kan extension $\mathbf{C}^{\mathbf{I}} \to \mathbf{C}^{\mathbf{J}}$ exists, because this will allow us to compose various kinds of colimits together.

The maximal assumptions imply the minimal ones, because we know that if $\mathbf{C}$ is a cofibrantly generated simplicial model category, then it has all homotopy Kan extensions (Example 8.3.31 and Example 8.5.26). So all of these assumptions hold for all of our categories of spaces, based spaces, and spectra.

We'll describe the proofs using both of these assumptions, because while the minimalist proofs tend to be simpler, the maximalist proofs using the bar construction give great intuition for what's really going on.

8.6.1 Changing the category C

We first consider functors $F : \mathbf{C} \to \mathbf{D}$ and their effect on the colimit or limit of a diagram $X : \mathbf{I} \to \mathbf{C}$. Recall from (1.6.10) and (1.6.11) that when colimits and limits exist in $\mathbf{C}$ and $\mathbf{D}$, there are canonical maps that interchange the colimit or limit with any functor F.

It turns out that there are similar maps for the homotopy colimit and homotopy limit, but we have to assume that F is a homotopy functor (i.e. it preserves weak equivalences). Recall from Definition 3.5.1 that the homotopy category of functors $\mathrm{Ho}\,\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ is defined by taking all functors and then inverting the natural transformations that are weak equivalences at every object.

Lemma 8.6.1. *Suppose* $\mathbf{C}$ *and* $\mathbf{D}$ *are categories with weak equivalences, in which all homotopy colimits and homotopy limits exist, and* F *is a homotopy functor. Then there are canonical interchange maps in the homotopy category of functors* $\mathrm{Ho}\,\mathrm{Fun}(\mathbf{C}^{\mathbf{I}}, \mathbf{D})$

$$\begin{aligned} \operatorname*{hocolim}_{a \in \mathbf{I}} F(X(a)) &\longrightarrow F\Big(\operatorname*{hocolim}_{a \in I} X(a)\Big), \\ F\Big(\operatorname*{holim}_{a \in I} X(a)\Big) &\longrightarrow \operatorname*{holim}_{a \in \mathbf{I}} F(X(a)). \end{aligned} \tag{8.6.2}$$

They are characterized as the unique maps making the following squares commute in $\operatorname{Ho}\operatorname{Fun}(\mathbf{C}^{\mathbf{I}}, \mathbf{D})$:

$$\begin{array}{ccc} \operatorname{hocolim} \circ F & \dashrightarrow & F \circ \operatorname{hocolim} \\ \downarrow & & \downarrow \\ \operatorname{colim} \circ F & \xrightarrow{(1.6.10)} & F \circ \operatorname{colim} \end{array} \qquad \begin{array}{ccc} F \circ \lim & \xrightarrow{(1.6.11)} & \lim \circ F \\ \downarrow & & \downarrow \\ F \circ \operatorname{holim} & \dashrightarrow & \operatorname{holim} \circ F. \end{array}$$

Proof. We do this for the homotopy colimit, since the other case is dual. Letting Q denote the cofibrant replacement of each diagram in $\mathbf{C}^{\mathbf{I}}$ by an equivalent diagram on which the homotopy colimit is equivalent to the colimit, the desired transformation can be described as

$$\begin{aligned} \operatorname{hocolim} \circ F &\xleftarrow{\sim} \operatorname{hocolim} \circ F \circ Q \\ &\to \operatorname{colim} \circ F \circ Q \\ &\to F \circ \operatorname{colim} \circ Q \\ &= F \circ \operatorname{hocolim}. \end{aligned}$$

To see that it is unique, let g be any other transformation and consider the diagram below, where the lower square is the original square pre-composed with Q:

$$\begin{array}{ccc} \operatorname{hocolim} \circ F & \overset{g}{\dashrightarrow} & F \circ \operatorname{hocolim} \\ \uparrow \sim & & \uparrow \sim \\ \operatorname{hocolim} \circ F \circ Q & \xrightarrow{g \circ \mathrm{id}_Q} & F \circ \operatorname{hocolim} \circ Q \\ \downarrow & & \downarrow \sim \\ \operatorname{colim} \circ F \circ Q & \xrightarrow{(1.6.10) \circ \mathrm{id}_Q} & F \circ \operatorname{colim} \circ Q \end{array}$$

This allows us to see that g is determined by the other maps in the diagram. □

We say that F **preserves homotopy colimits** if the first of the maps in (8.6.2) is a weak equivalence for every diagram X. Similarly F preserves homotopy limits if the second map in (8.6.2) is always a weak equivalence.

Proposition 8.6.3. *In any model category, the left derived functor $\mathbb{L}F$ of any left Quillen functor $F : \mathbf{C} \to \mathbf{D}$ preserves homotopy colimits. Similarly, the right-derived functor of any right Quillen functor preserves homotopy limits.*

This is the derived version of the fact that left adjoints preserve colimits and right adjoints preserve limits (Theorem 1.6.9).

Proof. We focus on the left Quillen case. This was actually already proven for any left-deformable left adjoint in Exercise 3.6.29. (We only used model categories in the statement because they're very common and they make the result less awkward to state.)

In the maximalist case that $F : \mathbf{C} \to \mathbf{D}$ is also a simplicial left Quillen functor between cofibrantly generated simplicial model categories, we can do better: this becomes a single map, not a zig-zag, that gives an isomorphism

$$F(B(QX, \mathbf{I}, *)) \cong B(F(QX), \mathbf{I}, *).$$

This is because F commutes with colimits and with tensors $\Delta[n] \otimes (-)$, and therefore it commutes with the formation of the bar construction:

$$F\left(\left(\coprod_{c_0 \to \cdots \to c_n} \Delta[n] \otimes QX(c_0)\right) / \sim\right) \cong \left(\coprod_{c_0 \to \cdots \to c_n} \Delta[n] \otimes F(QX(c_0))\right) / \sim .$$

See Exercise 8.7.36 for more details on how to define this canonical map and to check that it is well-defined. □

Example 8.6.4. Suspension spectra commute with homotopy colimits:

$$\operatorname{hocolim} \Sigma^\infty_+ X \simeq \Sigma^\infty_+ \left(\operatorname{hocolim}^{(u)} X\right),$$
$$\operatorname{hocolim} \Sigma^\infty X \simeq \Sigma^\infty \left(\operatorname{hocolim}^{(b)} X\right).$$

These are actually isomorphisms of spectra, provided the diagram X is pointwise cofibrant and we use the bar construction to define the homotopy colimit.

Similarly, the infinite loop space commutes with homotopy limits,

$$\operatorname{holim} \Omega^\infty E \simeq \Omega^\infty \left(\operatorname{holim} E\right).$$

Again, this is an isomorphism if E is pointwise fibrant (every $E(i)$ is an Ω-spectrum) and we take ev_0 as our model for Ω^∞.

Composing these together, the functor $\Omega^\infty \Sigma^\infty$ turns homotopy pushout squares of spaces into homotopy pullback squares of spaces.

We can also use this to sketch the proof of a useful general result about when functors commute with homotopy colimits and homotopy limits.

Lemma 8.6.5. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be cofibrantly generated simplicial model categories and let* F *be a simplicial homotopy functor. Then* F *commutes with all homotopy colimits iff it commutes with coproducts and homotopy pushouts (up to equivalence).*

Proof sketch. The assumption that F is simplicial makes the proof easier because then we have a canonical map $B(F(X), \mathbf{I}, *) \to F(B(X, \mathbf{I}, *))$ (see Exericse 8.7.36), that we just have to prove is an equivalence.

If F commutes with all homotopy colimits then it certainly commutes with these special cases. Conversely, if F commutes with coproducts and homotopy pushouts, then it commutes with sequential homotopy colimits as well, since we can model a mapping telescope as a homotopy pushout of maps between three coproducts, as in Exercise 1.7.34. Then we model the homotopy colimit by taking pointwise cofibrant replacement of X and then using the bar construction $B(X, \mathbf{I}, *)$. By induction on n we show that the maps $\mathrm{Sk}^n B(F(X), \mathbf{I}, *) \to F(\mathrm{Sk}^n B(X, \mathbf{I}, *))$ are equivalences for all $n \geq 0$, which we prove using (8.2.31) and the fact that F preserves coproducts and homotopy pushouts. Then since F commutes with filtered colimits, the map $B(F(X), \mathbf{I}, *) \to F(B(X, \mathbf{I}, *))$ is an equivalence as well. □

A similar argument gives the following.

Lemma 8.6.6. *For a simplicial homotopy functor F between cofibrantly generated simplicial model categories, F commutes with all homotopy limits iff it commutes with products and homotopy pullbacks. F commutes with all finite cell homotopy limits (Definition A.2.29) iff it commutes with all pullbacks.*

8.6.2 Changing the category I

For any functor of indexing categories $\alpha : \mathbf{I} \to \mathbf{J}$, there is a restriction functor $\alpha^* : \mathbf{C}^{\mathbf{J}} \to \mathbf{C}^{\mathbf{I}}$ that takes each diagram $X : \mathbf{J} \to \mathbf{C}$ to the composite $X \circ \alpha : \mathbf{I} \to \mathbf{C}$. There are canonical maps

$$\begin{aligned} \operatorname*{colim}_{a \in \mathbf{I}} \alpha^* X(a) &\longrightarrow \operatorname*{colim}_{b \in \mathbf{J}} X(b), & \lim_{b \in \mathbf{J}} X(b) &\longrightarrow \lim_{a \in \mathbf{I}} \alpha^* X(a), \\ \operatorname*{hocolim}_{a \in \mathbf{I}} \alpha^* X(a) &\longrightarrow \operatorname*{hocolim}_{b \in \mathbf{J}} X(b), & \operatorname*{holim}_{b \in \mathbf{J}} X(b) &\longrightarrow \operatorname*{holim}_{a \in \mathbf{I}} \alpha^* X(a). \end{aligned} \tag{8.6.7}$$

See Exercise 8.7.13. These canonical maps are usually not equivalences, but they become equivalences under certain conditions on the functor $\alpha : \mathbf{I} \to \mathbf{J}$.

Definition 8.6.8. Let $\alpha : \mathbf{I} \to \mathbf{J}$ be a functor of small categories. For each $j \in \mathbf{J}$, let $(j \downarrow \alpha) = j/\alpha$ be the comma category or slice category. It has an object for each object $i \in \mathbf{I}$ and map $j \to \alpha(i)$, and a morphism for each morphism $f : i_0 \to i_1$ in $\mathbf{I}$ that forms a commuting triangle

$$\begin{array}{ccc} & j & \\ \swarrow & & \searrow \\ \alpha(i_0) & \xrightarrow{\alpha(f)} & \alpha(i_1). \end{array}$$

We say that α is **terminal**[10] if the classifying space $B(j \downarrow \alpha)$ is path-connected for every $j \in \mathbf{J}$, and **homotopy terminal** if $B(j \downarrow \alpha)$ is contractible for every $j \in \mathbf{J}$. Intuitively, α is homotopy terminal when its image contains a contractible space of objects "lying to the right" of each object $j \in \mathbf{J}$.

Similarly we let $(\alpha \downarrow j) = \alpha/j$ be the comma category or slice category with an object for each object $i \in \mathbf{I}$ and map $j \to \alpha(i)$, and a morphism for each morphism $f : i_0 \to i_1$ in $\mathbf{I}$ that forms a commuting triangle

$$\begin{array}{ccc} \alpha(i_0) & \xrightarrow{\alpha(f)} & \alpha(i_1) \\ & \searrow \quad \swarrow & \\ & j. & \end{array}$$

We say that α is **initial** if the classifying space $B(\alpha \downarrow j)$ is path-connected for every $j \in \mathbf{J}$, and **homotopy initial** if $B(\alpha \downarrow j)$ is contractible for every $j \in \mathbf{J}$. Intuitively, α is homotopy initial when its image contains a contractible space of objects "lying to the left" of each object $j \in \mathbf{J}$.

Lemma 8.6.9. *We have isomorphisms*

$$B(j \downarrow \alpha) \cong B(\mathbf{J}(j,-), \mathbf{I}, *) \quad \text{and} \quad B(\alpha \downarrow j) \cong B(*, \mathbf{I}, \mathbf{J}(-, j)).$$

Proof. An n-tuple of composable arrows in $(j \downarrow \alpha)$ is an n-tuple of composable arrows $i_0 \to \cdots \to i_n$ in $\mathbf{I}$ together with a commuting diagram

$$\begin{array}{ccccccccc} & & j & & & & & & \\ & \swarrow & \downarrow & \searrow & & & & \searrow & \\ \alpha(i_0) & \longrightarrow & \alpha(i_1) & \longrightarrow & \cdots & \longrightarrow & \alpha(i_n), & & \end{array}$$

but this data is determined by giving just the first map $j \to \alpha(i_0)$, which is a morphism in $J(j,-)$. This allows us to identity the bar constructions together. □

Remark 8.6.10. We are doing the case where $\mathbf{I}$ is discrete. In the enriched case, the comma category $(j \downarrow \alpha)$ does not quite make sense, so we simply define "homotopy terminal" to mean that $B(\mathbf{J}(j,-), \mathbf{I}, *)$ is contractible.

Theorem 8.6.11. *In any category* $\mathbf{C}$*,*

- *If α is terminal then* $\operatorname{colim}_{a \in \mathbf{I}} \alpha^* X(a) \to \operatorname{colim}_{b \in \mathbf{J}} X(b)$ *is an isomorphism.*
- *If α is initial then* $\lim_{b \in \mathbf{J}} X(b) \to \lim_{a \in \mathbf{I}} \alpha^* X(a)$ *is an isomorphism.*

[10] This is sometimes called being final or left cofinal, but that terminology seems more confusing.

In any cofibrantly generated simplicial model category **C***,*

- *If* α *is homotopy terminal then* $\operatorname*{hocolim}_{a\in\mathbf{I}} \alpha^* X(a) \to \operatorname*{hocolim}_{b\in\mathbf{J}} X(b)$ *is a weak equivalence.*
- *If* α *is homotopy initial then* $\operatorname*{holim}_{b\in\mathbf{J}} X(b) \to \operatorname*{holim}_{a\in\mathbf{I}} \alpha^* X(a)$ *is a weak equivalence.*

We actually don't have a "minimalist" version of this result. The proof uses the bar construction in an essential way.

Proof. The first two cases are an exercise. For the case of homotopy colimits, we first make X pointwise cofibrant in the projective model structure (which exists by Exercise 5.8.37), so that the homotopy colimit is computed by the bar construction. If α is homotopy terminal, then we put the canonical map $B(\alpha^* X, \mathbf{I}, *) \to B(X, \mathbf{J}, *)$ into a commuting diagram

$$\begin{array}{ccc}
B(X,\mathbf{I},*) & \longrightarrow & B(X,\mathbf{J},*) \\
{\scriptstyle\sim}\uparrow & & \uparrow{\scriptstyle\sim} \\
B(B(X,\mathbf{J},\mathbf{J}),\mathbf{I},*) & \longrightarrow & B(B(X,\mathbf{J},\mathbf{J}),\mathbf{J},*) \\
{\scriptstyle\cong}\updownarrow & & {\scriptstyle\cong}\updownarrow \\
B(X,\mathbf{J},B(\mathbf{J},\mathbf{I},*)) & \xrightarrow{\sim} & B(X,\mathbf{J},B(\mathbf{J},\mathbf{J},*))
\end{array}$$

where for simplicity we have suppressed notation for α^*. The vertical equivalences are by the bar lemma (Lemma 8.3.15), and the horizontal equivalence is because α is homotopy terminal. (And of course both of these also use the homotopy invariance of the bar construction, Theorem 8.3.12.) Therefore the canonical map along the top row is also a weak equivalence.

For homotopy limits, we first make X is pointwise fibrant so that the homotopy limits can be computed by the cobar construction. Then the following commuting diagram shows that when α is homotopy initial, the canonical map of homotopy limits is an equivalence.

$$\begin{array}{ccc}
F(*,\mathbf{J},X) & \longrightarrow & F(*,\mathbf{I},X) \\
{\scriptstyle\sim}\uparrow & & \uparrow{\scriptstyle\sim} \\
F(*,\mathbf{J},F(\mathbf{J},\mathbf{J},X)) & \longrightarrow & F(*,\mathbf{I},F(\mathbf{J},\mathbf{J},X)) \\
{\scriptstyle\cong}\updownarrow & & {\scriptstyle\cong}\updownarrow \\
F(B(*,\mathbf{J},\mathbf{J}),\mathbf{J},X) & \xrightarrow{\sim} & F(B(*,\mathbf{I},\mathbf{J}),\mathbf{J},X)
\end{array}$$

□

Corollary 8.6.12 (Quillen Theorem A)**.** *If* $\alpha\colon \mathbf{I} \to \mathbf{J}$ *is a functor such that each of the spaces* $B(j \downarrow \alpha)$ *is contractible, then* α *induces a homotopy equivalence* $B\mathbf{I} \simeq B\mathbf{J}$*.*

The same is also true when $B(\alpha \downarrow j)$ is contractible, since $B\mathbf{C}^{\mathrm{op}} \cong B\mathbf{C}$.

Example 8.6.13. If $\mathbf{I}$ has a terminal object, the inclusion of the terminal object $* \to \mathbf{I}$ is homotopy terminal, using Exercise 7.7.13. Therefore the induced map

$$X(*) \to \operatorname*{hocolim}_{i \in \mathbf{I}} X(i)$$

is an equivalence. We've seen this already for some particular choices of indexing category $\mathbf{I}$, but Theorem 8.6.11 tells us that this happens for any $\mathbf{I}$ that has a terminal object.

Example 8.6.14. If $\mathbf{I}$ is the sequential category $\{\bullet \to \bullet \to \cdots\}$, then the diagonal $\mathbf{I} \to \mathbf{I} \times \mathbf{I}$ is homotopy terminal. Therefore a homotopy colimit over a grid is equivalent to the homotopy colimit along the diagonal. We've used this fact for ordinary colimits many times!

Example 8.6.15. If $\mathbf{I}$ is the sequential category $\{\bullet \to \bullet \to \cdots\}$ and $\mathbf{I}' \subseteq \mathbf{I}$ is any full subcategory that has infinitely many objects, then $\mathbf{I}' \to \mathbf{I}$ is homotopy terminal. So a sequential homotopy colimit can be calculated using any subsequence.

8.6.3 Fubini Theorems

Next we move to the more multiplicative properties of homotopy colimits and homotopy limits. If $\mathbf{I}$ and $\mathbf{J}$ are two small categories, let $\mathbf{I} \times \mathbf{J}$ be the product category with objects the ordered pairs $(i, j) \in \operatorname{ob}\mathbf{I} \times \operatorname{ob}\mathbf{J}$, and morphisms the product

$$(\mathbf{I} \times \mathbf{J})((i_0, j_0), (i_1, j_1)) = \mathbf{I}(i_0, i_1) \times \mathbf{J}(j_0, j_1).$$

A diagram $X : \mathbf{I} \times \mathbf{J} \to \mathbf{C}$ can be interpreted as an $\mathbf{I}$-diagram of $\mathbf{J}$-diagrams, so if we take the homotopy colimit over $\mathbf{J}$, this forms an $\mathbf{I}$-diagram and we can take its homotopy colimit.

Proposition 8.6.16. *Suppose* $\mathbf{C}$ *is a category with weak equivalences that has all homotopy colimits and homotopy left Kan extensions. Then the iterated homotopy colimit is equivalent to the homotopy colimit over* $\mathbf{I} \times \mathbf{J}$*:*

$$\operatorname*{hocolim}_{i \in \mathbf{I}} \left(\operatorname*{hocolim}_{j \in \mathbf{J}} X(i, j) \right) \simeq \operatorname*{hocolim}_{(i,j) \in \mathbf{I} \times \mathbf{J}} X(i, j).$$

Similarly, if $\mathbf{C}$ *has all homotopy limits and homotopy right Kan extensions, then the iterated homotopy limit is equivalent to the homotopy limit,*

$$\operatorname*{holim}_{i \in \mathbf{I}} \left(\operatorname*{holim}_{j \in \mathbf{J}} X(i, j) \right) \simeq \operatorname*{holim}_{(i,j) \in \mathbf{I} \times \mathbf{J}} X(i, j).$$

Proof. We prove the statement for homotopy colimits, since the statement for homotopy limits is similar. Each side describes a functor $\mathrm{Ho}(\mathbf{C}^{\mathbf{I}\times\mathbf{J}}) \to \mathrm{Ho}\,\mathbf{C}$. On the right-hand side, the right adjoint is the functor $\mathrm{Ho}\,\mathbf{C} \to \mathrm{Ho}(\mathbf{C}^{\mathbf{I}\times\mathbf{J}})$ that takes each space Y to the constant diagram $\mathrm{const}(Y)\colon \mathbf{I}\times\mathbf{J} \to \mathbf{C}$.

On the left-hand side, we are composing the homotopy left Kan extension along the projection map $\mathbf{I}\times\mathbf{J} \to \mathbf{I}$ and the homotopy colimit along $\mathbf{I}$. For the corresponding (strict) left Kan extension and colimit functors

$$\mathbf{C}^{\mathbf{I}\times\mathbf{J}} \longrightarrow \mathbf{C}^{\mathbf{I}} \longrightarrow \mathbf{C},$$

the right adjoints send Y to the constant diagram $\mathrm{const}(Y)\colon \mathbf{J} \to \mathbf{C}$, and then the pullback or restriction this diagram along the functor $\mathbf{I}\times\mathbf{J} \to \mathbf{I}$ gives a constant diagram $\mathrm{const}(Y)\colon \mathbf{I}\times\mathbf{J} \to \mathbf{C}$. These right adjoins preserve *all* weak equivalences, so they are composable in the sense of Lemma 3.5.13. Therefore when we think of these as functors

$$\mathrm{Ho}(\mathbf{C}^{\mathbf{I}\times\mathbf{J}}) \longrightarrow \mathrm{Ho}(\mathbf{C}^{\mathbf{I}}) \longrightarrow \mathrm{Ho}\,\mathbf{C},$$

the composite of the right-derived functors is the right-derived functor of the composite. The composite of the right-derived functors gives the right adjoint of $\operatorname{hocolim}_{\mathbf{I}}\Big(\operatorname{hocolim}_{\mathbf{J}}(-)\Big)$, while the right-derived composite is just the constant diagram functor $\mathrm{const}(-)$. This finishes the proof that the right adjoints of $\operatorname{hocolim}_{\mathbf{I}}\Big(\operatorname{hocolim}_{\mathbf{J}}(-)\Big)$ and $\operatorname{hocolim}_{\mathbf{I}\times\mathbf{J}}(-)$ agree, so these two operations (in the homotopy category of diagrams) must be isomorphic.

For a more concrete version of the proof that holds if $\mathbf{C}$ is a cofibrantly generated simplicial model category, we make X pointwise cofibrant, then use the bar construction. Writing out the left-hand side as the realization of a bisimplicial set and taking the diagonal simplicial set, we get the expression for the right-hand side. Therefore the two realizations are homeomorphic. □

Corollary 8.6.17. *Homotopy colimits commute with each other:*

$$\operatorname*{hocolim}_{i\in\mathbf{I}}\Big(\operatorname*{hocolim}_{j\in\mathbf{J}} X(i,j)\Big) \simeq \operatorname*{hocolim}_{j\in\mathbf{J}}\Big(\operatorname*{hocolim}_{i\in\mathbf{I}} X(i,j)\Big).$$

Homotopy limits also commute with each other.

For example, if we wedge together two diagrams of spectra, the homotopy colimit of the resulting diagram is the wedge of the homotopy colimits,

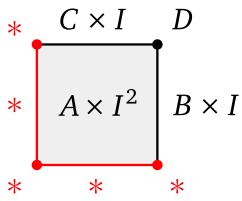
$$\operatorname{hocolim}\big(X(c)\vee Y(c)\big) \cong \Big(\operatorname{hocolim} X(c)\Big)\vee\Big(\operatorname{hocolim} Y(c)\Big).$$

Similarly if we want to take the homotopy cofiber of a map of diagrams $X \to Y$, we can take the homotopy cofiber before or after taking the homotopy colimit and the answers agree:

$$\operatorname{hocolim} C(X(i)\to Y(i)) \cong C\Big(\operatorname{hocolim} X(i) \to \operatorname{hocolim} Y(i)\Big).$$

As an added bonus, when we use the bar construction to model the homotopy colimit, these are *isomorphisms* of spectra, not just stable equivalences.

Example 8.6.18. Given a cube-shaped diagram $\{0 \to 1\}^n \to \mathbf{Top}_*$ or $\mathbf{Sp}$, the **total homotopy cofiber** is defined by iteratively taking the homotopy cofiber in each direction of the cube, n times, until we are left with a single space. Since homotopy colimits commute, it does not matter which order we take the homotopy cofibers in, we always get the same result. (And the results are the same up to homeomorphism if we use the usual model for the homotopy cofiber. See Exercise 8.7.23.)



The "Fubini theorem" for homotopy colimits can be generalized to a twisted product of categories. Suppose $\mathbf{J}\colon \mathbf{I} \to \mathbf{Cat}$ is any functor. So each $a \in \mathbf{I}$ is sent to a category $\mathbf{J}(a)$, and each morphism $f : a \to b$ is sent to a functor $\mathbf{J}(f)\colon \mathbf{J}(a) \to \mathbf{J}(b)$, strictly respecting composition. We define the **covariant Grothendieck construction** to be the category

$$\mathbf{I} \ltimes \mathbf{J} = \int_{\mathbf{I}} \mathbf{J}$$

that has one object for each ordered pair $(a \in \mathbf{I}, j \in \mathbf{J}(a))$, and a morphism $(a, j) \to (b, k)$ for every morphism $f : a \to b$ and morphism $\mathbf{J}(f)(j) \to k$ in $\mathbf{J}(b)$. It is left as an exercise to define the composition.

We like to use the notation $\mathbf{I} \ltimes \mathbf{J}$ for this category, for two reasons. First, it emphasizes that this is a generalization of the product $\mathbf{I} \times \mathbf{J}$, which we get when the diagram $\mathbf{J}$ is constant. Second, when $\mathbf{I}$ is a group H and $\mathbf{J}$ is a group N with H-action through automorphisms, the category $\mathbf{I} \ltimes \mathbf{J}$ is the same thing as the semidirect product group $H \ltimes N$.

Proposition 8.6.19 (Thomason)**.** *Suppose* $\mathbf{C}$ *is a category with weak equivalences that has all homotopy colimits and homotopy left Kan extensions. Then for any diagram* $X : \mathbf{I} \ltimes \mathbf{J} \to \mathbf{C}$*, the homotopy colimits along the categories* $\mathbf{J}(a)$ *form a diagram on* $\mathbf{I}$*, and the homotopy colimit of these spaces along* $\mathbf{I}$ *is equivalent to the homotopy colimit of the original diagram along all of* $\mathbf{I} \ltimes \mathbf{J}$*:*

$$\operatorname*{hocolim}_{a \in \mathbf{I}} \left(\operatorname*{hocolim}_{j \in \mathbf{J}(a)} X(a, j) \right) \simeq \operatorname*{hocolim}_{(a,j) \in \mathbf{I} \ltimes \mathbf{J}} X(a, j).$$

Proof. As in the proof of Proposition 8.6.16, we think of the left-hand side as the homotopy left Kan extension along the projection functor $\mathbf{I} \ltimes \mathbf{J} \to \mathbf{I}$, followed by the homotopy colimit along $\mathbf{I}$. Then it is enough to compare the right adjoints. One of them takes each space Y to the constant diagram $\mathrm{const}(Y)\colon \mathbf{I} \to \mathbf{C}$, then to the constant diagram $\mathrm{const}(Y)\colon \mathbf{I} \ltimes \mathbf{J} \to \mathbf{C}$, so that all together it agrees with the other right adjoint. □

Remark 8.6.20. The concrete version of the proof actually doesn't work here. The categorical nerve on the left-hand side gives us strings of n morphisms in $\mathbf{J}(a)$ and n

morphisms in $\mathbf{I}$ starting at $a \in \mathbf{I}$, while on the right-hand side we get strings of n morphisms in $\mathbf{I} \ltimes \mathbf{J}$. These are related, but they aren't the same. Intuitively, the first one gives a square grid of morphisms in $\mathbf{I} \ltimes \mathbf{J}$, while the second gives only the triangle in the lower half of that grid.

We leave it to the reader to formulate the dual version for homotopy limits. One has to use the **contravariant Grothendieck construction** $\int^{\mathbf{I}} \mathbf{J}$ in which a morphism $(a, j) \to (b, k)$ consists of a *backwards* morphism $f : b \to a$ and morphism $j \to \mathbf{J}(f)(k)$ in $\mathbf{J}(a)$.

Corollary 8.6.21 (Quillen Theorem B)**.** *If $\alpha \colon \mathbf{I} \to \mathbf{J}$ is a functor such that each map $j \to j'$ in $\mathbf{J}$ induces a weak equivalence $B(j' \downarrow \alpha) \to B(j \downarrow \alpha)$, then for each $j \in \mathbf{J}$ there is a fiber sequence*

$$B(j \downarrow \alpha) \to B\mathbf{I} \to B\mathbf{J}.$$

Similarly if each arrow induces a weak equivalence $B(\alpha \downarrow j) \to B(\alpha \downarrow j')$, then for each $j \in \mathbf{J}$ there is a fiber sequence

$$B(\alpha \downarrow j) \to B\mathbf{I} \to B\mathbf{J}.$$

Proof. Since $B\mathbf{C}^{\mathrm{op}} \cong B\mathbf{C}$, it suffices to prove the second version. By Quillen's Lemma (Theorem 7.4.33), the homotopy colimit of the spaces $B(j \downarrow \alpha)$ to $B(\mathbf{J})$ is a quasifibration. By Thomason's theorem on homotopy colimits, this homotopy colimit is $B(\mathbf{I} \ltimes (\alpha \downarrow -))$. So we just have to show that the map of categories $\mathbf{I} \to \mathbf{I} \ltimes (\alpha \downarrow -)$ induces an equivalence on classifying spaces. This follows because it is the left adjoint of the forgetful functor $\mathbf{I} \ltimes (\alpha \downarrow -) \to \mathbf{I}$, and adjunctions give equivalences on classifying spaces by Exercise 7.7.12. □

There are other results that could be called a "Fubini theorem." For instance, if $\mathbf{C}$ is a cofibrantly generated symmetric monoidal simplicial model category, and $X : \mathbf{I} \to \mathbf{C}$ and $Y : \mathbf{J} \to \mathbf{C}$ are two diagrams in $\mathbf{C}$, then we may form the **external tensor product** diagram

$$X \otimes Y : \mathbf{I} \times \mathbf{J} \to \mathbf{C}, \qquad (X \otimes Y)(i, j) = X(i) \otimes Y(j).$$

Then by comparing the formulas for the bar constructions, we see that

Proposition 8.6.22. *We have a natural weak equivalence*

$$\Big(\operatorname*{hocolim}_{i \in \mathbf{I}} X(i)\Big) \otimes \Big(\operatorname*{hocolim}_{j \in \mathbf{J}} Y(j)\Big) \simeq \operatorname*{hocolim}_{(i,j) \in \mathbf{I} \times \mathbf{J}} \big(X(i) \otimes Y(j)\big).$$

Finally, there is the closely related result that homotopy orbits respect composition. As before, let $\mathbf{C}$ be any category that has all homotopy colimits and homotopy left Kan extensions. Any object of $\mathbf{C}$ with G-action, is just a diagram $X : G \to \mathbf{C}$, where we think of G as a category with one object. The colimit of this diagram is the orbits X_G. If we take

a normal subgroup $N \trianglelefteq G$, then the N-orbits X_N have a leftover action by the quotient group G/N, which allows us to take the G/N-orbits. It is an elementary exercise to see that the composite operation is the same as the G-orbits,

$$(X_N)_{G/N} \cong X_G,$$

and the next result gives the homotopical version of this.

Proposition 8.6.23. *Suppose* $\mathbf{C}$ *is a category with weak equivalences that has all homotopy colimits and homotopy left Kan extensions. Then for objects* X *with* G*-action, there is an equivalence of homotopy orbits*

$$(X_{hN})_{hG/N} \simeq X_{hG}.$$

Similarly, if $\mathbf{C}$ *has all homotopy limits and homotopy right Kan extensions then there is an equivalence of homotopy fixed points* $(X^{hN})^{hG/N} \simeq X^{hG}$.

Note that when $G = H \ltimes N$ for some other subgroup H, this is a special case of Proposition 8.6.19, and when $G = H \times N$, this is a special case of the Fubini theorem (Proposition 8.6.16), giving $X_{h(H\times N)} \simeq (X_{hN})_{hH} \simeq (X_{hH})_{hN}$.

Proof. The proof is almost the same as in Proposition 8.6.16 and Proposition 8.6.19. The key point is that the functor that takes X with G-action to X_N with G/N-action is also a left Kan extension, so when we compare the right adjoints of the two sides, we again get the "constant diagram" functor that takes each object Y to the same object Y with G acting only through identity morphisms. □

As before, the concrete version of this proof doesn't work. For example, in unbased spaces, we are not able to get some homeomorphism of the form

$$(X \times_N EN) \times_{G/N} E(G/N) \ncong X \times_G EG.$$

So we have to rely on the abstract argument using uniqueness of left-derived functors.

8.6.4 Realization is a homotopy colimit

Another important fact is that the geometric realization itself is an example of a homotopy colimit.

Proposition 8.6.24. *In any cofibrantly generated simplicial model category* $\mathbf{C}$*, for any Reedy cofibrant simplicial object* $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}$*, there is a canonical equivalence*

$$|X| \simeq \operatorname*{hocolim}_{[i]\in\mathbf{\Delta}^{\mathrm{op}}} X_i.$$

Dually, for a Reedy fibrant cosimplicial object $Y : \mathbf{\Delta} \to \mathbf{C}$ *there is a canonical equivalence*

$$\mathrm{Tot}(Y) \simeq \operatorname*{holim}_{[i]\in\mathbf{\Delta}} Y^i.$$

In other words, the left-derived or "homotopically correct" realization is the same thing as the homotopy colimit along $\mathbf{\Delta}^{\mathrm{op}}$.

Proof. We do the case of the realization, since the totalization is dual. The realization and the homotopy colimit are coends of X with another diagram:

$$|X| = \Delta[-] \otimes_{\mathbf{\Delta}} X_-, \qquad \operatorname*{hocolim}_{[i]\in\mathbf{\Delta}^{\mathrm{op}}} X_i \simeq B(*, \mathbf{\Delta}, \mathbf{\Delta}) \otimes_{\mathbf{\Delta}} X_-.$$

Here the coend is defined using the tensor of $\mathbf{C}$ with $\mathbf{sSet}$, and the bar construction $B(*, \mathbf{\Delta}, \mathbf{\Delta})$ is computed in the category of simplicial sets, so it is the same thing as the two-sided nerve $N_\bullet(*, \mathbf{\Delta}, \mathbf{\Delta})$. We can easily form the zig-zag of equivalences of diagrams $\mathbf{\Delta} \to \mathbf{sSet}$

$$\Delta[-] \xrightarrow{\sim} * \xleftarrow{\sim} B(*, \mathbf{\Delta}, \mathbf{\Delta}).$$

It is a zig-zag of equivalences using the bar lemma (Lemma 8.3.15). Now the idea is to take the coend of each of these maps with X, and to cofibrantly replace things so that we still get equivalences at the end. Let QX be a cofibrant replacement of X in the projective model structure on functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}$, which exists by Exercise 5.8.37. Any equivalence of diagrams induces an equivalence after applying $- \otimes_{\mathbf{\Delta}} QX$, as we can check, one cell of QX at a time. We therefore get a commuting diagram in which the outside maps give the desired weak equivalence:

$$\begin{array}{ccccc}
\Delta[-] \otimes_{\mathbf{\Delta}} QX & \xrightarrow{\sim} & * \otimes_{\mathbf{\Delta}} QX & \xleftarrow{\sim} & B(*, \mathbf{\Delta}, \mathbf{\Delta}) \otimes_{\mathbf{\Delta}} QX \\
\downarrow{\scriptstyle\sim} & & \downarrow & & \downarrow{\scriptstyle\sim} \\
\Delta[-] \otimes_{\mathbf{\Delta}} X & \longrightarrow & * \otimes_{\mathbf{\Delta}} X & \longleftarrow & B(*, \mathbf{\Delta}, \mathbf{\Delta}) \otimes_{\mathbf{\Delta}} X.
\end{array}$$

It is canonical in the sense that it is an equivalence of left-derived functors lying over the strict colimit $* \otimes_{\mathbf{\Delta}} X$, and there is only one such equivalence in the homotopy category by Corollary 3.5.3. □

We can improve this zig-zag to an actual map $\operatorname*{hocolim}_{[i]\in\mathbf{\Delta}^{\mathrm{op}}} X_i \to |X_\bullet|$ called the "last-vertex map." See Exercise 8.7.33.

The fact that the realization is a homotopy colimit can be used to relate the so-called "fat realization" to the usual realization.

Definition 8.6.25. For a simplicial space $X_\bullet$, or more generally a simplicial object in any cofibrantly generated simplicial model category $\mathbf{C}$, the **fat realization** $\|X_\bullet\|$ is defined just as the realization is defined in (7.1.14) and (8.2.2), except that we only mod out by the face relations, not the degeneracies:

$$\|X_\bullet\| = \left(\coprod_{n\geq 0} \Delta[n] \otimes X_n \right) / \, (d^i t, x) = (t, d_i x). \tag{8.6.26}$$

This has a canonical map to the "thin" realization $|X_\bullet|$ by applying the remaining degeneracy relations.

Proposition 8.6.27. *In any cofibrantly generated simplicial model category* $\mathbf{C}$*, for any notion of cofibration in* $\mathbf{C}$ *satisfying the assumptions in Theorem 8.2.27, the fat realization* $\|X_\bullet\|$ *preserves weak equivalences when* $X_\bullet$ *is pointwise cofibrant. And when* $X_\bullet$ *is Reedy cofibrant, the canonical map* $\|X_\bullet\| \to |X_\bullet|$ *is a weak equivalence.*

Proof. We prove that the fat realization preserves equivalences by defining the n-skeleton as in Definition 8.2.29 and then rearranging the universal property to give pushout squares

$$\begin{array}{ccc} \partial\Delta[n] \otimes X_n & \longrightarrow & \Delta[n] \otimes X_n \\ \downarrow & & \downarrow \\ \mathrm{Sk}^{n-1}\|X_\bullet\| & \longrightarrow & \mathrm{Sk}^n\|X_\bullet\|. \end{array} \tag{8.6.28}$$

As soon as the levels X_n are cofibrant, the map along the top is a cofibration, and so this square is a homotopy pushout and the colimit of the skeleta $\mathrm{Sk}^n\|X_\bullet\|$ is a homotopy colimit, which allows us to check that $\|-\|$ preserves equivalences.

Now let $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \subseteq \mathbf{\Delta}^{\mathrm{op}}$ be the subcategory of injective maps, as in Exercise 7.7.3. The fat realization is the coend $\Delta[-] \otimes_{\mathbf{\Delta}_{\mathrm{inj}}} X_-$, and the same argument as in Proposition 8.6.24 identifies $\|X_\bullet\|$ with the homotopy colimit of X along the subcategory $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}}$. So by the change-of-indexing-category theorem (Theorem 8.6.11), we only have to check that the inclusion $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \subseteq \mathbf{\Delta}^{\mathrm{op}}$ is homotopy terminal. Equivalently, the inclusion $\mathbf{\Delta}_{\mathrm{inj}} \subseteq \mathbf{\Delta}$ is homotopy initial.

So fix $n \geq 0$ and let $\mathbf{C}_n$ be the category whose objects are totally ordered maps $[k] \to [n]$, and whose morphisms are injective totally ordered maps $[k_1] \to [k_2]$ commuting with the map to $[n]$. To prove that $B\mathbf{C}_n$ is contractible, we only have to give a zig-zag of natural transformations between the identity functor $\mathbf{C}_n \to \mathbf{C}_n$ and a constant functor. One way to do this is to define a functor $F : \mathbf{C}_n \to \mathbf{C}_n$ that takes each map $f : [k] \to [n]$ to a new map $F(f) : [k+n+1] \to [n]$ that adds $(n+1)$ new points to the domain, mapping to the $(n+1)$ points in $[n]$, and placing each one before one of the preimages $f^{-1}(i)$ in the total ordering on $[k]$. Then F receives a natural transformation from the identity by including $[k] \to [k+n+1]$ in the obvious way. F also receives a natural transformation from the constant functor on the identity map $[n] \to [n]$, by restricting the map $F(f)$ to the $(n+1)$ new points, which just give the identity map $[n] \to [n]$. Therefore $B\mathbf{C}_n \simeq *$, which finishes the proof. □

Corollary 8.6.29. *For simplicial objects in* **Top**, $\|X_\bullet\|$ *always preserves weak equivalences, and if* $X_\bullet$ *is Reedy h-cofibrant then the canonical map* $\|X_\bullet\| \to |X_\bullet|$ *is a weak equivalence.*

8.6.5 The space of maps out of a homotopy colimit

Proposition 8.6.30. *In any cofibrantly generated simplicial model category* $\mathbf{C}$*, there are weak equivalences of simplicial sets*

$$\mathbb{R}\mathbf{C}_\bullet\left(\operatorname*{hocolim}_{i\in\mathbf{I}} X(i), Y\right) \simeq \mathbb{R}\mathrm{Map}^{\mathbf{I}}_\bullet(X, \mathrm{const}(Y)) \simeq \operatorname*{holim}_{i\in\mathbf{I}^{\mathrm{op}}} \mathbb{R}\mathbf{C}_\bullet(X(i), Y)$$

$$\mathbb{R}\mathbf{C}_\bullet\left(Y, \operatorname*{holim}_{i\in\mathbf{I}} X(i)\right) \simeq \mathbb{R}\mathrm{Map}^{\mathbf{I}}_\bullet(\mathrm{const}(Y), X) \simeq \operatorname*{holim}_{i\in\mathbf{I}} \mathbb{R}\mathbf{C}_\bullet(Y, X(i))$$

where $\mathbb{R}$ *means the right-derived functor, so we make the input cofibrant and the output fibrant first.*

Proof. Make X pointwise cofibrant and Y fibrant. Using the same technique as in Lemma 8.5.10, we form isomorphisms of simplicial sets

$$\mathbf{C}_\bullet(B(X, \mathbf{I}, *), Y) \cong F(X, \mathbf{I}, \mathrm{const}(Y)) \cong F(*, \mathbf{I}^{\mathrm{op}}, \mathbf{C}_\bullet(X, Y))$$

where the $\mathbf{C}_\bullet(X, Y)$ denotes the diagram $\mathbf{I}^{\mathrm{op}} \to \mathbf{sSet}$ that sends $i \in \mathbf{I}$ to $\mathbf{C}_\bullet(X(i), Y)$. Under these assumptions all of the functors that appear are equivalent to their derived functors, which proves the claim. The second claim is similar. □

This shows that the homotopy colimit satisfies an enriched version of the universal property of the colimit, up to weak equivalence. This corresponds to the fact that homotopy colimits are the same thing as colimits in the underlying ∞-category (Example 10.3.20).

8.6.6 Commuting colimits and limits

Homotopy colimits usually don't commute with homotopy limits, but there are a couple of important cases where they do. To set this up, as in Section 8.6.3, consider a diagram $X : \mathbf{I} \times \mathbf{J} \to \mathbf{C}$. Then there is a canonical interchange map

$$\operatorname*{hocolim}_{a\in\mathbf{I}} \left(\operatorname*{holim}_{b\in\mathbf{J}} X(a, b)\right) \longrightarrow \operatorname*{holim}_{b\in\mathbf{J}} \left(\operatorname*{hocolim}_{a\in\mathbf{I}} X(a, b)\right) \tag{8.6.31}$$

for the same reason as in Lemma 8.6.1.

This interchange map is usually not an equivalence. For instance, if $\mathbf{I}$ and $\mathbf{J}$ each have two objects and no morphisms, then the interchange map is of the form

$$(A \amalg B) \times (C \amalg D) \longrightarrow (A \times C) \amalg (B \times D),$$

which is not an equivalence in spaces or spectra. (It is an equivalence in abelian groups though.)

One interesting case where the interchange is an equivalence is when the colimit is filtered in the sense of Exercise 6.6.31, and when the homotopy limit is finite cell in the sense of Definition A.2.29.

Theorem 8.6.32. *In* **Top**, **Top**$_*$, **sSet**, **sSet**$_*$, *and any of our categories of spectra, filtered homotopy colimits commute with finite cell homotopy limits.*

In contrast to the previous results, this does not seem to be true for every model category, or every category where homotopy colimits and limits exist. There has to be some way of describing the homotopy type using a small set of generators, analogous to the way we describe weak equivalences in the above settings using homotopy groups. We do know that this fact holds in any combinatorial model category (Definition 10.4.26), and therefore in every presentable ∞-category by [nLa25], which uses [Dug01, Prop 7.3].

Proof. Note that by Exercises 7.7.38 and 8.7.7, the filtered homotopy colimit in each of these settings commutes with the formation of homotopy groups.

By Lemma 8.6.6, it suffices to prove that filtered homotopy colimits commute with homotopy pullbacks. Since a homotopy pullback square is one in which the induced map of homotopy fibers is an equivalence, it suffices to prove that any filtered homotopy colimit of fiber sequences is a fiber sequence. We do this in all of the above cases using the algebraic fact that a filtered colimit of exact sequences is exact, so that by the 5-lemma, the filtered colimit of the homotopy fibers has the same homotopy groups as the homotopy fiber of the filtered colimits. But there is a natural map between them inducing this isomorphism on homotopy groups, so they are equivalent. □

When we pass to spectra, we can use stability to do a little better:

Theorem 8.6.33. *In any of our categories of spectra,* all *homotopy colimits commute with finite cell homotopy limits.*

Proof. By Lemma 8.6.6, it suffices to prove that homotopy colimits commute with homotopy pullbacks. But a homotopy pullback square is a homotopy pushout square, and homotopy colimits preserve homotopy pushout squares by Corollary 8.6.17, so they must preserve homotopy pullback squares as well! □

Some authors define **sifted homotopy colimits** to be homotopy colimits along categories with the property that the homotopy colimit will commute with all finite products, in any sufficiently nice model category (say a cofibrantly generated simplicial model category). So this gives another case where the interchange is an equivalence, by definition. The homotopy colimit along $\mathbf{\Delta}^{\mathrm{op}}$ is sifted, because by Proposition 8.6.24 it is equivalent to the geometric realization, which we know commutes with finite products.

For the rest of this section, let's restrict attention to homotopy colimits of topological spaces. The category $\mathbf{\Delta}^{\mathrm{op}}$ is not filtered, and in general the geometric realization does not commute with all homotopy pullbacks. This might seem strange, because we know from Theorem 7.1.31 that it commutes with all *strict* pullbacks. The issue is that if we take a strict pullback along a map of simplicial spaces $E_\bullet \to B_\bullet$ such that each $E_n \to B_n$ is a fibration, then at each simplicial level we have a homotopy pullback, but on the realizations we don't necessarily have a homotopy pullback, because there is no reason for $|E_\bullet| \to |B_\bullet|$ to be a fibration.

This does work when $E_\bullet$ and $B_\bullet$ are discrete and the map between them is a Kan fibration, by Theorem 7.1.38. More generally, the **π_*-Kan condition** guarantees that the geometric realization preserves the homotopy pullback. It is complicated to state, see [BF78, Rez14] for more details. One special case of this is:

Proposition 8.6.34. *Geometric realization of simplicial spaces commutes with homotopy pullbacks*

$$|A_\bullet \times^h_{B_\bullet} E_\bullet| \xrightarrow{\sim} |A_\bullet| \times^h_{|B_\bullet|} |E_\bullet|$$

if every one of the spaces B_n is connected.

Corollary 8.6.35. *For a simplicial space $B_\bullet$, the canonical map interchanging realization with the based loopspace*

$$|\Omega B_\bullet| \xrightarrow{\sim} \Omega |B_\bullet|$$

is an equivalence if every one of the spaces B_n is connected.

These in turn are proven from a more basic statement called **descent for homotopy colimits**. Descent gives us another sitation where homotopy colimits commute with homotopy pullbacks: it says that the homotopy colimit of a collection of fiber sequences of spaces $F_i \to E_i \to B_i$ is a fiber sequence, provided that the maps of base spaces $B_i \to B_j$ are all weak equivalences, or alternatively that the maps of homotopy fibers $F_i \to F_j$ are all weak equivalences. See Section A.2.1 for more details and a proof.

8.7 Exercises

8.7.1. If $\otimes\colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ is a Quillen tensor, prove that the composite of $\otimes$ with any Quillen left adjoint, is again a Quillen tensor. The composition could be of the form $F(X) \otimes Y$, or $X \otimes F(Y)$, or even $F(X \otimes Y)$.

This exercise is helpful for checking the various examples form simplicial model categories.

8.7.2. (a) Prove that every topological model category (Remark 8.1.17) is a simplicial model category (Definition 8.1.1). (Exercise 8.7.1 is helpful for this.)

(b) Explain why this process cannot be reversed. In particular, if we have a Quillen tensor $\mathbf{sSet} \times \mathbf{C} \to \mathbf{C}$ and we compose with singular simplices $S_\bullet$, we get a functor $\mathbf{Top} \times \mathbf{C} \to \mathbf{C}$, but it is usually not a Quillen tensor anymore.

8.7.3. Generalize the definition of "simplicial model category" from Definition 8.1.1 and "topological model category" from Remark 8.1.17. Replace simplicial sets or topological spaces by an arbitrary symmetric monoidal model category $\mathbf{V}$, and formulate the more general definition of a $\mathbf{V}$-model category $\mathbf{C}$. If $\mathbf{V}$ is a symmetric monoidal category of spectra, we call the result a **spectral model category**.

Prove that if $\mathbf{V} \to \mathbf{V}'$ is a strong symmetric monoidal Quillen left adjoint, then every $\mathbf{V}'$-model category is a $\mathbf{V}$-model category in a natural way. Therefore every spectral model category is also a topological model category, and every topological model category is a simplicial model category.

8.7.4. Suppose that $\mathbf{I}$ is a category enriched in unbased spaces, and that X and Y are diagrams of based spaces on $\mathbf{I}$, so that the actions have the form

$$X(a) \wedge \mathbf{I}(a,b)_+ \longrightarrow X(b), \qquad \mathbf{I}(a,b)_+ \wedge Y(b) \longrightarrow Y(a).$$

(a) Prove the claim in Example 8.3.3 that there is a homeomorphism

$$B^\wedge(X, \mathbf{I}_+, Y) \cong B^\times(X, \mathbf{I}, Y)/B^\times(*, \mathbf{I}, *).$$

(b) If everything is pointwise Quillen cofibrant, argue that $B^\times(*, \mathbf{I}, *)$ is in fact a subcomplex of $B^\times(X, \mathbf{I}, Y)$, and therefore we have a cofiber sequence

$$B^\times(*, \mathbf{I}, *) \longrightarrow B^\times(X, \mathbf{I}, Y) \longrightarrow B^\wedge(X, \mathbf{I}_+, Y).$$

In the special case that $Y = *$, deduce a cofiber sequence

$$\operatorname*{hocolim}^{(u)}_{c \in \mathbf{I}} (*) \longrightarrow \operatorname*{hocolim}^{(u)}_{c \in \mathbf{I}} X(c) \longrightarrow \operatorname*{hocolim}^{(b)}_{c \in \mathbf{I}} X(c).$$

(c) As a special case, argue that when G is a Quillen cofibrant topological monoid or group acting on a based space X, the unbased and based variants of the homotopy orbits fit into a cofiber sequence

$$BG \longrightarrow X_{uhG} \longrightarrow X_{bhG}.$$

(Every topological group G can be replaced by a Quillen cofibrant topological monoid $G' \xrightarrow{\sim} G$ by Theorem 6.5.20.)

8.7.5. Adapt the cyclic bar construction from Exercise 7.7.29 to any category $\mathbf{I}$ enriched in $\mathbf{V}$, and any $(\mathbf{I}, \mathbf{I})$-bimodule X in $\mathbf{C}$. As a special case, taking $\mathbf{V} = \mathbf{C}$ to be the category of spectra, for any ring spectrum R we get the **topological Hochschild homology**

$$\mathrm{THH}(R) = B^{\mathrm{cyc}}(R),$$

Prove the homotopy invariance theorem (Theorem 8.3.12), and conclude that THH(R) preserves equivalences whenever $\mathbb{S} \to R$ is a cofibration of spectra.

8.7.6. Generalize Exercise 7.7.36 from **Top** to any simplicial model category **C**. That is, when **I** is a simple directed tree, show that the formula for the simple homotopy colimit $S(X, \mathbf{I}, *)$ from Exercise 7.7.36 gives the homotopy colimit of any diagram $X : \mathbf{I} \to \mathbf{C}$. In particular, the mapping telescope is always a model for the sequential homotopy colimit.

8.7.7. Use Exercise 7.7.38 to prove that for any filtered homotopy colimit of spectra, the stable homotopy groups commute with the colimit, and therefore so do the extraordinary homology groups:

$$E_k\left(\operatorname*{hocolim}_{c\in\mathbf{I}} X_c\right) \cong \operatorname*{colim}_{c\in\mathbf{I}} E_k(X_c), \qquad \pi_k\left(\operatorname*{hocolim}_{c\in\mathbf{I}} X_c\right) \cong \operatorname*{colim}_{c\in\mathbf{I}} \pi_k(X_c).$$

8.7.8. Let X be a sequential spectrum with a left action by a finite group G. We defined the homotopy orbits and homotopy fixed points as

$$X_{hG} = B(*, G, X) = EG_+ \wedge_G X,$$

$$X^{hG} = F(*, G, X) = F^G(EG_+, X).$$

In this exercise we define the **Tate construction** or **Tate fixed points** X^{tG}.

(a) Explain why the category of spectra with commuting left and right G-actions, $\mathbf{Sp}^{(G,G)}$, is a model category, with weak equivalences and fibrations measured by forgetting the group actions. (You could interpret these as diagrams and use Exercise 5.8.37, or interpret them as modules and use Theorem 6.5.6.)

(b) Restricting to the case with just one G-action, suppose $X, Y \in \mathbf{Sp}^G$ and that the G-action on Y is trivial. Explain how an equivariant map $X \to Y$ induces a canonical map $X_{hG} \to Y$, and how an equivariant map $Y \to X$ induces a canonical map $Y \to X^{hG}$. Prove that this also works even if the map $Y \to X$ is only a map in the homotopy category $\mathrm{Ho}\,\mathbf{Sp}^G$.

(c) Choose a $G \times G^{\mathrm{op}}$-representation V that contains an embedded copy of G as a $G \times G^{\mathrm{op}}$-set. Define a $G \times G^{\mathrm{op}}$-equivariant map

$$c : S^V \to G_+ \wedge S^V$$

that is degree 1 on each sphere, by collapsing away to the basepoint the complement of a small ball about each point in the embedded copy of G.

(d) Define the **pretransfer** to be the map in the homotopy category $\mathbf{Sp}^{(G,G)}$ given by the zig-zag

$$\mathbb{S} \xrightarrow{\sim} \Omega^V\Sigma^\infty S^V \xrightarrow{c} \Omega^V\Sigma^\infty(G_+ \wedge S^V) \xleftarrow{\sim} G_+ \wedge \mathbb{S}.$$

Prove that this agrees in the homotopy category with the zig-zag

$$\mathbb{S} \longrightarrow F(G_+, \mathbb{S}) \xleftarrow{\sim} G_+ \wedge \mathbb{S}$$

where the first map comes from the collapse $G \to *$ and the second is the inclusion of the wedge into the product, which is $G \times G^{\mathrm{op}}$-equivariant. The pretransfer is therefore independent of the choice of V and embedding $G \to V$.

We then define the **transfer** or **additive norm map** to be the map in the homotopy category of spectra *with just one G-action*

$$T : X_{hG} \to X$$

$$\mathbb{S} \wedge_G (EG_+ \wedge X) \xrightarrow{\text{pretr}\wedge\text{id}} (G_+ \wedge \mathbb{S}) \wedge_G (EG_+ \wedge X) \xleftrightarrow{\cong} EG_+ \wedge X \xrightarrow{\sim} X.$$

By part (b), this induces a map $X_{hG} \to X^{hG}$. We define X^{tG} to be its cofiber:

$$X_{hG} \xrightarrow{T} X^{hG} \longrightarrow X^{tG}.$$

8.7.9. Suppose that G is a topological group, and that $X = \Sigma^\infty_+ G = \mathbb{S}[G]$ is the spherical group ring on G.

(a) Show that $X_{hG} \simeq \mathbb{S}$.

(b) If G is a finite (discrete) group, show that $X^{hG} \simeq \mathbb{S}$. (Hint: First show that X is equivalent to $\prod_G \mathbb{S} = F(G_+, \mathbb{S})$.)

(c) When G is finite, show that the equivalence $X_{hG} \simeq X^{hG}$ is given by the additive norm map of Exercise 8.7.8. In other words, $X^{tG} \simeq *$.

8.7.10. Let G be a discrete group. As in Definition A.1.1, we define a **free G-cell complex** to be a space with left G-action built from $\varnothing$ by successively attaching maps of the form $G \times S^{n-1} \to G \times D^n$, with G acting only on the first coordinate by multiplication on the left. A **free based G-cell complex** is built in the same way but starting from the one-point space $* = G/G$ instead of $\varnothing$.

(a) When X is a free G-cell complex, prove that the map from the unbased homotopy orbits to the strict orbits $X_{uhG} \to X_G$ is an equivalence.

(b) When X is a free based G-cell complex, prove that the map from the based homotopy orbits to the strict orbits $X_{bhG} \to X_G$ is an equivalence. What do the unbased homotopy orbits look like in this case?

(Hint: Try using model structures, not the bar construction, to do this exercise.)

8.7.11. As in Exercise 7.7.41, let $C_2 \cong \mathbb{Z}/2$ be the cyclic group of order 2. Let X be the C_2-CW complex built by attaching one free 2-cell $C_2 \times D^2$ to the point, making a wedge $S^2 \vee S^2$, and then attaching one free 3-cell whose attaching map on the first copy of D^3 is a map of $S^2 \to S^2 \vee S^2$ of degree $(1,-2)$.

(a) Prove that X is a Moore space $M(\mathbb{Z}/3,2)$. In other words, $\pi_1(X) = 0$, $H_2(X) \cong \mathbb{Z}/3$, and all other reduced homology groups are zero.

(b) Use Exercise 8.7.10 to show that the based homotopy C_2-orbits of X are equivalent to the strict orbits, and prove that these strict orbits are contractible.

(c) Conclude that the map of unbased homotopy orbits $*_{uhC_2} \to X_{uhC_2}$ is not an equivalence, but the map of based homotopy orbits $*_{bhC_2} \to X_{bhC_2}$ are an equivalence.

(d) Use this to show that a map of cofiber sequences $A \to X \to X/A$ that is an equivalence on A and on X/A does not necessarily induce an equivalence on X.

8.7.12. Suppose that $\mathbf{C}$ is a cofibrantly generated simplicial model category (using the transfinite version of the small-object argument from Exercise 5.8.42, and let $\mathbf{I}$ be any small simplicially enriched category. Prove that the category of simplicial diagrams $\mathbf{C}^{\mathbf{I}}$ has a projective model structure.

Here we are thinking of those diagrams whose action maps are of the form $\mathbf{I}_\bullet(a,b) \otimes X(a) \to X(b)$, where $\otimes\colon \mathbf{sSet} \times \mathbf{C} \to \mathbf{C}$ is the tensoring of $\mathbf{C}$ with simplicial sets.

(We already proved this in Exercise 5.8.37 when $\mathbf{I}$ is discrete, and in Exercise 7.7.17 when $\mathbf{C} = \mathbf{sSet}$. In general, the only tricky part is the small-object argument. Think about why you can get the desired factorization even if you have a transfinite composition of cofibrations, not just generating cofibrations.)

8.7.13. Pick one of the four canonical maps in (8.6.7) and define it two different ways:

- explicitly, by identifying each term $F^*X(a) = X(F(a))$ in the diagram $\mathbf{I} \to \mathbf{C}$ with the corresponding term $X(F(a))$ in the diagram $\mathbf{J} \to \mathbf{C}$, and
- abstractly, using the counit $F_! F^* \to \mathrm{id}$ of the adjunction between F^* and left Kan extension $F_!$, the unit $\mathrm{id} \to F_* F^*$ of the adjunction between F^* and right Kan extension F_*, and the fact that natural transformations of functors induce canonical natural transformations of left- or right-derived functors.

8.7.14. Show that for the two parts of Theorem 8.6.11 that involve colimits and homotopy colimits, the converse holds. So if the canonical map of homotopy colimits is an equivalence for every diagram $X\colon \mathbf{J} \to \mathbf{C}$, then F must be homotopy terminal.

8.7.15. For any collection X_α of spaces or spectra, define the **weak product** to be the homotopy colimit of all of the finite products:

$$\bigoplus_{\alpha\in A} X_\alpha := \operatorname*{hocolim}_{S\subseteq A,\ |S|<\infty} \prod_{\alpha\in S} X_\alpha.$$

(a) Prove that there is a map from the wedge sum of spectra to the weak product of spectra, and this is always an equivalence. (You might want to use the fact that the collection of finite subsets of A defines a filtered colimit.)

(b) Prove that there is a map from the weak product of spaces to the product of spaces, and this is an equivalence if for each $n \geq 0$ there are only finitely many spaces X_α such that $\pi_n(X_\alpha) \neq 0$.

8.7.16. Prove that in any symmetric monoidal simplicial model category, the tensor commutes with homotopy colimits and the cotensor commutes with homotopy limits:

$$\operatorname*{hocolim}_{i\in\mathbf{I}} K \otimes X(i) \simeq K \otimes \left(\operatorname*{hocolim}_{i\in\mathbf{I}} X(i)\right),$$

$$\operatorname*{holim}_{i\in\mathbf{I}} X(i)^K \simeq \left(\operatorname*{holim}_{i\in\mathbf{I}} X(i)\right)^K.$$

Is this also true for strict colimits and limits?

8.7.17. Prove that in spectra, *all* homotopy colimits commute with finite cell homotopy limits. (By Lemma 8.6.6, it suffices to show that homotopy colimits commute with pullbacks. Does stability help?)

Prove using the same method that the smash product functor $Y \wedge (-)$ commutes with all homotopy colimits and all finite cell homotopy limits.

8.7.18. In this exercise we give useful cofiber sequences for simplicial spectra and homotopy colimits of spectra. (These same results also apply to spaces though!)

(a) For any simplicial spectrum $X_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Sp}$, prove that there are cofiber sequences

$$\mathrm{Sk}^{n-1}|X_\bullet| \longrightarrow \mathrm{Sk}^n|X_\bullet| \longrightarrow \Sigma^n(X_n/L_nX).$$

Actually, $X_\bullet$ has to be Reedy cofibrant for this to work. Explain why. (By Exercise 8.7.37, every simplicial object can be replaced by an equivalent one that is Reedy cofibrant.)

(b) If $\mathbf{I}$ is a discrete category, for any diagram of spectra $X \colon \mathbf{I} \to \mathbf{Sp}$, prove that there is a filtration of the homotopy colimit $B(X,\mathbf{I},*)$ and cofiber sequences of the form

$$\mathrm{Sk}^{n-1}B(X,\mathbf{I},*) \longrightarrow \mathrm{Sk}^nB(X,\mathbf{I},*) \longrightarrow \bigvee_{c_0\to\cdots\to c_n} \Sigma^n X(c_0),$$

where the maps $c_0 \to \cdots \to c_n$ range over all composites of non-identity morphisms. (The spectra $X(c)$ don't have to be made cofibrant for this to be true – why?)

(c) Dualize the previous two points to cosimplicial objects and homotopy limits.

8.7.19. In this exercise we use Proposition 8.2.31 to get connectivity estimates on the bar construction.

(a) If X and Y are both $(a-1)$-connected spaces, and we form a bar construction $B(X,G,Y)$ using a well-based topological monoid G and the Cartesian product, prove that $B(X,G,Y)$ is also $(a-1)$-connected.

(b) Now suppose we use the smash product instead. Prove that if X is $(a-1)$-connected and Y is $(b-1)$-connected then $B^{\wedge}(X,G,Y)$ is $(a+b-1)$-connected.

(c) Generalize the above two parts to the categorical bar constructions $B(X,\mathbf{I},Y)$ and $B^{\wedge}(X,\mathbf{I},Y)$. (You'll want to make a uniform connectivity bound across all of the spaces $X(c)$.)

8.7.20. Suppose $\mathbf{I}$ is a discrete category, $X : \mathbf{I} \to \mathbf{Sp}$ is a diagram of spectra, and each of the spectra $X(i)$ is $(n-1)$-connected, so $\pi_k(X(i)) = 0$ for all $k < n$.

(a) Prove that the homotopy colimit is $(n-1)$-connected, and that the *lowest* homotopy group π_n commutes with the colimit:

$$\pi_n\left(\operatorname*{hocolim}_{i\in\mathbf{I}} X(i)\right) \cong \operatorname*{colim}_{i\in\mathbf{I}} \pi_n(X(i)).$$

(Hint: Use Exercise 8.7.18 to show you can represent every element by the 1-skeleton, which has an explicit description in Example 8.3.23.)

(b) What happens for π_{n+1}? Can you give a counterexample to the above formula?

8.7.21. Again suppose that $\mathbf{I}$ is a discrete category, $X : \mathbf{I} \to \mathbf{Sp}$ is a diagram of spectra, and each of the spectra $X(i)$ is $(n-1)$-connected.

(a) Prove that if the diagram is a homotopy pullback, then the homotopy limit is $(n-2)$-connected. What happens for larger diagrams with finitely many morphisms?

(b) For sequential homotopy limits, prove that we have a short exact sequence for *all* homotopy groups

$$0 \longrightarrow \lim_{i\to\infty}{}^1\, \pi_{k+1}(X(i)) \longrightarrow \pi_k\left(\operatorname*{holim}_{i\to\infty} X(i)\right) \longrightarrow \lim_{i\to\infty} \pi_k(X(i)) \longrightarrow 0.$$

and that therefore the homotopy limit is only $(n-2)$-connected, and $(n-1)$-connected iff the $\lim^1$ of the system on π_n vanishes. (Hint: You already know this result for spaces from (1.5.23).)

8.7.22. Let $\mathbf{I}$ be a discrete catgory. Deduce from Exercise 8.7.20 that:

(a) Any map $X \to Y$ of diagrams of spectra on $\mathbf{I}$ such that each map $X(i) \to Y(i)$ is $(n-1)$-connected, induces an $(n-1)$-connected map on homotopy colimits.

(b) For any extraordinary homology theory E in which the coefficients vanish in negative degrees, $E_k(*, \varnothing) = 0$ for $k < 0$, if each spectrum $X(i)$ is $(n-1)$-connected then we get an isomorphism on E-homology

$$E_n\left(\operatorname*{hocolim}_{i\in\mathbf{I}} X(i)\right) \cong \operatorname*{colim}_{i\in\mathbf{I}} E_n(X(i)).$$

In particular, we get an isomorphism on ordinary homology

$$H_n\left(\operatorname*{hocolim}_{i\in\mathbf{I}} X(i); G\right) \cong \operatorname*{colim}_{i\in\mathbf{I}} H_n(X(i); G).$$

(c) Formulate and prove the corresponding claims for finite cell homotopy limits and sequential homotopy limits.

8.7.23. Let $X : \{0 \to 1\}^{\times n} \to \mathbf{Top}_*$ or $\mathbf{Sp}$ be any cube-shaped diagram.[11] Let us label the objects in the cube category $\{0 \to 1\}^{\times n}$ as subsets $S \subseteq \{1, \dots, n\}$, which tells us which coordinates are "1." Let $P(n) \subseteq \{0 \to 1\}^{\times n}$ be the punctured cube, the full subcategory on the proper subsets $S \subsetneq \{1, \dots, n\}$.

Prove that the following spaces are all equivalent. We call any one of them the **total homotopy cofiber** of the cube:

(a) The iterated homotopy cofiber as defined in Example 8.6.18.

(b) The strict cofiber of the map $\operatorname*{hocolim}_{S\in P(n)} X(S) \to \operatorname*{hocolim}_{S\in\{0\to 1\}^{\times n}} X(S)$, in other words the map of Bousfield–Kan homotopy colimits from the punctured cube to the whole cube.

(c) The homotopy cofiber of the map $\operatorname*{hocolim}_{S\in P(n)} X(S) \to X(\{1, \dots, n\})$.

(d) The homotopy colimit of the larger diagram $\widetilde{X}$ on $P(n+1)$ which is X along $P(n)$, and which is the one-point space or zero spectrum $*$ along all of the remaining objects.

8.7.24. Prove that a homotopy colimit along a cube-shaped diagram can be equivalently be described by taking a space $X(S) \times I^{S^c}$ for each set $S \subseteq \{0 \to 1\}^{\times n}$, and gluing them together along the faces of the cubes $I^{S^c} = \prod_{j\notin S} I$ in an appropriate way, generalizing the picture from Example 7.4.15.

[11] If you want to define the total homotopy cofiber for a diagram of unbased spaces, it's best to add disjoint basepoints to them first.

8.7.25. Dualize the above exercises to the homotopy limit and total homotopy fiber. In the category of spectra, explain why for a commuting square, the total homotopy cofiber is the double suspension of the total homotopy fiber, and so one is contractible iff the other is. This gives another proof of Corollary 2.4.17.

8.7.26. Prove that the homotopy colimit construction $B(QX, \mathbf{I}, *)$ preserves E-local equivalences of spectra, if E is any extraordinary homology theory. Therefore the canonical map

$$\operatorname*{hocolim}_{i\in\mathbf{I}} X(i) \longrightarrow \operatorname*{hocolim}_{i\in\mathbf{I}} X(i)_E$$

is an E-local equivalence, so the map of E-localizations

$$\left(\operatorname*{hocolim}_{i\in\mathbf{I}} X(i)\right)_E \longrightarrow \left(\operatorname*{hocolim}_{i\in\mathbf{I}} X(i)_E\right)_E$$

is a stable equivalence. In other words "you can localize before or after taking the hocolim and you get the right answer."

8.7.27. In this exercise we show that finitary homotopy functors on $\mathbf{Top}_*$ are equivalent to plain old homotopy functors on the category of based finite CW complexes $\mathbf{FinCW}_*$, resolving a claim we made in Section 6.4.3. See Definition 6.4.12 for the definitions of "finitary" and "homotopy functor."

(a) If $F\colon \mathbf{FinCW}_* \to \mathbf{Top}_*$ is any homotopy functor, define a functor $G\colon \mathbf{Top}_* \to \mathbf{Top}_*$, by sending X to its CW replacement $QX \to X$, and then taking the homotopy colimit of F over all the finite subcomplexes of QX. Prove that G is a finitary homotopy functor whose restriction to $\mathbf{FinCW}_*$ is naturally equivalent to F.

(b) Conversely, if we start with a can be extended to a finitary homotopy functor $H\colon \mathbf{Top}_* \to \mathbf{Top}_*$, prove that if we restrict to $\mathbf{FinCW}_*$ and then perform the above procedure, the resulting functor is naturally equivalent to H.

8.7.28. Prove Lemma 6.4.15 using the following idea. Replace each functor $F\colon \mathbf{FinCW}_* \to \mathbf{Top}_*$ by the functor $F'(K) = |S_\bullet F(K)|$, and then replace F' by the quotient functor $G(K) = F'(K)/F'(*)$. Prove that this is naturally equivalent to F, and that if F is strictly reduced then both F' and G are strictly reduced as well. Explain why this gives an equivalence of homotopy categories between reduced functors and strictly reduced functors.

8.7.29. Suppose $F\colon \mathbf{Top} \to \mathbf{Top}$ is a functor. Define a new functor F' by the formula

$$F'(X) = \left|[n] \mapsto F(\mathrm{Map}(\Delta^n, X))\right|.$$

In other words, we apply F to the spaces of continuous maps from Δ^n to X, as n varies this defines a simplicial space, and we take its realization.

(a) (*) Prove that F' is a simplicial functor by the following rule: we send

$$S_n\mathrm{Map}(X,Y)\times\mathrm{Map}(\Delta^n,X)\longrightarrow\mathrm{Map}(\Delta^n,Y)$$

by sending each pair $(f:\Delta^n\times X\to Y, g:\Delta^n\to X)$ to the composite

$$\Delta^n\xrightarrow{(\mathrm{id},\mathrm{id})}\Delta^n\times\Delta^n\xrightarrow{\mathrm{id}\times g}\Delta^n\times X\xrightarrow{f}Y.$$

Then for each value of f, as g varies this defines a continuous map $\mathrm{Map}(\Delta^n,X)\to\mathrm{Map}(\Delta^n,Y)$, so we may apply F to it, giving a map

$$S_n\mathrm{Map}(X,Y)\times F(\mathrm{Map}(\Delta^n,X))\longrightarrow F(\mathrm{Map}(\Delta^n,Y)).$$

Now taking realizations and using Milnor's theorem we get

$$|S_\bullet\mathrm{Map}(X,Y)|\times|F(\mathrm{Map}(\Delta^\bullet,X))|\longrightarrow|F(\mathrm{Map}(\Delta^\bullet,Y))|.$$

Prove that this makes F' a simplicial functor (Definition 8.1.34), whose underlying functor sends each 0-simplex of maps $f:X\to Y$ to the map

$$|F(\mathrm{Map}(\Delta^\bullet,X))|\longrightarrow|F(\mathrm{Map}(\Delta^\bullet,Y))|$$

that just applies f at every level.

(b) Prove that there is a natural transformation $F\to F'$, and it is an equivalence if F is a homotopy functor and the simplicial object defining F' is Reedy cofibrant.

(c) Use singular simplices and geometric realization to prove that every homotopy functor F admits a zig-zag of weak equivalences to a functor F' that is simplicially enriched.

8.7.30. Generalize Exercise 7.7.21 from **Top** to any category **C**: the coend $X\otimes_{\mathbf{I}}\mathbf{I}(-,c)$ and the end $\mathrm{Map}^{\mathbf{I}}(\mathbf{I}(c,-),X)$ are both isomorphic to $X(c)$.

Then use this to prove that if $F:\mathbf{C}\to\mathbf{D}$ is an embedding of categories, i.e. $\mathbf{C}(a,b)\to\mathbf{D}(F(a),F(b))$ is a bijection, then the left Kan extension of X from **C** to **D**, agrees with the original diagram X along **C**. So it is in fact an extension. (The same is true for topological categories if $\mathbf{C}(a,b)\to\mathbf{D}(F(a),F(b))$ is a homeomorphism.)

Then prove the same thing for right Kan extensions.

8.7.31. (*) Use the action of each orthogonal group $O(n)$ on the sphere S^n to define an action by the infinite orthogonal group O on the fibrant sphere spectrum $f\mathbb{S}$, as defined in Example 2.2.6. Prove that the resulting homotopy orbit spectrum $\mathbb{S}_{hO}$ can be written as the colimit of homotopy orbit spectra $(F_nS^0)_{hO(n)}$, and use this to form a stable equivalence to the Thom spectrum MO from Example 2.1.20.

8.7.32. In Proposition 8.6.24 we proved that in any cofibrantly generated simplicial model category $\mathbf{C}$, for any simplicial object $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}$, the homotopy colimit of X is equivalent to the realization $|X|$. What is the homotopy *limit* of X equivalent to? (Theorem 8.6.11 is helpful here.)

8.7.33. In this exercise we describe a direct map between the realization $|X_\bullet|$ and the homotopy colimit of $X_\bullet$ over $\mathbf{\Delta}^{\mathrm{op}}$ that is a weak equivalence, not just a zig-zag as in the proof of Proposition 8.6.24.

(a) Define a map of simplicial sets

$$N_\bullet(*, \mathbf{\Delta}, \mathbf{\Delta}(-,[k])) \to \Delta[k]_\bullet$$

by taking each string of $(n+1)$ maps $[k_0] \to [k_1] \to \cdots \to [k_n] \to [k]$ to the map $[n] \to [k]$ that records the images of each "last vertex" $k_i \in [k_i]$ in $[k]$. Prove that this commutes with faces and degeneracies and is therefore a map of simplicial sets.

(b) Conclude using the proof of Proposition 8.6.24 that applying $(-) \otimes_{\mathbf{\Delta}} X$ to this map gives a weak equivalence $\operatorname*{hocolim}_{[i]\in\mathbf{\Delta}^{\mathrm{op}}} X_i \xrightarrow{\sim} |X_\bullet|$.

In the case that $\mathbf{C}$ is topologically enriched, we can also think of this as giving a natural equivalence of $\mathbf{\Delta}$-diagrams $B(*, \mathbf{\Delta}, \mathbf{\Delta}) \to \Delta^{(-)}$.

8.7.34. [EKMM97, VII.3.2]

(a) If $\mathbf{C}$ is a topological model category, and $K_\bullet$ is a simplicial set, prove that the topological tensor $|K_\bullet| \otimes X$ is always isomorphic to the realization of the simplicial object of $\mathbf{C}$ whose nth level is

$$K_n \otimes X := \coprod_{k\in K_n} X.$$

(Hint: Write down the formula for the realization, and remember that $\otimes$ preserves colimits!)

(b) What is the corresponding statement for any simplicial model category?

8.7.35. Let $\mathbf{C}$ be any simplicial model category. Prove that for every $K_\bullet \in \mathbf{sSet}$ and $X, Y \in \mathbf{C}$, there are natural isomorphims of simplicial sets

$$\mathbf{C}_\bullet(K_\bullet \otimes X, Y) \cong \mathrm{Map}_\bullet(K_\bullet, \mathbf{C}_\bullet(X,Y)) \cong \mathbf{C}_\bullet(X, Y^{K_\bullet}).$$

(Hint: Consider a map from the standard n-simplex $\Delta[n]$ into any one of these simplicial sets, and then rearrange to form a map $(\Delta[n] \times K) \otimes X \to Y$.)

8.7.36. (*) In this exercise we explain a different way to think about simplicial functors, and more generally functors that are enriched in some other category $\mathbf{V}$, that is helpful for defining the maps that commute F with homotopy colimits in Proposition 8.6.3.

Let $\mathbf{V}$ be any symmetric monoidal category, and let $\mathbf{C}$ be a category with a tensor product functor $\otimes\colon \mathbf{V}\times\mathbf{C}\to\mathbf{C}$, satisying the same coherences as for a simplicial model category. Assume that the tensor has a right adjoint in the second slot $\mathrm{Map}\colon \mathbf{C}^{\mathrm{op}}\times\mathbf{C}\to\mathbf{V}$, so that we have natural bijections

$$f : A\otimes X\to Y \qquad \leftrightarrow \qquad \widetilde{f}: A\to \mathrm{Map}(X,Y).$$

Therefore $\mathbf{C}$ is also enriched in $\mathbf{V}$, with mapping objects given by $\mathrm{Map}(X,Y)$ and composition defined in Proposition 4.1.38. We have already seen that this composition is associative and unital, using the coherences for the tensor product.

(a) Now suppose $\mathbf{D}$ also has a tensor product $\otimes\colon \mathbf{V}\times\mathbf{D}\to\mathbf{D}$ with all of the same properties, and that $F\colon \mathbf{C}\to\mathbf{D}$ is a functor. Show how to go back and forth between an action of F on the mapping spaces, and an assembly map for F:

$$F_{X,Y}\colon \mathrm{Map}(X,Y)\to\mathrm{Map}(F(X),F(Y)) \quad \forall\, X,Y\in\mathbf{C}$$

$$\text{vs.} \qquad \alpha_{A,X}\colon A\otimes F(X)\to F(A\otimes X) \quad \forall\, A\in\mathbf{V}, X\in\mathbf{C}.$$

(b) Show that $F_{X,X}$ respects the identity element iff $\alpha_{I,X}$ is canonically identified with the identity map of $F(X)$.

(c) Show that $F_{X,Y}\otimes F_{Y,Z}$ agrees with $F_{X,Z}$ along the composition maps iff $\mathrm{id}_A\otimes\alpha_{B,X}$ and $\alpha_{A,B\otimes X}$ compose to give $\alpha_{A\otimes B,X}$.

(d) If the tensor products also have right adjoints in the other slot, prove that the assembly data for F with the above properties is equivalent to coassembly data

$$c_{A,X}\colon F(X^A)\to F(X)^A \quad \forall\, A\in\mathbf{V}, X\in\mathbf{C}$$

such that $c_{I,X}$ is the identity of $F(X)$ and c_{A,X^B} and $c_{B,X}$ compose to give $c_{A\otimes B,X}$.

(e) If F is a left adjoint, prove that it is a $\mathbf{V}$-enriched left adjoint iff the maps $\alpha_{A,X}$ are isomorphisms. Similarly if F is a right adjoint, it is a $\mathbf{V}$-enriched right adjoint iff the maps $c_{A,X}$ are isomorphisms.

In summary, being enriched is equivalent to having assembly or coassembly maps that satisfy axioms analogous to the axioms for a module over a ring. This is what allows us to move bar and cobar contructions in and out of the functor F in Proposition 8.6.3.

8.7.37. (*) [GJ99, Ch VII],[Hir03, Ch. 15] In this exercise we show that for any model category $\mathbf{C}$, the category of simplicial objects $\mathbf{sC} = \mathbf{C}^{\Delta^{op}}$ has a **Reedy model structure**, in which the weak equivalences are the level equivalences. A map of simplicial objects $f : X_\bullet \to Y_\bullet$ is a **level equivalence** if each map $X_n \to Y_n$ is a weak equivalence in the model category $\mathbf{C}$.

(a) For each map of simplicial objects $f : X_\bullet \to Y_\bullet$, define a relative latching map

$$L_n f : X_n \cup_{L_n X} L_n Y \longrightarrow Y_n.$$

We say that f is a **Reedy cofibration** if this relative latching map is a cofibration in $\mathbf{C}$. If f is both a Reedy cofibration and level equivalence, show that the maps

$$X_n \longrightarrow X_n \cup_{L_n X} L_n Y \longrightarrow Y_n$$

are both weak equivalences. (You might find the filtration from Lemma 8.2.33 helpful.)

(b) Define a matching object $M_n X$ for any simplicial object $X_\bullet$ as the limit of X_i over all injective totally ordered maps $[i] \to [n]$ for $i < n$. Show that this is also a punctured cube-shaped limit. (This is actually easier than Proposition 8.2.18 because you are just enumerating all the faces of the n-simplex.)

(c) Show that there is a natural matching map $X_n \to M_n X$, and more generally for each map of simplicial objects $f : X_\bullet \to Y_\bullet$ a relative matching map

$$M_n f : X_n \longrightarrow M_n X \times_{M_n Y} Y_n.$$

We say that f is a **Reedy fibration** if this relative matching map is a fibration in $\mathbf{C}$. If f is both a Reedy fibration and level equivalence, show that the maps

$$X_n \longrightarrow M_n X \times_{M_n Y} Y_n \longrightarrow Y_n$$

are both weak equivalences. (Consider dualizing the filtration from Lemma 8.2.33.)

(d) Prove that the level equivalences satisfy 2-out-of-3, and that the Reedy cofibrations, level equivalences, and Reedy fibrations are all preserved by retracts.

(e) Show that in every square of simplicial objects

$$\begin{array}{ccc} A_\bullet & \longrightarrow & X_\bullet \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B_\bullet & \longrightarrow & Y_\bullet, \end{array}$$

a lift as shown can be constructed one simplicial level at a time, by finding a lift in the square

$$\begin{array}{ccc} A_n \cup_{L_nA} L_nB & \longrightarrow & X_n \\ {\scriptstyle L_ni}\downarrow & \nearrow & \downarrow{\scriptstyle M_np} \\ B_n & \longrightarrow & M_nX \times_{M_nY} Y_n. \end{array}$$

Use this and the previous parts to show that if i is a Reedy cofibration, p is a Reedy fibration, and one of them is a level equivalence, then the lift exists.

(f) Construct a factorization of any map $f : A_\bullet \to Y_\bullet$ into a Reedy cofibration $A_\bullet \to X_\bullet$ and a Reedy fibration $X_\bullet \to Y_\bullet$, by inductively factoring a map of the form

$$A_n \cup_{L_nA} L_nX \longrightarrow M_nX \times_{M_nY} Y_n$$

and defining the new space thus created to be X_n. Show that we can arrange for either $A_\bullet \to X_\bullet$ or $X_\bullet \to Y_\bullet$ to be a level equivalence as well, using the previous parts of this exercise.

8.7.38. Using the Reedy model structure of the previous exercise, show that if $\mathbf{C}$ is a simplicial model category, the realization functor $\mathbf{sC} \to \mathbf{C}$ is left Quillen. Therefore it preserves cofibrant objects and equivalences between them, giving an alternate proof of Theorem 8.2.25.

8.7.39. Let $\mathbf{C}$ be any cofibrantly generated model category. Then the category of simplicial objects $\mathbf{sC} = \mathbf{C}^{\Delta^{\mathrm{op}}}$ has a projective model structure in which the weak equivalences and fibrations are measured at each simplicial level, by Exercise 5.8.37. Prove that the identity functor $\mathbf{sC} \to \mathbf{sC}$ gives a Quillen equivalence between the projective model structure and the Reedy model structure. (Which direction is the Quillen left adjoint?)

8.7.40. Suppose $\mathbf{C}$ is a category, $\mathbf{V}$ is a symmetric monoidal category, and we have a tensor $\otimes : \mathbf{V} \times \mathbf{C} \to \mathbf{C}$ with the same coherence properties as Definition 8.1.1, as in Exercise 8.7.3. In other words, $\mathbf{C}$ is enriched, tensored, and cotensored over $\mathbf{V}$. So we also have a mapping space $\mathrm{Map}(X, Y) \in \mathbf{V}$ and cotensor $Y^K \in \mathbf{C}$ when $K \in \mathbf{V}$ and $X, Y \in \mathbf{C}$.

Suppose $\mathbf{I}$ is a category enriched in $\mathbf{V}$, and let us suppose that there exists a constant diagram $\mathrm{const}(I) : \mathbf{I}^{\mathrm{op}} \to \mathbf{V}$, whose maps are of the form

$$\mathbf{I}(a, b) \otimes \mathrm{const}(I)(b) = \mathbf{I}(a, b) \otimes I \longrightarrow I = \mathrm{const}(I)(a).$$

(This does not exist in general, for instance when $\mathbf{I}$ is enriched in based spaces.)

(a) For each $\mathbf{V}$-diagram $X : \mathbf{I} \to \mathbf{C}$, prove that the coend $X \otimes_{\mathbf{I}} \mathrm{const}(I)$ is an **enriched colimit** in the sense that for every object $Y \in \mathbf{C}$ there is an isomorphism of objects of $\mathbf{V}$

$$\mathrm{Map}(X \otimes_{\mathbf{I}} \mathrm{const}(I), Y) \cong \mathrm{Map}^{\mathbf{I}}(X, \mathrm{const}(Y)).$$

In other words, the "space" of maps out of the colimit is isomorphic to the "space" of maps from X to the constant diagram on Y.

(b) Similarly, for each $\mathbf{V}$-diagram $Y : \mathbf{I} \to \mathbf{C}$, define a covariant version of $\mathrm{const}(I)$ and then prove that the end $\mathrm{Map}^{\mathbf{I}}(\mathrm{const}(I), Y)$ is an **enriched limit** in the sense that for every object $X \in \mathbf{C}$ there is an isomorphism of objects of $\mathbf{V}$

$$\mathrm{Map}(X, \mathrm{Map}^{\mathbf{I}}(\mathrm{const}(I), Y)) \cong \mathrm{Map}^{\mathbf{I}}(\mathrm{const}(X), Y).$$

In other words, the "space" of maps to the limit is isomorphic to the "space" of maps from the constant diagram on X to Y.

Since the constant diagram does not in general exist, we often ask instead for a **weighted colimit**; this is defined as $X \otimes_{\mathbf{I}} Y$, where $Y : \mathbf{I}^{\mathrm{op}} \to \mathbf{V}$ is a diagram of "weights." The same argument shows a more general universal property for these weighted colimits:

$$\mathrm{Map}(X \otimes_{\mathbf{I}} Y, Z) \cong \mathrm{Map}^{\mathbf{I}}(X, Y).$$

See [Kel05] for an extensive discussion.

8.7.41. Explain how to prove the right-induced model structure for $\mathbf{V}$-categories (Theorem A.2.20), by following the same strategy as the right-induced model structure for $\mathrm{Alg}(\mathbf{V})$ from Section 6.5.3.

8.7.42. Recall from Definition 6.1.27 that an h-cofibration of diagram spectra is a map $X \to Y$ that has the homotopy extension property in the category of spectra, whereas a levelwise unbased h-cofibration of spectra is one that has the unbased version of the homotopy extension property at each spectrum level separately.

(a) Verify that the hypotheses of Theorem 8.2.27 hold for the levelwise unbased h-cofibrations. (Most of this is already done in Exercise 6.6.15.)

(b) Verify that the hypotheses of Theorem 8.2.27 hold for the h-cofibrations of diagram spectra. For the pushout-product part, you may assume the following result of Schänzl and Vogt [SV02, 4.2 and Cor 2.9]:

Proposition 8.7.1. *If $i : A \to X$ is an unbased Hurewicz cofibration of topological spaces, and $g : B \to Y$ is an h-cofibration of spectra, then the pushout-product*

$$i \square g : (A_+ \wedge Y) \cup_{(A_+ \wedge B)} (B_+ \wedge X) \longrightarrow B_+ \wedge Y$$

is again an h-cofibration of spectra.

8.7.43. In Corollary 8.3.20 we showed that $B(M,R,N)$ is the left-derived functor of $M \wedge_R N$, when M and N have cofibrant underlying spectra, and $\mathbb{S} \to R$ is a cofibration of spectra. Explain why the proof of Corollary 6.5.11 shows that the left-derived functor is also $(QM) \wedge_R N$, if Q is a cofibrant replacement of M *as an R-module*, and R and N are arbitrary.

8.7.44. (*) [EKMM97, VII.2.9] Suppose $\mathbf{C}$ is a closed symmetric monoidal category with tensor $\otimes$, and that there is a strong symmetric monoidal functor $F\colon \mathbf{Top} \to \mathbf{C}$, which allows us to tensor any topological space K with an object $X \in \mathbf{C}$, giving $F(K)\otimes X \in \mathbf{C}$. You might want to imagine that $\mathbf{C}$ is the category of spectra. Let us refer to the adjoints of the tensor as

$$\mathrm{Map}(-,-)\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Top}, \qquad (-)^{(-)}\colon \mathbf{Top}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C}.$$

(a) Let R be an algebra in $\mathbf{C}$. Prove that the category of modules also has a tensor

$$\otimes\colon \mathbf{Top} \times \mathrm{Mod}(R) \to \mathrm{Mod}(R),$$

that has a right adjoint in each variable. Specifically, we want one of the right adjoints to be the functor

$$\mathrm{Mod}(R)^{\mathrm{op}} \times \mathrm{Mod}(R) \to \mathbf{Top}$$

$$(M,N) \mapsto \mathrm{Map}^R(M,N)$$

where $\mathrm{Map}^R(M,N)$ is the subspace of $\mathrm{Map}(M,N)$ consisting of those maps that commute with the R-action. (Hint: The tensor should be just $F(K)\otimes -$ again.)

(b) Similarly, construct a tensor

$$\otimes_{\mathrm{Alg}}\colon \mathbf{Top} \times \mathrm{Alg}(C) \to \mathrm{Alg}(C)$$

that has a right adjoint in each slot. We want one of the right adjoints to give

$$\mathrm{Map}_{\mathrm{Alg}}(R,S) \subseteq \mathrm{Map}(R,S),$$

the subspace of all maps $R \to S$ that are maps of algebras.

(Hint: Generalizing the colimit of algebras in Lemma 6.5.13, define the tensor $K \otimes_{\mathrm{Alg}} R$ to be the coproduct over n of all expressions of the form $(F(K)\otimes R)^{\otimes n}$, modulo the relations that identify every sub-expression of the form $(F(K)\otimes R)^{\otimes 2}$ with $(F(K)\otimes R)^{\otimes 1}$ along the maps shown, and that similarly insert a copy of $(F(K)\otimes R)^{\otimes 0}$ anywhere and then identity to $(F(K)\otimes R)^{\otimes 1}$ as shown.)

$$\begin{array}{ccc} F(K\times K)\otimes R\otimes R & \xrightarrow{\cong} & (F(K)\otimes R)^{\otimes 2} \\ {\scriptstyle F(\Delta_K)\otimes\mathrm{id}\otimes\mathrm{id}}\uparrow & & \\ F(K)\otimes R\otimes R & \xrightarrow{\mathrm{id}\otimes\mu_R} & (F(K)\otimes R)^{\otimes 1} \end{array} \qquad \begin{array}{ccc} *\otimes I & \xrightarrow{\cong} & (F(K)\otimes R)^{\otimes 0} \\ {\scriptstyle F(K\to *)\otimes\mathrm{id}}\uparrow & & \\ F(K)\otimes I & \xrightarrow{\mathrm{id}\otimes\eta_R} & (F(K)\otimes R)^{\otimes 1} \end{array}$$

(c) Do the same for CAlg(**C**). (You only have to apply one more relation to define the tensor $K \otimes_{\mathrm{CAlg}} R$, permuting the copies of $(F(K) \otimes R)$ around.) Explain why we can think of $K \otimes_{\mathrm{CAlg}} R$ as a "configuration space" of points in K labeled by elements of R, where the labels multiply when the points collide together.

(d) Explain how these results extend if we replace **Top** by **sSet**, or by any Cartesian monoidal category **V**.

8.7.45. [EKMM97, VII.2.9] Suppose **C** is a topological symmetric monoidal model category. So it satisfies the assumptions of Exercise 8.7.44, and in addition **C** has a model structure such that the tensor $F(-) \otimes (-) \colon \mathbf{Top} \times \mathbf{C} \to \mathbf{C}$ is a Quillen tensor.

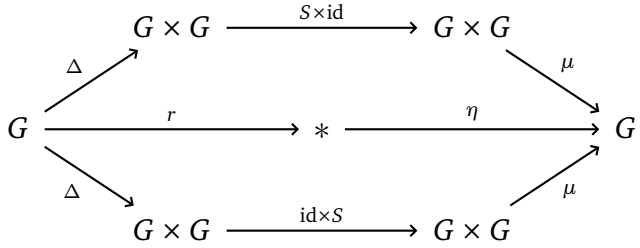
(a) Let R be an algebra in **C**. If the right-induced model structure exists on the category of modules Mod(R) (see Theorem 6.5.2), prove that it is also a topological model category.

(b) Similarly, if the right-induced model structure exists on the category of algebras Alg(**C**), prove it is also a topological model category.

(c) Do the same for CAlg(**C**).

(d) Explain how these results extend to any simplicial model category, or more generally any **V**-model category, so long as the tensor product in **V** is the Cartesian product $\times$.

8.7.46. Suppose **C** is a symmetric monoidal category, but the tensor product is the Cartesian product $\times$. Then as in Exercise 4.4.8, a **group object** in **C** is a monoid (or algebra) G with an inversion map $S \colon G \to G$ such that the following diagram commutes.



A morphism of groups is a morphism of monoids between them, so groups form a subcategory Grp(**C**) $\subseteq$ Alg(**C**). When the multiplication is commutative we denote this subcategory by CGrp(**C**) $\subseteq$ CAlg(**C**).

(a) Prove that the map S is unique, if it exists. So being a group is a *property*, not extra data. (Hint: How did you prove that inverses are unique in your first course on group theory?)

(b) Prove that the subcategory of group objects Grp(**C**) $\subseteq$ Alg(**C**) is closed under all colimits and limits in Alg(**C**), as constructed in Lemma 6.5.13. So the colimit in groups is the same as the colimit in monoids.

(c) If $\mathbf{C}$ is a topological model category and G is a group in $\mathbf{C}$, prove that the tensor $K \otimes_{\mathrm{Alg}} G$ defined in Exercise 8.7.44 is also a group. So $\mathrm{Grp}(\mathbf{C})$ is also enriched, tensored, (and cotensored) over spaces.

(This is less complicated than it appears. You have a "word" whose letters are pairs $(k_i, g_i) \in F(K) \times G$, and two letters can be multiplied when they have the same value of k_i. What is the inverse of this word?)

(d) Do the same for abelian groups $\mathrm{CGrp}(\mathbf{C})$, showing that they are closed under colimits, limits, and tensors and cotensors with topological spaces.

This exercise supplies most of the ingredients for showing that $\mathrm{Grp}(\mathbf{C})$ and $\mathrm{CGrp}(\mathbf{C})$ are also (topological) model categories. The only remaining parts are the last two bullet points of Theorem 6.5.2, which have to be checked in individual examples. See e.g. Proposition 9.1.3 and Proposition 9.1.7.

8.7.47. (*) Let $\mathbf{C}$ be a symmetric monoidal simplicial category. We have seen in Exercise 8.7.44 that the category of algebras $\mathrm{Alg}(\mathbf{C})$ is also simplicially enriched and tensored. So, if we have a simplicial object in algebras, $X_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Alg}(\mathbf{C})$, then we can define its realization $|X_\bullet|$ as in Definition 8.2.1. However, there are two ways to implement this:

- Use the tensor and colimits in $\mathrm{Alg}(\mathbf{C})$ from Exercise 8.7.44. Call this $|X_\bullet|_{\mathrm{Alg}(\mathbf{C})}$.
- Use the tensor and colimits in $\mathbf{C}$, and give the result an algebra structure using Proposition 8.2.9. Call this $|X_\bullet|_{\mathbf{C}}$.

In other words, we could define the realization by taking the tensors $\Delta^n \otimes (-)$ and the coequalizer in $\mathrm{Alg}(\mathbf{C})$, or in the underlying category $\mathbf{C}$.

Prove that these give canonically isomorphic results, so that $|X_\bullet|_{\mathrm{Alg}(\mathbf{C})} \cong |X_\bullet|_{\mathbf{C}}$. (See also [BGH22, Section 5.7], [GKRW23, Lem 8.17], [BDS22, Prop 5.5.5].)

(The same is also true in commutative algebras, or as we will see in Exercise 9.6.33, in algebras over any operad.)

8.7.48. Using the isomorphism $|X_\bullet|_{\mathrm{CAlg}(\mathbf{C})} \cong |X_\bullet|_{\mathbf{C}}$ from the previous exercise, prove that the homotopy pushout of a diagram of commutative ring spectra $X \leftarrow R \to Y$, in other words the homotopy colimit of this diagram in $\mathrm{CAlg}(\mathbf{Sp})$, is computed as the two-sided bar construction $B(X, R, Y)$.

Chapter 9

Homotopy-coherent abelian groups

This chapter is concerned with the relationship between spectra and abelian groups. We have already seen that each abelian group G gives an Eilenberg–Mac Lane spectrum HG, and that maps in the stable homotopy category $[X, Y]_s$ form an abelian group. Both of these relationships can be extended, if we are willing to generalize what we mean by "abelian group" to something more homotopy coherent.

A **grouplike E_∞ space** is a space X with a multiplication that is associative, commutative, and has an identity and inverses, but only up to a coherent collection of homotopies. One might call it an "abelian group up to homotopy." It's probably more accurate to call it a "homotopy-coherent abelian group." The related notion of a **grouplike A_∞ space** is only associative, unital, and with inverses, in other words it is a homotopy-coherent group.

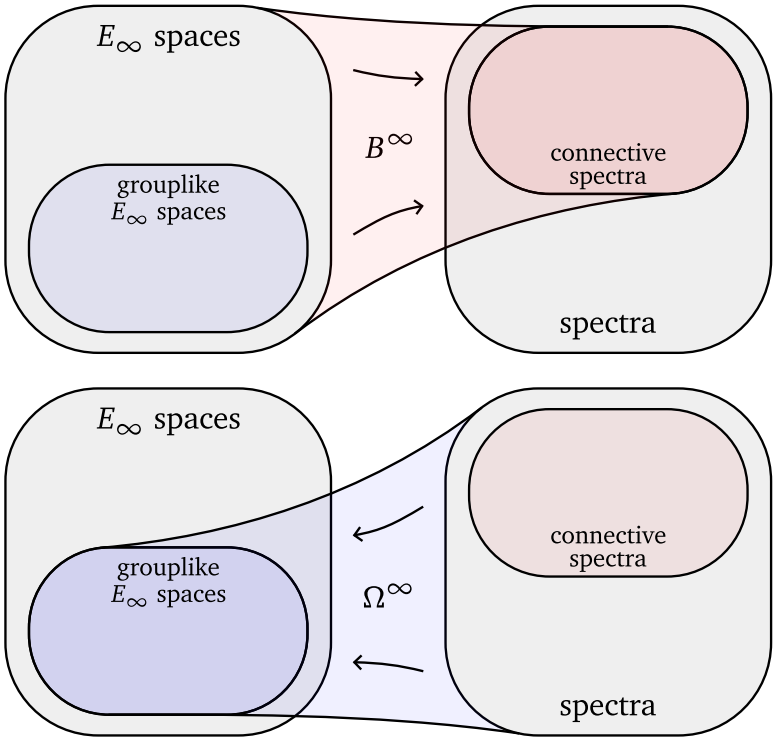
In this chapter we explain how define these homotopy-coherent groups, using the notion of an **operad**. An operad is a collection of abstract functions, each of which has several inputs but only one output, together with a formal rule for how to compose these functions. The precise definition is somewhat complicated, but it is a powerful framework for organizing a large collection of multiplications and their compositions that only agree up to homotopy, rather than agreeing strictly.

We also give a second definition of homotopy-coherent groups using an approach due to Graeme Segal. The "Segal" versions are equivalent to the versions defined using operads, but they are much simpler to work with, because they can be stated entirely using functions with a single input.

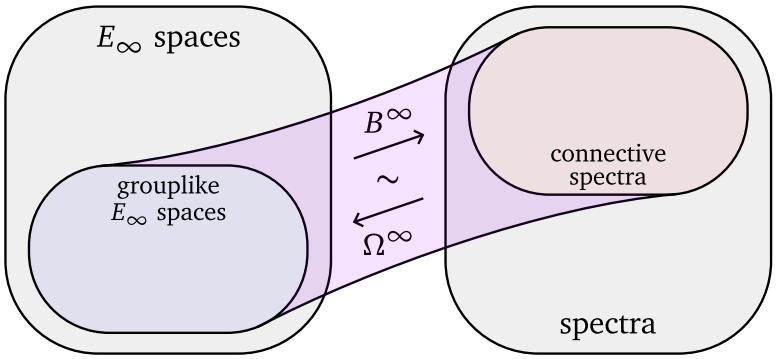
We then give the **recognition principle** that homotopy-coherent groups, or grouplike A_∞ spaces, are essentially the same thing as loop spaces ΩX. Furthermore homotopy-coherent abelian groups, or grouplike E_∞ spaces are essentially the same thing as infinite loop spaces $\Omega^\infty E$. (Actually, we only give the proof of this recogition principle in the Segal setting, where it is simpler.) So this notion of homotopy-coherent algebra gives us a new way to define interesting examples of spectra!

To give a little more detail, we send each E_∞ space (or homotopy-coherent abelian monoid) X to a spectrum called the **delooping spectrum** $B^\infty X$. This is formally somewhat like the suspension spectrum $\Sigma^\infty X$, but be warned that the homotopy type is different. $\Sigma^\infty X$ is a *suspension spectrum* whose 0th level is X, whereas $B^\infty X$ is an Ω*-spectrum* whose 0th level is the group completion of X. As a special case, if G is a discrete abelian group, its delooping spectrum $B^\infty G$ is an Eilenberg–Mac Lane spectrum HG, not the suspension spectrum $\Sigma^\infty G$.

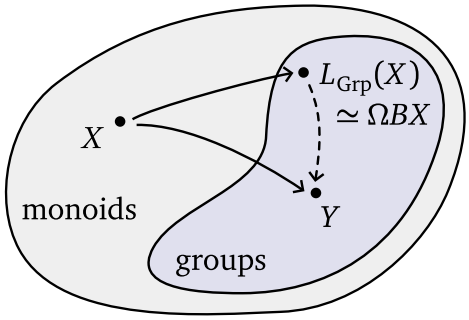
Going the other way, for each spectrum E, the infinite loop space $\Omega^\infty E$ is always grouplike E_∞ space, aka a homotopy-coherent abelian group. This gives us an adjunction $(B^\infty, \Omega^\infty)$ between E_∞ spaces and spectra.



The recognition principle tells us that these operations give an equivalence of homotopy categories, between connective spectra and grouplike E_∞ spaces (homotopy-coherent abelian groups). In other words, spectra are almost the same thing as homotopy-coherent abelian groups, the only difference is that spectra can have negative homotopy groups.

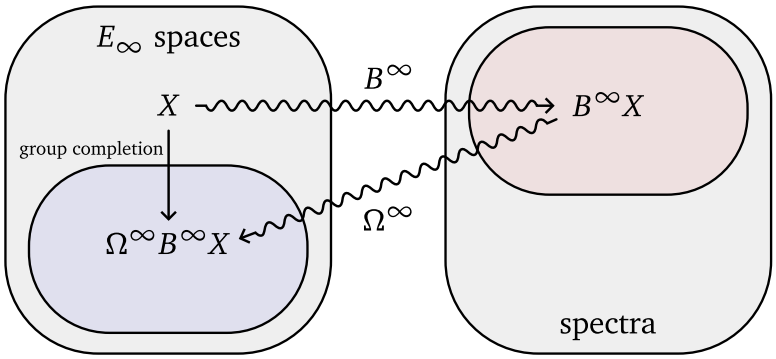


We also discuss the notion of **group completion**. This is the universal way of making each monoid into a group by adding inverses. There is a homotopical version of group completion that applies to monoids up to weak equivalence, or to homotopy-coherent monoids or abelian monoids. It is defined differently in all of these cases, but it turns out that they are all equivalent.



The homotopical group completion can always be modeled as the map from X to loops of its bar construction, ΩBX. For topological monoids, there is an issue with this statement because $X \to \Omega BX$ is not actually a map of monoids, so we only get the weak statement then $X \to \Omega BX$ is *weakly equivalent* to the map from X to its group completion. However, when we define B and Ω for homotopy-coherent monoids, it turns out that $X \to \Omega BX$ *is* actually a map of homotopy-coherent monoids, and this map *is* the group completion of X.

Group completion is an important notion because many of the spectra that we care about don't actually arise from homotopy-coherent abelian groups, but rather from homotopy-coherent abelian monoids (i.e. E_∞ spaces) that we have to group complete. In this case the E_∞ space X group completes to $\Omega^\infty B^\infty X$, and both of these deloop to give the same spectrum $B^\infty X$, whose infinite loopspace is the group completion $\Omega^\infty B^\infty X$.



The relationship between spectra and E_∞ spaces can be exploited to produce new examples of spectra. If $\mathbf{C}$ is any symmetric monoidal category, its classifying space $B\mathbf{C}$ an E_∞ space. Its associated spectrum $B^\infty(B\mathbf{C})$ is called the **algebraic K-theory** $K(\mathbf{C})$, because the term "K-theory" is used any time formal inverses are added. When $\mathbf{C}$ is the category of real or complex vector spaces, this produces the connective versions of the spectra ko and ku that classify stabilized vector bundles.

In other words, K-theory spectra are a natural generalization of Eilenberg–Mac Lane spectra, where elements of a group G are replaced by objects in a symmetric monoidal category.

9.1 The homotopy theory of strict monoids and groups

As a prologue, before introducing A_∞ and E_∞ spaces, we first cover the simpler theory of topological monoids up to weak equivalence, including the theory of group completion. This builds some intuition for what happens when we pass to the homotopy-coherent versions later in the chapter.

9.1.1 Topological monoids and groups

Recall that:

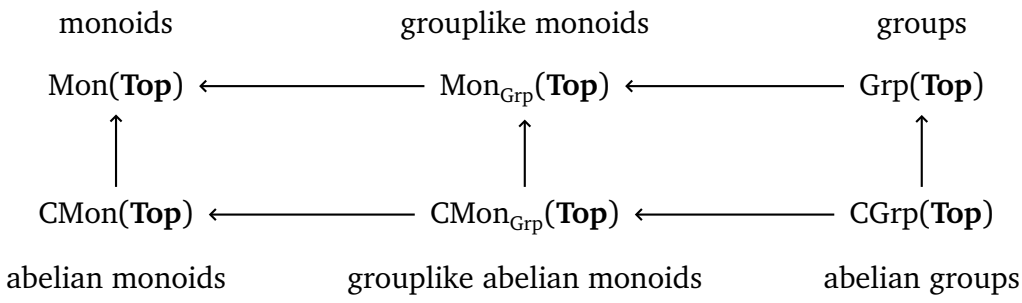
- A **topological monoid** is a monoid object in **Top** (Definition 4.1.12). Concretely, it is a topological space M with an associative and unital multiplication $\mu\colon M \times M \to M$ that is also continuous.
- A **topological group** is a group object in **Top** (Exercise 8.7.45). Concretely, it is a topological space G that is also a group, such that the multiplication $\mu\colon G \times G \to G$ and inversion $S\colon G \to G$ are continuous.

- A **topological abelian monoid** is a commutative monoid object in **Top** (Definition 4.1.12). So it is a topological monoid M whose multiplication is also commutative.
- A **topological abelian group** is a commutative group object in **Top** (Exercise 8.7.45). So it is a topological group A whose multiplication is also commutative.

In each of these cases, a morphism is a continuous function that is also homomorphism with respect to the multiplication μ and unit map $\eta\colon * \to M$.

When M is a topological monoid, the set $\pi_0(M)$ becomes a monoid as well, by multiplying together representatives in each path component. We say that M is **grouplike** if $\pi_0(M)$ has all inverses, so that it is not just a monoid but a group as well. This is equivalent to asking for each map $g\cdot(-)\colon M \to M$ to be a weak equivalence, and is also equivalent to asking that for each $g \in M$ there is an $h \in M$ such that gh and hg are both in the path component of the identity element $1 \in M$. See Exercise 9.6.1.

Summarizing, we have the following categories and forgetful functors. (In fact, each of these functors is the inclusion of a full subcategory, because a map of groups or commutative monoids is the same thing as a map of the underlying monoids.)



Example 9.1.1. Recall from Exercise 4.4.14 that a **permutative category** $(\mathbf{C}, \otimes, I)$ is a symmetric monoidal category in which the associator and unitor isomorphisms are actually equalities,

$$(A\otimes B)\otimes C = A\otimes(B\otimes C), \qquad I\otimes A = A = A\otimes I.$$

It follows that the classifying space $B\mathbf{C}$ is a topological monoid, with multiplication induced by the tensor product

$$B\mathbf{C}\times B\mathbf{C} \cong B(\mathbf{C}\times\mathbf{C}) \xrightarrow{B(\otimes)} B\mathbf{C}.$$

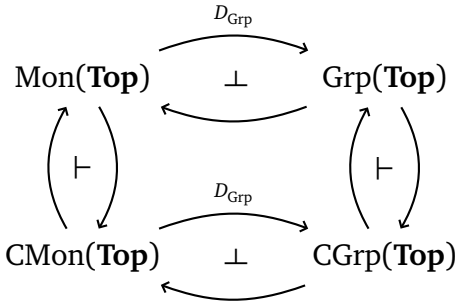
You can check that $B\mathbf{C}$ is usually not an abelian monoid, because $\otimes$ is only symmetric up to isomorphism, not strictly symmetric. We will see later in Example 9.3.15 that this makes $B\mathbf{C}$ into the homotopy-coherent version of an abelian monoid, called an E_∞ space.

Example 9.1.2. The space of maps $\mathrm{Map}_*(S^n, S^n) = \Omega^n S^n$ is a topological monoid under composition. Its identity element is the identity map, and if we restrict to the components of the maps of degree ± 1, we get a grouplike monoid. The homotopy groups of $\mathrm{Map}_*(S^n, S^n)$ are the homotopy groups of S^n, but shifted down n degrees, and also its lowest homotopy group is a $\mathbb{Z}/2$ instead of a $\mathbb{Z}$.

More generally, for any unbased space X the space of maps $\mathrm{Map}(X, X)$ is a topological monoid, and for any based space X the space of based maps $\mathrm{Map}_*(X, X)$ is a topological monoid. Restricting to those maps that are homotopy equivalences makes a grouplike monoid, see Example 9.1.21 for more.

Let us ignore the "grouplike" condition for the moment and focus on monoids, commutative monoids, groups, and commutative groups.

Proposition 9.1.3. *All four of these categories have right-induced model structures, and the forgetful functors between them are right Quillen, giving Quillen adjunctions as shown.*



Recall that a right-induced model structure means that the weak equivalences and the fibrations are determined by forgetting the monoid structure. So a homomorphism of monoids is weak equivalence iff it is a weak equivalence after forgetting the multiplication, and similarly for fibrations. The cofibrations are more complicated: they are generated by taking a free group or free monoid on the generating cofibrations $S^{n-1} \to D^n$ in **Top** and gluing the resulting maps together.

Proof. The proof is by the recipe given in Theorem 6.5.2, and in fact we have already constructed this model structure for Mon(**Top**) in Theorem 6.5.20. The remaining cases are similar, but we will do the case of abelian groups in detail.

- The forgetful functor $U\colon \mathrm{CGrp}(\mathbf{Top}) \to \mathbf{Top}$ is clear. Its left adjoint $F_{\mathrm{CGrp}}\colon \mathbf{Top} \to \mathrm{CGrp}(\mathbf{Top})$ is defined as the tensor $K \otimes_{\mathrm{CAlg}} \mathbb{Z}$ from Exercise 8.7.45. Concretely, this is a quotient of
$$\coprod_{n\geq 0} (K \times \mathbb{Z})^{\times n}$$

in which we allow ourselves to permute the terms $(k_i, m_i) \in K \times \mathbb{Z}$, add two adjacent terms when they have the same point of K, and delete any term in which $m_i = 0$.

You should think of this as the "free topological abelian group on the topological space K," and the above expressions are words of the form $k_1^{m_1} k_2^{m_2} \cdots k_n^{m_n}$.

- Limits and colimits in CGrp(**Top**) are the same as in CMon(**Top**) = CAlg(**Top**) by Exercise 8.7.45(b). We constructed these explicitly in Lemma 6.5.24.
- A **free cofibration** in CGrp(**Top**) is defined to be retract of a relative $F_{\mathrm{CGrp}}(I)$-cell complex, where $I = \{S^{n-1} \to D^n\}_{n \geq 0}$ are the generating cofibrations in **Top**.
- The maps $F_{\mathrm{CGrp}}(I)$ and $F_{\mathrm{CGrp}}(J)$ each satisfy the countable smallness condition. By Exercise 1.7.31, it is enough to show that each map in $F_{\mathrm{CGrp}}(I)$ or $F_{\mathrm{CGrp}}(J)$ is a closed inclusion. To show this for $F_{\mathrm{CGrp}}(S^{n-1}) \to F_{\mathrm{CGrp}}(D^n)$, let $U \subseteq D^n$ be an open neighborhood of S^{n-1} that retracts onto S^{n-1}. Then $F_{\mathrm{CGrp}}(U)$ is an open subset of $F_{\mathrm{CGrp}}(D^n)$ that retracts onto $F_{\mathrm{CGrp}}(S^{n-1})$, so by Lemma 1.2.3, the map $F_{\mathrm{CGrp}}(S^{n-1}) \to F_{\mathrm{CGrp}}(D^n)$ is a closed inclusion.
- By the same trick as in the last point, the map $F_{\mathrm{CGrp}}(D^n) \to F_{\mathrm{CGrp}}(D^n \times I)$ has a deformation retract, and is therefore a weak equivalence. It quickly follows that every $F_{\mathrm{CGrp}}(J)$-cell complex is a weak equivalence as well.

For the remaining three categories, the only difference in the proof is the definition of the free functor T, which we leave to Exercise 9.6.3. The forgetful functors preserve weak equivalences and fibrations, so we only need to know that they have left adjoints, and we also leave this to Exercise 9.6.3. □

We can therefore pass freely between the different notions while preserving weak equivalences. The left adjoints in particular are very interesting: the vertical ones are a homotopical version of abelianization, while the horizontal ones are a homotopical version of group completion. We study this group completion in more detail in Section 9.1.3, after recalling the simplicial version of monoids and groups.

Remark 9.1.4. We are only studying topological abelian groups up to weak equivalence. These are fairly well-understood by homotopy theorists. If one wants to understand topological abelian groups up to homeomorphism, the situation is much harder because they do not form an abelian category. In other words, they don't have a good notion of kernel and cokernel. There is interesting recent work on frameworks for algebra that get around this, such as the pyknotic sets of [BH19] and condensed mathematics of [CS19].

9.1.2 Simplicial monoids and groups

We can also do a simplicial version of all of the above topological constructions.

Definition 9.1.5. A **simplicial monoid** is a monoid object in **sSet**, or equivalently a simplicial object in the category of monoids. Spelling this out, this is a simplicial set simplicial set $M_\bullet$ with either

- a map of simplicial sets $\mu\colon M_\bullet \times M_\bullet \to M_\bullet$ that is associative and has a unit map of simplicial sets $* \to M_\bullet$, or equivalently
- the structure of a monoid on each set M_n such that each face and degeneracy map is a monoid homomorphism $d_i\colon M_n \to M_{n-1}$ and $s_i\colon M_n \to M_{n+1}$.

Similarly a **simplicial group** is a group object in **sSet**, or equivalently a simplicial set in which the levels G_n are groups and the face and degeneracy maps are homomorphisms. A **simplicial abelian group** is the same except that the multiplication is always commutative. We use the notation:

$$\begin{aligned} \mathrm{Mon}(\mathbf{sSet}) &= \mathbf{sMon} && \text{simplicial monoids} \\ \mathrm{Grp}(\mathbf{sSet}) &= \mathbf{sGrp} && \text{simplicial groups} \\ \mathrm{CMon}(\mathbf{sSet}) &= \mathbf{sCMon} && \text{simplicial abelian monoids} \\ \mathrm{CGrp}(\mathbf{sSet}) &= \mathbf{sAb}^{1} && \text{simplicial abelian groups} \end{aligned}$$

Lemma 9.1.6. *For any simplicial monoid $M_\bullet$, the realization $|M_\bullet|$ is a topological monoid. For any topological monoid M, the singular simplices $S_\bullet M$ are a simplicial monoid. This defines an adjunction*

$$\mathrm{Mon}(\mathbf{sSet}) \underset{S_\bullet}{\overset{|-|}{\rightleftarrows}} \mathrm{Mon}(\mathbf{Top}).$$

The same is true for simplicial groups, and for abelian monoids and groups as well.

Proof. The multiplication $M_\bullet \times M_\bullet \to M_\bullet$ gives on realization a map of topological spaces

$$|M_\bullet| \times |M_\bullet| \cong |M_\bullet \times M_\bullet| \to |M_\bullet|,$$

where the isomorphism comes from Milnor's theorem (Theorem 7.1.31). Similarly the unit map $* \to M_\bullet$ gives a unit map $* \to |M_\bullet|$. The maps of simplicial sets that we started with satisfied the axioms for a monoid, and so these maps of realizations also satisfy the axioms for a monoid.

Applying the same reasoning to the singular simplices, we see that it also preserves monoids, and the usual adjunction between **sSet** and **Top** respects these monoid structures. The argument for groups and abelian groups is essentially the same. □

[1] We could say **sCGrp**, but **sAb** is widely used and easier to parse.

Proposition 9.1.7. *Each of the four categories* Mon(**sSet**)*,* Grp(**sSet**)*,* CMon(**sSet**)*, and* CGrp(**sSet**) *has a right-induced model structure from* **sSet***. The forgetful functors between them are all right Quillen, and geometric realization and singular simplices give a Quillen equivalence between each of these model structures and the corresponding model structure for* **Top** *from Proposition 9.1.3.*

Proof. The proofs are similar to Proposition 9.1.3 but easier. We remark that the free monoid functor $F_{\mathrm{Mon}}\colon \mathbf{sSet} \to \mathrm{Mon}(\mathbf{sSet})$ takes each simplicial set $X_\bullet$ and then takes the free monoid $F(X_n)$ on each of the sets X_n. The other four "free" functors similarly take the free group, abelian monoid, or abelian group on each of the sets X_n. □

For the next lemma let $I = \{\partial\Delta[k]_\bullet \to \Delta[k]_\bullet\}$ be the generating cofibrations for the Quillen model structure on **sSet**, so that the maps

$$F_{\mathrm{Mon}}(I) = \{F_{\mathrm{Mon}}(\partial\Delta[k]_\bullet) \to F_{\mathrm{Mon}}(\Delta[k]_\bullet)$$

are the generating cofibrations for Mon(**sSet**).

Lemma 9.1.8. *If $* \to M_\bullet$ is a $F_{\mathrm{Mon}}(I)$-cell complex of simplicial monoids, then each M_n is a free monoid on some set S_n.*[2] *The corresponding statement for simplicial groups, abelian monoids, and abelian groups also holds.*

Proof. We only prove this for monoids. $M_\bullet$ is formed by taking iterated pushouts in the category of simplicial monoids along maps of the form $F(\partial\Delta[k]_\bullet) \to F(\Delta[k]_\bullet)$, where F denotes free monoid. At each simplicial level, this is a map of free monoids on an injective map of generating sets $\partial\Delta[k]_n \to \Delta[k]_n$. It follows by comparing universal properties that any pushout along such a map, in the category of monoids, will send free monoids to free monoids. Therefore M_n is a free monoid. □

Lemma 9.1.9. *Every simplicial group $G_\bullet$ is a Kan complex.*

The proof of this is rather short and can be found in [GJ99, I.3.4]. All it requires is an explicit formula which fills each horn with a simplex. In essence, the formula takes the elements in G_{n-1} given by each face of the horn, applies degeneracy maps to give elements of G_n, and then uses the group structure to multiply these elements and their inverses together in a very specific order, so that when we pass back to each of the faces in the horn, there is cancellation in the formula and we are left with our original element of G_{n-1}.

[2] We are not claiming that the sets S_n form a simplicial set, and in general they don't.

9.1.3 Group completion

If M is a discrete monoid, its **discrete group completion** is a discrete group $D_{\mathrm{Grp}}(M)$ that is universal among groups G receiving a monoid map M. It is straightforward to show that such a group exists:

- One can start with a presentation of M by generators and relations. Then $D_{\mathrm{Grp}}(M)$ is the group that has the same presentation.
- Or, one can define elements of $D_{\mathrm{Grp}}(M)$ as words $g_1 g_2 \cdots g_n$ where each g_i is either an element of M or a formal inverse, and we are allowed to multiply adjacent elements, cancel identity elements, and cancel any pair of the form $g g^{-1}$ or $g^{-1} g$.[3]
- If M is abelian, the group completion $D_{\mathrm{Grp}}(M)$ can be defined as equivalence classes of formal differences of elements of M. See Exercise 9.6.2.

Example 9.1.10. The monoid $(\mathbb{N}, +)$ has one generator and no relations. Therefore its group completion has one generator and no relations, so it is $(\mathbb{Z}, +)$.

When we pass from discrete monoids to topological monoids, the definition of D_{Grp} still makes sense, giving the left adjoint of the forgetful functor $\mathrm{Grp}(\mathbf{Top}) \to \mathrm{Mon}(\mathbf{Top})$ (Exercise 9.6.3). We could also do these to every level of a simplicial monoid to get a simplicial group, and this is the left adjoint of the forgetful functor $\mathrm{Grp}(\mathbf{sSet}) \to \mathrm{Mon}(\mathbf{sSet})$. However, these "strict group completion" functors do not preserve weak equivalences. We therefore want something more homotopy-invariant.

Definition 9.1.11. The **homotopical group completion**

$$L_{\mathrm{Grp}}(-) \colon \mathrm{Mon}(\mathbf{Top}) \to \mathrm{Grp}(\mathbf{Top})$$

is the left-derived functor of the discrete group completion D_{Grp}. We also use the same term for the corresponding left-derived functor on simplicial monoids:

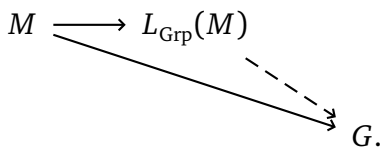
$$L_{\mathrm{Grp}}(-) \colon \mathrm{Mon}(\mathbf{sSet}) \to \mathrm{Grp}(\mathbf{sSet}).$$

This is just like how we defined the homotopy colimit in terms of the colimit (Definition 3.4.29). The homotopical group completion is the universal way of doing the group completion in a homotopy-invariant way.

By definition, this functor $L_{\mathrm{Grp}}(-)$ sends monoids to groups, preserves weak equivalences, and the associated map of homotopy categories $\mathrm{Ho}\,\mathrm{Mon}(\mathbf{Top}) \to \mathrm{Ho}\,\mathrm{Grp}(\mathbf{Top})$ is the left adjoint of the forgetful functor.

[3] In fact, this is just the Gabriel-Zisman localization from Definition 3.1.9, applied to a category with one object, where we choose to invert every morphism in that category.

Spelling this out, we send every topological monoid M to a topological group $L_{\mathrm{Grp}}(M)$, with a zig-zag of maps of monoids $M \to L_{\mathrm{Grp}}(M)$. (Every map in the zig-zag is a map of topological monoids, and the backwards maps are also weak equivalences.) Furthermore, for any zig-zag of monoids from M to a topological group G, there is a unique zig-zag of topological groups $L_{\mathrm{Grp}}(M) \to G$ making the following diagram commute:



We know that this left-derived functor exists, because D_{Grp} is a left Quillen functor by Proposition 9.1.3. The left-derived functor $L_{\mathrm{Grp}}(M)$ can be computed by taking a cofibrant replacement of M as a monoid (which intuitively is a "presentation" of M), and then applying D_{Grp} (which intuitively is forming topological group with the same "presentation.") From this we can observe right away:

Lemma 9.1.12. *If M is a discrete free monoid $F_{\mathrm{Mon}}(S)$, then $L_{\mathrm{Grp}}(M) \simeq D_{\mathrm{Grp}}(M)$ is the corresponding discrete free group $F_{\mathrm{Grp}}(S)$.*

But be careful! When M is discrete and not free, $L_{\mathrm{Grp}}(M)$ is usually not discrete anymore! In other words, the homotopical group completion adds topology even when the monoid we started with was discrete. In fact, *every* topological group up to equivalence is obtained by group-completing some discrete monoid, by a theorem of McDuff [McD79].

How do we describe the homotopy type of $L_{\mathrm{Grp}}(M)$ in general? The next few results will show that $L_{\mathrm{Grp}}(M)$ is always weakly equivalent to ΩBM. This is a very useful result that will appear many times throughout the chapter.

Recall that we may form the bar construction $BM = B(*, M, *)$. This makes sense whether M is a topological or a simplicial monoid, and gives equivalent answers when we pass back and forth, except that in the topological case we should always assume that M is well-based, so that the passage from M to BM preserves weak equivalences by Theorem 7.3.10. There is a canonical map of based spaces

$$M \to \Omega BM, \tag{9.1.13}$$

whose adjunct $\Sigma M \to BM$ arises by identifying M with the space of 1-cells $N_1(*, M, *) = * \times M \times *$, giving a map $\Delta^1 \times M \to BM$ whose endpoints all go to the basepoint 0-cell.

Lemma 9.1.14. *If M is grouplike and well-based then (9.1.13) is a weak equivalence.*

Proof. We form the commuting diagram

$$\begin{array}{ccccc} M & \longrightarrow & EM & \longrightarrow & BM \\ \downarrow & & \downarrow\sim & & \|\sim \\ \Omega BM & \longrightarrow & PBM & \longrightarrow & BM. \end{array}$$

The bottom row is always a fiber sequence. The top row is a fiber sequence when M is grouplike by Proposition 7.4.35. Since the vertical maps in the middle and right are equivalences, the map on the left is an equivalence too. □

To prove that the map $M \to \Omega BM$ is equivalent to the group completion, we next have to describe the homotopy type of BM in the special case that M is a free monoid.

Lemma 9.1.15. *If $M = F_{\mathrm{Mon}}(S)$ is a discrete free monoid on the set S, then BM is an Eilenberg-Maclane space $K(G,1)$, where $G = F_{\mathrm{Grp}}(S)$ is the corresponding discrete free group.*

In other words, BM is homotopy equivalent to a wedge of circles, one for each element of the set S, and the map $BM \to BG$ is a weak equivalence.

Proof. We follow the proof in [DK80, Prop 2.9]. Let S be the set of free generators of M. The topological version of BM is a CW complex with one n-cell for each list of n non-identity elements $w_1, \dots, w_n \in M$, each of which is a nonempty "word" whose letters are elements of S. For each $k \geq 1$, let $BM^{(k)}$ be the subcomplex of those cells for which the total number of letters in the words w_1 through w_n is $\leq k$. When $k = 1$, this is exactly the wedge of circles indexed by S, so we just have to show that each $BM^{(k)}$ deformation retracts onto the previous complex $BM^{(k-1)}$ when $k \geq 2$.

Each cell in $BM^{(k)} \setminus BM^{(k-1)}$ is a simplex determined by a sequence of nonempty words $w_1, \dots, w_n \in M$ that together have exactly k letters. These group together according to which string of k letters in S we use, and each grouping consists of a unique k-simplex and some of its faces. The remaining faces are the ones already in $BM^{(k-1)}$, and these form the subcomplex of Δ^k given by the union of the 0th and kth faces. Therefore we can form $BM^{(k)}$ from $BM^{(k-1)}$ by taking a pushout along a coproduct of maps of the form $\Delta^{k-1} \cup_{\Delta^{k-2}} \Delta^{k-1} \to \Delta^k$. Each of these maps has a deformation retraction, and applying these all at the same time gives a deformation retraction of $BM^{(k)}$ back to $BM^{(k-1)}$, finishing the proof. □

Lemma 9.1.16. *If $M_\bullet$ is a simplicial monoid and each M_n is a free monoid, then $B|M_\bullet| \to B(D_{\mathrm{Grp}}(|M_\bullet|))$ is a weak equivalence.*

Proof. The bar construction $B(*, |M_\bullet|, *)$ is the realization of a bisimplicial object, which at level (m,n) is $N_n(*, M_m, *)$. The map $M \to D_{\mathrm{Grp}}(M)$ induces an equivalence on the

realization in the n-direction by Lemma 9.1.15. Then we can take the realization in the m-direction (which preserves equivalences by Example 8.2.23 and Theorem 8.2.25) and conclude that we have an equivalence on the entire realization. □

Proposition 9.1.17. *For any simplicial monoid or well-based topological monoid M, the map $BM \to B(L_{\mathrm{Grp}}(M))$ is a weak equivalence.*

Proof. Without loss of generality $M_\bullet$ is a simplicial monoid. The derived group completion is computed by taking the cofibrant replacement of $M_\bullet$ and then applying D_{Grp}. By Lemma 9.1.8, this cofibrant replacement $QM_\bullet$ is levelwise free, so by Lemma 9.1.16 the group completion preserves the homotopy type of the classifying space. □

Theorem 9.1.18 (Homotopy type of the group completion)**.** *If M is a simplicial monoid or well-based topological monoid, then*

$$L_{\mathrm{Grp}}(M) \simeq \Omega BM,$$

as a space or simplicial set receiving a map from M.

Proof. We apply the map (9.1.13) to both M and $L_{\mathrm{Grp}}(M)$, forming the commuting square

$$\begin{array}{ccc} M & \longrightarrow & L_{\mathrm{Grp}}(M) \\ {\scriptstyle (9.1.13)}\downarrow & & \downarrow{\scriptstyle \sim\ (9.1.13)} \\ \Omega BM & \xrightarrow{\sim} & \Omega B(L_{\mathrm{Grp}}(M)). \end{array}$$

The map on the right is an equivalence by Lemma 9.1.14 and the map on the bottom is an equivalence by Proposition 9.1.17. □

Note that we are claiming nothing about the group structure: ΩBM is not a even monoid, and so it doesn't even make sense to ask whether $M \to \Omega BM$ is a homomorphism. This is discussed more in Exercise 9.6.6. We leave it to Exercise 9.6.9 to show that $M \to \Omega BM$ agrees with the group completion as a map of "H-spaces," in other words spaces that have a multiplication that has a unit up to homotopy.

Even though $M \to \Omega BM$ is not a map of monoids, Theorem 9.1.18 is still a very useful result. For instance it immediately tells us:

Corollary 9.1.19. *If M is a grouplike monoid then $M \to L_{\mathrm{Grp}}(M)$ is a weak equivalence.*

In other words, a monoid is weakly equivalent to a group iff it is grouplike. This is a really simple way to check whether something is equivalent to a group. It also motivates us to abandon the study of groups up to weak equivalence, because grouplike monoids are good enough, and it's much easier to work with grouplike monoids.

Example 9.1.20. If we take the space $\mathrm{Map}_*(S^n, S^n)_{\pm 1}$ of degree-one based maps $S^n \to S^n$ from Example 9.1.2, this is a grouplike monoid. It is therefore possible to change it up to weak equivalence to be a topological group! But that topological group is quite nasty to describe, so it's usually better to work with the more explicit space $\mathrm{Map}_*(S^n, S^n)_{\pm 1}$.

Example 9.1.21. Recall from Example 7.4.36 that for any topological space F, we may define the space of homotopy automorphisms $\mathrm{hAut}(F) \subseteq \mathrm{Map}(F, F)$. Then $\mathrm{hAut}(F)$ is a topological monoid, and it is grouplike because every homotopy equivalence has an inverse up to homotopy. Therefore by Corollary 9.1.19, $\mathrm{hAut}(F)$ is equivalent to an actual topological group.

If we then construct the universal fibration out of $\mathrm{hAut}(F)$ as in Example 7.4.36, we find that we get a *fiber bundle* with fiber equivalent to F. This proves the bizarre fact that *every fibration is weakly equivalent to a fiber bundle*.

This concludes the basic theorems about group completion of a monoid. We finish this subsection by describing the corresponding results for abelian monoids and abelian groups. Note that when M is abelian, the multiplication $\mu\colon M \times M \to M$ is a monoid homomorphism. Therefore we can think of M has a monoid object in $\mathrm{CMon}(\mathbf{Top})$, using the multiplication μ for both the commutative monoid structure and the monoid structure. This defines a forgetful functor

$$\mathrm{CMon}(\mathbf{Top}) \to \mathrm{Mon}(\mathrm{CMon}(\mathbf{Top})). \tag{9.1.22}$$

However, by the Eckmann-Hilton argument (Lemma 1.4.5), when a space has two multiplications that interchange in this way, the two multiplications *have* to agree! So the category $\mathrm{Mon}(\mathrm{CMon}(\mathbf{Top}))$ really only consists of spaces with a single multiplication. In other words, (9.1.22) is an *isomorphism* of categories.

Why is it a good idea to think of a commutative multiplication as two multiplications? Because we can take all of the above constructions and run them with the category **Top** replaced by $\mathrm{CMon}(\mathbf{Top})$. So instead of BM being an object of **Top**, it's an object of $\mathrm{CMon}(\mathbf{Top})$. In other words, BM is also a commutative monoid! And ΩBM is also a commutative monoid by pointwise multiplication of two loops. Furthermore the map $M \to \Omega BM$ is a homomorphism of commutative monoids.

Theorem 9.1.23 (Homotopy type of the abelian group completion)**.** *If M is a simplicial abelian monoid or well-based topological abelian monoid, then $L_{\mathrm{CGrp}}(M)$ is weakly equivalent to ΩBM, as a commutative monoid receiving a map from M.*

Here L_{CGrp} is the left-derived functor of the left adjoint of the forgetful functor $\mathrm{CGrp} \to \mathrm{CMon}$. The proof of the above theorem is almost exactly the same as Theorem 9.1.18, there is just one step that needs to be carried out differently:

Lemma 9.1.24. *If $M = \oplus_S \mathbb{N}$ is a discrete free abelian monoid, then BM is an Eilenberg-Maclane space $K(G,1)$, where $G = \oplus_S \mathbb{Z}$ is the corresponding free abelian group.*

In other words, BM is homotopy equivalent to a (possibly infinite-dimensional) torus, with one coordinate direction for each element of S.

Proof. This proof is inspired by a mathoverflow post by Tyler Lawson [Law12]. Consider the category $\mathbf{I}$ whose objects are elements of $\bigoplus_S \mathbb{Z}$, and with a unique morphism $(m_s)_{s\in S} \to (n_s)_{s\in S}$ when $m_s \leq n_s$ for all $s \in S$, and no morphisms otherwise. In other words, this category comes from a partial ordering on $\bigoplus_S \mathbb{Z}$.

We first prove that $B\mathbf{I}$ is contractible. It suffices to prove this when S is finite, because the entire complex is a filtered colimit of these subcomplexes for which S is finite. (To put this another way, each sphere in $B\mathbf{I}$ lands in a subcomplex where S is finite, so if it extends to a disc in that subcomplex, it certainly extends to a disc in $B\mathbf{I}$.)

Now that S is finite, we write $B\mathbf{I}$ as the colimit of a sequence of subcomplexes, where the kth subcomplex is the classifying space of the subcategory in which we insist that $n_s \geq -k$ for all $s \in S$. Each of these categories has an initial object, namely the constant tuple $(-k)_{s\in S}$, so its classifying space is contractible.

Now we observe that the action of the group $\bigoplus_S \mathbb{Z}$ on $B\mathbf{I}$ is a covering space action, and the quotient by this is identified with the bar construction BM. □

As before, we conclude that abelian groups are the same thing as grouplike abelian monoids, up to weak equivalence. We also get an agreement between the two forms of group completion if M is abelian:

Corollary 9.1.25. *The canonical map $L_{\mathrm{Grp}}(M) \to L_{\mathrm{CGrp}}(M)$ is an equivalence for every abelian monoid M. In other words the map $M \to L_{\mathrm{CGrp}}(M)$ also satisfies the universal property of the non-abelian group completion.*

9.1.4 Homology of the group completion

We just proved that the homotopical group completion of a monoid M always has the homotopy type of ΩBM. This is a useful theoretical result, but sometimes we want a more practical result describing the algebraic invariants of ΩBM, such as its homotopy groups or homology groups.

You might expect that the homotopy groups of ΩBM would be easy to describe, because when M is grouplike, B simply shifts homotopy groups up, and Ω always shifts homotopy groups down. However, when M is not grouplike, there is no formula for the homotopy groups of ΩBM in terms of the homotopy groups of M. In fact, we will see later that if

there were such a formula, it would give a formula for the stable homotopy groups of spheres in terms of the symmetric groups Σ_n!

Luckily, the homology $H_*(\Omega BM)$ does have a nice formula in terms of $H_*(M)$, provided we make some mild assumptions about M.

Theorem 9.1.26 (McDuff–Segal group completion theorem)**.** *[MS76, RW13] If M is a well-based topological monoid, R is any ring, and $\pi_0(M)$ is central in $H_*(M;R)$, then the natural map*

$$H_*(M;R)[\pi_0(M)^{-1}] \to H_*(\Omega BM;R)$$

is an isomorphism.

Before proving this, we first clarify the statement and give several examples. (See also Remark 9.1.35 for more general versions of the theorem.)

In the statement of Theorem 9.1.26, $H_*(M;R)$ is considered to be a graded ring under the Pontryagin product, and $\pi_0(M)$ is considered to be a subset of this ring, namely the R-module generators of $H_0(M;R)$. This subset is **central** when each element $[a] \in \pi_0(M)$ commutes with every other element of the ring, $[a][b] = [b][a]$. This happens in particular if the multiplication map $M \times M \to M$ is commutative up to homotopy.

The expression $H_*(M;R)[\pi_0(M)^{-1}]$ refers to the localization of the ring in which we make $\pi_0(M)$ invertible. The ring $H_*(M;R)$ is only associative, not commutative in general, and localization of associative rings can get quite complicated, in fact they resemble the Gabriel-Zisman localization from Definition 3.1.9. But under the assumption that $\pi_0(M)$ is central, the localization is fairly simple. The following construction can be found in any standard algebra text, such as [Lan65, DF04].

Definition 9.1.27. For any ring R, and any central subset $S \subseteq R$ that is closed under multiplication, the **localization** $R[S^{-1}]$ is the set of equivalence classes of "fractions" a/b with $a \in R$ and $b \in S$, where a/b is equivalent to c/d iff $ade = bce$ for some fifth element $e \in S$. The addition and multiplication is defined in the usual way from grade school:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{ad}, \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

It is left to Exercise 9.6.10 to check that this defines a ring, and that it is universal among rings receiving a homomorphism from R such that each element of S goes to something invertible.

Now we can parse the statement of the McDuff–Segal group completion theorem: by Exercise 9.6.9, the map $M \to \Omega BM$ is a map of H-spaces and therefore gives a ring homomorphism $H_*(M;R) \to H_*(\Omega BM;R)$. Therefore by the universal property of localization it extends to a ring homomorphism $H_*(M;R)[\pi_0(M)^{-1}] \to H_*(\Omega BM;R)$. The theorem claims that this extended map is an isomorphism.

Example 9.1.28. If $M = \mathbb{N}$ with the discrete topology, then $\Omega BM \simeq \mathbb{Z}$. The map on homology with $\mathbb{Z}$ coeffiecients is $\mathbb{Z}[t] \to \mathbb{Z}[t, t^{-1}]$, which indeed agrees with the localization of $\mathbb{Z}[t]$ that inverts all of the powers of t.

Example 9.1.29. If $\pi_0(M) = *$ then M is grouplike, so $M \to \Omega BM$ is an equivalence. This is consistent with the McDuff-Segal theorem which says that nothing is inverted, so $H_*(M) \to H_*(\Omega BM)$ is an isomorphism.

Example 9.1.30. Let $M = F_{\mathrm{Mon}}(S^1)$, the free monoid on the unbased space S^1. Up to homeomorphism, M is the disjoint union of tori

$$M \cong * \amalg S^1 \amalg (S^1 \times S^1) \amalg (S^1 \times S^1 \times S^1) \amalg \cdots$$

and so its homology is homology is a direct sum of exterior algebras which we denote $\Lambda[x_1, \ldots, x_n]$. The multiplication on homology is the tensor product $\Lambda[x_1, \ldots, x_m] \otimes \Lambda[x_{m+1}, \ldots, x_{m+n}] \to \Lambda[x_1, \ldots, x_{m+n}]$. The subset $\pi_0(M)$ is not central in $H_*(M)$ since the inclusions of $\Lambda[x_1]$ and $\Lambda[x_2]$ into $\Lambda[x_1, x_2]$ do not agree. Therefore the McDuff–Segal theorem does not apply. However we do still know for formal reasons that the group completion is the free group $F_{\mathrm{Grp}}(S^1)$.

Example 9.1.31. We could also form the quotient in Mon(**Top**) of $F_{\mathrm{Mon}}(S^1)$ by $F_{\mathrm{Mon}}(*)$. This quotients the $(n-1)$-skeleton of each of the above n-tori to a point, so that we get a wedge of spheres

$$M \simeq S^1 \vee S^2 \vee S^3 \vee \cdots .$$

In this case $\pi_0(M) = *$, so this monoid is already grouplike, and the group completion of M is equivalent to M again.

Example 9.1.32. Again let us take

$$M \cong * \amalg S^1 \amalg (S^1 \times S^1) \amalg (S^1 \times S^1 \times S^1) \amalg \cdots$$

but with a different multiplication. We let S^1 be the abelian group $\mathbb{R}/\mathbb{Z}$, and we define $G = \mathbb{Z} \times (S^1)^\infty$ as the product group. Then we let $M \leq G$ be the submonoid of points of the form $(n, t_1, \ldots, t_n, 0, 0, \ldots)$ with $n \geq 0$ and each $t_i \in \mathbb{R}/\mathbb{Z}$. So now M is an abelian monoid, and its homology is a direct sum of exterior algebras which we denote $t^n \cdot \Lambda[x_1, \ldots, x_n]$. Taking into account how the monoid multiplication treats the connected components, we compute the homology ring of the entire space as the ring

$$\mathbb{Z}[t] \otimes \Lambda[tx_1, t^2x_2, t^3x_3, \ldots].$$

Therefore after inverting t, the homology of the group completion of M is

$$\mathbb{Z}[t, t^{-1}] \otimes \Lambda[x_1, x_2, x_3, \ldots].$$

In fact, the group completion of M is the original group $G = \mathbb{Z} \times (S^1)^\infty$, which also has this homology. (This will follow from our proof of the group completion theorem below.)

Example 9.1.33. Recall from Example 9.1.1 that for any permutative category $\mathbf{C}$, the classifying space $B\mathbf{C}$ is a topological monoid. Its group completion $\Omega B(B\mathbf{C})$ is often called the **algebraic K-theory space** $\Omega^\infty K(\mathbf{C})$ of the category $\mathbf{C}$.

For example, if we take the symmetric monoidal category of finite sets and isomorphisms $(\mathbf{Fin}^\cong, \amalg, \varnothing)$ and make it into a permutative category using Exercise 4.4.14, then we get a topological monoid $B\mathbf{Fin}^\cong$. Since $\mathbf{Fin}^\cong$ is equivalent to a disjoint union of one-object categories with morphisms the symmetric group Σ_n, we get a homotopy equivalence

$$B\mathbf{Fin}^\cong \simeq \coprod_{n\geq 0} B\Sigma_n.$$

The space $B\mathbf{Fin}^\cong$ is a topological monoid, and along this equivalence, the multiplication on $B\Sigma_m \times B\Sigma_n$ maps to $B\Sigma_{m+n}$ by the group homomorphism $\Sigma_m \times \Sigma_n \to \Sigma_{m+n}$ that takes two permutations to the corresponding block permutation.

The hypotheses of the McDuff–Segal theorem hold because the two maps $\Sigma_n \rightrightarrows \Sigma_{m+n}$ that include Σ_n as either the first n letters or the last n letters, differ by a conjugation in Σ_{m+n}, and so they agree on homology. (In fact, the resulting maps $B\Sigma_n \rightrightarrows B\Sigma_{m+n}$ agree up to unbased homotopy. See Exercise 7.7.43.)

The theorem tells us that the group completion $\Omega B(B\mathbf{Fin}^\cong)$ has homology given by $\mathbb{Z}[t,t^{-1}]\otimes H_*(B\Sigma_\infty)$, because inverting t has the effect of taking the colimit over $H_*(B\Sigma_n)$ as $n\to\infty$. In other words, the K-theory space of $\mathbf{Fin}^\cong$ has homology given by the homology of the infinite symmetric group. Its homotopy turns out to be the stable homotopy groups of spheres $\pi_*(\mathbb{S})$, see Theorem 9.5.5.

Now we are ready to give the proof of the group completion theorem. We start with a preliminary result, which we prove in Section A.2.2.

Proposition 9.1.34. *[MS76, Prop 2] If M is a well-based monoid acting on the space X, and each $x\in M$ induces a homology isomorphism $H_*(X;\mathbb{Z})\to H_*(X;\mathbb{Z})$, then the map $B(*,M,X)\to B(*,M,*)$ is a **homology fibration**, in the sense that the map from its fiber to its homotopy fiber also induces an isomorphism on $H_*(-;\mathbb{Z})$.*

Proof of Theorem 9.1.26. Without loss of generality $\pi_0(M)$ is generated by some finite set of elements $[m_1], \ldots, [m_n]$. (If the theorem is proven in this case, it can be deduced in the general case by taking filtered colimits.) Multiplying the elements m_i together gives a single element $m = m_1\cdots m_n$, such that inverting m is enough to invert every element of $\pi_0(M)$. We define a space M_∞ as the mapping telescope of the maps $M\to M$ that multiply on the right by m:

$$M_\infty = \operatorname{hocolim}\left(M \xrightarrow{\cdot m} M \xrightarrow{\cdot m} M \xrightarrow{\cdot m} \cdots \right)$$

It is straightforward to show that the homology of M_∞ is $H_*(M;R)[\pi_0(M)^{-1}]$.

We extend the map $M \to \Omega BM$ to a zig-zag $M_\infty \to \Omega BM$ as follows. Notice that M_∞ still has a well-defined left action by M by multiplying on the left, so we may form the Borel construction $EM \times_M M_\infty = B(*, M, M_\infty)$. This is contractible because it can be rewritten as the homotopy colimit of the contractible spaces $B(*, M, M)$, and it has a projection map to $BM = B(*, M, *)$. The homotopy fiber of this map is therefore equivalent to ΩBM, so the inclusion of the strict fiber into the homotopy fiber gives a map $M_\infty \to \Omega BM$ in the homotopy category, extending the canonical map $M \to \Omega BM$.

It remains to show that $M_\infty \to \Omega BM$ is an isomorphism on homology with R coefficients. By the universal coefficient theorem, it suffices to take $R = \mathbb{Z}$. Then $M_\infty \to \Omega BM$ is an isomorphism on homology by Proposition 9.1.34, so long as we can prove that the left action of M on M_∞ is through maps that give isomorphisms on $H_*(M_\infty; \mathbb{Z})$. To prove this, we first pick any $x \in M$ and observe that the left action of x on M_∞, when restricted to any copy of M in the mapping telescope, can be inverted on homology: we pick a $y \in M$ such that $xy \sim m^n$ for some $n \geq 0$, and then see that the inverse of the left x-action is given by the right y-action and then multiplication by the inverse of m^n on homology. This uses the fact that x and y are central in $H_*(M)$. It also uses the fact that we can work one copy of M at a time.[4] □

Remark 9.1.35. The McDuff–Segal theorem (Theorem 9.1.26) also holds with local coefficients, if we make the slightly stronger assumption that M is commutative up to homotopy. Given any coefficient system A on ΩBM, writing A also for the pullback of this local coefficient system to M_∞, we have an isomorphism

$$H_*(M_\infty; A) \to H_*(\Omega BM; A).$$

A different way of saying this is that the map $M_\infty \to \Omega BM$ is acyclic. See [RW13] for a proof of how to extend McDuff and Segal's argument to this case. One can also pass from homology to extraordinary homology $E_*(M)$ where E is any associative ring spectrum, and one gets a similar group-completion theorem provided $\pi_0(M)$ is central in $E_*(M)$; see [Nik17] for a proof of this using technology from Lurie.

Warning 9.1.36. Some authors, e.g. [May75], define a "group completion" to be a map of H-spaces $M \to M'$ that satisfies the conclusion of Theorem 9.1.26. This is a useful *ad hoc* definition for computations, but you should be careful: it is only the group completion of M when $\pi_0(M)$ is central in $H_*(M; R)$ for all R. When that assumption fails, it is not the correct definition. The correct definition of the group completion is the universal approximation up to weak equivalence by a group (Definition 9.1.11).

[4] The action of x on the entire cylinder M_∞ isn't a homotopy equivalence in general, even if M is homotopy-commutative, because the homotopy inverses on each copy of M don't usually patch together to give a well-defined homotopy inverse on all of M_∞. If this always worked, then M_∞ would be weakly equivalent to ΩBM, but this is impossible by Exercise 9.6.11.

9.2 Abelian groups and spectra

Now that we understand a fair bit about simplicial and topological monoids, let's restrict attention to abelian groups. In this section we show how to make every topological abelian group or chain complex into a spectrum.

9.2.1 The Dold-Kan correspondence

The Dold-Kan correspondence is a famous result that tells us that simplicial abelian groups are the same thing as chain complexes, up to isomorphism. As a result, topological abelian groups are the same thing as chain complexes, up to weak equivalence.

Definition 9.2.1. For any simplicial abelian group $A_\bullet$, the **Moore complex** A_* is the chain complex

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \longrightarrow \cdots$$

such that ∂_n is the alternating sum of face maps, $\partial_n(a) = \sum_{i=0}^n d_i(a)$. The **normalized Moore complex** is the subcomplex $NA_* \leq A_*$ in which

$$NA_n = \bigcap_{i=1}^n \ker d_i.$$

On this smaller abelian group, the boundary map is now given by just the face map d_0, since all the other face maps vanish. The following lemma explains why this is a reasonable definition to make.

Lemma 9.2.2. *In the special case that $A_\bullet = \mathbb{Z}[K_\bullet]$ is the free simplicial abelian group on the simplicial set $K_\bullet$, NA_n is isomorphic to the free abelian group on the nondegenerate points K_n^{nd}, with boundary map the alternating sum $\partial_n(a) = \sum_{i=0}^n d_i(a)$.*

Proof. Using the latching cube from Proposition 8.2.18, we see that the face maps d_1 through d_n project onto the n degenerate copies of K_{n-1} inside K_n. Therefore their joint kernel can be identified with the free abelian group on the nondegenerate points. We make this identification by the composite

$$\bigcap_{i=1}^n \ker d_i \to \mathbb{Z}[K_n] \to \mathbb{Z}[K_n]/\mathbb{Z}[L_nK] \cong \mathbb{Z}[K_n^{nd}],$$

and check that along this isomorphism the alternating sum of the differentials on $\mathbb{Z}[K_n/L_nK]$ agrees with the differential d_0 on $\bigcap_{i=1}^n \ker d_i$. □

Lemma 9.2.3. *The inclusion $NA_* \to A_*$ is a chain homotopy equivalence.*

We skip the proof since we won't use it; see [GJ99, III.2.4].

Theorem 9.2.4 (Dold-Kan correspondence)**.** *The normalized chain complex functor*

$$N : \mathbf{sAb} \to \mathbf{Ch}_{\geq 0}(\mathbb{Z})$$

is an equivalence of categories. There is also a natural isomorphism between the homotopy groups of the space $|A_\bullet|$ and the homology groups of the complex NA_*. Therefore the weak equivalences in the two categories coincide.

We emphasize how strong this result is: we have an equivalence of categories, not just a Quillen equivalence or an equivalence of homotopy categories. Chain complexes and simplicial abelian groups aren't just the same thing up to weak equivalence or quasi-isomorphism, they are the same thing up to *isomorphism*. If you go forward and back, you get the same simplicial abelian group or chain complex back, up to isomorphism.

Proof. As with Brown representability, this proof can be found in many sources, for instance [GJ99, III.2] or [Wei13, 8.4.4], and so we give only a moderately detailed sketch. We go back from chain complexes to abelian groups by sending each chain complex C_* to the simplicial abelian group $\Gamma C_n = \bigoplus_{\beta : [n] \twoheadrightarrow [k]} C_k$, where the sum is taken over all surjective maps of totally ordered sets $[n] \twoheadrightarrow [k]$. For each map $\alpha : [m] \to [n]$, the composition $\beta \circ \alpha : [m] \to [k]$ may not be surjective, but its image is isomorphic as a totally ordered set to $[k']$ for some $k' \leq k$, and by restricting to this we get a smaller map of totally ordered sets $[m'] \twoheadrightarrow [k']$. We let α act on the sum by sending the summand corresponding to $[n] \to [k]$ to the summand for $[m'] \to [k']$, using the map $[k'] \to [k]$ to send C_k to $C_{k'}$. It is straightforward to check that this defines a simplicial abelian group. One can then define explicit isomorphisms $N\Gamma \cong \mathrm{id}$ and $\Gamma N \cong \mathrm{id}$ as in [GJ99, III.2] or [Wei13, 8.4.4].

The author finds it a bit more illuminating to think about why each simplicial abelian group $A_\bullet$ must be of the form $\bigoplus_{\beta : [n] \twoheadrightarrow [k]} C_k$ for some abelian groups C_k, by using the latching cube from Proposition 8.2.18. Inductively, the latching object has been shown to be $\bigoplus_{\beta : [n] \twoheadrightarrow [k]} C_k$ for all surjective maps $[n] \to [k]$ for $k < n$, in other words only the identity map $[n] \to [n]$ is missing. Furthermore we assume we have a retract from A_n back onto this direct sum, where the map from A_n to each C_k is obtained by combinations of face maps, never using d_0. We therefore have that the latching map

$$\bigoplus_{\beta : [n] \twoheadrightarrow [k], k < n} C_k \to A_n$$

is the inclusion of a summand, and we define C_n to be the complementary summand. It is isomorphic to the cokernel of the latching map, but it is also isomorphic to the intersection of the kernels of d_1 through d_n, because that joint kernel is exactly the elements that go

to zero under the retract that we have inductively defined. Now when we go to the next level, we define the latching map for A_{n+1} by taking the colimit of a larger punctured cube, and by rearranging the universal properties we see that it is again a direct sum of the desired form. By the proof of Proposition 8.2.18, we can always map back to any one of the C_k using face maps and never using d_0, and the induction is complete.

The structure of these latching cubes remembers all of the degeneracy maps and face maps of A_n, except for the 0th face map d_0. And d_0 is determined by what it does on each C_n, where it lands in C_{n-1} in the previous level. So this data is exactly enough to recover the simplicial abelian group $A_\bullet$.

Since $A_\bullet$ is a simplicial abelian group, it is a Kan complex by Lemma 9.1.9, and so it is fibrant as a simplicial set. Therefore each element of a homotopy groups of $|A_\bullet|$ based at the identity element $1 \in |A_\bullet|$ can be represented as a map out of any simplicial set that we choose, so long as its realization is equivalent to S^{n-1}. It is convenient to take $\Delta[n]/\partial\Delta[n]$. If we do so, then the element is represented by a single map $\Delta[n] \to A_\bullet$ that sends everything in the boundary to the identity element. This corresponds to an element of A_n for which every face map is zero. Since d_1 through d_n vanish, the element is in NA_n, and since d_0 vanishes it is a cycle in the chain complex NA_*. In other words, every map from an n-sphere to $|A_\bullet|$ is represented by an n-cycle in NA_*.

Similarly, two of these spheres $\alpha, \beta \colon \Delta[n]/\partial \to A_\bullet$ give the same element of $\pi_n(|A_\bullet|)$ precisely when there is a map $\Delta[n+1] \to A_\bullet$ sending the first two faces to α and β, and all other faces to zero. A similar but more elaborate manipulation shows that this is equivalent to having an element of C_{n+1} whose image under d_0 is the difference of the two cycles $[\alpha] - [\beta]$. This gives the isomorphism $\pi_n(|A_\bullet|) \cong H_n(NA_*)$, □

What's going on here? You should think of a simplicial abelian group as a highly redundant object: it has lots of needlessly repeated information. The same group C_k shows up multiple times at each simplicial level, starting in level k, and of all the possible maps we could have between these multiple copies of the group, we get only the identity map of C_k and the same map $C_k \to C_{k-1}$. Once we remove the redundancy and write down the minimal set of data we need to specify everything else, all we have left is a chain complex.

On the other hand, you shouldn't let this discourage you from using simplicial abelian groups. Even though they're highly redundant, they organize the information in such a way that you can do very different things than you could with a chain complex. Such as taking a geometric realization and getting a topological space!

Example 9.2.5. Let X be a topological space and let $C_*(X;\mathbb{Z})$ be its singular chain complex. This is exactly the Moore complex of the free simplicial abelian group $\mathbb{Z}[S_\bullet(X)]$. Passing to topological abelian groups, this corresponds (up to equivalence) to the free abelian group on X, in other words the tensor $X \otimes_{\mathrm{CAlg}} \mathbb{Z}$ defined in Exercise 8.7.45.

Therefore, along the Dold-Kan correspondence, the chains of a space X correspond to the free abelian group on X!

The Dold-Kan correspondence is an equivalence of categories along which the weak equivalences agree. In fact, the cofibrations and fibrations are preserved as well. To formulate this, recall from Exercise 5.8.1 that an **equivalence of model categories** is an equivalence of the underlying categories $\mathbf{C} \simeq \mathbf{D}$ along which a map in $\mathbf{C}$ is a cofibration, weak equivalence, or fibration iff its image in $\mathbf{D}$ is such. In an equivalence of model categories, both of the functors are left Quillen, and both are right Quillen.

Proposition 9.2.6. *The Dold-Kan correspondence is an equivalence of model categories, provided we give* **sAb** *the right-induced model structure from* **sSet***, and we give* $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$ *the projective model structure from Example 5.2.11.*

Proof. We have already seen that the weak equivalences agree because the homotopy groups of $A_\bullet$ are naturally isomorphic to the homology groups of NA_*. To prove the cofibrations agree, we check that the generating cofibrations I in each category go to cofibrations in the other. For **sAb**, the generating cofibrations are of the form $\mathbb{Z}[\partial\Delta[n]] \to \mathbb{Z}[\Delta[n]]$. On normalized chain complexes, this goes to a map of chain complexes that adds an extra $\mathbb{Z}$ at level n, and is therefore a pushout along one of the generating cofibrations $(0 \to \mathbb{Z}) \to (\mathbb{Z} \to \mathbb{Z})$. Going the other way, the map of chain complexes $(0 \to \mathbb{Z}) \to (\mathbb{Z} \to \mathbb{Z})$ corresponds to the map of simplicial abelian groups which is zero below simplicial level $(k-1)$, is the identity $\mathbb{Z} \to \mathbb{Z}$ at simplicial level $(k-1)$, is the inclusion $\mathbb{Z}^{\oplus(k-1)} \to \mathbb{Z}^{\oplus(k-1)} \oplus \mathbb{Z}$ at simplicial level k, and is a union of degenerate images of this map at all higher simplicial levels. This is a free abelian group on a cofibration of simplicial sets, namely $\Delta[k-1]/\mathrm{Sk}^{k-2} \to \Delta[k]/\mathrm{Sk}^{k-2}$, and so it is a cofibration of simplicial abelian groups.

Finally, the fibrations must agree because they are the maps with the right-lifting property along the acyclic cofibrations. □

The hiccup occurs when we examine monoidal structures. We may define a tensor product in **sAb** by taking the tensor product $\otimes_{\mathbb{Z}}$ at each simplicial level. For chain complexes C_* and D_*, the tensor product is defined as in Exercise 4.4.1 as

$$(C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes D_j, \quad d(a \otimes b) = (da) \otimes b + (-1)^{|a|} a \otimes (db).$$

These extend to symmetric monoidal structures on **sAb** and $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$.

Proposition 9.2.7. *The units of these symmetric monoidal structures agree, and the functor N is lax symmetric monoidal, with structure map $\nabla\colon N(A_\bullet) \otimes N(B_\bullet) \to N(A_\bullet \otimes B_\bullet)$ a quasi-isomorphism of chain complexes.*

Since we have an equivalence of categories, this is the same thing as saying that Γ is oplax symmetric monoidal, and its structure map $\nabla^*\colon \Gamma(C_* \otimes D_*) \to \Gamma(C_*) \otimes \Gamma(D_*)$ is a weak equivalence of simplicial abelian groups.

Proof. The map $\nabla\colon N(A_\bullet)\otimes N(B_\bullet) \to N(A_\bullet \otimes B_\bullet)$ is traditionally called the **Eilenberg–Zilber map** or the shuffle map. Geometrically, it corresponds to taking a product of simplices $\Delta^m \times \Delta^n$ and decomposing it into $(m+n)$ simplices, the ones that are nondegenerate in the product simplicial set $\Delta[m] \times \Delta[n]$. There is one such simplex for each ascending sequence of pairs $(0,0) = (i_0, j_0) < \cdots < (i_{m+n}, j_{m+n}) = (m,n)$ such that at each stage either i_a or j_a increases by one, and the other stays put. This is the same thing as a choice of subset of size m

$$\mu = \{\mu_1, \dots, \mu_m\} \subseteq \{0, \dots, m+n-1\},$$

which tells us in which m slots the first index increases, and its complementary subset $\nu = \{\nu_1, \dots, \nu_n\}$ which tells us when the second index increases.

The Eilenberg–Zilber map sends each product of simplices to the sum of these corresponding pieces, with appropriate signs. If you carefully write this out, you arrive at the formula

$$\nabla(a \otimes b) = \sum_{(\mu,\nu)} \operatorname{sign}(\mu, \nu)(s_\nu a) \otimes (s_\mu b),$$

see e.g. [SS03a, 2.3]. Here we think of the pair of subsets (μ, ν) as a "shuffle permutation" of $\{0, \dots, m+n-1\}$, in other words a permutation that preserves the ordering of each of the sets $\{1, \dots, m\}$ and $\{m+1, \dots, m+n\}$, sending them to the sets μ and ν respectively. The term s_ν is shorthand for the composition of degeneracy maps $s_{\nu_n} \cdots s_{\nu_1}$, and similarly $s_\mu = s_{\mu_m} \cdots s_{\mu_1}$. A straightforward verification shows that this gives a strong symmetric monoidal functor, both on the Moore complex A_* and the normalized version NA_*.

On any free simplicial abelian groups $\mathbb{Z}[X_\bullet]$ and $\mathbb{Z}[Y_\bullet]$, the Eilenberg–Zilber map passes between two different cell complex structures on the same topological space $|X_\bullet| \times |Y_\bullet| \cong |X_\bullet \times Y_\bullet|$ and respects the orientation of each cell, from which we compute that it is an isomorphism on homology. It follows that the map is a quasi-isomorphism on every simplicial abelian group. (One can also directly define a chain homotopy inverse, see [SS03a, 2.3].) □

Remark 9.2.8. The Eilenberg–Zilber map is often used to define the cross product on singular homology; you might have already seen it when you learned the cross product. We avoided using it in Theorem 1.3.11 because we had cellular homology available. In cellular homology, the cross product is much easier to define, because a product of cells is always a cell.

In summary, the tensor product is not preserved up to isomorphism, only up to equivalence! However, if we think of Γ as the left adjoint and N as the right adjoint, then (Γ, N) defines

a lax symmetric monoidal Quillen adjunction in the sense of Exercise 6.6.27. This is still enough to pass back and forth between the two settings:

Proposition 9.2.9. *The map induced by N on algebras, commutative algebras, and modules, has an inverse up to weak equivalence.*

This is left as an exercise, or see [SS03a, Thm 1.1]. A monoid or algebra object in **sAb** is usually called a **simplicial ring**, while a monoid or algebra object in $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$ is called a **differential graded algebra (DGA)**. So we now know that differential graded algebra corresponds to a simplicial ring, up to equivalence, and the modules over these two algebra objects also correspond up to equivalence.

Remark 9.2.10. It turns out that there is a second monoidal structure of the form $\Delta\colon N(A_\bullet \otimes B_\bullet) \to N(A_\bullet) \otimes N(B_\bullet)$ called the **Alexander–Whitney map**, see e.g. [SS03a, 2.3]. You may have seen this when you defined the cross product and cup product on singular *cohomology*, for instance in [Hat02, Ch. 3].

The Alexander-Whitney map makes N oplax monoidal but not oplax symmetric monoidal. Equivalently, the dual map Δ^* makes Γ lax monoidal but not lax symmetric monoidal. So it can be used to turn differential graded algebras into simplicial rings very easily, but it can't be applied to the commutative versions; for those we have to use N and just use the fact that some simplicial commutative ring $R_\bullet$ has $NR_\bullet$ *weakly equivalent* to our desired commutative differential graded algebra.

Remark 9.2.11. You can think of the failure of N to be strong monoidal as a failure of Milnor's theorem (Theorem 7.1.31) at the level of cellular chain complexes. The tensor product in chain complexes corresponds intuitively to products of cell complexes, where a product of two cells is a cell. The tensor product in simplicial abelian groups corresponds to products of simplicial sets, in which a product of two simplices is *not* a simplex, but rather a signed sum of $\binom{m+n}{m}$ simplices. The Eilenberg–Zilber map takes each product of simplices "as cells" and rewrites it as a product "as simplices."

This intuition becomes more precise when we restrict attention to *free* simplicial abelian groups $\mathbb{Z}[X_\bullet]$ and $\mathbb{Z}[Y_\bullet]$. Then the tensor product in chain complexes corresponds to the cellular chain complex of the product CW complex $|X_\bullet| \times |Y_\bullet|$, while the tensor product in simplicial abelian groups corresponds to the cellular chain complex of the CW complex $|X_\bullet \times Y_\bullet|$. These two CW complexes are homeomorphic, but their cellular chain complexes are not isomorphic, only equivalent!

You could also think about $|X_\bullet| \times |Y_\bullet|$ as the realization of a bisimplicial set, which has a product cell $\Delta^m \times \Delta^n$ for each pair of cells in the two factors. The cell complex $|X_\bullet \times Y_\bullet|$ is instead the realization of the corresponding diagonal simplicial set, with one cell Δ^n for each pair of n-cells in the two factors. Passing to chain complexes, we have a bicomplex and its diagonal complex. A variant of the Eilenberg–Zilber map tells us that the total

complex of this bicomplex is equivalent, but not isomorphic, to its diagonal. (This is not true for all bicomplexes, only the ones that are products of two complexes.)

9.2.2 Delooping an abelian group

Now that we've shown that nonnegatively-graded chain complexes $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$ are the same thing as (simplicial or topological) abelian groups up to equivalence, we can turn all of these objects into spectra. The idea is simple: just apply the bar construction to A repeatedly, forming a spectrum that we call $B^\infty A$. Each time we take this bar construction we form a "delooping" in the following sense.

Definition 9.2.12. For any based space X, a **delooping** of X is a based space Y and a weak equivalence $X \simeq \Omega Y$. Since Ω shifts homotopy groups down by 1, we have:

$$\begin{aligned} \pi_0(Y) &\cong ? \\ \pi_1(Y) &\cong \pi_0(X) \\ \pi_2(Y) &\cong \pi_1(X) \\ \vdots &\cong \vdots \end{aligned}$$

So if Y is a connected delooping of X, then the homotopy groups of Y are those of X shifted up by one, with no additional data added beneath because $\pi_0(Y) = *$.

Example 9.2.13. By Lemma 9.1.14, for any well-based topological group or grouplike monoid G, the space BG is a connected delooping of G. In other words, BG is connected and $G \simeq \Omega BG$. So BG has the same homotopy groups as G, shifted up by one.

For instance, the circle group S^1 has only one homotopy group, a $\mathbb{Z}$ in degree one. Therefore BS^1 has only one homotopy group, a $\mathbb{Z}$ in degree two. By the uniqueness of Eilenberg–Mac Lane spaces, therefore $BS^1 \simeq K(\mathbb{Z}, 2) \simeq \mathbb{CP}^\infty$. In other words, $\mathbb{CP}^\infty$ is a delooping of S^1.

If A is a topological abelian group, then we have already observed that BA is again a topological abelian group. So, other than our sense of propriety, there is nothing stopping us from applying the bar construction to BA, giving a space B^2A. This is again a topological abelian group, so we can apply B again, and so on ad infinitum!

In fact, we can do this if A is just a topological abelian monoid. The only thing we lose in that case is that we only get a map $A \to \Omega BA$, not an equivalence, so this isn't really a delooping in the sense of Definition 9.2.12.

Definition 9.2.14. The **delooping spectrum** $B^\infty A$ of a topological abelian monoid A is the sequential spectrum whose nth level is $B^n A$, and the adjunct bonding map is the canonical map $B^n A \to \Omega B(B^n A)$. We usually assume that A is well-based so that this has the homotopy type of its left-derived functor $\mathbb{L}B^\infty$.

By Lemma 9.1.14, when A is grouplike (and well-based) this is an Ω-spectrum, and so it really is a sequence of deloopings of A. By the properties of Ω-spectra from Section 2.2, the stable homotopy groups of $B^\infty A$ are just the homotopy groups of the space A, and the infinite loopspace is just A again, $A \simeq \Omega^\infty B^\infty A$.

If A is not grouplike, then the map $A \to \Omega BA$ is not an equivalence, but all the rest of the adjunct bonding maps are still equivalences. This is because BA, B^2A, and so on are all *connected*, and so they are *automatically* grouplike. So $BA \to \Omega B^2 A$ is an equivalence, and so on. In other words, $B^\infty A$ is always a positive Ω-spectrum in the sense of Theorem 6.4.2. In total we have proven:

Lemma 9.2.15. *If A is any well-based topological abelian monoid, then $B^\infty A$ is a positive Ω-spectrum whose infinite loopspace is the group completion of A.*

Example 9.2.16. Supppose A is a plain old discrete abelian group. Then $\pi_0(A) = A$ and all of the higher homotopy groups vanish. By delooping A repeatedly, we see that $B^n A$ has $\pi_n \cong A$, and all other homotopy groups are zero. In other words, $B^n A$ is an Eilenberg–Mac Lane space $K(A,n)$, and $B^\infty A$ is a model for the Eilenberg–Mac Lane spectrum:

$$HA \simeq B^\infty A.$$

Corollary 9.2.17. *If A is a discrete abelian group and $n \geq 0$, the Eilenberg–Mac Lane space $K(A,n)$ can be given the structure of a topological abelian group, such that the equivalences $K(A,n) \to \Omega K(A,n+1)$ are homomorphisms of topological abelian groups.*

Actually, this is the only interesting case. Any other topological abelian group turns out to be a product of Eilenberg–Mac Lane spaces, so its delooping is a product of shifted Eilenberg–Mac Lane spectra.

Lemma 9.2.18. *Any topological abelian group A is equivalent in* CGrp(**Top**) *to a product of Eilenberg–Mac Lane spaces $K(\pi_n(A), n) = B^n(\pi_n(A))$.*

Proof. It is easiest to prove this using the Dold-Kan correspondence (Theorem 9.2.4). We only have to show that every chain complex admits a zig-zag of quasi-isomorphisms to a sum of one-term chain complexes. This is straightforward, using the fact that $\mathbb{Z}$ is a principal ideal domain and so every abelian group can be presented by a two-term resolution as in (1.3.9). □

Corollary 9.2.19. *The delooping spectrum $B^\infty A$ of any topological abelian group is equivalent to the sum (or product) of the Eilenberg–Mac Lane spectra $\Sigma^n H(\pi_n(A))$.*

Proof. We have to do a little work here because the bar construction only preserves finite products. The projection of A to the product $K(\pi_0(A),0) \times \cdots \times K(\pi_n(A),n)$ is $(n-1)$-connected, so on B^k it induces an $(n+k-1)$-connected map of spaces, and therefore on

B^∞ it induces an $(n-1)$-connected map of spectra. From this we can deduce that the projection of A to the infinite product induces a map of spectra that is $(n-1)$-connected for all n, and is therefore a stable equivalence. □

Note that when adding together one Eilenberg–Mac Lane spectrum for each integer, we can take either the wedge sum or the product, and we get the same result, by Exercise 3.6.25.

In summary, we have constructed a string of functors that takes each chain complex to a spectrum:

$$\begin{array}{ccccccc} \mathbf{Ch}_{\geq 0}(\mathbb{Z}) & \xrightarrow{\Gamma} & \mathbf{sAb} & \xrightarrow{|-|} & \mathrm{CGrp}(\mathbf{Top}) & \xrightarrow{B^\infty} & \mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}} \\ C_* & \longmapsto & \Gamma C_* & \longmapsto & |\Gamma C_*| & \longmapsto & B^\infty|\Gamma C_*|. \end{array}$$

Furthermore the resulting spectrum is a product of Eilenberg–Mac Lane spectra, and its stable homotopy groups are exactly the homology groups of the original chain complex C_*. We will also see soon that the spectrum is also a module over the ring spectrum $H\mathbb{Z}$ in a natural way. This is how homological algebra "sits inside" stable homotopy theory.

One might be upset about this, because this comes nowhere close to capturing all spectra. But actually, this is a win for spectra. It illustrates how much broader the category of spectra is compared to the category of chain complexes. There are so many more spectra, and the ones that aren't products of Eilenberg–Mac Lane spectra are the ones that the world of chain complexes can't see.

9.2.3 Modeling the delooping using configuration spaces

There is another way to think about the delooping spectrum that is more geometric. Recall from Exercise 8.7.44 that the category of topological abelian monoids $\mathrm{CMon}(\mathbf{Top})$ is enriched, tensored, and cotensored over **Top**. In particular, the space of continuous homomorphisms $\mathrm{Hom}(A,A')$ has a topology, and there is a tensoring functor

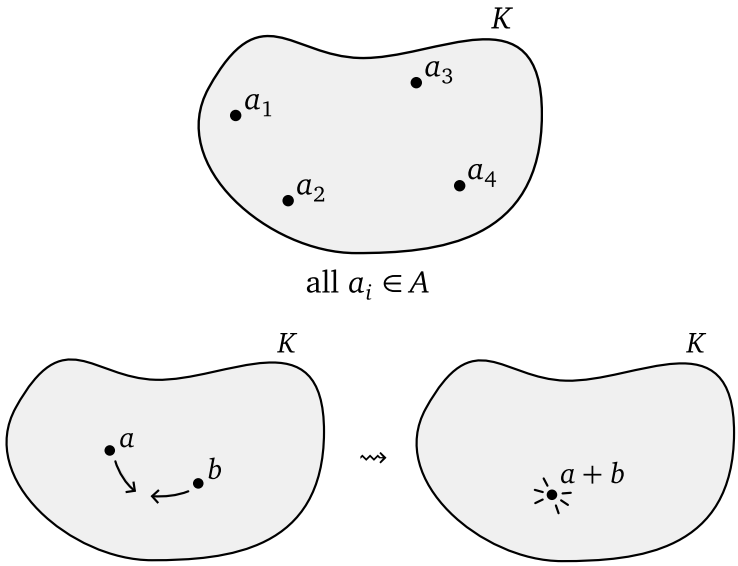
$$\otimes_{\mathrm{CMon}} : \mathbf{Top} \times \mathrm{CMon}(\mathbf{Top}) \to \mathrm{CMon}(\mathbf{Top})$$

such that for each unbased space K, maps of abelian groups $K \otimes_{\mathrm{CMon}} A \to A'$ correspond to maps of unbased spaces $K \to \mathrm{Hom}(A,A')$. Concretely, $K \otimes_{\mathrm{CMon}} A$ is formed as the disjoint union of products

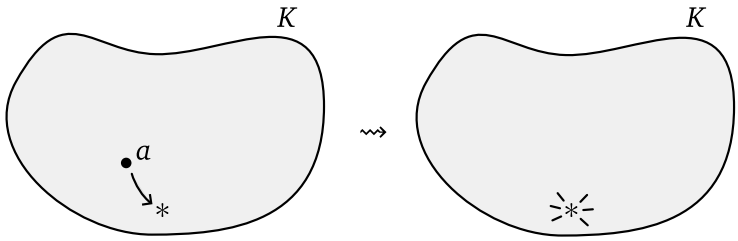
$$\coprod_{n\geq 0}\left(\prod_{i=1}^{n}(K\times A)\right),$$

modulo relations of the form $(k,a)(\ell,b) = (\ell,b)(k,a)$, $(k,a)(k,b) = (k,a+b)$, and $(k,0) = \varnothing$, applied inside any longer string of pairs of the form $(k_1,a_1)\cdots(k_n,a_n)$. Informally, this is a "configuration space" of points in K labeled by elements of A, where points can collide

and add their labels, and points disappear if they are labeled by the identity element $0 \in A$.



There is a based variant of this construction: the set of group homomorphisms $\mathrm{Hom}(A, A')$ is a based space, and there is a based tensor product $K \tilde{\otimes}_{\mathrm{CMon}} A$, such that maps $K \tilde{\otimes}_{\mathrm{CMon}} A \to A'$ correspond to based maps $K \to \mathrm{Hom}(A, A')$. Concretely, $K \tilde{\otimes}_{\mathrm{CMon}} A$ is formed as above but with the additional relation $(*, a) = 0$. More formally, it is the quotient of $K \otimes_{\mathrm{CMon}} A$ by $* \otimes_{\mathrm{CMon}} A = A$ in the category $\mathrm{CMon}(\mathbf{Top})$. And pictorially, this is just a configuration space in which points that go to $* \in K$ disappear:



Remark 9.2.20. The category of topological abelian *groups* $\mathrm{CGrp}(\mathbf{Top})$ also has a tensor and reduced tensor with exactly the same formula, by Exercise 8.7.46. Indeed, when A is an abelian group, the above constructions always output abelian groups.

This might seem like a wild thing to think about, but it is actually the construction that we have been considering all along:

Lemma 9.2.21. *If A is a topological abelian monoid, the iterated bar construction $B^n A$ is homeomorphic to the reduced tensor $S^n \tilde{\otimes}_{\mathrm{CMon}} A$.*

Proof. Since we have $S^n \tilde{\otimes}_{\mathrm{CMon}} A \cong S^1 \tilde{\otimes}_{\mathrm{CMon}} (S^{n-1} \tilde{\otimes}_{\mathrm{CMon}} A)$, it suffices to prove this when $n = 1$. In that case the topological tensor is the realization of the simplicial tensor by Exercise 8.7.34, which is the realization of $S^1_\bullet \otimes_{\mathrm{CMon}} A$. And this last simplicial set is exactly the simplicial nerve of A, so its realization is the bar construction BA. □

Along this correspondence, the loopspace corresponds to the cotensor of A with S^n, and the map $A \to \Omega^n B^n A$ is exactly the unit of this adjunction.

Corollary 9.2.22. *The delooping spectrum $B^\infty A$ is an orthogonal spectrum.*

The $O(n)$-action is the usual one on S^n in the tensor product $S^n \otimes_{\mathrm{CMon}} A$. Thinking geometrically, it applies the $O(n)$-action to the points in S^n that are labeled by elements of A, moving the points around.

Lemma 9.2.21 allows us to think of the delooping spectrum $B^\infty A$ as a kind of suspension spectrum of A, where instead of using the smash product of topological spaces $S^n \wedge A$, we use the tensor of topological spaces with abelian groups, $S^n \tilde{\otimes}_{\mathrm{CMon}} A$. It's important to remember that this is only a conceptual similarity. The spectrum $B^\infty A$ is *not* stably equivalent to $\Sigma^\infty A$.

In fact, while the homology of $\Sigma^\infty A$ is just the homology of A, the homology of $B^\infty A$ is only the "generators" of the homology of A. To explain what this means, notice that A has a commutative multiplication, and so $H_*(A)$ is a ring under the Pontryagin product. For simplicity, let's assume it is a free graded-commutative algebra on some generators x_i. Then by a spectral sequence argument, $H_*(BA)$ is also a free graded-commutative algebra on the same generators, except that each generator has been bumped up in degree by one, so that if it generated a polynomial algebra before then now it generates an exterior algebra, and vice-versa. See **??**.

As we apply the bar construction, we get a map $H_i(A) \to H_{i+1}(BA)$, and this map turns out to be an isomorphism on the generators but zero on all of their products, essentially for degree reasons. Therefore in the colimit, all of the products of the generators "boil away," and we are left with just the generators themselves. We will return to this in more detail in **??**.

9.2.4 Delooping is lax symmetric monoidal

Next we discuss monoidal structure on CGrp(**Top**) and CGrp(**sSet**) = **sAb**. These categories already have Cartesian products, which are also coproducts by Lemma 6.5.27.[5] In other words, a continuous homomorphism out of $A \times C$ is the same thing as a continuous

[5] Strictly speaking, Lemma 6.5.27 only proves that $\times$ is the coproduct in the category of commutative monoids, but it quickly follows that it is also the coproduct in commutative groups.

homomorphism out of A and C separately, when A and C are both topological abelian groups.

There is another more important monoidal structure in which we take the tensor product:

Definition 9.2.23. For any two topological abelian groups A and C, the tensor product $A \otimes_{\mathbb{Z}} C$ is defined as the quotient in **Top** of the space

$$\coprod_{n\geq 0}\left(\prod_{i=1}^{n}(A\times C)\right)$$

by all relations of the form $(a+a',c)=(a,c)(a',c)$, $(a,c+c')=(a,c)(a,c')$, $(a,c)(a',c')=(a',c')(a,c)$, and $(a,0)=(0,c)=0$, applied inside any longer string of pairs of the form $(a_1,c_1)\cdots(a_n,c_n)$.[6] In other words, it is the usual construction of the tensor product, with the most sensible way of giving it a topology.

We sometimes write this just as $A\otimes C$. Note the similarity of this definition to the tensor of A with a space K from Section 9.2.3. In fact, if C is the free group on the topological space K, then we have an isomorphism

$$A\otimes_{\mathbb{Z}} C \cong K\otimes_{\mathrm{CMon}} A.$$

For simplicial abelian groups $A_\bullet$ and $C_\bullet$, recall that we defined $A_\bullet\otimes_{\mathbb{Z}} C_\bullet$ more simply as the tensor product on each simplicial level, $A_n\otimes_{\mathbb{Z}} C_n$.

Lemma 9.2.24. *Geometric realization* **sAb** $\to$ CGrp(**Top**) *is strong symmetric monoidal, in other words it preserves these tensor products up to canonical isomorphism.*

Proof. The tensor product of the simplicial abelian groups $A_\bullet$ and $C_\bullet$ is equivalently defined as the quotient of

$$\coprod_{n\geq 0}\left(\prod_{i=1}^{n}(A_\bullet\times C_\bullet)\right)$$

by all relations of the form $(a+a',c)=(a,c)(a',c)$, $(a,c+c')=(a,c)(a,c')$, $(a,c)(a',c')=(a',c')(a,c)$, and $(a,0)=(0,c)=0$, because this can all be done at each simplicial level separately, and at any one simplicial level this is just the definition of the tensor product. Since the realization preserves coproducts, finite products, and coequalizers, the realization of this gives the same formula as in Definition 9.2.23. □

Letting B denote the bar construction, we have

Lemma 9.2.25. *$A\otimes BC\cong B(A\otimes C)\cong BA\otimes C$, and more generally $B^mA\otimes B^nC\cong B^{m+n}(A\otimes C)$.*

[6] Alternatively, one can apply F_{CGrp} to the space $A\times C$ and then mod out by only the relations $(a+a',c)=(a,c)(a',c)$ and $(a,c+c')=(a,c)(a,c')$.

Proof. This is by comparing definitions: all three correspond to the quotient of

$$\coprod_{n\geq 0}\left(\prod_{i=1}^{n}(A\times S^1\times C)\right)$$

by the same relations. The more general formula follows from the same argument with S^m and S^n in the place of S^1. □

Proposition 9.2.26. *The delooping spectrum $B^\infty\colon \mathrm{CGrp}(\mathbf{Top})\to \mathbf{Sp}^O_{\mathbf{Top}}$ is lax symmetric monoidal.*

Proof. We construct maps of the form

$$B^\infty A\wedge B^\infty C\to B^\infty(A\otimes C)$$

for topological abelian groups A and C. It suffices to give a bimorphism of spectra in the sense of Definition 6.2.9. We construct the maps

$$B^m A\wedge B^n C\to B^m A\otimes B^n C\to B^{m+n}(A\otimes C)$$

by the canonical map from the smash product to the tensor product, and then the isomorphism of the previous lemma. It is then a long bookkeeping exercise to see that these create a bimorphism of spectra, and the resulting isomorphisms are coherent. □

This proves part of the black box that we promised in Example 4.1.31, that the Eilenberg–Mac Lane spectrum functor $H(-)$ is lax symmetric monoidal. As a special case, we have proven that

Proposition 9.2.27. *$H\mathbb{Z}=B^\infty\mathbb{Z}$ is a ring spectrum. More generally, the Eilenberg–Mac Lane spectrum functor $H(-)\colon \mathbf{Ab}\to \mathbf{Sp}^O_{\mathbf{Top}}$ is lax symmetric monoidal.*

Even more generally, since $B^\infty(-)$ is lax symmetric monoidal, it takes topological rings to ring spectra and modules to module spectra. So for every topological abelian group A, the delooping spectrum $B^\infty A$ is an $H\mathbb{Z}$-module spectrum in a canonical way.

Corollary 9.2.28. *Every DGA creates a ring spectrum via $B^\infty|\Gamma(-)|$, and every commutative DGA is equivalent to the Moore complex of a simplicial commutative ring, which then creates the ring spectrum via $B^\infty|-|$. Every module over any such ring becomes a module over its corresponding ring spectrum.*

We will see later that this gives an equivalence between chain complexes and $H\mathbb{Z}$-module spectra, and more generally between modules over any DGA and modules over the corresponding ring spectrum.

9.3 Operads and homotopy-coherent algebra

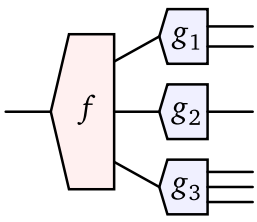
Now that we have reviewed the theory of monoids and groups, we are ready to tackle the homotopy-coherent versions of monoids and groups. Amazingly, once we pass from abelian groups to their homotopy-coherent versions, we are suddenly able to see all of the richness of stable homotopy theory: the delooping spectra $B^\infty A$ capture all possible connective spectra!

9.3.1 Definition of an operad

The key concept we need is an operad. Informally, an operad is a collection of operations that have multiple inputs and one output, and that can be composed with each other. Operads allow us to do homotopy-coherent algebra because the operations organize into topological spaces, and so two different operations can have a homotopy between them.

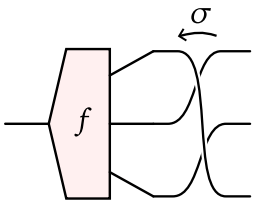
To motivate the definition, let's think a bit about the definition of a category. In a category, a morphism is a formal, abstract function from one object to another. It doesn't actually have to be a function. It's just a symbol. The composition in the category is a multiplication rule on these symbols, that behaves formally like composition of actual functions.

Let's do the same thing for functions that take multiple inputs, and give one output. So we should think about a category where there is a formal, abstract notion of an n-ary function, that takes n inputs and gives one output. It's not actually a function, it's just a symbol. Associated to this symbol are n objects of the category where the n inputs live, and one object where the output lives. And then we can ask for a composition rule on these new kind of symbols, that behaves just like the composition of multivariable functions.



How exactly does that composition behave? Well, if we have one function f with n inputs, and n functions $(g_1, \ldots, g_n)$ such that g_i has k_i inputs, then there should be a composition $f \circ (g_1, \ldots, g_n)$ that takes $k_1 + \cdots + k_n$ inputs, and returns a single output, as shown. This composition should be associative when we compose three successive waves of functions together, and there should be an identity function with one input.

If you think about it a little more, the fact that we write the inputs in a certain order also shouldn't matter. A given function from a pair of inputs (x, y) to an output $f(x, y)$ also defines a function from the pair (y, x) to the output $f(x, y)$. So these symbols should also have a symmetric group action that "permutes the inputs around."



Now we can derive the definition of an operad. Just specialize to the case that our category has only one object, so the n inputs and one output of each function always lie in the same set X. So now we have a single set of formal n-ary functions for each $n \geq 0$, and some abstract composition maps relating them together. (We'll return to the case of many objects in Definition 9.4.43.)

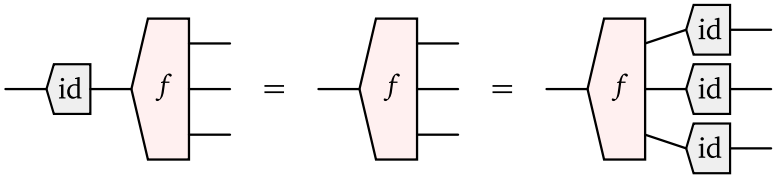
Definition 9.3.1 ([May72, Def 1.1])**.** A symmetric operad of topological spaces, or just an **operad** for short, is

- a sequence of topological spaces $\mathcal{O}(0)$, $\mathcal{O}(1)$, ...,
- a continuous right action of the symmetric group Σ_n on $\mathcal{O}(n)$ for each $n \geq 0$,
- an identity element $\mathrm{id} \in \mathcal{O}(1)$, and
- continuous composition maps

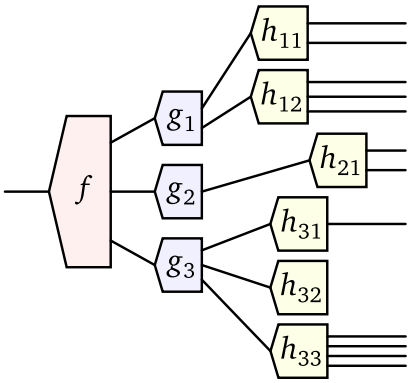
$$\circ\colon \mathcal{O}(n) \times \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_n) \longrightarrow \mathcal{O}\Big(\sum_i k_i\Big),$$

such that the following equations hold whenever they make sense:

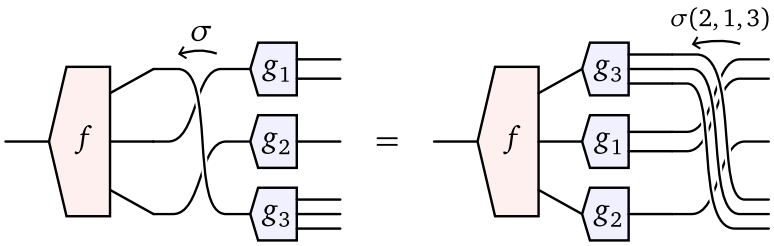
- (Identity) $\mathrm{id} \circ f = f = f \circ (\mathrm{id}, \ldots, \mathrm{id})$,



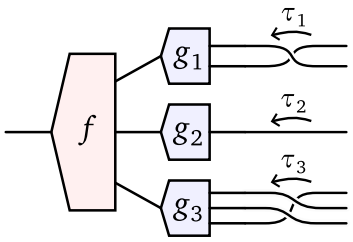
- (Associativity) $(f \circ (g_1, \ldots, g_n)) \circ (h_{11}, \ldots, h_{1k_1}, h_{21}, \ldots, h_{2k_2}, \ldots, h_{n1}, \ldots, h_{nk_n})$
$= f \circ (g_1 \circ (h_{11}, \ldots, h_{1k_1}), g_2 \circ (h_{21}, \ldots, h_{2k_2}), \ldots, g_n \circ (h_{n1}, \ldots, h_{nk_n}))$,



- (Symmetry) $(f\sigma)\circ(g_1,\dots,g_n) = (f\circ(g_{\sigma^{-1}(1)},\dots,g_{\sigma^{-1}(n)}))\sigma(k_1,\dots,k_n)$,



and $f\circ(g_1\tau_1,\dots,g_n\tau_n) = (f\circ(g_1,\dots,g_n))(\tau_1 \amalg \cdots \amalg \tau_n)$.



Here we have $f \in \mathcal{O}(n)$, $g_i \in \mathcal{O}(k_i)$ for $1 \le i \le n$ and k_i any integer, and $h_{ij} \in \mathcal{O}(\ell_{ij})$ for $1 \le j \le k_i$ and ℓ_{ij} any integer. We also have $\sigma \in \Sigma_n$, $\tau_i \in \Sigma_{k_i}$, $\sigma(k_1,\dots,k_n)$ is the permutation of $(\sum_i k_i)$ letters that moves the blocks of size k_1, …, k_n according to the permutation σ, and $(\tau_1 \amalg \cdots \amalg \tau_n)$ is the permutation of $(\sum_i k_i)$ letters that applies each τ_i to its own block of size k_i inside the set of size $\sum_i k_i$.

The three axioms express an identity property of the operation $\mathrm{id} \in \mathcal{O}(1)$, an associativity property for the composition, and a symmetry property that the σ-action "permutes the inputs" of f, as illustrated above.

A map of operads $\mathcal{O} \to \mathcal{O}'$ consists of maps of spaces $\mathcal{O}(n) \to \mathcal{O}'(n)$ for all n, commuting with the Σ_n-action and the composition map, and preserving the identity. The category of operads is denoted $\mathrm{Op}(\mathbf{Top})$. We similarly define $\mathrm{Op}(\mathbf{C})$ for any symmetric monoidal category $\mathbf{C}$ by asking that the levels $\mathcal{O}(n)$ be objects of $\mathbf{C}$, and using the tensor product in $\mathbf{C}$ instead of the Cartesian product when defining the composition map:

$$\circ\colon \mathcal{O}(n)\otimes\mathcal{O}(k_1)\otimes\cdots\otimes\mathcal{O}(k_n) \longrightarrow \mathcal{O}\Big(\sum_i k_i\Big).$$

We focus on the case of operads in topological spaces. We call the elements $f \in \mathcal{O}(n)$ "operations," and we think of them as abstract functions that take n inputs and give one output. They are abstract because there isn't actually a function from some actual set to another set, just a symbol that takes the place of a function, and a formal rule for how to compose these symbols together. Since the $\mathcal{O}(n)$ are topological spaces, we can also deform one operation to another, by following a path in the space $\mathcal{O}(n)$.

Now let's be honest. Definition 9.3.1 is intimidating. It is the natural end result of our intuition about how functions should compose, but somehow when we write it out in this way, we encounter a mess of notation and symmetric group actions. At times this definition can be hard to use, and we can get lost in the shuffle of these permutations.[7] It's useful to reformulate this so that it more closely matches our intuition, and hides the symmetric group actions, while still being explicit enough to check directly in examples.

Definition 9.3.2 ([Lur17, 2.1.1.1])**.** A **coordinate-free operad** is

- an assignment of each finite set I to a topological space $\mathcal{O}(I)$,
- an identity element $\mathrm{id}_i \in \mathcal{O}(\{i\})$ for each singleton $\{i\}$, and
- composition maps
$$\circ\colon \mathcal{O}(J) \times \prod_{j\in J} \mathcal{O}(\alpha^{-1}(j)) \longrightarrow \mathcal{O}(I)$$
for each map of finite sets $\alpha\colon I \to J$,

such that the following equations hold whenever they make sense:

- (Identity) $\mathrm{id}_j \circ f = f$ when $f \in \mathcal{O}(I)$ and the composition is along the unique map $I \to \{j\}$, and $f = f \circ (\mathrm{id}_i)_{i\in I}$ with composition along the identity map $I \to I$,
- (Associativity) for two composable maps $\alpha\colon I \to J$ and $\beta : J \to K$ and elements $f \in \mathcal{O}(K)$, $g_k \in \mathcal{O}(\beta^{-1}(k))$, and $h_j \in \mathcal{O}(\alpha^{-1}(j))$ we have
$$(f \circ (g_k)_{k\in K}) \circ (h_j)_{j\in J} = f \circ (g_k \circ (h_j)_{j\in\beta^{-1}(k)})_{k\in K}.$$

A map of operads $\mathcal{O} \to \mathcal{O}'$ again consists of maps $\mathcal{O}(I) \to \mathcal{O}'(I)$ commuting with the composition map and preserving the identity elements. We similarly define coordinate-free operads in any symmetric monoidal category $\mathbf{C}$ by asking that the levels $\mathcal{O}(I)$ be objects of $\mathbf{C}$, instead of topological spaces.

The symmetric group actions and their axiom have disappeared! In fact, they have been subsumed into the composition map, which is now performing the work of both the Σ_n-action and the composition in the original definition of an operad.

More explicitly, given a coordinate-free operad $\mathcal{O}$, we recover an ordinary operad by taking the spaces $\mathcal{O}(\{1,\dots,n\})$. Each permutation σ sends f to $f\sigma := f \circ (\mathrm{id})_{i\in I}$, composed along the function $\sigma\colon I \to I$, and we define the composition $f \circ (g_1,\dots,g_n)$ to be the coordinate-free composition along the map of finite sets $\{1,\dots,\sum_i k_i\} \to \{1,\dots,n\}$ that sends the first k_1 letters to 1, the next k_2 letters to 2, and so on.

[7] Pun intended.

Proposition 9.3.3. *This procedure defines an equivalence of categories between coordinate-free operads and operads.*

So we will also use Op(**Top**) to refer to the category of coordinate-free operads, when it is convenient for us to do so. The proof of Proposition 9.3.3 is left to Exercise 9.6.15.

Why should this be true? Doesn't a coordinate-free operad have way more information? It does, but whenever there is a bijection $I \cong J$, the spaces $\mathcal{O}(I)$ and $\mathcal{O}(J)$ are homeomorphic. So the extra information is actually redundant. But it is sometimes easier to work with coordinate-free operads in practice, because all of the permutations are handled implicitly in the composition along the functions $I \to J$.

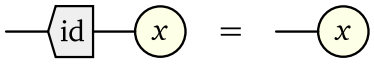
We are almost ready to give examples of operads. But we should first explain what it means to concretely realize the abstract family of operations in $\mathcal{O}$.

Definition 9.3.4 ([May72, Lem 1.4])**.** An **algebra** over the operad $\mathcal{O}$ consists of

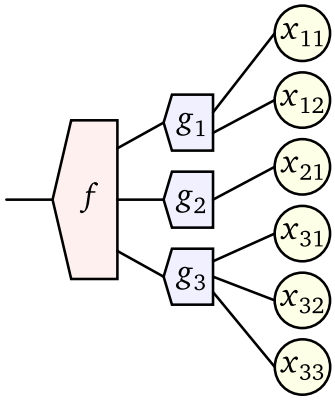
- a topological space X and
- action maps $\mathcal{O}(n) \times X^{\times n} \to X$

such that the following analogues of the identity, associativity, and symmetry axioms of an operad hold:

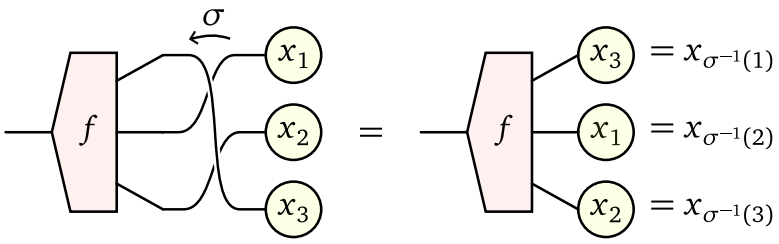
- (Identity) $\mathrm{id}(x) = x$,



- (Associativity) $(f \circ (g_1, \dots, g_n))(x_{11}, \dots, x_{1k_1}, x_{21}, \dots, x_{2k_2}, \dots, x_{n1}, \dots, x_{nk_n})$
$= f(g_1(x_{11}, \dots, x_{1k_1}), g_2(x_{21}, \dots, x_{2k_2}), \dots, g_n(x_{n1}, \dots, x_{nk_n}))$,



- (Symmetry) $(f\sigma)(x_1,\dots,x_n) = f(x_{\sigma^{-1}(1)},\dots,x_{\sigma^{-1}(n)})$.



Again, to be more specific, $f \in \mathcal{O}(n)$ and $f(x_1,\dots,x_n)$ denotes the output of the action map on the element $(f, x_1,\dots,x_n) \in \mathcal{O}(n)\times X^{\times n}$. Furthermore, as before we have $\sigma \in \Sigma_n$, $g_i \in \mathcal{O}(k_i)$ for $1 \le i \le n$, and the remaining elements x_i and x_{ij} are all points in X.

In the coordinate-free case, we ask for action maps $\mathcal{O}(I)\times X^I \to X$ such that

- (Identity) $\mathrm{id}_a(x) = x$, and
- (Associativity) for any $\alpha\colon I \to J$, $f \in \mathcal{O}(J)$, $g_j \in \mathcal{O}(\alpha^{-1}(j))$, and $(x_i)_{i\in I} \in X^I$ we have
$$(f \circ (g_j)_{j\in J})(x_i)_{i\in I} = f(g_j(x_i)_{i\in\alpha^{-1}(j)})_{j\in J}.$$

A map of $\mathcal{O}$-algebras is a map of spaces $X \to X'$ commuting with the action maps. The **category of $\mathcal{O}$-algebras** is denoted $\mathrm{Alg}_{\mathcal{O}}(\mathbf{Top})$.

This can all be generalized to any symmetric monoidal category $\mathbf{C}$, where an algebra over an operad in $\mathrm{Op}(\mathbf{C})$ is an object $X \in \mathbf{C}$ with action maps $\mathcal{O}(n)\otimes X^{\otimes n} \to X$ satisfying the same properties. The category of such algebras is denoted $\mathrm{Alg}_{\mathcal{O}}(\mathbf{C})$.

9.3.2 Examples of operads

Example 9.3.5. The **associative operad** $\mathrm{Assoc}(n) = \Sigma_n$. We can think of this as an operad in sets, $\mathrm{Assoc} \in \mathrm{Op}(\mathbf{Set})$, or consider Σ_n as a discrete space so that $\mathrm{Assoc} \in \mathrm{Op}(\mathbf{Top})$. The right action of Σ_n is by multiplication on the right, the identity operation is the unique element of Σ_1, and the composition $f \circ (g_1,\dots,g_n)$ is the permutation of $\sum_i k_i$ letters that first applies each g_i to the corresponding block of k_i letters, and then uses f to permute the blocks around.

It is straightforward to check that this defines an operad, or to define the corresponding coordinate-free version in which $\mathrm{Assoc}(I)$ is the set of permutations of the set I.

This is called the "associative operad" because it describes all possible operations you can generate out of a single associative and unital multiplication $X\times X \to X$. You can

compose the multiplication with itself, and because it is associative, this gives you a unique operation $X^{\times n} \to X$ for each value of $n \geq 0$. But then you could also permute the inputs before you multiply them, and in general this changes the answer, because we didn't assume the multiplication was commutative. So a single associative multiplication actually creates $n!$ different ways to multiply any n inputs together, which leads us to define $\mathrm{Assoc}(n) = \Sigma_n$. The following lemma checks that we really did capture associative and unital multiplications by doing this.

Lemma 9.3.6. *An algebra over* Assoc *is the same thing as a topological monoid.*

So we have an isomorphism of categories $\mathrm{Alg}_{\mathrm{Assoc}}(\mathbf{Top}) \cong \mathrm{Alg}(\mathbf{Top}) = \mathrm{Mon}(\mathbf{Top})$.

Proof. Any topological monoid X gives an Assoc-algebra by having each $\sigma \in \Sigma_n$ act by the function $X^{\times n} \to X$ that applies σ to the inputs and then applies the unique n-fold multiplication that comes from the product on X.

Conversely, if X is an Assoc-algebra, then the action of the identity permutation $e \in \Sigma_2$ defines a two-fold multiplication $\mu : X \times X \to X$, and the action of the unique element $e \in \Sigma_0$ defines a unit map $\eta : * \to X$. The action of $e \in \Sigma_3$ then decomposes in two different ways as $\mu(\mu, \mathrm{id})$ or $\mu(\mathrm{id}, \mu)$. Since these are equal, μ must be associative.

It is then straightforward to check that the action of every other $\sigma \in \Sigma_n$ must be by the function $X^{\times n} \to X$ that applies σ to the inputs and then applies the unique n-fold multiplication that comes from the product on X. Therefore all of the Assoc-algebra structure boils down to just the multiplication that is associative and unital. □

Example 9.3.7. The **commutative operad** Comm has $\mathrm{Comm}(n) = *$, the one-point space. The Σ_n-action and composition maps are uniquely defined, and the operad axioms are trivial to verify.

This is called the "commutative operad" because it describes all possible operations you can generate out of a single commutative, associative, and unital multiplication $X \times X \to X$. As before, you get a unique n-fold multiplication $X^{\times n} \to X$ by composing the two-fold multiplication by itself. But now it's commutative, so switching the inputs doesn't change the result. So there is only one way to multiply n elements together, which leads us to define $\mathrm{Comm}(n) = *$.

Lemma 9.3.8. *An algebra over* Comm *is the same thing as a topological abelian monoid.*

So we have an isomorphism of categories $\mathrm{Alg}_{\mathrm{Comm}}(\mathbf{Top}) \cong \mathrm{CAlg}(\mathbf{Top}) = \mathrm{CMon}(\mathbf{Top})$. The proof is essentially the same as in Lemma 9.3.6.

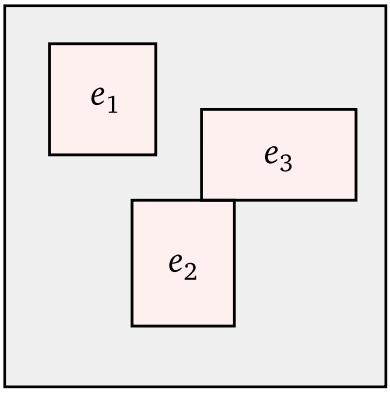
We also have the following trivial but important examples.

Example 9.3.9. The **trivial operad** has $\mathcal{O}(1) = *$ and $\mathcal{O}(n) = \varnothing$ otherwise. An algebra over this operad is just a topological space X, with no other structure. The **trivial pointed operad** has $\mathcal{O}(0) = *$, $\mathcal{O}(1) = *$, and $\mathcal{O}(n) = \varnothing$ otherwise. An algebra over this operad is a based space X, because the unique operation in $\mathcal{O}(0)$ gives a basepoint $* \to X$.

The next example is the most famous one. To match historical convention, we will let k refer to the operad level, so that n can be the dimension of the cubes.

Let I^n be the n-dimensional cube. We say that an embedding $I^n \to I^n$ is **rectilinear** if it is a product of linear embeddings of the form $i(x) = ax + b$. Giving a rectilinear embedding is the same as giving its image $[a_1, b_1] \times \cdots \times [a_n, b_n]$, which is determined by the $2n$ numbers $a_1, b_1, \ldots, a_n, b_n$. So we think of the space of rectilinear embeddings as a subspace of I^{2n}.

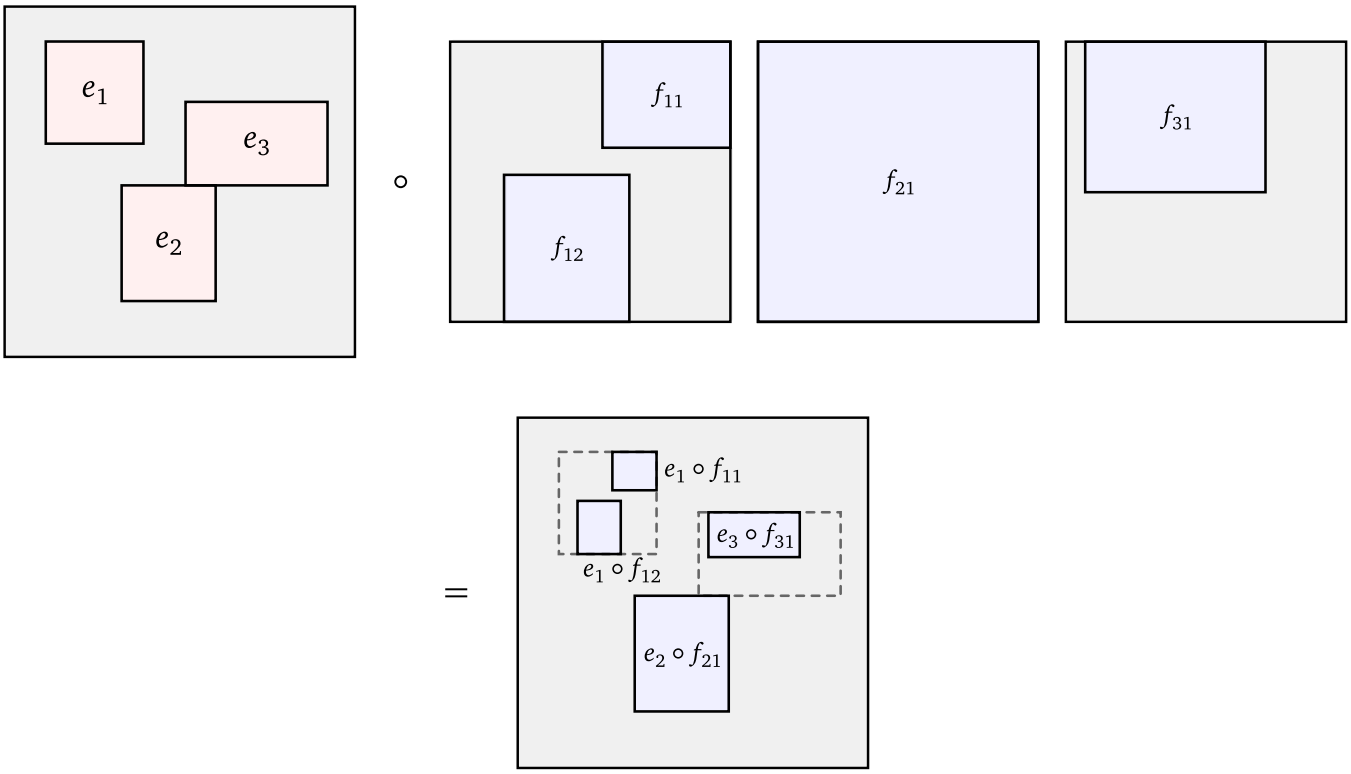
Example 9.3.10. For each $n \geq 0$, the **little n-cubes operad** $\mathscr{C}_n$, also called the **n-cubitos operad**, has $\mathscr{C}_n(k)$ equal to the space of k-tuples of rectilinear embeddings $(e_1, \ldots, e_k)$ such that the interiors $e_i((0,1)^n)$ are disjoint. This is topologized as a subspace of I^{2nk}.



The Σ_k-action relabels the cubes. In symbols, $(e_1, \ldots, e_k)\sigma = (e_{\sigma^{-1}(1)}, \ldots, e_{\sigma^{-1}(k)})$. The identity operation is the identity embedding $I^n \to I^n$. The composition $(e_1, \ldots, e_k) \circ ((f_{11}, \ldots, f_{1\ell_1}), \ldots, (f_{k1}, \ldots, f_{k\ell_1}))$ is simply the list of compositions

$$(e_1 \circ f_{11}, \ldots, e_1 \circ f_{1\ell_1}, e_2 \circ f_{21}, \ldots, e_2 \circ f_{2\ell_2}, \ldots, e_k \circ f_{k1}, \ldots, e_k \circ f_{k\ell_k}).$$

This is illustrated below.



The corresponding coordinate-free version has $\mathscr{C}_n(I)$ the set of I-tuples of rectilinear embeddings with disjoint interiors. It is straightforward to check that this defines an operad.

When $n = 0$, this is just the trivial pointed operad from Example 9.3.9. When $n = 1$ this is called the **little intervals operad**, and its operations are given by non-overlapping open intervals $(e_1, \ldots, e_k)$ in I, which do not have to be written in order from left to right (so e_1 does not have to be the leftmost interval).

Example 9.3.11. Every n-fold loop space $\Omega^n X$ is an $\mathscr{C}_n$-algebra in a canonical way. Think of $\Omega^n X$ as the space of maps of pairs $(I^n, \partial I^n) \to (X, *)$. Then each set of k little n-cubes defines a map $(\Omega^n X)^{\times k} \to \Omega^n X$, by defining the new map $I^n \to X$ to be the basepoint outside each of the little cubes, and a rescaled version of each of the k given maps $I^n \to X$ inside each of the little cubes. On connected components $\pi_0(\Omega^n X)$, this is just the addition on $\pi_n(X)$ that we defined in Section 1.4! It is an exercise to check that this defines an algebra over the operad $\mathscr{C}_n$. We will also see soon that these are close to being the only examples of $\mathscr{C}_n$-algebras.

Let's think about this more when $n = 1$. The little intervals operad $\mathscr{C}_1$ is parametrizing the ways that we can concatenate loops. You have actually been working with this operad, implicitly, ever since we defined the fundamental group in your first class on algebraic

topology. There is a canonical way to compose two loops in a space, using the two embeddings $[0, 1/2]$ and $[1/2, 1]$. This corresponds to a particular point in the space $\mathscr{C}_1(2)$, because $\mathscr{C}_1(2)$ is the space of all rules for composing two loops. When we use this particular multiplication to compose three loops α, β, and γ, the compositions $(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$ don't agree, they only agree up to a homotopy. In other words, they give two points in $\mathscr{C}_1(3)$, and they aren't equal, but they are still connected by a path.

When we first learn about the fundamental group, the existence of this homotopy is enough for us. But now we are moving into the world of homotopy-coherent algebra, and in this new world, it isn't enough that each component of $\mathscr{C}_1(3)$ is path-connected. We need it to be *contractible*. And the language of operads is exactly what we need to talk meaningfully about "a space of operations being contractible."

Lemma 9.3.12. *The space $\mathscr{C}_1(k)$ is a union of contractible components for all $k \geq 0$. In other words, the map of operads $\mathscr{C}_1 \to$ Assoc that passes to connected components, is a homotopy equivalence at each operad level.*

The proof is left to Exercise 9.6.18. The operads $\mathscr{C}_n$ for $n \geq 2$ don't have contractible components, but if we take the colimit as $n \to \infty$, we get contractibility again:

Example 9.3.13. The **little ∞-cubes operad** $\mathscr{C}_\infty$ is the colimit of the operads $\mathscr{C}_n$ as $n \to \infty$. Here we define the map $\mathscr{C}_n \to \mathscr{C}_{n+1}$ by multiplying all of the embeddings $I^n \to I^n$ by the identity of I to give embeddings $I^{n+1} \to I^{n+1}$. So an operation in $\mathscr{C}_\infty(k)$ is k embeddings of $I^n \to I^n$ for some value of n, modulo the relation that we may increase n as much as desired.

Note that if X is a strict infinite loop space, in the sense that there are homeomorphisms $X \cong \Omega X_1$ and $X_n \cong \Omega X_{n+1}$ for all $n \geq 1$, then X is an $\mathscr{C}_\infty$-algebra, because the action of $\mathscr{C}_n$ on $X \cong \Omega^n X_n$ extends to an $\mathscr{C}_{n+1}$-action on $X \cong \Omega^{n+1} X_{n+1}$.

Lemma 9.3.14. *The spaces $\mathscr{C}_\infty(k)$ are weakly contractible. In other words, the map of operads $\mathscr{C}_\infty \to$ Comm that passes to connected components, is a homotopy equivalence at each operad level.*

The proof is also left to Exercise 9.6.18.

Example 9.3.15. The **Barratt-Eccles operad** is at level n the space $\mathscr{E}(n) = E\Sigma_n = B(*, \Sigma_n, \Sigma_n)$. To describe the rest of the structure, we start with the associative operad, which is Σ_n at each level. Then we take the category whose objects are Σ_n and that has a unique isomorphism between every pair of objects. The classifying space of this category is identified with $E\Sigma_n$, and now the composition maps in Assoc induce composition maps on these classifying spaces.

Suppose $\mathbf{C}$ is a permutative category, as in Example 9.1.1 or Exercise 4.4.14. Then the classifying space $B\mathbf{C}$ is an algebra over the Barratt-Eccles operad: each element of

Σ_n acts by permuting the copies of $\mathbf{C}^{\times n}$ and then multiplying them, and the canonical isomorphisms between these products extend these to a map $E\Sigma_n \times (B\mathbf{C})^{\times n} \to B\mathbf{C}$, which defines the operad action.

9.3.3 A_∞ and E_∞ spaces

Now we can make our first definition of homotopy-coherent monoids, groups, and abelian groups.

Definition 9.3.16. An A_∞ **operad** is any operad $\mathcal{O}$ that is equivalent to the associative operad, in the sense that each space $\mathcal{O}(n)$ has a weak equivalence to Σ_n as a discrete space, and this defines a map of operads $\mathcal{O} \to \mathrm{Assoc}$ that is a weak equivalence at each level.

An A_∞ **space** is any algebra X over any A_∞ operad. This makes $\pi_0(X)$ into a discrete monoid, and we say that X is **grouplike** if $\pi_0(X)$ is a group.

This definition suggests that any two A_∞ operads have the same algebras up to weak equivalence, so that the notion of "A_∞ space" doesn't depend on the particular choice of A_∞ operad. This is true by Theorem 9.3.25 below.

Definition 9.3.17. An E_∞ **operad** is an operad $\mathcal{O}$ such that each space $\mathcal{O}(n)$ is a weakly contractible, and for any cell complex X, the balanced product $\mathcal{O}(n) \times_{\Sigma_n} X^{\times n}$ is equivalent to the corresponding homotopy orbits $\mathcal{O}(n) \times^h_{\Sigma_n} X^{\times n}$. (For instance, it is enough if $\mathcal{O}$ is Σ**-cofibrant**, meaning $\mathcal{O}(n)$ is a retract of a free Σ_n-cell complex.)

An E_∞ **space** is any algebra X over any E_∞ operad. We say that X is grouplike if the induced multiplication on $\pi_0(X)$ makes it into an (abelian) group.

Remark 9.3.18. The term A_∞ is supposed to represent that an algebra is "associative up to all (infinitely many) coherent higher homotopies." Similarly E_∞ means "everything (in other words associative and commutative) up to all coherent higher homotopies."

Definition 9.3.19. Our first definition of homotopy-coherent monoids and groups is:

- a **homotopy-coherent monoid** is an A_∞ space.
- a **homotopy-coherent group** is a grouplike A_∞ space.
- a **homotopy-coherent abelian monoid** is an E_∞ space.
- a **homotopy-coherent abelian group** is a grouplike E_∞ space.

Remark 9.3.20. One might expect to define homotopy-coherent groups using a larger operad that also specifies the inversion map $X \to X$, and all of the compositions of this

map with the multiplications. This is possible, but it is a complicated definition that is harder to use, and ultimately unnecessary. The mere existence of these inverses up to homotopy is enough to guarantee that (up to equivalence) they can be made into strict inverses!

Example 9.3.21. The associative operad Assoc and the little intervals operad $\mathscr{C}_1$ are both A_∞ operads.

For any based space X, its loop space ΩX is a grouplike A_∞ space, because it has an action by the little intervals operad $\mathscr{C}_1$, and on $\pi_0(\Omega X) \cong \pi_1(X)$ this is the usual concatenation of loops, which has inverses. In other words, ΩX is a homotopy-coherent group.

Example 9.3.22. The little ∞-cubes operad $\mathscr{C}_\infty$ is an E_∞ operad. This uses the fact that $\mathscr{C}_n(k)$ is Σ_k-equivariantly homotopy equivalent to a smooth manifold with smooth Σ_k-action, which is a free Σ_k-CW complex by Illman's triangulation theorem [Ill83], and so the orbits $\mathscr{C}_n(k) \times_{\Sigma_k} X^{\times k}$ are in fact homotopy orbits by Exercise 5.8.22.

The commutative operad Comm is *not* an E_∞ operad, because the orbits $* \times_{\Sigma_n} X^{\times n}$ are not equivalent to the homotopy orbits in general.

The Barratt-Eccles operad is also an E_∞ operad. Therefore for any permutative category $\mathbf{C}$, the classifying space $B\mathbf{C}$ is an E_∞ space. And for any spectrum E, $\Omega^\infty E$ is equivalent to a grouplike E_∞ space, because we can arrange so that E is a strict Ω-spectrum and so its 0th space has an action by the little ∞-cubes operad $\mathscr{C}_\infty$. Again on π_0 this is given by concatenation of loops, so it has inverses. In other words, $\Omega^\infty E$ is a homotopy-coherent abelian group.

Example 9.3.23. There is a **linear isometries operad** $\mathscr{L}$ whose space of k-fold operations $\mathscr{L}(k)$ consists of all linear isometries $(\mathbb{R}^\infty)^{\times k} \to \mathbb{R}^\infty$, where $\mathbb{R}^\infty = \bigoplus_{n=1}^\infty \mathbb{R}$ with its usual inner product $\langle x, y\rangle = \sum_{n=1}^\infty x_n y_n$. A linear isometry is a injective linear transformation (not an isomorphism!) that preserves this inner product.

The space of linear isometries $\mathbb{R}^\infty \to \mathbb{R}^\infty$ is contractible, see [LMSM86, II.1.5]. From this we can check that $\mathscr{L}$ is an E_∞ operad.

Note that a bijective linear isometry is not the same thing as an element of the infinite orthogonal group $O = \operatorname{colim}_{n\to\infty} O(n)$. The group O is the subset of those bijective linear isometries $\mathbb{R}^\infty \cong \mathbb{R}^\infty$ that are just an isometry of the first n coordinates $\mathbb{R}^n \cong \mathbb{R}^n$ for some $n \geq 0$, plus the identity on the remaining coordinates.

In fact, O is not just a topological group, but is also an algebra over the linear isometries operad $\mathscr{L}$, and is therefore also an E_∞ space. The action of $\mu\colon (\mathbb{R}^\infty)^{\times k} \to \mathbb{R}^\infty$ turns each k-tuple of elements of O into a single element of O, defined to the the identity on the complement of the image of μ, and the given k-tuple on the image of μ. This is of the required form because any finite-dimensional subspace of $\mathbb{R}^\infty$ is contained in the span of the first n coordinates for sufficiently large n. This action is also through group

homomorphisms, so we can take the classifying space and see that this also gives BO the structure of an $\mathscr{L}$-algebra.

There are many more examples like this that are relevant to geometric topology:

- $\mathrm{TOP} = \operatorname{colim}_{n\to\infty} \mathrm{TOP}(n)$, where $\mathrm{TOP}(n)$ is the group of homeomorphisms of $\mathbb{R}^n$, or (equivalently) the subgroup of those homeomorphisms that fix the origin,
- $\mathrm{PL} = \operatorname{colim}_{n\to\infty} \mathrm{PL}(n)$, where $\mathrm{PL}(n)$ is the group of piecewise-linear homeomorphisms of $\mathbb{R}^n$ that fix the origin,
- $G = \operatorname{colim}_{n\to\infty} G(n)$, where $G(n)$ is the group of based homotopy equivalences $S^n \to S^n$,
- classifying spaces of these such as $B\mathrm{TOP}$,
- and various quotients along the inclusions $O \to \mathrm{PL} \to \mathrm{TOP} \to G$, such as G/TOP.

For instance, the surgery exact sequence classifies all topological manifolds M homotopy equivalent to a given CW complex X using an exact sequence in which one of the terms is $[X, G/\mathrm{TOP}]$. It is useful to know in this setting that G/TOP is an E_∞ space, and so it deloops to a so-called L**-theory** spectrum $L(*)$. See e.g. [Lüc02, Wei94] for more information on examples like these.

9.3.4 The main results

We conclude this section by describing the main theorems concerning A_∞ and E_∞ spaces. The proofs are somewhat technical, and will not be used directly in the remainder of the chapter, so many of them are deferred to Section A.3.

Theorem 9.3.24 (Model structure on $\mathscr{O}$-algebras)**.** *For each operad $\mathscr{O}$, there is a right-induced model structure on* $\mathrm{Alg}_{\mathscr{O}}(\mathbf{Top})$*.*

In other words, a weak equivalence or fibration in $\mathrm{Alg}_{\mathscr{O}}(\mathbf{Top})$ is a map of algebras $X \to Y$ that is a weak homotopy equivalence or Serre fibration, respectively, after forgetting the $\mathscr{O}$-action.

Theorem 9.3.25 (Independence of operad)**.** *If $\mathscr{O}$ and $\mathscr{O}'$ are any two A_∞ operads, or any two E_∞ operads, there is a zig-zag of Quillen equivalences* $\mathrm{Alg}_{\mathscr{O}}(\mathbf{Top}) \simeq \mathrm{Alg}_{\mathscr{O}'}(\mathbf{Top})$*.*

So, the definition of an A_∞ or E_∞ space does not actually depend on which operad we use.

Corollary 9.3.26 (Rectification of A_∞ spaces)**.** *A_∞ spaces are equivalent to topological monoids, and grouplike A_∞ spaces are equivalent to topological groups.*

So homotopy-coherent monoids are the same thing as monoids, up to equivalence!

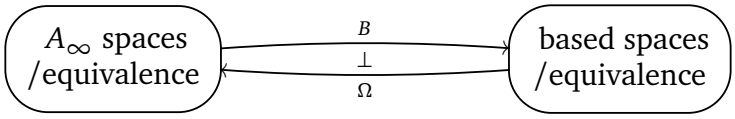
However this is not true for abelian monoids. Notice that for any operad $\mathcal{O}$, there is a unique map $\mathcal{O} \to \mathrm{Comm}$, and this allows us to treat every abelian monoid as an $\mathcal{O}$-algebra.

Proposition 9.3.27. *The forgetful map from topological abelian monoids to E_∞ spaces is not an equivalence.*

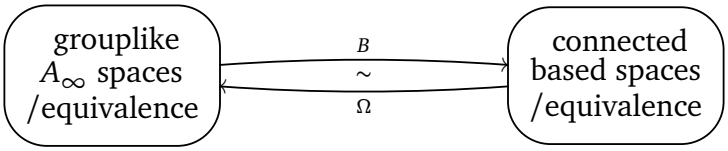
Finally, we state the recognition principle. It says that homotopy-coherent groups are essentially the same thing as loop spaces, and homotopy-coherent abelian groups are essentially the same thing as connective spectra. (We actually won't prove this version of the recognition principle in the book. We will prove the simpler Segal version in the next section.)

Theorem 9.3.28 (May recognition principle, unstable version)**.** *There is a Quillen adjunction between $\mathrm{Alg}_{\mathscr{C}_n}(\mathbf{Top})$ and $\mathbf{Top}_*$ whose right adjoint sends each based space X to the $\mathscr{C}_n$-algebra $\Omega^n X$. On homotopy categories, this adjunction is an equivalence when we restrict to the grouplike $\mathscr{C}_n$-algebras and the $(n-1)$-connected based spaces.*

In the special case that $n = 1$, this theorem says that we have an adjunction between A_∞ spaces (modeled as algebras over the little intervals operad) and based spaces:



We call the left adjoint "B" because it is a more general version of the bar construction that makes sense for any A_∞ space X. When X is actually a strictly associative monoid, BX is equivalent to the bar construction that we already know. We also get an equivalence between grouplike A_∞ spaces and connected based spaces:



In other words, X is equivalent to a loop space ΩY iff it is a grouplike A_∞ space. Homotopy-coherent groups are the same thing as loop spaces!

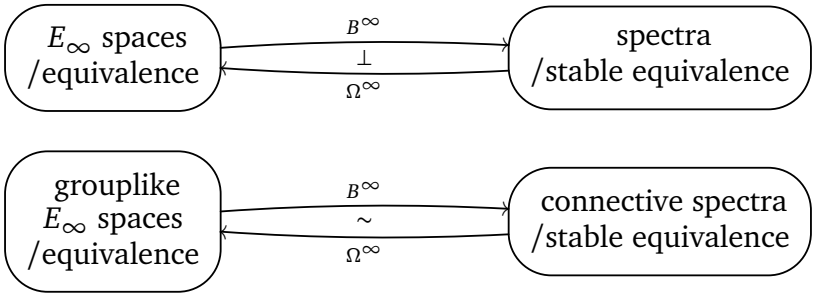
The derived unit of the adjunction $X \to \Omega BX$ is sometimes called the A_∞ **group completion**, because it is the universal approximation of an A_∞ space by a grouplike one. See Corollary 9.4.24 below.

When $n > 1$, we think of the left adjoint of Ω^n as a generalized version of the n-fold bar construction B^n. Again, when the $\mathscr{C}_n$-algebra X happens to be a commutative monoid, then this left adjoint is equivalent to the n-fold bar construction that we already know.

Theorem 9.3.29 (May recognition principle, stable version)**.** *There is an adjunction between the homotopy categories* $\mathrm{Ho}\,\mathrm{Alg}_{\mathscr{C}_\infty}(\mathbf{Top})$ *and* $\mathrm{Ho}\,\mathbf{Sp}$ *whose right adjoint sends each spectrum* X *to the homotopy colimit of the* $\mathscr{C}_n$*-algebras* $\Omega^n X_n$ *as* $n \to \infty$*. This is an equivalence of homotopy categories when we restrict to the grouplike* $\mathscr{C}_\infty$*-algebras and connective spectra.*

Informally, the left adjoint of this is a version of the delooping spectrum B^∞. So we can make sense of this whenever X is an E_∞ space, and then B^∞ and Ω^∞ form an adjunction between E_∞ spaces and spectra, inducing an equivalence between grouplike E_∞ spaces and connective spectra. In other words, homotopy-coherent abelian groups are all equivalent to infinite loop spaces!

This is illustrated at the beginning of the chapter, but we also include a simpler illustration here:



As before, the derived unit of the adjunction $X \to \Omega^\infty B^\infty X$ is sometimes called the E_∞ **group completion**, because it is the universal approximation of an E_∞ space by a grouplike one.

As mentioned before, we prove these results (and sketch the proof of the recognition principle) in Section A.3. We don't need them right now, because we are the most interested in using the delooping B^∞ to turn E_∞ spaces that come from *categories* into spectra. And it turns out there is a much cleaner way to do this, which we describe next.

9.4 The Segal approach to homotopy-coherent algebra

There is more than one way to define homotopy-coherent algebras. In this section we describe an alternate approach that originally comes from the work of Graeme Segal (pronounced "seagull"). This approach is much simpler, and makes it much easier to prove the main theorems. The disadvantage is that it is less general: it only works for A_∞ and E_∞ algebras, not for more general kinds of homotopy coherence. And it only works in *Cartesian* monoidal categories, not in any symmetric monoidal category. So it works for spaces with the Cartesian product, but not for spectra with the smash product.

But it is so much easier.

9.4.1 Segal A_∞ spaces

To see the essential idea, let's think about how we formed the bar construction BM of a topological monoid M in Definition 7.3.7. We started by defining a simplicial space $N_\bullet(M) = N_\bullet(*, M, *)$ called the nerve of M, whose nth level is $M^{\times n}$. The face maps of $N_\bullet(M)$ are given by multiplying adjacent terms, and the degeneracy maps are given by inserting units. In other words, the faces and degeneracies give the monoid structure on M! Now the idea is to consider simplicial spaces that are only *weakly equivalent* to something of the form $N_\bullet(M)$, and those will be the homotopy-coherent monoids.

To set this up, as in Chapter 7, let $\mathbf{\Delta}$ denote the simplex category. Its objects are the finite sets $[n] = \{0, 1, \ldots, n\}$ and its morphisms are the functions that preserve the total ordering. This category contains a special class of morphisms $\rho^n_j : [1] \to [n]$, one for each $1 \le j \le n$, that we call the **Segal maps**:

$$\rho^n_j : \{0,1\} \to \{0,1,\ldots,n\}, \qquad 0 \mapsto j-1, \;\; 1 \mapsto j.$$

These correspond to the 1-cells in the "spine" of the n-simplex, which is the path that runs through the vertices in order from 0 to n.

Definition 9.4.1. A **Segal A_∞ space**, or Segal monoid, is a simplicial space $X_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Top}$ such that

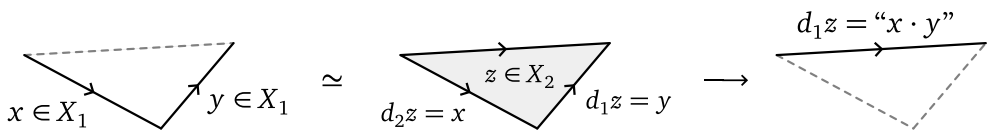
- $X_0 = *$ (in other words $X_\bullet$ is **reduced**), and
- for each $n \ge 0$, the Segal maps for $1 \le j \le n$ induce a weak equivalence

$$X_n \xrightarrow{\sim} \textstyle\prod_{j=1}^n X_1.$$

This last condition is sometimes called the **Segal condition**. These conditions ensure that $X_\bullet$ resembles the nerve of a monoid, because each level X_n is weakly equivalent to the n-fold product $(X_1)^{\times n}$.

We think of X_1 as the "underlying space." The rest of the diagram gives a homotopy-coherent monoid structure on X_1: the degeneracy map $X_0 = * \to X_1$ gives the unit, and the Segal maps plus the remaining face map give a zig-zag that defines the multiplication in the homotopy category

$$X_1 \times X_1 \xleftarrow[\sim]{(d_2,d_0)} X_2 \xrightarrow{d_1} X_1.$$



It is an exercise to see that this makes $\pi_0(X_1)$ into a monoid (Exercise 9.6.23). We say that $X_\bullet$ is **grouplike** if the monoid $\pi_0(X_1)$ has inverses, so that it is actually a group.

Example 9.4.2. If M is a topological monoid, then the nerve $N_\bullet(M)$ defines a Segal A_∞ space, which is grouplike iff M is grouplike. In this case the Segal maps induce a *homeomorphism*

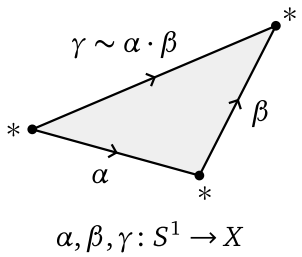
$$M^{\times n} \xrightarrow{\sim} \prod_{j=1}^{n} M.$$

Conversely, in any Segal A_∞ space $X_\bullet$, if the Segal maps give a homeomorphism, then $X_\bullet$ must be $N_\bullet(M)$ for some monoid M (Exercise 9.6.24). This illustrates how monoids sit inside the larger category of Segal A_∞ spaces.

Example 9.4.2 is universal, in the sense that every Segal A_∞ space is weakly equivalent to one of the form $N_\bullet(M)$. See Corollary 9.4.41.

Example 9.4.3. If Y is any based space, we show how to make the loopspace ΩY into a Segal A_∞ space. For each $n \geq 0$, let $X_n \subseteq \mathrm{Map}(\Delta^n, Y)$ be the subspace of all maps $\Delta^n \to Y$ such that every vertex of Δ^n goes to the basepoint. The face and degeneracy maps act in the usual way, by composing with the coface maps $\Delta^{n-1} \to \Delta^n$ and the codegeneracy maps $\Delta^{n+1} \to \Delta^n$, making $X_\bullet$ into a simplicial space.

Essentially by definition we have a homeomorphism $X_1 \cong \Omega Y$. If we go to the next level, X_2 is the space of all choices of two loops, plus a homotopy from their composite to a single loop, as shown. The Segal map restricts X_2 to the two loops formed by the spine of Δ^2. This gives a homotopy equivalence $X_2 \simeq (\Omega Y)^{\times 2}$, because the inclusion of the spine $\Delta^1 \cup_{\Delta^0} \Delta^1$ into Δ^2 is a homotopy equivalence.



We similarly get $X_n \simeq (\Omega Y)^{\times n}$ along the Segal maps for all $n \geq 0$, so that $X_\bullet$ is a Segal A_∞ space. We will usually refer to this space as $\Omega_\bullet Y$.

Example 9.4.3 is universal, in the sense that every *grouplike* Segal A_∞ space is weakly equivalent to one of the form $\Omega_\bullet Y$. See Proposition 9.4.21.

Remark 9.4.4. In the definition of a Segal A_∞ space we asked for a functor to **Top**, but the condition that $X_0 = *$ actually implies that the spaces X_n all have a canonical basepoint. So it is equivalent if we ask for a functor to $\mathbf{Top}_*$ instead. The same applies to Segal E_∞ spaces, which we define next.

9.4.2 Segal E_∞ spaces

Now let $\mathbf{Fin}_*$ be a skeleton of the category of finite pointed sets. Its objects are $\langle n \rangle = \mathbf{n}_+ = \{*, 1, \ldots, n\}$, and its morphisms are all maps that send $*$ to $*$. In this case the **Segal maps**

are the maps $\rho_j^n \colon \mathbf{n}_+ \to \mathbf{1}_+$ that send $j \mapsto 1$ and all other points to $*$.

Definition 9.4.5. A **Segal E_∞ space**, more commonly known as a **special Γ-space**, is a functor $X(-) \colon \mathbf{Fin}_* \to \mathbf{Top}$ such that

- $X(\mathbf{0}_+) = *$, and
- for each $n \geq 0$, the Segal maps for $1 \leq j \leq n$ induce a weak equivalence

$$X(\mathbf{n}_+) \xrightarrow{\sim} \textstyle\prod_{j=1}^n X(\mathbf{1}_+).$$

Again we think of $X(\mathbf{1}_+)$ as the "underlying space," and the rest of the diagram gives a homotopy-coherent abelian monoid structure on $X(\mathbf{1}_+)$. For instance, the unique map of finite based sets $\mathbf{0}_+ \to \mathbf{1}_+$ gives a unit map $X(\mathbf{0}_+) = * \to X(\mathbf{1}_+)$. Combining the Segal maps with the map $\mathbf{2}_+ \to \mathbf{1}_+$ that folds the two points into one, we get a zig-zag that defines the multiplication in the homotopy category

$$X(\mathbf{1}_+) \times X(\mathbf{1}_+) \xleftarrow{\sim} X(\mathbf{2}_+) \longrightarrow X(\mathbf{1}_+).$$

It is an exercise to see that this makes $\pi_0(X(\mathbf{1}_+))$ into an abelian monoid (Exercise 9.6.24). We say that $X(-)$ is **grouplike** (or a **very special Γ-space**) if the monoid $\pi_0(X(\mathbf{1}_+))$ is in fact a group.

Remark 9.4.6. It is often convenient to let $\mathbf{Fin}_*$ instead denote the category of all finite based sets. The corresponding notion of Segal E_∞ space has a space $X(I_+)$ for each finite set I, and the Segal maps must give an equivalence $X(I_+) \simeq \prod_{i \in I} X(\{i\}_+) \cong \prod_{i \in I} X(\mathbf{1}_+)$. These two notions give equivalent categories: a Segal space in one sense corresponds to a Segal space in the other sense that is unique up to isomorphism. So it is harmless to pass back and forth between them as needed.

Example 9.4.7. If A is a topological abelian monoid, we make it into a Segal E_∞ space by setting $X(\mathbf{n}_+) = A^{\times n}$, and each map of finite based sets $\alpha \colon \mathbf{m}_+ \to \mathbf{n}_+$ induces a map $\alpha_* \colon A^{\times m} \to A^{\times n}$ by multiplying the preimages together:

$$\alpha_*(a_1, \ldots, a_m) = \left(\prod_{i \in \alpha^{-1}(1)} a_i, \ldots, \prod_{i \in \alpha^{-1}(n)} a_i \right).$$

As before, a Segal E_∞ space is isomorphic to one of this form iff the Segal maps induce a *homeomorphism* $X(\mathbf{n}_+) \cong X(\mathbf{1}_+)^{\times n}$. This illustrates why a Segal E_∞ space is a weaker notion than a topological abelian monoid.

Example 9.4.7 is *not* universal! Not every Segal E_∞ space is weakly equivalent to one of this form. You should expect this, because we've already said that E_∞ spaces are not equivalent to strictly commutative monoids. See Exercise 9.6.14.

Example 9.4.8. This is the most important example. If $(\mathbf{C}, \otimes, I)$ is a (small) symmetric monoidal category, we show how to make the classifying space $B\mathbf{C} = |N_\bullet \mathbf{C}|$ into a Segal E_∞ space. In fact, we'll show how $\mathbf{C}$ defines a functor $\mathbf{Fin}_* \to \mathbf{Cat}$. Taking the nerve and the geometric realization gives the desired functor $\mathbf{Fin}_* \to \mathbf{Top}$.

The idea is to send $\mathbf{n}_+$ to the n-fold product category $\mathbf{C}^{\times n}$, and to have each map $\mathbf{m}_+ \to \mathbf{n}_+$ act by tensoring together the objects, analogous to Example 9.4.7. The issue is that this doesn't actually define a diagram on $\mathbf{Fin}_*$, because tensoring a set of objects in different orders gives the same thing up to isomorphism, but not the same thing on the nose.

To address this, we instead define $\mathbf{C}(\mathbf{n}_+)$ as the category whose objects are assignments of each subset $S \subseteq \mathbf{n} = \{1, \dots, n\}$ to an object $x_S \in \mathbf{C}$, and every pair of disjoint subsets S and T to an isomorphism $f_{S,T} : x_S \otimes x_T \cong x_{S \amalg T}$, such that:

- $x_\varnothing = I$,
- when $S = \varnothing$ the isomorphism $f_{\varnothing,T}$ is the canonical unitor isomorphism $I \otimes x_T \cong x_T$, and similarly for $f_{S,\varnothing}$, and
- for any triple of disjoint subsets the following diagram of isomorphisms commutes:

$$\begin{array}{ccc} x_S \otimes x_T \otimes x_U & \xrightarrow{\mathrm{id} \otimes f_{T,U}} & x_S \otimes x_{T \amalg U} \\ \downarrow{\scriptstyle f_{S,T} \otimes \mathrm{id}} & & \downarrow{\scriptstyle f_{S,T \amalg U}} \\ x_{S \amalg T} \otimes x_U & \xrightarrow{f_{S \amalg T, U}} & x_{S \amalg T \amalg U}. \end{array}$$

A morphism in $\mathbf{C}(\mathbf{n}_+)$ consists of morphisms $x_S \to x'_S$ that commute with the isomorphisms $f_{S,T}$. It is not hard to see that an object is determined up to isomorphism by the objects $x_{\{i\}}$ for the singletons $\{i\} \subseteq \mathbf{n}$, and a morphism is determined by what it does on the objects $x_{\{i\}}$. This allows us to show that $\mathbf{C}(\mathbf{n}_+)$ is equivalent (not isomorphic) to the n-fold product category $\mathbf{C}^{\times n}$. For low values of n we can say even more: $\mathbf{C}(\mathbf{1}_+)$ is isomorphic to $\mathbf{C}$ itself, while $\mathbf{C}(\mathbf{0}_+)$ is isomorphic to the one-point category $*$.

Each map $\alpha : \mathbf{m}_+ \to \mathbf{n}_+$ acts on $(x_S)_{S \subseteq \mathbf{m}}$ by sending it to $(x_{\alpha^{-1}(T)})_{T \subseteq \mathbf{n}}$. Informally, it is sending each singleton $\{i\} \subseteq \mathbf{n}$ to the tensor of the objects $x_{\{j\}}$ for all $j \in \alpha^{-1}(i)$, but we don't actually have to tensor them because we have already chosen an object that is isomorphic to that tensor product: the object $x_{\alpha^{-1}(i)}$.

We check that this rule extends to morphisms in a reasonable way and defines a strict diagram $\mathbf{Fin}_* \to \mathbf{Cat}$, not just a diagram that commutes up to isomorphisms. It has the Segal condition in the sense that the Segal maps give an equivalence of categories $\mathbf{C}(\mathbf{n}_+) \simeq \mathbf{C}^{\times n}$. Since the nerve and geometric realization preserve finite products, after taking the nerve and the realization, we therefore have a Segal E_∞ space $|N_\bullet \mathbf{C}(-)|$.

Example 9.4.8 was the motivating example in Segal's work, and indeed it is universal, as described in the following proposition. We omit the proof of this, see [Man10, Thm 1.4]:

Proposition 9.4.9. *Every Segal E_∞ space is weakly equivalent to one of the form $|N_\bullet \mathbf{C}(-)|$ for some small symmetric monoidal category $(\mathbf{C}, \otimes, I)$.*

This is a wonderful result because it tells us that symmetric monoidal categories are also a combinatorial way to describe homotopy-coherent commutative multiplications!

Definition 9.4.10. Our second definition of homotopy-coherent monoids and groups is:

- a **homotopy-coherent monoid** is a Segal A_∞ space $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Top}$.
- a **homotopy-coherent group** is a grouplike Segal A_∞ space $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Top}$.
- a **homotopy-coherent abelian monoid** is a Segal E_∞ space $\mathbf{Fin}_* \to \mathbf{Top}$.
- a **homotopy-coherent abelian group** is a grouplike Segal E_∞ space $\mathbf{Fin}_* \to \mathbf{Top}$.

This suggests that every Segal E_∞ space should be an A_∞ space with more structure. In other words, it should have an underlying A_∞ space. To see why this is true, recall from Example 7.1.17 that the **simplicial circle** is the based simplicial set $S^1_\bullet = \Delta[1]/\partial\Delta[1]$. Let's think of this as a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Fin}_*$, so that we can compose it with other functors.

Definition 9.4.11. If $X : \mathbf{Fin}_* \to \mathbf{Top}$ is a Segal E_∞ space, the **underlying A_∞ space** $X(S^1_\bullet)$ is the composite

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{S^1_\bullet} \mathbf{Fin}_* \xrightarrow{X} \mathbf{Top}.$$

Here we are thinking of X as defined on all finite based sets, not just the sets of the form $\mathbf{n}_+$, as in Remark 9.4.6.

More generally, for any based simplicial set $A_\bullet$ such that each level A_n is a finite set, we define $X(A_\bullet)$ to be the simplicial space given by the composite

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{A_\bullet} \mathbf{Fin}_* \xrightarrow{X} \mathbf{Top}.$$

When X comes from a commutative monoid as in Example 9.4.7, the space $X(A_\bullet)$ is sometimes called the **reduced Loday construction** of X along $A_\bullet$.

If you think about it, $X(S^1_\bullet)$ captures exactly those multiplications in X that are the "associative" ones: the ones coming from maps of finite sets $\mathbf{m}_+ \to \mathbf{n}_+$ that only combine adjacent elements, and only delete elements off the left and right-hand sides. See Exercise 7.7.44.

We will need to know a couple of technical lemmas about these spaces $X(A_\bullet)$.

Lemma 9.4.12. *There is a natural homeomorphism*

$$|X(A_\bullet)| \cong X(-) \times_{\mathbf{Fin}_*} \mathrm{Map}(-, |A_\bullet|).$$

In other words, the space $|X(A_\bullet)|$ is the coend (Definition 7.4.1) of the covariant functor $\mathbf{Fin}_* \to \mathbf{Top}_*$ given by $X(-)$, and the contravariant functor $\mathbf{Fin}_*^{\mathrm{op}} \to \mathbf{Top}_*$ given by $\mathrm{Map}(-,|A_\bullet|)$. This is remarkable because it shows that $|X(A_\bullet)|$ only depends on the topological space $|A_\bullet|$, not the simplicial set $A_\bullet$!

Proof. Realization commutes with finite products, colimits, and products with a fixed space, giving a homeomorphism

$$X(-) \times_{\mathbf{Fin}_*} \mathrm{Map}_*(-,|A_\bullet|) \cong \left|X(-) \times_{\mathbf{Fin}_*} \mathrm{Map}_*(-,A_\bullet)\right|.$$

We then notice that for fixed p, $\mathrm{Map}_*(-,A_p)$ is a free diagram on $\mathbf{Fin}_*^{\mathrm{op}}$, and therefore the coend simplifies to $X(A_p)$. The maps we get as p varies are exactly those coming from the simplicial structure of $A_\bullet$, so the entire construction simplifies to $|X(A_\bullet)|$. See also [MMO25, 2.30] for a more expansive version of this proof. □

Lemma 9.4.13. *If $X(\mathbf{0}_+) = *$ and K is any based space, then there is a canonical homeomorphism between the coend defined using the Cartesian product and the coend defined using the smash product:*

$$X(-) \times_{\mathbf{Fin}_*} \mathrm{Map}_*(-,K) \cong X(-) \wedge_{\mathbf{Fin}_*} \mathrm{Map}_*(-,K).$$

Proof. Both are quotients of the same disjoint union of products $X(\mathbf{n}_+) \times \mathrm{Map}_*(\mathbf{n}_+,K)$, so we just have to check they have the same equivalence relation. The extra relations from the smash product identify to the basepoint any pair in which either the point in $X(\mathbf{n}_+)$ or the point in $\mathrm{Map}_*(\mathbf{n}_+,K)$ is the basepoint. We just have to check that these relations have already been applied in the Cartesian product coend, gluing everything to the unique point in $X(\mathbf{0}_+) \times \mathrm{Map}_*(\mathbf{0}_+,K) = *$.

If our pair in $X(\mathbf{n}_+) \times \mathrm{Map}_*(\mathbf{n}_+,K)$ has first coordinate the basepoint of $X(\mathbf{n}_+)$, that coordinate is in the image of $X(\mathbf{0}_+) = *$ along the map $\mathbf{0}_+ \to \mathbf{n}_+$, so our pair is therefore identified to $X(\mathbf{0}_+) \times \mathrm{Map}_*(\mathbf{0}_+,K) = *$. Similarly if the second coordinate is the basepoint of $\mathrm{Map}_*(\mathbf{n}_+,K)$, it arises from $\mathrm{Map}_*(\mathbf{0}_+,K) = *$ by composing with the projection $\mathbf{n}_+ \to \mathbf{0}_+$, so again our pair is identified to $X(\mathbf{0}_+) \times \mathrm{Map}_*(\mathbf{0}_+,K) = *$. This completes the verification. □

9.4.3 Delooping a Segal A_∞ or E_∞ space

The Segal version of the bar construction B and the delooping spectrum B^∞ are significantly easier than their counterparts in the world of operads. We give them here in the main text, rather than relegating them to an appendix.

Definition 9.4.14. For any Segal A_∞ space $X_\bullet$, the corresponding **bar construction** BX is just the realization of $X_\bullet$ as a simplicial space:

$$BX := |X_\bullet|.$$

In fact, what we actually care about is the left-derived functor $\mathbb{L}B$, which agrees with this realization when $X_\bullet$ is Reedy cofibrant (Definition 7.2.11). This is automatically true if $X_\bullet$ came from a bisimplicial set $X_{\bullet\bullet}$ by taking realization in one direction. We can also make $X_\bullet$ Reedy cofibrant using either the projective model structure or the Reedy model structure on simplicial spaces, though it takes a bit of thought to see why you can do this and keep the condition that $X_0 = *$.[8] (See also Exercise 8.7.28.)

Theorem 9.4.15 (Segal recognition principle, unstable version)**.** *The realization B and the loopspace construction $\Omega_\bullet$ from Example 9.4.3 define an adjunction*

$$\mathbf{Top}_{\mathrm{Seg}}^{\mathbf{\Delta}^{\mathrm{op}}} \underset{\Omega_\bullet}{\overset{B}{\rightleftarrows}} \mathbf{Top}_*.$$

The corresponding adjunction of homotopy categories

$$\mathrm{Ho}\,\mathbf{Top}_{\mathrm{Seg}}^{\mathbf{\Delta}^{\mathrm{op}}} \underset{\Omega_\bullet}{\overset{\mathbb{L}B}{\rightleftarrows}} \mathrm{Ho}\,\mathbf{Top}_*$$

becomes an equivalence when restricted to grouplike A_∞ spaces on the left and connected spaces on the right.

The proof of the first part is immediate: we simply check that B and $\Omega_\bullet$ are adjoint, and so we get an adjunction of homotopy categories by Proposition 3.5.20. For the second part, we know that $\Omega_\bullet$ preserves and reflects equivalences of connected spaces. (In other words, if $f : Y \to Y'$ is a map of connected spaces then f is an equivalence iff $\Omega f : \Omega Y \to \Omega Y'$ is an equivalence.) Therefore by Exercise 6.6.16, it suffices to prove that the unit map $X_\bullet \to \Omega_\bullet BX$ is an equivalence when $X_\bullet$ is grouplike (and cofibrant). We prove this in Proposition 9.4.21 below.

Now we turn to the E_∞ case.

Definition 9.4.16. For any Segal E_∞ space $X(-)$, its **delooping spectrum** $B^\infty X$ is the orthogonal spectrum with nth level

$$\begin{aligned} B^n X &= |X(S^n_\bullet)| \\ &\cong X(-) \wedge_{\mathbf{Fin}_*} \mathrm{Map}_*(-, S^n), \end{aligned}$$

[8] Note that these do not give a model structure on Segal A_∞ spaces, because a colimit or limit of simplicial spaces does not necessarily preserve the condition that the Segal map is an equivalence. But the model structure on all simplicial spaces is still a useful tool here.

with $O(n)$ acting on S^n in the usual way. The bonding map $B^n X \wedge S^1 \to B^{n+1}X$ is the map of coequalizers

$$\big(X(-) \wedge_{\mathbf{Fin}_*} \mathrm{Map}_*(-, S^n)\big) \wedge S^1 \longrightarrow X(-) \wedge_{\mathbf{Fin}_*} \mathrm{Map}_*(-, S^{n+1})$$

defined on each term by the usual interchange $\mathrm{Map}_*(-,S^n) \wedge S^1 \to \mathrm{Map}_*(-,S^{n+1})$.

Again, B^∞ is "correct" or agrees with its left-derived functor $\mathbb{L}B^\infty$ as soon as each of the simplicial spaces $X(S^n_\bullet)$ is Reedy cofibrant. This follows if $X(-)$ is Reedy cofibrant in the sense outlined in Exercise 9.6.22. It is also automatically true if $X(-)$ came from a functor $\mathbf{Fin}_* \to \mathbf{sSet}$ by taking the geometric realization.

We will need to describe the bonding maps of $B^\infty X$ in another way. Note that for any Segal E_∞ space $X_\bullet$, the canonical map

$$|X(S^0_\bullet)| \wedge S^1 \cong X_1 \wedge S^1 \longrightarrow |X(S^1_\bullet)|$$

can be identified with the inclusion of the 1-skeleton of $|X(S^1_\bullet)|$ in the sense of Definition 8.2.29. For larger values of n, consider the bisimplicial object $X(S^n_\bullet \wedge S^1_\bullet)$. By Example 7.2.5, its realization can be computed in two ways: we either take the diagonal to get a simplicial object, whose realization is $|X(S^{n+1}_\bullet)|$, or we take the realization in each simplicial direction separately. This latter description allows us to take the 1-skeleton in just one of the simplicial directions, giving a map

$$|X(S^n_\bullet)| \wedge S^1 \longrightarrow |X(S^n_\bullet \wedge S^1_\bullet)| \cong |X(S^{n+1}_\bullet)|.$$

Lemma 9.4.17. *This coincides with the bonding map from Definition 9.4.16.*

Proof. Consider the following commuting diagram of bisimplicial spaces, in which the vertical maps arise from the action of the sets $\mathrm{Map}(\mathbf{m}_+, \mathbf{n}_+)$ on $X(\mathbf{m}_+)$, and the horizontal maps are assembly maps that for each point $y \in S^1_q$ apply the map of simplicial sets $S^n_\bullet \to S^n_\bullet \wedge S^1_q$, $x \mapsto (x,y)$, inside of Map_* or X:

$$\begin{array}{ccc} X(-) \wedge_{\mathbf{Fin}_*} \mathrm{Map}_*(-, S^n_\bullet) \wedge S^1_\bullet & \longrightarrow & X(-) \wedge_{\mathbf{Fin}_*} \mathrm{Map}_*(-, S^n_\bullet \wedge S^1_\bullet) \\ \downarrow & & \downarrow \\ X(S^n_\bullet) \wedge S^1_\bullet & \longrightarrow & X(S^n_\bullet \wedge S^1_\bullet), \end{array}$$

On the geometric realization, the vertical maps are the canonical isomorphisms $|X(-) \wedge_{\mathbf{Fin}_*} \mathrm{Map}_*(-, S^n_\bullet)| \cong |X(S^n_\bullet)|$. The top horizontal gives the first definition of the bonding map, as we can check on each piece of the bi-realization of the form

$$\Delta^p \times \Delta^q \times (X(-) \wedge_{\mathbf{Fin}_*} \mathrm{Map}_*(-, S^n_p) \wedge S^1_q).$$

Similarly the bottom horizontal gives the second definition of the bonding map, because on the piece

$$\Delta^p \times \Delta^1 \times X(S^n_p) \wedge S^1_1$$

it agrees with the inclusion of the 1-skeleton of $|X(S^n_\bullet \wedge S^1_\bullet)|$. Therefore along the canonical isomorphisms the two models of the bonding map coincide. □

Example 9.4.18. If X is the Segal E_∞ space coming from a commutative monoid A, so that $X(\mathbf{n}_+) = A^{\times n}$, then the above lemma allows us to show that $B^\infty X$ *coincides* with the delooping spectrum $B^\infty A$ from Definition 9.2.14.

Theorem 9.4.19 (Segal recognition principle, stable version)**.** *The delooping spectrum is a left adjoint on homotopy categories, whose right adjoint sends each spectrum E to its infinite loopspace $\Omega^\infty E$ with some E_∞ structure on it:*

$$\mathrm{Ho}\,\mathbf{Top}^{\mathbf{Fin}_*}_{\mathrm{Seg}} \underset{\Omega^\infty}{\overset{\mathbb{L}B^\infty}{\rightleftarrows}} \mathrm{Ho}\,\mathbf{Sp}^O. \qquad (\perp)$$

Furthermore, this becomes an equivalence when restricted to grouplike E_∞ spaces on the left and connective spectra on the right.

Again, we prove the adjunction part here, and save the proof of the equivalence part until Proposition 9.4.25 in the next subsection.

Proof. We first notice that B^∞ defines not just an orthogonal spectrum but a functor on all finite CW complexes. To be more specific, it defines a $\mathscr{W}$-spectrum in the sense of Definition 6.4.16. The value of this spectrum on the finite CW complex K is given by

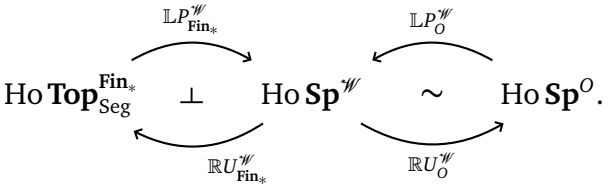
$$X(-) \wedge_{\mathbf{Fin}_*} \mathrm{Map}(-, K).$$

This is a $\mathscr{W}$-spectrum whose underlying orthogonal spectrum is $B^\infty X$ by Lemma 9.4.12. We may also call this operation "prolongation" and denote it by $P^{\mathscr{W}}_{\mathbf{Fin}_*}$, since this is the left adjoint of the forgetful functor $U^{\mathscr{W}}_{\mathbf{Fin}_*}$ that takes every functor $\mathscr{W} \to \mathbf{Top}_*$ and restricts attention to the finite CW complexes that are discrete finite sets $\mathbf{n}_+$.

It is not hard to show that this defines a Quillen adjunction between diagrams on $\mathbf{Fin}_*$ with the projective model structure, and $\mathscr{W}$-spectra with the absolute stable model structure of Proposition 6.4.26. So we have factored B^∞ into a left Quillen adjoint followed by a right Quillen equivalence:

$$\mathbf{Top}^{\mathbf{Fin}_*} \underset{U^{\mathscr{W}}_{\mathbf{Fin}_*}}{\overset{P^{\mathscr{W}}_{\mathbf{Fin}_*}}{\rightleftarrows}} \mathbf{Sp}^{\mathscr{W}} \underset{U^{\mathscr{W}}_{O}}{\overset{P^{\mathscr{W}}_{O}}{\rightleftarrows}} \mathbf{Sp}^O. \qquad (\perp \quad \sim)$$

Taking the derived functors, we get adjunctions of homotopy categories



Here we have restricted from the homotopy category of all diagrams $\mathrm{Ho}\,\mathbf{Top}^{\mathbf{Fin}_*}$ to the smaller category of those that satisfy the definition of a Segal E_∞ space, $\mathrm{Ho}\,\mathbf{Top}^{\mathbf{Fin}_*}_{\mathrm{Seg}}$. This is possible because the right-derived functor $\mathbb{R}U^{\mathscr{W}}_{\mathbf{Fin}_*}$ makes each $\mathscr{W}$-spectrum X into an absolute Ω-spectrum, in other words an excisive functor, and then restricts attention to its value on the spaces $\mathbf{n}_+$. By convention (Definition 6.4.16) this gives a diagram on $\mathbf{Fin}_*$ in which $X(\mathbf{0}_+) = *$, and by the excision condition we see that $X(\mathbf{n}_+)$ is identified with the product of $X((\mathbf{n}-\mathbf{1})_+)$ and $X(\mathbf{1}_+)$, from which we can deduce that it has the Segal condition. Therefore this functor actually lands in the subcategory $\mathbf{Top}^{\mathbf{Fin}_*}_{\mathrm{Seg}}$, so using Proposition 3.5.20 the derived functors also form an adjunction out of that subcategory.[9]

Therefore the composite $\mathbb{R}U^{\mathscr{W}}_{O} \circ \mathbb{L}P^{\mathscr{W}}_{\mathbf{Fin}_*}$ is a left adjoint. We check that this is equivalent to the left-derived functor $\mathbb{L}B^\infty$. You might think that this is automatically true, but it is not, see for instance Example 3.5.19. However we have the map

$$\mathbb{L}B^\infty = B^\infty \circ Q \cong U^{\mathscr{W}}_{O} \circ P^{\mathscr{W}}_{\mathbf{Fin}_*} \circ Q \xrightarrow{\sim} U^{\mathscr{W}}_{O} \circ R \circ P^{\mathscr{W}}_{\mathbf{Fin}_*} \circ Q \cong \mathbb{R}U^{\mathscr{W}}_{O} \circ \mathbb{L}P^{\mathscr{W}}_{\mathbf{Fin}_*},$$

where Q denotes a cofibrant replacement in $\mathbf{Top}^{\mathbf{Fin}_*}_{\mathrm{Seg}}$ and R denotes fibrant replacement in $\mathbf{Sp}^{\mathscr{W}}$. The map in the middle is an equivalence because the functor $U^{\mathscr{W}}_{O}$ preserves all weak equivalences by Lemma 6.4.21. So $\mathbb{L}B^\infty$ is a left adjoint as well.

Finally, the right adjoint is $\mathbb{R}U^{\mathscr{W}}_{\mathbf{Fin}_*} \circ \mathbb{L}P^{\mathscr{W}}_{O}$, which takes each orthogonal spectrum X to a larger $\mathscr{W}$-spectrum that has X as its underlying spectrum (up to stable equivalence), makes the $\mathscr{W}$-spectrum version of X into an absolute Ω-spectrum, and then restricts this to all finite sets. This gives a diagram on $\mathbf{Fin}_*$ whose value at $\mathbf{1}_+ \cong S^0$ is the 0th level of the underlying spectrum, which is the 0th level of X, after taking fibrant replacement. In other words, it is equivalent to $\Omega^\infty X$. □

Remark 9.4.20. We have stated these results as adjunctions of homotopy categories, not as Quillen adjunctions. However, in the next chapter when we introduce ∞-categories, it will be clear that these proofs actually give adjunctions of ∞-categories, which are as good as Quillen adjunctions. See Exercise 10.8.12.

[9] This is one place where it's very convenient to not be tied down to model categories, and to allow yourself to talk about derived functors in any category with weak equivalences.

9.4.4 Proof of the equivalence; A_∞ and E_∞ group completion

In the previous subsection, we defined an adjunction (B, Ω) between Segal A_∞ spaces and based spaces, and an adjunction $(B^\infty, \Omega^\infty)$ between Segal E_∞ spaces and spectra.

In this subsection we prove that these are *equivalences* of homotopy categories, when we restrict to grouplike A_∞ spaces and connected spaces, respectively grouplike E_∞ spaces and connective spectra. As a consequence, the derived unit map $X \to \Omega BX$ is the group completion of the A_∞ space X, and the derived unit map $X \to \Omega^\infty B^\infty X$ is the group completion of the E_∞ space X.

Proposition 9.4.21. *For any grouplike, Reedy cofibrant Segal A_∞ space $X_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Top}$, the unit map $X_\bullet \to \Omega_\bullet BX$ of the adjunction from Theorem 9.4.15 is an equivalence.*

As before, when we proved that $M \to \Omega BM$ is an equivalence for M grouplike, the key ingredient is descent for homotopy colimits. In this case, we need the version from Theorem A.2.6, which tells us that if a map of simplicial spaces $E_\bullet \to B_\bullet$ has "constant homotopy fiber," then the map of realizations $|E_\bullet| \to |B_\bullet|$ has that same homotopy fiber as well.

Proof. By the Segal condition, it suffices to check this on X_1. Following the original proof in [Seg74, Prop 1.5], we define the path space $PX_\bullet$ as in Exercise 7.7.14. It has $PX_n := X_{1+n}$, with each map $\alpha \colon [m] \to [n]$ acting through the map $0 \amalg \alpha \colon [1+m] \to [1+n]$ that sends $0 \mapsto 0$ and $i \mapsto \alpha(i-1)+1$ when $i > 0$. By Exercise 7.7.14(d), since $X_0 = *$, the realization $|PX_\bullet|$ is contractible.

The face maps $d_0 \colon X_{1+n} \to X_n$ give a map of simplicial spaces $PX_\bullet \to X_\bullet$, and we check that this map satisfies the hypotheses of Theorem A.2.6. In other words, we show that all of the maps d_0 have the same homotopy fiber, in the sense that each face or degeneracy map $\alpha \colon [m] \to [n]$ the commuting square

$$\begin{array}{ccc} X_{1+n} & \xrightarrow{(0 \amalg \alpha)^*} & X_{1+m} \\ \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0} \\ X_n & \xrightarrow{\alpha^*} & X_m \end{array} \tag{9.4.22}$$

is a homotopy pullback square. Along the Segal maps, the corners and vertical maps of this square are identified with

$$\begin{array}{ccc} X_1 \times X_1^n & \dashrightarrow & X_1 \times X_1^m \\ \downarrow{\scriptstyle *\times \mathrm{id}} & & \downarrow{\scriptstyle *\times \mathrm{id}} \\ X_1^n & \dashrightarrow & X_1^m, \end{array}$$

although this makes the horizontal maps only defined in the homotopy category. When α is a degeneracy map, we can do a little better and the Segal maps give an equivalence to a square of the form

$$\begin{array}{ccc} X_1 \times * \times X_1^n & \xrightarrow{\mathrm{id}\times i\times \mathrm{id}} & X_1 \times X_1 \times X_1^n \\ \downarrow{\scriptstyle *\times\mathrm{id}\times\mathrm{id}} & & \downarrow{\scriptstyle *\times\mathrm{id}\times\mathrm{id}} \\ * \times * \times X_1^n & \xrightarrow{\mathrm{id}\times i\times \mathrm{id}} & * \times X_1 \times X_1^n, \end{array}$$

where $i\colon * \to X_1$ is the inclusion of the basepoint. This shows that (9.4.22) is a homotopy pullback. In the second case, when α is a face map other than d_0 the square is equivalent to

$$\begin{array}{ccc} X_1 \times X_2 \times X_1^{n-2} & \xrightarrow{\mathrm{id}\times d_1\times \mathrm{id}} & X_1 \times X_1 \times X_1^{n-2} \\ \downarrow{\scriptstyle *\times\mathrm{id}\times\mathrm{id}} & & \downarrow{\scriptstyle *\times\mathrm{id}\times\mathrm{id}} \\ * \times X_2 \times X_1^{n-2} & \xrightarrow{\mathrm{id}\times d_1\times \mathrm{id}} & * \times X_1 \times X_1^{n-2}, \end{array}$$

where $d_1\colon X_2 \to X_1$ is the middle face map, which models the multiplication map of X_1 up to equivalence. So again (9.4.22) is a homotopy pullback square. In the last case of $\alpha = d_0$, the same analysis simplifies the square to

$$\begin{array}{ccc} X_2 \times X_1^{n-1} & \xrightarrow{d_1\times \mathrm{id}} & X_1 \times X_1^{n-1} \\ \downarrow{\scriptstyle d_0\times\mathrm{id}} & & \downarrow{\scriptstyle *\times\mathrm{id}} \\ X_1 \times X_1^{n-1} & \xrightarrow{*\times \mathrm{id}} & * \times X_1^{n-1}. \end{array}$$

Letting $\mu\colon X_1 \times X_1 \to X_1$ be any representative of the homotopy class of the multiplication map, this square is a homotopy pullback if the simpler square

$$\begin{array}{ccc} X_1 \times X_1 & \xrightarrow{\mu} & X_1 \\ \downarrow{\scriptstyle *\times\mathrm{id}} & & \downarrow \\ X_1 & \longrightarrow & * \end{array}$$

is a homotopy pullback, and this holds when $\pi_0(X_1)$ is a group by Exercise 9.6.1(d).

By Theorem A.2.6 we therefore have that when $X_\bullet$ is grouplike, the homotopy fiber of $|PX_\bullet| \to |X_\bullet|$ is equivalent to X_1, the equivalence given by the standard inclusion of $X_1 = PX_0$ into the realization. We now form the map of fiber sequences

$$\begin{array}{ccccc} X_1 & \longrightarrow & |PX_\bullet| & \longrightarrow & |X_\bullet| \\ \downarrow & & \downarrow{\scriptstyle \sim} & & \| \\ \Omega|X_\bullet| & \longrightarrow & P|X_\bullet| & \longrightarrow & |X_\bullet| \end{array}$$

where the middle vertical is the explicit simplicial homotopy from Exercise 7.7.14(c), composed with the projection from $|PX_\bullet|$ to $|X_\bullet|$. Explicitly, each point $x \in PX_0 = X_1$ gives a degenerate path $s_1 x \in PX_1 = X_2$, which is sent by this homotopy to the path in $|PX_\bullet|$ given by the 1-simplex $f_{\mathrm{id}}^*(s_1 x) \in X_2$, where $f_{\mathrm{id}} : [2] \to [2]$ is the map $0, 1 \mapsto 0$ and $2 \mapsto 2$. This projects down to the path in $|X_\bullet|$ given by the 1-simplex $d_0 f_{\mathrm{id}}^*(s_1 x) \in X_1$, which by comparing definitions is just $x \in X_1$. Therefore the induced map $X_1 \to \Omega|X_\bullet|$ is the canonical one that sends each point of X_1 to the loop that traverses its corresponding 1-cell in the realization. This is exactly the unit map that we wanted to show is an equivalence, so the proof is complete. □

Corollary 9.4.23. *For any path-connected based space Y, the counit map $(\mathbb{L}B)\Omega_\bullet Y \to Y$ is an equivalence. When Y is not path-connected, the map is an equivalence from $(\mathbb{L}B)\Omega_\bullet Y$ to the path component of the basepoint of Y.*

Proof. By Exercise 9.6.26, the counit map is the universal approximation of each based space by a path-connected based space. It is straightforward to check that the inclusion of the path component of the basepoint is another such universal approximation, but universal approximations are unique and so these must agree. □

The A_∞ **group completion** of $X_\bullet$ is defined to be the derived unit map $X_\bullet \to \Omega_\bullet(\mathbb{L}B)X$, or just $X_\bullet \to \Omega_\bullet BX$ when $X_\bullet$ is cofibrant. This is justified by the following corollary.

Corollary 9.4.24. *The derived unit map $X_\bullet \to \Omega_\bullet(\mathbb{L}B)X$ is the universal approximation of $X_\bullet$ up to weak equivalence by a grouplike Segal A_∞ space. And when $X_\bullet = N_\bullet(M)$, it is equivalent to the homotopical group completion $N_\bullet(M) \to N_\bullet(L_{\mathrm{Grp}}(M))$ from Definition 9.1.11.*

Proof. The first claim is proven in Exercise 9.6.26. The second claim follows from Proposition 9.4.21 using the same argument as in Theorem 9.1.18. □

Next we use this A_∞ proof to do the E_∞ case.

Proposition 9.4.25. *For any Reedy cofibrant Segal E_∞ space $X_\bullet : \mathbf{Fin}_* \to \mathbf{Top}$, the spectrum $B^\infty X$ is a positive Ω-spectrum,*

$$|X(S^k_\bullet)| \xrightarrow{\sim} \Omega|X(S^{k+1}_\bullet)| \qquad \text{when } k \geq 1,$$

and furthermore $|X(S^k_\bullet)|$ is a $(k-1)$-connected space for all $k \geq 0$.

Proof. We start by proving the connectivity claim. Since $P^{\mathscr{W}}_{\mathbf{Fin}_*}$ is a left Kan extension, by Theorem 8.3.19 its left-derived functor can be given by the categorical bar construction as in Example 8.3.31:

$$P^{\mathscr{W}}_{\mathbf{Fin}_*} X \simeq B^\wedge(X, \mathbf{Fin}_*, \mathscr{W}).$$

Using Exercise 8.7.19, since $\mathrm{Map}_*(\mathbf{m}_+, S^k)$ is $(k-1)$-connected for every $k \geq 0$, the bar construction $B^\wedge(X(-), \mathbf{Fin}_*, \mathrm{Map}_*(-, S^k))$ is also $(k-1)$-connected. But this is equivalent to $B^k X$, so the connectivity claim is proven.

Now by Lemma 9.4.17, the bonding map $|X(S^k_\bullet)| \to \Omega|X(S^{k+1}_\bullet)|$ is the canonical map $Y_1 \to \Omega|Y_\bullet|$ for the Segal A_∞ space $Y(\mathbf{n}_+) = X(S^k_\bullet \wedge \mathbf{n}_+)$. When $k \geq 1$, Y_1 is connected, so by the A_∞ group completion result (Proposition 9.4.21), this map is an equivalence. □

Note that the remaining bonding map $|X(S^0_\bullet)| \to \Omega|X(S^1_\bullet)|$ is just the unit of the A_∞ adjunction, $X_\bullet \to \Omega_\bullet BX$. Therefore when $X_\bullet$ is grouplike (and cofibrant), we can conclude that $B^\infty X$ is an Ω-spectrum, and that the unit map of the E_∞ adjunction is an equivalence of spaces:

$$X_\bullet \xrightarrow{\sim} \Omega^\infty_\bullet B^\infty X.$$

Since Ω^∞ preserves and reflects weak equivalences of connective spectra, this finishes the proof of the equivalence claim from Theorem 9.4.19.

Corollary 9.4.26. *For any connective spectrum Y, the derived counit map $(\mathbb{L}B^\infty)\Omega^\infty_\bullet Y \to Y$ is an equivalence. When Y is not connective, the map is a universal approximation of Y by a connective spectrum, in other words the connective cover of Y.*

As before, we say the E_∞ **group completion** of $X_\bullet$ is the derived unit map $X_\bullet \to \Omega^\infty_\bullet(\mathbb{L}B^\infty)X$, or just $X_\bullet \to \Omega^\infty_\bullet B^\infty X$ when $X_\bullet$ is cofibrant.

Corollary 9.4.27. *This is the universal approximation of $X_\bullet$ up to weak equivalence by a grouplike Segal E_∞ space, and on the underlying A_∞ spaces it agrees with the A_∞ group completion map from Corollary 9.4.24.*

Again, both of these corollaries follow from the formal argument in Exercise 9.6.26. (Except for the last part of the second corollary, which follows by inspecting the definition.)

Remark 9.4.28. The McDuff–Segal group completion theorem (Theorem 9.1.26) also applies to the A_∞ and E_∞ group completions that we have defined here. This is because every A_∞ space is equivalent to one that comes from a strict monoid by Corollary 9.4.41.

Remark 9.4.29. We have given the proof of the group completion property for $X \to \Omega BX$ that comes from [Seg74]. In [BF78], Bousfield and Friedlander give a different proof, using the π_*-Kan condition in the place of descent for homotopy colimits. In [May75, Section 15] and [MMO25, Section 8] the authors prove that the conclusion of the McDuff–Segal group completion theorem holds for $X \to \Omega BX$. Under some mild commutativity hypotheses this also shows that $X \to \Omega BX$ is equivalent to the group completion. See Warning 9.1.36.

9.4.5 Categories of operators

At this point we have used the term "E_∞ space" to refer to two completely different notions:

- algebras over an E_∞-operad $\mathcal{O}$, and
- reduced diagrams $\mathbf{Fin}_* \to \mathbf{Top}_*$ satisfying the Segal condition.

In this section we explain why these two notions are equivalent, following the work of May and Thomason [MT78] and its reworking in [MMO25]. The key idea is to introduce a third notion of E_∞ space that sits in between these. The following definition is a variant of the one found in [MT78, MMO25], that is adapted to more closely resemble [Lur17].

Definition 9.4.30. An **inert morphism** $\alpha\colon \mathbf{m}_+ \to \mathbf{n}_+$ is a map of finite sets such that $\alpha^{-1}(i)$ is a singleton for each $1 \le i \le n$. Let $\Pi \subseteq \mathbf{Fin}_*$ be the full subcategory of all the finite based sets $\mathbf{n}_+$ and the inert maps between them.

Intuitively, Π is the subcategory generated by the permutations and the "projection" maps that just send some of the points of $\mathbf{n}_+$ to the basepoint. All together, these are the maps that don't perform any multiplications, hence the term "inert." Each of the Segal maps is an inert morphism, so it still makes sense to ask that whether a diagram $X : \Pi \to \mathbf{Top}_*$ has the Segal condition.

Warning 9.4.31. This is the unbased definition of Π. The based version of Π from [MT78, MMO25] is defined by allowing $\alpha^{-1}(j)$ to be a singleton *or empty*. In other words, the injective maps $\mathbf{m}_+ \to \mathbf{n}_+$ are also included. The based version of Π adds canonical basepoints to our algebras – this is fine when considering operads such that $\mathcal{O}(0) = *$, but it doesn't work for more general operads.

Definition 9.4.32. An (unbased) **category of operators** is a topological category $\mathcal{D}$ that sits in between Π and $\mathbf{Fin}_*$, in the sense that there are functors

$$\Pi \longrightarrow \mathcal{D} \longrightarrow \mathbf{Fin}_*$$

that are the identity on the set of objects, and whose composition is the inclusion $\Pi \subseteq \mathbf{Fin}_*$.

If $\mathcal{D}$ is a category of operators, a **$\mathcal{D}$-algebra** is a diagram $X : \mathcal{D} \to \mathbf{Top}$, whose terms we refer to as $X_0, X_1, \ldots$, such that

- $X_0 = *$, and
- for each $n \ge 0$ the Segal maps induce a weak equivalence $X_n \simeq \prod_{j=1}^n X_1$.

A weak equivalence of such algebras is a map of diagrams $X \to X'$ that gives an equivalence $X_n \simeq X'_n$ for all $n \geq 0$, or equivalently just for $n = 1$. We let $\mathrm{Alg}_{\mathscr{D}}(\mathbf{Top})$ denote the category of $\mathscr{D}$-algebras.

Example 9.4.33. The categories $\mathbf{\Delta}^{\mathrm{op}}$ and $\mathbf{Fin}_*$ are both categories of operators, and their algebras are precisely the Segal A_∞ and E_∞ spaces, respectively.

Example 9.4.34. The category Π itself is a category of operators. Every unbased space Y defines a Π-algebra RY by the rule $RY(\mathbf{n}_+) = Y^{\times n}$, with each inert map $\alpha\colon \mathbf{m}_+ \to \mathbf{n}_+$ acting by sending the tuple $(y_1, \ldots, y_m)$ to the tuple $(y_{\alpha^{-1}(1)}, \ldots, y_{\alpha^{-1}(n)})$.[10]

It is straightforward to see that this is a right adjoint to the functor $LX = \mathrm{ev}_1 X$, which takes each Π-algebra X to its first term X_1. Furthermore, because of the Segal condition the unit of this adjunction $X \to RLX$ is always an equivalence of Π-algebras, and the counit $LRY \to Y$ is an isomorphism of unbased spaces. So Π-algebras up to equivalence are the same thing as unbased spaces.

Definition 9.4.35. We say that a $\mathscr{D}$-algebra X is in **standard form** if its underlying Π-algebra is of the form RX_1. In other words, the spaces of the diagram are just $X_1^{\times n}$, and the inert morphisms in $\Pi \subseteq \mathscr{D}$ act in a standard way. The rest of $\mathscr{D}$ is still allowed to act however it likes.

Example 9.4.36. For each coordinate-free operad $\mathscr{O}$ of topological spaces, we define a category of operators $\mathscr{O}^\otimes$ in which

$$\mathscr{O}^\otimes(\mathbf{m}_+, \mathbf{n}_+) = \coprod_{\alpha\colon \mathbf{m}_+ \to \mathbf{n}_+} \prod_{j \in \mathbf{n}} \mathscr{O}(\alpha^{-1}(j)),$$

and composition coming from the multiplication of $\mathscr{O}$. The projection $\mathscr{O}^\otimes \to \mathbf{Fin}_*$ remembers only the morphism α. The inclusion $\Pi \to \mathscr{O}^\otimes$ sends each inert map $\rho\colon \mathbf{m}_+ \to \mathbf{n}_+$ to the point in $\mathscr{O}^\otimes(\mathbf{m}_+, \mathbf{n}_+)$ in which the map of finite sets is ρ, and for each $j \in \mathbf{n}$ we select the identity element $\mathrm{id} \in \mathscr{O}(\rho^{-1}(j)) \cong \mathscr{O}(1)$.[11]

This is an important example, so you should think about it a little. A map in $\mathscr{O}^\otimes$ is a map of finite based sets $\mathbf{m}_+ \to \mathbf{n}_+$ together with "decorations" by operations in $\mathscr{O}(\alpha^{-1}(j))$, one for each $j \in \mathbf{n} = \{1, \ldots, n\}$. The decorations tell us which operation to use when combining points of $\mathbf{m}$ together to go to a single point of $\mathbf{n}$.

Now it should be clear why each $\mathscr{O}$-algebra X defines an $\mathscr{O}^\otimes$-algebra. We define the $\mathscr{O}^\otimes$-algebra by taking the nth term to be $X^{\times n}$, and we define the $\mathscr{O}^\otimes$-action

$$X^{\times m} \times \mathscr{O}^\otimes(\mathbf{m}_+, \mathbf{n}_+) \longrightarrow X^{\times n}$$

[10] This also works for the based version of Π if we assume that Y is a based space. When defining the action of $\alpha\colon \mathbf{m}_+ \to \mathbf{n}_+$, we take the basepoint $* \in Y$ in each coordinate j for which $\alpha^{-1}(j) = \varnothing$.

[11] When we use the based version of Π, we have to assume that $\mathscr{O}(0) = *$. Then when defining $\Pi \to \mathscr{O}^\otimes$, we take this unique point for each $j \in \mathbf{n}$ such that when $\rho^{-1}(j) = \varnothing$.

by taking for each α the product over each $j \in \mathbf{n}$ of the $\mathcal{O}$-action maps

$$X^{\times \alpha^{-1}(j)} \times \mathcal{O}(\alpha^{-1}(j)) \longrightarrow X.$$

It is straightforward to check that this defines an $\mathcal{O}^{\otimes}$-algebra in standard form. In fact, these are all of them, by Exercise 9.6.27:

Lemma 9.4.37. *There is an isomorphism of categories between algebras over the operad $\mathcal{O}$, and algebras over the category of operators $\mathcal{O}^{\otimes}$ that are in standard form.*

Now the main technical result is the following. It requires the use of monads, so we prove it in Proposition A.3.37:

Proposition 9.4.38. *Let $\mathcal{O}$ be an operad. As in Definition 9.3.17, assume that the balanced product $\mathcal{O}(n) \times_{\Sigma_n} X^{\times n}$ is always equivalent to the corresponding homotopy orbits $\mathcal{O}(n) \times^h_{\Sigma_n} X^{\times n}$.*

Then there is a weak equivalence between the identity functor on $\mathrm{Alg}_{\mathcal{O}^{\otimes}}(\mathbf{Top})$ and a functor that sends every $\mathcal{O}^{\otimes}$-algebra to one that is in standard form.

In particular, this holds for any A_∞ or E_∞ operad. It does not hold for the commutative operad, because $\mathrm{Comm}^{\otimes}$-algebras are Segal E_∞ spaces, whereas Comm-algebras are strictly commutative monoids, and these are not the same thing up to weak equivalence.

Now let $\mathcal{O}$ be any E_∞ operad. The map $\mathcal{O}^{\otimes} \to \mathbf{Fin}_*$ is a pointwise equivalence of topological categories, meaning that it is an equivalence on each mapping space (because the spaces $\mathcal{O}(k)$ are contractible). Therefore by Exercise 5.8.16 we have a Quillen equivalence between diagrams over the two categories, so we get in particular an equivalence of homotopy categories

$$\mathrm{Ho}\,\mathbf{Top}^{\mathcal{O}^{\otimes}} \simeq \mathrm{Ho}\,\mathbf{Top}^{\mathbf{Fin}_*}.$$

Along this equivalence the Segal conditions correspond, and any diagram in which X_0 is contractible can be replaced by an equivalent diagram with $X_0 = *$ using the technique of Exercise 8.7.28. Therefore we get an equivalence of homotopy categories

$$\mathrm{Ho}\,\mathbf{Top}^{\mathcal{O}^{\otimes}}_{\mathrm{Seg}} \simeq \mathrm{Ho}\,\mathbf{Top}^{\mathbf{Fin}_*}_{\mathrm{Seg}}.$$

Together with Proposition 9.4.38, this proves:

Theorem 9.4.39. *For any E_∞ operad $\mathcal{O}$, we have an equivalence of homotopy categories*

$$\mathrm{Ho}\,\mathrm{Alg}_{\mathcal{O}}(\mathbf{Top}) \simeq \mathrm{Ho}\,\mathbf{Top}^{\mathbf{Fin}_*}_{\mathrm{Seg}}.$$

In other words, our two notions of "E_∞ space" are equivalent to each other. May and Thomason go further in [MT78] and also show that the corresponding delooping spectra

B^∞ are the same, a result known as the "uniqueness of delooping machines."[12] There is no need to prove this here, because we never gave a precise definition of B^∞ in the operad case.

To prove the same comparison result for A_∞ spaces, we let $\mathcal{O}$ be any A_∞ operad, and modify the definition of $\mathcal{O}^\otimes$ to only use maps in the subcategory $\mathbf{\Delta}^{\mathrm{op}} \subseteq \mathbf{Fin}_*$. This "$A_\infty$ category of operators" $\mathcal{O}^\otimes_{A_\infty}$ is weakly equivalent to the category $\mathbf{\Delta}^{\mathrm{op}}$, so as before we get a Quillen equivalence between $\mathbf{\Delta}^{\mathrm{op}}$-diagrams and algebras over $\mathcal{O}^\otimes_{A_\infty}$, preserving the Segal condition. Putting this all together, this proves:

Theorem 9.4.40. *For any A_∞ operad $\mathcal{O}$, we have an equivalence of homotopy categories*

$$\mathrm{Ho}\,\mathrm{Alg}_{\mathbf{O}}(\mathbf{Top}) \simeq \mathrm{Ho}\,\mathbf{Top}^{\mathbf{\Delta}^{\mathrm{op}}}_{\mathrm{Seg}}.$$

In other words, the two notions of A_∞ spaces that we have seen are equivalent to each other. Taking $\mathcal{O}$ to be the associative operad and tracing through the construction, we see that each strict monoid M gives the $\mathcal{O}^\otimes_{A_\infty}$-algebra in standard form whose nth term is $M^{\times n}$. Along the isomorphism $\mathbf{O}^\otimes_{A_\infty} \cong \mathbf{\Delta}^{\mathrm{op}}$, this corresponds to the Segal A_∞ space $N_\bullet(M)$.

Corollary 9.4.41. *Segal A_∞ spaces are equivalent to topological monoids: each one is weakly equivalent to $N_\bullet(M)$ for some well-based topological monoid M.*

Example 9.4.42. There is a notion of a **non-symmetric operad**, which is like a symmetric operad except that we drop the Σ_n-action and the symmetry axiom. In the coordinate-free version, we have a space $\mathcal{O}(I)$ for each *totally ordered* finite set I, and we only allow compositions along maps of finite sets $I \to J$ that preserve the total ordering.

Every non-symmetric operad becomes a symmetric operad, by multiplying the nth level $\mathcal{O}(n)$ by the symmetric group Σ_n. But there are not many examples of symmetric operads that arise this way. Essentially the only example of interest is the associative operad Assoc, and the A_∞-operads. When we think of them as non-symmetric operads, we only take the connected component of the identity permutation $1 \in \Sigma_n$, so that $\mathrm{Assoc}(n) = *$ and $\mathcal{O}(n) \simeq *$ for any A_∞-operad $\mathcal{O}$.

However, one advantage of thinking this way is that the proof of Theorem 9.4.40 makes more sense: the non-symmetric category of operators should have a map to $\mathbf{\Delta}^{\mathrm{op}}$ instead of $\mathbf{Fin}_*$, because there are no symmetric group actions anymore.

[12] Their proof proceeds by running both a May-style delooping and a Segal-style delooping at the same time, producing a bispectrum in the sense of Exercise 5.8.35, and then using the fact that for an Ω-bispectrum the two ordinary spectra sitting inside must be equivalent to each other.

9.4.6 Multicategories

In this section we describe a "theory of everything" for multiplicative structures in homotopy theory. All of the structures on topological spaces that we have seen so far, including diagrams of spaces, algebras over an operad, and even modules over an algebra, can be unified using a single concept.

Definition 9.4.43. A symmetric **multicategory**, also known as a symmetric **colored operad**, is

- a collection of objects or "colors" $\mathrm{ob}\,\mathscr{M}$,
- for each finite set I and function $a\colon I \to \mathrm{ob}\,\mathscr{M}$, in other words an indexed set of objects $\{a_i\}_{i\in I}$, and for each object b, a topological space of **multimorphisms** $\mathscr{M}(\{a_i\}_{i\in I}; b)$,
- an identity element $\mathrm{id}_{a,i} \in \mathscr{M}(\{a\}_{i\in\{i\}}; a)$ for each object a and singlelton $\{i\}$, and
- continuous composition maps

$$\circ\colon \mathscr{O}(\{b_j\}_{j\in J}; c) \times \prod_{j\in J} \mathscr{O}(\{a_i\}_{i\in\alpha^{-1}(j)}; b_j) \longrightarrow \mathscr{O}(\{a_i\}_{i\in I}; c)$$

  for each map of finite sets $\alpha\colon I \to J$ and indexed families of objects $\{a_i\}_{i\in I}$, $\{b_j\}_{j\in J}$, and c,

such that the following equations hold whenever they make sense:

- (Identity) $\mathrm{id}_{b,j} \circ f = f$ when $f \in \mathscr{M}(\{a_i\}_{i\in I}; b)$, $\mathrm{id}_{b,j} \in \mathscr{M}(\{b\}_{j\in\{j\}}; b)$ and the composition is along the unique map $I \to \{j\}$, and $f = f \circ (\mathrm{id}_{a_i,i})_{i\in I}$ when $f \in \mathscr{M}(\{a_i\}_{i\in I}; b)$, $\mathrm{id}_{a_i,i} \in \mathscr{M}(\{a_i\}_{i\in\{i\}}; a_i)$ and the composition is along the identity map $I \to I$,
- (Associativity) for two composable maps $\alpha\colon I \to J$ and $\beta\colon J \to K$ and elements $f \in \mathscr{O}(\{c_k\}_{k\in K}; d)$, $g_k \in \mathscr{O}(\{b_j\}_{j\in\beta^{-1}(k)}; c_k)$, and $h_j \in \mathscr{O}(\{a_i\}_{i\in\alpha^{-1}(j)}; b_j)$ we have

$$(f \circ (g_k)_{k\in K}) \circ (h_j)_{j\in J} = f \circ (g_k \circ (h_j)_{j\in\beta^{-1}(k)})_{k\in K}.$$

A **multifunctor** or map of multicategories $\mathscr{M} \to \mathscr{M}'$ consists of a function $F\colon \mathrm{ob}\,\mathscr{M} \to \mathrm{ob}\,\mathscr{M}'$ and continuous maps $\mathscr{M}(\{a_i\}_{i\in I}; b) \to \mathscr{M}'(\{F(a_i)\}_{i\in I}; F(b))$, commuting with the composition and preserving the identity elements. The category of multicategories is denoted Mult(**Top**). We similarly define Mult(**C**) for any symmetric monoidal category **C** by asking that the objects $\mathscr{M}(\{a_i\}_{i\in I}; b)$ be objects of **C**, and using the tensor product in **C** instead of the Cartesian product when defining the composition map.

As before, these have a non-coordinate-free version in which we write the morphism sets as $\mathscr{M}(a_1, \ldots, a_n; b)$ and then we have compositions and Σ_n-actions, and we sometimes use this notation when it is convenient, for instance just writing $\mathscr{M}(a; a)$ instead of $\mathscr{M}(\{a\}_{i\in\{i\}}; a)$.

Example 9.4.44. Of course, an operad $\mathscr{O}$ is the same thing as a multicategory $\mathscr{M}$ with one object. A map between two of these is just a map of operads.

Example 9.4.45. Every topological category $\mathbf{I}$ defines a multicategory in which there are only n-ary morphisms for $n = 1$. So $\mathscr{M}(a_1, \ldots, a_n; b)$ is empty unless $n = 1$, in which case it is just the space $\mathbf{I}(a_1, b)$. A map between two of these is just a topological functor.

Example 9.4.46. Every symmetric monoidal category $(\mathbf{C}, \otimes, I)$ defines a multicategory $\mathscr{M}_{\mathbf{C}}$. The set of multimorphisms $\mathscr{M}_{\mathbf{C}}(\{a_i\}_{i\in I}; b)$ is the set of morphisms out of the tensor product, $\mathbf{C}(\bigotimes_{i\in I} a_i, b)$. If $\mathbf{C}$ is enriched in topological spaces then we use this topology as well. A map between two of these multicategories $\mathscr{M}_{\mathbf{C}} \to \mathscr{M}_{\mathbf{D}}$ is the same thing as a lax symmetric monoidal functor $\mathbf{C} \to \mathbf{D}$.

Definition 9.4.47. Let $\mathscr{M}$ be a multicategory. An **algebra** or **multi-diagram** over $\mathscr{M}$ consists of

- a topological space $X(a)$ for each object $a \in \operatorname{ob}\mathscr{M}$ and
- action maps $\mathscr{M}(\{a_i\}_{i\in I}; b) \times \prod_{i\in I} X(a_i) \to X(b)$

such that

- (Identity) $\mathrm{id}_{a,i}(x) = x$, and
- (Associativity) for any $\alpha\colon I \to J$, $f \in \mathscr{M}(\{b_j\}_{j\in J}; c)$, $g_j \in \mathscr{M}(\{a_i\}_{i\in\alpha^{-1}(j))}; b_j)$, and $x_i \in X(a_i)$ we have

$$(f \circ (g_j)_{j\in J})(x_i)_{i\in I} = f(g_j(x_i)_{i\in\alpha^{-1}(j)})_{j\in J}.$$

A map of $\mathscr{M}$-algebras consists of maps $X(a) \to X'(a)$ commuting with the action maps. The **category of $\mathscr{M}$-algebras** is denoted $\mathrm{Alg}_{\mathscr{M}}(\mathbf{Top})$. As before, this can be generalized to any symmetric monoidal category $\mathbf{C}$, giving a category of algebras $\mathrm{Alg}_{\mathscr{M}}(\mathbf{C})$.

Think of this as a more general, "multiplicative" kind of diagram. In an ordinary diagram, we have morphisms $a \to b$ that act by continuous maps $X(a) \to X(b)$. In a multi-diagram, we can also have multimorphisms $\{a_i\}_{i\in I} \to b$ that act by continuous multiplication maps $\prod_{i\in I} X(a_i) \to X(b)$.

All of the theories of homotopy-coherent multiplication that we have seen are special kinds of multi-diagrams:

Example 9.4.48. If $\mathcal{O}$ is an operad, and we think of it as a multicategory with one object, then a multi-diagram over this multicategory is the same thing as an $\mathcal{O}$-algebra. So A_∞ and E_∞ spaces in the first sense are examples of multi-diagrams.

Example 9.4.49. If $\mathbf{I}$ is a topological category and we think of it as a multicategory with only 1-ary morphisms, then a multi-diagram over this is the same thing as an ordinary diagram over $\mathbf{I}$. So A_∞ and E_∞ spaces in the second (Segal) sense are also examples of multi-diagrams. And algebras over a category of operators are also a kind of multi-diagram.

Example 9.4.50. We can recast multi-diagrams in the following way. Consider the multicategory $\mathcal{M}_{\mathbf{Top}}$ that is defined out of the symmetric monoidal category of topological spaces. So it has an object for each space X, and the set of multimorphisms $\mathcal{M}_{\mathbf{Top}}(\{A_i\}; B)$ is the space $\mathrm{Map}(\prod_{i\in I} A_i, B)$. Then a multi-diagram on any other multicategory $\mathcal{M}$ is just a map of multicategories $\mathcal{M} \to \mathcal{M}_{\mathbf{Top}}$.

Example 9.4.51. If $\mathcal{O}$ is an operad and X is an $\mathcal{O}$-algebra, an **$\mathcal{O}$-module** over X is a space Y and action maps

$$\mathcal{O}(n) \times X^{\times(n-1)} \times Y \to Y$$

that satisfy the same identities as in Definition 9.3.4, only that for every tuple of coordinates $(x_1, \ldots, x_n)$ one of the coordinates must be in Y. This can be recast by defining a multicategory $\mathcal{M}$ with two objects x and y, and taking the n-ary morphisms to be

- $\mathcal{O}(n)$ if the target object is x and all of the source objects are x,
- $\mathcal{O}(n)$ if target object is y and exactly one of the source objects is y, and
- $\varnothing$ otherwise.

In other words,

$$\mathcal{M}(x, \ldots, x; x) = \mathcal{O}(n), \qquad \mathcal{M}(x, \ldots, x, y; y) = \mathcal{O}(n),$$
$$\mathcal{M}(x, \ldots, x, y; x) = \varnothing, \qquad \mathcal{M}(x, \ldots, x, y, y; y) = \varnothing, \ldots$$

Then an $\mathcal{M}$-algebra is the same thing as an $\mathcal{O}$-algebra X and an $\mathcal{O}$-module Y over X.

Example 9.4.52. As a special case of Example 9.4.51, when $\mathcal{O} = \mathrm{Comm}$ we get a multicategory $\mathscr{C}\mathcal{M}$ that has two objects x and y, and n-ary morphisms

- $*$ if the target object is x and all of the source objects are x,
- $*$ if target object is y and exactly one of the source objects is y, and
- $\varnothing$ otherwise.

Then a $\mathscr{C}\mathscr{M}$-algebra (or a Comm-module) is the same thing as a commutative algebra or monoid X, and an X-module Y.

Example 9.4.53. As a special case of Example 9.4.51, when $\mathscr{O} = \mathrm{Assoc}$, we get the "bimodule multicategory" $\mathscr{B}\mathscr{M}$. Again this has two objects x and y, and n-ary morphisms

- Σ_n if the target object is x and all of the source objects are x,
- Σ_n if target object is y and exactly one of the source objects is y, and
- $\varnothing$ otherwise.

Then a $\mathscr{B}\mathscr{M}$-algebra or an Assoc-module is the same thing as an associative algebra or monoid X, and an X-bimodule Y. The fact that we can rearrange the terms $(x_1, \ldots, x_{n-1}, y)$ in any order before we multiply them means that Y ends up getting multiplication actions on both the left and right sides.

Example 9.4.54. [Lur17, 4.2.1.1] We can also define a **left module multicategory** $\mathscr{L}\mathscr{M}$ by hand. This is the sub-multicategory of $\mathscr{B}\mathscr{M}$ in which we have two objects x and y, and multimorphisms

- Σ_n if the target object is x and all of the source objects are x,
- if the target is y, and exactly one of the source objects is y, which is in the ith slot, then the morphisms are the subset $S \subseteq \Sigma_n$ corresponding to those permutations that send i to n, and
- $\varnothing$ otherwise.

We check that these are closed under compositions and therefore give a sub-multicategory. An $\mathscr{L}\mathscr{M}$-algebra is the same thing as an associative algebra or monoid X, and a left X-module Y. The fact that we can only rearrange the terms $(x_1, \ldots, x_{n-1}, y)$ so that the y is at the end before multiplying them, means that we only get a multiplication on the left-hand side.[13]

The same proofs as in Corollary A.3.22 and Proposition A.3.26 show that

Proposition 9.4.55. *For each topological multicategory $\mathscr{M}$, the category of algebras $\mathrm{Alg}_{\mathscr{M}}(\mathbf{Top})$ has a model structure in which a weak equivalence or fibration of algebras $X \to X'$ is a map such that each of the maps $X(a) \to X'(a)$ is a weak equivalence or fibration, respectively.*

Any map of multicategories $\phi : \mathscr{M} \to \mathscr{M}'$ induces a Quillen adjunction $(\phi_!, \phi^*)$ between $\mathrm{Alg}_{\mathscr{M}}(\mathbf{Top})$ and $\mathrm{Alg}_{\mathscr{M}'}(\mathbf{Top})$.

[13] This is not a special case of Example 9.4.51, and can't be defined for any operad $\mathscr{O}$! It can only be defined for those operads that come from non-symmetric operads, as defined in Example 9.4.42.

Every multicategory also has an associated category of operators:

Example 9.4.56. For each topological multicategory $\mathscr{M}$, we define a category $\mathscr{M}^{\otimes}$ in which the objects are finite sets I and maps $I \to \operatorname{ob}\mathscr{M}$, and the morphisms are

$$\mathscr{M}^{\otimes}(\{a_i\}_{i\in I}, \{b_j\}_{j\in J}) = \coprod_{\alpha\colon I_+\to J_+} \prod_{j\in J} \mathscr{M}(\{a_i\}_{i\in\alpha^{-1}(j)}; b_j).$$

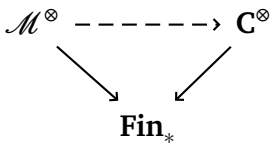
As in Example 9.4.36, the composition in this category is defined using the composition in $\mathscr{M}$. The projection $\mathscr{M}^{\otimes} \to \mathbf{Fin}_*$ remembers only the morphism α.

As before, a map in $\mathscr{M}^{\otimes}$ is a map between collections of objects $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$, in which the objects a_i are first sorted into groups $\{a_i\}_{i\in\alpha^{-1}(j)}$, and then each of these groups is sent to a single object b_j by a single multi-morphism in $\mathscr{M}$.

We call $\mathscr{M}^{\otimes}$ the **category of operators** associated to $\mathscr{M}$. Actually, it is not quite a category of operators in the sense of Definition 9.4.32, because it has too many objects. But we can still define an **inert morphism** in $\mathscr{M}^{\otimes}$ to be any morphism that consists of an inert morphism $\alpha\colon I_+ \to J_+$, and the choice of the identity element in each of the spaces $\mathscr{M}(\{a_i\}, a_i)$. In other words, an inert morphism is anything obtained from permutation and projection maps applied to a tuple of objects $\{a_i\}_{i\in I}$.

In the special case that $\mathscr{M}$ came from a symmetric monoidal category $(\mathbf{C}, \otimes, 1)$, we call its category of operators $\mathbf{C}^{\otimes}$. It has an object for each finite family of objects $\{a_i\}_{i\in I}$, and the morphisms are maps of finite based sets $\alpha\colon I_+ \to J_+$ together with maps of the form $\bigotimes_{i\in\alpha^{-1}(j)} a_i \to b_j$ for each $j \in J$. The inert morphisms are the inert maps $I_+ \to J_+$ together with identity maps $a_i = a_i$.

This is a neat way of expressing the symmetric monoidal structure of $\mathbf{C}$ using only a single category, no functors or natural isomorphisms are involved. We will see later that in fact the category $\mathbf{C}^{\otimes}$ and its projection functor to $\mathbf{Fin}_*$ completely describes the symmetric monoidal structure on $\mathbf{C}$ (Exercise 10.8.27), and a multi-diagram in $\mathbf{C}$ can be elegantly described as a functor $\mathscr{M}^{\otimes} \to \mathbf{C}^{\otimes}$ that commutes with the projection to finite sets, and that preserves the inert morphisms (see Exercise 9.6.28):



So now we can study any sort of multi-diagram using just ordinary diagrams! And this can be done in any symmetric monoidal category $\mathbf{C}$, not just a category like **Top** where the tensor product is the Cartesian product! The only tradeoff is redundancy: the category of operators $\mathscr{M}^{\otimes}$ is much larger and more redundant than $\mathscr{M}$, because every set of multi-morphisms $\mathscr{M}(\{a_i\}_{i\in I}; b)$ occurs infinitely many times in $\mathscr{M}^{\otimes}$, not just once.

However, this is really really useful. It's so useful that Lurie uses this idea to define multi-diagrams, and therefore all sorts of homotopy-coherent algebras, in the setting of ∞-categories. See Section 10.7.

Remark 9.4.57. Ultimately, the reason why the Segal approach is simpler than the operadic approach is that the Segal approach manages to describe everything using ordinary diagrams instead of multi-diagrams. Using categories of operators, however, we can express every multi-diagram as some sort of ordinary diagram. Again, the tradeoff for this is redundancy: to describe an $\mathcal{O}$-algebra using ordinary categories, one needs to form a diagram in which each of the action maps $\mathcal{O}(n) \times X^{\times n} \to X$ occurs infinitely many times.

Remark 9.4.58. Multicategories are not actually a "theory of everything" because they only encode structures that have several inputs and one output. Sometimes you want operations with one input and multiple outputs, such as a coalgebra. Or even multiple inputs and multiple outputs. The PROPs of Boardman–Vogt can be used for this, but the reader is warned that they are less technically developed than operads and multicategories, because there are much more serious technical issues that arise when one tries to study coalgebras in spaces and spectra. See for instance [HS14, PS19].

9.5 Introduction to algebraic K-theory

Perhaps the most important point of the previous section is that for each symmetric monoidal category $(\mathbf{C}, \otimes, I)$, the classifying space $B\mathbf{C} = |N_\bullet \mathbf{C}|$ has the structure of a Segal E_∞ space, and therefore it can be turned into a spectrum $B^\infty(B\mathbf{C})$. We call this the **Segal K-theory** of $\mathbf{C}$.

This is a form of **algebraic K-theory**, a deep and difficult subject that is very important both to pure algebra and to homotopy theory. In this section we give the reader a small taste of this theory and some of its variants. Much more extensive information can be found in [Wei13].

To get the overall idea, K-theory is a universal way to take a moduli space of some sort of mathematical object, and formally add inverses to it. In other words, it is the group completion of a moduli space. And when the mathematical objects form cofiber sequences, we also ask that K-theory treats the cofiber sequences as if they were split. In this way, K-theory is like the reduced Euler characteristic $\widetilde{\chi}(X) = \chi(X) - 1$ of a based finite CW complex X, which has the additivity formula $\widetilde{\chi}(X) = \widetilde{\chi}(A) + \widetilde{\chi}(X/A)$. Or like the dimension of a free R-module, which also has the formula $\dim(M) = \dim(N) + \dim(M/N)$.

9.5.1 K-theory of a ring

Algebraic K-theory first arose as a rule that assigns rings to abelian groups.

Let R be any ring, and let Proj_R be the set of finitely generated projective left R-modules. Let $(\mathrm{Proj}_R/\cong)$ be the set of isomorphism classes. We make $(\mathrm{Proj}_R/\cong)$ into a monoid by taking the direct sum $\oplus$.

Definition 9.5.1. The 0th algebraic K-theory $K_0(R)$ is the discrete group completion (Section 9.1.3) of the monoid $((\mathrm{Proj}_R/\cong), \oplus, 0)$.

This is a functor of the ring R, since for every ring homomorphism $R \to S$ the extension of scalars $S \otimes_R (-)$ gives a well-defined map $(\mathrm{Proj}_R/\cong) \to ((\mathrm{Proj}_S/\cong)$, which gives a homomorphism $K_0(R) \to K_0(S)$.

Example 9.5.2. If every projective R-module is free, then $K_0(R) = \mathbb{Z}$. Each module P is sent to its rank, as an element of $\mathbb{Z}$. Since $K_0(R)$ is a group completion, it also contains "formal negatives" of R-modules as well, which is what gives us the negative integers and not just the positive ones.

There are many justifications for studying the group $K_0(R)$. One is that, when X is a finite cell complex and $C^0(X)$ is the set of continuous functions $X \to \mathbb{R}$, the finitely generated projective modules over $C^0(X)$ correspond to the finite-dimensional vector bundles over X, a result known as the Serre–Swan theorem [Wei13, EI.4.9]. Therefore the group $K_0(X)$ is actually isomorphic to $KO^0(X)$, the group completion of the set of isomorphism classes of vector bundles on X. In other words, *algebraic K-theory is a generalization of topological K-theory*.

Another motivation for $K_0(R)$ is that when R is a number ring, $K_0(R)$ turns out to be $\mathbb{Z}$ plus the ideal class group of R, which is a fundamental invariant in number theory.

A third motivation that is closer to the topologist's heart is as follows: when $R = \mathbb{Z}[\pi_1(X)]$ is a group ring on a fundamental group, there is a "Wall finiteness obsruction" that determines when X is homotopy equivalent to a finite complex, and this obstruction is an element of the quotient group

$$\widetilde{K}_0(\mathbb{Z}[\pi_1(X)]) \cong K_0(\mathbb{Z}[\pi_1(X)])/K_0(\mathbb{Z}).$$

This is one important example of an algebraic invariant of spaces that is *not* an extraordinary homology theory![14] Again, see [Wei13] for much more information.

[14] Although the K-theory spectrum itself gives an extraordinary homology theory by the rule $Y \mapsto \pi_*(Y \wedge K(\mathbb{Z}[\pi_1(X)])$, the rule that assigns X to this K-theory spectrum $K(\mathbb{Z}[\pi_1(X)])$ is not excisive, so it is not really a homology theory "in the X variable."

We can extend K_0 to a sequence of groups K_0, K_1, K_2, by thinking of the symmetric monoidal category of all finitely generated projective modules and isomorphisms between them, $\mathrm{Proj}_R^{\cong}$, and not just the isomorphism classes. Note that the classifying space $B\mathrm{Proj}_R^{\cong}$ has the homotopy type

$$B\mathrm{Proj}_R^{\cong} \simeq \coprod_{[P]\in(\mathrm{Proj}_R/\cong)} B\mathrm{Aut}(P),$$

where $\mathrm{Aut}(P) \subseteq \mathrm{Hom}_R(P,P)$ is the set of all R-linear automorphisms of P. When every R-module is free, this can be written more simply as

$$B\mathrm{Proj}_R^{\cong} \simeq \coprod_{n\geq 0} BGL_n(R).$$

Definition 9.5.3. The **algebraic K-theory spectrum** $K(R)$ is the delooping spectrum of the classifying space of this category of finitely generated projective R-modules and isomorphisms:

$$K(R) := B^{\infty}(B\mathrm{Proj}_R^{\cong}).$$

The nth algebraic K-theory group $K_n(R)$ is the nth homotopy group of this spectrum.

By our discussion of group completion, $K_n(R)$ is also the nth homotopy group of the space

$$\Omega^{\infty}K(R) = \Omega^{\infty}B^{\infty}(B\mathrm{Proj}_R^{\cong}) \simeq \Omega B(B\mathrm{Proj}_R^{\cong}),$$

which we call the algebraic K-theory space. Both the space and the spectrum have the property that the nth homotopy group is the group $K_n(R)$.

On K_0, this gives the same group that we gave in Definition 9.5.1. The group $K_1(R)$ turns out be the abelianization of the infinite general linear group $GL_{\infty}(R)^{\mathrm{ab}}$, where $GL_{\infty}(R)$ is the colimit of the automorphism groups of the modules $R^{\oplus n}$ as $n \to \infty$. Again when R is a number ring this has special significance, because it is the group of units $R^{\times}$. On the other hand, when $R = \mathbb{Z}[\pi_1(X)]$ this is the group where the "Whitehead torsion" lives, an invariant which helps us study high-dimensional manifolds by telling us when a cobordism is trivial.[15]

The higher K-groups contain deeper arithmetic information. For instance, they are related to special values of ζ-functions [Zhu15]. And they are unfortunately difficult to compute: we do not even know all of the groups $K_n(\mathbb{Z})$. Proving that $K_{4n}(\mathbb{Z}) = 0$ is equivalent to the Vandiver conjecture in number theory [Wei13, Section VI.10].

It was an impressive achievement to even define these groups. The first definition by Quillen was that the K-theory space is the "plus construction"

$$\Omega^{\infty}K(R) := K_0(R) \times BGL_{\infty}(R)^{+}.$$

[15] Actually the Whitehead torsion is an element of $K_1(\mathbb{Z}[\pi_1(X)]) = GL_{\infty}(\mathbb{Z}[\pi_1(X)])^{\mathrm{ab}}$ modulo the subgroup generated by the trivial units, i.e. the 1×1 invertible matrices with entries in $\pm\pi_1(X)$.

That is, the K-theory space has connected components $K_0(R)$, and each component is obtained from the classifying space $BGL_\infty(R)$ by constructing an acyclic map $BGL_\infty(R) \to BGL_\infty(R)^+$ that kills the commutator subgroup of π_1. (See Exercise 9.6.31.) The fact that this agrees with our group-completion definition is essentially a result of the McDuff–Segal group completion theorem (Theorem 9.1.26).

No matter which definition we use, the homology of the K-theory space is the homology of the infinite general linear group:

$$H_*(\Omega^\infty K(R)) \cong H_*(BGL_\infty(R)) \cong \operatorname*{colim}_{n\to\infty} H_*(BGL_n(R)).$$

And so, a final motivation for K-theory is that it is a spectrum that captures the limiting or stable behavior of the homology $H_*(BGL_n(R))$ as $n \to \infty$. This is perhaps the most important motivation for K-theory for someone who is interested in studying group homology and homological stability.

Remark 9.5.4. You might wonder why we used the finitely generated projective R-modules when defining $K(R)$, and we didn't just take *all* R-modules. The reason is that if we took all R-modules, the K-theory spectrum would be contractible! This is a result of a trick called the **Eilenberg swindle**.

To see how this works on K_0, we prove that the class of any projective module P is zero by taking a complementary projective module Q such that $P \oplus Q$ is free, and forming the isomorphism

$$P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots.$$

Letting F be the sum of countably many copies of $P \oplus Q$, we conclude that $[P]+[F]=[F]$ in this infinite version of K_0, which implies that $[P]=0$.

9.5.2 Examples of K-theory of a symmetric monoidal category

Of course, we can define K-theory in exactly the same way for any symmetric monoidal category $\mathbf{C}$. We let $K(\mathbf{C})$ denote the spectrum $B^\infty(B\mathbf{C})$, and let $\Omega^\infty K(\mathbf{C})$ denote its infinite loopspace, which is equivalent to $\Omega B(B\mathbf{C})$. Usually we restrict $\mathbf{C}$ to its subcategory of isomorphisms $\mathbf{C}^\cong$ before doing this construction. This fits the philosophy of K-theory because it is "the moduli space of all objects in $\mathbf{C}$, with inverses added."

The group $K_n(\mathbf{C})$ is defined to be the nth homotopy group of either of these objects. If we arrange so that $\mathbf{C}$ consists entirely of isomorphisms, then the group $K_0(\mathbf{C})$ is the free abelian group on the isomorphism classes $[X]$ of objects of $\mathbf{C}$, up to the relation that $[X \otimes Y] = [X]+[Y]$. (This follows from Exercises 9.6.2 and 9.6.7).

There are many fascinating examples this philosophy can be applied to. Perhaps the most basic one is the category of finite sets and isomorphisms under disjoint union, $(\mathbf{Fin}^\cong, \amalg, \varnothing)$.

Theorem 9.5.5 (Barratt–Priddy–(Segal)–Quillen)**.** *The algebraic K-theory of finite sets is the sphere spectrum, $K(\mathbf{Fin}^{\cong}) \simeq \mathbb{S}$.*

We omit the proof, at least in this version of the book. The equivalence is given by any map from $\mathbb{S}$ that takes the generator of π_0 to the element of $K_0(\mathbf{Fin}^{\cong})$ given by the one-point set.

Remark 9.5.6. Homotopy theorists sometimes joke that "the definition of the integers is wrong, it should have been $\mathbb{S}$ instead." The reason for this is that we defined the integers by taking all finite sets, passing to isomorphism classes, and then adding inverses (group completing). But this is unnatural. It is far more natural to take the moduli space of all finite sets $B\mathbf{Fin}^{\cong}$ and group complete *first*. By the Barratt–Priddy–Quillen theorem, this gives the space $\Omega^\infty\mathbb{S}$, or the spectrum $\mathbb{S}$. If we want to pass to isomorphism classes at the very end, in other words take π_0, we can do so and we get $\mathbb{Z}$ back. But with this "correct" definition of $\mathbb{Z}$ we now have higher homotopy groups, which represent the additive higher symmetries of finite sets: $\pi_1(\mathbb{S}) \cong \mathbb{Z}/2$, $\pi_2(\mathbb{S}) \cong \mathbb{Z}/2$, $\pi_3(\mathbb{S}) \cong \mathbb{Z}/24$, and so on.

The next example is that Eilenberg–Mac Lane spectra HA are also K-theory spectra.

Example 9.5.7. Let A be a discrete abelian group. Then it is a symmetric monoidal category in a trivial way, where the objects are the elements of A, and there are no morphisms other than the identity morphisms. The classifying space of this category is A again, as a discrete topological space. The K-theory of this category is the delooping spectrum $B^\infty A$, as we have observed, and therefore it is the Eilenberg–Mac Lane spectrum HA.

We could also modify this by letting $\mathscr{B}A$ be the category with one object and morphisms given by elements of A. It is then an exercise to see that this is a symmetric monoidal category, and its K-theory is equivalent to ΣHA.

We could also modify the above example to allow a topology on A, and if we do it right then we end up with the delooping spectrum $B^\infty A$, which is a product of Eilenberg–Mac Lane spectra by Corollary 9.2.19.

Example 9.5.8. Let $\mathbf{C}$ be the category of finite-dimensional real inner product spaces and isomorphisms between them.[16] We make this topological, so that each morphism space is homeomorphic to the orthogonal group $O(n)$, see Exercise 9.6.21. The Segal K-theory of the symmetric monoidal category $\mathbf{C}$ then turns out to be ko, the connective cover of KO. Repeating this example with complex vector spaces gives ku instead!

[16] For definiteness we make the set of objects into a set, for instance by assuming that the vector spaces are subspaces of a fixed copy of $\mathbb{R}^\infty$.

9.5.3 K-theory of DGAs, ring spectra, and spaces

One might want to define algebraic K-theory for ring objects in chain complexes or spectra. As before, it turns out we'll get zero unless we restrict to the finite modules, so we define those first:

Definition 9.5.9. If R is a ring, a **perfect R-module** is an R-module chain complex that is quasi-isomorphic to a bounded complex in which each term is finitely generated projective.

If R is a ring spectrum, a **perfect R-module** is a retract in $\mathrm{Ho}\,\mathrm{Mod}(R)$ of a cofibrant R-module that is built by attaching finitely many cells of the form $R \wedge F_i(S^{n-1}_+ \to D^n_+)$.

If R is a DGA, a **perfect R-module** is defined just like R-modules for ring spectra, using the map of complexes $R \otimes (0 \to \mathbb{Z}) \to R \otimes (\mathbb{Z} \to \mathbb{Z})$ as the notion of "cell." So it is a retract up to homotopy of a finite sequence of "cell" attachments.

In each case, these turn out to be the "compact" objects, in the sense that the maps in the homotopy category out of a perfect R-module P commute with infinite sums, $[P, \bigoplus M_\alpha] \cong \bigoplus [P, M_\alpha]$.

The category of perfect R-modules is denoted Perf_R. This is a symmetric monoidal category under direct sum $\oplus$, or wedge sum $\vee$ in the spectrum case.

You might think that now we're going to take the Segal K-theory of Perf_R with this symmetric monoidal structure. But this turns out to be too big. For instance, when R is an ordinary ring, it doesn't give the same answer as $K(R)$ as defined above. To fix this, we need a new form of K-theory that doesn't just add inverses, but is also additive over cofiber sequences, just like the Euler characteristic χ.

Definition 9.5.10 ([Wal85])**.** A **Waldhausen category** is a category $\mathbf{C}$ and two classes of morphisms called the "cofibrations" and "weak equivalences," such that

- every isomorphism is both a cofibration and a weak equivalence,
- there is a chosen zero object $*$ and every object is cofibrant,
- $\mathbf{C}$ has all pushouts along cofibrations (homotopy pushouts),
- the pushout of a cofibration is a cofibration, and
- a weak equivalence of homotopy pushout diagrams induces a weak equivalence on their pushouts.

An **exact functor $\mathbf{C} \to \mathbf{D}$** is a functor between Waldhausen categories that preserves the zero object, cofibrations, weak equivalences, and pushouts along cofibrations.

Example 9.5.11. If $\mathbf{M}$ is any model category that is pointed (the initial and terminal objects are isomorphic), then the subcategory of cofibrant objects $\mathbf{M}^c$ is a Waldhausen category. We usually restrict to a further subcategory of objects that have some finiteness condition, so that the K-theory is nontrivial. For example, we might take the (cofibrant) perfect R-modules Perf_R^c, which for an ordinary ring R means the perfect chain complexes in $\mathbf{Ch}_{\geq 0}(R)$, and for a ring spectrum R means the perfect R-module chain complexes in $\mathrm{Mod}(R)$.

Intuitively, a Waldhausen category is a category with a notion of "cofiber sequence" $A \to B \to C$. We define a form of K-theory for these categories that universally splits these cofiber sequences, making B equivalent to the sum of A and C.

Definition 9.5.12 ([Wal85])**.** The $S_\bullet$**-construction** (pronounced "S-dot construction") of a Waldhausen category $\mathbf{C}$ is the simplicial object in Waldhausen categories $S_\bullet\mathbf{C}$ whose kth level is the category $S_k\mathbf{C}$ whose objects are grids of the form

$$\begin{array}{ccccccccc}
* = X_{0,0} & \longrightarrow & X_{0,1} & \longrightarrow & X_{0,2} & \longrightarrow & X_{0,3} & \longrightarrow \cdots \longrightarrow & X_{0,k} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
* = X_{1,1} & \longrightarrow & X_{1,2} & \longrightarrow & X_{1,3} & \longrightarrow & \cdots & \longrightarrow & X_{1,k} \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & * = X_{2,2} & \longrightarrow & X_{2,3} & \longrightarrow & \cdots & \longrightarrow & X_{2,k} \\
& & & & \downarrow & & \downarrow & & \downarrow \\
& & & & * = X_{3,3} & \longrightarrow & \ddots & & \vdots \\
& & & & & & \downarrow & & \downarrow \\
& & & & & & * = X_{k-1,k-1} & \longrightarrow & X_{k-1,k} \\
& & & & & & & & \downarrow \\
& & & & & & & & * = X_{k,k}
\end{array}$$

in which every horizontal map is a cofibration, and every square is a pushout. The collection of such grids forms a Waldhausen category, in which

- A morphism $X \to Y$ is a collection of maps $X_{i,j} \to Y_{i,j}$ forming a commuting diagram,
- A weak equivalence is a map such that each map $X_{i,j} \to Y_{i,j}$ is a weak equivalence,
- A cofibration is a map such that each map $X_{0,j} \to Y_{0,j}$ and each pushout map $Y_{0,j} \cup_{X_{0,j}} X_{0,j+1} \to Y_{0,j+1}$ is a cofibration, and
- the chosen zero object is the grid in which every entry is $*$.

We have face and degeneracy functors between the $S_k\mathbf{C}$ for varying k: the ith face map deletes the ith row and column of the grid, while the ith degeneracy duplicates the ith row and column and inserts identity maps.

We let $w_\bullet S_k\mathbf{C}$ denote the nerve of the subcategory of weak equivalences. As we vary k this gives a bisimplicial set, and we let $|w_\bullet S_\bullet\mathbf{C}|$ denote its realization. The Waldhausen K-theory space is the loopspace

$$\Omega^\infty K(\mathbf{C}) = \Omega|w_\bullet S_\bullet\mathbf{C}|.$$

We can go further and create a spectrum by iterating the $S_\bullet$ construction. The **Waldhausen K-theory spectrum** $K(\mathbf{C})$ is defined by taking spectrum level n to be the realization of the multisimplicial set $w_\bullet S_\bullet \dots S_\bullet\mathbf{C} = w_\bullet S_\bullet^{(n)}\mathbf{C}$. So spectrum level 0 is just the classifying space of the weak equivalences $|w_\bullet\mathbf{C}|$, and spectrum level 1 is $|w_\bullet S_\bullet\mathbf{C}|$. The bonding maps arise from the identification $\mathbf{C} \cong S_1\mathbf{C}$, as we did for Segal K-theory in Lemma 9.4.17.

Remark 9.5.13. You should think of the $S_\bullet$-construction as a generalization of the bar construction of a monoid. The terms $X_{0,1}$, $X_{1,2}$, …, $X_{k-1,k}$ are like the elements g_1, g_2, …, g_k in the nerve $N_\bullet(M)$. The face maps "multiply" them, by replacing g_i and g_{i+1} by the middle term h of some cofiber sequence of the form $g_i \to h \to g_{i+1}$. Therefore the Waldhausen K-theory spectrum $|w_\bullet S_\bullet^{(\infty)}\mathbf{C}|$ is a generalization of the delooping spectrum $B^\infty X$ from Definition 9.2.14 and Definition 9.4.16.

Remark 9.5.14. You should also think of $S_k\mathbf{C}$ as the category of "flags," or one-dimensional diagrams $X_{0,1} \to X_{0,2} \to \dots \to X_{0,k}$, together with choices of quotient $X_{i,j} \cong X_j/X_i$. The quotients are unique up to isomorphism, making $S_k\mathbf{C}$ equivalent to the category of flags of cofibrations of length k in $\mathbf{C}$. This point of view makes it easier to contemplate what happens when we apply the $S_\bullet$-construction n times: $S_{k_1}S_{k_2}\dots S_{k_n}\mathbf{C}$ is a category of n-dimensional grids of size $k_1 \times k_2 \times \dots \times k_n$, together with compatible choices of quotients, such that for each little cube in the grid of dimension up to n, the map from the punctured cube to the final vertex is a cofibration. (This is similar to the Reedy cofibration condition in Definition 8.2.15.)

Proposition 9.5.15 ([Wal85])**.** *$K(\mathbf{C})$ is a symmetric spectrum and a positive Ω-spectrum, so its infinite loopspace is $\Omega|w_\bullet S_\bullet\mathbf{C}|$. On K_0, this gives the free abelian group on set of weak equivalence classes of objects $[X]$ of $\mathbf{C}$, up to the relation that $[B] = [A] + [C]$ whenever there is a cofiber sequence $A \to B \to C$.*

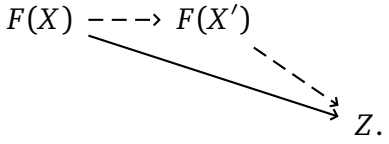
The next result tells us exactly how Waldhausen K-theory is a generalization of Segal K-theory.

Proposition 9.5.16 ([Wal85, BO20])**.** *If $\mathbf{C}$ is a cocartesian monoidal category, it is a Waldhausen category in which a map is a cofibration when it is isomorphic to the inclusion of a summand $X \to X \oplus Y$, and the weak equivalences are the isomorphisms. Its Waldhausen K-theory is equivalent to its Segal K-theory.*

Remark 9.5.17. A recent result of Calle and Chan extends this construction to *every* symmetric monoidal category, not just those where the tensor product is the coproduct [CC25]. So Waldhausen K-theory is in fact more general than Segal K-theory.

It is straightforward to see that each exact functor $F : \mathbf{C} \to \mathbf{D}$ induces a map of Waldhausen K-theory spectra $K(\mathbf{C}) \to K(\mathbf{D})$, making K-theory into a functor. To state when this map is equivalence, suppose that F satisfies the following three conditions:

- A morphism $f : X \to Y$ is a weak equivalence in $\mathbf{C}$ iff $F(f) : F(X) \to F(Y)$ is a weak equivalence in $\mathbf{D}$.
- For any $X \in \mathbf{C}$, $Z \in \mathbf{D}$, and morphism $F(X) \to Z$ in $\mathbf{D}$, there is a cofibration $X \to X'$ in $\mathbf{C}$ and weak equivalence $F(X') \to Z$ in $\mathbf{D}$ such that the following diagram commutes:



- $\mathbf{C}$ has a cylinder functor as defined in [Wei13, IV.8.8]. It will be enough for us that this is true when $\mathbf{C}$ is a subcategory of the cofibrant objects $\mathbf{M}^c$ in some simplicial model category $\mathbf{M}$, and $\mathbf{C}$ is closed under the tensor with the interval $\Delta[1] \otimes (-)$. This allows us to form mapping cylinders in $\mathbf{C}$ that interact with cofibrations and weak equivalences in all of the ways that one might hope for.

Proposition 9.5.18 (Waldhausen Approximation Theorem)**.** *Under these assumptions, the map of K-theory spectra $K(\mathbf{C}) \to K(\mathbf{D})$ is a stable equivalence.*

Example 9.5.19. The algebraic K-theory of a DGA R is defined to be the Waldhausen K-theory of the category of perfect R-module chain complexes. By Proposition 9.5.18, quasi-isomorphic DGAs give stably equivalent K-theory spectra.

Example 9.5.20. The algebraic K-theory of a ring spectrum R is defined to be the Waldhausen K-theory of the category of perfect R-module spectra. Again by Proposition 9.5.18, any stable equivalence of ring spectra $R \xrightarrow{\sim} S$ gives a stable equivalence of K-theory spectra $K(R) \xrightarrow{\sim} K(S)$. More importantly, Proposition 9.5.18 also allows us to move between different symmetric monoidal categories of spectra to use in defining Perf_R, and show that the resulting symmetric spectrum $K(R)$ is preserved up to stable equivalence. So the definition of K-theory does not depend on which model of spectra we use in our Waldhausen category.

Example 9.5.21. When R is an ordinary ring, we can take the finitely generated projective R-modules, or the perfect R-module chain complexes, or the perfect module spectra over

the ring spectrum HR. All three of these give equivalent K-theory spectra, so we can use $K(R)$ to refer to any of these constructions. For the first two the relevant result is called the **Gillet–Waldhausen theorem** [Wei13, V.2.2]. For the last two, one proves this from the adjunctions of categories in Section 9.2 that connect $\mathbf{Ch}_{\geq 0}(R)$ to $\mathrm{Mod}(HR)$ and from **??**.

Example 9.5.22. The collection of all based finite CW complexes forms a Waldhausen category. Its K-theory is called $A(*)$, and is equivalent to the K-theory of the sphere spectrum $K(\mathbb{S})$.

More generally, for any space X we can build a Waldhausen category $R^{fd}(X)$ whose objects are spaces Y containing X as a retract, that are equivalent to X with finitely many cells attached, or a retract up to homotopy of such a space. The Waldhausen K-theory of $R^{fd}(X)$ is called $A(X)$, the **algebraic K-theory of spaces**, also called Waldhausen A-theory. When X is path-connected, this turns out to be equivalent to $K(\Sigma^\infty_+\Omega X)$, and on K_0 and K_1 it agrees with $K_0(\mathbb{Z}[\pi_1(X)])$ and $K_1(\mathbb{Z}[\pi_1(X)])$. The higher homotopy groups of $A(X)$ are not isomorphic to the higher K-groups $K_n(\mathbb{Z}[\pi_1(X)])$.

The homotopy groups of $A(X)$ turn out to be the correct home for the higher versions of the Wall finiteness obstruction in $K_0(\mathbb{Z}[\pi_1(X)])$ and the Whitehead torsion in $K_0(\mathbb{Z}[\pi_1(X)])$. The structure of the ring spectrum $\Sigma^\infty_+\Omega X$ is needed in those higher groups; the group ring $\mathbb{Z}[\pi_1(X)] = \pi_0(\Sigma^\infty_+\Omega X)$ is not enough!

9.5.4 Multiplicative properties

In this chapter we've turned E_∞ spaces X into spectra $B^\infty X$. We should think of the homotopy-coherent operation on X as "addition," because it corresponds to the addition operation on an infinite loopspace, which corresponds to the addition in the homotopy groups of a spectrum.

One might wonder what happens when X has a second operation called "multiplication," that distributes over the addition. Does this create a ring spectrum structure on $B^\infty X$?

In Proposition 9.2.26 we saw that this is true in the special case that X is actually an abelian group: a ring structure on X becomes a ring structure on $B^\infty X$. In this section we describe how this generalizes from abelian groups to Segal E_∞ spaces and to Waldhausen categories. In light of Proposition 9.4.9, it is no loss of generality to assume that our Segal E_∞ space comes from a symmetric monoidal category.

Definition 9.5.23. If $\mathbf{C}_1$, $\mathbf{C}_2$, and $\mathbf{D}$ are symmetric monoidal categories, a **biexact functor** $F\colon \mathbf{C}_1 \times \mathbf{C}_2 \to \mathbf{D}$ is a functor that has the structure of a strong symmetric monoidal functor in each variable separately, that are compatible in the sense described in [EM06]. For technical reasons we also assume that each category has a distinuished choice of unit

object, and that if either of the inputs to F is the distinguished unit, then the output is also the distinguished unit.

The compatibility is easiest to state if the symmetric monoidal structure is the coproduct in each category, with unit the initial object $\varnothing$. Then a functor F is biexact when it preserves coproducts in each variable separately, and also $F(\varnothing, Y) = \varnothing = F(X, \varnothing)$.

When $\mathbf{C}_1$, $\mathbf{C}_2$, and $\mathbf{D}$ are Waldhausen categories, a **biexact functor** $F : \mathbf{C}_1 \times \mathbf{C}_2 \to \mathbf{D}$ is a functor that is exact (Definition 9.5.10) in each variable separately. This also implies that $F(*, Y) = * = F(X, *)$.

Being exact in each variable separately means $F(X, -)$ and $F(-, Y)$ are exact. Think of these biexact functors as "multiplications," because they preserve "additions" in each slot, which is a form of the distributive property $(a + b)c = ac + bc$. (And in the case of Waldhausen categories, they preserve all cofiber sequences in each slot as well.)

Now, the idea is that K-theory will take such "multiplication functors" to multiplications in spectra, in other words maps of the form $K(\mathbf{C}_1) \wedge K(\mathbf{C}_2) \to K(\mathbf{D})$. This correspondence should also respect compositions of biexact functors, so that for instance if $\mathbf{C}$ is a category with a biexact functor $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$ that is associative, then the resulting multiplication on $K(\mathbf{C})$ is associative as well.

The right way to set this up is to ask for K-theory to be a multifunctor, or map of multicategories, as defined in Definition 9.4.43. Recall from Example 9.4.46 that symmetric spectra $\mathbf{Sp}^\Sigma$ form a multicategory, in which an n-ary morphism from $(X_1, \dots, X_n) \to Y$ is map of spectra from the smash product $X_1 \wedge \cdots \wedge X_n \to Y$. On the other hand, there is a multicategory $\mathscr{W}ald$ in which the objects are Waldhausen categories and an n-ary morphism $(\mathbf{C}_1, \dots, \mathbf{C}_n) \to \mathbf{D}$ is a multiexact functor $\mathbf{C}_1 \times \cdots \times \mathbf{C}_n \to \mathbf{D}$, which is defined just like a biexact functor but is exact in each of the n slots separately.

The claim that K-theory is a multifunctor says that each multiexact functor as above goes to a map of spectra $K(\mathbf{C}_1) \wedge \cdots \wedge K(\mathbf{C}_n) \to K(\mathbf{D})$, preserving compositions and the symmetric group actions that permute the factors $\mathbf{C}_i$ around.

In the case of Waldhausen K-theory, we define this structure by taking each biexact functor $F : \mathbf{C}_1 \times \mathbf{C}_2 \to \mathbf{D}$ to the bimorphism of spectra that at level (m, n) comes from the map of multisimplicial categories

$$S_\bullet^{(m)}\mathbf{C}_1 \times S_\bullet^{(n)}\mathbf{C}_n \to S_\bullet^{(m+n)}\mathbf{D}$$

that takes each m-dimensional grid of objects in $\mathbf{C}_1$ and n-dimensional grid in $\mathbf{C}_2$, and creates an $(m + n)$-dimensional grid in $\mathbf{D}$ by multiplying the corresponding objects in the two grids together, using the biexact functor F.

Theorem 9.5.24 ([Zak18])**.** *This gives a well-defined map of symmetric spectra $K(\mathbf{C}_1) \wedge K(\mathbf{C}_2) \to K(\mathbf{D})$. The n-fold variant of this construction for all $n \geq 0$ makes Waldhausen*

K-theory into a multifunctor, from the multicategory of Waldhausen categories and multiexact functors to the multicategory of symmetric spectra.

Corollary 9.5.25. *Any algebraic structure on a Waldhausen category* $\mathbf{C}$ *defined through multiexact functors, such as a unital and associative multiplication, gives the same algebraic structure on the K-theory spectrum* $K(\mathbf{C})$*.*

Example 9.5.26. If R is a commutative ring, DGA, or ring spectrum, the tensor product over R defines a biexact functor $\otimes_R \colon \mathrm{Perf}_R \times \mathrm{Perf}_R \to \mathrm{Perf}_R$, giving a map of K-theory spectra $K(R) \wedge K(R) \to K(R)$. With some effort one extends this to an algebra over the Barratt–Eccles operad $\mathscr{O}$ in the multicategory of Waldhausen categories $\mathscr{Wald}$ (in other words a map of muticategories $\mathscr{O} \to \mathscr{Wald}$), using the canonical isomorphisms that make $\otimes_R$ associative and commutative up to isomorphism. This makes $K(R)$ into an E_∞ ring spectrum, in other words an E_∞ algebra in spectra. We will soon see that this is just as good as making $K(R)$ into a commutative ring spectrum.

As you can see, being a multifunctor is a very useful property, because it transports algebraic structure from the category level to the spectrum level. We would like Segal K-theory to have this property as well. It actually doesn't, unfortunately, but it does if we modify it a little bit.

Definition 9.5.27. The **Elmendorf–Mandell** K**-theory** $K^{\mathrm{EM}}(\mathbf{C})$ of the symmetric monoidal category $\mathbf{C}$ is the spectrum $B^\infty_{\mathrm{EM}}(\mathbf{C})$ that at spectrum level n comes from the quotient of the Segal K-theory simplicial set

$$\mathbf{C}(S^n_\bullet) \twoheadrightarrow \mathbf{C}^{\mathrm{EM}}(S^n_\bullet)$$

defined on simplicial level k as follows. On the left-hand side, by Example 9.4.8, we have a category in which an object is an assignment of each subset $S \subseteq S^n_k \setminus \{*\} = \{1,\dots,k\}^{\times n}$ to an object $x_S \in \mathbf{C}$, and each pair of disjoint subsets to an isomorphism $x_S \otimes x_T \cong x_{S \amalg T}$, satisfying some compatibilities. We define the right-hand side by restricting attention to product subsets of the form

$$S_1 \times \cdots \times S_n \subseteq \{1,\dots,k\}^{\times n}, \qquad S_i \subseteq \{1,\dots,k\} \ \forall\ i,$$

and isomorphisms of the form

$$x_{\dots \times S_{i-1} \times S_i \times S_{i+1} \times \dots} \otimes x_{\dots \times S_{i-1} \times S'_i \times S_{i+1} \times \dots} \cong x_{\dots \times S_{i-1} \times (S_i \amalg S'_i) \times S_{i+1} \times \dots}.$$

See [EM06, Construction 4.4] for more details.[17]

[17] In [EM06] the authors use morphisms $x_S \otimes x_T \to x_{S \amalg T}$ that are not assumed to be isomorphisms. In our discussion we will restrict our attention to the version of the construction in which these are isomorphisms.

Lemma 9.5.28 ([EM06, Thm 4.6])**.** *For every symmetric monoidal category* $\mathbf{C}$*, the natural map* $K^{\mathrm{Seg}}(\mathbf{C}) \to K^{\mathrm{EM}}(\mathbf{C})$ *is a level equivalence of symmetric spectra, and therefore* $K^{\mathrm{EM}}(\mathbf{C})$ *is also a positive* Ω*-spectrum whose 0th level is the homotopical group completion of* $B\mathbf{C}$*.*

We can send every biexact functor $F : \mathbf{C}_1 \times \mathbf{C}_2 \to \mathbf{D}$ to the bimorphism of spectra that at level (m, n) comes from the map of simplicial categories

$$\mathbf{C}^{\mathrm{EM}}(S^m_\bullet) \times \mathbf{C}^{\mathrm{EM}}(S^n_\bullet) \longrightarrow \mathbf{C}^{\mathrm{EM}}(S^{m+n}_\bullet)$$

by taking the two families of objects $x_{S_1 \times \cdots \times S_m}$ and $y_{T_1 \times \cdots \times T_n}$ to the family of products $F(x_{S_1 \times \cdots \times S_m}, y_{T_1 \times \cdots \times T_n})$, for each product subset $S_1 \times \cdots \times S_m \times T_1 \times \cdots \times T_n \subseteq \{1, \ldots, k\}^{m+n}$.

Theorem 9.5.29 ([EM06])**.** *This gives a well-defined map of symmetric spectra* $K(\mathbf{C}_1) \wedge K(\mathbf{C}_2) \to K(\mathbf{D})$*. The n-fold variant of this construction for all* $n \geq 0$ *makes Elmendorf–Mandell K-theory into a multifunctor, from the multicategory of symmetric monoidal categories (with distinguished choice of unit object) to the multicategory of symmetric spectra.*

Corollary 9.5.30. *Any algebraic structure on a symmetric monoidal category* $\mathbf{C}$ *defined through multiexact functors, such as a unital and associative multiplication, gives the same algebraic structure on the Elmendorf–Mandell K-theory spectrum* $K(\mathbf{C})$*.*

Theorem 9.5.31 ([BO20])**.** *For Waldhausen categories that arise from cocartesian monoidal categories* $(\mathbf{C}, \oplus, 1)$ *as in Proposition 9.5.16, the equivalence between Waldhausen and Elmendorf–Mandell K-theories of* $\mathbf{C}$ *respects the multiplicative structure, and so when* $K(\mathbf{C})$ *is an algebra this map is an equivalence of algebras.*

This can also be generalized further to a multiplicative K-theory of multicategories. See [JY24] for much more information.

Example 9.5.32. The same construction as in Example 9.5.26 on the category of finite sets and isomorphisms (or alternatively on the Waldhausen category of all finite based sets) makes $K(\mathbf{Fin}^{\cong})$ into an E_∞ ring spectrum. In fact, it is equivalent to the sphere spectrum $\mathbb{S}$ with its usual structure as a commutative ring spectrum. This is the "multiplicative" version of the Barratt–Priddy–Quillen theorem.

9.6 Exercises

9.6.1. Prove that the following are equivalent for a topological monoid M:

(a) M is **grouplike**, i.e. $\pi_0(M)$ is a group.

(b) M has inverses up to homotopy: for each $g \in M$ there is an $h \in M$ such that gh and hg are both in the path component of the identity element $1 \in M$.

(c) For each $g \in M$, the left-multiplication map $g \cdot (-) \colon M \to M$ is a weak equivalence.

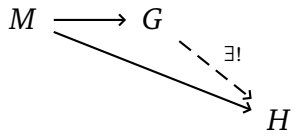
(d) The **shear map**

$$M \times M \longrightarrow M \times M$$
$$(x, y) \mapsto (x, xy)$$

is a weak equivalence.

In fact, we don't even need to assume that M is a strict monoid – it is enough if it is a monoid in the homotopy category $\mathrm{Ho}\,\mathbf{Top}_*$. This is useful for the proof of Theorem 9.4.15.

9.6.2. Suppose M is a discrete abelian monoid, so an element of CMon(**Set**). Prove that there is an abelian group G that is universal among all abelian groups receiving a homomorphism from M. So there is a homomorphism $M \to G$, and any homomorphism $M \to H$ induces a unique group homomorphism as shown.



In fact, we may construct it as the set of pairs $(a, b) \in M^2$, which we think of as a formal differences $a - b$, up to a relation similar to the one in Definition 9.1.27. Alternatively, we may take the free abelian group on the elements of M, and then mod out by the relations $[a + b] = [a] + [b]$.

Now generalize this to non-abelian monoids and groups. (The first description above won't do, you'll need the second description.) Show that when the monoids happen to be abelian, we get the same answer.

9.6.3. (a) For each of the three remaining categories in Proposition 9.1.3, construct the free functor F that is left adjoint to the forgetful functor U to topological spaces.

(b) Define the left adjoint of the forgetful functor CGrp(**Top**) → CMon(**Top**). Then define the left adjoints for the remaining forgetful functors in Proposition 9.1.3. (You are essentially taking Exercise 9.6.2 and adding a topology.)

9.6.4. Explain why Theorem 5.3.16 gives a model structure on modules over any topological monoid M. Show that each monoid homomorphism $M \to M'$ induces a Quillen adjunction on their categories of modules.

9.6.5. Prove that the model structure on **sMon** from Proposition 9.1.7 is simplicial, and the model structures on Mon(**Top**) from Proposition 9.1.3 is topological (and therefore also simplicial). Essentially the same argument works for the other model structures

in these propositions as well. So all of these categories are in fact simplicial model categories, and the techniques of the previous chapter apply to them.

9.6.6. Define the Moore loop space $\Omega_{(m)}Y$, see e.g. [CM95, Section 5.1]. It is similar to the loop space except that the loops have lengths assigned to them, and the lengths add when we compose the loops. Unfortunately, even with this fix, the canonical map $G \to \Omega_{(m)}BG$ is still not a map of monoids, because there is no way to choose the lengths consistently.

9.6.7. Prove that if we take a homotopical group completion $M \to L_{\mathrm{Grp}}(M)$ and apply π_0, we get a discrete group completion $\pi_0(M) \to D_{\mathrm{Grp}}(\pi_0(M)) \cong \pi_0(L_{\mathrm{Grp}}(M))$. Use this to explain why the definition of K_0 from Definition 9.5.3 agrees with the simpler definition from Definition 9.5.1 that is just the group completion of isomorphism classes of R-modules under direct sum.

9.6.8. Prove that for any well-based topological monoid M, $\pi_1(BM)$ is identified with the discrete group completion of the monoid $\pi_0(M)$.

9.6.9. In this exercise we give a very weak sense in which $M \to \Omega BM$ is a homomorphism. An **H-space** or Hopf space is a based space X with a unital multiplication in the homotopy category $h\mathbf{Top}_*$. In other words, there is a map of based spaces $\mu : X \times X \to X$ such that $\mu(*, -)$ and $\mu(-, *)$ are homotopic through maps of based spaces to the identity of X. A **map of H-spaces** is a based map $f : X \to Y$ that commutes with the multiplications up to homotopy, so $\mu_Y(f(-), f(-))$ is homotopic to $f\mu_X(-,-)$ as maps $X \times X \to Y$.

(a) Explain why every topological monoid is an H-space by neglect of structure. What's an example of an H-space that isn't a monoid? (For instance, it might fail to be associative, or to be strictly unital.)

(b) Show that the composition of loops makes the based loop space ΩY into an H-space, and this defines a functor from based spaces to H-spaces.

(c) Prove that the canonical map $M \to \Omega BM$ for a topological monoid M is a map of H-spaces.

(d) Upgrade the proof of Theorem 9.1.18 to show that $M \to \Omega BM$ is equivalent to $M \to L_{\mathrm{Grp}}(M)$ as a map of H-spaces.

(e) Show that every H-space X has a multiplication on the homology $H_*(X)$, generalizing the Pontryagin product. Show that every map of H-spaces induces a map on homology $H_*(X) \to H_*(Y)$ that commutes strictly with this multiplication. (So it is a ring map, if multiplications on $H_*(X)$ and $H_*(Y)$ are associative.)

This is needed for the statement of the McDuff-Segal group completion theorem, because it shows that $H_*(M) \to H_*(\Omega BM)$ is indeed a ring map.

9.6.10. Recall the definition of the localization $R[S^{-1}]$ from Definition 9.1.27, when R is a ring and S is a central subset of R that is closed under multiplication.

(a) Check that $R[S^{-1}]$ is indeed a ring with a homomorphism $R \to R[S^{-1}]$ that sends each element of S to an invertible element (i.e. a unit).

(b) Prove that this is universal: given any other ring homomorphism $R \to T$ such that S goes to units in T, prove that it factors in a unique way through $R[S^{-1}]$.

9.6.11. In this exercise we explain why the McDuff–Segal theorem only gives a homology isomorphism and not a homotopy equivalence $M_\infty \simeq \Omega BM$.

(a) Prove that in any topological monoid M, the fundamental group $\pi_1(M, 1)$ based at the unit element $1 \in M$ must be abelian.

(b) Use the example of $M = B\mathbf{Fin}^{\cong} \simeq \coprod_{n\geq 0} B\Sigma_n$ to show that the fundamental group at other points of M may be non-abelian.

(c) When $M = B\mathbf{Fin}^{\cong}$, explain why the space $M_\infty \simeq \mathbb{Z} \times B\Sigma_\infty$ can't be equivalent to a topological monoid, and therefore can't be equivalent to ΩBM.

9.6.12. In this exercise we prove the **Dold–Thom theorem**: for a based connected cell complex X, the homotopy groups of the infinite symmetric product $\mathrm{Sym}^\infty(X)$ are naturally isomorphic to the homology of X.

(a) Define the n-fold symmetric product $\mathrm{Sym}^n(X)$ to be the quotient of $X^{\times n}$ by the symmetric group action, and let $\mathrm{Sym}^\infty(X)$ be the colimit of these spaces along maps $\mathrm{Sym}^n(X) \to \mathrm{Sym}^{n+1}(X)$ that include the basepoint of X in the new coordinate.

Prove that $\mathrm{Sym}^\infty(X)$ is a commutative topological monoid, and is isomorphic to the quotient $F_{\mathrm{CMon}}(X)/F_{\mathrm{CMon}}(*)$ in the category of topological monoids. In other words it is the universal commutative monoid on X *that uses the given basepoint* $* \in X$ *as its unit*. (Hint: Universal properties are your friends!)

(b) Use the previous part to show that $\mathrm{Sym}^\infty(X)$ preserves weak equivalences when X is a based cell complex. (You could also use the techniques from Section A.1.2 to do this part.)

(c) If X is connected, prove that $\mathrm{Sym}^\infty(X)$ is grouplike, and therefore it is equivalent to its homotopical group completion.

(d) Conclude that when X is any based cell complex, $\mathrm{Sym}^\infty(X)$ is equivalent to the quotient $F_{\mathrm{CGrp}}(X)/F_{\mathrm{CGrp}}(*)$ in the category of topological abelian groups. Now use simplicial abelian groups and the Dold–Kan correspondence to show that $\pi_*(\mathrm{Sym}^\infty(X)) \cong H_*(X)$, proving the Dold–Thom theorem.

(e) Conclude that $\mathrm{Sym}^\infty(S^n)$ is another model for $K(\mathbb{Z}, n)$. Can you describe it explicitly from the configuration space point of view? How is it related to the n-fold bar construction?

9.6.13. Define the simplicial version of the delooping spectrum, which we call $N^\infty A_\bullet$. It should take each simplicial abelian group $A_\bullet$ to a sequential spectrum of simplicial sets

$$N^\infty : \mathbf{sAb} \longrightarrow \mathbf{Sp}^{\mathbb{N}}_{\mathbf{sSet}},$$

and on geometric realization it should agree with the delooping spectrum $B^\infty |A_\bullet|$ from Definition 9.2.14.

9.6.14. In this exercise we prove Proposition 9.3.27, that not every E_∞ space can be rectified to a strictly commutative topological monoid.

(a) Show that $F_{\mathrm{CMon}}(S^1)$ is not a product of Eilenberg–Mac Lane spaces.

(b) Assume that $QS^0 = \Omega^\infty \mathbb{S}$ is equivalent to a product of Eilenberg–Mac Lane spaces and derive a contradiction. (Does this need to go in the spectral sequences chapter?)

9.6.15. (*) In this exercise we develop a feel for coordinate-free operads. Let $\mathcal{O}$ be any coordinate-free operad (Definition 9.3.2).

(a) Prove that every bijection $I \cong J$ gives in a canonical way an isomorphism $\mathcal{O}(I) \cong \mathcal{O}(J)$.

(b) Prove that the procedure in Proposition 9.3.3 does in fact produce an ordinary operad.

(c) Prove that this procedure is fully faithful, in other words the set of maps of operads doesn't change when we pass from coordinate-free operads to ordinary operads.

(d) Prove that every ordinary operad comes from a coordinate-free operad, and therefore this procedure is essentially surjective as well.

9.6.16. Let M be a topological monoid. Check that we can define an operad in which

$$\mathcal{O}(n) = \begin{cases} M & n = 1, \\ \varnothing & n \neq 1, \end{cases}$$

and the composition of $\mathcal{O}$ comes from the multiplication of M. What does it mean when X is an algebra over this operad?

9.6.17. Let X be a topological space. Define the **endomorphism operad** End_X by $\mathrm{End}_X(n) = \mathrm{Map}(X^{\times n}, X)$, with Σ_n acting by permuting the inputs, and composition by composition of functions. Check that this is an operad, and that a map of operads $\mathscr{O} \to \mathrm{End}_X$ is the same thing as making X an algebra over $\mathscr{O}$.

9.6.18. In this exercise we analyze the homotopy type of the spaces $\mathscr{C}_n(k)$ using good old-fashioned algebraic topology.

(a) Prove that each component of $\mathscr{C}_1(k)$ is contractible. (Don't overthink this – a straight-line homotopy will work just fine.)

(b) Give a precise definition of the space $\mathrm{Conf}_k(I^n)$ of ordered k-tuples of distinct points in I^n, and prove that $\mathscr{C}_n(k)$ is homotopy equivalent to $\mathrm{Conf}_k(I^n)$.

(c) Using this equivalence, prove that $\mathscr{C}_n(1)$ is contractible, and the map $\mathscr{C}_n(k) \to \mathscr{C}_n(k-1)$ that forgets the kth cube has homotopy fiber equivalent to I^n minus $(k-1)$ points, or equivalently $(S^{n-1})^{\vee(k-1)}$.

(d) Conclude that each space $\mathscr{C}_n(k)$ is $(n-2)$-connected and therefore each space $\mathscr{C}_\infty(k)$ is weakly contractible.

9.6.19. Suppose we take the recognition theorem (Theorem 9.3.28) as a black box, and let B denote the left adjoint. Prove that, if we decide to model A_∞ spaces as strictly associative topological monoids, then the unit of the adjunction $X \to \Omega BX$ must be equivalent to the group completion $X \to L_{\mathrm{Grp}}(X)$ as a morphism in the comma category $(X \downarrow \mathrm{Ho}\,\mathrm{Mon}(\mathbf{Top}))$.

(Hint: Prove that it satisfies the universal property of the group completion. You might also try to follow the strategy that we used to prove Theorem 9.1.18.)

9.6.20. Explain why Segal A_∞ and E_∞ spaces can be defined in any category $\mathbf{C}$ with finite products, but not in a general symmetric monoidal category $\mathbf{C}$.

9.6.21. Show how to generalize Example 9.4.8 to handle the case that the symmetric monoidal category $\mathbf{C}$ has a topology on its mapping spaces, and the tensor product is a topological functor. Does this also generalize Example 9.4.7, or do we have to topologize something else to make that work?

9.6.22. Let us say that a functor $X : \mathbf{Fin}_* \to \mathbf{Top}$ is **Reedy cofibrant** if for each $n \geq 0$, the map

$$L_n(X) = \operatorname*{colim}_{S \subsetneq \mathbf{n}} X(S_+) \longrightarrow X(\mathbf{n}_+)$$

has the homotopy extension property. Here the colimit is being taken along subset inclusions $S \subseteq T$.

If X is Reedy cofibrant in the above sense, prove that for every finite simplicial based set $A_\bullet$, the simplicial space $X(S^n_\bullet)$ is Reedy cofibrant.

9.6.23. Prove the claim that for any Segal A_∞ space $X_\bullet$, the zig-zag defined in Definition 9.4.1 makes $\pi_0(X_1)$ into a monoid.

9.6.24. Prove that a Segal A_∞ or E_∞ space in which the Segal maps induce homeomorphisms $X_n \cong X_1^{\times n}$ must be of the form given in Example 9.4.2 or Example 9.4.7.

9.6.25. In this exercise we give a more explicit description of the E_∞ structure on the infinite loopspace of a spectrum. Let E be any sequential, symmetric, or orthogonal spectrum.

(a) Consider the diagram $\mathbf{Fin}_* \to \mathbf{Sp}$ given by

$$\mathbf{n}_+ \mapsto \mathbf{n}_+ \wedge E = \bigvee_{i=1}^{n} E,$$

and then apply some fixed infinite loopspace functor $\Omega^\infty : \mathbf{Sp} \to \mathbf{Top}_*$ to get a diagram $X : \mathbf{Fin}_* \to \mathbf{Top}_*$. Prove that this defines a Segal E_∞ space.

(b) Prove that the Segal E_∞ space X from part (a) is weakly equivalent to the Segal E_∞ space from Theorem 9.4.19, produced by prolonging E to make a $\mathscr{W}$-spectrum, then restricting to the finite sets $\mathbf{n}_+$.

9.6.26. This exercise supplies the formal part of the proof that $X_\bullet \to \Omega_\bullet(\mathbb{L}B)X$ is a universal approximation of a Segal A_∞ space $X_\bullet$ by a grouplike one. It is a general criterion for recognizing "universal approximations" in many contexts, and we have already seen it when we proved that derived functors are unique (Proposition 3.5.2).

(a) Suppose $\mathbf{C}$ is any category with weak equivalences, $\mathbf{A} \subseteq \mathbf{C}$ is any full subcategory, and $L : \mathbf{C} \to \mathbf{A}$ is any homotopical functor, with a natural transformation $\mathrm{id} \to L$. Assume that

- $X \to LX$ is a weak equivalence for every $X \in \mathbf{A}$, and
- $L(X \to LX)$ is a weak equivalence for every $X \in \mathbf{C}$.

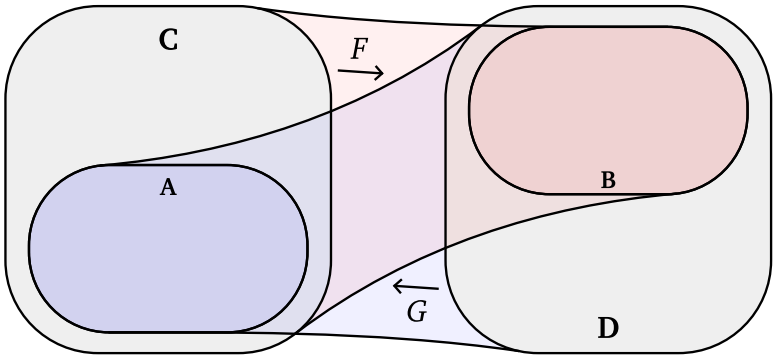
Prove that $X \to LX$ is the universal approximation of X by an object in $\mathbf{A}$ up to weak equivalence, in the sense that every map from X to an object $Y \in \mathbf{A}$ factors uniquely through LX in the homotopy category. Or equivalently, $L : \mathrm{Ho}\,\mathbf{C} \to \mathrm{Ho}\,\mathbf{A}$ is the left adjoint of the inclusion $\mathrm{Ho}\,\mathbf{A} \to \mathrm{Ho}\,\mathbf{C}$. (Hint: Adapt the proof of Proposition 3.5.2.)

(b) Describe the dual result when the natural transformation goes in the direction $L \to \mathrm{id}$.

(c) Suppose $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ are adjoint functors that have derived functors $\mathbb{L}F$ and $\mathbb{R}G$. Assume that

- F factors through a full subcategory $\mathbf{B} \subseteq \mathbf{D}$,

- G factors through a full subcategory $\mathbf{A} \subseteq \mathbf{C}$,
- the derived unit $X \to (\mathbb{R}G)(\mathbb{L}F)X$ is an equivalence for every $X \in \mathbf{A}$, and
- the derived counit $(\mathbb{L}F)(\mathbb{R}G)Y \to Y$ is an equivalence for every $Y \in \mathbf{B}$.

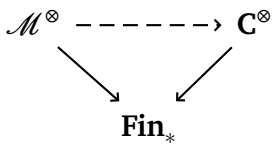


Prove using parts (a) and (b) that the derived unit is the universal approximation of X by an object of $\mathbf{A}$, and the derived counit is a universal approximation of Y by an object of $\mathbf{B}$.

(d) Show using Exercise 6.6.16 that the fourth assumption in part (c) is satisfied if we know the other three assumptions, and we know that $\mathbb{R}G$ preserves and reflects weak equivalences.

9.6.27. Prove the claim in Lemma 9.4.37 that $\mathcal{O}$-algebras are the same thing as $\mathcal{O}^{\otimes}$-algebras in standard form. As the main step, explain how the action of a morphism in $\mathcal{O}^{\otimes}(\mathbf{m}_+, \mathbf{n}_+)$ is determined by its constituent parts in $\mathcal{O}^{\otimes}(\{\alpha^{-1}(j)\}_+, \{j\}_+)$, using the fact that the inert morphisms in $\Pi \subseteq \mathcal{O}^{\otimes}$ act by product projections.

9.6.28. Let $\mathbf{C}$ be a symmetric monoidal category, $\mathcal{M}$ a multicategory, and let $\mathcal{M}^{\otimes}$ and $\mathbf{C}^{\otimes}$ be their categories of operators as defined in Example 9.4.56. Prove that a multi-diagram in $\mathbf{C}$ indexed by $\mathcal{M}$ is the same thing as a functor



that preserves inert morphisms. (Hint: Focus on the inert morphisms first, because they help you see where all the objects go.)

9.6.29. Use Proposition 9.5.18 to prove that the definition of the K-theory of a ring spectrum doesn't depend on which symmetric monoidal model category of spectra from (8.1.36) that we use.

9.6.30. This exercise sets up some group-theory preliminaries for the plus construction. Let G be any group and let $[G,G] \leq G$ be its commutator subgroup, the subgroup generated by all expressions of the form $aba^{-1}b^{-1}$ with $a, b \in G$.

(a) Prove that $[G,G]$ is a normal subgroup of G, and the quotient $G/[G,G]$ is the abelianization G^{ab}.

(b) We say that a subgroup $P \leq G$ is **perfect** if $P = [P,P]$. Show that perfect subgroups need not be normal, but for any G there is a maximal perfect subgroup P, which is normal.

9.6.31. (*) [Quillen's +-construction] The following is based on the plus-construction exercise in [Wei13]. Let X be any based connected CW complex with fundamental group G, and let $P \trianglelefteq G$ be any subgroup that is both normal and perfect, meaning that P is equal to its commutator subgroup $[P,P]$. We say that a map $f : X \to Y$ is a **plus construction** relative to P if Y is also connected, if f induces on π_1 the map $G \to G/P$, and f is an **acyclic map**, meaning it gives an isomorphism on homology with all local coefficient systems.

(a) Show that we can attach a 2-cell to X for each element of P, and that on homology all this does is add $\mathbb{Z}$ summands to $H_2(X)$. Then show each of these $\mathbb{Z}$-summands is in the image of the Hurewicz homomorphism $\pi_2 \to H_2$, so that we may attach a 3-cell to kill each of the 2-cells we added, and arrive at a space X^+ such that the map $X \to X^+$ is an isomorphism on homology and on π_1 is the projection $G \to G/P$.

(b) Show also that for any covering space $\widetilde{X}^+ \to X^+$ with pullback $\widetilde{X} \to X$, the map of covering spaces $\widetilde{X} \to \widetilde{X}^+$ is an isomorphism on homology as well. This implies that $X \to X^+$ is an isomorphism on homology with all local coefficients, and proves that a plus-construction for X exists.

(c) Now we prove it is unique. Show that if $g : X \to Y$ is any other map whose map on π_1 contains P in the kernel, then g must factor through $f : X \to X^+$. And if g is a plus construction then the factorization $h : X^+ \to Y$ is an equivalence.

(d) In the proof of the McDuff–Segal group completion theorem, explain why $M_\infty^+ \simeq \Omega BM$, where P is the commutator subgroup of $\pi_1(M_\infty^+)$. Conclude that the group completion of the category of finite sets is

$$\Omega B(B\mathbf{Fin}^{\cong}) \simeq \Omega B\left(\coprod_{n\geq 0} B\Sigma_n\right) \simeq \mathbb{Z} \times B\Sigma_\infty^+$$

where the plus construction is taken with respect to the commutator subgroup of Σ_∞.

The next few exercises are concerned with the notion of a monad from Definition A.3.7, and reflexive coequalizers from Definition A.3.15.

9.6.32. Prove that the functor $F : \mathbf{Top} \to \mathbf{Top}$ by $F(X) = X^{\times n}$ preserves reflextive coequalizers. Then observe that your proof works in any symmetric monoidal category $\mathbf{C}$ in which the tensor $\otimes$ preserves colimits in each slot.

(Hint: Write the n-fold product of the coequalizers as the quotient of $Y^{\times n}$ by n different relations, one for each slot. Then use the reflection map j to show that these n relations are equivalent to a single relation, applied to every slot simultaneously.)

9.6.33. (*) Generalize Exercise 8.7.47 by showing that the geometric realization in $\mathcal{O}$-algebras agrees with the geometric realization of algebras, if $\mathcal{O}$ is any operad in topological spaces. (The proof is given in [BGH22, Sec 5.7], if you need a hint as to how to get started.)

9.6.34. Generalize Γ-spaces by taking diagrams on $\mathbf{Fin}_*$ in unbased spaces, letting $B = X_0$, and asking that the map to the *fiber* product $X_n \to X_1 \times_B \ldots \times_B X_1$ be an equivalence. Show that these lead to *parametrized* spectra, a sequence of spaces containing B as a retract and bonding maps that use the fiberwise version of reduced suspension.

Chapter 10

An introduction to ∞-categories*

In Chapter 8 we introduced simplicial model categories, so that we could study homotopy coherence. In other words, instead of making a definition by naming elements and naming which equations between them must hold, we only ask for *homotopies*, and then homotopies between them, and so on.

Simplicial model categories are one setting to study homotopy coherence. They are very highly structured: we have all kinds of gadgets like bar constructions, tensors and cotensors, cofibrations and fibrations. The objects have explicit descriptions, and we can write out examples. Working in a simplicial model category has a very "generators and relations" feel to it.

There is another way to study homotopy coherence that is in some sense the polar opposite. The idea is to have as *little* structure as possible, and to have all of the complexity of a homotopy colimit or a bar construction built into the machinery in such a way that you don't touch it directly. This second framework is called an ∞-category, or an $(\infty, 1)$-category.

An ∞-category $\mathbf{C}$ is like a category, except that the collection of morphisms $\mathbf{C}(a, b)$ forms a *homotopy type*, rather than a set. In other words, $\mathbf{C}(a, b)$ is a topological space or a simplicial set, but considered up to weak equivalence. We will see a couple of different ways to define this precisely.

∞-categories are another natural setting for the study of homotopy coherence. In fact, they have homotopy coherence baked into their definition: because $\mathbf{C}(a, b)$ is only defined up to weak equivalence, it only makes sense to ask whether two morphisms $f, f' \in \mathbf{C}(a, b)$ are homotopic, not equal. So the associativity axiom for a category is phrased as saying that $(fg)h$ and $f(gh)$ are homotopic, rather than equal, and then we need more coherence homotopies between these.

Every model category, and indeed every category $\mathbf{C}$ with a class of weak equivalences

W, gives an ∞-category called $N(\mathbf{C})[W^{-1}]$. Every functor on $\mathbf{C}$ that preserves weak equivalences, such as a homotopy colimit, passes to a functor on the ∞-category. We can therefore use model categories to concretely model and geometrically visualize various things in the land of ∞-categories. For example, the Bousfield–Kan homotopy colimit is a concrete model for the colimit in the associated ∞-category.

∞-categories play a central role in modern homotopy theory, because they are an intermediate structure between a model category $\mathbf{C}$ and its homotopy category $\mathrm{Ho}\,\mathbf{C}$. When we pass from a model category to its ∞-category, we turn the weak equivalences into isomorphisms, but we still retain enough information to form things like homotopy colimits, which we couldn't do if we went all the way to $\mathrm{Ho}\,\mathbf{C}$ (see Warning 3.4.10). In other words, ∞-categories are like model categories with some of the "inessential" data thrown out, and the "essential" data left intact.

The tradeoff of working with ∞-categories is that every notion needs to be homotopy-coherent instead of strict, and this adds complexity. You can't work with ∞-categories in an explicit, "generators and relations" type way. You have to appeal more to universal properties and formal arguments.

Which is better, simplicial model categories, or ∞-categories? In the author's opinion, this question itself is misleading. It's best to learn some of both, so that you can play their strengths off of each other.

In this chapter we introduce the two most common models for ∞-categories, namely simplicial categories and quasicategories. We show how Dwyer–Kan localization allows us to translate all of the theorems from the earlier chapters into theorems about ∞-categories. We then cover adjunctions, colimits, and presentable ∞-categories, which are a particularly nice kind of ∞-category that we have a very tight grip on. In the later part of the chapter we build towards the more complex notion of a symmetric monoidal ∞-category, and explain why every symmetric monoidal model category gives one of these. This allows us to think of the smash product of spectra as a structure on the ∞-category of spectra, and to state Lurie's universal characterization of the smash product.

Throughout this chapter we avoid giving proofs, citing most of the results out to the existing literature, including [Lur09, Lur17, Cis19, Lan21]. In the next chapter when we cover ring spectra, we will return to giving proofs for everything, since most of the theorems have a version that can be stated and proven using model categories.

10.1 Simplicial categories and quasicategories

In this section we define ∞-categories in two different ways, and build some intuition for what the definitions are doing. Our goal is to convince the reader that it is not scary

to learn the definitions and to work with ∞-categories in a precise way.

10.1.1 First definition: simplicial categories

Recall that the intuitive definition of an ∞-category is a category $\mathbf{C}$ where between any two objects a and b we get a *homotopy type* of morphisms $\mathbf{C}(a,b)$. So $\mathbf{C}(a,b)$ is a topological space, but we don't care about the space up to homeomorphism, we only care about it up to weak equivalence. Or equivalently, we have a simplicial set of morphisms $\mathbf{C}_\bullet(a,b)$, and we consider it up to weak equivalence.

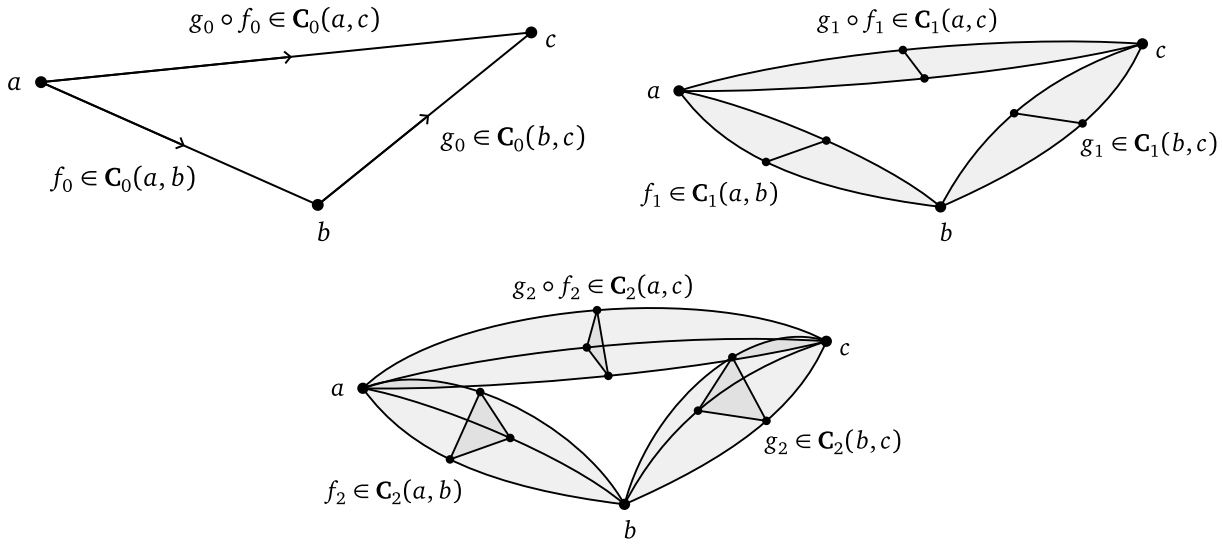
We call these things "∞-categories" because there is a notion of

- a morphism (a point in the space $\mathbf{C}(a,b)$),
- a morphism between morphisms (a path in $\mathbf{C}(a,b)$ between two points),
- a morphism between those (a homotopy of paths),
- and so on, ad infinitum.

We also call these $(\infty,1)$**-categories** because after the 1-morphisms, all of the higher morphisms are invertible. (To make the distinction clearer, we will now say **1-category** to refer to a category in the usual sense.)

Now let's make this precise.

Definition 10.1.1. A **simplicial category** $\mathbf{C}$ is a category enriched in simplicial sets, as in Proposition 8.1.21. We will sometimes write $\mathbf{C}_\bullet$ when we want to emphasize the simplicial structure.



Writing this out, a simplicial category $\mathbf{C}_\bullet$ has a set of objects $\operatorname{ob}\mathbf{C}$, and a simplicial set of morphisms $\mathbf{C}_\bullet(a,b)$ between every pair of objects. This is the same thing as a simplicial object in categories, $\mathbf{C}_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Cat}$, *with the extra property* that it is constant on objects. In other words, we get a category $\mathbf{C}_n$ for each $n \geq 0$, but they all have the same objects, and the face and degeneracy functors between all induce the identity map on the underlying collection of objects. The picture above shows the categories $\mathbf{C}_0$, $\mathbf{C}_1$, and $\mathbf{C}_2$.

Definition 10.1.2. If $\mathbf{C}$ is a simplicial category, the **homotopy category** $\pi_0\mathbf{C}$ has the same objects, but the morphisms sets are the sets of connected components $\pi_0(\mathbf{C}_\bullet(a,b))$, as defined in Definition 8.1.26.

Writing this out, two morphisms $f, g \in \mathbf{C}_0(a,b)$ are identified in $\pi_0\mathbf{C}$ iff there is a zig-zag of 1-simplices in $\mathbf{C}_1(a,b)$ that go from f to g. It might not be possible to go from f to g by a single 1-simplex, though it is always possible if $\mathbf{C}_\bullet(a,b)$ is a Kan complex, by the proof of Lemma 8.1.27. At any rate, since $\pi_0\mathbf{C}$ is the category of homotopy classes of morphisms, an isomorphism in $\pi_0\mathbf{C}$ is the same thing as a "homotopy equivalence" in $\mathbf{C}$.

Definition 10.1.3. A **simplicial functor** $F : \mathbf{C} \to \mathbf{D}$ consists of a function of objects that we also call $F : \operatorname{ob}\mathbf{C} \to \operatorname{ob}\mathbf{D}$, and maps of simplicial sets

$$F : \mathbf{C}_\bullet(a,b) \to \mathbf{D}_\bullet(F(a), F(b))$$

that respect composition and identity morphisms, as in Definition 8.1.34. A **Dwyer–Kan equivalence**, or **DK-equivalence**, is a simplicial functor $F : \mathbf{C} \to \mathbf{D}$ such that

- Each map $\mathbf{C}_\bullet(a,b) \to \mathbf{D}_\bullet(F(a), F(b))$ is a weak equivalence of simplicial sets, in other words it gives a weak equivalence on the realization (Definition 7.1.37), and
- The functor $\pi_0 F : \pi_0\mathbf{C} \to \pi_0\mathbf{D}$ is essentially surjective.

Our **first definition of ∞-category** is: an ∞-category is a simplicial category, considered up to DK-equivalence. So if two simplicial categories are related by a zig-zag of DK-equivalences, we consider them to be the "same" ∞-category. (Just like when we defined spectra: two spectra are the "same" if they are connected by a zig-zag of stable equivalences.)

A simplicial functor that only satisfies the first condition in Definition 10.1.3 is called a **DK-embedding**, or more simply a **fully faithful** map of ∞-categories.

The second condition in Definition 10.1.3 is a homotopical version of being essentially surjective. It tells us that every object $d \in \mathbf{D}$ is homotopy equivalent to an object of the form $F(a)$ for $a \in \mathbf{C}$. There are morphisms $F(a) \to d$ and $d \to F(a)$ whose compositions are homotopic to the identity.

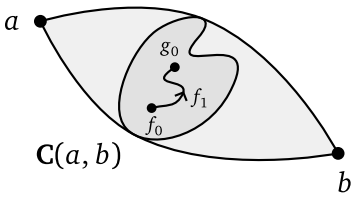
Putting this all together, a DK-equivalence is an up-to-homotopy version of an equivalence of categories from Definition 3.1.18. This is summarized in the following table.

Isomorphism of categories	**Equivalence of categories**	**Dwyer–Kan equivalence**
bijection on objects	bijection on *isomorphism classes* of objects	bijection on *homotopy equivalence classes* of objects
bijection on all morphisms	bijection on morphisms between each pair of objects	*weak equivalence* on morphisms between each pair of objects

When we made this definition, we asked for $\mathbf{C}$ to be enriched in "spaces," and by "spaces," we meant simplicial sets. However, it would have been just as easy to ask for $\mathbf{C}$ to be enriched in topological spaces, as shown below on the right.

Then F should be a topological functor, which gives continuous maps

$$\mathbf{C}(a,b) \to \mathbf{D}(F(a),F(b)).$$



We define $\pi_0\mathbf{C}$ and DK-equivalences in the same way. (We saw these already in Exercise 7.7.27.)

Applying geometric realization and singular simplices to the morphism sets $\mathbf{C}(a,b)$ brings us back and forth between these two settings, and we get the same category back, up to pointwise equivalence (same objects and equivalent mapping spaces). Therefore:

Lemma 10.1.4. *The first definition of ∞-category works equally well with simplicial sets or with topological spaces.*

So we can be a little loose and just say that $\mathbf{C}$ is enriched in "spaces," without specifying whether "spaces" means simplicial sets or topological spaces. Since we are working up to weak equivalence, you don't need to worry which one we choose.[1]

Example 10.1.5. The category **Top** of all topological spaces is enriched in spaces, and is therefore an ∞-category. The subcategory of all CW complexes $\mathbf{CW} = \mathbf{Top}^{\mathrm{cf}}$ is also enriched in spaces, and is therefore also an ∞-category. The inclusion $\mathbf{Top}^{\mathrm{cf}} \subseteq \mathbf{Top}$ is *not* a Dwyer–Kan equivalence. Not every space is homotopy equivalent to a CW complex![2]

We therefore have two different ∞-categories, and only one of them is going to be called "the ∞-category of spaces." (Can you guess which one?)

[1] To be more precise, any time we have a category enriched in topological spaces, we will implicitly apply singular simplices and think of it as a category enriched in simplicial sets.

[2] This is an important point. A Dwyer–Kan equivalence is a bijection on homotopy equivalence classes of objects, not on weak equivalence classes of objects! If you want to invert the weak equivalences, you have to localize – see Section 10.2.

Example 10.1.6. As we mentioned above, we will call a category in the usual sense a "1-category," to distinguish it from an ∞-category. Every 1-category is an ∞-category in a natural way. Just form a simplicial category where the simplicial hom spaces $\mathbf{C}_\bullet(a,b)$ are all discrete (i.e. constant simplicial sets). Every equivalence of 1-categories gives a Dwyer–Kan equivalence and therefore an equivalence of ∞-categories.

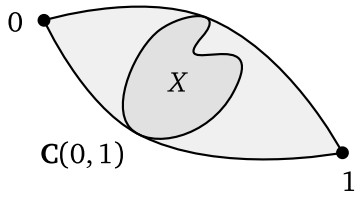
Note that **Top**, regared as a 1-category, is different from **Top** with its enrichment in spaces. In the first case, we have the category $\mathbf{Top}^{disc}$ of topological spaces and continuous maps, but *no topology* on the set of continuous maps. In the second case, we have the category $\mathbf{Top}^{enr}$ of topological spaces, and for each pair of spaces we have the *space* of continuous maps between them.

When living in ∞-land, we sometimes use the word "isomorphism" to refer to a homotopy equivalence in the simplicial category $\mathbf{C}$. So in the example above, an "isomorphism" in $\mathbf{Top}^{disc}$ is a homeomorphism, while an "isomorphism" in $\mathbf{Top}^{enr}$ is a homotopy equivalence. However, we won't make a habit of this when talking about simplicial categories, because there is already a notion of isomorphism that is stronger than this, namely a map with a strict inverse. We'll continue to say "homotopy equivalence" for now. But when we introduce quasicategories later, the corresponding maps in that setting will be called "isomorphisms," because in a quasicategory there is no such thing as a strict inverse.

Example 10.1.7. We can make several toy examples of ∞-categories.[3] For instance, there is an empty ∞-category $\varnothing$ with no objects, and a one-point ∞-category $*$ that has one object, whose automorphisms are the simplicial set $*$ that has one point at every simplicial level. These are initial and terminal, respectively, in the category of simplicial categories and simplicial functors.

Those examples both come from 1-categories, but here's one that doesn't. For any space X, we form a category $\mathbf{C} = \mathbf{U}(X)$ that has two objects 0 and 1, and only one nontrivial space of morphisms, which is $\mathbf{C}(0,1) = X$. In other words,

$$\mathbf{C}(0,0) = \{\mathrm{id}_0\}, \qquad \mathbf{C}(0,1) = X, \qquad \mathbf{C}(1,0) = \varnothing, \qquad \mathbf{C}(1,1) = \{\mathrm{id}_1\}.$$



We could also do a variant of this with three objects 0, 1, and 2, and the only nontrivial morphisms are from 0 to 1 and from 0 to 2. Again, we can pick any two spaces there.

[3] It is often said that "you can't construct ∞-categories explicitly." That's usually true for the second definition that we give below (quasicategories), but in this first definition (simplicial categories), we can give many explicit examples.

We could also make an example with a single object 0. Then $\mathbf{C}(0,0)$ has to be a monoid in the category of simplicial sets: there has to be a multiplication on it that is associative and has an identity element. Similarly, if we took our category with two objects 0 and 1, and allowed some nontrivial morphisms from 0 to 0, then $\mathbf{C}(0,0)$ has to be a monoid that acts on the space of morphisms $\mathbf{C}(0,1)$.

Example 10.1.8. This is the most important example. If $\mathbf{C}_\bullet$ is any simplicial model category, we take the subcategory of bifibrant objects $\mathbf{C}_\bullet^{\mathrm{cf}}$, as a category enriched in simplicial sets. This is the **underlying ∞-category** of $\mathbf{C}_\bullet$.[4]

Why the bifibrant objects? Well, Dwyer–Kan equivalences only see objects up to homotopy equivalence. But that's not what we want. We want the objects up to weak equivalence. But the set of all objects up to weak equivalence is the same as the set of bifibrant objects up to homotopy equivalence, by Whitehead's theorem (Corollary 5.2.28). It's also important that the simplicial sets $\mathbf{C}_\bullet(-,-)$ have the correct homotopy type, and this happens only when the first input is cofibrant and the second input is fibrant. So if we only consider bifibrant objects, then $\mathbf{C}_\bullet(-,-)$ is guaranteed to have the correct homotopy type.

We could also define the underlying ∞-category by taking every object of $\mathbf{C}$, and defining the morphisms to be $\mathbf{C}_\bullet(QRX, QRY)$, where Q and R are cofibrant and fibrant replacements, respectively. This is Dwyer–Kan equivalent to $\mathbf{C}_\bullet^{\mathrm{cf}}$.

By the fundamental theorem of model categories (Theorem 5.2.27), the homotopy category $\pi_0\mathbf{C}_\bullet^{\mathrm{cf}}$ is equivalent to $\mathrm{Ho}\,\mathbf{C}$, the homotopy category of $\mathbf{C}$. So you can think of $\mathbf{C}_\bullet^{\mathrm{cf}}$ as an enrichment of the homotopy category: instead of just taking the set of homotopy classes of maps between two objects X and Y, we're taking the (derived) space of all maps from X to Y. That's the underlying ∞-category of a model category.

It turns out that any Quillen equivalence will give an equivalence of these underlying ∞-categories (Corollary 10.2.13). So we can define the **∞-category of spaces** $\mathscr{S}$ to be the underlying ∞-category of either **Top** or **sSet**. In other words, $\mathscr{S}$ is the simplicial category of CW complexes and continuous maps between them, or the simplicial category of Kan complexes and the simplicial homs between them. Those are Dwyer–Kan equivalent to each other. (So for those keeping score at home, $\mathbf{Top}^{\mathrm{cf}}$ is the correct ∞-category of spaces, not **Top**.)

As an example, the space of maps from S^1 to S^1 in the ∞-category of spaces $\mathscr{S}$ is equivalent to $LS^1 \simeq S^1 \times \mathbb{Z}$. Similarly we can define the **∞-category of based spaces** $\mathscr{S}_*$ to be the underlying ∞-category of either $\mathbf{Top}_*$ or $\mathbf{sSet}_*$.

The **∞-category of spectra** Sp is the underlying ∞-category of any of the models of spectra we have seen: $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}}$, $\mathbf{Sp}^{\Sigma}_{\mathbf{Top}}$, $\mathbf{Sp}^{O}_{\mathbf{Top}}$, $\mathscr{M}$, $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{sSet}}$, or $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}}$. So it has an object for

[4] More precisely, it is the simplicial category that models the underlying ∞-category. The quasicategory version is in Example 10.1.55 and Definition 10.2.8.

each cellular Ω-spectrum, and the morphisms are given by the mapping spaces $\mathrm{Map}_*(X,Y)$ from Definition 2.3.12. Equivalently, it has an object for every spectrum, and the mapping spaces are

$$\mathrm{Sp}(X,Y) = \mathrm{Map}_*(QRX, QRY).$$

Taking π_0 gives the stable homotopy category Ho **Sp**, because

$$[X,Y]_s = \pi_0 \mathrm{Map}_*(QRX, QRY).$$

As an example, the space of maps from $\mathbb{S}$ to $\mathbb{S}$ in the ∞-category of spectra Sp is equivalent to the space $\Omega^\infty\mathbb{S}$, whose homotopy groups are the stable homotopy groups of spheres. Taking π_0 gives the maps in the stable homotopy category, $[\mathbb{S},\mathbb{S}]_s \cong \mathbb{Z}$.

The following model structure is an important tool in this setting. We refer the reader to [Ber07] for the proof.

Theorem 10.1.9 (Bergner model structure)**.** *[Ber07, Thm 1.1] There is a model structure on the category of simplicial categories and simplicial functors, in which*

- *the cofibrations are generated by the functor $\varnothing \to *$ and the functors*

  $$\{U(\partial\Delta[n]) \to U(\Delta[n])\}_{n\geq 0},$$

  where $U(X)$ is the ∞-category with two objects and one nontrivial morphism space from Example 10.1.7,
- *a weak equivalence is a Dwyer–Kan equivalence (Definition 10.1.3), and*
- *a fibration is a simplicial functor $F\colon \mathbf{C} \to \mathbf{D}$ such that*
  - *each map $\mathbf{C}_\bullet(a,b) \to \mathbf{D}_\bullet(F(a),F(b))$ is a Kan fibration, and*
  - *every homotopy equivalence $e\colon F(a) \to b$ in $\mathbf{D}_0(F(a),b)$, lifts to a homotopy equivalence $d\colon a \to a'$ in $\mathbf{C}_0(a,a')$, with $F(d) = e$.*

Note that this is not the same as the right-induced model structure on simplicial categories from Theorem A.2.20: in that one the object set is fixed, the weak equivalences are the pointwise equivalences, and the fibrations are the pointwise fibrations. In the Bergner model structure the object set is allowed to vary, and the weak equivalences are the Dwyer–Kan equivalences.

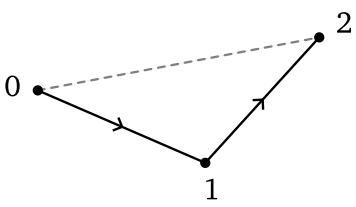
Be careful when trying to describe the cofibrant categories in the Bergner model structure. The generating cofibrations are very simple, but when we take a pushout along such a cofibration, we take the pushout *in the category of simplicial categories*. This is complicated. For instance, in the simplest possible case that the category has only one object, the pushout category already looks like the pushout we described in Lemma 6.5.16.

10.1.2 Second definition: quasicategories

Now, the first definition of ∞-category from Definition 10.1.3 is best if you want to understand what an ∞-category really is, or describe the homotopy type of any one mapping space $\mathbf{C}(X,Y)$ in the ∞-category. However, it's hard to use for serious technical work.

The problem is that cofibrant replacements and colimits are too complicated. For instance, we would like to define the category of functors, $\mathrm{Fun}(\mathbf{C},\mathbf{D})$, in a way that respects Dwyer–Kan equivalences. But this requires us to take the cofibrant replacement of $\mathbf{C}$, and then we don't have much intuition for what the objects of the resulting category $\mathrm{Fun}(\mathbf{C},\mathbf{D})$ actually look like. The natural transformations are even harder to understand, because even if $\mathbf{C}$ is cofibrant, the product $\mathbf{C}\times\{0\to 1\}$ isn't cofibrant again!

The next definition of "∞-category" doesn't have these problems. It was originally introduced by Boardman and Vogt [BV73], but it wasn't until the famous work of Joyal and then Lurie [Joy02, Joy08, Lur09, Lur17] that it started being used extensively in homotopy theory.



Recall from Definition 7.1.35 that the ith **horn** of $\Delta[n]$ is the simplicial set $\Lambda_i[n]\subseteq\Delta[n]$ corresponding to the union of all of the faces of $\Delta[n]$, except for the ith face. This makes sense for $0\le i\le n$, but if we restrict to $0<i<n$ then we call $\Lambda_i[n]$ an **inner horn**. The lowest-dimensional example is $\Lambda_1[2]$, pictured to the right.

Definition 10.1.10. A **quasicategory** is a large[5] simplicial set $C_\bullet$ with the property that every inner horn in $C_\bullet$ can be filled:

$$\begin{array}{ccc}\Lambda_i[n]_\bullet & \longrightarrow & C_\bullet \\ \downarrow & \nearrow & \\ \Delta[n]_\bullet & & \end{array} \tag{10.1.11}$$

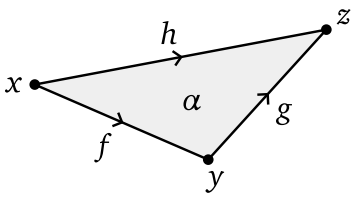
An **object** of C is a 0-simplex $x\in C_0$. A **morphism** is a 1-simplex $f\in C_1$.

We usually write morphisms as $f: x\to y$, where $x=d_1 f$ and $y=d_0 f$.

The **identity morphism** of x is the degenerate 1-simplex $\mathrm{id}_x := s_0 x\in C_1$.

[5]This just means that the levels X_n don't have to be sets, they could be larger than that. Don't worry too much about issues of size until you understand the theory well and want to dig a little deeper.

An inner horn $\Lambda_1[2] \to C_\bullet$ is the same thing as a choice of two morphisms of the form $f : x \to y$ and $g : y \to z$. We call each 2-simplex $\alpha : \Delta[2] \to C_\bullet$ that fills this horn a possible **composition** of f and g, as shown on the right. (We think of the 2-simplex as the composition, not just the edge h!)



So C_2 is the set of all compositions of pairs of morphisms. More generally, we think of C_n as the set of all possible n-fold compositions of morphisms $x_0 \to x_1 \to \cdots \to x_n$.

The axioms tell us that any two morphisms $f : x \to y$ and $g : y \to z$ have a composition, but they don't say that the composition is unique. It usually isn't! However, if we keep track of them in the right way, then compositions are unique up to a contractible choice. We therefore have something that is like a category, but a little bit weaker. A quasi-category!

A **functor**, or map of quasicategories, is just a map of simplicial sets $F : C \to D$. So it consists of functions $C_n \to D_n$ that commute with the face and degeneracy maps.

Our **second definition of ∞-category** is: an ∞-category is a quasicategory. We will consider two of them to be "the same" if there is a categorical equivalence between them, in the sense of Definition 10.1.24 below. For now, think of a categorical equivalence as a functor $F : C \to D$ that has an inverse up to some notion of natural isomorphism.

We sometimes restrict attention to **locally small quasicategories**, which have the property that for every finite set $S \subseteq C_0$, the set of all n-fold compositions (simplices in C_n) with vertices lying in S forms a set, not a proper class [Lur09, 5.4.1.7]. Similarly a **small quasicategory** is one in which the entire collection C_n is a set for every $n \geq 0$.

Intuitively, quasicategories are a way of interpolating between spaces and categories. So every space should be a quasicategory, and every 1-category should be a quasicategory.

Example 10.1.12. If X is any topological space, its singular simplices $S_\bullet X$ are a Kan complex, in other words every one of its horns can be filled, not just the inner horns (Definition 7.1.37). Therefore $S_\bullet X$ is a quasicategory. The objects are the points of X, the morphisms are the continuous paths, and the compositions are the compositions of paths up to homotopy.

Example 10.1.13. For any 1-category $\mathbf{C}$, the nerve $N(\mathbf{C})$ from Example 7.1.10 is a quasicategory. In fact, the inner horn fillers in (10.1.11) are unique, because the only way we can fill them in is by literally taking the compositions of the morphisms in the 1-category $\mathbf{C}$. So the objects of the quasicategory $N(\mathbf{C})$ are the objects of $\mathbf{C}$, the morphisms of $N(\mathbf{C})$ are the morphisms of $\mathbf{C}$, and the compositions in $N(\mathbf{C})$ are the (uniquely defined) compositions in $\mathbf{C}$.

To write this out a little more, if we have an inner horn $\Lambda_i[n] \to N(\mathbf{C})$, we can pass to the vertices, giving a sequence of objects $c_0, c_1, \ldots, c_n$. The 1-simplices between adjacent

vertices give a sequence of morphisms

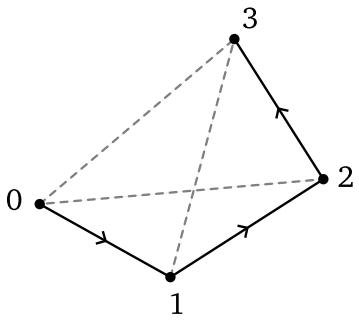
$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \longrightarrow \cdots \xrightarrow{f_n} c_n.$$

The rest of the horn $\Lambda_i[n]$ gives some, but not all, of the possible compositions of these morphisms. For instance, $\Lambda_1[3]$ gives the 2-simplices (f_2, f_3), $(f_1, f_3 \circ f_2)$, and (f_1, f_2), but not the last 2-simplex $(f_2 \circ f_1, f_3)$ or the 3-simplex (f_1, f_2, f_3). However, since $\mathbf{C}$ is a 1-category, its morphisms can be composed in a unique way, and so the only way to fill this in is with the 3-simplex (f_1, f_2, f_3).

Recall from Definition 9.4.1 that the **spine** of the n-simplex $\Delta[n]$ is the subcomplex

$$\Delta[1] \cup_{\Delta[0]} \Delta[1] \cup_{\Delta[0]} \cdots \cup_{\Delta[0]} \Delta[1] \subseteq \Delta[n]$$

consisting of all the vertices, and the edges between adjacent vertices. In a quasicategory, when we're thinking of an n-simplex as a "composition" of n morphisms, the spine of the simplex gives the list of n morphisms and none of their compositions.



The above proof can be generalized to show that in any nerve of a 1-category $C = N(\mathbf{C})$, every *spine*

$$\Delta[1] \cup_{\Delta[0]} \Delta[1] \cup_{\Delta[0]} \cdots \cup_{\Delta[0]} \Delta[1] \to C$$

extends uniquely to an n-simplex $\Delta[n] \to C$.

Lemma 10.1.14. *A 1-category is the same thing as a simplicial set in which every inner horn* (10.1.11) *can be filled uniquely, which is true iff every spine can be filled uniquely. Furthermore, maps of simplicial sets between these are the same thing as functors of 1-categories.*

In other words, the nerve N : **Cat** $\to$ **sSet** is fully faithful. When we think of each 1-category as a quasicategory, the set of functors between two 1-categories doesn't change!

Now, the definition of a quasicategory is not too complicated, but in order to parse it correctly, we have to make a mental shift. Up until this point, we've been thinking of simplicial sets as a model for topological spaces, so we've been considering them up to **weak homotopy equivalence**, in other words maps that give an equivalence on the realizations $|X_\bullet| \simeq |Y_\bullet|$.

However, at this point we want to think of simplicial sets as a generalization of categories, in light of Lemma 10.1.14. In particular, the simplicial sets

$$\Delta[0] = N(\{0\}), \qquad \Delta[1] = N(\{0 \to 1\})$$

are now categories. Let's recall that the one-point category $\{0\}$ is not equivalent to the "walking arrow" category $\{0 \to 1\}$. The first has one isomorphism class, the second has two isomorphism classes.

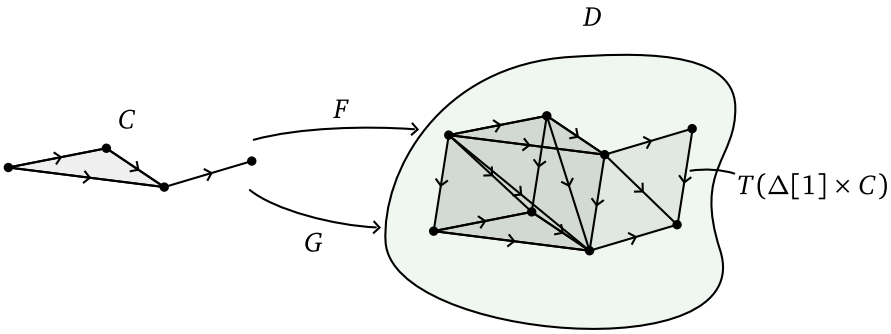
So, $\Delta[1]$ and $\Delta[0]$ should *not be equivalent anymore*!

Instead, $\Delta[0]$ should be equivalent to $N(\{0 \cong 1\})$, the nerve of the category with a single nontrivial isomorphism. This is a much larger simplicial set than $\Delta[1]$, because it has two nondegenerate simplices in every dimension.

To make this mental shift, we need to stop thinking of 1-simplices as homotopies, and remember that they have a direction. They are no longer invertible, the way they were when we worked with simplicial sets and the Kan model structure.

In particular, a map $\Delta[1] \times C \to D$ is not the correct definition of a "natural isomorphism." It's only a natural transformation:

Definition 10.1.15. For any two quasicategories C and D, a **natural transformation** of functors $F, G \colon C \rightrightarrows D$ is a map of simplicial sets $T \colon \Delta[1] \times C \to D$ that on each vertex of $\Delta[1]$ gives F and G. Formally, $T(d^1 \times \mathrm{id}_C) = F$ and $T(d^0 \times \mathrm{id}_C) = G$.



More generally, the **functor category** $\mathrm{Fun}(C, D)$ is the simplicial set of maps $\mathrm{Map}_\bullet(C, D)$ that we defined in Exercise 7.7.15. The n-simplices $\mathrm{Fun}_n(C, D)$ are the maps of simplicial sets

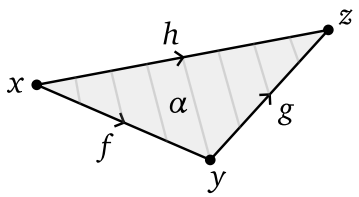
$$\Delta[n] \times C \to D.$$

So the 0-simplices are the functors $C \to D$, the 1-simplices are the natural transformations, and the higher simplices describe all possible coherent n-fold compositions of natural transformations.

Example 10.1.16. For any quasicategory C, the **opposite category** C^{op} is obtained by taking the simplicial set with the same levels but with face and degeneracy maps reversed: d_i on the opposite simplicial set is d_{n-i} on the original simplicial set, and similarly for degeneracies. When C is the nerve of a 1-category, C^{op} is the nerve of its opposite category.

There is a weak homotopy equivalence $C \simeq C^{\mathrm{op}}$, but in general there will *not* be a categorical equivalence between them!

Once we're comfortable with the fact that not all of the 1-simplices are invertible, we introduce the next point of philosophy: all of the 2-simplices *are* going to be invertible. So any time we have composition $\alpha \in C_2$ of the morphisms f and g, we think of it as an invertible "homotopy" between $g \circ f$ and the remaining edge $h = d_1\alpha$, as shown on the right.



$\alpha : h \sim g \circ f$

This motivates the following definition.

Definition 10.1.17. For any simplicial set C, the **homotopy category** (or "fundamental category") is the 1-category hC in which

- the objects are the 0-simplices C_0,
- the morphisms are generated by the nondegenerate 1-simplices C_1^{nd}, and
- each 2-simplex gives a relation $h \sim g \circ f$.

To be more precise, the morphisms are equivalence classes of strings of composable 1-cells, including possibly the identity morphisms $s_0 x \in C_1$, but we allow ourselves to remove or add identity 1-cells, and for each 2-simplex we are allowed to apply the equivalence relation $h \sim g \circ f$ in the middle of any such word.

Lemma 10.1.18. *We have an adjunction*

$$\mathbf{sSet} \underset{N(-)}{\overset{h(-)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Cat}.$$

So when C is a simplicial set and D is a 1-category, functors of 1-categories $hC \to D$ are the same thing as maps of simplicial sets $C \to N(D)$.

Proof. Left to Exercise 10.8.4. □

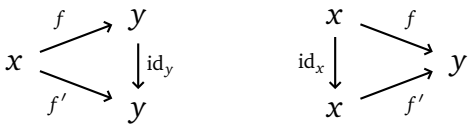
In particular, the homotopy category of the nerve of D just gives the category D back:

$$hN(D) \cong D.$$

On the other hand, if C is a simplicial set, then we have a map $C \to N(hC)$ that approximates C as well as possible by something that is the nerve of a 1-category.

If we adopt the practice of blurring the distinction between a category D and its nerve $N(D)$, then we can rewrite this as: each quasicategory has a forgetful map to its homotopy category $C \to hC$, and this is an isomorphism when C is a 1-category.

Lemma 10.1.19. *When C is a quasicategory, the homotopy category hC can be equivalently defined by taking the morphisms $hC(x,y)$ to be equivalence classes of morphisms $f : x \to y$ in C, up to either of the following two equivalence relations $f \sim f'$:*

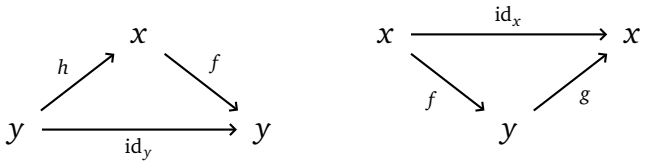


The proof of this is elementary but not short, see [Lur09, 1.2.3.9] or [Rez22, 11.13].

Corollary 10.1.20. *If C is a quasicategory, then a morphism in the homotopy category hC can always be represented by a single 1-simplex $f \in C_1$. Furthermore, if $f, g, h \in C_1$ and h agrees with $g \circ f$ in the homotopy category, then there exists a single 2-simplex $a \in C_2$ such that $d_0 a = g$, $d_1 a = h$, and $d_2 a = f$.*

Definition 10.1.21. In any simplicial set C, an **isomorphism** in C is a morphism $f \in C_1$ that becomes an isomorphism in the homotopy category hC.

When C is a quasicategory, by Corollary 10.1.20, $f : x \to y$ is an isomorphism iff there exist morphisms $g, h : y \to x$ and compositions $g \circ f \sim \mathrm{id}_x$ and $f \circ h \sim \mathrm{id}_y$ as shown.



Also, it is always possible to select $g = h$.

For instance, in the nerve of any 1-category $N(\mathbf{C})$, an isomorphism in this sense is the same thing as an isomorphism in the 1-category $\mathbf{C}$. In a general quasicategory, it is perhaps closer to what you would call a "homotopy equivalence."

Lemma 10.1.22. *[DS11, Prop 2.2] If C is a quasicategory, then $f : \Delta[1] \to C$ specifies an isomorphism in C iff it extends to a map $\widetilde{f} : N(\{0 \cong 1\}) \to C$.*

So when $f \in C_1$ is an isomorphism, not only do we get an inverse morphism $g \in C_1$ and compositions $f \circ g \sim \mathrm{id}_y$ and $\mathrm{id}_x \sim g \circ f$, but we also get coherences relating all the homotopies $g \circ f \circ g \sim g$, $f \circ g \circ f \sim f$, $g \circ f \circ g \circ f \sim \mathrm{id}_x$, and so on. These are encoded by the infinite set of nondegenerate simplices of the simplicial set $N(\{0 \cong 1\})$. An isomorphism carries an infinite amount of structure!

Lemma 10.1.23. *[Rez22, Thm 35.2], [Lan21, 2.2.2] More generally, for any simplicial set X and quasicategory C, a map*

$$F : X \times \Delta[1] \to C$$

sends each 1-simplex of the form $\{x\} \times \Delta[1]$ *to an isomorphism in* C *iff it extends to a map*

$$\widetilde{F} : X \times N(\{0 \cong 1\}) \to C.$$

We call such a map F a **natural isomorphism** of functors $X \to C$. We will see shortly that $\mathrm{Fun}(X, C)$ is a quasicategory. The morphisms in this quasicategory are given by maps $X \times \Delta[1] \to C$, and the isomorphisms in this quasicategory are exactly the natural isomorphisms that we have just defined.

We are now ready to describe the new, stricter notion of "weak equivalence."

Definition 10.1.24. A map of simplicial sets $X \to Y$ is a **Joyal equivalence**, or a **categorical equivalence**, if for every quasicategory C, the induced map of homotopy categories

$$h\,\mathrm{Fun}(Y, C) \to h\,\mathrm{Fun}(X, C)$$

is an equivalence of 1-categories. (So it is a bijection on isomorphism classes of objects, and a bijection on the morphisms between each pair of objects.)

Expanding this out, $X \to Y$ is a Joyal equivalence if for each quasicategory C, the set of functors $X \to C$ up to natural isomorphism is identified with the set of functors $Y \to C$ up to natural isomorphism, and the natural transformations between such functors are also identified, up to natural isomorphism.

The most basic example is an inner horn inclusion. Let J be the set of all inner horn inclusions,

$$J = \{\, \Lambda_i[n] \to \Delta[n] \,\}_{n \geq 1,\ 0 < i < n}, \tag{10.1.25}$$

Recall from Definition 5.1.19 that a J-cofibration is a retract of a relative J-cell complex. In quasicategory theory, this is called an **inner anodyne** map. It is any map built by successively adding inner horn fillings, and then possibly taking a retract of the result.

Proposition 10.1.26. *[Cis19, 3.2.4 and 3.3.7] Every inner anodyne map is a Joyal equivalence. In particular, every inner horn inclusion* $\Lambda_i[n] \to \Delta[n]$ *is a Joyal equivalence.*

In addition, if f *is a cofibration of simplicial sets and* g *is an inner anodyne map then the pushout-product* $f \square g$ *is again inner anodyne, and therefore a Joyal equivalence.*

Unfortunately, this does not describe all of the Joyal equivalences, only some of them. It is not so easy to understand what they look like in general, for arbitrary simplicial sets X and Y. However, if X and Y happen to be quasicategories, then it's easier, because we can take $C = X$ or $C = Y$. The following is then an exercise (Exercise 10.8.7).

Lemma 10.1.27. *If* C *and* D *are quasicategories, a Joyal equivalence (aka a categorical equivalence) is a functor* $F : C \to D$ *such that there exists an inverse functor* $G : D \to C$ *and natural isomorphisms* $G \circ F \cong \mathrm{id}_C$, $F \circ G \cong \mathrm{id}_D$.

As we discussed after Lemma 10.1.23, the natural isomorphism $\mathrm{id}_C \cong G \circ F$ can be given by a natural transformation $\mathrm{id}_C \to G \circ F$, so a map of simplicial sets $C \times \Delta[1] \to C$, that at each object $x \in C_0$ picks out an isomorphism $x \cong GF(x)$ in C.

Example 10.1.28. If $\mathbf{C}$ and $\mathbf{D}$ are 1-categories, a map $F : N(\mathbf{C}) \to N(\mathbf{D})$ is a Joyal equivalence iff when regarded as a functor $F : \mathbf{C} \to \mathbf{D}$, it is an equivalence of 1-categories.

The following model structure will be a powerful black box for us. It was first described in work of Joyal [Joy08, 2.6], and a proof can be found in [Lur09, 2.2.5.1] or [Cis19, 3.3.7]. A particularly short version is in [DS11, App C].

Theorem 10.1.29 (Joyal model structure)**.** *The category of (large) simplicial sets* **sSet** *has a model structure in which*

- *the cofibrations are the same as the Kan model structure, the maps that are injective at each level (Definition 7.1.37),*
- *the weak equivalences are the Joyal equivalences (Definition 10.1.24), and*
- *the fibrations are the Joyal fibrations, which have the right-lifting property with respect to the maps that are cofibrations and Joyal equivalences.*

Furthermore,

- *This model category* $\mathbf{sSet}_{\text{Joyal}}$ *is cofibrantly generated.*[6]
- *The identity functor* $\mathbf{sSet}_{\text{Joyal}} \to \mathbf{sSet}_{\text{Kan}}$ *is left Quillen.*
- *Every object in* $\mathbf{sSet}_{\text{Joyal}}$ *is cofibrant, and an object is fibrant iff it is a quasicategory.*
- $\mathbf{sSet}_{\text{Joyal}}$ *is a symmetric monoidal model category under the Cartesian product.*

Remark 10.1.30. Because the fibrant objects are the quasicategories, we can make fibrant replacements in this model structure by using the small-object argument from Theorem 5.1.13 with the set of all inner horn inclusions from (10.1.25). In other words, we can make each simplicial set X into a quasicategory fX, by attaching fillers for all possible inner horns in X, and doing this countably many times. This *doesn't change* the vertices X_0 or the edges X_1, it only adds in higher simplices. So intuitively, all we are doing is "adding in all possible compositions" for the morphisms that are already in X_1!

Warning 10.1.31. A fibration in the Joyal structure is *not* the same thing as a map which has lifts along each inner horn:

$$\begin{array}{ccc} \Lambda_i[n]_\bullet & \longrightarrow & X_\bullet \\ \downarrow & \nearrow & \downarrow \\ \Delta[n]_\bullet & \longrightarrow & Y_\bullet \end{array} \tag{10.1.32}$$

[6] However, according to [DS11], no one has yet written down a manageable set of generating acyclic cofibrations.

We distinguish these using the following definition.

Definition 10.1.33. An **inner fibration** is a map of simplicial sets that has a lift for each square of the form (10.1.32). In other words, it is a J-fibration, if J is the set of inner horn inclusions from (10.1.25).

An **isofibration** is a map such that every isomorphism $f : x \to y$ in Y can be lifted to an isomorphism $\bar{f}$ in X, starting at any chosen lift $\bar{x}$ of x, or ending at any chosen lift $\bar{y}$ of y.

It follows from Proposition 10.1.26 that *if* $X \to Y$ is a Joyal fibration, *then* it is an inner fibration. The converse only holds in special cases:

Lemma 10.1.34.
- *A map $X \to *$ is a Joyal fibration iff it is an inner fibration, i.e. X is a quasicategory. Note that $X \to *$ is always an isofibration.*
- *If Y is a quasicategory, a map $X \to Y$ is a Joyal fibration iff it is both an inner fibration and an isofibration. When these hold, X must be a quasicategory as well.*

We can use Theorem 10.1.29 as a black box and deduce many important corollaries, using our knowledge of model categories and Quillen tensors:

Corollary 10.1.35.
- *If C is a simplicial set and D is a quasicategory, then* $\mathrm{Fun}(C, D)$ *is a quasicategory. In this case, every Joyal equivalence in each slot gives a Joyal equivalence on* $\mathrm{Fun}(C, D)$*. [Lur09, 1.2.7.3]*
- *If C is a quasicategory then* $\mathrm{Fun}(\Delta[2], C) \to \mathrm{Fun}(\Lambda_1[2], C)$ *is an acyclic Kan fibration,*[7] *so its fiber is a contractible Kan complex. In other words, the space of compositions of a given pair of morphisms $f : x \to y$ and $g : y \to z$ is contractible.*
- *Every Joyal equivalence is a weak homotopy equivalence. Between Kan complexes, every weak homotopy equivalence is a Joyal equivalence.*
- *Two functors $F, G : C \rightrightarrows D$ are identified in the homotopy category* $\mathrm{Ho}\,\mathbf{sSet}_{\mathrm{Joyal}}$ *iff they extend to a functor $C \times N(\{0 \cong 1\}) \to D$, iff they are naturally isomorphic.*

We say that a quasicategory C is an **∞-groupoid**, or an **$(\infty, 0)$-category**, if every one of its morphisms is invertible.

Lemma 10.1.36. *C is an ∞-groupoid iff it is a Kan complex.*

We already know that between Kan complexes, a Joyal equivalence is the same thing as a weak homotopy equivalence. Therefore, spaces up to weak equivalence are the same thing as ∞-groupoids up to Joyal equivalence. This is sometimes called the **homotopy hypothesis**.

[7] Note that acyclic Joyal fibrations are the same thing as acyclic Kan fibrations, because the cofibrations in $\mathbf{sSet}_{\mathrm{Joyal}}$ and $\mathbf{sSet}_{\mathrm{Kan}}$ coincide.

Example 10.1.37. For any quasicategory C, the **core** $C^{\simeq}$ is the sub-simplicial set of C consisting of all objects, all isomorphisms, and all possible compositions of the isomorphisms. (So it contains every n-simplex whose spine consists of isomorphisms. Equivalently, it contains the n-simplices in which every 1-simplex is an isomorphism.) It is straightforward to see that $C^{\simeq}$ is also a quasicategory, so it is an ∞-groupoid.

For any quasicategory C, the **full localization** is the Kan complex $C[C_1^{-1}]$ obtained by taking the Kan fibrant replacement of C. In other words, there is a weak homotopy equivalence (not a Joyal equivalence!) of the form $C \to C[C_1^{-1}]$. So $C[C_1^{-1}]$ is an ∞-groupoid, and we should think of it as the universal ∞-groupoid receiving a map from C. We will later interpret this as the localization in which we have turned every morphism in C_1 into an isomorphism.

Finally, we explain why quasicategories carry out the philosophy of what an ∞-category should be: the morphisms between any two objects x and y form a space, that is well-defined up to weak homotopy equivalence.

Definition 10.1.38. Let C be any quasicategory and let $x, y \in C_0$ be objects of C. We define the cylindrical **mapping space** $\mathrm{Map}_C(x,y)$ to be the pullback of simplicial sets

$$\begin{array}{ccc} \mathrm{Map}_C(x,y) & \longrightarrow & \mathrm{Fun}(\Delta[1], C) \\ \downarrow & & \downarrow{\scriptstyle d^1, d^0} \\ * & \xrightarrow{x,y} & \mathrm{Fun}(\Delta[0] \amalg \Delta[0], C). \end{array}$$

In other words, it is the simplicial set whose n-simplices are those maps

$$\Delta[n] \times \Delta[1] \to C$$

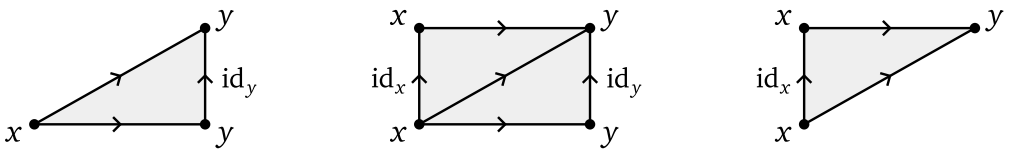
that along the two ends $\Delta[n] \times (* \amalg *)$ are just the degenerate n-simplices at the points x and y.

We could also define the **right mapping space** $\mathrm{Map}_C^R(x,y)$ to be the simplicial set whose n-simplices are those maps $\Delta[n+1] \to C$ that send the final vertex to y, and the face spanned by all the remaining vertices to the degenerate n-simplex at x. The face and degeneracy maps act on the first n vertices of $\Delta[n+1]$.

Or, we could define the **left mapping space** $\mathrm{Map}_C^L(x,y)$ to be the simplicial set whose n-simplices are those maps $\Delta[n+1] \to C$ that send the initial vertex to x, and the face spanned by all the remaining vertices to the degenerate n-simplex at y. The face and degeneracy maps act on the last n vertices of $\Delta[n+1]$.

In each of these mapping spaces, a 0-simplex is a morphism $f : x \to y$. However, the 1-simplices are different. A 1-simplex in the cylinder mapping space from f to g is given

by a rectangle as shown, while in the right or left mapping space a 1-simplex is given by a single triangle:



$\mathrm{Map}_C^L(x,y)$ $\mathrm{Map}_C(x,y)$ $\mathrm{Map}_C^R(x,y)$

These are closely related, because the rectangle can be collapsed down onto either of these triangles. In general we can give a collapse map

$$\Delta[n]\times\Delta[1]\to\Delta[n+1]$$

by the map of posets

$$\{0\to 1\to\cdots\to n\}\times\{0\to 1\}\to\{0\to 1\to\cdots\to(n+1)\}$$

that takes $(i,0)$ to i and $(i,1)$ to $(n+1)$ for all $0\le i\le n$. This commutes with all faces and degeneracies, giving a map of simplicial sets

$$\mathrm{Map}_C^R(x,y)\longrightarrow\mathrm{Map}_C(x,y).$$

A similar trick with the left mapping space gives a map $\mathrm{Map}_C^L(x,y)\to\mathrm{Map}_C(x,y)$.

Lemma 10.1.39. *[DS11, Cor 4.7], [Lur09, 4.2.1.8] For any quasicategory C, all three of these mapping spaces are Kan complexes, and these maps between them are weak homotopy equivalences.*

If we pass to the connected components $\pi_0\mathrm{Map}_C(x,y)$, for any of these mapping spaces, we get exactly the morphisms from x to y in the homotopy category hC. Therefore Lemma 10.1.39 is a more highly structured version of Lemma 10.1.19, which gave us different criteria for when two maps f and f' agree in hC.

Warning 10.1.40. There is no strictly associative composition

$$\mathrm{Map}_C(x,y)\times\mathrm{Map}_C(y,z)\longrightarrow\mathrm{Map}_C(x,z),$$

only a composition that is well-defined up to homotopy. To define it, notice that an n-simplex on the left-hand side specifies a map $\Delta[n]\times\Lambda_1[2]\to C$, this can be filled to a map $\Delta[n]\times\Delta[2]\to C$, even if we've already specified the filling on $\partial\Delta[n]\times\Delta[2]$. Repeating this for all n-simplices one value of $n\ge 0$ at a time defines the composition. A similar horn-filling argument shows that it is welll-defined up to homotopy.

Since we cannot make the above composition associative, these mapping spaces do *not* give a category enriched in simplicial sets. (If only it were that easy to go back and forth between quasicategories and simplicial categories...)

Lemma 10.1.41. *A map of quasicategories $F : C \to D$ is a Joyal equivalence iff the map of homotopy categories $hC \to hD$ is essentially surjective and F is **fully faithful**, meaning the maps induced by F*

$$\mathrm{Map}_C(x, y) \longrightarrow \mathrm{Map}_D(F(x), F(y))$$

are weak homotopy equivalences of Kan complexes.

To summarize the mapping spaces we have encountered so far:

- Between any two quasicategories C and D, we have a quasicategory of functors $\mathrm{Fun}(C, D)$, and
- within any one quasicategory C, between any two objects x and y, we have a Kan complex of morphisms $\mathrm{Map}_C(x, y)$.

So $\mathrm{Map}_C(x, y)$ is just a space, but $\mathrm{Fun}(C, D)$ is more than a space, it's a quasicategory! Since the maps between two quasicategories are more than just a space, the collection of all quasicategories is more than just an ∞-category. It forms something called an $(\infty, 2)$**-category**. On your first pass through the subject, you should ignore this, and think of them as just an ∞-category. We accomplish this by restricting $\mathrm{Fun}(C, D)$ to the core, $\mathrm{Fun}(C, D)^{\simeq}$. This is the simplicial set of functors and natural isomorphisms between them, and this is a Kan complex.

10.1.3 Passing between simplicial categories and quasicategories

The astute reader will notice that we gave *no examples* of quasicategories, other than the quasicategory associated to a space X (Example 10.1.12), and the nerve of a 1-category $N(\mathbf{C})$ (Example 10.1.13).

There is a reason for this: it's very difficult to explicitly construct examples. Most of our examples are constructed indirectly out of some simpler kind of input.

For instance, it turns out that every (pointwise fibrant) simplicial category $\mathbf{C}_\bullet$ gives a quasicategory $N^s(\mathbf{C}_\bullet)$, called the **simplicial nerve** or the **homotopy-coherent nerve**. This extends the construction of the nerve from discrete 1-categories to simplicial 1-categories, giving a much larger class of examples. In fact, this gives *all possible examples*: every quasicategory, up to Joyal equivalence, is $N^s(\mathbf{C}_\bullet)$ for some simplicial category $\mathbf{C}_\bullet$.

To understand how this works, recall that the geometric realization and singular simplices are functors

$$\mathbf{sSet} \underset{S_\bullet(-)}{\overset{|-|}{\rightleftarrows}} \mathbf{Top} \quad \bot \quad .$$

We defined these by choosing a sequence of topological spaces Δ^n. Then we defined $S_n Y$ to be the set of maps in **Top** from Δ^n to Y, and we defined $|X_\bullet|$ by taking products $X_n \times \Delta^n$ and gluing them together along face and degeneracy maps.

It turns out that every adjunction of the form

$$\mathbf{sSet} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{C}$$

arises this way: you pick the objects $L(\Delta[n])$ in the category **C**, and they have to form a diagram on $\mathbf{\Delta}$, in other words, a cosimplicial object in **C**. Once this is done, you define $R(Y)$ to be the simplicial set whose nth level is the set of maps $L(\Delta[n]) \to Y$ in **C**. And $L(X_\bullet)$ is the object of **C** formed out of the products $X_n \times L(\Delta^n) = \coprod_{X_n} L(\Delta^n)$ by modding out by face and degeneracy maps.

Lemma 10.1.42. *For any category* **C** *with all colimits, this procedure always defines an adjunction, and every adjunction from* **sSet** *arises this way.*

So, to define an adjunction between simplicial sets and simplicial categories of the form

$$\mathbf{sSet} \underset{N^s}{\overset{\mathfrak{C}}{\rightleftarrows}} \mathbf{sCat},$$

all we have to do is name some simplicial categories $\mathfrak{C}(\Delta[n])$, and check that they have face and degeneracy maps between them!

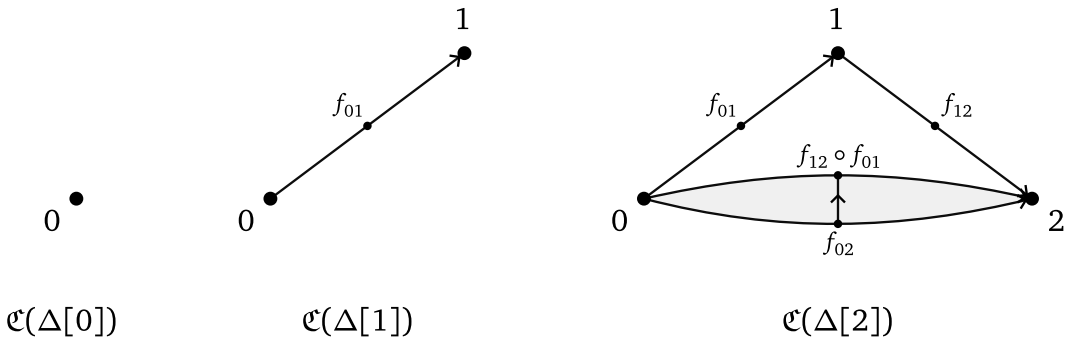
We could pick the category $\{0 \to 1 \to \cdots \to n\}$, whose nerve is the standard n-simplex. This is a perfectly fine category, but it's discrete: it doesn't have any higher simplices in its mapping spaces. The set of maps from this category to some other simplicial category $\mathbf{C}_\bullet$ is going to give us $N_n(\mathbf{C}_0)$, in other words the nerve of the category $\mathbf{C}_0$. This doesn't see any of the higher simplices in $\mathbf{C}_\bullet$, so it can't be the right answer.

What should we do instead? If we think through the small examples, the category $\{0\}$ is still the right choice for $\mathfrak{C}(\Delta[0])$, and $\{0 \to 1\}$ is the right choice for $\mathfrak{C}(\Delta[1])$, but once we get to $n = 2$, we should take $\mathfrak{C}(\Delta[2])$ to be a category with objects 0, 1, and 2, morphisms

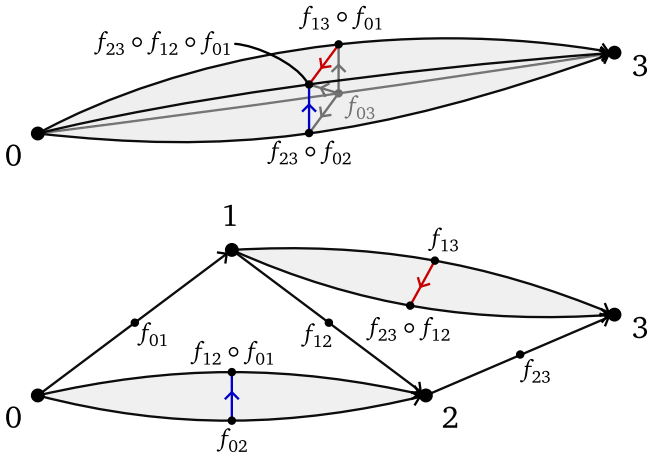
$$\begin{array}{ccccc} & & 1 & & \\ & \overset{f_{01}}{\nearrow} & & \overset{f_{12}}{\searrow} & \\ 0 & & \xrightarrow{\quad f_{02} \quad} & & 2, \end{array}$$

then a homotopy or 1-simplex of morphisms $0 \to 2$ from f_{02} to $f_{12} \circ f_{01}$. That's the most natural choice because nothing is made to strictly agree, it only agrees up to homotopy.

Continuing the pattern, $\mathfrak{C}(\Delta[n])$ should have objects $0, \ldots, n$, morphisms $f_{ij} : i \to j$ for all $i < j$, and the smallest system of homotopies we can write down between the composites that makes all of the mapping spaces contractible. Between any two vertices i and j, the set of all composite morphisms that go from i to j is identified with all possible subsets $S \subseteq \{i+1, i+2, \ldots, j-1\}$, because S tells us which vertices we "stop at" on the way from i to j. These form the vertices of a cube of dimension $(j-i-1)$. We need to extend these vertices to some contractible simplicial set. Well, the simplest way to do that is to actually take a cube, in other words a product of copies of the 1-simplex, $(\Delta[1])^{\times(j-i-1)}$. Writing this down carefully leads to the following definition.



$\mathfrak{C}(\Delta[0])$ $\mathfrak{C}(\Delta[1])$ $\mathfrak{C}(\Delta[2])$



$\mathfrak{C}(\Delta[3])$

Definition 10.1.43. The category $\mathfrak{C}(\Delta[n])$ has objects $\{0, 1, \ldots, n\}$. For each pair of objects i and j, let $[i, j] \subseteq \{0, \ldots, n\}$ be the subset of those k such that $i \leq k \leq j$. Note that when $j < i$, the set $[i, j]$ is empty.

Let $P_{i,j}$ be the set of all subsets of $[i, j]$ that include i and j. We consider $P_{i,j}$ to be a (cube-shaped) partially ordered set, by inclusion of subsets.

We define the simplicial set of morphisms $\mathfrak{C}(\Delta[n])(i,j)$ to be the nerve of $P_{i,j}$. In other words, it has a p-simplex for each sequence of inclusions

$$\{i,j\} \subseteq S_0 \subseteq S_1 \subseteq \cdots \subseteq S_p \subseteq [i,j].$$

The simplicial structure is by deleting and duplicating the sets S_a, and the composition

$$\mathfrak{C}(\Delta[n])(i,j) \times \mathfrak{C}(\Delta[n])(j,k) \to \mathfrak{C}(\Delta[n])(i,k)$$

takes unions of the sets in $[i,j]$ and $[j,k]$ to give sets in $[i,k]$.

Definition 10.1.44. For each simplicial category $\mathbf{C}_\bullet$, the **simplicial nerve** $N^s(\mathbf{C}_\bullet)$ is the simplicial set whose nth level is the set of maps of simplicial categories $\mathfrak{C}(\Delta[n]) \to \mathbf{C}_\bullet$.

For each simplicial set $X_\bullet$, the **categorical realization** is the simplicial category $\mathfrak{C}(X)$ obtained by the formula for realization as in (8.2.2)

$$\coprod_{k,\ell\geq 0} \coprod_{\alpha\in\mathbf{\Delta}([k],[\ell])} \mathfrak{C}(\Delta[k]) \times X_\ell \rightrightarrows \coprod_{k\geq 0} \mathfrak{C}(\Delta[k]) \times X_k \longrightarrow \mathfrak{C}(X_\bullet). \tag{10.1.45}$$

Example 10.1.46. Think through this definition and convince yourself that

- the 0-simplices of $N^s(\mathbf{C}_\bullet)$ are the objects of $\mathbf{C}_\bullet$,
- the 1-simplices of $N^s(\mathbf{C}_\bullet)$ are the 0-simplices of morphisms in $\mathbf{C}_\bullet$, and
- the 2-simplices of $N^s(\mathbf{C}_\bullet)$ are the 1-simplices of morphisms in $\mathbf{C}_\bullet$ from a single morphism f_{02} to a composite $f_{12} \circ f_{01}$.

Note the grading shift! Morphisms in a simplicial category are 0-simplices, but they become 1-simplices in the associated quasicategory.

If $\mathbf{C}_\bullet = \mathbf{C}_0$ is a discrete simplicial category, in other words a 1-category, the simplicial nerve is the same thing as the nerve. See Exercise 10.8.8.

Theorem 10.1.47. *The categorical realization and simplicial nerve define a Quillen equivalence*

$$\mathbf{sSet}_{\text{Joyal}} \underset{N^s}{\overset{\mathfrak{C}}{\rightleftarrows}} \mathbf{sCat}_{\text{Bergner}}$$

In fact, a map of simplicial sets $X \to Y$ is a Joyal equivalence iff $\mathfrak{C}(X) \to \mathfrak{C}(Y)$ is a DK-equivalence.

Compare this to Theorem 7.1.39, the Quillen equivalence between simplicial sets with the Kan model structure and topological spaces. The only thing that has changed here is the model structure on **sSet**, and what category we're taking the "realization" in.

Corollary 10.1.48. *If the simplicial category* $\mathbf{C}_\bullet$ *is pointwise fibrant (its mapping spaces* $\mathbf{C}_\bullet(a,b)$ *are Kan complexes), then the simplicial nerve* $N^s(\mathbf{C}_\bullet)$ *is a quasicategory. Moreover, every quasicategory is equivalent to one of this form.*

Lemma 10.1.49. *More generally, if* $F : \mathbf{C}_\bullet \to \mathbf{D}_\bullet$ *is any simplicial functor such that each of the maps* $\mathbf{C}_\bullet(a,b) \to \mathbf{D}_\bullet(F(a),F(b))$ *is a Kan fibration, then* $N^s(F) : N^s(\mathbf{C}_\bullet) \to N^s(\mathbf{D}_\bullet)$ *is an inner fibration of simplicial sets.*

Definition 10.1.50. We define the **∞-category of spaces** $\mathscr{S}$ as

$$\mathscr{S} := N^s(\mathbf{Kan}) \simeq N^s(\mathbf{CW}),$$

with the usual enrichments in simplicial sets. So it is the simplicial nerve of the category of all CW complexes, or all Kan complexes. These are equivalent because **CW** and **Kan** are simplicial categories in which the mapping spaces are Kan complexes, and the singular simplices and geometric realization give DK-equivalences of simplicial categories $\mathbf{CW} \to \mathbf{Kan} \to \mathbf{CW}$. So we get equivalences of quasicategories after applying N^s.

Example 10.1.51. If $\mathbf{C}$ is any topological category, then its singular simplices $S_\bullet\mathbf{C}$ form a pointwise fibrant simplicial category, and so $N^s(S_\bullet\mathbf{C})$ is a quasicategory. Writing this out, the n-simplices in this quasicategory are the maps from the topological category $|\mathfrak{C}(\Delta[n])|$ to $\mathbf{C}$. And the morphism spaces of the topological category $|\mathfrak{C}(\Delta[n])|$ are literally cubes!

So every topological category $\mathbf{C}$ gives a quasicategory. There is a 0-simplex for every object, a 1-simplex for every morphism, a 2-simplex for every continuous path from a single morphism f_{02} to a composite $f_{12} \circ f_{01}$, a 3-simplex for every square of morphisms of a particular form, and so on. In particular, this gives a really nice description of the ∞-category of spaces $\mathscr{S} \simeq N^s(\mathbf{CW})$.

One might wonder what happens to the mapping spaces when we go from simplicial categories to quasicategories. Each of the two definitions has a notion of "mapping space" between two objects, but do they agree?

Proposition 10.1.52. *If* $\mathbf{C}_\bullet$ *is a fibrant simplicial category and* $C = N^s(\mathbf{C}_\bullet)$ *is its simplicial nerve, then there are canonical weak equivalences*

$$\mathrm{Map}_C(x,y) \simeq \mathbf{C}_\bullet(x,y)$$

that respect the identity and composition maps up to homotopy. (In other words, this gives an isomorphism of categories enriched in the homotopy category of spaces $\mathrm{Ho}\,\mathbf{sSet}$*.)*

Proof: [Lur09, 2.2.0.1] and [DS11].

This is an important result for us, because our entire philosophy is that an ∞-category is a category with a homotopy type of morphisms $C(x,y)$ between every pair of objects. This would not be a very useful philosophy if the homotopy type changed as we passed from one definition to the next!

Example 10.1.53. If we build a simplicial category with two objects 0 and 1 and one nontrivial space of maps $\mathbf{C}_\bullet(0,1)$, as in Example 10.1.7, then as long as we make that space a Kan complex, the simplicial nerve of $\mathbf{C}_\bullet$ will give a quasicategory. It will again have two objects, and the space of maps between them will be the same as $\mathbf{C}_\bullet(0,1)$, by Proposition 10.1.52.

Definition 10.1.54. We define the (locally small) quasicategory of all (small) quasicategories as follows. Recall that for quasicategories C and D the simplicial set of all maps between them is called $\mathrm{Fun}(C,D)$. Taking the core, we get a Kan complex $\mathrm{Fun}(C,D)^{\simeq}$. As C and D vary these form a fibrant simplicial category $\mathbf{qCat}_\bullet$, so we can apply N^s and get a single quasicategory

$$\mathrm{Cat}_\infty := N^s(\mathbf{qCat}_\bullet).$$

This is usually called the **∞-category of all ∞-categories** ([Lur09, 3.0.0.1]).

So Cat_∞ is a quasicategory, whose objects are small quasicategories. By Proposition 10.1.52, the mapping spaces are equivalent to the space of all functors $C \to D$ and natural isomorphisms between them:

$$\mathrm{Map}_{\mathbf{Cat}_\infty}(C,D) \simeq \mathrm{Fun}(C,D)^{\simeq}.$$

We threw out all of the non-invertible natural transformations.

10.1.4 From simplicial model categories to ∞-categories

If you look at the definition of the ∞-category of spaces $\mathscr{S}$ from Definition 10.1.50, you might guess how this should generalize to any simplicial model category.

Example 10.1.55. For any simplicial model category $\mathbf{C}_\bullet$, we can now define its **underlying ∞-category** (or more precisely, its underlying quasicategory) to be the simplicial nerve of the bifibrant objects,

$$N^s(\mathbf{C}_\bullet^{\mathrm{cf}}).$$

In [Lur09, Lur17] this is denoted $N^s(\mathbf{C}_\bullet^o)$. This is consistent with Example 10.1.8, since N^s is how we move from simplicial categories to quasicategories, and the simplicial sets in $\mathbf{C}_\bullet^{\mathrm{cf}}$ are all Kan complexes by Lemma 8.1.25.

As a special case, the **∞-category of spaces** $\mathscr{S}$ is defined as

$$\mathscr{S} \simeq N^s(\mathbf{Top}^{\mathrm{cf}}) \simeq N^s(\mathbf{sSet}^{\mathrm{cf}}),$$

the **∞-category of based spaces** $\mathscr{S}_*$ is defined as

$$\mathscr{S}_* \simeq N^s(\mathbf{Top}_*^{\mathrm{cf}}) \simeq N^s(\mathbf{sSet}_*^{\mathrm{cf}}),$$

and the **∞-category of spectra** Sp is defined as

$$\mathrm{Sp} \simeq N^s((\mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}})^{\mathrm{cf}}) \simeq N^s((\mathbf{Sp}^{\mathbb{N}}_{\mathbf{sSet}})^{\mathrm{cf}}) \simeq N^s((\mathbf{Sp}^{\Sigma}_{\mathbf{Top}})^{\mathrm{cf}}) \simeq N^s((\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}})^{\mathrm{cf}}) \simeq N^s((\mathbf{Sp}^{O})^{\mathrm{cf}}) \simeq \cdots$$

We will see in Corollary 10.2.13 that Quillen equivalences of simplicial model categories $\mathbf{C}_\bullet \simeq \mathbf{D}_\bullet$ give equivalences of quasicategories $N^s(\mathbf{C}_\bullet^{\mathrm{cf}}) \simeq N^s(\mathbf{D}_\bullet^{\mathrm{cf}})$, so that all of these models for Sp really are equivalent (Corollary 10.2.14). In Definition 10.1.50 we were able to observe this directly, because singular simplices and geometric realization both preserve bifibrant objects. This doesn't usually happen with Quillen equivalences though. To make the argument run in general, we'll have to rely on the magic of localization.

Example 10.1.56. For any simplicial set K, the **∞-category of presheaves** on K is defined to be the functor category

$$\mathscr{P}(K) := \mathrm{Fun}(K^{\mathrm{op}}, \mathscr{S}) = \mathrm{Fun}(K^{\mathrm{op}}, N^s(\mathbf{Kan})).$$

So an object of $\mathscr{P}(K)$ is a map of simplicial sets $K^{\mathrm{op}} \to \mathscr{S}$, a 1-simplex is a natural transformation $K^{\mathrm{op}} \times \Delta[1] \to \mathscr{S}$, and so on. The quasicategory $\mathscr{P}(K)$ plays an important role in the theory, as we will see soon.

It will be useful to have a simplicial model category that gives the presheaf category $\mathscr{P}(K)$. It should be the model category of diagrams of spaces $\mathbf{Top}^{\mathbf{I}}$ or diagrams of simplicial sets $\mathbf{sSet}^{\mathbf{I}}$. The following result makes this precise.

Proposition 10.1.57. *[Lur09, 4.2.4.4] If* $\mathbf{C}$ *is any of our model categories of spaces, based spaces, or spectra, and* $\mathbf{I}$ *is any small simplicial category, there is a canonical equivalence of* ∞*-categories*

$$N^s((\mathbf{C}^{\mathbf{I}})^{\mathrm{cf}}) \simeq \mathrm{Fun}(N^s(f\mathbf{I}), N^s(\mathbf{C}^{\mathrm{cf}})),$$

that is natural in $\mathbf{I}$ *and* $\mathbf{C}$.

This is a little easier to state if, instead of fibrantly replacing $\mathbf{I}$ to make $N^s(\mathbf{I})$ correct, we take a simplicial set K and a map of simplicial categories $\mathfrak{C}[K] \to \mathbf{I}$ that is a DK-equivalence. (That guarantees that K is equivalent to $N^s(f\mathbf{I})$.) Then we take the evaluation map $\mathbf{C}^{\mathbf{I}} \times \mathbf{I} \to \mathbf{C}$, apply the simplicial nerve to get a map

$$N^s(\mathbf{C}^{\mathbf{I}}) \times N^s(\mathbf{I}) \cong N^s(\mathbf{C}^{\mathbf{I}} \times \mathbf{I}) \longrightarrow N^s(\mathbf{C}),$$

take its adjoint to get a map $N^s(\mathbf{C}^{\mathbf{I}}) \to \mathrm{Fun}(N^s(\mathbf{I}), N^s(\mathbf{C}))$, then compose with $K \to N^s(\mathbf{I})$ and restrict to the cofibrant and fibrant objects in the projective model structure to get

$$N^s((\mathbf{C}^{\mathbf{I}})^{\mathrm{cf}}) \longrightarrow \mathrm{Fun}(K, N^s(\mathbf{C}^{\mathrm{cf}})).$$

This is the map that is an equivalence.

Remark 10.1.58. In fact, this holds much more generally when **C** is any combinatorial model category, see Definition 10.4.26. (**Top** and $\mathbf{Sp}^O_{\mathbf{Top}}$ are not combinatorial, but they are Quillen equivalent to combinatorial model categories, which is why Proposition 10.1.57 still applies in those cases.)

Remark 10.1.59. You can think of Proposition 10.1.57 as a rectification result. The fact that the map is essentially surjective means that every homotopy-coherent diagram in **C** can be strictified to an actual diagram. (A diagram in $\mathrm{Fun}(N(\mathbf{I}), N^s(\mathbf{C}))$ is really a homotopy-coherent diagram in **C**, because each string of morphisms in **I** is sent to a string of morphisms in **C** together with coherent homotopies between their compositions, by the definition of the simplicial nerve N^s.)

Corollary 10.1.60. *For any small simplicial category* **I** *and small simplicial set* K *that is Joyal equivalent to* $N^s(f\mathbf{I})$*, the* ∞*-category of presheaves* $\mathscr{P}(K)$ *is modelled by either of the simplicial model categories* $\mathbf{Top}^{\mathbf{I}}$ *or* $\mathbf{sSet}^{\mathbf{I}}$.

10.1.5 Segal categories

There are more models of ∞-categories beyond simplicial categories and quasicategories. We only briefly mention one of them, Segal categories, because it is useful for building examples. As you may recall, it is nearly impossible to define examples of quasicategories directly. But simplicial categories are sometimes also difficult to use for examples, because they require compositions that are strictly associative, and sometimes in nature that doesn't happen. Segal categories are a middle road where we weaken the definition of simplicial category a little bit. They are the many-objects version of Segal A_∞ spaces, or Segal monoids, that we gave in Definition 9.4.1.

Definition 10.1.61. A **Segal category** is a simplicial space $X_\bullet$, in other words a bisimplicial set or a simplicial topological space, such that

- X_0 is a discrete space, and
- The Segal maps from Definition 9.4.1 induce a weak equivalence $X_n \xrightarrow{\sim} X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$.

As usual, we can use either definition of "space" here because taking singular simplices and geometric realization preserves the above two conditions.

Example 10.1.62. Every simplicial category $\mathbf{C}_\bullet$ defines a Segal category by taking its nerve $N_\bullet\mathbf{C}_\bullet$. This is a bisimplicial set such that level (n,k) is a choice of $(n+1)$ objects $c_0,\ldots,c_n$ and k-simplices of morphisms between them, in other words an element of the set

$$\mathbf{C}_k(c_0,c_1)\times\cdots\times\mathbf{C}_k(c_{n-1},c_n).$$

This is a Segal category because when $n = 0$ this is just the discrete set of objects (changing k has no effect), and for larger values of n the Segal maps induce an isomorphism. In fact, every Segal category in which the Segal maps are isomorphisms, arises in this way.

Example 10.1.63. [Rez22, 43.5] Every quasicategory C defines a Segal category by taking an extended version of the mapping space from Definition 10.1.38: at level n we take a disjoint union over all objects $c_0, \dots, c_n$ of the space $\mathrm{Map}_C(x_0, \dots, x_n)$, defined as the pullback

$$\begin{array}{ccc} \mathrm{Map}_C(x_0,\dots,x_n) & \longrightarrow & \mathrm{Fun}(\Delta[n], C) \\ \downarrow & & \downarrow{\scriptstyle \text{restrict to vertices}} \\ * & \xrightarrow{x_0,\dots,x_n} & \mathrm{Fun}(\Delta[0] \amalg \cdots \amalg \Delta[0], C). \end{array}$$

The Segal maps are equivalences because each one rearranges into $\mathrm{Fun}(-, C)$ applied to the spine inclusion of $\Delta[n]$. The spine inclusion is inner anodyne, and therefore this Segal map is an acyclic Kan fibration.

Proposition 10.1.64. *These operations have inverses up to equivalence, and in particular, every Segal category produces a simplicial category or a quasicategory.*

In fact, there are multiple such ways to define these equivalences. (You could think of the map from Segal categories back to simplicial categories as a many-objects version of the rectification theorem from Corollary 9.4.41.) We refer the reader to the survey [Ber10] for details.

10.2 The magic of Dwyer–Kan localization

The theory of ∞-categories really gets going when we gain the ability to localize.

Localization is when we take a class of morphisms and invert them. We've seen this twice already in this book. The first time was in Chapter 3, when we took a 1-category $\mathbf{C}$ and a class of morphisms W, and formed a 1-category $\mathbf{C}\langle W^{-1}\rangle = \mathrm{Ho}\,\mathbf{C}$ in which every morphism of W is turned into an isomorphism. We might call this the "Gabriel-Zisman localization" $\mathbf{C} \to \mathbf{C}\langle W^{-1}\rangle$.

Then we saw left Bousfield localization, where we have a model category $\mathbf{C}$ with one class of weak equivalences V, and we create a new model structure with the same cofibrations, but where the weak equivalences are some larger class of maps W. In this case we didn't actually invert anything, because we have the same 1-category $\mathbf{C}$ as before. But we have more weak equivalences, so on homotopy categories we get another Gabriel-Zisman localization $\mathbf{C}\langle V^{-1}\rangle \to \mathbf{C}\langle W^{-1}\rangle$.

In this section we'll define localization in any ∞-category. For any quasicategory C and class of morphisms $W \subseteq C_1$, we'll define a quasicategory $C[W^{-1}]$ that universally turns the morphisms of W into isomorphisms. In other words, they all gain inverses up to homotopy.

We call this the "Dwyer–Kan localization" after its original inventors, but it is also called the ∞-localization, the quasi-localization, or the simplicial localization. It is the correct ∞-categorical version of the Gabriel-Zisman localization.

However, the Dwyer–Kan localization *does not agree* with the Gabriel-Zisman localization, *even if C is the nerve of a 1-category*! Because now we're not making the morphisms of W into strict isomorphisms, we're making them into isomorphisms up to homotopy. This creates new homotopies, and homotopies between them, and so forth. So *even if we start with a 1-category, the Dwyer–Kan localization $C[W^{-1}]$ is no longer a 1-category*! In symbols:

$$N(\mathbf{C})[W^{-1}] \not\simeq N(\mathbf{C}\langle W^{-1}\rangle).$$

In fact, every ∞-category is equivalent to the Dwyer–Kan localization of some 1-category [BK12]. The localization magically creates all the rest of the higher morphisms.

Dwyer–Kan localization really is magic, because it allows us to work effectively with ∞-categories by thinking entirely in terms of 1-categories. This leads to huge simplifications and much more direct connections between ∞-category theory and the more classical techniques of model categories and bar constructions that we saw earlier in the book.

10.2.1 Dwyer–Kan localization of a quasicategory

We first define Dwyer–Kan localization in the setting of quasicategories, because that's where it's easiest. Suppose C is a quasicategory and $W \subseteq C_1$ is any set of morphisms. For any other quasicategory D, recall that the quasicategory $\mathrm{Fun}(C, D)$ consists of all functors $C \to D$, and all n-fold compositions of natural transformations between them.

Definition 10.2.1. Let $\mathrm{Fun}^W(C, D) \subseteq \mathrm{Fun}(C, D)$ be the full subcategory whose objects are those functors $C \to D$ that send every morphism in W to an isomorphism in D.

By "full subcategory," we mean that we take all of the natural transformations and compositions between these. It is easy to see that this defines a quasicategory again.

Definition 10.2.2. If C is a quasicategory and $W \subseteq C_1$, the **Dwyer–Kan localization**, or DK-localization for short, is any quasicategory $C[W^{-1}]$ equipped with a map $\delta : C \to C[W^{-1}]$ such that

- each morphism in W is sent to an isomorphism in $C[W^{-1}]$, and

- for any other quasicategory D, composition with δ induces a Joyal equivalence

$$\mathrm{Fun}(C[W^{-1}], D) \xrightarrow{\sim} \mathrm{Fun}^W(C, D).$$

The localization must be unique up to Joyal equivalence under C, if it exists. That's because the above property can be used to construct a Joyal equivalence between any two localizations, commuting with the map in from C. So now we only have to convince ourselves that such a thing exists.

Proposition 10.2.3. *For any quasicategory C and any subset $W \subseteq C_1$, the Dwyer–Kan localization $C[W^{-1}]$ exists. In fact, it can be made functorial, so that every functor $F : C \to D$ such that $F(W_C) \subseteq W_D$ gives a map*

$$C[W_C^{-1}] \to D[W_D^{-1}],$$

strictly respecting composition and identity functors.

Proof. We sketch the proof from [Rez22], because it is really illuminating. We change notation slightly and let W be any simplicial set with a map to C that on 1-simplices hits the desired set of morphisms. For instance, W could just be a disjoint union of copies of $\Delta[1]$, one for each morphism that we want to invert.

Let $W \to W'$ be any fibrant replacement in the Kan model structure. In other words, it is a cofibration and weak homotopy equivalence to a Kan complex W'. For instance, if W is a disjoint union of copies of $\Delta[1]$, we could take W' to be a disjoint union of copies of $N\{0 \cong 1\}$. Intuitively, W' is W with "all of its inverses added."

Then we take the pushout $C \cup_W W'$, and take its fibrant replacement in the Joyal model structure, giving $C[W^{-1}] := f(C \cup_W W')$. Intuitively, the Joyal fibrant replacement is "adding in all possible compositions." So after this fibrant replacement, we have an n-simplex (in fact, many n-simplices) for every list of n composable morphisms $(f_1, \ldots, f_n)$ consisting of both the new inverse morphisms in W' and the old morphisms in C.

To see this is the desired localization, notice that every morphism in W now has the required inverses to become an isomorphism. Since $\mathbf{sSet}_{\mathrm{Joyal}}$ is a symmetric monoidal model category, we get a homotopy pullback square of quasicategories

$$\begin{array}{ccc} \mathrm{Fun}(f(C \cup_W W'), D) & \longrightarrow & \mathrm{Fun}(C, D) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(W', D) & \longrightarrow & \mathrm{Fun}(W, D). \end{array}$$

If we restrict the two functor categories on the right to those functors sending W to isomorphisms in D, then this remains a homotopy pullback square, but now its bottom horizontal

map becomes an equivalence. Therefore the top horizontal map is an equivalence as well, which proves that $f(C \cup_W W')$ has the universal property.

Finally, this definition of $C[W^{-1}]$ is a functor because the small-object argument constructs functorial factorizations. □

Remark 10.2.4. Another proof of this uses Lurie's model category of "marked simplicial sets" $\mathbf{sSet}^+$ [Lur09, §3.1]. A marked simplicial set is just a simplicial set K with a choice of edges $W \subseteq K_1$, containing at least the degenerate edges.

Example 10.2.5. If we take a quasicategory C and invert the full set of morphisms C_1, we get a Kan complex $C[C_1^{-1}]$, that is just the fibrant replacement of C in the Kan model structure on simplicial sets. This is the full localization from Example 10.1.37.

Along the equivalence between Kan complexes and topological spaces, this corresponds to taking the classifying space, or the geometric realization $|C|$. So we get the slogan that "passing to the classifying space is the same thing as taking the Dwyer–Kan localization along all of the morphisms."

Example 10.2.6. If $\mathbf{C}$ is a 1-category with one object, then its morphisms form a discrete monoid M. If we localize $N(\mathbf{C}) = N(M)$ by inverting all the morphisms, we get a quasicategory with one object where every morphism is an isomorphism. Taking categorical realization, we get a simplicial or topological with one object, and the morphisms form a grouplike monoid G. If we trace through the universal properties, we see there is a monoid map $M \to G$, and any map from M to a grouplike monoid factors up to weak equivalence through G. In other words, we get the homotopical group completion of M!

So we can think of Dwyer–Kan localization as the many-objects version of the homotopical group completion from Definition 9.1.11. As was the case back then, this can create a topology or higher homotopy groups in the mapping spaces $\mathbf{C}(a, b)$ even though the category $\mathbf{C}$ we started with was discrete.

We reiterate at this point that if we ever have a 1-category, the Dwyer–Kan localization $N(\mathbf{C})[W^{-1}]$ is not the Gabriel-Zisman localization $\operatorname{Ho}\mathbf{C} = \mathbf{C}\langle W^{-1}\rangle$ that we constructed in Definition 3.1.9. The Gabriel-Zisman localization is always a 1-category, but the Dwyer–Kan localization can have "more homotopy type" in its mapping spaces. They are related though:

Lemma 10.2.7. *The homotopy category of the Dwyer–Kan localization $h(C[W^{-1}])$ is identified with the Gabriel-Zisman localization $(hC)\langle W^{-1}\rangle$.*

10.2.2 From categories with weak equivalences to ∞-categories

Now we're ready for the magic part.

Definition 10.2.8. For any 1-category $\mathbf{C}$ and any class of morphisms W that we call the "weak equivalences" of $\mathbf{C}$ (with no assumptions whatsoever on W), we define the **underlying ∞-category** of $(\mathbf{C}, W)$ to be the DK-localization of the nerve, $N(\mathbf{C})[W^{-1}]$.

More generally, if $\mathbf{C}_\bullet$ is a simplicial category and W is any class of morphisms in $\mathbf{C}_0$, the underlying ∞-category of $(\mathbf{C}_\bullet, W)$ is defined to be the DK-localization of the simplicial nerve, $N^s(\mathbf{C}_\bullet)[W^{-1}]$.

So now we can assign an ∞-category to each model category. We don't even need the simplicial enrichment! The weak equivalences create the enrichment for us.

If the model category happens to be simplicial, the following result shows us that the space of maps from X to Y that we magically created using localization agrees with the "correct" homotopy type for the space of maps, in other words $\mathbf{C}_\bullet(QRX, QRY)$.

Proposition 10.2.9 (Dwyer–Kan)**.** *[Lur17, 1.3.4.7], [Lan21, 3.2.25] If* $\mathbf{C}$ *is a model category and* W *is its set of weak equivalences, then the inclusion of the cofibrant and/or fibrant objects give equivalences of* ∞*-categories:*[8]

$$\begin{array}{ccccc} & & N(\mathbf{C})[W^{-1}] & & \\ & \overset{\sim}{\nearrow} & & \overset{\sim}{\nwarrow} & \\ N(\mathbf{C}^{\mathrm{c}})[W^{-1}] & & & & N(\mathbf{C}^{\mathrm{f}})[W^{-1}] \\ & \underset{\sim}{\nwarrow} & & \underset{\sim}{\nearrow} & \\ & & N(\mathbf{C}^{\mathrm{cf}})[W^{-1}]. & & \end{array}$$

If $\mathbf{C}_\bullet$ *is a simplicial model category, then the same is true when we use the simplicial enrichments, and these are equivalent to the unenriched versions:*

$$\begin{array}{ccccccc} & & N^s(\mathbf{C}_\bullet)[W^{-1}] & \xleftarrow{\sim} & N(\mathbf{C}_0)[W^{-1}] & & \\ & \overset{\sim}{\nearrow} & & \overset{\sim}{\nwarrow} \quad \overset{\sim}{\nearrow} & & \overset{\sim}{\nwarrow} & \\ N^s(\mathbf{C}^{\mathrm{c}}_\bullet)[W^{-1}] & & N^s(\mathbf{C}^{\mathrm{f}}_\bullet)[W^{-1}] & & N(\mathbf{C}^{\mathrm{c}}_0)[W^{-1}] & & N(\mathbf{C}^{\mathrm{f}}_0)[W^{-1}] \\ & \underset{\sim}{\nwarrow} & & \underset{\sim}{\nearrow} \quad \underset{\sim}{\nwarrow} & & \underset{\sim}{\nearrow} & \\ N^s(\mathbf{C}^{\mathrm{cf}}_\bullet) & \xrightarrow{\sim} & N^s(\mathbf{C}^{\mathrm{cf}}_\bullet)[W^{-1}] & \xleftarrow{\sim} & N(\mathbf{C}^{\mathrm{cf}}_0)[W^{-1}] & & \end{array}$$

So the two competing definitions of "underlying ∞-category" that we have seen for $\mathbf{C}_\bullet$ are canonically equivalent to each other:

$$N^s(\mathbf{C}^{\mathrm{cf}}_\bullet) \simeq N(\mathbf{C}_0)[W^{-1}].$$

[8] Strictly speaking, W is not a set of morphisms in $\mathbf{C}^{\mathrm{cf}}$, and we should be saying $W \cap \mathbf{C}^{\mathrm{cf}}$, but we say W just to keep the diagram from getting too cluttered.

So now, because of the magic of localization, there's no loss of generality if we just assume we have a 1-category $\mathbf{C}$ with weak equivalences W_C. Here are a few facts about the resulting ∞-category $N(\mathbf{C})[W_C^{-1}]$ that are straightforward to prove. (Recall that a functor is **homotopical** when it preserves all weak equivalences.)

Lemma 10.2.10. *For any 1-categories* $\mathbf{C}$ *and* $\mathbf{D}$ *with weak equivalences* W_C *and* W_D *respectively,*

- *Any homotopical functor* $F\colon \mathbf{C} \to \mathbf{D}$ *induces a map of* ∞*-categories*
$$N(\mathbf{C})[W_C^{-1}] \to N(\mathbf{D})[W_D^{-1}].$$
- *Any natural transformation* $F \to G$ *of homotopical functors* $\mathbf{C} \to \mathbf{D}$ *induces a natural transformation*
$$N(\mathbf{C})[W_C^{-1}] \times \Delta[1] \to N(\mathbf{D})[W_D^{-1}].$$
If all of the maps $F(x) \to G(x)$ *are weak equivalences in* $\mathbf{D}$*, then this is a natural isomorphism of functors of* ∞*-categories.*
- *If* $F\colon \mathbf{C} \to \mathbf{D}$ *and* $G\colon \mathbf{D} \to \mathbf{C}$ *are homotopical, and if the composites* $F \circ G$ *and* $G \circ F$ *have zig-zags of natural weak equivalences to the identity functor, then* F *induces an equivalence of* ∞*-categories*
$$N(\mathbf{C})[W_C^{-1}] \simeq N(\mathbf{D})[W_D^{-1}].$$

We cannot overstate how useful this lemma is, when we approach ∞-categories from our point of view. It allows us to concretely model and understand many things in the world of ∞-categories using only 1-categories. And it transforms everything we have proven so far in the book into a statement about ∞-categories.

Example 10.2.11. The suspension functor Σ induces an equivalence on the ∞-category of spectra:
$$\Sigma\colon N(\mathbf{Sp})[W^{-1}] \xrightarrow{\sim} N(\mathbf{Sp})[W^{-1}].$$
Here we took the nerve of the 1-category of sequential spectra and inverted the stable equivalences. By Proposition 10.2.9, this is a model for the ∞-category of spectra Sp that we defined in Example 10.1.55. More generally, smashing with any cofibrant spectrum X gives a map of ∞-categories (not an equivalence)
$$X \wedge (-)\colon N(\mathbf{Sp})[W^{-1}] \longrightarrow N(\mathbf{Sp})[W^{-1}].$$
Since $X \wedge (-)$ is a simplicial functor, we could have obtained the same thing (up to equivalence) by using the simplicial enrichment:
$$X \wedge (-)\colon N^s(\mathbf{Sp}_\bullet)[W^{-1}] \longrightarrow N^s(\mathbf{Sp}_\bullet)[W^{-1}].$$

But this doesn't work with the simpicial category of bifibrant spectra, $N^s(\mathbf{Sp}^{\mathrm{cf}}_\bullet)$, because smash products don't preserve the property of being fibrant. So we really do need Dwyer–Kan localization to make examples like this work.

Example 10.2.12. Suppose $\mathbf{C}$ is a model category with weak equivalences V, and there is a left Bousfield localization of $\mathbf{C}$ with weak equivalences $W \supseteq V$. Then on the underlying ∞-categories, the map

$$N(\mathbf{C})[V^{-1}] \longrightarrow N(\mathbf{C})[W^{-1}]$$

is also a Dwyer–Kan localization. This is because, in general, Dwyer–Kan localizations can be composed. In other words, inverting V and then inverting W is the same thing as just inverting W. (You can prove this by comparing their universal properties.)

So in particular, if we take spectra and localize with respect to a homology theory E, this gives a map of ∞-categories

$$\mathrm{Sp} = N(\mathbf{Sp})[W^{-1}] \longrightarrow N(\mathbf{Sp})[W_E^{-1}].$$

We call $N(\mathbf{Sp})[W_E^{-1}]$ the **E-local ∞-category of spectra**, Sp_E. It is the correct ∞-categorical version of the E-local stable homotopy category from Definition 3.3.1.

Corollary 10.2.13. *For every left Quillen equivalence of model categories $F : \mathbf{C} \to \mathbf{D}$, the left-derived functor $\mathbb{L}F$ induces an equivalence of the underlying ∞-categories*

$$N(\mathbf{C})[W_C^{-1}] \simeq N(\mathbf{D})[W_D^{-1}].$$

More generally, every pair of adjoint functors (F, G) such that F is left-deformable and G is right-deformable, gives an adjunction of ∞-categories. The only reason we can't state this now is that we haven't defined adjunctions of ∞-categories yet. See Corollary 10.3.18.

Corollary 10.2.14. *Every model category of spectra that we have introduced in this book gives the same ∞-category* Sp*, up to equivalence. Every left- or right-derived functor that we have defined induces a map of ∞-categories, and every Quillen equivalence gives an equivalence of ∞-categories.*

In other words, everything that we've done in the previous chapters becomes some statement about the underlying ∞-category. All we need to do is preserve weak equivalences, and then we're working in the land of ∞-categories, even if we don't realize it.

Example 10.2.15. The equivalence between monoids and A_∞ spaces (Corollary 9.3.26) is an equivalence of ∞-categories. The equivalence between the operadic and Segal versions of A_∞ spaces (Theorem 9.4.40) is also an equivalence of ∞-categories. This is really useful because the 1-categorical description of this equivalence is by a zig-zag of Quillen equivalences. In quasicategories, it becomes *a single functor*.

The same method gives an equivalence of ∞-categories between grouplike A_∞ spaces and connected based spaces. This is very useful because these subcategories are too small to still be model categories, so they no longer have a Quillen equivalence between them. We only have homotopical functors B and Ω that are inverses up to natural weak equivalence.

Let's extend the magic to categories of diagrams now. The following is a corollary of Proposition 10.1.57.

Corollary 10.2.16. *If* $\mathbf{C}$ *is any of our model categories of spaces, based spaces, or spectra,* $\mathbf{I}$ *is any small simplicial category, and* K *is any small simplicial set such that* $\mathfrak{C}[K] \simeq \mathbf{I}$*, then there is an equivalence of* ∞*-categories*

$$N(\mathbf{C}^{\mathbf{I}})[W_{\mathbf{I}}^{-1}] \simeq \mathrm{Fun}(K, N(\mathbf{C})[W^{-1}]).$$

Here W *is the weak equivalences in* $\mathbf{C}$ *and* $W_{\mathbf{I}}$ *is the pointwise weak equivalences in* $\mathbf{C}^{\mathbf{I}}$.

In other words, the ∞-category of diagrams up to pointwise weak equivalence, $N(\mathbf{C}^{\mathbf{I}})[W_{\mathbf{I}}^{-1}]$, is equivalent to diagrams in the ∞-category, $\mathrm{Fun}(K, N(\mathbf{C})[W^{-1}])$.

In the special case that $\mathbf{I}$ is a 1-category, we can describe this as before by taking the evaluation map $\mathbf{C}^{\mathbf{I}} \times \mathbf{I} \to \mathbf{C}$, applying the ordinary nerve to get a map

$$N(\mathbf{C}^{\mathbf{I}}) \times N(\mathbf{I}) \cong N(\mathbf{C}^{\mathbf{I}} \times \mathbf{I}) \longrightarrow N(\mathbf{C}),$$

taking its adjoint to get a map $N(\mathbf{C}^{\mathbf{I}}) \to \mathrm{Fun}(N(\mathbf{I}), N(\mathbf{C}))$, then applying Dwyer–Kan localizations as shown, and observing that the dashed arrow must exist by the universal property of the localization:

$$\begin{array}{ccc} N(\mathbf{C}^{\mathbf{I}}) & \longrightarrow & \mathrm{Fun}(K, N(\mathbf{C})) \\ \downarrow & & \downarrow \\ N(\mathbf{C}^{\mathbf{I}})[W_{\mathbf{I}}^{-1}] & \dashrightarrow & \mathrm{Fun}(K, N(\mathbf{C})[W^{-1}]). \end{array}$$

Again, this is also true more generally when $\mathbf{C}$ is any combinatorial model category (see Definition 10.4.26).

Example 10.2.17. The homotopy colimit is a functor $\mathbf{C}^{\mathbf{I}} \to \mathbf{C}$ that preserves weak equivalences. Therefore this gives a map of quasicategories $N(\mathbf{C}^{\mathbf{I}})[W_{\mathbf{I}}^{-1}] \to N(\mathbf{C})[W^{-1}]$. By Corollary 10.2.16, this rearranges into a map of quasicategories

$$\mathrm{Fun}(K, N(\mathbf{C})[W^{-1}]) \longrightarrow N(\mathbf{C})[W^{-1}].$$

We will see soon that this describes the colimit inside the ∞-category $N(\mathbf{C})[W^{-1}]$.

Finally, we extend the magic to $\mathbf{Cat}_\infty$, the ∞-category of all ∞-categories from Definition 10.1.54. This doesn't quite come about from a simplicial model category: we take the bifibrant objects in $\mathbf{sSet}_{\text{Joyal}}$, but that isn't quite a simplicial model category because it has a Quillen tensor with $\mathbf{sSet}_{\text{Joyal}}$, not with $\mathbf{sSet}_{\text{Kan}}$. Then we take the simplicial mapping spaces $\mathrm{Fun}(C,D)$, restrict to the cores $\mathrm{Fun}(C,D)^{\simeq}$, observe that they form a simplicial category, and define $\mathbf{Cat}_\infty$ to be the associated quasicategory.

Lemma 10.2.18. *[Lur09, §3.1], [Lan21, 3.2.26]* $\mathbf{Cat}_\infty$ *is equivalent to the Dwyer-Kan localization of* $\mathbf{sSet}$ *at the class of Joyal equivalences. In other words,* $\mathbf{Cat}_\infty \simeq N(\mathbf{sSet})[W_{\text{Joyal}}^{-1}]$.

10.2.3 Dwyer–Kan localization of a simplicial category

Our focus has been on Dwyer–Kan localizations of quasicategories, because they are easier to work with. But it is also possible to directly describe the Dwyer–Kan localization of a simplicial category $\mathbf{C}_\bullet$ at a class of weak equivalences $W \subseteq \mathbf{C}_0$.

Unfortunately, the universal property is significantly harder to state in this case, because it involves maps out of $\mathbf{C}_\bullet[W^{-1}]$, and as we discussed before, describing the space of maps between two simplicial categories is a challenge. To get around this, we simply *define* the DK-localization to be any map that has the universal property of the DK-localization when we pass to quasicategories:

Definition 10.2.19. If $\mathbf{C}_\bullet$ is a simplicial category W is a class of morphisms in $\mathbf{C}_0$, the **Dwyer–Kan localization** $\mathbf{C}_\bullet[W^{-1}]$ is defined to be any simplicial category equipped with a map $\delta\colon \mathbf{C}_\bullet \to \mathbf{C}_\bullet[W^{-1}]$, such that after fibrantly replacing and taking the simplicial nerve, we get the Dwyer–Kan localization of the quasicategory $N^s(f\,\mathbf{C}_\bullet)$ at the image of W:

$$N^s(f\,\mathbf{C}_\bullet[W^{-1}]) \simeq N^s(f\,\mathbf{C}_\bullet)[W^{-1}].$$

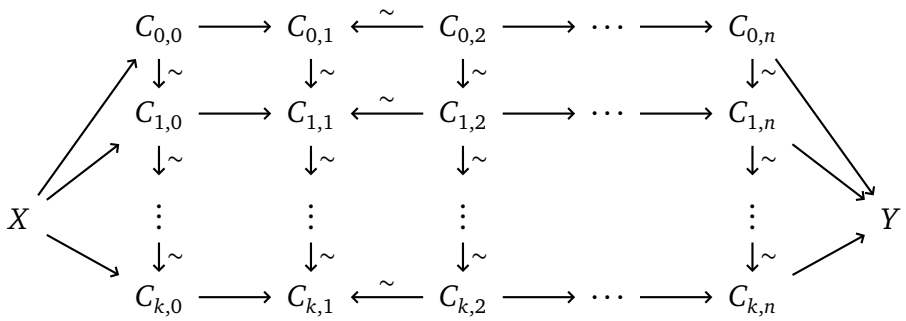
In particular, every morphism in W becomes an isomorphism in the homotopy category $\pi_0(\mathbf{C}_\bullet[W^{-1}])$, and $\mathbf{C}_\bullet[W^{-1}]$ is in some sense universal among simplicial categories with this property. Although to measure that, you have to go to quasicategories.

Example 10.2.20. It is a theorem of Dwyer and Kan that we can always construct $\mathbf{C}_\bullet[W^{-1}]$ using the **hammock localization** $L^H(\mathbf{C}_\bullet, W)$.

We first describe this in the special case that $\mathbf{C}_\bullet = \mathbf{C}_0$ is discrete. Then $L^H(\mathbf{C}, W)$ is a simplicial category. At simplicial level 0 it is almost the construction of the homotopy category $\mathrm{Ho}\,\mathbf{C}$ from Definition 3.1.1, except that we drop one of the relations. So, the morphisms are equivalence classes of zig-zags, where the backwards-pointing maps have to be weak equivalences, and we may compose maps pointing in the same direction, or delete identity maps. However, we may *not* cancel copies of the same map pointing in

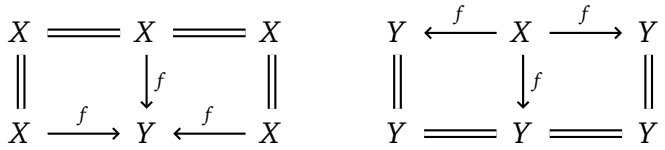
opposite directions. (This is because we are not trying to make strict inverses. We only want inverses up to homotopy!)

At simplicial level k, the category $L^H(\mathbf{C}, W)_k$ can be described as all "hammocks." That is, grids of maps that are k maps tall and any number of maps wide, where the vertical maps are in W, and any column of horizontal maps can point backwards if they are also in W as well:



These are considered up to essentially the same relations as above: we may compose adjacent columns pointing in the same direction, and cancel columns of identity maps. As k varies, we get a simplicial set by deleting or duplicating rows. And then this forms a simplicial category by sticking two hammocks together to form a larger hammock.

In this simplicial category, every morphism f in W now has an inverse up to homotopy, given by the same morphism pointing in the opposite direction. The following hammocks give the homotopies between each composite of f with this formal inverse, and the identity map.



More generally, if $\mathbf{C}_\bullet$ is a simplicial category, we can repeat this construction for each category $\mathbf{C}_n$, using the same class of weak equivalences W (or really its image in $\mathbf{C}_n$ under the degeneracy maps). This produces a bisimplicial category, and taking its diagonal we get a simplicial category that we call $L^H(\mathbf{C}_\bullet, W)$.

This localization is not nearly as useful on a technical level as the one we saw in the last section, but it is a convenient way to name specific elements in the ∞-category underlying $(\mathbf{C}, W)$. Once we have two morphisms in the homotopy category represented by zig-zags, we know exactly how to describe a path between them, by a hammock as above.

Example 10.2.21. It is also a theorem of Dwyer and Kan that the localization can be constructed explicitly by taking any simplicial category $W_\bullet$ with a map to $\mathbf{C}_\bullet$ that on 0-simplices has W as its image, taking cofibrant replacements $QW_\bullet \to Q\mathbf{C}_\bullet$ in the Bergner

model structure, and then taking the 1-categorical localization at each simplicial level,

$$Q\mathbf{C}_n[QW_n^{-1}],$$

forming the desired simplicial category. This description is more complicated to write out explicitly, but on the other hand it is more closely related to the group completion given in Definition 9.1.11.

Chronologically, the localizations in the above two examples were defined long before the localization of a quasicategory. But the localization in a quasicategory is really much easier to define and use. This is one reason why the development of quasicategories really pushed the theory forward.

10.3 Adjunctions and colimits in ∞-categories

The adjunction is one of the most fundamental concepts in category theory. If we can understand these, then we understand colimits and Kan extensions as a special case. And later on when we discuss presentable ∞-categories, left adjoint functors and colimits will play an especially important role.

In the world of ∞-categories, it is both very simple and very complicated to say what an adjunction is. The simple part is that an adjunction between $F\colon C \to D$ and $G\colon D \to C$ is a natural weak equivalence of spaces

$$\mathrm{Map}_D(F(x), y) \simeq \mathrm{Map}_C(x, G(y)).$$

The complicated part is saying what "natural" means, especially when working in the setting of quasicategories.

The sticky point is that we need to understand what the mapping space should be as a functor $C^{\mathrm{op}} \times C \to \mathscr{S}$, where $\mathscr{S} = N^s(\mathbf{Kan})$ is the ∞-category of spaces. This turns out to be quite complicated if we do it entirely in the setting of quasicategories: it requires us to develop straightening and unstraightening first, see Example 10.6.49 below. But this chapter is meant to be introductory, so we will use the more quick-and-dirty definition using simplicial categories instead.

10.3.1 The functorial mapping space

Our first goal is to define a functor $C^{\mathrm{op}} \times C \to \mathscr{S}$ for each quasicategory C, sending (x, y) to $\mathrm{Map}_C(x, y)$. As we remarked above, this is quite difficult, because the mapping spaces $\mathrm{Map}_C(-,-)$ don't have a strictly associative composition. The quick fix is to work in a setting where they do.

Definition 10.3.1. For any simplicial category $\mathbf{C}_\bullet$, the **functorial mapping space** is the simplicial functor

$$\mathbf{C}_\bullet^{\mathrm{op}} \times \mathbf{C}_\bullet \to \mathbf{sSet}_\bullet$$
$$(x, y) \mapsto \mathbf{C}_\bullet(x, y).$$

The spaces $\mathbf{C}_\bullet^{\mathrm{op}}(x, w) \times \mathbf{C}_\bullet(y, z)$ act on these by composition:

$$\begin{aligned} &\mathbf{C}_\bullet(x, y) \times \left(\mathbf{C}_\bullet^{\mathrm{op}}(x, w) \times \mathbf{C}_\bullet(y, z)\right) \\ \cong\ &\mathbf{C}_\bullet(w, x) \times \mathbf{C}_\bullet(x, y) \times \mathbf{C}_\bullet(y, z) \to \mathbf{C}_\bullet(w, z). \end{aligned}$$

If $\mathbf{C}_\bullet$ is pointwise fibrant (all the simplicial sets $\mathbf{C}_\bullet(a, b)$ are Kan complexes), then taking the simplicial nerve of this defines a functor of quasicategories

$$N^s(\mathscr{C}_\bullet)^{\mathrm{op}} \times N^s(\mathscr{C}_\bullet) \cong N^s(\mathscr{C}_\bullet^{\mathrm{op}} \times \mathscr{C}_\bullet) \to N^s(\mathbf{Kan}) = \mathscr{S}.$$

Remark 10.3.2. If $\mathbf{C}_\bullet$ fails to be fibrant then this is easy to fix: let $f : \mathbf{sSet} \to \mathbf{sSet}$ be any fibrant replacement functor such that f is strong symmetric monoidal, and $\mathrm{id} \to f$ is a symmetric monoidal natural tranformation. In particular, $f(X \times Y) \cong fX \times fY$. For instance, $fX = S_\bullet|X|$ works, as does Kan's functor Ex^∞ from [GJ99]. Then we can take $f\mathbf{C}_\bullet(x, y)$ instead of $\mathbf{C}_\bullet(x, y)$ in the definition of the functorial mapping space.

Definition 10.3.3. For any quasicategory C, the **functorial mapping space** is the map of quasicategories

$$C^{\mathrm{op}} \times C \to \mathscr{S}$$

obtained by the above formula on any pointwise fibrant simplicial category $\mathbf{C}_\bullet$ such that $N^s(\mathbf{C}_\bullet) \simeq C$.

This is well-defined up to natural isomorphism. The adjoint of this functor has a special name.

Definition 10.3.4. For any quasicategory C, the **Yoneda embedding** is the functor

$$C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathscr{S}) = \mathscr{P}(C)$$

given by the adjoint of the functorial mapping space.

Informally, the Yoneda embedding sends each object $x \in C_0$ to the functor $\mathrm{Map}_C(-, x)$, and then maps $x \to y$ give maps $\mathrm{Map}_C(-, x) \to \mathrm{Map}_C(-, y)$ by composition. Formally, that "composition" is defined using the functorial mapping space $\mathrm{Map}_C(-,-) \colon C^{\mathrm{op}} \times C \to \mathscr{S}$.

There is also a similar Yoneda embedding $C^{\mathrm{op}} \to \mathrm{Fun}(C, \mathscr{S})$ given by the other adjoint, which sends each object $x \in C_0$ to the functor $\mathrm{Map}_C(x, -)$. It is related to this one by taking the opposite category, so every result we state about one immediately gives a corresponding result about the other.

Lemma 10.3.5 (Yoneda Lemma)**.** *[Lan21, Prop 4.2.10] If $x \in C_0$ is an object and $F : C \to \mathscr{S}$ is a functor then the canonical map*

$$\begin{aligned} \mathrm{Map}_{\mathrm{Fun}(C,\mathscr{S})}(\mathrm{Map}_C(x,-),F(-)) &\to \mathrm{Map}_{\mathrm{Fun}(\{x\},\mathscr{S})}(\mathrm{Map}_C(x,-),F(-)) && \text{evaluate at } x \\ &\cong \mathrm{Map}_{\mathscr{S}}(\mathrm{Map}_C(x,x),F(x)) \\ &\to \mathrm{Map}_{\mathscr{S}}(*,F(x)) && \text{evaluate at } \mathrm{id}_x \\ &\simeq F(x) \end{aligned}$$

is an equivalence.

Note that this definition is insensitive to which version of the mapping space we use. Because no matter which definition we take for $\mathrm{Map}_{\mathscr{S}}(*,F(x))$, it is always equivalent to the space $F(x)$.

Corollary 10.3.6. *The Yoneda embedding $C^{\mathrm{op}} \to \mathrm{Fun}(C,\mathscr{S})$ is indeed an embedding: it is fully faithful in the sense that the induced map of spaces*

$$\mathrm{Map}_C(y,x) \to \mathrm{Map}_{\mathrm{Fun}(C,\mathscr{S})}(\mathrm{Map}_C(x,-),\mathrm{Map}_C(y,-))$$

is an equivalence.

Corollary 10.3.7. *The object that represents a functor is unique up to canonical isomorphism: natural isomorphisms of functors $C \to \mathscr{S}$ of the form*

$$\mathrm{Map}_C(x,-) \cong \mathrm{Map}_C(x',-)$$

come from isomorphisms $x \cong x'$ in C that are unique in hC.

As a result, to name an object of an ∞-category up to canonical isomorphism, it is enough to name the functor that it represents, just like how it worked for any 1-category.

Example 10.3.8. If C is the nerve of a 1-category, then the functor $C^{\mathrm{op}} \times C \to \mathscr{S}$ factors through the 1-category of sets, and it is the usual set of maps $C(x,y)$. The Yoneda lemma given just above becomes the familiar 1-categorical version. However, the Yoneda embedding is still different: we are embedding C into the category of presheaves of spaces $\mathscr{P}(C) = \mathrm{Fun}(C^{\mathrm{op}},\mathscr{S})$, which is larger than the category of presheaves of sets $\mathrm{Fun}(C^{\mathrm{op}},\mathbf{Set})$.

10.3.2 Definition of an adjunction

Now that we have made $\mathrm{Map}_C(-,-)$ into a functor on $C^{\mathrm{op}} \times C$, it is easy to define an adjunction.

Definition 10.3.9. An **adjunction** of quasicategories consists of two functors $F : C \to D$ and $G : D \to C$ and a natural isomorphism of functors $C^{\mathrm{op}} \times D \to \mathscr{S}$

$$\mathrm{Map}_D(F(-),-) \cong \mathrm{Map}_C(-,G(-)).$$

Expanding this out, the data of the natural isomorphism is given by a single natural transformation $C^{\mathrm{op}} \times D \times \Delta[1] \to \mathscr{S}$ that for each object $(x,y) \in C^{\mathrm{op}} \times D$ is an isomorphism in $\mathscr{S}$, in other words a weak homotopy equivalence of spaces.

Example 10.3.10. Taking connected components of all of the spaces, we see that every adjunction of ∞-categories between C and D gives an adjunction of 1-categories between the homotopy categories hC and hD. So adjunctions of ∞-categories are "lifted" or "enriched" versions of adjunctions of homotopy categories.

Example 10.3.11. If C and D come from 1-categories then an adjunction in the sense of Definition 10.3.9 is the same thing as an adjunction of 1-categories.

Lemma 10.3.12. *Adjoints are unique. In particular any two right adjoints G and G' of the same functor F are naturally isomorphic.*

We can define adjunctions explicitly when we work in simplicial categories. If F and G happen to come from simplicial functors, then all we need to build an adjunction is a zig-zag of weak equivalences of simplicial functors

$$\mathbf{D}_\bullet(F(-),-) \simeq \mathbf{C}_\bullet(-,G(-)).$$

Each term of this zig-zag is a simplicial natural transformation:

Definition 10.3.13. A simplicial natural transformation of functors $\alpha, \beta : \mathbf{I}_\bullet \rightrightarrows \mathbf{J}_\bullet$ is a simplicial functor $\mathbf{I}_\bullet \times \{0 \to 1\} \to \mathbf{J}_\bullet$ that on $\mathbf{I}_\bullet \times \{0\}$ and $\mathbf{I}_\bullet \times \{1\}$ recovers the original two functors.

The important thing to know is that the data of a natural transformation is given at simplicial level 0, by a morphism for each input object, but you have to check naturality at every simplicial level, not just simplicial level 0.

We'll focus on examples that come from topological spaces, because those are even easier:

Lemma 10.3.14. *Every natural transformation of topological functors automatically respects the enrichment. So if we turn them into simplicial functors by taking singular simplices of the mapping spaces, it becomes a simplicial natural transformation.*

Example 10.3.15. If K, X, and Y are CW complexes then we get a natural homeomorphism

$$\mathrm{Map}(K \times X, Y) \cong \mathrm{Map}(X, \mathrm{Map}(K,Y)).$$

Therefore we get an adjunction between $K \times (-)$ and $\mathrm{Map}(K,-)$ on the underlying ∞-category of **Top**, as a category enriched in topological spaces.

We would like to say that we also get an adunction on the usual ∞-category of spaces, which corresponds to the subcategory $\mathbf{Top}^{\mathrm{cf}} = \mathbf{CW}$. However, this is much more difficult than it appears. For one, you have to use a cofibrant replacement functor Q so that you still land in CW complexes. This means that we get a weak equivalence of mapping spaces

$$\mathrm{Map}(K \times X, Y) \cong \mathrm{Map}(X, \mathrm{Map}(K, Y)) \xleftarrow{\sim} \mathrm{Map}(X, Q\mathrm{Map}(K, Y)).$$

But then there is another problem: Q is not a topologically enriched functor, so $Q\mathrm{Map}(K, -)$ is not a valid candidate for the right adjoint of $K \times (-)$ in this setting. In spectra this gets even worse, because you have both cofibrant and fibrant replacements, and finding ways to enrich them makes the formulas very complicated.

Example 10.3.16. Let $\mathbf{Kan}_\bullet$ be the simplicial category of Kan complexes, with the usual simplicial sets of maps between them, and let $\mathbf{qCat}_\bullet$ be the larger simplicial category of all quasicategories. (These are simplicial categories. If we take the simplicial nerve then we get quasicategories $\mathscr{S}$ and $\mathbf{Cat}_\infty$, respectively.)

The inclusion between these is the "inclusion of ∞-groupoids into ∞-categories." This functor has both a left and a right adjoint. The left adjoint sends every quasicategory D to its full localization $D[D_1^{-1}]$ from Example 10.1.37. The right adjoint sends every quasicategory D to its core $D^{\simeq}$.

As in the previous example, the left adjoint will be clunky and difficult to prove right now. But the right adjoint is easier: for every Kan complex C and quasicategory D, we have an isomorphism of simplicial sets

$$\mathrm{Map}(C, D^{\simeq}) \cong \mathrm{Fun}(C, D).$$

(Actually Map and Fun are defined in exactly the same way, as the internal hom in $\mathbf{sSet}$. We just give them different names to emphasize that the first is the mapping space in $\mathbf{Kan}_\bullet$ and the second is the mapping space in $\mathbf{qCat}_\bullet$.) This isomorphism strictly commutes with the action of the simplicial mapping spaces, so it defines an adjunction of ∞-categories between the inclusion $\mathbf{Kan}_\bullet \to \mathbf{qCat}_\bullet$ and the core $\mathbf{qCat}_\bullet \to \mathbf{Kan}_\bullet$. Taking the simplicial nerve, we get an adjunction between the inclusion $\mathscr{S} \to \mathbf{Cat}_\infty$ and the core $\mathbf{Cat}_\infty \to \mathscr{S}$.

10.3.3 Using Dwyer–Kan localization to define adjunctions

We've just given a very concrete definition of an adjunction of ∞-categories, but it is very hard to actually use in examples like $\mathbf{Top}$ or $\mathbf{Sp}$. It is much easier to use the magic of Dwyer–Kan localization instead. This works because we can rearrange the definition of an adjunction into something that only uses natural transformations:

Lemma 10.3.17. *[Lur25, 6.2.1.1] An adjunction between the functors of quasicategories $F : C \to D$ and $G : D \to C$ is specified by two natural transformations*

$$\eta : \mathrm{id}_C \to G \circ F, \qquad \epsilon : F \circ G \to \mathrm{id}_D,$$

such that the first diagram below commutes in the homotopy category $h\,\mathrm{Fun}(C,D)$*, and the second diagram commutes in the homotopy category* $h\,\mathrm{Fun}(D,C)$*.*

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow^{=} & \downarrow{\scriptstyle \epsilon F} \\ & & F, \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow^{=} & \downarrow{\scriptstyle G\epsilon} \\ & & G. \end{array}$$

In fact, we only need the natural transformation η to define a natural map $\mathrm{Map}_D(F(x),y) \to \mathrm{Map}_C(x,G(y))$. The other transformation and the triangle identities are used to check that this natural map is a weak equivalence of spaces.

The proof of Proposition 3.5.20 verifies the hypotheses of Lemma 10.3.17 whenever F is a left-derived functor and G is the right-derived right adjoint. We conclude:

Corollary 10.3.18. *[DHKS04] For every adjunction of categories with weak equivalences*

$$\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D},$$

if F *is has a left-derived functor and* G *has a right-derived functor (Definition 3.4.15) then the derived functors* $\mathbb{L}F$ *and* $\mathbb{R}G$ *induce an adjunction of* ∞*-categories*

$$N(\mathbf{C})[W_C^{-1}] \underset{\mathbb{R}G}{\overset{\mathbb{L}F}{\rightleftarrows}} N(\mathbf{D})[W_D^{-1}].$$

As a special case, every Quillen adjunction of model categories gives an adjunction of ∞-categories. On the homotopy categories, this is the adjunction that we proved all the way back in Proposition 3.5.20. (And when F and G form a Quillen equivalence, this is an equivalence of ∞-categories, as in Corollary 10.2.13.)

Example 10.3.19. The disjoint basepoint and forgetful functors give an adjunction between the ∞-categories of spaces and based spaces:

$$\mathscr{S} \underset{U}{\overset{(-)_+}{\rightleftarrows}} \mathscr{S}_*.$$

The left-derived suspension and loopspace functors give an adjunction on the ∞-category of spaces

$$\mathscr{S} \underset{\Omega}{\overset{\mathbb{L}\Sigma}{\rightleftarrows}} \mathscr{S},$$

and the left-derived suspension spectrum and infinite loopspace functors give another adjunction between spaces and spectra:

$$\mathscr{S} \underset{\Omega^\infty}{\overset{\mathbb{L}\Sigma^\infty}{\rightleftarrows}} \mathrm{Sp}.$$

If we pass to the homotopy category, this recovers some adjunctions between $h\mathscr{S} = \mathrm{Ho}\,\mathbf{Top}$ and $h\mathrm{Sp} = \mathrm{Ho}\,\mathbf{Sp}$ that we've already seen in Chapter 3.

Example 10.3.20. If $\mathbf{C}$ is any of our model categories of spaces, based spaces, or spectra and $\mathbf{I}$ is any discrete 1-category, then we can consider the composite

$$N(\mathbf{C})[W^{-1}] \xrightarrow{\text{const}} N(\mathbf{C}^{\mathbf{I}})[W_{\mathbf{I}}^{-1}] \xrightarrow{\sim} \mathrm{Fun}(N(\mathbf{I}), N(\mathbf{C})[W^{-1}]).$$

The first map is the constant diagram functor, and the second map is the equivalence from Corollary 10.2.16. The composite of these is just the operation that collapses $N(\mathbf{I})$ to a point, giving a constant diagram at some point of $N(\mathbf{C})[W^{-1}]$.

The second of these functors is an equivalence. The first of these functors has a left adjoint by by Corollary 10.3.18, given by the left-derived functor of the colimit $\mathbf{C}^{\mathbf{I}} \to \mathbf{C}$. But we know by Corollary 8.3.22 that this is the homotopy colimit, or to be specific the Bousfield–Kan homotopy colimit $B(QX, \mathbf{I}, *)$. Putting this all together, the left adjoint of the above composite is given by the composite

$$\mathrm{Fun}(N(\mathbf{I}), N(\mathbf{C})[W^{-1}]) \xleftarrow{\sim} N(\mathbf{C}^{\mathbf{I}})[W_{\mathbf{I}}^{-1}] \xrightarrow{B(Q-,\mathbf{I},*)} N(\mathbf{C})[W^{-1}].$$

In summary, when $C = N(\mathbf{C})[W^{-1}]$ is the ∞-category of spaces, based spaces, or spectra, we can explicitly describe the left adjoint of the constant diagram functor $C \to \mathrm{Fun}(N(\mathbf{I}), C)$. The left adjoint functor $\mathrm{Fun}(N(\mathbf{I}), C) \to C$ is composed of the equivalence from Corollary 10.2.16 and the Bousfield–Kan homotopy colimit. So it has two steps. The first step turns the "homotopy-coherent diagram" in $\mathrm{Fun}(N(\mathbf{I}), C)$ into a strict diagram in $\mathbf{C}^{\mathbf{I}}$. The second step takes the homotopy colimit of that strict diagram using the bar construction.

Example 10.3.21. Returning to Example 10.3.16, we want to describe the left adjoint of the inclusion $\mathscr{S} \to \mathbf{Cat}_\infty$. Each of these is a localization of the same category $\mathbf{sSet}$ by a different set of weak equivalences:

$$\mathscr{S} \simeq N(\mathbf{sSet})[W_{\mathrm{Kan}}^{-1}], \qquad \mathbf{Cat}_\infty \simeq N(\mathbf{sSet})[W_{\mathrm{Joyal}}^{-1}].$$

The inclusion $\mathscr{S} \to \mathbf{Cat}_\infty$ can be modeled by the right-derived functor of the identity functor $\mathbf{sSet}_{\mathrm{Kan}} \to \mathbf{sSet}_{\mathrm{Joyal}}$, which is right Quillen. So by Corollary 10.3.18, the left adjoint

at the level of ∞-categories is the left-derived identity functor $\mathbf{sSet}_{\text{Joyal}} \to \mathbf{sSet}_{\text{Kan}}$, which is just the identity functor. It is equivalent to the Kan fibrant replacement $C \to C[C^{-1}]$, and this latter functor lands inside the bifibrant objects in $\mathbf{sSet}_{\text{Kan}}$. This shows that the full localization $C \to C[C^{-1}]$ is indeed the left adjoint of the inclusion $\mathscr{S} \to \mathbf{Cat}_\infty$.

We could also verify the right adjoint in the same way. The inclusion of Kan complexes into quasicategories is not the left-derived functor of the identity $\mathbf{sSet}_{\text{Kan}} \to \mathbf{sSet}_{\text{Joyal}}$, so we first restrict the domain to Kan complexes. Then the inclusion $\mathbf{sSet}^{\text{cf}}_{\text{Kan}} = \mathbf{Kan} \to \mathbf{sSet}_{\text{Joyal}}$ is the functor we want, and it sends all Kan equivalences to Joyal equivalences, so it is its own left-derived functor. The right adjoint is the core $D \mapsto D^{\simeq}$, as before, and this preserves all Joyal equivalences as well, so by Corollary 10.3.18 this is the right adjoint of the inclusion $\mathscr{S} \to \mathbf{Cat}_\infty$.

10.3.4 Colimits and limits

Now that we know what an adjunction is, we can define colimits and limits inside a quasicategory C.

If C is a quasicategory, then a **diagram** in C is a map of simplicial sets $X : K \to C$, where K is any small simplicial set.[9] The colimit of the diagram $X : K \to C$ will be given by a single object $(\operatorname{colim} X) \in C_0$, together with some data that describes how maps out of $(\operatorname{colim} X)$ are identified with maps out of the original diagram X.

Definition 10.3.22. For any quasicategory C and small simplicial set K, we define the constant diagram functor $C \to \operatorname{Fun}(K, C)$ by sending every n-simplex $\alpha : \Delta[n] \to C$ to the n-simplex of functors $K \to C$ given by

$$K \times \Delta[n] \longrightarrow \{*\} \times \Delta[n] \cong \Delta[n] \xrightarrow{\alpha} C.$$

The **colimit** is the left adjoint of this constant diagram functor and the **limit** is the right adjoint. By definition, these are functors $\operatorname{Fun}(K, C) \to C$ equipped with natural isomorphisms

$$\operatorname{Map}_C(\operatorname{colim} X, z) \cong \operatorname{Map}_{\operatorname{Fun}(K,C)}(X, \operatorname{const}(z))$$
$$\operatorname{Map}_{\operatorname{Fun}(K,C)}(\operatorname{const}(y), X) \cong \operatorname{Map}_C(y, \lim X).$$

(An isomorphism of functors to $\mathscr{S}$ means concretely that the spaces produced by these functors are weakly equivalent.)

Example 10.3.23. If C is (the nerve of) a 1-category, then the above universal property becomes the universal property of the ordinary colimit. In other words, colimits in a 1-category are a special case of colimits in ∞-categories.

[9] K doesn't have to be a 1-category, or even a quasicategory! When C is the simplicial nerve of some simplicial category $\mathbf{C}_\bullet$, this notion is classically called a "homotopy-coherent diagram."

Example 10.3.24. Let $\mathbf{C}_\bullet$ be any of our model categories of spaces, based spaces, or spectra, let $\mathbf{I}$ be any discrete category, and let $X : N(\mathbf{I}) \to N^s(\mathbf{C}_\bullet)$ be any diagram. Then we computed the colimit in Example 10.3.20 as the composite

$$\mathrm{Fun}(N(\mathbf{I}), N(\mathbf{C})[W^{-1}]) \xleftarrow{\sim} N(\mathbf{C}^{\mathbf{I}})[W_{\mathbf{I}}^{-1}] \xrightarrow{B(Q-,\mathbf{I},*)} N(\mathbf{C})[W^{-1}].$$

The first of these steps turns the "homotopy-coherent diagram" in $\mathrm{Fun}(N(\mathbf{I}), C)$ into a strict diagram in $\mathbf{C}^{\mathbf{I}}$, and then the second step takes the homotopy colimit of that strict diagram using the bar construction. This is a general formula for the colimit in any of these ∞-categories!

The same argument shows that the limit functor $\mathrm{Fun}(N(\mathbf{I}), C) \to C$ is also given by the homotopy limit in this way. With more work, we could also get a similar conclusion for any combinatorial model category $\mathbf{C}$, and any simplicial category $\mathbf{I}$. As a slogan:

In any well-behaved model category $\mathbf{C}$*, the colimit in the underlying* ∞*-category* $N(\mathbf{C})[W^{-1}]$ *is the homotopy colimit in* $\mathbf{C}$.

Morally, it should also be possible to prove that the homotopy colimit is the ∞-categorical colimit using Proposition 8.6.30, but in practice this requires some awkward cofibrant and fibrant replacements, and also a more careful analysis about how the mapping spaces in the category $\mathrm{Fun}(N^s(\mathbf{I}), \mathscr{S})$ compare to the mapping spaces in Proposition 8.6.30. This all goes well beyond the scope of this chapter.

Summarizing the previous two examples, for any diagram in any model category $\mathbf{C}$, we used to have two distinct notions: the *strict colmit* of the diagram, vs. the *homotopy colimit* of the diagram. In the world of ∞-categories, these two notions become the same: they are both examples of a colimit in an ∞-category. The difference now is that the strict colimit is the colimit in the underlying 1-category $N(\mathbf{C})$, while the homotopy colimit is the colimit in the underlying ∞-category $N(\mathbf{C})[W^{-1}]$.

There are also a few important examples of colimits and limits that can be done directly in any quasicategory C, without needing to assume that C came from some well-behaved model category so that we could take the homotopy colimit. (And this is good because not every ∞-category is "large" enough to come from a model category.)

Example 10.3.25. If $K = \varnothing$, then $X : \varnothing \to C$ is an empty diagram. The colimit of the empty diagram is an **initial object** of C. By Definition 10.3.22, x is an initial object of C if for all objects $z \in C_0$, $\mathrm{Map}_C(x, z) \simeq *$.

Dually, the limit of the empty diagram is a **terminal object** of C. This is an object y such that for all objects z, $\mathrm{Map}_C(z, y) \simeq *$.

For instance, in the ∞-category of spaces $\mathscr{S}$, the empty space is initial and the one-point space $*$ is terminal. In the ∞-category of spectra Sp, the zero spectrum is both initial and terminal.

More generally, in any simplicial model category $\mathbf{C}_\bullet$, the initial object becomes the initial object in the underlying ∞-category $N^s(\mathbf{C}_\bullet)[W^{-1}]$, while the terminal object becomes the terminal object in $N^s(\mathbf{C}_\bullet)[W^{-1}]$. We can see this by comparing to the other model for the underlying ∞-category $N^s(\mathbf{C}_\bullet^{\mathrm{cf}})$.[10]

Example 10.3.26. If K is discrete, then X is just a collection of spaces $X(k)$ for $k \in K$, and no maps between them. The colimit of X is the **coproduct** of the objects $X(k)$. By the above definitions and [Lan21, 4.3.24], x is a coproduct of the $X(k)$ if there are maps $X(k) \to x$ that for each z induce a weak homotopy equivalence

$$\mathrm{Map}_C(x,z) \xrightarrow{\sim} \prod_{k\in K} \mathrm{Map}_C(X(k),z).$$

Similarly the limit is the **product**, which is equipped with maps $x \to X(k)$ that for each y induce a weak homotopy equivalence

$$\mathrm{Map}_C(y,x) \xrightarrow{\sim} \prod_{k\in K} \mathrm{Map}_C(y,X(k)).$$

Again, this agrees with the coproduct in spaces, spectra, or in any combinatorial model category (Definition 10.4.26).

Example 10.3.27. If $K = \Lambda^0[2]$, in other words the nerve of the 1-category $\{\bullet \leftarrow \bullet \to \bullet\}$, then a diagram $X : K \to C$ is just a pushout diagram $b \leftarrow a \to c$. The colimit of X is the **pushout**. By the above definitions and [Lan21, 4.3.25], x is a pushout of this diagram if $b \leftarrow a \to c$ extends to a square

$$\overline{X} : \Delta[1] \times \Delta[1] \to C$$

with last vertex x, and for each y the resulting homotopy-commuting square

$$\begin{array}{ccc} \mathrm{Map}_C(x,y) & \longrightarrow & \mathrm{Map}_C(c,y) \\ \downarrow & & \downarrow \\ \mathrm{Map}_C(b,y) & \longrightarrow & \mathrm{Map}_C(a,y) \end{array}$$

is a homotopy pullback square. By Example 10.3.20, this agrees with the homotopy pushout in spaces, spectra, or in any combinatorial model category (Definition 10.4.26).

Dually, if $K = \Lambda^2[2]$, the nerve of the 1-category $\{\bullet \to \bullet \leftarrow \bullet\}$, then a diagram $X : K \to C$ is a pullback diagram $b \to d \leftarrow c$, and its limit is the **pullback**. Taking the op of the above, x is a pullback of this diagram if X extends to a square

$$\overline{X} : \Delta[1] \times \Delta[1] \to C$$

[10] We do have to take the fibrant replacement of the initial object and the cofibrant replacement of the terminal object for them to fit into this other model for the underlying ∞-category of $\mathbf{C}_\bullet$, but these replacements don't change the homotopy type of the mapping space.

with first vertex x, and for each y the resulting homotopy-commuting square

$$\begin{array}{ccc} \mathrm{Map}_C(y,x) & \longrightarrow & \mathrm{Map}_C(y,c) \\ \downarrow & & \downarrow \\ \mathrm{Map}_C(y,b) & \longrightarrow & \mathrm{Map}_C(y,d) \end{array}$$

is a homotopy pullback square. Again by Example 10.3.20, this agrees with the homotopy pullback in spaces, spectra, or in any combinatorial model category.

It turns out that these are the most important examples:

Proposition 10.3.28. *If C is a quasicategory and has all coproducts (including the initial object) and all pushouts, then C has all colimits along (small) simplicial sets. In other words, C is **cocomplete**.*

Dually, if C has all products and all pullbacks then it has all limits, so C is **complete**. (This follows immediately by applying the above proposition to C^{op}.)

That handles existence of colimits. If you ever want a formula for the colimit, well, that's not what ∞-categories are good at. That's what simplicial model categories are for! They are modeled by the homotopy colimit, as discussed in Example 10.3.24.

Example 10.3.29. We constructed a quasicategory $\mathbf{Cat}_\infty$ of all quasicategories in Definition 10.1.54, and recast it in Lemma 10.2.18 as the underlying ∞-category of the model category $\mathbf{sSet}_{\mathrm{Joyal}}$. It's useful to think about what colimits and limits look like here.

It turns out that this is another setting where the colimit in the ∞-category is modeled by the left-derived colimit in the model category, since $\mathbf{sSet}_{\mathrm{Joyal}}$ is a combinatorial model category. So the coproduct in $\mathbf{Cat}_\infty$ of a set of quasicategories C_α is just their coproduct as simplicial sets $\coprod_\alpha C_\alpha$, since these are cofibrant objects so their disjoint union has the correct homotopy type. Similarly the product of the quasicategories C_α in $\mathbf{Cat}_\infty$ is just their product as simplicial sets $\prod_\alpha C_\alpha$, since they are fibrant objects so their product has the correct homotopy type.

To form the pushout of three quasicategories $D \leftarrow C \rightarrow E$ in $\mathbf{Cat}_\infty$, you replace D up to Joyal equivalence so that $C \rightarrow D$ is injective, and then take the pushout in **sSet**. It is a straightforward exercise in monoidal model structures to show this can be given by

$$D \cup_C (C \times N(\{0 \cong 1\})) \cup_C E,$$

a "double mapping cylinder" in quasicategories where $N(\{0 \cong 1\})$ plays the role of the interval. (Then you take the Joyal fibrant replacement so that the result is a quasicategory again.) Intuitively, we are making a larger category in which the images of each object

$x \in C$ in D and E are joined by an isomorphism, and then we are adding in all composites of these isomorphisms with the existing maps in D and E.

On the other hand, the pullback of the quasicategories $C \to E \leftarrow D$ is the pullback $C \times_E \mathrm{Fun}(N(\{0 \cong 1\}), E) \times_E D$, in other words the category of pairs of objects in C and D and an isomorphism between their images in E. This is already a quasicategory, so no further fibrant replacement is needed. Intuitively, this is the category of pairs of objects in C and D and an isomorphism between their images in E.

10.3.5 Applying functors to colimits

We want to understand when a functor of quasicategories $F : C \to D$ sends colimits in C to colimits in D. To make sense of this, we have to rearrange the definition of a colimit to some kind of cone on the diagram $X : K \to C$.

Definition 10.3.30. For simplicial sets K and L, the **cylindrical join** (or fat join) is the quotient

$$K \diamond L = (K \times \Delta[1] \times L)/\sim$$

where the equivalence relation collapses $K \times L$ times the initial vertex of $\Delta[1]$ down to K, and collapses $K \times L$ times the terminal vertex of $\Delta[1]$ down to L.

As a special case, the **cylindrical cone** $K^{\triangleright} = K \diamond \{\infty\}$ is the product $K \times \Delta[1]$ modulo the relation that collapses $K \times \{1\}$ to the single point $\{\infty\}$. And the dual cylindrical done $K^{\triangleleft} = \{\infty\} \diamond K$ is the quotient of $K \times \Delta[1]$ by the relation that collapses $K \times \{0\}$ to the single point $\{\infty\}$.

From Definition 10.3.22, any time $X : K \to C$ has a colimit, we can plug in the identity map of $(\mathrm{colim}\, X)$ into the isomorphism

$$\mathrm{Map}_C(\mathrm{colim}\, X, \mathrm{colim}\, X) \cong \mathrm{Map}_{\mathrm{Fun}(K,C)}(X, \mathrm{const}(\mathrm{colim}\, X))$$

and get a map from X into the constant functor on $(\mathrm{colim}\, X)$. This rearranges into a map of simplicial sets

$$\overline{X} : K \diamond \{\infty\} \to C$$

that extends X and that sends the cone point to $(\mathrm{colim}\, X)$. Conversely, any such extension with cone point y gives a natural map

$$\mathrm{Map}_C(y, z) \to \mathrm{Map}_{\mathrm{Fun}(K,C)}(X, \mathrm{const}(z)). \tag{10.3.31}$$

When this map is an equivalence (i.e. an isomorphism in $\mathscr{S}$), $y = \mathrm{colim}\, X$. So we can think of the cone $\overline{X}$ as giving all of the essential data needed to identify some object of C as the colimit of X.

Definition 10.3.32. A **colimit cone** of $X : K \to C$ is any extension $\overline{X} : K \diamond \{\infty\} \to C$ such that the induced map (10.3.31) is an equivalence.

Dually, a limit can be represented by a diagram $\overline{X} : \{\infty\} \diamond K \to C$ that extends X, that we call a **limit cone**.

Example 10.3.33. When $K = \varnothing$, the cone on K is just a single point. This corresponds to the fact that an initial object is given by just one object of C, and no other data. When K is discrete, the cone $K \diamond \{\infty\}$ is a union of 1-simplices whose terminal vertices are all identified with ∞. This corresponds to the fact that a coproduct is given by an object y and just one morphism from each of the factors to y.

We can now make sense of when a functor of quasicategories preserves colimits or limits.

Definition 10.3.34. If C and D are quasicategories with all colimits and $F : C \to D$ is a functor, we say that F **preserves colimits** if for each colimit cone $\overline{X} : K^{\triangleright} \to C$, the composite

$$K^{\triangleright} \xrightarrow{\overline{X}} C \xrightarrow{F} D$$

is a colimit cone in D. Dually, if C and D have all limits then we say that F **preserves limits** if it preserves every limit cone.

Remark 10.3.35. There is another notion of join $K * L$, whose vertices are the union of the vertices of K and L, and whose simplices are the product of simplices

$$(K * L)_n = \coprod_{[n] \cong [i] \amalg [j]} K_i \times L_j,$$

using the convention that $K_{-1} = L_{-1} = \varnothing$. There is a canonical Joyal equivalence $K \diamond L \to K * L$ that is the identity on the subspace $K \amalg L$ and that is natural in K and L. Using this fact, one can prove that colimits can be represented equally well by maps out of the cone

$$\overline{X} : K * \{\infty\} \to C$$

instead of using the cylindrical cone $K \diamond \{\infty\}$, and a functor preserves colimits iff it preserves this kind of cone. In fact, we already saw this in Example 10.3.27: the square $\Delta[1] \times \Delta[1]$ is isomorphic to the cone $\Lambda^0[2] * \{\infty\}$, not the cylindrical cone $\Lambda^0[2] \diamond \{\infty\}$.

The following is a large source of examples:

Lemma 10.3.36. *Left adjoints preserve colimits and right adjoints preserve limits.*

Example 10.3.37. By Corollary 10.3.18, left Quillen functors of model categories give left adjoints on the associated ∞-categories. Therefore they preserve all colimits. This is consistent with the fact that the colimit in this setting is the homotopy colimit, and left Quillen functors preserve those by Proposition 8.6.3.

Lemma 10.3.38. *[Lan21, 4.3.33] Colimits and limits in a functor category* $\mathrm{Fun}(K, C)$ *are computed "pointwise." In other words, the evaluation at each vertex of* K *gives a functor* $\mathrm{Fun}(K, C) \to C$ *that preserves all colimits and limits.*

We know that left adjoints preserve colimits. It turns out that there are essentially no other examples. As soon as C is "presentable," then preserving colimits is the same thing as being a left adjoint. We will state this in Lemma 10.4.33 after we explain what it means to be presentable.

Example 10.3.39. One of our most treasured adjunctions goes between **Set** and **Top**, by giving each set a discrete topology, and taking each space to its underlying set. This is a good example of an adjunction of 1-categories that does *not* give an adjunction of ∞-categories between **Set** and $\mathscr{S} = N(\mathbf{Top})[W^{-1}]$!

One way to see this is to see that the discrete topology functor **Set** $\to$ **Top** preserves strict colimits, but does not send colimits to homotopy colimits. (It does send filtered colimits to filtered homotopy colimits though.) Another way to see this is that for a set S and space X, the space of maps $S \to X$ is the product space $\prod_S X$, but the set of maps is the product *set* $\prod_S UX$, with the discrete topology. These are not equivalent spaces, so these two functors cannot be adjoint as functors of ∞-categories!

10.4 Presentable ∞-categories

The motivation behind ∞-categories is that they are a more powerful way to describe "homotopy theories." What is a homotopy theory? Before we define it precisely, let's give some intuition. The category of spaces should be a homotopy theory. The category of spectra should be one too. So should modules over a ring spectrum, diagrams of spaces, and so on. In a "homotopy theory," you should have objects up to some notion of weak equivalence, and you should be able to build larger objects out of smaller ones, forming things like cofibers and fibers.

One candidate for a "homotopy theory" is a model category, considered up to Quillen equivalence. Another candidate is an ∞-category, in other words a quasicategory considered up to Joyal equivalence.

When we compare these two candidates, we realize there is a mismatch. Some ∞-categories are too small, or too unstructured, to have come from a model category. For instance, suppose we just take two objects and a single space of morphisms between them, as in Example 10.1.7. That defines an ∞-category, but it can't possibly come from a model category, and so it seems silly to think of it as a homotopy theory.

In this section we single out a special class of "presentable" ∞-categories. These all come from model categories, and they agree nicely with our intuition for what a "homotopy

theory" is. As an added bonus, it is much easier to understand the category of colimit-preserving functors $\mathrm{LFun}(C, D)$ between two presentable ∞-categories.

As an application, in this new setting of presentable ∞-categories, we will see that the ∞-category of spaces $\mathscr{S}$ and the ∞-category of spectra Sp have universal properties (Corollary 10.4.37 and Corollary 10.4.53). We will use this later to give a new definition of the smash product of spectra. And we will also show how to construct an especially well-behaved kind of Dwyer–Kan localization in this setting, called a Bousfield localization, or a reflective localization.

10.4.1 Filtered colimits and compact objects

To understand what it means to be presentable, we have to first have to review filtered colimits and compact objects, both in a 1-category and in an ∞-category. This will require a bit more set theory than usual, so buckle up!

First we give the intuition. Our overall goal in this section is to define **κ-compact** objects when κ is a "regular" cardinal number. The slogan is:

A κ-compact object in a 1-category **C** *behaves like a set with $< \kappa$ elements.*

A κ-compact object in an ∞-category C behaves like a cell complex with $< \kappa$ cells.

So for example, an $\aleph_0$-compact object behaves like a finite set, or a finite cell complex.

Now, the notion of "κ-compact" is going to make sense in any category with all colimits, even if it doesn't have a notion of "cell complex." So the definition is going to be phrased entirely in terms of colimits. An object X will be κ-compact when any map from X into a sufficiently large colimit fails to hit every "piece" of the colimit. The notion of "sufficiently large colimit" is captured by another notion, a **κ-filtered colimit**. So we will have to define this first.

Now that we've given the intution, let's start giving some definitions. Recall that an **ordinal number** is a well-ordered set in which each element is equal to the set of elements that precede it. They are the canonical representatives for isomorphism classes of well-ordered sets. The ordinals themselves are well-ordered, and $\alpha < \beta$ when α is an "initial segment" of β. The first few ordinal numbers are

$$0 = \emptyset, \;\; 1 = \{0\}, \;\; 2 = \{0,1\}, \;\; 3 = \{0,1,2\}, \;\ldots,$$
$$\omega = \{0,1,2,\ldots\}, \;\; \omega+1 = \{0,1,2,\ldots,\omega\}, \;\ldots.$$

A **cardinal number** is an ordinal number that is minimal among all ordinal numbers that have a given cardinality. So the ordinals 0, 1, 2, and ω are also cardinal numbers. On the other hand, $\omega+1$ is not a cardinal, because it has the same cardinality as ω. We let $\aleph_0$

denote ω as a cardinal number, and let $\aleph_1$ denote the next cardinal number.[11]

Now we are ready to define κ-filtered colimits. Recall from Exercise 6.6.31 that a **filtered** category is a 1-category $\mathbf{I}$ in which every diagram with finitely many objects and morphisms has a **right cone**. In other words, there is an additional object of $\mathbf{I}$ that receives a map from every term of the diagram, commuting with all of the morphisms in the diagram. For instance, the usual sequential colimit category

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

is filtered. (This is also the ordinal number ω regarded as a category.) On the other hand, the pushout category $\{\bullet \leftarrow \bullet \rightarrow \bullet\}$ is not filtered.

We can generalize this by allowing all diagrams with cardinality less than some fixed cardinal κ. This only works well if κ is regular in the following sense.

Definition 10.4.1. An infinite cardinal κ is **regular** if it cannot be written as a union of less than κ sets, each of which has cardinality $< \kappa$. Equivalently, if we write κ as an ordinal, it is impossible to pick a cofinal subset (i.e. an unbounded subset) of cardinality $< \kappa$.

For instance, $\aleph_0$ is regular, because a finite union of finite sets is always finite. And $\aleph_1$ is regular, because a countable union of countable sets is always countable.

Definition 10.4.2. [Lur09, A.1.1.1 and 5.3.1.7] Let κ be a regular cardinal.

- A **κ-filtered 1-category** is a 1-category $\mathbf{I}$ such that, for any other 1-category $\mathbf{J}$ with $< \kappa$ morphisms (including the identity morphisms!) and any diagram $\mathbf{J} \to \mathbf{I}$, the diagram has a right cone. A **κ-filtered colimit** in a 1-category is a colimit along any κ-filtered 1-category.
- A **κ-filtered ∞-category** is a quasicategory K such that, for any simplicial set J with $< \kappa$ nondegenerate points in total, every diagram $J \to K$ has an extension to the cone $J * \{\infty\} \to C$. A **κ-filtered colimit** in an ∞-category is a colimit along any κ-filtered ∞-category.

When $\kappa = \aleph_0$, we usually drop the κ from the notation. So a filtered 1-category or ∞-category is one in which every finite diagram has a right cone.

Intuitively, a κ-filtered colimit is a colimit that "looks trivial" when you wear glasses that can only see $< \kappa$ objects at a time. Because when you look through those glasses, any sub-diagram that you can actually see must cone off to a point. That sub-diagram is trivial,

[11] There is a next cardinal because the cardinals are well-ordered as well. We don't know whether $\aleph_1$ is the cardinality of $\mathbb{R}$, but that's fine. We only need to know that a set is countable iff it has cardinality $< \aleph_1$.

in the sense that it contributes nothing to the colimit, other than the one point that it cones off to. So from your point of view, the colimit looks like it's going to all cone off to one object of the diagram. You're wrong though: the colimit can still be non-trivial. You just have to put on bigger glasses to see that.

Example 10.4.3. The usual sequential colimit category is given by the partially ordered set $\mathbb{N}$, or equivalently by the ordinal ω. It looks trivial when we only look at finitely many objects at a time, so it is filtered. In other words, $\mathbb{N}$ is an $\aleph_0$-filtered 1-category.

If we put on bigger glasses and look at countably many objects at a time, then we can see that the colimit along $\mathbb{N}$ is nontrivial. Formally, the entire category $\mathbb{N}$ is a diagram with $< \aleph_1$ morphisms, and it does not have a right cone in $\mathbb{N}$. So we say that $\mathbb{N}$ is not an $\aleph_1$-filtered 1-category.

In general, if κ is a regular cardinal, then when we regard it as an ordinal number, κ itself is always a κ-filtered 1-category. (Think about why we needed κ to be a regular cardinal for this to be true!) But κ is not λ-filtered for any larger regular cardinal $\lambda > \kappa$.

Example 10.4.4. This is the most important example. Let X be any CW complex, and let $\mathbf{I}$ be the 1-category that indexes the finite subcomplexes of X. So an object of $\mathbf{I}$ is a finite subcomplex $A_\alpha \subseteq X$, and a morphism is an inclusion of subcomplexes. (Notice that $\mathbf{I}$ comes from a partially ordered set. Morphisms $A_\alpha \to A_\beta$ are unique when they exist.)

Then $\mathbf{I}$ is a filtered 1-category, because any time we are given finitely many finite subcomplexes of X, we can take their union to get another finite subcomplex of X.

If instead we take the category that indexes all of the *countable* subcomplexes of X, that category is $\aleph_1$-filtered. And more generally, for any regular cardinal κ, the category indexing all of the subcomplexes with $< \kappa$ cells is κ-filtered. Because for any set of $< \kappa$ subcomplexes with $< \kappa$ cells each, their union is again a subcomplex with $< \kappa$ cells.

Remark 10.4.5. It follows from Exercise 10.8.13 that if $C = N(\mathbf{C})$ is the nerve of a 1-category $\mathbf{C}$, then C is a κ-filtered ∞-category iff $\mathbf{C}$ is a κ-filtered 1-category. So when we're working with a 1-category we can just say "κ-filtered" and we don't have to specify which version of the definition we're using. Also, this means that the previous two examples are examples of κ-filtered ∞-categories.

Example 10.4.6. Any 1-category with a terminal object, or any ∞-category with a terminal object in the sense of Example 10.3.25, is κ-filtered for every regular cardinal κ. This is because every diagram can be coned off to the terminal object.

Example 10.4.7. If β is an ordinal number, we can think of it as a category with a unique morphism $\alpha \to \gamma$ whenever $\alpha \leq \gamma < \beta$. Then β is a κ-filtered ordinal in the sense of Exercise 5.8.42 iff it is a κ-filtered 1-category in the sense of Definition 10.4.2.[12]

[12]Some authors declare that an ordinal can only be κ-filtered if it is a limit ordinal. That would cause

Now that we've introduced κ-filtered colimits, we can move on to our main goal of defining κ-compact objects.

Definition 10.4.8. [Lur09, A.1.1.1 and 5.3.4.5], [Lan21, 5.2.3] Let κ be a regular cardinal.

- Let $\mathbf{C}$ be a 1-category. An object $x \in \mathbf{C}$ is **κ-compact** if the functor $\mathbf{C}(x,-)\colon \mathbf{C} \to \mathbf{Set}$ preserves all colimits along κ-filtered 1-categories.
- Let C be a quasicategory. An object $x \in C_0$ is **κ-compact** if the functor $\mathrm{Map}_C(x,-)\colon C \to \mathscr{S}$ preserves all colimits along κ-filtered ∞-categories.

Again when $\kappa = \aleph_0$, we drop the κ from the notation. A compact object is one in which $\mathrm{Map}_C(x,-)$ preserves all filtered colimits, in whichever setting we are working in.

Example 10.4.9. In the 1-category **Set**, a set S is κ-compact iff it has $< \kappa$ elements. This is because if we take any map from S to a κ-filtered colimit $(\mathrm{colim}\, X_\alpha)$, the images of the points in S form a sub-diagram of size $< \kappa$, so they must cone off inside the diagram by definition, and this means that the map from S factors through one of the terms of the diagram, $S \to X_\alpha$. A similar argument shows that the lift is unique, up to mapping further forward in the diagram, so that the natural map $\left(\mathrm{colim}\, \mathrm{Map}_{\mathbf{Set}}(S, X_\alpha)\right) \to \mathrm{Map}_{\mathbf{Set}}(S, \mathrm{colim}\, X_\alpha)$ is a bijection.

Example 10.4.10. If we think of **sSet** as a 1-category, then a simplicial set K is κ-compact iff it has $< \kappa$ nondegenerate points. (See Exercise 10.8.16 for other equivalent versions of this.) More generally, if $\mathbf{I}$ is a 1-category with $< \kappa$ morphisms, then an object of $\mathbf{Set}^{\mathbf{I}}$ is κ-compact iff it is built as a coproduct of $< \kappa$ free diagrams, modulo $< \kappa$ relations.

Example 10.4.11. The notion of κ-compact objects in **Top** is not well-behaved. The problem is that the definition of κ-compact asks about *arbitrary* filtered colimits, but we only get good properties in **Top** if we restrict attention to colimits along inclusions of subcomplexes, or transfinite sequences of closed inclusions. Fortunately, in the associated ∞-category the compact objects are much better behaved.

Remark 10.4.12. As before, when $C = N(\mathbf{C})$ is the nerve of a 1-category $\mathbf{C}$, then they have the same objects, and an object is κ-compact in $\mathbf{C}$ iff it is κ-compact in C. So in particular, κ-compact objects in **Set** are sets with $< \kappa$ elements, regardless of whether we think of **Set** as a 1-category or as an ∞-category. (The proof of this uses the fact that $\mathbf{Set} \to \mathscr{S}$ preserves all filtered colimits, and therefore preserves κ-filtered colimits for any regular cardinal κ.)

the two definitions of "filtered" to disagree, and we think that's unnecessarily confusing. Although, this is a minor point either way, because the two definitions would only disagree for successor ordinals, and successor ordinals play no role in the small object argument, nor do they matter when measuring whether an object is compact.

Remark 10.4.13. In a 1-category, κ-compactess is stronger than the smallness condition of Exercise 5.8.42, because it involves not just ordinals but general filtered colimits. (It turns out that κ-compactness in a 1-category or ∞-category can be equivalently phrased using only colimits along κ-filtered partially ordered sets [Lur09, 5.3.1.16], but it is not enough to use only colimits along ordinals.)

Now let's move to examples in ∞-categories.

Example 10.4.14. In the ∞-category of spaces $\mathscr{S}$, any simplicial set K with $< \kappa$ non-degenerate points (see Exercise 10.8.16) is κ-compact. Or if we think of the objects as topological spaces, any CW complex or cell complex with $< \kappa$ cells is κ-compact. The proof that these objects are compact uses a variant of the small-object argument.

Actually, this is not quite all of the κ-compact objects in $\mathscr{S}$, because we can also take retracts of these. In other words, if K is a cell complex with $< \kappa$ cells, and if X is a retract of K in the homotopy category $h\mathscr{S}$, then X will be κ-compact as well. When $\kappa \geq \aleph_1$, this doesn't add any new examples, because we can model X as the mapping telescope of the maps $K \to X \to K \to X \to K \to \cdots$, and this simplifies to a mapping telescope of the form $K \to K \to K \to \cdots$, so it still has $< \kappa$ cells. But when $\kappa = \aleph_0$, in other words we are thinking about finite cell complexes, then this proof shows that X has countably many cells, not finitely many cells. In fact, it may not be possible to find a finite complex that is equivalent to it.

Definition 10.4.15. An object X of $\mathscr{S} \simeq \mathrm{Ho}\,\mathbf{CW}$ is **finitely dominated** if it is a retract in $\mathrm{Ho}\,\mathbf{CW}$ of a finite CW complex. In particular, X is a CW complex, and there is a finite CW complex K and maps $X \to K \to X$ whose composite is homotopic to the identity map of X.

Lemma 10.4.16. *An object in $\mathscr{S}$ is κ-compact iff it is a retract in the homotopy category $h\mathscr{S} \simeq \mathrm{Ho}\,\mathbf{sSet}$ of a simplicial set with $< \kappa$ nondegenerate points. Or if we think of the objects as topological spaces, a space is κ-compact iff it is a retract in the homotopy category of a CW complex with $< \kappa$ cells.*

So when $\kappa \geq \aleph_1$, an object is κ-compact iff it is equivalent to a CW complex with $< \kappa$ cells. When $\kappa = \aleph_0$, an object is κ-compact iff it is finitely dominated.

Remark 10.4.17. As κ increases, there are fewer and fewer κ-filtered colimits, and therefore there are more and more κ-compact objects.

10.4.2 Definition and examples of presentable ∞-categories

A presentable ∞-category is a particularly nice kind of ∞-category that we have a tighter grip on. Spaces, spectra, and diagrams of spaces or spectra are all examples of this nice class of ∞-categories.

Definition 10.4.18. [Lur09, 5.5.1.1], [Lan21, 5.2.3], [BGH22, 7.11.1] A **presentable ∞-category** is a quasicategory C that is

- locally small (for any finite $S \subseteq C_0$, the set of simplices with vertices in S is a set),
- cocomplete (for each small simplicial set K, every diagram $K \to C$ has a colimit), and
- there is a regular cardinal κ and a subcategory $C' \subseteq C$ that is
  - small (the vertices C'_0 are a set, and therefore the simplices C'_n are also a set),
  - a **full subcategory** (meaning every simplex in C with vertices in C' is actually a simplex in C'),
  - C' consists entirely of κ-compact objects, and
  - C' is **colimit-dense**: every object of C is isomorphic to the colimit of some diagram $K \to C'$.

  In summary, C' is a small full colimit-dense subcategory of κ-compact objects.

Usually, we can take C' to be the subcategory of "all cell complexes with $< \kappa$ cells," in some sense. Except, we can't literally take all of them. We have to take some representative sample of them that is small enough to form a set.

There are many equivalent formulations of this. For instance, C is presentable iff it is cocomplete and **accessible**, which in turn means that there is a regular cardinal κ such that C has all κ-*filtered* colimits, and contains a small full subcategory of κ-compact objects $C'' \subseteq C$, such that every object of C is a κ-*filtered* colimit of objects in C'' [Lur09, 5.4.2.2(3)]. More plainly, if the subcategory C' can be chosen to generate C under all colimits, then we can pick a (possibly larger) subcategory C'' that generates C using only κ-filtered colimits.

When C is the nerve of a 1-category $\mathbf{C}$, Definition 10.4.18 becomes the notion of a locally presentable 1-category, or just a presentable 1-category for short.

Example 10.4.19. A **presentable 1-category** is a 1-category $\mathbf{C}$ that is

- locally small (each $\mathbf{C}(a, b)$ is a set),
- cocomplete (for each small 1-category $\mathbf{I}$, every diagram $\mathbf{I} \to \mathbf{C}$ has a colimit), and
- there is a regular cardinal κ and a small full colimit-dense subcategory $\mathbf{C}' \subseteq \mathbf{C}$ consisting of κ-compact objects.

Example 10.4.20. The category **Set** is presentable. (Since this is a 1-category, it doesn't matter whether we think of it as an ∞-category or not, the definition of presentable is equivalent.) We take the subcategory **Fin** ⊆ **Set** to have one set of each finite size. Every set S is the filtered colimit of its finite subsets, so **Fin** is colimit-dense, and that is the only difficult part of the definition to check.

Example 10.4.21. The category of discrete groups and homomorphisms **Grp** is presentable. We take the subcategory **FinPres** ⊆ **Grp** to have one object for each isomorphism class of finitely presented groups. This is a set, and every group G is the filtered colimit of the groups G_α obtained by taking any finite subset of the generators and any finite subset of the relations that use only the chosen generators. We get similar conclusions for abelian groups, rings, and other sorts of algebraic structures.

Example 10.4.22. The category **sSet** is presentable. We take the subcategory $\mathbf{sSet}_{\mathbf{Fin}} \subseteq$ **sSet** to consist of all finite simplicial sets K, meaning there are finitely many nondegenerate points in total, together with a chosen embedding of the levels K_n into a fixed copy of $\mathbb{N}$, so that the collection of all the simplicial sets in $\mathbf{sSet}_{\mathbf{Fin}}$ still forms a set. Again, every simplicial set X is the filtered colimit of its finite simplicial subsets, and the other properties are easy to check.

Example 10.4.23. More generally, for any (small) 1-category **I**, the category of diagrams $\mathbf{Set}^{\mathbf{I}}$ is presentable, and for any (small) simplicial category **I**, the categories of diagrams $\mathbf{sSet}^{\mathbf{I}}$ and $\mathbf{sSet}^{\mathbf{I}}_*$ are presentable. The size of the κ we use depends on the number of morphisms in **I**.

Now we want to give examples of presentable ∞-categories. There is one that we can do easily.

Example 10.4.24. The ∞-category of spaces $\mathscr{S}$ is presentable. To prove this, we show that every Kan complex K is the colimit of the constant diagram $K \to \mathscr{S}$ sending everything to the one-point space $*$, see Exercise 10.8.14. Therefore we can take the small full subcategory $C' \subseteq \mathscr{S}$ to have *only one object*! We don't actually have to use the subcategory of all finite cell complexes, or all finitely dominated cell complexes, embedded in some sufficiently large fixed set. (Though of course that would also work, and we would need to use it if we wanted to generate $\mathscr{S}$ under *filtered* colimits.)

Example 10.4.25. The full subcategory $\mathscr{S}^{\mathrm{fin}} \subseteq \mathscr{S}$ of all spaces homotopy equivalent to a finite CW complex, is *not* presentable, because it doesn't have all colimits. The subcategory of finitely dominated spaces is also not presentable.

To show that an ∞-category is presentable, it is convenient to use the following kind of model category:

Definition 10.4.26. A **combinatorial model category** is a model category **C** that is

- presentable as a 1-category (Example 10.4.19), and
- cofibrantly generated by sets of maps I and J (Definition 5.2.8).

Example 10.4.27. The category of simplicial sets **sSet** is a combinatorial model category. So are the categories of based simplicial sets $\mathbf{sSet}_*$, sequential spectra of simplicial sets $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{sSet}}$, and symmetric spectra of simplicial sets $\mathbf{Sp}^{\Sigma}_{\mathbf{sSet}}$. We can also take diagrams in any of these categories, indexed by a (small) simplicial category **I**, and the resulting projective model structure also gives a combinatorial model category.

The categories of spaces **Top**, based spaces **Top**, and our other models of spectra ($\mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}}$, $\mathbf{Sp}^{O}$, $\mathscr{M}$, ...) all fail to be combinatorial model categories, because the underlying category isn't presentable.

Theorem 10.4.28 (Lurie, Dugger)**.** *[Lur09, A.3.7.6] [Dug01, Cor 1.2] If* **C** *is a combinatorial model category, its underlying* ∞*-category* $N^s(\mathbf{C}^{\mathrm{cf}}_{\bullet}) \simeq N(\mathbf{C})[W^{-1}]$ *is presentable. Conversely, if* C *is a presentable* ∞*-category, then it is Joyal equivalent to the underlying* ∞*-category of some combinatorial model category* **C**.

This is actually better than it seems, because the notion of being a presentable ∞-category is invariant under equivalences of ∞-categories. So any model category that is *Quillen equivalent* to a combinatorial model category, also gives a presentable ∞-category. And furthermore, every presentable ∞-category arises this way.

Example 10.4.29. The ∞-categories of spaces $\mathscr{S}$, based spaces $\mathscr{S}_*$, and spectra Sp are all presentable.

Lemma 10.4.30. *[Lur09, 5.5.3.6] If* C *is a presentable* ∞*-category and* K *is any small simplicial set then* $\mathrm{Fun}(K, C)$ *is also presentable.*

Example 10.4.31. It follows from Lemma 10.4.30 that for any small simplicial set K, the ∞-categories of diagrams or functors

$$\mathrm{Fun}(K, \mathscr{S}), \quad \mathrm{Fun}(K, \mathscr{S}_*), \quad \mathrm{Fun}(K, \mathrm{Sp})$$

are also presentable. (This also follows from Theorem 10.4.28 since these can all be modeled using combinatorial model categories.) In particular, the ∞-category of presheaves $\mathscr{P}(K) = \mathrm{Fun}(K^{\mathrm{op}}, \mathscr{S})$ is presentable.

10.4.3 Consequences of presentability

Now that we have many examples of presentable ∞-categories, let's talk about properties. Presentability allows us to construct things using colimits that would be "too big" in an arbitrary category. Here are a couple of examples of this:

Lemma 10.4.32. *[Lan21, 5.2.5] Presentable ∞-categories have all small limits, in addition to having all small colimits.*

Lemma 10.4.33. *[Lan21, 5.2.2] If C is presentable and D is any quasicategory, then the functor $F : C \to D$ has a right adjoint iff it preserves colimits.*

To give the idea of the proofs, if $\mathbf{C}$ is a 1-category and $X : \mathbf{I} \to \mathbf{C}$ is a diagram, the limit of X ought to be the colimit of all of the left cones of X, in other words all the objects Y_i with commuting maps into the diagram X. This is a cool idea: you can construct any limit as a colimit!

Unfortunately this doesn't make sense, because the collection of all left cones is too large to be a set in general. But in a presentable 1-category it does make sense: we can just take those cones in which Y_i is in the small subcategory $\mathbf{C}'$. If the larger colimit did exist, it would be equivalent to this smaller one, because for every cone whose cone point Y is not in $\mathbf{C}'$, we write Y as a colimit of objects in $\mathbf{C}'$, and that gives a larger collection of cones whose colimit is the original cone with cone point Y. So the colimit we want is "equivalent" to a colimit that actually exists, using the subcategory $\mathbf{C}'$.

The construction of the right adjoint G in Lemma 10.4.33 is similar: the value of $G(Y)$ should be the colimit over all objects $Z \in \mathbf{C}$ with maps $F(Z) \to Y$, and we can make this make sense by restricting attention to $Z \in \mathbf{C}'$.

A nice consequence of the above is that presentable ∞-categories are always "tensored and cotensored over spaces," just like a simplicial model category:

Lemma 10.4.34. *[Lur09, 5.5.1.7] When C is a presentable ∞-category, there is a tensoring functor $\otimes : \mathscr{S} \times C \to C$ and a natural weak equivalence*

$$\mathrm{Map}_C(K \otimes X, Y) \simeq \mathrm{Map}(K, \mathrm{Map}_C(X, Y)).$$

There is also a cotensoring functor $(-)^{(-)} : \mathscr{S}^{\mathrm{op}} \times C \to C$ and a natural weak equivalence

$$\mathrm{Map}_C(X, Y^K) \simeq \mathrm{Map}(K, \mathrm{Map}_C(X, Y)).$$

The idea is that $K \otimes X$ can be defined as the colimit of the constant diagram along K at the object X, while X^K can be defined as the limit of that same diagram.

Another reason why we like presentable ∞-categories so much is that it's much easier to describe functors between them, and use this to give universal properties for the categories we care about. Ultimately, we'll use this idea to define the smash product of spectra in a new way. For now, let's just setup the framework and prove the universal property for the ∞-category of spaces.

Definition 10.4.35. Let C and D be quasicategories with all colimits, and suppose that C is presentable. A functor $F : C \to D$ is **right-exact** if it preserves all colimits in the sense of

Definition 10.3.34. Equivalently, if it preserves all coproducts and pushouts. Equivalently, if it is a left adjoint.

Let $\mathrm{LFun}(C,D) \subseteq \mathrm{Fun}(C,D)$ be the full subcategory of all left adjoint (right exact) functors. This is a quasicategory because it is a full subcategory of a quasicategory.

We could go even further and define the **category of presentable ∞-categories** Pr^L to be the simplicial category whose objects are presentable ∞-categories and whose simplicial mapping spaces are $\mathrm{LFun}(C,D)$. As with the collection of all ∞-categories **qCat** from Definition 10.1.54, Pr^L is really an $(\infty,2)$-category, because each of the simplicial sets $\mathrm{LFun}(C,D)$ is a quasicategory, not a Kan complex. And unlike $\mathbf{Cat}_\infty$, we really do want to keep it that way, and not restrict to the core $\mathrm{LFun}(C,D)^{\simeq}$.[13]

Recall that for every small simplicial set X, there is a canonical map $X \to \mathscr{P}(X)$ called the Yoneda embedding (Definition 10.3.4).

Proposition 10.4.36. *[Lur09, 5.1.5.6] For any small simplicial set X, and quasicategory C that has all colimits, composition with the Yoneda embedding induces a Joyal equivalence*

$$\mathrm{LFun}(\mathscr{P}(X), C) \simeq \mathrm{Fun}(X, C).$$

Corollary 10.4.37. *The ∞-category of spaces $\mathscr{S}$ is the free presentable ∞-category on a point. In other words, for every quasicategory C with all colimits, evaluation at $* \in \mathscr{S}$ gives an equivalence of quasicategories*

$$\mathrm{LFun}(\mathscr{S}, C) \simeq \mathrm{Fun}(*, C) \cong C.$$

Morally, the reason for this is that $\mathscr{P}(X)$ is "freely generated by colimits" out of the simplicial set X. The left adjoint functors preserve these colimits, so they are determined by where the generators of $\mathscr{P}(X)$ go. And the generators of $\mathscr{P}(X)$ are just the points in X.

10.4.4 Pointed and stable ∞-categories

We have finally given the universal property of the ∞-category of spaces $\mathscr{S}$: it is the free presentable ∞-category on the one-point space $*$. It turns out that based spaces $\mathscr{S}_*$ and spectra Sp have similar universal properties. We just have to change the setting and add a little more structure to our ∞-categories first.

Definition 10.4.38. [Lur17, 1.1.1.1] A **pointed ∞-category** is a quasicategory C that has a **zero object** $0 \in C$, in other words an object that is both initial and terminal. Equivalently, C has an initial object $\varnothing \in C_0$ and a terminal object $* \in C_0$, and the canonical map $\varnothing \to *$ is an isomorphism in C.

[13] Don't worry. We won't develop $(\infty,2)$-categories in this chapter, because we won't need to. We only need the mapping $(\infty,1)$-categories $\mathrm{LFun}(C,D)$ to state all of the universal properties that we want.

Example 10.4.39. [Lur09, 7.2.2.1] For any quasicategory C with a final object $*$, we can let the quasicategory C_* denote the subcategory of $\mathrm{Fun}(\Delta[1], C)$ consisting of those morphisms in which the source is a final object of C (not necessarily the chosen final object $*$). Then C_* is a pointed ∞-category. If C happens to be pointed, then the forgetful map $C_* \to C$ is a Joyal equivalence.

For instance, doing this to $\mathscr{S} = N^s(\mathbf{Kan})$ gives us a quasicategory that is equivalent (though not isomorphic) to $N^s(\mathbf{Kan}_*)$. So by a slight abuse of notation, we will use $\mathscr{S}_*$ to refer to both $N^s(\mathbf{Kan})_*$ and $N^s(\mathbf{Kan}_*)$.

Lemma 10.4.40. *There is a canonical equivalence* $\mathrm{Fun}(C, D)_* \simeq \mathrm{Fun}(C, D_*)$.

This follows because both sides are identified with the same full subcategory of $\mathrm{Fun}(C \times \Delta[1], D)$, using Lemma 10.3.38 to see that the final object in $\mathrm{Fun}(C, D)$ is the constant functor to the final object $* \in D$.

Example 10.4.41. We can modify the presheaf category $\mathscr{P}(K)$ from Example 10.1.56 to be presheaves of pointed spaces, $\mathrm{Fun}(K^{\mathrm{op}}, \mathscr{S}_*)$. By Lemma 10.4.40, this is equivalent to $\mathscr{P}(K)_*$, in other words presheaves of unbased spaces with a map from the constant presheaf at $*$. This is the ∞-category underlying the model category of "unbased diagrams of based spaces" that we considered all the way back in Lemma 5.3.43.

We can now give the universal property for based spaces, proven in Exercise 10.8.17. Notice that left adjoint functors automatically preserve the zero object, so they are always "pointed" functors.

Proposition 10.4.42. *For any small simplicial set X and pointed presentable ∞-category C, there is a natural Joyal equivalence*

$$\mathrm{LFun}(\mathscr{P}(X)_*, C) \simeq \mathrm{Fun}(X, C).$$

Corollary 10.4.43. *The ∞-category of based spaces $\mathscr{S}_*$ is the free presentable pointed ∞-category on a single object S^0. In other words, for every pointed ∞-category C, evaluation at $S^0 \in \mathscr{S}_*$ gives a natural equivalence of quasicategories*

$$\mathrm{LFun}(\mathscr{S}_*, C) \simeq C.$$

Now we can move to the setting where Sp has a universal property.

Definition 10.4.44. [Lur17, 1.1.1.9 and 1.1.3.4], [BGH22, Def 4.1.1] A **stable ∞-category** is a quasicategory C that

- has a zero object $0 \in C$, so C is pointed,
- has all finite limits and colimits, and

- a commuting square $\Delta[1] \times \Delta[1] \to C$ is a pushout iff it is a pullback.

In any stable ∞-category C, we can define the cofiber of any map $x \to y$ as the pushout in a square of the form

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & z, \end{array} \tag{10.4.45}$$

and similarly define the fiber of any map $y \to z$ as the pullback in any square of this form. As a special case, we can define the suspension Σx as the pushout $0 \cup_x 0$, and the loop space Ωx as the pullback $0 \times_x 0$.

Lemma 10.4.46. *In any stable ∞-category C, these define equivalences of quasicategories $\Sigma\colon C \to C$ and $\Omega\colon C \to C$ that are inverses up to isomorphism.*

Example 10.4.47. Of course, the ∞-category of spectra Sp is the primary example of a stable ∞-category. But there are many others: the underlying ∞-category of modules over a ring spectrum $\mathrm{Mod}(R)$ is another example, as is the underlying ∞-category of unbounded chain complexes over a ring $\mathbf{Ch}(R)$.

There are now two ways to state the universal property: we could either use presheaves valued in spectra, following the pattern of the previous examples, or we could take a new approach and form a category of "spectrum objects" $\mathrm{Sp}(C)$ in any pointed quasicategory C with all finite limits. The first approach looks like this:

Proposition 10.4.48. *[Lur09, 1.4.2.9 and 1.4.4.5] For any small simplicial set X and stable ∞-category C, there is a natural Joyal equivalence*

$$\mathrm{LFun}(\mathrm{Fun}(X^{\mathrm{op}}, \mathrm{Sp}), C) \simeq \mathrm{Fun}(X, C).$$

The second approach looks like this:

Definition 10.4.49. [Lur17, §1.4] Let S be a pointed quasicategory with all finite colimits, and let C be a quasicategory with all finite limits.

- Let $\mathrm{Exc}_*(S, C) \subseteq \mathrm{Fun}(S, C)$ be the full subcategory of all functors that are **reduced** (sending the zero object to the zero object) and **excisive** (sending pushout squares in S to pullback squares in C).
- Let $\mathscr{S}^{\mathrm{fin}} \subseteq \mathscr{S}$ be the full subcategory on the finite CW complexes, or equivalently the simplicial sets with finitely many nondegenerate vertices in total.
- The **stabilization** of C, or the category of **spectrum objects** of C, is the quasicategory

$$\mathrm{Sp}(C) := \mathrm{Exc}_*(\mathscr{S}^{\mathrm{fin}}, C).$$

So a spectrum object in C is a reduced excisive functor from finite CW complexes to C. We saw the model category version of this back in Section 6.4.3. As was the case then, you can recover the usual notion of "spectrum" out of this by evaluating the functor $X : \mathscr{S}^{\text{fin}} \to C$ on the spheres S^n. This produces a sequence of spaces $X(S^n)$, which turn out to have bonding maps $\Sigma X(S^n) \to X(S^{n+1})$ that are well-defined up to homotopy. So we get a (somewhat weaker) version of the usual definition of a spectrum.

Lemma 10.4.50. *If C has all finite limits then $\mathrm{Sp}(C)$ is a stable ∞-category, with a map $\Omega^\infty : \mathrm{Sp}(C) \to C$ that evaluates each functor at S^0. Furthermore C is stable iff this map is an equivalence.*

Lemma 10.4.51. *We have*

$$\mathrm{Sp}(\mathscr{S}) \simeq \mathrm{Sp}(\mathscr{S}_*) \simeq \mathrm{Sp}(\mathrm{Sp}) \simeq \mathrm{Sp}.$$

So all three of these ∞-categories stabilize to make the ∞-category of spectra.

Proposition 10.4.52. *[Lur17, 1.4.4.5] For presentable ∞-category C and presentable stable ∞-category D, there is a natural Joyal equivalence*

$$\mathrm{LFun}(\mathrm{Sp}(C), D) \simeq \mathrm{LFun}(C, D).$$

At the end, both approaches give the corollary that we want:

Corollary 10.4.53. *[Lur17, 1.4.4.6] The ∞-category of spectra Sp is the free presentable stable ∞-category on a single object $\mathbb{S}$. In other words, for every presentable stable ∞-category C, evaluation at $\mathbb{S} \in \mathrm{Sp}$ gives a natural equivalence of quasicategories*

$$\mathrm{LFun}(\mathrm{Sp}, C) \simeq C.$$

10.4.5 Bousfield localization of an ∞-category

There is a special kind of Dwyer–Kan localization called a Bousfield localization, or a reflective localization. It has the additional property that the functor $C \to C[W^{-1}]$ has a fully faithful right adjoint. We introduce these here because it is much easier to work with them when C is presentable.

Definition 10.4.54. A (left) **Bousfield localization**, also called a **reflective localization**, is a Dwyer–Kan localization $L : C \to D \simeq C[W^{-1}]$ that has a fully faithful right adjoint $R : D \to C$.

So this is a special kind of localization in which the right adjoint puts the localized category D inside the original category C, and then L replaces each object $x \in C_0$ by a "nearest" object $Lx \in D_0$. Just like when we took the Bousfield localization of a model category in Definition 5.7.1, we think of Lx as the universal approximation of x by an object in this subcategory.

Lemma 10.4.55. *[Lan21, 5.1.19 and 5.1.23] For L to be a Bousfield localization, it suffices to check that either*

- *$L : C \to D$ is a Dwyer–Kan localization and its right adjoint R exists, or*
- *$L : C \to D$ is a functor with a fully faithful right adjoint R.*

The version of this for 1-categories already appeared in Exercise 3.6.21.

We also have the following characterization. It is the ∞-version of the argument in Proposition 3.5.2 and Exercise 9.6.26, and has a similar proof:

Lemma 10.4.56. *[Lur09, 5.2.7.4(3)] A functor $L : C \to C$ defines a Bousfield localization onto its image in C iff there exists a natural tranformation $\alpha : \mathrm{id} \to L$ such that for each object $x \in C_0$ the two morphisms $\alpha_{Lx}, L(\alpha_x) : Lx \rightrightarrows LLx$ are isomorphisms in C.*

Bousfield localizations are incredibly common, so much that some authors take "localization" to exclusively mean Bousfield localization (for example [Lur09, Lur17]). So watch out!

Example 10.4.57. Every left Bousfield localization of a model category $(\mathbf{C}, C, V, F)$ (see Definition 5.7.1) induces a Bousfield localization on the associated ∞-category

$$N(\mathbf{C})[V^{-1}] \to N(\mathbf{C})[W^{-1}].$$

This follows from Lemma 10.4.55 since the right adjoint exists.

So for example if we take the ∞-category of spectra and localize at a homology theory E, the resulting map of ∞-categories $\mathrm{Sp} \to \mathrm{Sp}[W_E^{-1}]$ is a Bousfield localization. Its fully faithful right adjoint includes the subcategory of E-local spectra into all spectra. This is the same situation that we saw back in Corollary 3.3.10, except we are now talking about the ∞*-category* of spectra, not just the homotopy category of spectra.

Example 10.4.58. The homotopical group completion of Definition 9.1.11 gives a Bousfield localization of ∞-categories

$$N(\mathrm{Mon}(\mathbf{Top}))[W^{-1}] \to N(\mathrm{Grp}(\mathbf{Top}))[W^{-1}].$$

The right adjoint is the inclusion of topological groups back into topological monoids.

Example 10.4.59. Passing from a space or spectrum to the nth stage of its Postnikov tower is an example of a Bousfield localization. The class of weak equivalences in this case is all maps $X \to Y$ such that $P_n X \to P_n Y$ is an equivalence, in other words all maps that induce isomorphisms in π_k for $k \leq n$. See Example 10.4.70 for more details.

Example 10.4.60. If $\mathbf{C}$ is a model category with weak equivalences W, the Dwyer–Kan localization $N(\mathbf{C}) \to N(\mathbf{C})[W^{-1}]$ is usually *not* a Bousfield localization. If it were, then every colimit in $\mathbf{C}$ would be equivalent to the corresponding homotopy colimit, and this is usually false.

10.4.6 Constructing Bousfield localizations

We want a procedure for constructing Bousfield localizations in a quasicategory C, or in a model category $\mathbf{C}$. The idea is to take some starting set of weak equivalences S_0 that we want to invert. We can't just Dwyer–Kan localize at S_0, because this usually doesn't give a Bousfield localization. Rather, we use S_0 to build some larger collection $S = \overline{S_0}$ that is "generated" by S_0, such that $C \to C[S^{-1}]$ is a Bousfield localization. This procedure will work any time that C is a presentable ∞-category, or when $\mathbf{C}$ is a well-behaved model category.

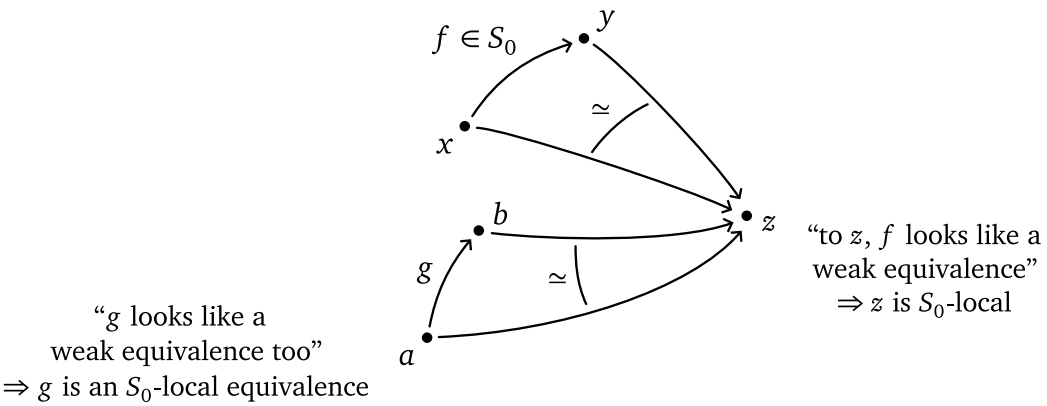
One way to go from the set S_0 to the class $S = \overline{S_0}$ is to take the "S_0-local equivalences" in the following sense.

Definition 10.4.61. [Lur09, 5.5.4.1] If C is a quasicategory and $S_0 \subseteq C_1$ is any collection of maps in C, we say that

- an object $z \in C_0$ is **S_0-local** if every morphism $x \to y$ in S_0 induces a weak equivalence of spaces $\mathrm{Map}_C(y,z) \to \mathrm{Map}_C(x,z)$.[14]
- a morphism $a \to b$ is an **S_0-local equivalence** if for every S_0-local object $z \in C_0$, the map induces a weak equivalence of spaces $\mathrm{Map}_C(b,z) \to \mathrm{Map}_C(a,z)$.

If $\mathbf{C}$ is a model category and S_0 is any collection of maps in $\mathbf{C}$, we define S_0-local objects and S_0-local equivalences in $\mathbf{C}$ in the same way, using the mapping spaces in the underlying ∞-category $N(\mathbf{C})[W^{-1}]$. (So when $\mathbf{C}$ is a simplicial model category, we can use the simplicial mapping spaces $\mathbf{C}_\bullet(-,-)$, provided we make the source cofibrant and the target fibrant.)

Intuitively, an S_0-local object is an object that "sees the maps in S_0 as equivalences," and an S_0-local equivalence is a map that "is an equivalence from the point of view of any S_0-local object," as shown below.



[14] We use the functorial mapping space from Definition 10.3.3 to define the action of $x \to y$ on $\mathrm{Map}_C(-,z)$.

Let $S = \overline{S_0}$ be the class of S_0-local equivalences. This turns out to give the same class of S-local objects and S-local equivalences, so we think of this as the "closure" of S_0 under the process of going back and forth between maps and objects.

The other way to go from the set S_0 to the class $S = \overline{S_0}$ is to take the closure under the following operations.

Definition 10.4.62. [Lur09, 5.5.4.5] A collection of morphisms $S \subseteq C_1$ in the quasi-category C is **strongly saturated** if it is closed under pushouts, homotopy colimits, and 2-out-of-3. So any pushout of a morphism in S is in S, any homotopy colimit in $\mathrm{Fun}(\Delta[1], C)$ of morphisms in S is in S, and if h is a composition of g and f, then when any two of them are in S, so is the third.

The intersection of any collection of strongly saturated classes is again strongly saturated, so for any collection of morphisms S_0, there is a smallest collection $S = \overline{S_0}$ that is strongly saturated and contains S_0. Intuitively, it's all of the morphisms that we can build from S_0 using 2-out-of-3, pushouts, and homotopy colimits. At a minimum, this always includes all of the weak equivalences in C.

Proposition 10.4.63. *[Lur09, 5.5.4.15]*

For any presentable ∞-category C and any small set of morphisms $S_0 \subseteq C_1$, there is a Bousfield localization $C \to C[S^{-1}]$ in which

- *the weak equivalences $S = \overline{S_0}$ are precisely the S_0-local equivalences, and this also coincides with the smallest strongly saturated class containing S_0,*
- *the right adjoint includes the full subcategory $C' \subseteq C$ of S_0-local objects (or equivalently the S-local objects), and*
- *the category $C' \simeq C[S^{-1}]$ is presentable.*

*An **accessible localization** is a Bousfield localization constructed in this way out of some set of morphisms S_0 (as opposed to a proper class of morphisms).*[15]

Now we can make a huge number of examples of Bousfield localizations, and also a huge number of examples of presentable ∞-categories. All you need is a presentable ∞-category to start (such as $\mathscr{P}(K) = \mathrm{Fun}(K^{\mathrm{op}}, \mathscr{S})$), take *any set of morphisms*, and then pass to the subcategory of local objects, and you have another presentable ∞-category!

[15] More generally, we say a functor is accessible if it preserves κ-filtered colimits for some regular cardinal κ [Lur09, 5.4.2.5 and 5.3.4.5]. An accessible localization is a reflective localization in which the left adjoint is accessible in this sense.

Example 10.4.64. In $\mathscr{S}$, $\mathscr{S}_*$, and Sp, for any homology theory E_*, the class of E_*-isomorphisms is strongly saturated. This is why we called this the class of E-local equivalences, because it's unchanged under the closure procedure from either Definition 10.4.61 or Definition 10.4.62.

There is also a model category version of Proposition 10.4.63. Instead of using presentable ∞-categories, we use combinatorial model categories.

Theorem 10.4.65. *[Bar10, Prop 2.2] If $(\mathbf{C}, V)$ is any left proper combinatorial model category with weak equivalences V, and if S is the class of S_0-local equivalences for some small set S_0 of morphisms in $\mathbf{C}$, then the left Bousfield localization model structure $(\mathbf{C}, S)$ is guaranteed to exist, and it is again left proper and combinatorial. If in addition $(\mathbf{C}, V)$ is a simplicial model category then so is $(\mathbf{C}, S)$.*

As in Section 5.7, the fibrant objects in this model structure are the fibrant objects in the original model structure that are also S_0-local. And the fibrant replacement in the new model structure is the universal approximation of each object X by an S_0-local object RX, up to V-equivalence.

There is also another class of "cellular" model categories where this theorem also holds. This is useful because some familiar examples like **Top** fail to be combinatorial, but they are still cellular, and the statement of the theorem is not invariant under Quillen equivalences. (It's asserting the existence of a model structure on **C** itself, not on some model category Quillen equivalent to **C**.)

Definition 10.4.66. A model category **C** is **cellular** if

- it is cofibrantly generated by sets of morphisms I and J,
- the domains and codomains of I and the domains of J are small relative to I (in the sense of Exercise 5.8.42), and
- every cofibration $X \to Y$ is an effective monomorphism, i.e. it is the equalizer of the two maps $Y \to Y \cup_X Y$.

Every cofibrantly generated model category that we have seen in this book satisfies this condition, including both **sSet** and **Top**, and all of our model categories of spectra.

Theorem 10.4.67. *[Hir03, Thm 4.1.1] Theorem 10.4.65 applies to cellular model categories as well.*

In other words, if $(\mathbf{C}, V)$ is a left proper cellular model category with weak equivalences V, and if S is the class of S_0-local equivalences for some small set S_0 of morphisms in $\mathbf{C}$, then the left Bousfield localization model structure $(\mathbf{C}, S)$ is guaranteed to exist, and it is again left proper and cellular. If in addition $(\mathbf{C}, V)$ is a simplicial model category then so is $(\mathbf{C}, S)$.

Example 10.4.68. This recovers Theorem 5.7.7, which says that each of the model categories **Top**, $\mathbf{Top}_*$, and $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}}$ has a left Bousfield localization along the E_*-isomorphisms, for any extraordinary homology theory E_*. (We checked that the E_*-isomorphisms are generated by some set of morphisms S_0 in Lemma 5.7.11.) And it extends that theorem to **sSet**, $\mathbf{sSet}_*$, and all of our other model categories of spectra.

Example 10.4.69. Consider the set of maps $S_0 = \{S^{n-1} \to D^n\}_{n\geq 0}$ in **Top**, or equivalently in the ∞-category of spaces $\mathscr{S}$. Then the S_0-local objects are the weakly contractible spaces, S_0-local equivalences $S = \overline{S_0}$ consists of every morphism in **Top**. The localization $\mathscr{S}[S^{-1}]$ is equivalent to the one-point ∞-category $*$. Or in other words, the model category localization $(\mathbf{Top}, S)$ is Quillen equivalent to the one-point model category $*$.

Example 10.4.70. In the previous example, if we fix an integer $k \geq 0$ and let $S_0 = \{S^{n-1} \to D^n\}_{n\geq k}$, then the S_0-local objects are the k-truncated spaces, which have $\pi_i = 0$ for all $i > k$. The S_0-local equivalences are the maps that are isomorphisms on π_i for all $i > k$. The localization $\mathscr{S}[S^{-1}]$ is equivalent to the subcategory of k-truncated spaces $\tau_{\leq k}\mathscr{S}$, and the localization functor $\mathscr{S} \to \tau_{\leq k}\mathscr{S}$ sends every space X to its Postnikov section $P_k X$.

10.4.7 Simpson's theorem

In the previous section we saw that we can construct a huge number of examples of presentable ∞-categories, by starting with any presheaf category $\mathscr{P}(K) = \mathrm{Fun}(K^{\mathrm{op}}, \mathscr{S})$, any set of morphisms S_0 in this category, and passing to the full subcategory of S_0-local objects. It turns out that every presentable ∞-category arises this way!

Theorem 10.4.71 (Simpson)**.** *[Lur09, 5.5.1.1] A quasicategory C is presentable iff it is a Bousfield localization of a presheaf category $\mathscr{P}(K) = \mathrm{Fun}(K^{\mathrm{op}}, \mathscr{S})$ along a class of morphisms of the form $\overline{S_0}$, where S_0 is a set (not a proper class) of morphisms.*

This is also sometimes called a "Giraud theorem" because it is analogous to a result of Giraud, as explained in [Lur09, §6.1].

Corollary 10.4.72. *If C is presentable and $C \to C[S^{-1}]$ is any Bousfield localization along $S = \overline{S_0}$ where S_0 is a set of morphisms, then $C[S^{-1}]$ is also presentable.*

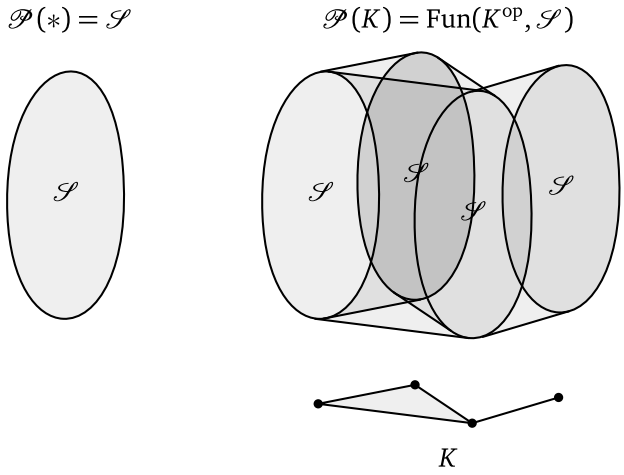
The idea of the proof is that if C is presentable and K is any chosen small full colimit-dense subcategory of C, then the Yoneda embedding and the restriction to K give a functor

$$C \longrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathscr{S}) \longrightarrow \mathrm{Fun}(K^{\mathrm{op}}, \mathscr{S}),$$

which turns out to be fully faithful, and is the inclusion of a subcategory of local objects. In other words, we can represent objects of C as diagrams because the maps out of each object in K are enough to "see" everything about each object of C.

Simpson's theorem goes some way towards explaining why we call these "presentable" ∞-categories. Every one of them is of the form $\mathscr{P}(K)[S^{-1}]$. Think of K as the "generators," and S as the "relations." By the universal properties in Proposition 10.4.36 and Definition 10.2.2, a left adjoint functor $\mathscr{P}(K)[S^{-1}] \to D$ is the same thing as a functor $K \to D$, whose induced functor $\mathscr{P}(K) \to D$ sends every map in S to an isomorphism in D. This is just like how the presentation of a group G describes all possible homomorphisms from G to any other group H.[16]

Remark 10.4.73. Another intuition about presentable ∞-categories is that they are "retracts of cell complexes, in which the cells are copies of the ∞-category of spaces $\mathscr{S}$." We glue together copies of $\mathscr{S}$, one for each vertex in K, to make the category $\mathrm{Fun}(K^{\mathrm{op}}, \mathscr{S})$. Then the localization $\mathscr{P}(K)[S^{-1}]$ is a "retract" of this cell complex, because it is both a subcategory and a quotient of $\mathscr{P}(K)$. Presentable ∞-categories inherit many of the good properties of $\mathscr{S}$ simply because they are made out of several copies of $\mathscr{S}$!



Example 10.4.74. The ∞-category of spaces $\mathscr{S}$ is presentable, so it can be built out of copies of $\mathscr{S}$ in this way. We only need one copy! More precisely, $\mathscr{S}$ is the trivial localization of presheaves on a point, $\mathscr{P}(*) = \mathrm{Fun}(*, \mathscr{S}) \cong \mathscr{S}$.

Example 10.4.75. Since pointed spaces form a presentable ∞-category $\mathscr{S}_*$, they must be an accessible localization of some presheaf category. To see this, let $\mathbf{I}$ be the 1-category with two objects 0 and 1, morphisms $f : 0 \to 1$ and $g : 1 \to 0$, and the one relation $gf = \mathrm{id}$. So an $\mathbf{I}$-diagram is a choice of two objects, one of which is a retract of the other, and $\mathbf{I} \cong \mathbf{I}^{\mathrm{op}}$. Let $\mathrm{Fun}(N(\mathscr{I})^{\mathrm{op}}, \mathscr{S})$ be the presheaf category, and let us localize it at the set of morphisms of the form $(S^{n-1} = S^{n-1}) \to (D^n = D^n)$, where the two diagrams are constant diagrams but the map between them is given by the inclusion $S^{n-1} \to D^n$. Then this localization is equivalent to the ∞-category of based spaces $\mathscr{S}_*$, by Exercise 10.8.19.

[16] This paragraph is based on an excellent MathOverflow post by Charles Rezk [Rez19].

Proposition 10.4.76. *[Lur17, 1.4.4.9] Simpson's theorem also holds for pointed presentable ∞-categories: they are all accessible localizations of* $\mathscr{P}(K)_* \simeq \mathrm{Fun}(K^{\mathrm{op}}, \mathscr{S}_*)$.

And every stable presentable ∞-category is an accessible localization of $\mathrm{Fun}(K^{\mathrm{op}}, \mathrm{Sp})$.

In fact, the quoted reference only proves this for stable ∞-categories, but an easier variant of the argument works for pointed ∞-categories as well. See Exercise 10.8.18.

Example 10.4.77. Since spectra are a stable presentable ∞-category Sp, they can be built out of presheaves on Sp, $\mathscr{S}_*$, or $\mathscr{S}$. It's easy to build them out of Sp, they're just $\mathrm{Fun}(*, \mathrm{Sp})$!

To build them out of based spaces, we take diagrams on the sphere category **S** from Definition 5.3.34, and then invert all of the stable equivalences in Definition 5.6.3. This gives an ∞-category, modeled by the model category in which we have taken the left Bousfield localization of the level model structure that inverts these maps. Inverting these maps is the same as inverting all stable equivalences, which we see using the stable model structure from Theorem 5.2.12. Therefore when we do this localization at the ∞-categorical level we get Sp!

Finally, we could build Sp out of copies of unbased spaces $\mathscr{S}$, by combining the above construction with the one from Example 10.4.75.

Example 10.4.78. We can now give a concrete topological model of every presentable ∞-category. Let **I** be any small topological category and let $\mathbf{Top}^{\mathbf{I}}$ be the model category of **I**-diagrams of spaces, with V the class of level equivalences. Then $(\mathbf{Top}^{\mathbf{I}}, V)$ is a cellular model category, so we may take any set of morphisms S_0 in $\mathbf{Top}^{\mathbf{I}}$, and form the left Bousfield localization $(\mathbf{Top}^{\mathbf{I}}, S)$ where $S = \overline{S_0}$. The model category $(\mathbf{Top}^{\mathbf{I}}, S)$ models a presentable ∞-category, and *every* presentable ∞-category arises in this way! (We could also take the subcategory of S_0-local diagrams in $\mathbf{Top}^{\mathbf{I}}$ and invert the level equivalences V, and this gives the same ∞-category.)

Example 10.4.79. Equivalently, we could let **I** be any small simplicially enriched category, let $\mathbf{sSet}^{\mathbf{I}}$ be the simplicial category of diagrams $\mathbf{I} \to \mathbf{sSet}$, and again let $S = \overline{S_0}$ be any class of weak equivalences that arises from a set of morphisms S_0. Then we get a model category $(\mathbf{sSet}^{\mathbf{I}}, S)$ that models a presentable ∞-category, and every presentable ∞-category arises this way.

Example 10.4.80. In the same pattern as the previous examples, every pointed presentable ∞-category can be modeled as $(\mathbf{Top}^{\mathbf{I}}_*, S)$, where $\mathbf{Top}^{\mathbf{I}}_*$ is a category of diagrams of based spaces indexed by an unbased topological category **I**, and S is again some class of S_0-local equivalences.

Similarly, every stable presentable ∞-category can be modeled as $(\mathbf{Sp}^{\mathbf{I}}, S)$, where $\mathbf{Sp}^{\mathbf{I}}$ is a category of diagrams of spectra indexed by an unbased topological category **I** (and we can use any model of spectra we want), and S is again some class of S_0-local equivalences.

In conclusion, every "homotopy theory" is a Bousfield localization of the homotopy theory of diagrams of spaces $\mathbf{Top}^{\mathbf{I}}$. So Section 5.3 was actually a lot more universal than it first appeared to be! If you want to understand every "homotopy theory," you only have to understand diagrams of spaces.

10.5 The Lurie tensor product

This is an important concept that we wish to highlight it in its own section. It is something that really exists purely in the world of ∞-categories, and is helpful for getting a more global perspective on what the ∞-categories of spaces $\mathscr{S}$ and spectra Sp are really doing.

The idea is that there is a special kind of product on presentable ∞-categories, called the Lurie tensor product. Here are two intuitions for it:

- If an ∞-category is like a set, then a presentable ∞-category is like an abelian group. (The "sums" are the colimits.) The Lurie tensor product $C \otimes D$ is like the tensor product of abelian groups, because it is the universal ∞-category that receives a "bilinear" or "biexact" map from C and D (preserving colimits in each slot).
- A presentable ∞-category is like a "retract of a cell complex" in which the cells are copies of the ∞-category of spaces $\mathscr{S}$. (We mentioned this in Remark 10.4.73. The copies of $\mathscr{S}$ are like the copies of $\mathbb{Z}$ that appear in the presentation of an abelian group.) From this point of view, the "sum" of two presentable ∞-categories C and D is their Cartesian product $C \times D$, because it has a cell for every cell in C *or* in D. The "tensor" of C and D is the Lurie tensor product $C \otimes D$, which has a cell for every *pair* of a cell in C and a cell in D.

10.5.1 Motivation and definition

First let's think about what the "sum" and "tensor" should be in ordinary ∞-categories. Given two quasicategories C and D, their "sum" is the coproduct $C \amalg D$. This is the sum because a functor out of $C \amalg D$ is a product of two functors:

$$\mathrm{Fun}(C \amalg D, E) \cong \mathrm{Fun}(C, E) \times \mathrm{Fun}(D, E).$$

Similarly the "tensor" is the cartesian product $C \times D$, because a functor out of $C \times D$ is a functor from C into functors out of D:

$$\mathrm{Fun}(C \times D, E) \cong \mathrm{Fun}(C, \mathrm{Fun}(D, E)) \cong \mathrm{Fun}(D, \mathrm{Fun}(C, E)).$$

In other words, a functor out of $C \times D$ is a "bifunctor" out of C and D.

What happens when we move to the setting of presentable ∞-categories, and functors that preserve colimits? Then the sum and tensor change! The new "sum" is now the product category $C \times D$, because we have the same identity as before but with right-exact functors:

$$\mathrm{LFun}(C \times D, E) \cong \mathrm{LFun}(C, E) \times \mathrm{LFun}(D, E).$$

For this we recall from Definition 10.4.35 that $\mathrm{LFun}(C, D) \subseteq \mathrm{Fun}(C, D)$ is the full subcategory of colimit-preserving functors, equivalently left-adjoint functors, from C to D. We can similarly let $\mathrm{RFun}(C, D)$ be the full subcategory of right-adjoint functors. We can prove the above identity by the manipulations

$$\begin{aligned}\mathrm{LFun}(C \times D, E) &\simeq \mathrm{RFun}(E, C \times D) \\ &\simeq \mathrm{RFun}(E, C) \times \mathrm{RFun}(E, D) \\ &\simeq \mathrm{LFun}(C, E) \times \mathrm{LFun}(D, E),\end{aligned}$$

which are justified by the following lemma.

Lemma 10.5.1. *[Lur09, 5.5.3.4] When C and D are presentable ∞-categories, there is a Joyal equivalence of quasicategories $\mathrm{LFun}(C, E) \simeq \mathrm{RFun}(E, C)$, which is natural in C and E.*

At any rate, now that we understand the sum, we can move on to the tensor. The new "tensor" is called the Lurie tensor product $C \otimes D$.

Definition 10.5.2. For any two presentable ∞-categories C and D, the **Lurie tensor product** is the presentable ∞-category $C \otimes D$ with the defining property that there is a Joyal equivalence

$$\mathrm{LFun}(C \otimes D, E) \simeq \mathrm{LFun}(C, \mathrm{LFun}(D, E)) \cong \mathrm{LFun}(D, \mathrm{LFun}(C, E))$$

for any presentable ∞-category E, that is natural in E. The presentable ∞-category $C \otimes D$ with this property exists by [Lur17, 4.8.1.3 and 4.8.1.15].

This is the same universal property as before, only now we're using left adjoint functors, instead of all functors. So the ∞-category $\mathrm{LFun}(C, \mathrm{LFun}(D, E))$ describes all functors $C \times D \to E$ *that preserve colimits in each slot*. As a functor of E, this is representable by $C \otimes D$, so there is a bifunctor $C \times D \to C \otimes D$ preserving colimits in each slot, and this is universal among all such bifunctors. This is why it makes sense to call this a "tensor product" of C and D. It distributes over addition in each slot, which lines up perfectly with how we think of a "multiplication" in ordinary algebra.

Remark 10.5.3. Think by analogy with the category of sets **Set** and the category of abelian groups **Ab**. A sum in **Set** is a disjoint union, and a tensor in **Set** is a cartesian product, because this is the universal operation on two sets that preserves disjoint unions in each

slot. When we move to **Ab**, a sum is now a cartesian product – it can't be the disjoint union $A \amalg B$, because that isn't even a group. You have to join the zero elements together, and then close the result under sums, and that gives you the product group $A \oplus B \cong A \times B$. Similarly, a tensor is the usual tensor product $A \otimes B$, which by definition is the universal group receiving a bilinear map from $A \times B$.

Remark 10.5.4. Lurie's description of this tensor product is completely steeped in the language of symmetric monoidal ∞-categories, so we recommend reading the rest of this chapter before looking at [Lur17, 4.8.1.3].

We also have the following less-symmetric description of $C \otimes D$:

Lemma 10.5.5. *[Lur17, 4.8.1.17] For presentable ∞-categories C and D we have*

$$C \otimes D \simeq \mathrm{RFun}(C^{\mathrm{op}}, D).$$

This makes sense even though C^{op} is not presentable, because we can still define RFun to be the category of right-adjoint functors.

10.5.2 Calculating the Lurie tensor product

Example 10.5.6. If $\mathscr{S}$ is the ∞-category of spaces, then

$$\begin{aligned}\mathrm{LFun}(\mathscr{S} \otimes \mathscr{S}, E) &\simeq \mathrm{LFun}(\mathscr{S}, \mathrm{LFun}(\mathscr{S}, E)) \\ &\simeq \mathrm{LFun}(\mathscr{S}, E)\end{aligned}$$

using Corollary 10.4.37. Therefore by the Yoneda lemma we have

$$\mathscr{S} \otimes \mathscr{S} \simeq \mathscr{S}.$$

So the Lurie tensor product of $\mathscr{S}$ with itself is $\mathscr{S}$ again! The resulting biexact functor

$$\mathscr{S} \times \mathscr{S} \longrightarrow \mathscr{S}$$

sends $(*, *) \mapsto *$. It follows that it must send each pair of spaces (X, Y) to the cartesian product space $X \times Y$. This is the universal biexact functor out of $\mathscr{S} \times \mathscr{S}$!

Example 10.5.7. More generally, for any presentable ∞-category C, we have $\mathscr{S} \otimes C \simeq C$. The resulting biexact functor $\mathscr{S} \times C \to C$ is the tensor product of objects of C with spaces from Lemma 10.4.34 [Lur17, 4.8.1.20].

Example 10.5.8. The tensor of two presheaf categories is a presheaf category:

$$\begin{aligned}\mathrm{LFun}(\mathscr{P}(K)\otimes\mathscr{P}(L),E) &\simeq \mathrm{LFun}(\mathscr{P}(K),\mathrm{LFun}(\mathscr{P}(L),E))\\ &\simeq \mathrm{Fun}(K,\mathrm{LFun}(\mathscr{P}(L),E))\\ &\simeq \mathrm{Fun}(K,\mathrm{Fun}(L,E))\\ &\cong \mathrm{Fun}(K\times L,E)\\ &\simeq \mathrm{LFun}(\mathscr{P}(K\times L),E)\\ \Rightarrow\ \mathscr{P}(K)\otimes\mathscr{P}(L) &\simeq \mathscr{P}(K\times L).\end{aligned}$$

This corresponds to the intuition that $\mathscr{P}(K)$ is a "cell complex" of ∞-categories with one "cell" $\mathscr{S}$ for each nondegenerate simplex in K. The tensor of these cell complexes should then be a cell complex with cells given by the product simplicial set $K\times L$!

Example 10.5.9. The tensor of two accessible Bousfield localizations $C\to C[S^{-1}]$ and $D\to D[T^{-1}]$ is again an accessible Bousfield localization:

$$\begin{aligned}\mathrm{LFun}(C[S^{-1}]\otimes D[T^{-1}],E) &\simeq \mathrm{LFun}(C[S^{-1}],\mathrm{LFun}(D[T^{-1}],E))\\ &\simeq \mathrm{LFun}(C,\mathrm{LFun}(D[T^{-1}],E))|_{S\times D^{\simeq}\mapsto E^{\simeq}}\\ &\simeq \mathrm{LFun}(C,\mathrm{LFun}(D,E))|_{(S\times D^{\simeq})\cup(C^{\simeq}\times T)\mapsto E^{\simeq}}\\ &\simeq \mathrm{LFun}(C\otimes D,E)|_{(S\times D^{\simeq})\cup(C^{\simeq}\times T)\mapsto E^{\simeq}}\\ &\simeq \mathrm{LFun}((C\otimes D)[(S\times D^{\simeq})\cup(C^{\simeq}\times T)],E)\\ \Rightarrow\ C[S^{-1}]\otimes D[T^{-1}] &\simeq (C\otimes D)[(S\times D^{\simeq})\cup(C^{\simeq}\times T)].\end{aligned}$$

In other words, we localize $C\otimes D$ at all morphisms that are the images of morphisms in $C\times D$ of the form (f,g), where either $f\in S$ and g is an isomorphism, or f is an isomorphism and $g\in T$. This is a "pushout-product" of the two classes of morphisms that we invert. And it is generated by the sets $S_0\times D'^{\simeq}$ and $C'^{\simeq}\times T_0$, where S_0 and T_0 are the sets that generate S and T, respectively, and C' and D' are the small subcategories that generate C and D, respectively. So this really is still an accessible localization.

Example 10.5.10. The tensor of based spaces with itself is based spaces:

$$\mathscr{S}_*\otimes\mathscr{S}_*\simeq\mathscr{S}_*$$

The resulting biexact functor $\mathscr{S}_*\times\mathscr{S}_*\to\mathscr{S}_*$ is the smash product $(X,Y)\mapsto X\wedge Y$. More generally, for any presentable ∞-category C, we have $\mathscr{S}_*\otimes C\simeq C_*$, where C_* is the category of pointed objects from Example 10.4.39 [Lur17, 4.8.1.21]. Therefore when C is pointed, we have $\mathscr{S}_*\otimes C\simeq C$.

Example 10.5.11. The tensor of spectra with itself is spectra:

$$\mathrm{Sp}\otimes\mathrm{Sp}\simeq\mathrm{Sp}$$

The resulting biexact functor $\mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ is the smash product $(X, Y) \mapsto X \wedge Y$. More generally, for any presentable ∞-category C, we have $\mathrm{Sp} \otimes C \simeq \mathrm{Sp}(C)$, where $\mathrm{Sp}(C)$ is the stabilization from Definition 10.4.49 [Lur17, 4.8.1.23]. Therefore when C is stable, we have $\mathrm{Sp} \otimes C \simeq C$.

Remark 10.5.12. If we think of $\mathscr{S}$ as analogous to the abelian group $\mathbb{Z}$, then $\mathscr{S}_*$ and Sp act like the abelian group $\mathbb{Q}$. We have $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$, and once an abelian group is a rational vector space, tensoring with $\mathbb{Q}$ is the same thing as tensoring with $\mathbb{Z}$, and it doesn't matter whether we do the tensor product relative to $\mathbb{Z}$ or $\mathbb{Q}$. The Lurie tensor product behaves the same way when we go from $\mathscr{S}$ to $\mathscr{S}_*$ or Sp.

In Corollary 10.7.24 we will give a universal property for the isomorphism $\mathrm{Sp} \otimes \mathrm{Sp} \simeq \mathrm{Sp}$, and therefore for the smash product functor $\mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$. But to make sense of this we have to learn what a symmetric monoidal ∞-category is, and that requires us to first learn what a "system" or "diagram" of ∞-categories is.

10.6 Systems of ∞-categories

Suppose we want to define a collection of ∞-categories $C(a)$, $C(b)$, ..., and functors between them. We could list the categories $C(a)$ and the functors $F\colon C(a) \to C(b)$ individually, but that's clearly not enough data. We need natural isomorphisms between their composites as well, and these all have to be coherent with each other somehow.

If you think about it, the right notion is to have a map $X\colon K \to \mathrm{Cat}_\infty = N^s(\mathbf{qCat})$. So K is a simplicial set, each vertex $k \in K_0$ goes to a quasicategory $X(k)$, each 1-simplex $f \in K_1$ goes to a functor $X(f)\colon X(k_0) \to X(k_1)$, each 2-simplex of K goes to a natural isomorphism $X(h) \cong X(g) \circ X(f)$, and so on.

As a special case, a diagram of spaces is a map of simplicial sets $X\colon K \to \mathscr{S} = N^s(\mathbf{Kan})$. So each vertex of K goes to a Kan complex $X(k)$, each 1-simplex of K goes to a map of Kan complexes $X(k_0) \to X(k_1)$, each 2-simplex of K goes to a homotopy, and so on.

This is called a **straightened** diagram. It's nice, but it comes with a serious problem: the terms of the diagram, in other words the spaces or ∞-categories $X(k)$, are wrapped up inside the definition of $N^s(\mathbf{qCat})$ or $N^s(\mathbf{Kan})$, and we can't access them directly. They show up as *vertices* of the simplicial set $N^s(\mathbf{qCat})$ or $N^s(\mathbf{Kan})$, so we don't directly see the simplices in the simplicial set $X(k)$. This is really inconvenient when we do constructions and manipulations with these diagrams.

It turns out there is a different definition, called an **unstraightened** diagram, that bundles everything together into a single big simplicial set E that contains the ∞-categories or spaces $X(k)$ as subsets. The big simplicial set E has a map back to K, and the map $E \to K$

is called a **cocartesian fibration**, or a **left fibration** in the case of a diagram of spaces. This is much more convenient for doing constructions and manipulations.

The two kinds of diagram are also equivalent to each other. The process of going from the functor $K \to \mathrm{Cat}_\infty$ to the cocartesian fibration $E \to K$ is called "unstraightening", and its inverse is called "straightening." So we can always jump back and forth between them. (Or, we could just spend all of our life with the unstraightened diagrams, because they're better.)

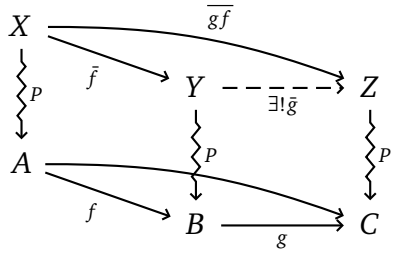
10.6.1 Cocartesian fibrations of 1-categories

It's easiest to understand straightening and unstraightening in the context of 1-categories first. The key idea is the notion of a "fibration" for 1-categories.

Recall that a Serre fibration is a map of topological spaces $p : E \to B$ such that for every path $f : I \to B$ starting at $a \in B$ and every point $x \in p^{-1}(a)$, there is a lifted path $\bar{f} : I \to E$ that starts at x.[17] When we make E and B into categories, a "path" becomes a morphism. Since morphisms aren't invertible in general, we actually get two kinds of fibrations, one that has lifts "forwards" along morphisms and another that has lifts "backwards" along morphisms.

To make this precise, let $P : \mathbf{E} \to \mathbf{B}$ be any functor of 1-categories. For each object in the base category $A \in \mathrm{ob}\,\mathbf{B}$, we let $\mathbf{E}_A$ denote the fiber category over A. It consists of all the objects that map to A, and all the morphisms that map to *the identity morphism* $\mathrm{id}_A : A \to A$.

Definition 10.6.1. A functor of 1-categories $P : \mathbf{E} \to \mathbf{B}$ is a **cocartesian fibration** (also called a **Grothendieck op-fibration**) if for every morphism $f : A \to B$ in the base category $\mathbf{B}$, and every object X in the fiber category $\mathbf{E}_A$, there is a morphism $\bar{f} : X \to Y$ in the total category $\mathbf{E}$ that has the following universal property:



In other words, $P(\bar{f}) = f$, and for any morphism $g : B \to C$ and any morphism $\overline{gf} : X \to Z$ such that $P(\overline{gf}) = gf$, there exists a unique morphism $\bar{g} : Y \to Z$ such that $P(\bar{g}) = g$ and $\overline{gf} = \bar{g}\bar{f}$. We call $\bar{f}$ a **cocartesian morphism**, or a **cocartesian lift** of f, when it has this property.

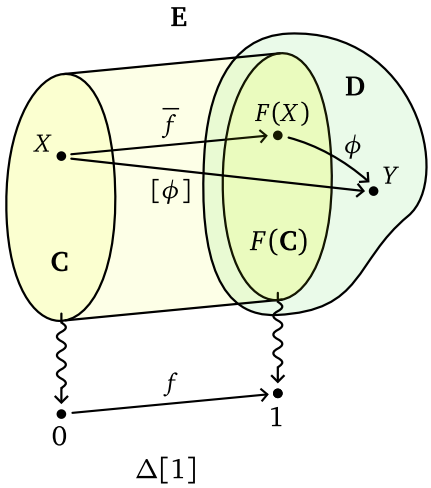
[17] And the same is true for any D^n-family of paths. And the lift isn't unique, but it is unique up to homotopy.

Unpacking this definition, this means that every morphism f in the base category **B** can be lifted to a morphism $\bar{f}$ in the total category **E** that is "universal" among all lifts of morphisms of the form gf. You can usually guess what this cocartesian lift will be, because it's the most canonical way to lift the morphism to **E**.

Example 10.6.2. We saw all the way back in Section 1.1 that the forgetful functor **Top** → **Set** is a cocartesian fibration. For any surjective maps of sets $f : X \to Y$, and any topology on X, the cocartesian lift $\bar{f}$ is the quotient map from X to Y with the quotient topology. This is the "universal" way to topologize Y along the map of sets $X \to Y$. (It is left as an exercise to describe the cocartesian lift when f is not surjective.)

Example 10.6.3. Let $F : \mathbf{C} \to \mathbf{D}$ be any functor of 1-categories. Then we can encode F into a cocartesian fibration.

Let $B = \{\bullet \to \bullet\}$, and let **E** be the disjoint union of **C** and **D**, plus a morphism $[\phi]$ from each object $X \in \mathbf{C}$ to each object $Y \in \mathbf{D}$ for each map of the form $\phi : F(X) \to Y$. These new morphisms from $X \in \mathbf{C}$ to $Y \in \mathbf{D}$ can be easily composed with morphisms $Y \to Y'$ in **D**, and they can also be composed with morphisms $X' \to X$ in **C** by applying F first. This makes **E** into a category, with a projection functor back to **B**.



This is exactly the Grothendieck construction from Proposition 8.6.19. It is a cocartesian fibration because the nontrivial morphism of **B** has a cocartesian lift to each object $X \in \mathbf{C}$, given by the map $X \to F(X)$ in **E** represented by the identity map $F(X) \to F(X)$ in **D**.

The previous example is very important. We could actually recover the functor F from the category **E**! The following lemmas explain why.

Lemma 10.6.4. *Cocartesian lifts are unique up to canonical isomorphism: for any two cocartesian lifts $X \to Y_1$ and $X \to Y_2$ of the same morphism $A \to B$, there is a canonical isomorphism $Y_1 \cong Y_2$ in the fiber category $\mathbf{E}_B$, along which the two lifts agree.*

The fact that $Y_1 \cong Y_2$ is in the fiber category $\mathbf{E}_Y$ means that it maps down to the identity morphism id_B. So in the example of **Top** → **Set**, this isomorphism must be the identity map, and this lemma says that the quotient topology is unique on the nose.

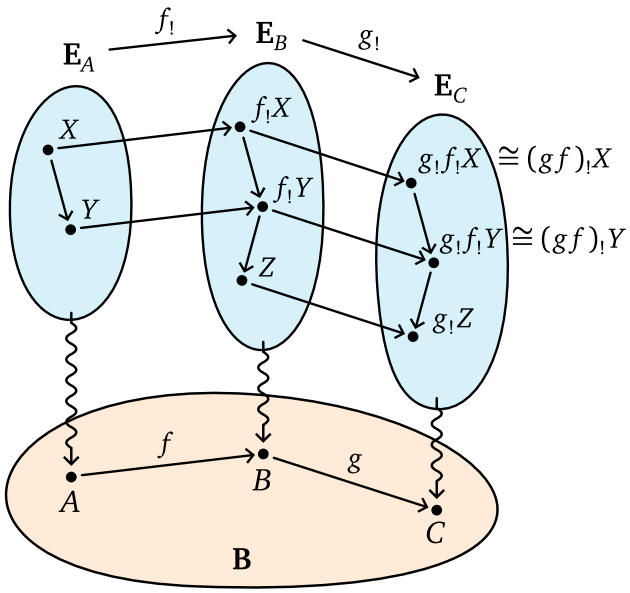
Lemma 10.6.5. *For any morphism $f : A \to B$ in the base category **B**, the cocartesian lifts to every object $X \in \mathbf{E}_A$ define a **pushforward functor** $f_! : \mathbf{E}_A \to \mathbf{E}_B$, that is well-defined up to canonical isomorphism.*

The functor $f_!$ is defined on each object $X \in \mathbf{E}_A$ by choosing a cocartesian lift $\bar{f} : X \to X'$, and defining $f_!(X) = X'$. On morphisms, for each morphism $\phi : X \to Y$ in $\mathbf{E}_A$, the universal

property of cocartesian lifts gives a unique morphism in $\mathbf{E}_B$ making the following square commute:

$$\begin{array}{ccc} X & \longrightarrow & f_!(X) \\ \downarrow{\scriptstyle \phi} & & \vdots \\ Y & \longrightarrow & f_!(Y) \end{array}$$

We define $f_!(\phi)$ to be this new morphism. It is left to Exercise 10.8.22 to verify that this indeed makes $f_!$ into a functor, and that it is well-defined up to canonical isomorphism.



Example 10.6.6. In the cocartesian fibration from Example 10.6.3, there is only one nontrivial arrow in the base category $\{\bullet \to \bullet\}$, and its pushforward functor is (canonically isomorphic to) the functor F that we used to build the fibration in the first place.

Example 10.6.7. We can make another example where $\mathbf{B}$ is the category of rings and ring homomorphisms, and $\mathbf{E}$ is the category of pairs (R, M) where R is a ring and M is a left R-module. A morphism in $\mathbf{E}$ from (R, M) to (S, N) consists of a ring homomorphism $\alpha : R \to S$ and a map of abelian groups $\phi : M \to N$ that is linear in the sense that $\phi(rm) = \alpha(r)\phi(m)$. So the fiber category $\mathbf{E}_R$ is just the category of left R-modules. The pushforward functor along $\alpha : R \to S$ is precisely the extension of scalars functor $S \otimes_R (-)$.

Remark 10.6.8. A pushforward functor can also be desribed as a functor from the product category

$$\bar{F} : \mathbf{E}_A \times \{0 \to 1\} \to \mathbf{E}$$

that is the identity of $\mathbf{E}_A$ on $\mathbf{E}_A \times \{0\}$, sends all of $\mathbf{E}_A \times \{1\}$ into $\mathbf{E}_B$, and sends every arrow of the form $\{X\} \times \{0 \to 1\}$ to a cocartesian lift of f. Such a functor is unique up to isomorphism, and the pushforward $f_!$ is just the resulting map $\mathbf{E}_A \times \{1\} \to \mathbf{E}_B$.

When we compose the pushforward functors along $f : A \to B$ and $g : B \to C$, we get a canonical isomorphism $(gf)_! \cong g_! f_!$ (see Exercise 10.8.22). So we can think of a cocartesian fibration as giving a collection of functors whose compositions agree up to canonical isomorphism! This is captured by the following definition.

Definition 10.6.9. An **indexed category** is a 1-category $\mathbf{B}$ and an assignment of

- a category $\mathbf{C}_A$ for each object $A \in \mathrm{ob}\,\mathbf{B}$,
- a functor $f_! : \mathbf{C}_A \to \mathbf{C}_B$ for each morphism $A \to B$ in $\mathbf{B}$,
- a natural isomorphism $(\mathrm{id}_A)_! \cong \mathrm{id}_{\mathbf{C}_A}$ for each object $A \in \mathrm{ob}\,\mathbf{B}$, and
- a natural isomorphism $(gf)_! \cong g_! f_!$ for each pair of functors $f : A \to B$ and $g : B \to C$,

such that the following three diagrams of natural isomorphisms commute.

$$\begin{array}{ccc} (hgf)_! & \xleftrightarrow{\cong} & (hg)_! f_! \\ \updownarrow{\scriptstyle\cong} & & \updownarrow{\scriptstyle\cong} \\ h_!(gf)_! & \xleftrightarrow{\cong} & h_! g_! f_! \end{array} \qquad \begin{array}{ccc} (\mathrm{id}_B)_! f_! & \xleftrightarrow{\cong} & \mathrm{id}_{\mathbf{C}_B} \circ f_! \\ \updownarrow{\scriptstyle\cong} & & \| \\ (\mathrm{id}_B \circ f)_! & = & f_! \end{array} \qquad \begin{array}{ccc} f_!(\mathrm{id}_A)_! & \xleftrightarrow{\cong} & f_! \circ \mathrm{id}_{\mathbf{C}_A} \\ \updownarrow{\scriptstyle\cong} & & \| \\ (f \circ \mathrm{id}_A)_! & = & f_! \end{array}$$

Remark 10.6.10. The reader familiar with 2-category theory might recognize that this is the same thing as a "pseudofunctor" from the 1-category $\mathbf{B}$ to the 2-category **Cat** of all categories, functors, and natural transformations. The "pseudo" in pseudofunctor refers to the fact that compositions in $\mathbf{B}$ doesn't go to compositions in **Cat** strictly, they only agree up to coherent natural isomorphisms.

Theorem 10.6.11. *For every cocartesian fibration $\mathbf{E} \to \mathbf{B}$, the pushforward functors $f_!$ and their canonical isomorphisms give a $\mathbf{B}$-indexed category. Conversely, every $\mathbf{B}$-indexed category comes from a cocartesian fibration over $\mathbf{B}$.*

That is, given a $\mathbf{B}$-indexed category we can define a cocartesian fibration $\mathbf{E}$ by the **Grothendieck construction**: an object in $\mathbf{E}$ is a pair (A, X) where $A \in \mathrm{ob}\,\mathbf{B}$ and $X \in \mathrm{ob}\,\mathbf{C}_A$, and a morphism in $\mathbf{E}$ from (A, X) to (B, Y) is a morphism $f : A \to B$ in $\mathbf{B}$ and a morphism $f_!(X) \to Y$ in $\mathbf{C}_B$. It is left as an exercise to see why the canonical isomorphisms between the pushforward functors allow us to make this into a category.[18]

It turns out that if we go back and forth between cocartesian fibrations and indexed categories in this way, we always get the same result back, up to an appropriate notion of equivalence. So the two notions are interchangeable! (We skip the precise definition of "equivalence" here because it is involved and we won't need it. See [Shu08] for more details.)

This is the 1-categorical version of "straightening and unstraightening:"

[18] We saw this in Proposition 8.6.19 already, but only in the special case that the isomorphisms $(\mathrm{id}_A)_! \cong \mathrm{id}_{\mathbf{C}_A}$ and $(gf)_! \cong g_! f_!$ are all equalities of functors.

- a straightened diagram on $\mathbf{B}$ is a $\mathbf{B}$-indexed category,
- an unstraightened diagram on $\mathbf{B}$ is a cocartesian fibration $\mathbf{E} \to \mathbf{B}$,

and these two notions are the same thing, up to equivalence.

Example 10.6.12. One of the simplest examples of an indexed category is when $\mathbf{B} = \{0 \to 1\}$, so that the nontrivial morphism of $\mathbf{B}$ is sent to some functor $F : \mathbf{C} \to \mathbf{D}$, and for simplicity we can just assign id_0 and id_1 to the actual identity functors of $\mathbf{C}$ and $\mathbf{D}$, respectively. This corresponds to the cocartesian fibration of Example 10.6.3.

Example 10.6.13. The cocartesian fibration of rings and modules from Example 10.6.7 corresponds to the indexed category in which $\mathbf{B}$ is the category of rings and ring homomorphisms, each ring R is sent to the category $\mathrm{Mod}(R)$ of left R-modules, and each ring homomorphism $\alpha : R \to S$ is sent to the extension of scalars functor $S \otimes_R (-)$.

Example 10.6.14. Every symmetric monoidal category $(\mathbf{C}, \otimes, I)$ defines a cocartesian fibration $\mathbf{C}^{\otimes} \to \mathbf{Fin}_*$, where $\mathbf{Fin}_*$ denotes the category of finite based sets. We gave this already in Example 9.4.56: an object of $\mathbf{C}^{\otimes}$ consists of a finite set I and an I-tuple of objects $\{a_i\}_{i \in I}$ in the category $\mathbf{C}$. A morphism $\{a_i\}_{i \in I} \to \{b_j\}_{j \in J}$ in $\mathbf{C}^{\otimes}$ is a map of finite based sets $\alpha : I_+ \to J_+$ together with maps of the form $\bigotimes_{i \in \alpha^{-1}(j)} a_i \to b_j$ for each $j \in J$.

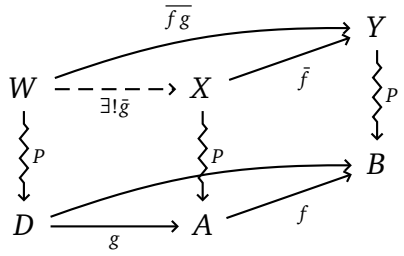
Thinking of this as an indexed category, each finite based set I_+ is sent to the product category $\mathbf{C}^{\times I}$, with one copy of $\mathbf{C}$ for each $i \in I$. The pushforward functor along $\alpha : I_+ \to J_+$ takes tensor products along each set $\alpha^{-1}(j)$. So it sends the I-tuple $\{a_i\}_{i \in I}$ to the J-tuple $\{\bigotimes_{i \in \alpha^{-1}(j)} a_i\}_{j \in J}$. Intuitively, this indexed category is encoding the fact that all of these tensor product functors and their compositions agree with each other, up to canonical isomorphism. See Exercise 10.8.27.

10.6.2 Cartesian and bicartesian fibrations of 1-categories

Now we can reverse the arrows in the universal property, and get a second kind of fibration that has "backwards" lifts along morphisms. This causes the fiber categories to have pullback functors between them.

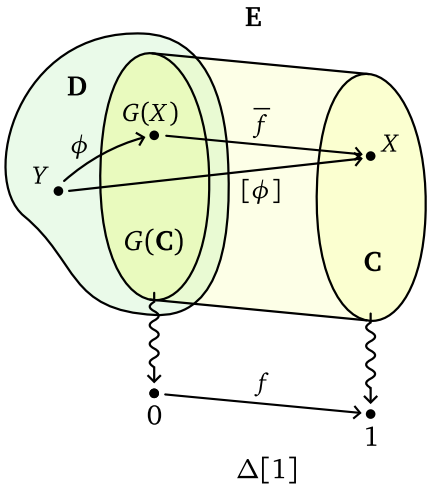
Definition 10.6.15. A functor of 1-categories $p : \mathbf{E} \to \mathbf{B}$ is a **cartesian fibration** (also called a **Grothendieck fibration**) when for every morphism $f : A \to B$ in the base category $\mathbf{B}$, and every object Y in the fiber category $\mathbf{E}_B$, there is a morphism $\bar{f} : X \to Y$ in the total

category **E** that has the following universal property:



In other words, $P(\bar{f}) = f$, and for any morphism $g : D \to A$ and any morphism $\overline{fg} : W \to Y$ such that $P(\overline{fg}) = fg$, there exists a unique morphism $\bar{g} : W \to X$ such that $P(\bar{g}) = g$ and $\overline{fg} = \bar{f}\,\bar{g}$. Again we call $\bar{f}$ a **cartesian morphism** when it has this property.

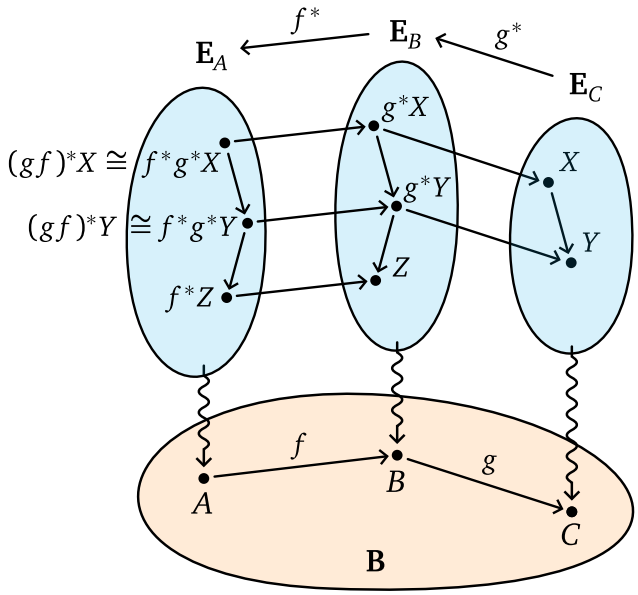
Example 10.6.16. The forgetful functor **Top** $\to$ **Set** is also a cartesian fibration. For any injective map of sets $f : X \to Y$, and any topology on Y, the cartesian lift $\bar{f}$ is the inclusion of X with the subspace topology. This is the "universal" way to topologize X along the map of sets $X \to Y$. (It is left as an exercise to describe the cartesian lift when f is not injective.)



Example 10.6.17. If $G : \mathbf{D} \to \mathbf{C}$ is any functor of 1-categories, then we can encode G into a cartesian fibration, dual to Example 10.6.3. We again let $B = \{\bullet \to \bullet\}$, and let **E** be the disjoint union of **C** and **D**, but now we add a morphism $[\phi]$ from $X \in \mathbf{C}$ to $Y \in \mathbf{D}$ for each morphism $\phi : X \to G(Y)$. Again this makes **E** into a category, with a projection functor back to **B**.

This is the contravariant form of the Grothendieck construction. It is a cartesian fibration because the nontrivial morphism of **B** has a cartesian lift to each object $Y \in \mathbf{D}$, given by the map $G(Y) \to Y$ in **E** represented by the identity map $G(Y) \to G(Y)$ in **C**.

Lemma 10.6.18. *Cartesian lifts are unique up to canonical isomorphism. For any morphism $f : A \to B$ in the base category* **B***, the cartesian lifts to every object $Y \in \mathbf{E}_B$ define a* ***pullback functor*** $f^* : \mathbf{E}_B \to \mathbf{E}_A$*, that is well-defined up to canonical isomorphism.*



When we compose the pullback functors along $f : A \to B$ and $g : B \to C$, we get a canonical isomorphism $(fg)^* \cong f^*g^*$, by the dual of Exercise 10.8.22. So we can also think of a cartesian fibration as giving a collection of functors whose compositions agree up to canonical isomorphism!

Theorem 10.6.19. *For every cartesian fibration $\mathbf{E} \to \mathbf{B}$, the pullback functors f^* and their canonical isomorphisms give a $\mathbf{B}^{\mathrm{op}}$-indexed category. Conversely, every $\mathbf{B}^{\mathrm{op}}$-indexed category comes from a cartesian fibration over $\mathbf{B}$.*

So now, given an indexed category over $\mathbf{B}$, we can always turn it into a cocartesian fibration over $\mathbf{B}$, or a cartesian fibration over $\mathbf{B}^{\mathrm{op}}$. These are different categories! In one, the horizontal morphisms flow in the same direction as the functors $\mathbf{C}_A \to \mathbf{C}_B$, and in the other the horizontal morphisms flow in the opposite direction as the functors. The vertical morphisms (within each fiber category) go the same direction in both categories. So we have flipped the direction of some, but not all, of the morphisms that generate the category.

The most interesting and important examples have both of these properties simultaneously.

Definition 10.6.20. A **bicartesian fibration** (or **bifibration**) is a functor $P : \mathbf{E} \to \mathbf{B}$ that is both a cocartesian and a cartesian fibration. So every arrow in the base category $f : A \to B$ can be lifted in either direction, and therefore it gives both a pushforward functor $f_! : \mathbf{E}_A \to \mathbf{E}_B$ and a pullback functor $f^* : \mathbf{E}_B \to \mathbf{E}_A$.

Example 10.6.21. If we have a left adjoint functor $F : \mathbf{C} \to \mathbf{D}$ with right adjoint $G : \mathbf{D} \to \mathbf{C}$, then there is a bicartesian fibration over $\mathbf{B} = \{\bullet \to \bullet\}$ in which the category $\mathbf{E}$ is again the disjoint union of $\mathbf{C}$ and $\mathbf{D}$, plus a morphism from $X \in \mathbf{C}$ to $Y \in \mathbf{D}$ for every map of

the form $F(X) \to Y$, equivalently for every map of the form $X \to G(Y)$. This is both the forwards Grothendieck construction for F, and the backwards Grothendieck construction for Y. They give the same category!

The previous example illustrates what happens in general. Whenever we have both pushforwards and pullbacks, they are always adjoint to each other.

Lemma 10.6.22. *In a bifibration, each pushforward functor $f_!$ is the left adjoint of the corresponding pullback functor f^*.*

The proof of this is left to Exercise 10.8.23.

In fact a very strong version of this statement is true: the bifibration gives you a *particular* choice of adjunction between $f_!$ and f^*, and these particular choices respect composition. So if we compose the given adjunctions for $(f_!, f^*)$ and $(g_!, g^*)$, we get the adjunction for $((gf)_!, (gf)^*)$. This means that bifibrations are a powerful and efficient way to write down a large collection of adjunctions between functors, without getting lost in the coherence between them. In practice, the definition of the total category $\mathbf{E}$ is much simpler than the definition of each individual adjunction!

Example 10.6.23. We can define a bifibration of "all diagrams of spaces over all indexing categories." We let $\mathbf{B}$ be the 1-category in which an object is a small category $\mathbf{I}$ and a morphism is a functor $\alpha : \mathbf{I} \to \mathbf{J}$. Let $\mathbf{E}$ be the 1-category in which an object is a pair $(\mathbf{I}, X)$ of a small category $\mathbf{I}$ and a diagram of spaces $X : \mathbf{I} \to \mathbf{Top}$. A morphism in $\mathbf{E}$ from $(\mathbf{I}, X)$ to $(\mathbf{J}, Y)$ is a functor $\alpha : \mathbf{I} \to \mathbf{J}$ and a map of diagrams $X \to Y$ "over" α. In other words, a map $X(i) \to Y(\alpha(i))$ for all $i \in \mathbf{I}$, such that every map $i \to i'$ gives a commuting square

$$\begin{array}{ccc} X(i) & \longrightarrow & Y(\alpha(i)) \\ \downarrow & & \downarrow \\ X(i') & \longrightarrow & Y(\alpha(i')). \end{array}$$

It is an exercise to check that this is a bifibration. The pullback functors α^* are exactly the restriction of diagram functors $(\alpha^* Y)(i) := Y(\alpha(i))$ from Definition 5.4.14, and the pushforwards $\alpha_!$ are the left Kan extensions from Definition 5.4.14. All of the data of every diagram X over every category $\mathbf{I}$, and every restriction and left Kan extension between them, can be captured in a single category![19]

Of course, we could also run this example with **Top** replaced by any other category. So for instance there is a bifibration of all diagrams of *spectra* over all indexing categories.

[19] In fact, we have been thinking this way the whole time, which is why we used the notation $(\alpha_!, \alpha^*)$ for this adjunction.

Example 10.6.24. We can define a bifibration of "all modules over all rings." We actually already defined this in Example 10.6.7: the base category **B** is the category of rings and ring homomorphisms, the objects in the total category **E** are pairs (R, M) where R is a ring and M is a left R-module, and the morphisms are defined so that the pushforward along $\alpha: R \to S$ is extension of scalars, and the pullback along α is the restriction of scalars.

Example 10.6.25. A large class of interesting bifibrations is given by **local systems**. A local system is a diagram whose indexing category is a groupoid. So the first version of this is if we take the bifibration from Example 10.6.23 and restrict attention to those indexing categories **I** that are groupoids (all morphisms are invertible). We get the category of "all diagrams over all groupoids." This is a bifibration in which the pullbacks α^* are restrictions and the pushforwards $\alpha_!$ are left Kan extensions. See Exercise 10.8.25 for more examples related to this one.

In any bicartesian fibration, since it is a cartesian fibration, it can be described up to equivalence by giving just the pullback functors f^* and the isomorphisms between them. The pushforward functors $f_!$ and the isomorphisms between them arise automatically from this data.

Corollary 10.6.26. *Bifibrations over* **B** *correspond to indexed categories over* $\mathbf{B}^{\mathrm{op}}$*, with the property that each of the functors* f^* *has a left adjoint* $f_!$*.*

Corollary 10.6.27. *Given an indexed category over* $\mathbf{B}^{\mathrm{op}}$ *consisting entirely of right adjoint functors* f^**, the left adjoints* $f_!$ *give an indexed category over* **B** *in a canonical way.*

This is a powerful version of "uniqueness of left adjoints." It allows us to go from entire systems of right adjoint functors to the entire system consisting of their left adjoints.

10.6.3 Cocartesian fibrations of ∞-categories

Now that we have built some intuition with 1-categories, we are ready for the ∞-categorical version of a cocartesian fibration. As before, this will be a single map of quasicategories, but it will have a lifting property that allows us to define pushforward functors between the fiber categories.

Definition 10.6.28. [Lur09, 2.4.2.1], [Lan21, 3.1.1], [BS18, 3.1] A **cocartesian fibration** is a map of quasicategories $p: E \to B$ that is an inner fibration (Definition 10.1.33) and has cocartesian lifts in the following sense: for each 1-simplex $f \in B_1$ there exists a 1-simplex $\bar{f} \in E_1$ such that $p(\bar{f}) = f$, and such that every diagram of the following form

has a lift, *when* the first edge $\Delta^{\{0,1\}} \subseteq \Lambda_0[n]$ goes to $\bar{f}$.

$$\begin{array}{ccc} \Lambda_0[n]_\bullet & \longrightarrow & E_\bullet \\ \downarrow & \nearrow & \downarrow \\ \Delta[n]_\bullet & \longrightarrow & B_\bullet \end{array}$$

So in summary, every *inner* horn $\Lambda_i[n] \to \Delta[n]$ has a lift, and in addition *some* of the left-hand horns $\Lambda_0[n]$ have lifts.

Example 10.6.29. When E and B are nerves of 1-categories, this coincides with the definition of a cocartesian fibration from Definition 10.6.1. See Exercise 10.8.24. So all of our examples from the previous sections become examples of cocartesian fibrations of ∞-categories.

We use the notation $E(a) = p^{-1}(a)$ for the fiber over a vertex $a \in B_0$. When p is a cocartesian fibration, $E(a)$ is always a quasicategory. (Think about why this is true!)

Definition 10.6.30. A map of cocartesian fibrations is a commuting square

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\phi} & B' \end{array}$$

such that Φ takes every cocartesian morphism of p to a cocartesian morphism of p'. Such a map is an **equivalence of cocartesian fibrations** if ϕ is an equivalence and either of the following two equivalent conditions holds: [Lur25, Theorem 023M]

- Φ is a Joyal equivalence on the total categories $E \simeq E'$.
- $\Phi(a)$ is a Joyal equivalence of fiber categories $E(a) \simeq E'(\phi(a))$ for all $a \in B_0$.

This is just like how Serre fibrations work: a map of two fibrations over the same base is an equivalence iff it induces an equivalence of fibers.

We can build examples from simplicial categories in the following way.

Definition 10.6.31. A **cocartesian fibration of simplicial categories** is a simplicial functor $P : \mathbf{E} \to \mathbf{B}$ such that

- the categories are pointwise fibrant, so each $\mathbf{E}_\bullet(x, y)$ and $\mathbf{B}_\bullet(a, b)$ is a Kan complex,
- the map P induces a Kan fibration $\mathbf{E}_\bullet(x, y) \to \mathbf{B}_\bullet(P(x), P(y))$, and

- for every morphism $f \in \mathbf{B}_0(a,b)$ and every object $x \in \operatorname{ob} P^{-1}(a)$, there exists a lift $\bar{f} \in \mathbf{E}(x,y)$ such that for every object $z \in \operatorname{ob}\mathbf{E}$ projecting to $c \in \operatorname{ob}\mathbf{B}$, composition with $\bar{f}$ and f induces a homotopy pullback square of simplicial sets

$$\begin{array}{ccc} \mathbf{E}_\bullet(y,z) & \xrightarrow{\circ\bar{f}} & \mathbf{E}_\bullet(x,z) \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P} \\ \mathbf{B}_\bullet(b,c) & \xrightarrow{\circ f} & \mathbf{B}_\bullet(a,c). \end{array}$$

Think about why this is the correct definition: in the case of a 1-category, the universal property of cocartesian arrows is exactly the statement that the above square is a strict pullback square.

Lemma 10.6.32. *[Lur09, 2.4.1.10] If $\mathbf{E} \to \mathbf{B}$ is a cocartesian fibration of simplicial categories, then the map of simplicial nerves $N^s(\mathbf{E}) \to N^s(\mathbf{B})$ is cocartesian fibration of quasicategories. Furthermore the cocartesian lifts $\bar{f}$ for the functor $\mathbf{E} \to \mathbf{B}$ go to cocartesian lifts for the functor of quasicategories $N^s(\mathbf{E}) \to N^s(\mathbf{B})$.*

This becomes particularly simple if $\mathbf{B}$ is discrete, in other words a 1-category. Then it just says that for every morphism $g\colon b \to c$ in $\mathbf{B}$, the space of morphisms in $\mathbf{E}$ that project down to the morphism g is equivalent to the space of morphisms in $\mathbf{E}$ that project to gf, and the equivalence is given by composing with $\bar{f}$.

Example 10.6.33. Suppose $\mathbf{C}_\bullet$ and $\mathbf{D}_\bullet$ are simplicial categories and $F\colon \mathbf{C}_\bullet \to \mathbf{D}_\bullet$ is a simplicial functor. Then as in Example 10.6.3, we can define a cocartesian fibration of simplicial categories $\mathbf{E}_\bullet \to \mathbf{B}_\bullet$ by taking $\mathbf{B}$ to be the 1-category $\{\bullet \to \bullet\}$ as before, and taking $\mathbf{E}_\bullet$ to be the disjoint union of $\mathbf{C}_\bullet$ and $\mathbf{D}_\bullet$, plus morphism spaces from $x \in \operatorname{ob}\mathbf{C}$ to $y \in \operatorname{ob}\mathbf{D}$ given by the formula $\mathbf{D}_\bullet(F(x),y)$. This satisfies Definition 10.6.31, because composition with the morphism $x \to F(x)$ induces an isomorphism of simplicial sets

$$\mathbf{E}_\bullet(F(x),z) := \mathbf{D}_\bullet(F(x),z) =: \mathbf{E}_\bullet(x,z),$$

and an isomorphism is certainly a weak homotopy equivalence.

Example 10.6.34. We can generalize the cocartesian fibration $\mathbf{C}^\otimes \to \mathbf{Fin}_*$ from Example 10.6.14 from the case where $\mathbf{C}$ is a 1-category to the case where $\mathbf{C}$ is a simplicial category. Assume that $(\mathbf{C}_\bullet, \otimes, I)$ is any symmetric monoidal simplicial category. So it has mapping simplicial sets $\mathbf{C}_\bullet(x,y)$, and a symmetric monoidal structure whose tensor product $\otimes\colon \mathbf{C}_\bullet \times \mathbf{C}_\bullet \to \mathbf{C}_\bullet$ is a simplicial functor, and whose isomorphisms are natural isomorphisms of simplicial functors. (Equivalently, we have a sequence of symmetric monoidal categories $(\mathbf{C}_n, \otimes, I)$ that have the same objects, and the same unit object I, with face and degeneracy functors between them that are strict symmetric monoidal.)

For instance, any symmetric monoidal model category $\mathbf{C}_\bullet$ with a strong symmetric monoidal Quillen left adjoint $\mathbf{sSet} \to \mathbf{C}$ will satisfy these assumptions. So spaces, simplicial sets, and various kinds of spectra all give examples.

Then we can form a simplicial category $\mathbf{C}_\bullet^\otimes$ and a functor $\mathbf{C}_\bullet^\otimes \to \mathbf{Fin}_*$ just as in Example 10.6.14. The objects are finite sets I and I-tuples of objects of $\mathbf{C}$ as before, $\{a_i\}_{i\in I}$. The morphisms from $\{a_i\}_{i\in I}$ to $\{b_j\}_{j\in J}$ in $\mathbf{C}_\bullet^\otimes$ are the simplicial set

$$\coprod_{\alpha\colon I_+\to J_+} \prod_{j\in J} \mathbf{C}_\bullet\left(\bigotimes_{i\in\alpha^{-1}(j)} a_i, b_j\right).$$

The composition comes from the tensor product and composition in $\mathbf{C}$. Note the similarity to Example 9.4.56. If all of the mapping spaces $\mathbf{C}_\bullet(x,y)$ are Kan complexes, then by Lemma 10.6.32, the map of simplicial nerves $N^s(\mathbf{C}_\bullet^\otimes) \to N(\mathbf{Fin}_*)$ will define a cocartesian fibration of quasicategories.

There is also criterion similar to the above that works for any map of quasicategories, not just any map of simplicial categories.

Proposition 10.6.35. *[Lur09, 2.4.4.3] A map of quasicategories $p\colon E \to B$ is a cocartesian fibration iff the following conditions hold:*

- *p is an inner fibration, and*
- *for each edge $f \in B_1$ starting at $a \in B_0$, there is a lift $\bar{f} \in E_1$ starting at any $x \in E(a)$, such that composition with $\bar{f}$ induces a homotopy pullback square*

$$\begin{array}{ccc} \mathrm{Map}_E(y,z) & \xrightarrow{\circ\bar{f}} & \mathrm{Map}_E(x,z) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ \mathrm{Map}_B(b,c) & \xrightarrow{\circ f} & \mathrm{Map}_B(a,b) \end{array}$$

  for each $z \in E_0$ projecting to $c \in B_0$.

As in the 1-category case, the purpose of a cocartesian fibration is to have pushforward functors. Recall that we are using $E(a) = p^{-1}(a)$ to denote the fibers of the cocartesian fibration $p\colon E \to B$. These are all quasicategories because p is an inner fibration.

Definition 10.6.36. [Lan21, 3.3.1] If $p\colon E \to B$ is a cocartesian fibration of simplicial sets, and $f \in B_1$ is a 1-simplex in the base from a to b, a **total cocartesian lift** along f is a map of simplicial sets $\bar{F}\colon E(a) \times \Delta[1] \to E$ such that

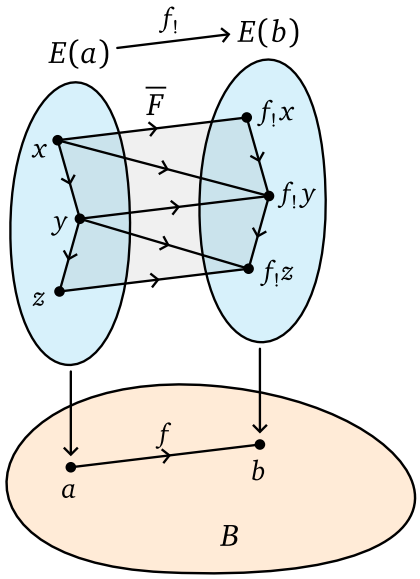
- On the left-hand side $E(a) \times \{0\}$, we get the identity of $E(a)$.

- The map consists entirely of lifts of f, in the sense that the following diagram commutes:

$$\begin{array}{ccc} E(a)\times\Delta[1] & \xrightarrow{\bar{F}} & E \\ \downarrow & & \downarrow p \\ \Delta[1] & \xrightarrow{f} & B. \end{array}$$

- The map consists entirely of cocartesian lifts of f, in the sense that each of the 1-simplices $\{\bar{x}\}\times\Delta[1]$ for $\bar{x}\in E(a)_0$ goes to a cocartesian lift of f.

The commuting diagram guarantees that on the right-hand side $E(a)\times\{1\}$, we get a map to the fiber category $E(b)$. We call this the associated **pushforward functor** $f_!\colon E(a)\to E(b)$.



We can extend this to a space of total cocartesian lifts, by defining an n-simplex to be a map $E(a)\times\Delta[n]\times\Delta[1]\to E$ that satisfies the first two conditions after projecting away the copy of $\Delta[n]$ to a point, and the last condition at every vertex of $\Delta[n]$.

Lemma 10.6.37. *[Lan21, 3.2.19] If p is a cocartesian fibration, then the space of total cocartesian lifts of any morphism f is a contractible Kan complex. Therefore pushforward functors $f_!$ exist and are unique up to contractible choice of natural isomorphism.*

As before, these pushforward functors assemble into a diagram of quasicategories $E(a)$ indexed by the quasicategory B. This is the content of the straightening/unstraightening theorem.

To state it, recall that $\mathbf{Cat}_\infty \simeq N(\mathbf{sSet})[W_{\text{Joyal}}^{-1}]$ is the ∞-category of all quasicategories, and let us denote by

$$\text{Cocart}_B \subseteq \text{Fun}(\Delta[1], N(\mathbf{sSet}))[W_{\text{Joyal}}^{-1}]$$

the full subcategory of the category of all functors of quasicategories, consisting only of those functors that are cocartesian fibrations and whose target is the fixed quasicategory B.

Theorem 10.6.38 (Straightening/unstraightening, cocartesian case)**.** *[Lur09, 3.2.0.1], [Lan21, §3.3] For each cocartesian fibration $E \to B$, there is an associated diagram $B \to \mathbf{Cat}_\infty$ that sends each vertex $a \in B_0$ to the fiber category $E(a)$ and each edge $f \in B_1$ to the pushforward functor $f_!$. This "straightening functor" defines an equivalence of ∞-categories*

$$\mathrm{Cocart}_B \simeq \mathrm{Fun}(B, \mathbf{Cat}_\infty).$$

The inverse functor is called "unstraightening," and sends each diagram $B \to \mathbf{Cat}_\infty$ to a version of the Grothendieck construction, which defines a cocartesian fibration $E \to B$. One way to define this is by following bundle theory, and defining a universal cocartesian fibration whose base is $\mathbf{Cat}_\infty$. Then given any functor $\phi : B \to \mathbf{Cat}_\infty$, you pull back the universal fibration along ϕ to get the desired fibration over B. The details of this are substantial to work out, see e.g. [RV18, 6.1.13].

We will use this theorem shortly to motivate the definition of a symmetric monoidal ∞-category: it should be a homotopy-coherent commutative monoid in $\mathbf{Cat}_\infty$, and should therefore be given by a functor $N(\mathbf{Fin}_*) \to \mathbf{Cat}_\infty$ with the Segal condition. However, such functors are hard to work with directly, so we work with a cocartesian fibration over $N(\mathbf{Fin}_*)$ instead. This is essentially the same thing, by the straightening/unstraightening theorem.

10.6.4 Other kinds of fibrations of ∞-categories

The definition of a cocartesian fibration dualizes nicely, as do all of the examples and the straightening/unstraightening theorem.

Definition 10.6.39. [Lan21, 3.1.1] A **cartesian fibration** is a map of quasicategories $p : E \to B$ that is an inner fibration (Definition 10.1.33) and has cartesian lifts in the following sense: for each 1-simplex $f \in B_1$ there exists a 1-simplex $\bar{f} \in X_1$ such that $p(\bar{f}) = f$, and such that every diagram of the following form has a lift, *when* the last edge $\Delta^{\{n-1,n\}} \subseteq \Lambda_n[n]$ goes to $\bar{f}$.

$$\begin{array}{ccc} \Lambda_n[n]_\bullet & \longrightarrow & E_\bullet \\ \downarrow & \nearrow & \downarrow \\ \Delta[n]_\bullet & \longrightarrow & B_\bullet \end{array}$$

This is equivalent to asking that $p^{\mathrm{op}} : E^{\mathrm{op}} \to B^{\mathrm{op}}$ is a cocartesian fibration. When p is both a cocartesian fibration and a cartesian fibration, we call it a **bicartesian fibration**.

Example 10.6.40. Again, when E and B are nerves of 1-categories, this definition of "cartesian fibration" coincides with the definition of a cartesian fibration from Definition 10.6.15.

As before, when $p: E \to B$ is a cartesian fibration, then the fibers $E(a)$ are all quasicategories, and for each morphism $f \in B_1$ from a to b there is a **pullback functor** $f^*: E(b) \to E(a)$. This is defined as in Definition 10.6.36, but using a map $\bar{F}: E(b) \times \Delta[1] \to E$ that on the right-hand side $E(b) \times \{1\}$ is the identity of $E(b)$, and on the left-hand side $E(b) \times \{0\}$ gives the pullback functor $f^*: E(b) \to E(a)$.

We let

$$\mathrm{Cart}_B \subseteq \mathrm{Fun}(\Delta[1], N(\mathbf{sSet}))[W_{\mathrm{Joyal}}^{-1}]$$

be the full subcategory of all functors of quasicategories, consisting only of those functors that are cartesian fibrations and whose target is the fixed quasicategory B.

Theorem 10.6.41 (Straightening/unstraightening, cartesian case)**.** *[Lur09, 3.2.0.1], [Lan21, §3.3] For each cartesian fibration $E \to B$, there is an associated diagram $B^{\mathrm{op}} \to \mathbf{Cat}_\infty$ that sends each vertex $a \in B_0$ to the fiber category $E(a)$ and each edge $f \in B_1$ to the pullback functor f^*. This "straightening functor" defines an equivalence of ∞-categories*

$$\mathrm{Cart}_B \simeq \mathrm{Fun}(B^{\mathrm{op}}, \mathbf{Cat}_\infty).$$

The notions of cocartesian and cartesian fibration get very interesting when they interact.

Lemma 10.6.42. *In a bicartesian fibration $p: E \to B$, each pushforward $f_!$ and pullback f^* form an adjunction of ∞-categories between $E(a)$ and $E(b)$.*

Lemma 10.6.43. *Straightening/unstraightening gives an equivalence between adjunctions of ∞-categories and bicartesian fibrations over $\Delta[1] = N(\{\bullet \to \bullet\})$.*

So if we have a functor $f^*: D \to C$, we can turn it into a cartesian fibration over $\Delta[1]$, and then the functor has a left adjoint iff this is also a cocartesian fibration. This is a powerful way to work with adjunctions of ∞-categories. It becomes essential whenever we want to study anything more complicated than just a single adjunction.

Corollary 10.6.44. *For any diagram of right adjoint functors $B^{\mathrm{op}} \to \mathbf{Cat}_\infty$, the left adjoints form a diagram $B \to \mathbf{Cat}_\infty$ in a canonical way.*

There are various interesting examples of cocartesian and cartesian fibrations of ∞-categories, including the ∞-categorical version of equivariant homotopy theory. These sometimes go under the umbrella term of **parametrized higher category theory**. See e.g. [BDG+16].

Another way to generalize these notions is to look at the special case where every 1-simplex in E is cocartesian.

Definition 10.6.45. [Lan21, 3.1.1] A **left fibration** is a map of quasicategories $p : E \to B$ that has lifts along every horn $\Lambda_i[n] \to \Delta[n]$ when $0 \leq i < n$. In other words, it is a cocartesian fibration in which *every* 1-simplex in E is a cocartesian edge.

Dually, a **right fibration** is a map that has lifts along every horn $\Lambda_i[n] \to \Delta[n]$ when $0 < i \leq n$. So it is a cartesian fibration in which every 1-simplex in E is cartesian.

Example 10.6.46. Every Kan fibration $E \to B$ is both a left and a right fibration. In particular, if we take any Serre fibration of topological spaces and then take its singular simplices, we get both a left and a right fibration. (Because we can lift paths in either direction!)

Lemma 10.6.47. *A cocartesian fibration is a left fibration iff the fibers $E(a)$ are all Kan complexes, or ∞-groupoids.*

By taking the opposite category, the same is true for cartesian fibrations and right fibrations.

Corollary 10.6.48. *The straightening/unstraightening equivalence restricts to an equivalence between left fibrations over B and the ∞-category* $\mathrm{Fun}(B, \mathscr{S})$. *Similarly we get an equivalence between right fibrations over B and* $\mathrm{Fun}(B^{\mathrm{op}}, \mathscr{S}) = \mathscr{P}(B)$.

As usual, it is difficult to write down any examples of any of this before we have the magic of localization. Here is one abstract example though.

Example 10.6.49. The functorial mapping space $C^{\mathrm{op}} \times C \to \mathscr{S}$ is defined in Definition 10.3.3 using simplicial categories, but we could also define it by defining the associated right fibration $E \to C \times C^{\mathrm{op}}$. This right fibration is sometimes called the "twisted arrow category" $Tw(C)$. It has an n-simplex for every map of simplicial sets

$$\Delta[n] * \Delta[n]^{\mathrm{op}} \to C$$

where $*$ is the join operation from Remark 10.3.35. Restricting to $\Delta[n]$ or $\Delta[n]^{\mathrm{op}}$ gives two functors $Tw(C) \to C$ and $Tw(C) \to C^{\mathrm{op}}$, and the resulting functor $Tw(C) \to C \times C^{\mathrm{op}}$ is a right fibration [Lan21, 4.2.4]. The corresponding functor $C^{\mathrm{op}} \times C \to \mathscr{S}$ is the functorial mapping space construction by [Lur09, 1.1.5.13].

10.6.5 Localizing a system of ∞-categories

We once again turn to the magic of Dwyer–Kan localization, so that we can get concrete examples of cocartesian fibrations of ∞-categories that are relevant to us. We already know how a cocartesian fibration of 1-categories works, so we just need to understand what happens when we invert some weak equivalences.

Proposition 10.6.50. *[Hin16, 2.1.4] Suppose $p: E \to B$ is a cocartesian fibration of quasicategories, and we have classes of weak equivalences $V \subseteq E_1$ and $W \subseteq B_1$ that we plan to invert. Let $V(a) = V \cap E(a)$ denote the subset of the weak equivalences in the total category that lie in a particular fiber category. Assume that*

- *for every $f \in W$, every cocartesian lift $\bar{f}$ is in V,*
- *for every $f \in B_1$, the pushforward $f_!$ preserves morphisms in V, and*
- *for every $f \in W$ from a to b, the pushforward $f_!$ induces an equivalence*

$$E(a)[V(a)^{-1}] \simeq E(b)[V(b)^{-1}].$$

Then the map of Dwyer–Kan localizations $E[V^{-1}] \to B[W^{-1}]$ is (up to Joyal equivalence) again a cocartesian fibration, whose fiber categories are the localizations $E(a)[V(a)^{-1}]$ and whose pushforwards $E(a)[V(a)^{-1}] \to E(b)[V(b)^{-1}]$ are induced by the pushforwards in the original cocartesian fibration.

In particular, if we don't invert anything in the base, then the only assumption we make about V is that the pushfowards $f_!$ preserve morphisms in V. (See also [Hau15, 4.21] for a version of this where we don't invert anything in the base.)

Example 10.6.51. If $(\mathbf{C}, W_{\mathbf{C}})$ and $(\mathbf{D}, W_{\mathbf{D}})$ are 1-categories with weak equivalences, and $F: \mathbf{C} \to \mathbf{D}$ is a homotopical functor (i.e. it preserves the weak equivalences), then we may make F into a cocartesian fibration of 1-categories $\mathbf{E} \to \{\bullet \to \bullet\}$. Taking the nerve, this gives a cocartesian fibration of ∞-categories $N(\mathbf{E}) \to \Delta[1]$. The set of weak equivalences $W_{\mathbf{C}} \amalg W_{\mathbf{D}} \subseteq N_1(\mathbf{E})$ satisfies the assumptions of Proposition 10.6.50, so we may invert them, and we get another cocartesian fibration of ∞-categories

$$N(\mathbf{E})[W^{-1}] \to \Delta[1].$$

You can check directly that the fiber categories are $N(\mathbf{C})[W_{\mathbf{C}}^{-1}]$ and $N(\mathbf{D})[W_{\mathbf{D}}^{-1}]$, and the pushforward functor here is exactly the functor induced by F. So this encodes the map induced by F on the underlying ∞-categories.

Proposition 10.6.50 is great, but it isn't as general as we would like. In practice, the pushforward functors $f_!$ don't preserve all of the weak equivalences, but they do preserve the weak equivalences on some subcategory of cofibrant objects. In other words, $f_!$ can be left-derived in the sense of Definition 3.4.15. The following proposition shows that we can also localize cocartesian fibrations under this assumption. For simplicity, we only describe it in the case where we are starting with a cocartesian fibration of 1-categories, and we don't invert anything in the base category. (The version in [NS18, Prop A.14] is more general than this.)

Proposition 10.6.52. *[NS18, Prop A.14] Suppose $P : \mathbf{E} \to \mathbf{B}$ is a cocartesian fibration of 1-categories, and we have a class of weak equivalences $V = \coprod_{a \in \mathbf{B}} V(a)$ that lies entirely in the fiber categories, i.e. every edge in V projects to an identity edge in B. Assume furthermore that*

- *for every morphism f in $\mathbf{B}$, the pushforward $f_!$ has a left-derived functor $\mathbb{L}f_!$, and*
- *for every composable morphisms f and g, the canonical map $\mathbb{L}g_!\mathbb{L}f_! \to \mathbb{L}(gf)_!$ is an equivalence. (In particular, this happens when we can consistently choose subcategories of cofibrant objects in each category $E(a)$, that are preserved by the pushforwards $f_!$, as in Lemma 3.5.13.)*

Then the map $N(\mathbf{E})[V^{-1}] \to N(\mathbf{B})$ is (up to Joyal equivalence) a cocartesian fibration of quasicategories, whose fiber categories are the localizations $N(\mathbf{E}(a))[V(a)^{-1}]$ and whose pushforwards $N(\mathbf{E}(a))[V(a)^{-1}] \to N(\mathbf{E}(b))[V(b)^{-1}]$ are the left-derived pushforwards $\mathbb{L}f_!$.

Remark 10.6.53. What is remarkable about a result like this is that not only does a cocartesian fibration exist whose fibers are the categories $N(\mathbf{E}(a))[V(a)^{-1}]$ and the pushforwards are the functors $\mathbb{L}f_!$, but this category can be obtained by taking the original fibration and localizing the whole thing at $V = \coprod_{a \in \mathbf{B}} V(a)$. If we don't make the above assumptions, this doesn't work, because localizing the whole category at V does more than just invert the weak equivalences $V(a)$ in each fiber – the localizing effect propagates in an uncontrollable way and we lose our grip on what the fiber categories look like.

Example 10.6.54. If $(\mathbf{C}, W_{\mathbf{C}})$ and $(\mathbf{D}, W_{\mathbf{D}})$ are model categories, and $F : \mathbf{C} \to \mathbf{D}$ is a left Quillen functor with right adjoint $G : \mathbf{D} \to \mathbf{C}$,[20] then we may take the bicartesian fibration $\mathbf{E} \to \{\bullet \to \bullet\}$ that encodes the adjunction (F, G), and invert the weak equivalences $W_{\mathbf{C}} \amalg W_{\mathbf{D}}$. By Proposition 10.6.52 this gives a bicartesian fibration again

$$N(\mathbf{E})[W^{-1}] \to \Delta[1].$$

The fiber categories are $N(\mathbf{C})[W_{\mathbf{C}}^{-1}]$ and $N(\mathbf{D})[W_{\mathbf{D}}^{-1}]$, the pushforward is the left-derived functor $\mathbb{L}F$, and the pullback is the right adjoint of this, which is therefore the right-derived functor $\mathbb{R}G$. This gives the adjunction between $\mathbb{L}F$ and $\mathbb{R}G$ from Corollary 10.3.18 by a different method.

Example 10.6.55. Let $\mathbf{E} \to \mathbf{B}$ be the bifibration of "all diagrams of spaces over all indexing categories" from Example 10.6.23. The base category $\mathbf{B}$ is all small categories $\mathbf{I}$ and functors between them, and the total category $\mathbf{E}$ is all pairs $(\mathbf{I}, X)$ of a small category and a diagram $X : \mathbf{I} \to \mathbf{Top}$. Let W be the class of all pointwise equivalences of diagrams $X \to X'$ over the same category $\mathbf{I}$. Then if we take the nerves and invert W, by Proposition 10.6.52, we get

[20] In fact, we only need these to be categories with weak equivalences, F to be a left-deformable left adjoint, and G a right-deformable right adjoint.

a cocartesian fibration of quasicategories $N(\mathbf{E})[W^{-1}] \to N(\mathbf{B})$. The fiber categories are the ∞-categories of diagrams $N(\mathbf{Top}^{\mathbf{I}})[W(\mathbf{I})^{-1}]$, and the pushforwards are the homotopy left Kan extensions $\mathbb{L}\alpha_!$. These have right adjoints, the derived restriction functors $\mathbb{R}\alpha^* \simeq \alpha^*$, so in fact we have a bicartesian fibration again.

This gives the "∞-category of all diagrams of spaces over all indexing categories." And of course, this example runs with **Top** replaced by any cofibrantly generated model category.

Example 10.6.56. Suppose $(\mathbf{C}, \otimes, I)$ is a symmetric monoidal model category with weak equivalences X. Then when we build the category of operators $\mathbf{C}^{\otimes} \to \mathbf{Fin}_*$ as in Example 10.6.14, the pushforward functors are all tensor products, and so they can all be compatibly left-derived, using the class of weak equivalences $W^{\otimes}$ that consists of all n-tuples of weak equivalences in each of the fiber categories $\mathbf{C}^{\otimes}_n \cong \mathbf{C}^{\times n}$. Therefore when we invert $W^{\otimes}$, by Proposition 10.6.52, we again get a cocartesian fibration $N(\mathbf{C}^{\otimes})[(W^{\otimes})^{-1}] \to N(\mathbf{Fin}_*)$. We will soon call this the "underlying symmetric monoidal ∞-category" of $((\mathbf{C}, W), \otimes, I)$.

Remark 10.6.57. There is a dual form of Proposition 10.6.52 in which we have a cartesian fibration and the pullbacks can all be right-derived. The theorem doesn't work though if we have a cocartesian fibration whose pushforwards can be right-derived.

10.7 Symmetric monoidal ∞-categories

Now that we understand how to write down a system of ∞-categories and functors between them that agree up to natural isomorphism, we can finally define symmetric monoidal ∞-categories.

A symmetric monoidal ∞-category is essentially a system of ∞-categories in which the categories themselves are of the form $C^{\times n}$ for $n \geq 0$, in other words all n-tuples of objects in a fixed ∞-category C. The functors between these correspond to all the ways to take subsets of the n-tuple and tensor the objects in that subset together. The isomorphisms between these functors represent the associativity and commutativity of the tensor product, in an "unbiased" or "coordinate-free" way.

As you might expect, every symmetric monoidal model category gives an example of a symmetric monoidal ∞-category, so $\mathcal{S}$, $\mathcal{S}_*$, and Sp are all important examples. The algebras in these ∞-categories correspond to the *homotopy-coherent* algebras in the original model category, a theme that we will explore more in the next chapter.

We can also put a symmetric monoidal structure on ∞-categories themselves, $\mathbf{Cat}_\infty$, and another symmetric monoidal structure on presentable ∞-categories Pr^L. The tensor product in each of these cases is the cartesian product, respectively the Lurie tensor product of Section 10.5.1. As an application, we can define the smash product of spectra using a universal property.

10.7.1 Definition of a symmetric monoidal ∞-category

Intuitively, a symmetric monoidal ∞-category should be a homotopy-coherent abelian monoid in $\mathbf{Cat}_\infty$. So it should be a quasicategory C with a tensor product functor $C \times C \to C$ that is associative and commutative up to coherent isomorphism, with functor $\{*\} \to C$ that specifies a unit for the tensor product, also up to coherent isomorphism.

How do we make this precise? From our work in Chapter 9, we know that the least painful way to proceed is to use the Segal notion of an E_∞ algebra. This should be a diagram of the form $N(\mathbf{Fin}_*) \to \mathbf{Cat}_\infty$ that satisfies the Segal condition. However, since we know the straightening/unstraightening theorem (Theorem 10.6.38), we understand that it is better to give a cocarteisan fibration into $N(\mathbf{Fin}_*)$ instead.

Recall from Definition 9.4.5 that the **Segal maps** in the category $\mathbf{Fin}_*$ are the maps $\rho^n_j : \mathbf{n}_+ \to \mathbf{1}_+$ that send $j \mapsto 1$ and all other points to $*$.

Definition 10.7.1. [Lur17, 2.0.0.7] A **symmetric monoidal ∞-category** is a cocartesian fibration of quasicategories $C^\otimes \to N(\mathbf{Fin}_*)$ such that, for each $n \geq 0$, the pushforward functors along the Segal maps ρ^n_j for $1 \leq j \leq n$ induce a Joyal equivalence

$$C^\otimes(\mathbf{n}_+) \xrightarrow{\sim} \prod_{j=1}^n C^\otimes(\mathbf{1}_+).$$

We call the fiber category $C = C^\otimes(\mathbf{1}_+)$ the "underlying ∞-category" of the symmetric monoidal ∞-category $C^\otimes$.

So there is an underlying ∞-category C, each fiber category $C^\otimes(\mathbf{n}_+)$ is equivalent to an n-fold product of copies of C in a consistent way, and then the remaining pushforward functors give the unit, the tensor product, and the isomorphisms between their compositions.

To be more specific, the unique map of finite based sets $\mathbf{0}_+ \to \mathbf{1}_+$ gives a functor

$$C^{\times 0} \simeq \{*\} \to C,$$

which gives the unit object of the tensor product $I \in C$. On the other hand, the map $\mathbf{2}_+ \to \mathbf{1}_+$ that folds the two points into one, combined with the equivalence from the Segal maps, gives a functor

$$\otimes : C \times C \to C,$$

and we call this the tensor product in C. Both of these are well-defined up to canonical isomorphism by Lemma 10.6.37. Furthermore the tensor is associative up to isomorphism, as we see using the map $\mathbf{3}_+ \to \mathbf{1}_+$ that folds the three points into one, and the various ways of factoring this into maps that fold two points together at a time. By a similar argument the tensor is commutative up to isomorphism, and has I as a unit up to isomorphism. The

rest of the maps over $\mathbf{Fin}_*$ give all of the higher coherences between these isomorphisms, so that we have an E_∞ monoid in quasicategories. (This is just like how we defined a Segal E_∞ space in Definition 9.4.5, it's just being applied to quasicategories now instead of topological spaces.)

Example 10.7.2. As always, if $C^\otimes$ is the nerve of a 1-category, then this is the same thing as a cocartesian fibration of 1-categories over $\mathbf{Fin}_*$. Every symmetric monoidal 1-category is an example of this by Example 10.6.14. And by Exercise 10.8.27, this is the same thing as a symmetric monoidal 1-category.

Example 10.7.3. More homotopically, if $\mathbf{C}$ is a symmetric monoidal model category, then it has an underlying symmetric monoidal ∞-category $N(\mathbf{C}^\otimes)[(W^\otimes)^{-1}] \to N(\mathbf{Fin}_*)$. This was constructed in Example 10.6.56 by taking the nerve of the category of operators $\mathbf{C}^\otimes$, and inverting the n-tuples of weak equivalences in each fiber. By inspecting this definition, we see that the underlying ∞-category is the usual one $N(\mathbf{C})[W^{-1}]$, and the tensor product is the left-derived tensor product $Q(-) \otimes Q(-)$.

In fact, this example works more generally if $\mathbf{C}$ is any category with weak equivalences satisfying the assumptions of Lemma 4.1.8. In brief, it only has to have some nice subcategory of "cofibrant" or "flat" objects $\mathbf{A}$, preserved by the tensor product, on which the tensor product has the correct homotopy type, and a cofibrant replacement functor that lands in $\mathbf{A}$. This is convenient because sometimes asking for a model structure is a little more than we can do.

So now our favorite examples of spaces $\mathscr{S}$, ∞-categories $\mathbf{Cat}_\infty$, based spaces $\mathscr{S}_*$, and spectra Sp, are all symmetric monoidal ∞-categories, using the Cartesian product in the first two cases and the smash product in the last two cases.

We can also get symmetric monoidal structures more abstractly in any ∞-category that has all finite products, or all finite coproducts.

Proposition 10.7.4. *[Lur17, 2.4.1.9, 2.4.2.7] For any quasicategory C with all finite products, there is a canonical symmetric monoidal structure on C in which the tensor product is the cartesian product. The same statement holds as well for finite coproducts.*

Therefore in the case of spaces $\mathscr{S}$ or ∞-categories $\mathbf{Cat}_\infty$, we don't need to use model categories to construct the symmetric monoidal structure. We can just use the fact that it is the cartesian product. For based spaces $\mathscr{S}_*$ and spectra Sp this doesn't work, because the smash product is neither the coproduct nor the product in those settings.

Lemma 10.7.5. *A symmetric monoidal structure on a quasicategory C induces in a canonical way a symmetric monoidal structure on the homotopy category hC.*

This follows from the fact that, more generally, for any cocartesian fibration $E \to N(\mathbf{B})$ in which the base is a 1-category $N(\mathbf{B})$, taking homotopy categories along the fibers gives

a cocartesian fibration of 1-categories $hE \to \mathbf{B}$. Using this lemma we obtain the symmetric monoidal structures on the homotopy category Ho **Top** and the stable homotopy category Ho **Sp** that we studied in detail earlier in this book. Remember that the symmetric monoidal ∞-category has more structure than the underlying symmetric monoidal homotopy category![21]

10.7.2 Symmetric monoidal functors

What should a map of symmetric monoidal ∞-categories be? There are two possible answers, one of which corresponds to strong symmetric monoidal functors, where $F(X) \otimes F(Y)$ is isomorphic to $F(X \otimes Y)$. The other answer corresponds to lax symmetric monoidal functors, where there is only a map $F(X) \otimes F(Y) \to F(X \otimes Y)$.

Recall from Definition 9.4.30 that an **inert morphism** in $\mathbf{Fin}_*$ is a map of finite based sets $I_+ \to J_+$ such that the preimage of each $j \in J$ is a singleton. These are the morphisms that in a symmetric monoidal ∞-category $C^\otimes$ correspond to the maps of fiber categories $C^{\times I} \to C^{\times J}$ that project away some of the factors and rearrange the others, but don't perform any multiplications or insert any unit maps. (That's why we call them "inert!")

Definition 10.7.6. Let $C^\otimes$ and $D^\otimes$ be two symmetric monoidal ∞-categories.

- A **lax symmetric monoidal functor** is a map of simplicial sets $F : C^\otimes \to D^\otimes$ over $N(\mathbf{Fin}_*)$, in other words it commutes with the projection to $N(\mathbf{Fin}_*)$, and in addition it must preserve every cocartesian lift of an inert morphism in $\mathbf{Fin}_*$.
- A **strong symmetric monoidal functor** is a map of simplicial sets $F : C^\otimes \to D^\otimes$ over $N(\mathbf{Fin}_*)$ that preserves *all* of the cocartesian morphisms. So it is a map of cocartesian fibrations in the sense of Definition 10.6.30.
- An **equivalence of symmetric monoidal ∞-categories** is a map of simplicial sets $F : C^\otimes \to D^\otimes$ over $N(\mathbf{Fin}_*)$ that is also Joyal equivalence. This implies that it preserves all cocartesian morphisms and induces an equivalence on every fiber category. So, equivalently, it is a strong symmetric monoidal functor that induces an equivalence on the underlying ∞-categories $C \simeq D$.

Example 10.7.7. When $C^\otimes$ and $D^\otimes$ arise from symmetric monoidal 1-categories, these capture the earlier notion of lax and strong symmetric monoidal functor from Definition 4.1.21.

To see why this is true, consider the cocartesian morphism from $(X, Y) \in C^\otimes(\mathbf{2}_+)$ to $X \otimes Y \in C^\otimes(\mathbf{1}_+)$. The functor F sends this to some morphism from $(F(X), F(Y)) \in D^\otimes(\mathbf{2}_+)$

[21] Implicitly, we have also been studying the symmetric monoidal ∞-category Sp already. Any time we are working with the model category of spectra, and not just the homotopy category of spectra, we are saying something about the ∞-category Sp, not just the 1-category $h\mathrm{Sp} = \mathrm{Ho}\,\mathbf{Sp}$.

to $F(X\otimes Y)\in D^{\otimes}(\mathbf{1}_+)$. By the universal property of cocartesian morphisms, this factors into the cocartesian morphism $(F(X),F(Y))\to F(X)\otimes F(Y)$ followed by a unique morphism $F(X)\otimes F(Y)\to F(X\otimes Y)$. This gives part of the structure of F as a lax symmetric monoidal functor; the morphism $I\to F(I)$ and the coherences between them follow by a similar argument.

If F preserves cocartesian morphisms, then the morphism $(F(X),F(Y))\to F(X\otimes Y)$ is already cocartesian. It follows that the second part of the factorization, $F(X)\otimes F(Y)\to F(X\otimes Y)$, must be an isomorphism. We use this to see that F is a strong symmetric monoidal functor.

Example 10.7.8. In Example 10.6.56 we associated to each symmetric monoidal model category $\mathbf{C}$ a symmetric monoidal ∞-category $N(\mathbf{C}^{\otimes})[(W^{\otimes})^{-1}]$. As you might expect, each strong symmetric monoidal Quillen left adjoint $F:\mathbf{C}\to\mathbf{D}$ gives a strong symmetric monoidal functor

$$F^{\otimes}: N(\mathbf{C}^{\otimes})[(W_{\mathbf{C}}^{\otimes})^{-1}]\longrightarrow N(\mathbf{D}^{\otimes})[(W_{\mathbf{D}}^{\otimes})^{-1}].$$

One way to see this is to apply F everywhere to form a functor of 1-catgories $F^{\otimes}: C^{\otimes}\to D^{\otimes}$, then restrict to the subcategory of cofibrant objects, and then take the nerve and invert the weak equivalences to get a map of localizations

$$F^{\otimes}: N((\mathbf{C}^{c})^{\otimes})[(W_{\mathbf{C}}^{\otimes})^{-1}]\longrightarrow N((\mathbf{D}^{c})^{\otimes})[(W_{\mathbf{D}}^{\otimes})^{-1}].$$

But after inverting the weak equivalences, the subcategory of cofibrant objects is equivalent to the whole category, so we also get a map $N(\mathbf{C}^{\otimes})[(W_{\mathbf{C}}^{\otimes})^{-1}]\longrightarrow N(\mathbf{D}^{\otimes})[(W_{\mathbf{D}}^{\otimes})^{-1}]$ as desired.

(Alternatively, we could take the Grothendieck construction of $F^{\otimes}$, which is a cocartesian fibration over the product category $\Delta[1]\times\mathbf{Fin}_*$, and apply Proposition 10.6.52. This gives a cocartesian fibration over $\Delta[1]$, and choosing a pushforward along this gives the desired map $N(\mathbf{C}^{\otimes})[(W_{\mathbf{C}}^{\otimes})^{-1}]\longrightarrow N(\mathbf{D}^{\otimes})[(W_{\mathbf{D}}^{\otimes})^{-1}]$.)

On the underlying ∞-category, this is just the left-derived functor $\mathbb{L}F$, so when F is a Quillen equivalence, this functor is an equivalence of symmetric monoidal ∞-categories.

Corollary 10.7.9. *All of our symmetric monoidal model categories of spectra give the same symmetric monoidal ∞-category* Sp*, up to equivalence. And all of the rest of the strong symmetric monoidal Quillen left adjoints from* (8.1.36) *give strong symmetric monoidal functors of ∞-categories.*

10.7.3 ∞-operads and algebras

One of the main things we like to do with a symmetric monoidal category is study algebras and modules over them. In Exercise 9.6.28 we recast this in a fancy way using categories

of operators: an algebra in a symmetric monoidal 1-category $\mathbf{C}$ is a map $\mathscr{M}^{\otimes} \to \mathbf{C}^{\otimes}$ over $\mathbf{Fin}_*$, where $\mathscr{M}$ is the multicategory or operad that parametrizes the kind of algebraic structure we want. And the functor has to preserve the inert morphisms (essentially the combinations of Segal maps) but not all of the cocartesian morphisms. This corresponds to the fact that an algebra will give an object X with a morphism $X \otimes X \to X$, not necessarily an isomorphism $X \otimes X \cong X$.

Let's make this precise. Recall again from Definition 9.4.30 that an **inert morphism** in $\mathbf{Fin}_*$ is a map of finite based sets $I_+ \to J_+$ such that the preimage of each $j \in J$ is a singleton.

Definition 10.7.10. [Lur17, 2.1.1.10] An **∞-multicategory** is a map of quasicategories $M^{\otimes} \to N(\mathbf{Fin}_*)$ that

- has cocartesian lifts for every *inert* morphism in $\mathbf{Fin}_*$, but not necessarily every morphism,
- the pushforward functors along the Segal maps ρ^n_j for $1 \leq j \leq n$ still induce a Joyal equivalence
$$M^{\otimes}(\mathbf{n}_+) \xrightarrow{\sim} \textstyle\prod_{j=1}^n M^{\otimes}(\mathbf{1}_+), \text{ and}$$
- for every morphism $\alpha\colon \mathbf{m}_+ \to \mathbf{n}_+$ and every pair of objects $x \in M^{\otimes}(\mathbf{m}_+)$ and $y \in M^{\otimes}(\mathbf{n}_+)$, if we let $\mathrm{Map}_{M^{\otimes}}(x,y)_\alpha$ denote the union of components of the mapping space that project down to α, then the Segal maps ρ^n_j for $1 \leq j \leq n$ and their cocartesian lifts to $y \to y_j$, $1 \leq j \leq n$, induce a weak homotopy equivalence on the space of morphisms
$$\mathrm{Map}_{M^{\otimes}}(x,y)_\alpha \xrightarrow{\sim} \textstyle\prod_{j=1}^n \mathrm{Map}_{M^{\otimes}}(x,y_j)_{\rho^n_j \circ \alpha}.$$

As a special case, an **∞-operad** is an ∞-multicategory in which the map $M^{\otimes} \to N(\mathbf{Fin}_*)$ is a bijection on isomorphism classes of objects. So if we use the skeletal model for $\mathbf{Fin}_*$, then the fiber quasicategories $M^{\otimes}(\mathbf{n}_+)$ each have a single isomorphism class of objects.

Warning 10.7.11. Many authors use "∞-operad" to refer to both of these notions. We believe this is misleading, because an operad is classically a multicategory with one object, and it's helpful to single those out amongst the larger class of all multicategories.

What's going on here is that we have less structure than a symmetric monoidal ∞-category. We still have an underlying ∞-category $M = M^{\otimes}(\mathbf{1}_+)$, and every fiber category is $M^{\times n}$. But now the fold map $\mathbf{n}_+ \to \mathbf{1}_+$ no longer has a cocartesian lift, so we no longer get a tensor product functor $M^{\times n} \to M$ for $n \geq 0$. Rather, we only have a notion of morphism over the map $\mathbf{n}_+ \to \mathbf{1}_+$, and we think of these as the "n-ary morphisms" of M. The final

axiom assures us that when we look at maps in $M^{\otimes}$ lying over an arbitrary map of finite sets $\alpha\colon \mathbf{m}_+ \to \mathbf{n}_+$, all we see are the $\alpha^{-1}(j)$-ary morphisms for each value of $j \in \mathbf{n}$.

So in total, an ∞-multicategory has a collection of objects, and for each n-tuple of input objects and single output object, a notion of n-ary morphism. It also has various coherent isomorphisms between their composition. In the special case where there is only one object, this is an ∞-operad.

Example 10.7.12. Every symmetric monoidal ∞-category $C^{\otimes}$ is an ∞-multicategory. The fact that every $\alpha\colon \mathbf{m}_+ \to \mathbf{n}_+$ has a cocartesian lift makes the last axiom of Definition 10.7.10 hold. In this case the n-ary morphisms from $\{a_1, \dots, a_n\}$ to b are given more simply by the morphisms from the tensor product $a_1 \otimes \cdots \otimes a_n$ to b.

Example 10.7.13. If $\mathcal{O}$ is any operad in topological spaces or in (fibrant) simplicial sets, then we may form its category of operators $\mathcal{O}^{\otimes} \to \mathbf{Fin}_*$ as in Example 9.4.36, and take the simplicial nerve, giving a map of quasicategories

$$N^s(\mathcal{O}^{\otimes}) \longrightarrow N(\mathbf{Fin}_*).$$

Using Lemma 10.6.32, we see that this satisfies the definition of an ∞-operad.

As a special case, taking $\mathcal{O} = \mathrm{Comm}$ to be the commutative operad, since $\mathrm{Comm}^{\otimes} \cong \mathbf{Fin}_*$, we get the ∞-operad given by the identity functor $N(\mathbf{Fin}_*) \to N(\mathbf{Fin}_*)$. And if we take $\mathcal{O} = \mathrm{Assoc}$ to be the associative operad, we get the ∞-operad given by the functor $N(\mathrm{Assoc}^{\otimes}) \to N(\mathbf{Fin}_*)$. Notice that each of these ∞-operads is the nerve of some map of 1-categories.

Definition 10.7.14. [Lur17, 2.1.2.7] As in Definition 10.7.6, a **map of ∞-multicategories** $M^{\otimes} \to N^{\otimes}$ is a map of simplicial sets that commutes with the projection to $N(\mathbf{Fin}_*)$, and preserves the cocartesian lifts of the inert morphisms. When $C^{\otimes}$ is a symmetric monoidal category, we call this an **M-algebra in C**.

The **∞-category of M-algebras in C**, denoted $\mathrm{Alg}_M(C)$, is the simplicial set in which an n-simplex is a map $\Delta[n] \times M^{\otimes} \to C^{\otimes}$ over $N(\mathbf{Fin}_*)$, that sends every vertex of $\Delta[n]$ times a cocartesian morphism in $M^{\otimes}$ to a cocartesian morphism in $C^{\otimes}$.

We also have the following special cases:

- When $M^{\otimes} = \mathrm{Assoc}^{\otimes}$, we call $\mathrm{Alg}_{\mathrm{Assoc}}(C)$ the **∞-category of algebras** $\mathrm{Alg}(C)$.
- When $M^{\otimes} = \mathrm{Comm}^{\otimes} = \mathbf{Fin}_*$, we call $\mathrm{Alg}_{\mathrm{Comm}}(C)$ the **∞-category of commutative algebras** $\mathrm{CAlg}(C)$. A commutative algebra is just a section of the functor $C^{\otimes} \to N(\mathbf{Fin}_*)$ that sends each inert morphism to some cocartesian lift.
- When $M^{\otimes} = LM^{\otimes}$ is the category of operators from Example 9.4.54, then $\mathrm{Alg}_M(C)$ is the "∞-category of all pairs of an algbra and a left module." If we take the full subcategory in which the algebra part is a fixed algebra R, we get the **∞-category of left R-modules** $\mathrm{LMod}_R(C)$.

- When $M^{\otimes} = D^{\otimes}$ is also a symmetric monoidal category, we call this the **∞-category of lax symmetric monoidal functors** $\mathrm{Fun}^{\mathrm{lax}}(D, C)$.
- The **∞-category of strong symmetric monoidal functors** $\mathrm{Fun}^{\otimes}(D, C)$ is the full subcategory of $\mathrm{Fun}^{\mathrm{lax}}(D, C)$ on those functors that are strong symmetric monoidal.

It follows from Proposition 10.1.26 above that these simplicial sets are indeed quasicategories.

Example 10.7.15. As usual, if C came from a symmetric monoidal 1-category $\mathbf{C}$, then $\mathrm{Alg}(C)$ and $\mathrm{CAlg}(C)$ are just the nerves of the 1-categories of algebras and commutative algebras in $\mathbf{C}$. More generally, if C came from a symmetric monoidal simplicial category $\mathbf{C}_\bullet$ as in Example 10.6.34, and $\mathscr{O}$ is any operad in simplicial sets, then every $\mathscr{O}$-algebra in $\mathbf{C}_\bullet$ gives an algebra over $N^s(\mathscr{O}^{\otimes})$ in the above sense. This gives a map of simplicial sets

$$N^s(\mathrm{Alg}_{\mathscr{O}}(\mathbf{C})) \longrightarrow \mathrm{Alg}_{N^s(\mathscr{O})}(N^s\mathbf{C})$$

that is usually not an isomorphism. See the following example.

Example 10.7.16. If $\mathbf{C}$ is a symmetric monoidal simplicial model category, then for every operad $\mathscr{O}$ in simplicial sets, we can compare $\mathscr{O}$-algebras in $\mathbf{C}$ with $N^s(\mathscr{O}^{\otimes})$-algebras in the underlying symmetric monoidal ∞-category $N(\mathbf{C}^{\otimes})[(W^{\otimes})^{-1}]$. As in the previous example, there is a canonical map

$$N(\mathrm{Alg}_{\mathscr{O}}(\mathbf{C}))[W^{-1}] \longrightarrow \mathrm{Alg}_{N^s(\mathscr{O})}(N(\mathbf{C}^{\otimes})[(W^{\otimes})^{-1}]),$$

but in general this is not a Joyal equivalence. You should think of the left-hand side as the "∞-category of strict $\mathscr{O}$-algebras in the model category," and the right-hand side as the "∞-category of homotopy-coherent $\mathscr{O}$-algebras." For instance, if $\mathscr{O} = \mathrm{Comm}$ and $\mathbf{C} = \mathbf{Top}$, then we are comparing strictly commutative monoids with E_∞ monoids, and these are not equivalent.

When the operad $\mathscr{O}$ is free enough that the orbits $\mathscr{O}(n) \times_{\Sigma_n} X^{\otimes n}$ are equivalent to the homotopy orbits, we expect $\mathscr{O}$-algebras to be "homotopy-coherent" already, and in these cases the two notions do line up:

Proposition 10.7.17. *[Hau25, Thm 4.10], [WY24, Thm 7.3.3] If $\mathscr{O}$ is a Σ-cofibrant simplicial or topological operad, and $\mathbf{C}$ is a symmetric monoidal simplicial or topological model category, respectively, such that $\mathrm{Alg}_{\mathscr{O}}(\mathbf{C})$ has a right-induced model structure, then the model category $\mathrm{Alg}_{\mathscr{O}}(\mathbf{C})$ has underlying ∞-category $\mathrm{Alg}_{N^s(\mathscr{O})}(N^s(\mathbf{C})[W^{-1}])$.*

Recall from Definition 9.3.17 that "Σ-cofibrant" means that each level $\mathscr{O}(n)$ is a retract of a free Σ_n-cell complex of spaces, or a simplicial set that has a free Σ_n-action at each level. See Theorem A.3.23 for several examples for which the remaining assumptions hold. In particular, E_∞-algebras in the model category **Top** correspond to commutative algebras in the ∞-category $\mathscr{S}$.

10.7.4 The universal property of the smash product

We conclude this chapter by giving Lurie's characterization of the smash product of spectra.

Recall from Section 10.5.1 that there is a "sum" and a "tensor" on ∞-categories, given by the disjoint union $C \amalg D$ and the cartesian product $C \times D$, respectively. Both of these define symmetric monoidal structures on $\mathbf{Cat}_\infty$, by Proposition 10.7.4. Or more concretely, each of these preserves all Joyal equivalences, even between simplicial sets that aren't quasicategories, and so it passes to a symmetric monoidal structure on $\mathbf{Cat}_\infty \simeq N(\mathbf{sSet})[W_{\text{Joyal}}^{-1}]$.

Example 10.7.18. Commutative algebras in $\mathbf{Cat}_\infty$ with the cartesian product turn out to be identified with symmetric monoidal ∞-categories.

Let's move to presentable ∞-categories Pr^L, and let's think of this as an $(\infty,1)$-category now, so we're going to take the cores of the mapping spaces, $\mathrm{LFun}(C,D)^{\simeq}$. In other words, the objects are presentable ∞-categories (Definition 10.4.18), and the spaces of morphisms consist of all the left adjoint functors, and natural isomorphisms between them.

As discussed in Section 10.5.1, the coproduct in this setting is now the cartesian product $C \times D$. This is the same as the product, so it induces a symmetric monoidal structure on Pr^L by Proposition 10.7.4. But the most interesting operation of all is the Lurie tensor product from Definition 10.5.2.

Theorem 10.7.19. *[Lur17, 4.8.1.15] There is a symmetric monoidal structure on* Pr^L *whose tensor product is the Lurie tensor product, and whose unit is the* ∞*-category of spaces* $\mathscr{S}$.

A commutative algebra for this tensor product is a symmetric monoidal ∞-category whose underlying ∞-category C is presentable, and whose tensor preserves colimits in each slot. This is called a **presentably symmetric monoidal ∞-category**.

Corollary 10.7.20. $\mathscr{S}$ *is a presentably symmetric monoidal* ∞*-category in a canonical way, with tensor the cartesian product.*

Now, it turns out that when we restrict attention to *pointed* presentable ∞-categories, we get a symmetric monoidal structure with the *same tensor product*! But the unit object is different, because $\mathscr{S}$ isn't pointed.

Proposition 10.7.21. *[Lur17, 4.8.2.11] The subcategory of pointed presentable* ∞*-categories* $\mathrm{Pr}^{\mathrm{Pt}}$ *is closed under the Lurie tensor product, and therefore inherits a symmetric monoidal structure, whose unit is the* ∞*-category of based spaces* $\mathscr{S}_*$.

Therefore pointed presentable ∞-categories are all $\mathscr{S}_*$-modules in a canonical way. In particular:

Corollary 10.7.22. *$\mathcal{S}_*$ is a presentably symmetric monoidal ∞-category in a canonical way, with tensor the smash product.*

Now we can run this story one more time with stable presentable ∞-categories. Again, they have a symmetric monoidal structure with the *same tensor product*, but a different unit:

Proposition 10.7.23. *[Lur17, 4.8.2.18] The subcategory of stable presentable ∞-categories* $\mathrm{Pr}^{\mathrm{St}}$ *is closed under the Lurie tensor product, and therefore inherits a symmetric monoidal structure, whose unit is the ∞-category of spectra* Sp.

Corollary 10.7.24. *[Lur17, 4.8.2.19]* Sp *is a presentably symmetric monoidal ∞-category in a canonical way. In other words, it has a symmetric monoidal structure with unit object $\mathbb{S}$, and tensor that preserves colimits separately in each variable.*

Moreover this symmetric monoidal structure is unique up to contractible choice: for any other such structure, there is a contractible Kan complex of strong symmetric monoidal functors between them that preserve colimits on the underlying category.

This is Lurie's universal characterization of the smash product of spectra. It is essentially the only product one can possibly form, that has the sphere spectrum as the unit, and that preserves colimits in each variable. *Any* other such product is *canonically* isomorphic to the smash product of spectra!

Example 10.7.25. The underlying symmetric monoidal ∞-category of symmetric or orthogonal spectra, $N^s(\mathbf{Sp}^\Sigma)^\otimes[W^{-1}]$ or $N^s(\mathbf{Sp}^O)^\otimes[W^{-1}]$, are both equivalent to Sp and satisfy this universal property. Therefore the concrete model of the smash product we saw in Section 6.2 is in fact characterized by Lurie's universal property!

This is a satisfying conclusion because it suggests that the actions by the symmetric or orthogonal groups, which at first seemed ad-hoc, were really correct thing to do. They didn't somehow pervert the structure of spectra, because they still give the correct smash product on the underlying ∞-category. Any other way of making the smash product work has to be equivalent to symmetric or orthogonal spectra!

10.8 Exercises

10.8.1. In a simplicial category $\mathbf{C}_\bullet$, prove that $f \in \mathbf{C}_0(X,Y)$ is a homotopy equivalence iff composing with f gives weak equivalences of simplicial sets

$$\mathbf{C}_\bullet(W,X) \xrightarrow{\sim} \mathbf{C}_\bullet(W,Y) \qquad \text{and} \qquad \mathbf{C}_\bullet(Y,Z) \xrightarrow{\sim} \mathbf{C}_\bullet(X,Z).$$

10.8.2. Prove that the 1-category of categories and functors **Cat** has a "natural" model structure in which

- a cofibration is a functor $F : C \to D$ that is injective on the set of objects,
- a weak equivalence is an equivalence of categories (so a functor $F : C \to D$ that is fully faithful and a bijection on the set of isomorphism classes of objects), and
- a fibration is an isofibration in the sense of Definition 10.1.33: it is a functor $F : \mathbf{C} \to \mathbf{D}$ such that every isomorphism $F(X) \cong Z$ in $\mathbf{D}$ can be lifted to an isomorphism $X \cong Y$ in $\mathbf{C}$.

(This exercise comes from [Joy08, 2.4].)

10.8.3. Formulate the 2-out-of-3 property for isomorphisms in any quasicategory C. Then prove that it always holds.

10.8.4. (a) Prove Lemma 10.1.18, that the homotopy category $h(-)$ is the left adjoint of the nerve $N(-)$.

(b) Prove that these give a Quillen adjunction between **sSet** with the Joyal model structure and **Cat** with the natural model structure of Exercise 10.8.2.

10.8.5. Suppose C and D are quasicategories.

(a) Prove that $C \times D$ is a quasicategory.

(b) Prove that $h(C \times D) \cong hC \times hD$. (In fact, this is true even if C and D are just simplicial sets.)

(c) Conclude that for quasicategories C, D, and E the composition of functors induces a functor

$$h\,\mathrm{Fun}(C,D) \times h\,\mathrm{Fun}(D,E) \to h\,\mathrm{Fun}(C,E),$$

and therefore if f is naturally isomorphic to f', then $g \circ f$ is naturally isomorphic to $g \circ f'$.

10.8.6. [Lan21, §2.3] Prove that the inclusion of the core $C^{\simeq} \to C$ induces on mapping spaces

$$\mathrm{Map}_{C^{\simeq}}(x,y) \to \mathrm{Map}_{C}(x,y)$$

an inclusion of a set of connected components, namely those components in which the vertices represent isomorphisms $x \cong y$ in C.

Prove that every equivalence of quasicategories $C \simeq D$ induces an equivalence of cores $C^{\simeq} \simeq D^{\simeq}$.

10.8.7. Using Exercise 10.8.5, prove that if C and D are quasicategories and $f : C \to D$ is a functor, then f is a Joyal equivalence (Definition 10.1.24) iff there is an inverse functor $g : D \to C$ such that $g \circ f$ is naturally isomorphic to id_X and $f \circ g$ is naturally isomorphic to id_Y.

10.8.8. This exercise analyzes how the simplicial nerve interacts with the ordinary nerve.

(a) If $\mathbf{C}_\bullet$ is a simplicial category, prove that there is an isomorphism of categories

$$hN^s(\mathbf{C}_\bullet) \cong N(\pi_0(\mathbf{C}_\bullet)).$$

So along the simplicial nerve, the two notions of "homotopy category" agree with each other.

(b) Prove that $\pi_0\mathfrak{C}(\Delta[n])$ is the discrete category $[n] = \{0 \to 1 \to \cdots \to n\}$.

(c) If $\mathbf{C}_\bullet = \mathbf{C}_0$ is a discrete simplicial category, prove that there is an isomorphism of simplicial sets

$$N^s(\mathbf{C}_\bullet) \cong N(\mathbf{C}_0).$$

So the simplicial nerve is just the ordinary nerve in this case.

10.8.9. In Example 10.1.56 we defined $\mathscr{P}(K) = \mathrm{Fun}(K^{\mathrm{op}}, \mathscr{S})$, where K is any simplicial set. Explain why there is essentially no loss of generality if we restrict attention to the case that K is a quasicategory.

10.8.10. Define the join $C * D$ for two 1-categories C and D. Like the join for simplicial sets, it should contain C and D as full subcategories, and then it should have a unique morphism from every object of C to every object of D.

Check that this defines a category $C * D$, and then prove that its nerve agrees with the join of the nerves

$$N(C * D) \cong N(C) * N(D),$$

as defined in Remark 10.3.35.

10.8.11. (a) Prove that if $F : C \to D$ is an equivalence of ∞-categories with inverse $G : D \to C$, then (F, G) also form an adjunction of ∞-categories.

(b) Explain how adjunctions of ∞-categories compose.

(c) Explain how parts (a) and (b) are being used implicitly in Example 10.3.20.

10.8.12. Lift the unstable and stable Segal recognition theorems (Theorem 9.4.15 and Theorem 9.4.19) to the following statement, using Corollary 10.3.18:

Theorem 10.8.1. *The realization B and the loopspace construction $\Omega_\bullet$ from Example 9.4.3 define an adjunction of ∞-categories*

$$\mathrm{Alg}(\mathscr{S}) \underset{\Omega_\bullet}{\overset{B}{\rightleftarrows}} \mathscr{S}_*.$$

that gives an equivalence between the ∞-categories of grouplike A_∞ spaces and connected spaces. Furthermore the delooping spectrum of Definition 9.4.16 gives an adjunction of ∞-categories

$$\mathrm{CAlg}(\mathscr{S}) \underset{\Omega^\infty}{\overset{\mathbb{L}B^\infty}{\rightleftarrows}} \mathrm{Sp}.$$

that gives an equivalence between the grouplike E_∞ spaces and connective spectra.

10.8.13. If $C = N(\mathbf{C})$ is the nerve of a 1-category and K is a small simplicial set, prove that:

(a) Functors $K \to C$ correspond to functors $hK \to \mathbf{C}$, where hK is the homotopy category of Definition 10.1.17.

(b) The functor category $\mathrm{Fun}(K, C)$ is isomorphic to the nerve of the 1-category of functors $hK \to \mathbf{C}$ and natural transformations between them.

(c) ∞-categorical colimits and limits of diagrams $K \to C$ coincide with the 1-categorical colimits and limits of the corresponding diagrams $hK \to \mathbf{C}$.

(d) The cone $K^\triangleright = K * \{\infty\}$ commutes with passage to the homotopy category: $h(K^\triangleright) \cong (hK)^\triangleright$.

This shows us that for ∞-categories that came from 1-categories, there is no interesting theory of colimits and limits for diagrams indexed by arbitrary simplicial sets: the only interesting examples are the diagrams indexed by 1-categories.

10.8.14. Let $\mathscr{S} \simeq N^s(\mathbf{sSet})[W^{-1}]$ be the ∞-category of spaces. Let K be any simplicial set, and consider the diagram $K \to \mathscr{S}$ that takes everything to the one-point space $* \in \mathscr{S}_0$. Prove that the colimit of this diagram is $K \in \mathscr{S}_0$, and the limit is again $* \in \mathscr{S}_0$. (Hint: You can turn this into a statement in the model category **sSet** first.)

10.8.15. Recall the construction of pushouts in $\mathbf{Cat}_\infty$ in Example 10.3.29. Suppose we take the pushout of $D \leftarrow * \rightarrow E$, where the single point $*$ maps to the initial object of D and E. Assume furthermore assume that D and E have no morphisms back to this initial object. Then prove that the pushout in $\mathbf{Cat}_\infty$ is the wedge sum $D \vee E$.

10.8.16. Let κ be a regular cardinal (Definition 10.4.1). Explain why the following conditions on a simplicial set J are equivalent:

- The union of the nondegenerate points $\bigcup_{n\geq 0} J_n^{\mathrm{nd}}$ has cardinality $< \kappa$.
- The realization $|J|$ has $< \kappa$ cells.

So it doesn't matter which one of these we use in the definition of a κ-filtered ∞-category (Definition 10.4.2). Show that furthermore if $\kappa \geq \aleph_1$ then these are equivalent to:

- The union of the sets $\bigcup_{n\geq 0} J_n$ has cardinality $< \kappa$.
- The set J_n has cardinality $< \kappa$ for every $n \geq 0$.
- Each set of nondegenerate points J_n^{nd} has cardinality $< \kappa$.

10.8.17. Show that the equivalence of Lemma 10.4.40 also gives an equivalence on the full subcategory of right-adjoint functors, $\mathrm{RFun}(C, D_*) \simeq \mathrm{RFun}(C, D)_*$. Use this to prove Corollary 10.4.43, by a string of manipulations similar to the one above Lemma 10.5.1.

10.8.18. Consider the operation $C \rightsquigarrow C_*$ from Example 10.4.39.

(a) Prove that there is a fiber sequence $\mathrm{Map}_{C_*}(x, y) \to \mathrm{Map}_C(x, y) \to \mathrm{Map}_C(*, y)$, which is natural in the sense that each map of quasicategories gives a map of fiber sequences.

(b) Prove that when C is pointed, the forgetful map $C_* \to C$ is an equivalence.

(c) Prove that $(-)_*$ preserves fully faithful functors.

(d) Prove that $(-)_*$ preserves adjunctions. So if C and D are adjoint then C_* and D_* are adjoint as well.

(e) Conclude that $(-)_*$ preserves Bousfield localizations, and use this to prove that if C is pointed and a localization of $\mathrm{Fun}(K^{\mathrm{op}}, \mathscr{S})$, then C is also a localization of $\mathrm{Fun}(K^{\mathrm{op}}, \mathscr{S}_*)$ for the same value of K.

10.8.19. As in Example 10.4.75, let $\mathbf{I}$ be the category that encodes retract diagrams. Let $\mathbf{Top}^{\mathbf{I}}$ be the category of $\mathbf{I}$-diagrams, in other words the category of pairs (X, A) where X is a space and A is a retract of X. Let $S_0 = \{(S^{n-1} = S^{n-1}) \to (D^n = D^n) : n \geq 0\}$.

(a) Prove that the S_0-local objects in $\mathbf{Top}^{\mathbf{I}}$ are the pairs (X, A) in which A is weakly contractible.

(b) Prove that the S_0-local equivalences are the maps $(X, A) \to (Y, B)$ that induce an equivalence on homotopy cofibers.

(c) Prove that the left Bousfield localization of $\mathbf{Top}^{\mathbf{I}}$ along S_0 is Quillen equivalent to $\mathbf{Top}_*$.

(d) Conclude using Example 10.4.57 that the ∞-categorical Bousfield localization of $\mathrm{Fun}(N(\mathscr{I})^{\mathrm{op}}, \mathscr{S})$ along S_0 is equivalent to $\mathscr{S}_*$.

10.8.20. Suppose $P : \mathbf{E} \to \mathbf{B}$ is any functor of 1-categories.

(a) Prove than any isomorphism $X \cong Y$ in $\mathbf{E}$ is automatically a cocartesian morphism.

(b) Prove that composing with an isomorphism doesn't change whether a given morphism $X \to Y$ in $\mathbf{E}$ is cocartesian. (This allows us to see why the notion is invariant under equivalences of categories.)

(c) Suppose $\bar{f} : X \to Y$ is a lift of $f : A \to B$, and f is an isomorphism. Prove that $\bar{f}$ is cocartesian iff it is an isomorphism.

10.8.21. Suppose $P : \mathbf{E} \to \mathbf{B}$ is a functor of 1-categories. Prove that the following are equivalent:

- Every morphism in $\mathbf{E}$ is cocartesian.
- P is a cocartesian fibration, and the fiber categories $\mathbf{E}_A$ are groupoids.

This is the 1-category version of left fibrations from Definition 10.6.45.

10.8.22. Let $\mathbf{E} \to \mathbf{B}$ be a cocartesian fibration of 1-categories.

(a) Prove that for any morphism $f : A \to B$ in the base category $\mathbf{B}$, the functor $f_!$ defined just after Lemma 10.6.5 is in fact a functor $\mathbf{E}_A \to \mathbf{E}_B$.

(b) Prove that for morphisms $f : A \to B$ and $g : B \to C$, we get a canonical isomorphism of pushforwards $(gf)_! \cong g_! f_!$.

(c) Prove that these canonical isomorphisms make the collection of all the pushforward functors $\{f_!\}_{f : A \to B}$ into an indexed category.

10.8.23. Let $\mathbf{E} \to \mathbf{B}$ be any functor, and let $f : A \to B$ be any morphism in $\mathbf{B}$ that has cocartesian lifts to every $X \in \mathbf{E}_A$, and cartesian lifts to every $Y \in \mathbf{E}_B$.

(a) Prove that there is a canonical map $X \to f^* f_! X$ for every $X \in \mathbf{E}_A$, and prove that this defines a natural transformation $\mathrm{id} \to f^* f_!$.

(b) Prove that this and the other canonical map $f_! f^* Y \to Y$ define an adjunction between $f_!$ and f^*. So pushforwards and pullbacks are always adjoint to each other.

10.8.24. Suppose $P : \mathbf{E} \to \mathbf{B}$ is any functor of 1-categories. Prove that the map of nerves $N(P) : N(\mathbf{E}) \to N(\mathbf{B})$ is a cocartesian fibration of ∞-categories (Definition 10.6.28) iff P is a cocartesian fibration of 1-categories.

10.8.25. Define the bifibration of **parametrized unbased spaces** $\mathscr{P}\mathbf{Top} \to \mathbf{Top}$ as follows. For each $A \in \mathbf{Top}$, we define an object of $\mathscr{P}\mathbf{Top}_A$ to be another space X and a map of spaces $X \to A$. Intuitively, this is a system of spaces X_a parametrized by the points $a \in A$. A morphism in $\mathscr{P}\mathbf{Top}$ from $(X \to A)$ to $(Y \to B)$ is a commuting square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & B. \end{array}$$

Prove this is a bifibration, and describe the pushforward and pullback functors.

10.8.26. Define the bifibration of **parametrized based spaces**, also known as **retractive spaces**, $\mathscr{R} \to \mathbf{Top}$, by defining an object of $\mathscr{R}_A$ to be a space X and maps $A \to X \to A$ that compose to the identity of A. Intuitively, this is a system of *based* spaces X_a, the basepoints being given by the map $A \to X$. A morphism in $\mathscr{R}$ from $(A \to X \to A)$ to $(B \to Y \to B)$ is a commuting diagram

$$\begin{array}{ccc} A & \longrightarrow & D \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & B. \end{array}$$

Prove this is a bifibration, and describe the pushforward and pullback functors.

10.8.27. Let $\mathbf{C}$ be a symmetric monoidal 1-category and let $\mathbf{C}^{\otimes} \to \mathbf{Fin}_*$ be the functor of 1-categories constructed in Example 10.6.14.

(a) Explain how to recover the category $\mathbf{C}$, the tensor product functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, the unit object $I : \{*\} \to \mathbf{C}$, and the structure isomorphisms α, ρ, and γ from Definition 4.1.1, all from the cocartesian fibration $\mathbf{C}^{\otimes} \to \mathbf{Fin}_*$.

(b) Now suppose $\mathbf{E} \to \mathbf{Fin}_*$ is any cocartesian fibration in "standard form:" the fiber categories are isomorphic to $\mathbf{C}^{\times n}$ for some fixed category $\mathbf{C}$, and along this isomorphism the Segal maps induce the projection maps $\mathbf{C}^{\times n} \to \mathbf{C}$. Prove that the rest of the cocartesian structure defines a symmetric monoidal structure on $\mathbf{C}$.

(c) Finally, explain why if we take any cocartesian fibration $\mathbf{E} \to \mathbf{Fin}_*$ such that the Segal maps induce an equivalence of categories $\mathbf{E}(\mathbf{n}_+) \simeq \prod_{j=1}^{n} \mathbf{E}(\mathbf{1}_+)$, then there is an equivalence of categories from $\mathbf{E}$ to another cocartesian fibration in standard form, and therefore $\mathbf{E}$ also specifies a symmetric monoidal category.

Chapter 11

Higher algebra (preview)

Now that we have both the smash product and the theory of simplicial objects up and running, we have everything we need to study the algebra of rings and modules in spectra, generalizing the study of DGAs and their modules in chain complexes. In this chapter we develop some of the basic theory of A_∞ ring spectra and their modules, and E_∞ ring spectra and their modules.

This is sometimes called "higher algebra." Higher algebra is the study of

- ring spectra and modules over them, or
- stable ∞-categories (Definition 10.4.44).

These are essentially equivalent, in a sense that we will discuss in **??**. Therefore we can develop most of higher algebra using only model categories. We will take this approach in this book, referring to the previous chapter only when the language of ∞-categories is especially clarifying, or when it would aid the reader to help connect these results to the literature. Of course, there are other works that take the opposite approach and work purely in ∞-categories, for example [Lur17, Cis19, Cno25].

One should think of the above two perspectives as analogous to the characterization of classical algebra as the study of

- rings and modules over them, or
- abelian categories,

which are also essentially equivalent by the Freyd–Mitchell embedding theorem.[1]

[1] Of course, algebra encompasses many other things as well, including groups and monoids, fields, solutions to polynomial equations, or more generally varieties and schemes, and so on. Some of these concepts have well-established "higher" analogues, while others are still under active development.

Returning our focus to ring spectra and their modules, the important basic results on these include:

- rectification theorems that tell us that A_∞ ring spectra are essentially the same thing as associative rings, and E_∞ ring spectra are the same thing as commutative rings,
- the construction of tensor products and bar constructions for modules over ring spectra, and more generally the many-object version of ring spectra called **spectral categories**,
- Morita theorems (which in [Lur17] are called recognition principles), that allow us to recognize any stable model category or stable ∞-category as the category of modules over a ring spectrum or spectral category,
- the Morita bicategory of all rings, bimodules, and tensor products, and the characterization of dualizable modules in this bicategory, and
- the use of t-structures, which allows us to see that stable ∞-categories are in some sense all given as iterated extensions of abelian categories.

One special case of the Morita theory is that the stable model category of chain complexes is equivalent to the category of module spectra over the ring $H\mathbb{Z}$. This theorem is a central theme of the book, because it explains how chain complexes of abelian groups sit inside the larger world of spectral or higher algebra. And we also have the converse statement that this homotopical world can also be filtered in a way that the successive quotients are something discrete and algebraic again (like the homotopy groups of a spectrum).

The most important ring spectrum is the sphere spectrum $\mathbb{S}$. Every spectrum is a module over $\mathbb{S}$ in a canonical way, just as every abelian group is a $\mathbb{Z}$-module in a canonical way. Therefore, we can think of spectral or higher algebra as the passage from $\mathbb{Z}$-coefficients to $\mathbb{S}$-coefficients. We sometimes write the smash product $\wedge$ as the tensor product over the sphere, $\otimes_{\mathbb{S}}$, and the smash product over $H\mathbb{Z}$ as the tensor product $\otimes_{\mathbb{Z}}$, in order to emphasize this transition.

In particular, we can take a construction like Hochschild homology in ordinary algebra, and replace the tensor products with smash products over $\mathbb{S}$, to get a different construction called topological Hochschild homology (THH). This turns out to be an essential ingredient to the study of algebraic K-theory, and many other constructions of interest in homotopy theory.

Chapter 12

Spectral sequences (preview)

A spectral sequence is a generalization of a long exact sequence. Like long exact sequences, spectral sequences are an important computational tool, and are ubitquitous in homotopy theory.

Despite the name, you actually don't need spectra to talk about spectral sequences. All you need is:

- a space, or spectrum, or chain complex, or other mathematical object X,
- an increasing filtration $\cdots \to X_{p-1} \to X_p \to \cdots \to X$ whose successive quotients X_p/X_{p-1} you know something about, and
- a functor E_* that takes each of the "short exact sequences"

  $$X_{p-1} \longrightarrow X_p \longrightarrow X_p/X_{p-1}$$

  to a long exact sequence.

The spectral sequence then gives you a method to go from the groups $E_*(X_p/X_{p-1})$, plus some extra data called "differentials," to the groups $E_*(X)$. So for instance, you can compute the homology of a space $H_*(X)$ from the homology of the quotients $H_*(X_p/X_{p-1})$, plus the differentials.

(We also get a spectral sequence if X has a *decreasing* filtration $X \to \cdots \to X_p \to X_{p-1} \to \cdots$ and the functor E_* takes the *fiber* sequences $F_p \to X_p \to X_{p-1}$ to long exact sequences.)

Every spectral sequence arises this way: Leray-Serre, Atiyah-Hirzebruch, homotopy orbits, double complex, Milnor-Moore, Adams, Adams-Novikov, Bökstedt, and so on. Our goal in this chapter is to explain this philosophy, do a few examples, and build intuition, so that the reader will be able to recognize when a spectral sequence should appear "in the wild," and to get the conventions right about indexing and differentials.

As far as actual calculations, we will not go far beyond the most basic examples – spectral sequence calculations are a vast subject that requires its own textbooks. We leave the discussion to other books in the area, including [Rav86]. The present chapter is meant to provide the foundational material that is usually taken on faith in a more computationally-oriented work.

This chapter can be read independently of much of the book. It is only placed at the end because at times we will want to discuss a spectral sequence that involves modules over ring spectra, which we developed in detail in the previous chapter. We also use spectral sequences to discuss the role of *rational* spectra in the world of spectra, and indicate a little about how the next layers of the stable homotopy category unfold. The reader interested in digging deeper in this direction is encouraged to learn some chromatic homotopy theory. See e.g. [Rav86, Rav92, Lur10].

Appendix A

Some technical results

A.1 Flat cofibrations and commutative ring spectra

In this section we introduce "flat cofibrations," a class of morphisms in $\mathbf{Sp}^\Sigma$ and $\mathbf{Sp}^O$ that is larger than the class of free cofibrations that we saw in the standard stable model structure (Theorem 6.1.36 and Theorem 6.1.37). Flat cofibrations are useful for proving that CAlg($\mathbf{Sp}$) has a model structure in Theorem 6.5.33, and for proving the rectification result that E_∞ ring spectra are equivalent to commutative ring spectra in **??**.

The idea is to take a free spectrum F_i on a space of the form $(\Sigma_i/H)\times D^n$ or $(O(i)/H)\times D^n$ when H is a closed subgroup, and then to divide out by the Σ_i or $O(i)$-action on this space and on the free spectrum term F_i. This is also called taking the "semifree spectrum" on the space $(\Sigma_i/H)\times D^n$ or $(O(i)/H)\times D^n$. The flat cofibrations are built out of these spectra in the same way that a free cofibration is built out of $F_i(S^{n-1}\to D^n)$.

A.1.1 A little bit of equivariant homotopy theory

To set this up, we first need to do a little equivariant homotopy theory. Let G be any topological group. When we consider subgroups $H\leq G$ we will always assume that the subgroup is closed.

Let $\mathbf{Top}^G$ be the category of topological spaces with continuous left G-action, and continuous maps that are equivariant, in other words they commute with the G-action. This is the same thing as the category of diagrams in $\mathbf{Top}$ indexed by a category $\mathbf{I}$ with one object, and with morphism space G.

Definition A.1.1. A **G-cell complex** X is an object of $\mathbf{Top}^G$ built up from $\varnothing$ by successively attaching maps of the form $G/H\times S^{n-1}\to G/H\times D^n$ for varying subgroups $H\leq G$, with G

acting only on the first coordinate by multiplication on the left. We also have the following variants:

- A **relative G-cell complex** $A \to X$ is built in the same way but starting from some G-space A,
- a **free G-cell complex** is built from $\varnothing$ using only $G \times S^{n-1} \to G \times D^n$. In particular, the G-action on the resulting space X is always free, and
- a **G-CW complex** is like a G-cell complex except that we only attach $G/H \times S^{n-1} \to G/H \times D^n$ at the nth stage, for each $n \geq 0$.

We can also combine these in various ways, talking for instance about a free G-CW complex, or a relative G-CW complex, and so on.

The intuitive way to think about a G-cell complex is that it is a cell complex in which the G-action permutes the cells, and when a cell is sent to itself by a group element, it always goes to itself by the identity map. This is because each cell is stabilized by a subgroup $H \leq G$, and then each coset of H sends this cell by a homeomorphism to some other cell. Not every G-action on a cell complex has this nice form, but this nice form is very useful for proving theorems. And by the model structure below, every space is *equivalent* to something in this nice form.

Theorem A.1.2. *As in Example 5.3.18, for any topological group G, the category* $\mathbf{Top}^G$ *has a* ***Borel model structure*** *in which*

- *the cofibrations are the retracts of free G-cell complexes,*
- *the weak equivalences are the Borel equivalences (weak equivalences after forgetting the G-action), and*
- *the fibrations are the maps that are Serre fibrations after forgetting the G-action.*

$\mathbf{Top}^G$ *also has a* ***Bredon model structure*** *in which*

- *the cofibrations are the retracts of G-cell complexes (not free anymore),*
- *the weak equivalences are the maps that induce equivalences on H-fixed points, $X^H \simeq Y^H$, for all closed subgroups $H \leq G$, and*
- *the fibrations are the maps that are Serre fibrations on H-fixed points, $X^H \to Y^H$, for all closed subgroups $H \leq G$.*

Proof. The proof is a straightforward adaptation of the proof of the projective model structure in Theorem 5.3.16 (and in the Borel case is actually just a special case of the projective model structure). □

A large source of examples of G-CW complexes comes from smooth actions of Lie groups. Recall that a **Lie group** is a smooth manifold G with a group structure such that the multiplication map $G \times G \to G$ and the inversion map $G \to G$ are smooth. A smooth G-action on a smooth manifold M is an action that is smooth as a map $G \times M \to M$.

Lemma A.1.3. *If G is a compact Lie group acting smoothly on M, then M is a G-CW complex. More generally, if $N \subseteq M$ is a smooth submanifold preserved by the G-action then $N \to M$ is a relative G-CW complex.*

Proof. This is a consequence of Illman's triangulation theorem [Ill83], which actually gives M not just a CW structure but an equivariant triangulation, which is a more restrictive kind of structure. □

Example A.1.4. Consider the two-sphere S^2 with the smooth S^1-action that rotates about a fixed axis. By Lemma A.1.3, this is an S^1-CW complex. We can also see this directly: it has two fixed 0-cells $S^1/S^1 \times D^0$ at the north and south poles, and then a single free 1-cell $S^1/\{e\} \times D^1$ that covers the rest of the sphere, and whose attaching map $S^1/\{e\} \times S^0 \to S^0$ collapses the S^1 to a point.

If we assume that G is discrete, then we can also define a model structure like this in any cofibrantly generated model category $\mathbf{C}$. We let $\mathbf{C}^G$ be the category of objects of $\mathbf{C}$ with an action of G through morphisms of $\mathbf{C}$, and the morphisms in $\mathbf{C}^G$ are the morphisms of $\mathbf{C}$ that commute with the G-action. As always, we say that such a morphism is **equivariant**.

For each $X \in \mathbf{C}$, we define $G/H \otimes X \in \mathbf{C}^G$ to be the coproduct of G/H copies of X, with $g \in G$ acting on the coproduct by sending the summand $\{k\} \otimes X$ to the summand $\{gk\} \otimes X$ by the identity of X. In particular, this means that $G/H \otimes I$ is an object of $\mathbf{C}^G$, defined as a coproduct of G/H copies of the unit object I. The H-fixed points X^H are the limit of the diagram formed out of X and its H-action, and we check that equivariant maps $G/H \otimes X \to Y$ correspond to nonequivariant maps $X \to Y^H$.

With these definitions in place, the following is similarly straightforward to prove:

Theorem A.1.5. *For any discrete group G and any cofibrantly generated model category $\mathbf{C}$[1] with generating cofibrations I, the category $\mathbf{C}^G$ has a **Borel model structure** in which*

- *the cofibrations are generated by $G \otimes I = \{\, G \otimes f \;:\; f \in I \,\}$,*
- *the weak equivalences are the Borel equivalences (weak equivalences after forgetting the G-action), and*
- *the fibrations are the maps that are fibrations in $\mathbf{C}$ after forgetting the G-action.*

[1] We do have to assume in the smallness condition that the factorization doesn't just exist but is also unique in the sense of Exercise 5.8.42. This allows us to prove that any non-equivariant factorization is actually equivariant, after possibly moving to a later term in the colimit system.

The generating acyclic cofibrations in this case are $G \otimes J = \{\ G \otimes f\ :\ f \in J\ \}$.

$\mathbf{C}^G$ *also has a* ***Bredon model structure*** *in which*

- *the cofibrations are generated by* $\{\ G/H \otimes f\ :\ f \in I,\ H \leq G\ \}$,
- *the weak equivalences are the maps that induce equivalences on H-fixed points,* $X^H \simeq Y^H$*, for all subgroups* $H \leq G$*, and*
- *the fibrations are the maps such that* $X^H \to Y^H$ *is a fibration in* $\mathbf{C}$*, for all closed subgroups* $H \leq G$.

The generating acyclic cofibrations in this case are $\{\ G/H \otimes f\ :\ f \in J,\ H \leq G\ \}$.

Remark A.1.6. We use the terms "Borel" and "Bredon" because of equivariant homology: Borel homology is a homology theory for G-spaces that preserves the maps that are equivalences on the underlying spaces, while Bredon homology is a different homology theory that preserves the maps that give equivalences $X^H \simeq Y^H$ for all $H \leq G$. So we decided to call these two kinds of equivalences "Borel equivalences" and "Bredon equivalences" to match. These are not totally standard names: although "Borel equivalence" is used fairly commonly, the second kind of equivalence is often just called an "equivariant equivalence" or "equivalence of G-spaces."

Example A.1.7. The category of based G-spaces, $\mathbf{Top}_*^G$, has both of these model structures. This could be deduced more easily using Exercise 5.8.8 because $\mathbf{Top}_*^G$ is a comma category in $\mathbf{Top}^G$.

Example A.1.8. The category $\mathbf{sSet}^G$ has both of these model structures. A cofibration in the first is a map such that each $X_n \to Y_n$ is injective with complement a free G-set, and a cofibration in the second is any map such that each $X_n \to Y_n$ is injective.

Example A.1.9. If $\mathbf{Sp}$ is any category of diagram spectra, then the category $\mathbf{Sp}^G$ has both of these model structures.[2] The cofibrations in the Bredon model structure are generated by $F_i(G/H_+ \wedge (S^{k-1} \to D^k)_+)$, where the G acts *only* on the G/H term, not on the disc or the free spectrum F_i, which in the orthogonal spectrum case looks like $F_i S^0(m) = \mathscr{I}(\mathbb{R}^i, \mathbb{R}^m)$. (See Definition 6.1.9 for a discussion of the category $\mathscr{I}$.)

Remark A.1.10. The Borel model structure is sometimes called "doubly naïve G-spectra" and the Bredon model structure is sometimes called "naïve G-spectra." That's because when we restrict attention to symmetric or orthogonal spectra, there is a third model structure called **genuine G-spectra**, whose cofibrations, weak equivalences, and fibrations are defined using the notion of suspension and desuspension by a G-representation, a

[2] This is *not* how we obtain symmetric and orthogonal spectra; those have actions by a different group G at each spectrum level.

sphere S^V that has a nontrivial G-action on it. Genuine G-spectra are an incredibly rich theory with many applications within and outside of homotopy theory, and they go beyond the scope of this book, see e.g. [HHR16, HHR21, BHM93]. We only mention them to clarify that the model structures we consider here are *not* the genuine model structure.

In equivariant homotopy theory we often want to change the group G. Given any homomorphism $\phi : H \to G$, we may restrict or compose with ϕ to turn any G-action into an H-action. This gives a functor that we call **restriction of group action** $\phi^* \colon \mathbf{C}^G \to \mathbf{C}^H$.

Lemma A.1.11. *In any category* $\mathbf{C}$ *that has all colimits and limits,* ϕ^* *has both a left adjoint* $\phi_!$ *and a right adjoint* ϕ_**, both of which are functors* $\mathbf{C}^H \to \mathbf{C}^G$*. When* $\mathbf{C}$ *is a cofibrantly generated model category and we use the Bredon model structure, both* $(\phi_!, \phi^*)$ *and* (ϕ^*, ϕ_*) *are Quillen adjunctions.*

When ϕ is injective, $\phi_!$ is often called **induction** and ϕ_* is called **coinduction**.

Proof. For any object X with H-action, we define $\phi_! X = G \otimes_H X$, in other words we take a coproduct of G copies of X and then quotient by the two maps $(G \times H) \otimes X \rightrightarrows X$ that multiply the H into G on the right, or multiply the H into X. It is straightforward to see this is a left adjoint. Similarly we define $\phi_* X = \mathrm{Hom}^H(G, X)$, in other words we take a product of G copies of X and then equalize the two maps $\prod_G X \rightrightarrows \prod_{G\times H} X$ with formulas dual to the ones above, and this gives the right adjoint of ϕ^*.

When we have the Bredon model structure, we check that ϕ^* preserves cofibrations and acyclic cofibrations, and also fibrations and acyclic fibrations. So it is both left and right Quillen. □

Example A.1.12. If $\phi : G \to \{1\}$ is the trivial homomorphism, then ϕ^* is the functor that gives every object the trivial G-action. Its left adjoint is the G-orbits $(-)_G$, or colimit along G. Its right adjoint is the G-fixed points $(-)^G$, or limit along G.

Example A.1.13. If $\phi : \{1\} \to G$ is the other trivial homomorphism, then ϕ^* is the functor that forgets the G-action. Its left adjoint is the free G-action $X \mapsto G \otimes X := \coprod_G X$, and its right adjoint is the cofree G-action $X \mapsto \mathrm{Hom}(G, X) := \prod_G X$.

Corollary A.1.14. *In either the Borel or Bredon model structure, the forgetful functor* $\mathbf{C}^G \to \mathbf{C}$ *and the colimit or* G*-orbits functor* $\mathbf{C}^G \to \mathbf{C}$ *both preserve cofibrations and weak equivalences of cofibrant objects.*

Proof. This follows from Lemma A.1.11 and Ken Brown's Lemma (Lemma 5.2.25). □

This means that there are two left-derived functors of the orbits $(-)_G$. The first is the homotopy orbits, which sends all Borel equivalences of G-spaces to weak equivalences.

The second is a more sophisticated kind of orbits that depends on the homotopy type of the fixed-point spaces X^H, and not just the homotopy type of X. These two functors are not the same: for the one-point space $*$ with trivial G-action, the first kind of orbits gives BG, while the second just gives $*$ again.

This illustrates the important point that the left-derived functor of F doesn't just depend on F. It also depends on which class of weak equivalences we use! The left-derived functor is unique, provided we specify *both F and the class of weak equivalences*.

We complete our brief tour of equivariant homotopy theory by discussing a bit about products.

Lemma A.1.15 ([Sch18, B.15])**.** *If $\mathbf{C}$ is a cofibrantly generated symmetric monoidal model category, then the functor $\mathbf{C}^H \times \mathbf{C}^K \to \mathbf{C}^{H\times K}$ given by the tensor product $\otimes$ is a Quillen tensor.*

Proof. The right adjoint $(\mathbf{C}^K)^{\mathrm{op}} \times \mathbf{C}^{H\times K} \to \mathbf{C}^H$ is computed by applying the right adjoint $\mathrm{Hom}(-,-)$ in $\mathbf{C}$, then taking the limit (or fixed points) along the "conjugation" K-action that composes with $k\cdot(-)$ in the target and $k^{-1}\cdot(-)$ in the source. The other right adjoint is similar.

Since $\otimes$ preserves colimits in each slot, we get an isomorphism of $H\times K$-objects of $\mathbf{C}$

$$((H/H')\otimes X)\otimes((K/K')\otimes Y)\cong(H/H'\times K/K')\otimes(X\otimes Y).$$

This is a coproduct of copies of $(H\times K)/L\otimes(X\otimes Y)$, one for each $(H\times K)$-orbit in the set $H/H'\times K/K'$. This formula makes it straightforward to check that pushout-products of generating cofibrations go to generating cofibrations, and similarly for generating acyclic cofibrations, as in Proposition 5.5.14. □

In the special case of topological spaces, we can say a little more:

Lemma A.1.16 ([Sch18, B.15])**.** *For any compact Lie group G, the functors $\times\colon \mathbf{Top}^G \times \mathbf{Top}^G \to \mathbf{Top}^G$ and $\wedge\colon \mathbf{Top}^G_* \times \mathbf{Top}^G_* \to \mathbf{Top}^G_*$ are also Quillen tensors for both model structures. In addition, a pushout-product of a Borel cofibration and a Bredon cofibration is a Borel cofibration.*

Proof. The proof is the same as Lemma A.1.15, except that we Lemma A.1.3 to argue that each product $G/H\times G/K$, with G acting diagonally, is a G-CW complex. And when the product is of the form $G\times G/K$, it is a free G-CW complex. □

The reader interested in a much more extensive survey of the basic structural results of equivariant homotopy theory can find them in [Sch18, Appendix B] and [HHR21]. The notes [May96] are also a good reference for the classical applications of equivariant stable homotopy theory.

A.1.2 Equivariant tensor powers and pushout-powers

One of the simplest ways to make a space with Σ_n-action is to take a space X with no group action and to take its n-fold Cartesian product, which we also call the Cartesian power, $X^{\times n}$. This has a left Σ_n-action by the usual formula

$$\sigma(x_1,\dots,x_n) = (x_{\sigma^{-1}(1)},\dots,x_{\sigma^{-1}(n)}).$$

In other words, this is the action that applies σ to shuffle the coordinates around. We can do the same construction as well in any symmetric monoidal category $\mathbf{C}$, giving a Σ_n-object of $\mathbf{C}$ that we call the n-fold **tensor power** $X^{\otimes n}$. In particular, we can form an n-fold smash power $X^{\wedge n}$ whenever X is a based space or a (symmetric or orthogonal) spectrum. Quotienting out by the Σ_n-action, or taking induction along the homomorphism $\Sigma_n \to \{e\}$ as in Lemma A.1.11, gives the **symmetric tensor power** $X^{\otimes n}/\Sigma_n$.

If $f : A \to X$ is any map of spaces, or more generally a morphism in any symmetric monoidal category $\mathbf{C}$, we can also give an n-fold pushout product $f^{\square n}$ a Σ_n-action. In other words, its source and target have Σ_n-actions and the map $f^{\square n}$ is Σ_n-equivariant. Recall from the discussion after Lemma 6.5.16 that $f^{\square n}$ can be defined from the cube-shaped diagram

$$\{S \subseteq \{1,\dots,n\}\} \to \mathbf{C}, \qquad S \mapsto A^{\otimes S^c} \otimes X^{\otimes S}, \tag{A.1.17}$$

in other words a tensor product with one copy of X for each $1 \le i \le n$ that is in S, and one copy of A for each $1 \le i \le n$ that is not in S. Each subset inclusion $S \subseteq T$ induces a map of tensor products by applying f in each slot that corresponds to an element of $T \setminus S$.

Definition A.1.18. The n-fold **pushout-power** $f^{\square n}$ is the map in $\mathbf{C}$ from the colimit of the punctured cube into the final vertex,

$$f^{\square n} \colon \operatorname*{colim}_{S \subsetneq \{1,\dots,n\}} A^{\otimes S^c} \otimes X^{\otimes S} \longrightarrow X^{\otimes n}.$$

We make this a Σ_n-equivariant map by letting σ act on the term $A^{\otimes S^c} \otimes X^{\otimes S}$ by sending it to the term $A^{\otimes \sigma(S)^c} \otimes X^{\otimes \sigma(S)}$ by the same formula as above. This induces a well-defined Σ_n-action on the colimit, and the map from this to $X^{\otimes n}$ is Σ_n-equivariant. The **symmetric pushout-power** is the quotient of this by the Σ_n-action, $f^{\square n}/\Sigma_n$.

When $A = \varnothing$ and the tensor product preserves colimits in each slot, the pushout-power is just the map $\varnothing \to X^{\otimes n}$, and the symmetric version is the map $\varnothing \to X^{\otimes n}/\Sigma_n$.

In the more general case, the twofold pushout-power is constructed from the the pushout

$$\begin{array}{ccc} X \otimes A & & \\ \uparrow & & \\ A \otimes A & \longrightarrow & A \otimes X \end{array}$$

by taking the map that swaps the coordinates of $A \otimes A$, and swaps the two objects $X \otimes A$ and $A \otimes X$ along the isomorphism $X \otimes A \cong A \otimes X$. This gives a Σ_2-action on the pushout, and the map from this to $X \otimes X$ is Σ_2-equivariant. Note that this is not a diagram in which the terms are spaces with G-action and the morphisms are G-equivariant maps. Rather, it is a more general kind of G-diagram in which the G-action can send one term of the diagram to a completely different term. See [DM16] for more details about this general notion.

Example A.1.19. When $\mathbf{C} = \mathbf{Top}$, n the concrete setting of topological spaces, if f is the inclusion of a closed subspace $A \subseteq X$, then the pushout-power $f^{\square n}$ is the inclusion into $X^{\times n}$ of the subspace of all points where at least one coordinate is in A.

The following lemma builds more intuition for what these pushout-powers look like, as Σ_n-equivariant spaces.

Lemma A.1.20. *In* $(\mathbf{Top}, \times, *)$*, the pushout-powers* $(S^{k-1} \to D^k)^{\square n}$ *and* $(D^{k-1} \to D^{k-1} \times I)^{\square n}$ *are relative* Σ_n*-CW complexes.*

For instance, if we take the 2-fold pushout-power $(S^0 \to D^1)^{\square 2}$, we get a square with a Σ_2-action that reflects across its diagonal, and the Σ_2-equivariant map that includes the boundary into the square. The square is obtained from its boundary by attaching two equivariant cells: a fixed 1-cell $(\Sigma_2/\Sigma_2) \times D^1$ that goes to the diagonal of the square, and a free 2-cell $(\Sigma_2/\{e\}) \times D^2$ that goes to the two triangles above and below the diagonal.

Proof of Lemma A.1.20. We can prove this by explicitly enumerating the cells in $(D^k)^{\times n} \cong I^{kn}$, checking that Σ_n permutes the cells, and then checking that the source of the pushout-power is a subcomplex. We get these cells by thinking of $I^{kn} = |\Delta[1]_\bullet^{\times kn}|$ as the realization of a product of simplicial intervals. Then it has a cell for each nondegenerate simplex in $\Delta[1]_\bullet^{\times kn}$, and the Σ_n-action comes from the Σ_n-action on each level of the simplicial set $\Delta[1]_\bullet^{\times kn}$, so the cells organize into orbits of the form $\Sigma_n/H \times \Delta^d$, which proves that I^{kn} is a Σ_n-CW complex.

Concretely, there is one d-cell for each partition of the kn coordinates of I^{kn} into d equivalence classes, plus a choice of total ordering on those equivalence classes. This gives a map $\Delta^d \to I^{kn}$ that takes the point with barycentric coordinates $(t_0, \ldots, t_d)$ to the point where all the coordinates in the first equivalence class are t_0, all the points in the second equivalence class are t_1, and so on.

Now it is straightforward to check that the domains of the two maps $(S^{k-1} \to D^k)^{\square n}$ and $(D^{k-1} \to D^{k-1} \times I)^{\square n}$ are Σ_n-equivariant subcomplexes, and the proof is complete. □

Now that we have a little intuition, let's work more formally in any symmetric monoidal category $\mathbf{C}$ in which the tensor preserves colimits in each slot.

There are two formal lemmas that we will use repeatedly to work with these pushout-powers. The first expresses the pushout-power $f^{\square n}$ as a relative cell complex, whenever the map f was a relative cell complex. Recall from Definition 5.1.1 that an I-cell complex is a countable composition of pushouts of coproducts of maps in I.[3]

Lemma A.1.21. *In any symmetric monoidal category* $\mathbf{C}$ *in which* $\otimes$ *preserves colimits in each slot, for any set of maps* I *in* $\mathbf{C}$ *and for any* I*-cell complex* $f : A \to X$*, the pushout-power* $f^{\square n}$ *is a cell complex in* $\mathbf{C}^{\Sigma_n}$ *whose cells are of the form*

$$\Sigma_n \otimes_{\Sigma_{k_1}\times\cdots\times\Sigma_{k_m}} (f_1^{\square k_1} \square \cdots \square f_m^{\square k_m}),$$

with each $f_i \in I$*, and with* $k_1 + \cdots + k_m = n$*.*

Proof. We first consider the case where $n = 2$ and we take a pushout-product of two different maps. Given two I-cell complexes

$$A = X^{(0)} \longrightarrow X^{(1)} \longrightarrow X^{(2)} \longrightarrow \cdots \longrightarrow X, \qquad B = Y^{(0)} \longrightarrow Y^{(1)} \longrightarrow Y^{(2)} \longrightarrow \cdots \longrightarrow Y,$$

we form the diagram of all possible tensor products

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & & \uparrow & \\
X^{(0)} \otimes Y^{(2)} & \longrightarrow & X^{(1)} \otimes Y^{(2)} & \longrightarrow & X^{(2)} \otimes Y^{(2)} & \longrightarrow & \cdots \\
 & \uparrow & & \uparrow & & \uparrow & \\
X^{(0)} \otimes Y^{(1)} & \longrightarrow & X^{(1)} \otimes Y^{(1)} & \longrightarrow & X^{(2)} \otimes Y^{(1)} & \longrightarrow & \cdots \\
 & \uparrow & & \uparrow & & \uparrow & \\
X^{(0)} \otimes Y^{(0)} & \longrightarrow & X^{(1)} \otimes Y^{(0)} & \longrightarrow & X^{(2)} \otimes Y^{(0)} & \longrightarrow & \cdots
\end{array}$$

Since the tensor preserves colimits in each slot, the colimit of the whole diagram is $X \otimes Y$, and the colimit of the left column and bottom row is $(A \otimes Y) \cup_{(A\otimes B)} (X \otimes B)$. This allows us to make the pushout-product $(A \to X)\square(B \to Y)$ into an $(I\square I)$-cell complex by taking the n-skeleton to be the colimit of all terms $X^{(i)} \otimes Y^{(j)}$ in which at least one of the numbers i or j is zero, or else $i + j \leq n$. An application of Exercise 5.8.28 shows that this new cell complex is a cell complex whose cells are of the form $f \square g$ with $f, g \in I$.

Now specialize to the case that the two cell complexes are the same, so we are taking a two-fold pushout-power $(A \to X)\square(A \to X)$ of a single I-cell complex $A \to X$. Then the

[3] The following lemma also holds for the transfinite version of cell complexes from Exercise 5.8.42. This is left to the reader as an exercise.

Σ_2-action sends each cell $f \square g$ to the cell $g \square f$, so these can be grouped together as the single Σ_2-equivariant cell $\Sigma_2 \otimes (f \square g)$. But only when $f \neq g$. Otherwise, Σ_2-action sends $f \square f$ to itself, so instead we get the pushout-power $f^{\square 2}$ with its own Σ_2-action.

Now we increase n. By induction, we get an $I^{\square n}$-cell complex structure on the map $(A \to X)^{\square n}$, in which the cells are of the form $f_1 \square \cdots \square f_n$ with $f_i \in I$. If we keep track of the Σ_n-action as before, we see that the subgroup of Σ_n that sends each cell to itself is of the form $\Sigma_{k_1} \times \cdots \times \Sigma_{k_m}$, where each k_i counts how many times the same morphism of I shows up in the product. The cosets of this subgroup send this term $f_1 \square \cdots \square f_n$ by isomorphisms to $\frac{n!}{k_1! \cdots k_m!}$ other terms, and once we group them together and write down the Σ_n-action on the entire orbit of cells, we get exactly the expression in the statement of the lemma. □

The second formal lemma expresses the map $f^{\otimes n} : A^{\otimes n} \to X^{\otimes n}$ in terms of pushout-powers $f^{\square q}$ for $q \leq n$.

Lemma A.1.22. *In any symmetric monoidal category* **C** *in which* $\otimes$ *preserves colimits in each slot, for any morphism* $f : A \to X$*, we have a* Σ_n*-equivariant filtration*

$$A^{\otimes n} = Q^n_0(f) \longrightarrow Q^n_1(f) \longrightarrow Q^n_2(f) \longrightarrow \cdots \longrightarrow Q^n_{n-1}(f) \longrightarrow Q^n_n(f) = X^{\otimes n}$$

in which the composite is $f^{\otimes n}$*, the* q*th map* $Q^n_{q-1}(f) \to Q^n_q(f)$ *is a pushout of*

$$\Sigma_n \otimes_{\Sigma_{n-q} \times \Sigma_q} \left(A^{\otimes (n-q)} \otimes f^{\square q} \right),$$

and the final map $Q^n_{n-1}(f) \to Q^n_n(f)$ *is exactly the pushout-power* $f^{\square n}$*.*

Proof. We define $Q^n_q(f)$ to be the colimit along the terms in the cube from (A.1.17) in which at most q copies of X show up. In other words, the terms indexed by those subsets S in which $|S| \leq q$. So by definition the last stage $Q^n_{n-1}(f) \to Q^n_n(f)$ is the pushout-power $f^{\square n}$.

The claim that $Q^n_{q-1}(f) \to Q^n_q(f)$ is a pushout of $\Sigma_n \otimes_{\Sigma_{n-q} \times \Sigma_q} \left(A^{\otimes (n-q)} \otimes f^{\square q} \right)$ is another comparison of universal properties. Intuitively, we go from all expressions with at most $(q-1)$ copies of X to those with at most q copies of X by adding in the $\binom{n}{q}$ expressions in which X appears exactly q times. These are joined to the previous expressions along maps of the form $A^{\otimes (n-q)} \otimes f^{\square q}$, by the same argument as in Lemma 8.2.33. Keeping track of the Σ_n-action on all of these pushouts gives $\Sigma_n \otimes_{\Sigma_{n-q} \times \Sigma_q} \left(A^{\otimes (n-q)} \otimes f^{\square q} \right)$. See also [Whi17, Lem A.1] and [BGH22, 5.6]. □

A.1.3 Pushout-powers of cofibrations

Now let's apply these formal lemmas to see how the pushout-power and tensor power behave in a cofibrantly generated symmetric monoidal model category **C**. Again, to build intuition, we first do the case of topological spaces.

Corollary A.1.23. *If we think of the pushout-power $f \mapsto f^{\square n}$ as a rule that assigns each morphism in* **Top** *with the usual (Quillen) model structure to a morphism in* $\mathbf{Top}^{\Sigma_n}$ *with the Bredon model structure from Theorem A.1.2, then this rule preserves all cofibrations and acyclic cofibrations.*

Proof. Lemma A.1.21 shows that this rule sends each I-cell complex to a complex whose cells are of the form $\Sigma_n \times_{\Sigma_{k_1}\times\cdots\times\Sigma_{k_m}} (f_1^{\square k_1}\square\cdots\square f_m^{\square k_m})$. Each map $f_i^{\square k_i}$ is a Σ_{k_i}-cell complex by Lemma A.1.20. Therefore their pushout-product is a Bredon $(\Sigma_{k_1}\times\cdots\times\Sigma_{k_m})$-cofibration by Lemma A.1.15, and so after inducing up to Σ_n we get a Bredon Σ_n-cofibration by Lemma A.1.11. The argument for acyclic cofibrations is the same, using J in the place of I. □

Corollary A.1.24. *In* **Top***, the symmetric pushout-power $f \mapsto f^{\square n}/\Sigma_n$ preserves all cofibrations and acyclic cofibrations.*

This is used in the proof of Theorem 6.5.33 to show that the category of commutative algebras in spaces, CAlg(**Top**), has a right-induced model structure. (An easier argument that gets around this is also given in Proposition 9.1.3, but this easier argument does not work for spectra.)

To show that CAlg(**Sp**) also has a right-induced model structure, we need to show that the pushout-powers $f^{\square n}/\Sigma_n$ preserve cofibrations and acyclic cofibrations, or at least preserve acyclic cofibrations between cofibrant objects.

The formal part of this argument is the following proposition. Let **C** be any cofibrantly generated symmetric monoidal model category.

Proposition A.1.25. *Suppose the symmetric pushout-power $(-)^{\square n}/\Sigma_n$ sends each generating cofibration to a cofibration. Then:*

- *if $f : A \to X$ is a cofibration then $f^{\square n}/\Sigma_n$ is a cofibration,*
- *if in addition A is cofibrant then the domain of $f^{\square n}/\Sigma_n$ is cofibrant, and*
- *if in addition to the above two assumptions, the map $f^{\otimes n}/\Sigma_n$ is a weak equivalence, then the symmetric pushout-power $f^{\square n}/\Sigma_n$ is also a weak equivalence.*

Proof. For the first claim, Lemma A.1.21 expresses the pushout-power $f^{\square n}$ as a cell complex in which the cells are of the form $\Sigma_n \times_{\Sigma_{k_1}\times\cdots\times\Sigma_{k_m}} (f_1^{\square k_1}\square\cdots\square f_m^{\square k_m})$, with the f_i generating cofibrations. Taking the quotient by Σ_n, we get a cell complex in which the cells are of the form

$$(f_1^{\square k_1}/\Sigma_{k_1})\square\cdots\square(f_m^{\square k_m}/\Sigma_{k_m}).$$

Each of the pieces $f_i^{\square k_i}/\Sigma_{k_i}$ is a cofibration by assumption, and **C** is a symmetric monoidal model category, so the whole pushout-product is again a cofibration. Composing these shows that $f^{\square n}/\Sigma_n$ is a cofibration.

For the second claim, Lemma A.1.22 gives a filtration of the map from $A^{\otimes n}$ to the domain of $f^{\square n}$. Taking Σ_n-orbits gives a filtration in which each map is a pushout of

$$(A^{\otimes(n-q)}/\Sigma_{n-q}) \otimes (f^{\square q}/\Sigma_q). \tag{A.1.26}$$

Since f is a cofibration, each of the maps $f^{\square q}/\Sigma_q$ is a cofibration by the first part of the proposition. And since A is cofibrant, $A^{\otimes(n-q)}/\Sigma_{n-q}$ is cofibrant by the same argument. Therefore each of the maps in the filbration is a cofibration, and composing them together shows that the domain of $f^{\square n}/\Sigma_n$ is cofibrant.

For the third claim, as in [Whi17, Lem A.3] we prove this by induction on n, and again use the filtration Q^n_q and mod out by Σ_n. The map from the 0th stage to the nth stage is $A^{\otimes n}/\Sigma_n \to X^{\otimes n}/\Sigma_n$, which is a weak equivalence by assumption. But when $q \leq n-1$, the map from the $(q-1)$st stage to the qth stage is a pushout of the map (A.1.26), which is a cofibration and also a weak equivalence by inductive hypothesis. Therefore all of these stages are weak equivalences, except possibly the last stage $f^{\square n}/\Sigma_n$. But since the composite of all the maps is a weak equivalence, by 2 out of 3, the final stage must be a weak equivalence as well. □

So, to finish the proof in Theorem 6.5.33 that CAlg(**Sp**) has a right-induced model structure, we only need to prove that the symmetric pushout-power $f^{\square n}/\Sigma_n$ sends generating cofibrations to cofibrations, and that the symmetric tensor power $X^{\wedge n}/\Sigma_n$ preserves weak equivalences of cofibrant objects.

But we have a big problem! The first claim *false* in $\mathbf{Sp}^\Sigma$ and $\mathbf{Sp}^O$, even if we use the positive cofibrations. When we take a map of the form $F_i(f : K \to L)$ and take its n-fold pushout-power, we get $F_{ni}(f^{\square n})$, but the Σ_n-action acts on the F_{ni} term, in addition to the $f^{\square n}$ term. So when we take the quotient, we don't get a cofibration in the usual stable model structure, which would be of the form $F_{ni}(g)$.[4]

The solution to this is to introduce a more general kind of cofibration that includes maps of the form $F_{ni}(f^{\square n})/\Sigma_n$.

A.1.4 Flat cofibrations and the flat model structure

We develop this in detail for orthogonal spectra on topological spaces, $\mathbf{Sp}^O$, but the proofs in the categories $\mathbf{Sp}^\Sigma$ and $\mathbf{Sp}^\Sigma_{\mathbf{sSet}}$ are essentially the same. To define the new notion of cofibration, we need to use a different adjunction between spectra and spaces. Let $\mathbf{S}^O = \mathscr{J}$ be the category from Definition 6.1.9, so that orthogonal spectra are $\mathbf{S}^O$-diagrams. In

[4] It turns out that before quotienting, we do get a cofibration in the genuine model structure, which is a useful fact when setting up the Hill–Hopkins–Ravenel multiplicative norm [HHR16]. But that doesn't help us here, because the orbits functor $(-)/\Sigma_n$ doesn't send the genuine cofibrations to cofibrations.

particular, a free orthogonal spectrum is defined by the formula

$$(F_i A)_n = \mathbf{S}^O(i,n) \wedge A$$

with left $O(n)$-action given by the isomorphism $O(n)_+ \cong \mathbf{S}^O(n,n)$ and then the composition in $\mathbf{S}^O$. (In this section we are writing the composition in $\mathbf{S}^O$ from right to left, so it is of the form $\mathbf{S}^O(j,k) \wedge \mathbf{S}^O(i,j) \to \mathbf{S}^O(i,k)$. This makes the action we just defined a left action.)

Definition A.1.27. Let $\mathrm{ev}_i^O \colon \mathbf{Sp}^O \to \mathbf{Top}_*^{O(i)}$ be the forgetful functor that takes each orthogonal spectrum X to its nth level X_i, but remembers the basepoint of X_i and its left $O(i)$-action. The left adjoint sends each based $O(i)$-space A to the **semifree spectrum** $\mathscr{G}_i A$, which has the formula

$$(\mathscr{G}_i A)_n = \mathbf{S}^O(i,n) \wedge_{O(i)} A,$$

where we apply the balanced product relation $(x, gy) \sim (xg, y)$ using the left $O(i)$-action on A and the right $O(i)$-action on $\mathbf{S}^O(i,n)$ coming from composition with $O(i)_+ \cong \mathbf{S}^O(i,i)$.

Example A.1.28. A semifree spectrum on a free $O(i)$-space is just a free spectrum, $\mathscr{G}_i(O(i)_+ \wedge A) \cong F_i A$.

Definition A.1.29. A **flat cofibration** of orthogonal spectra is a retract of a relative I-cell complex, where I is the set of maps

$$I = \left\{ \mathscr{G}_i((O(i)/H) \times S^{k-1})_+ \longrightarrow \mathscr{G}_i((O(i)/H) \times D^k)_+ \,:\, k, i \geq 0,\ H \leq O(i) \right\}.$$

Here as always H only ranges over the closed subgroups of $O(i)$. We also let J denote the set of maps

$$\begin{aligned} J = &\left\{ \mathscr{G}_i((O(i)/H) \times D^k \times \{0\})_+ \longrightarrow \mathscr{G}_i((O(i)/H) \times D^k \times I)_+ \,:\, k, i \geq 0,\ H \leq O(i) \right\} \\ &\cup \left\{ \mathscr{G}_i[EO(i) \times (O(i)/H) \longrightarrow CEO(i) \times (O(i)/H)]_+ \,\square\, \left(S_+^{k-1} \longrightarrow D_+^k\right) \,:\, k, i \geq 0,\ H \leq O(i) \right\}. \end{aligned}$$

Here $EO(i)$ denotes any contractible free $O(i)$-cell complex, for instance the one built by the bar construction $B(O(i), O(i), *)$ in Example 7.3.12, or one built by the small-object argument in the Borel model structure on $O(i)$-spaces. And $CEO(i)$ means the cone on $EO(i)$, so any $O(i)$-cell complex that is Bredon equivalent to $*$ and such that $EO(i) \to CEO(i)$ is a relative $O(i)$-cell complex. This guarantees that the map of products $EO(i) \times (O(i)/H) \to CEO(i) \times (O(i)/H)$ is both a Bredon $O(i)$-cofibration and a Borel weak equivalence. We understand what happens to such things when we take the semifree spectrum:

Lemma A.1.30. *The operation $\mathscr{G}_i(-)$ sends all Borel equivalences of based $O(i)$-spaces to level equivalences of spectra, and all Bredon $O(i)$-cofibrations of spaces to flat cofibrations of spectra.*

Proof. For the first claim, the spaces $\mathbf{S}^O(i,m)$ are based free $O(i)$-CW complexes using Lemma A.1.3. (The based space $\mathbf{S}^O(i,m)$ is a quotient of a disc bundle by its boundary sphere bundle, so this is actually an example of the relative form of the lemma.) This implies that $\mathbf{S}^O(i,m)\wedge_{O(i)}(-)$ sends Borel $O(i)$-equivalences to weak equivalences, by an inductive argument that proceeds one free $O(i)$-cell at a time.

The second claim follows since $\mathscr{G}_i$ is a left adjoint, so it preserves cell complexes, and each generating Bredon $O(i)$-cofibration $((O(i)/H)\times S^{k-1})_+ \to ((O(i)/H)\times D^k)_+$ goes to a flat cofibration by definition. □

The second set of maps in J in Definition A.1.29 allows us to build a model structure that has Bredon-style cofibrations and Borel-style weak equivalences. This process is called "mixing" in [Shi04, BDS22].

Lemma A.1.31. *The maps in J are all flat cofibrations and level equivalences. A map of spectra $X\to Y$ is a J-fibration iff each $X_i^H\to Y_i^H$ is a Serre fibration and the square*

$$\begin{array}{ccc} X_i^H & \longrightarrow & X_i^{hH} \\ \downarrow{\scriptstyle p_i^H} & & \downarrow{\scriptstyle p_i^{hH}} \\ Y_i^H & \longrightarrow & Y_i^{hH} \end{array}$$

is a homotopy pullback square, for all $i\geq 0$ and $H\leq O(i)$.

Proof. The first claim follows from Lemma A.1.30. For the second claim, the proof follows the same pattern as the proof of Lemma 5.6.6. A map $X\to Y$ has the right-lifting property with respect to the first set of maps in J iff each $X_i^H\to Y_i^H$ is a Serre fibration. So we may assume this holds for the rest of the proof.

Using Proposition 5.5.12, the right-lifting property for the second set of maps in J rearranges to the claim that the map of spaces $\mathrm{Map}_{\square}^{O(i)}(EO(i)\times(O(i)/H)\longrightarrow(O(i)/H),p_i)$ has the right-lifting property with respect to $S_+^{n-1}\to D_+^n$, in other words it is an acyclic Serre fibration. But this map is automatically a fibration, using the fact that $\wedge\colon \mathbf{Top}_*\times\mathbf{Top}_*^{O(i)}\to\mathbf{Top}_*^{O(i)}$ is a Quillen tensor. So we just have to determine when it is a weak equivalence as well. It rearranges into

$$\begin{aligned}&\mathrm{Map}^{O(i)}(CEO(i)\times(O(i)/H),X_i)\longrightarrow\\ &\mathrm{Map}^{O(i)}(EO(i)\times(O(i)/H),X_i)\times_{\mathrm{Map}^{O(i)}(EO(i)\times(O(i)/H),Y_i)}\mathrm{Map}^{O(i)}(CEO(i)\times(O(i)/H),Y_i).\end{aligned}$$

Since $EO(i)\times(O(i)/H)\to CEO(i)\times(O(i)/H)$ is a Bredon $O(i)$-cofibration, the second projection map in the fiber product is a Serre fibration, so this fiber product is a homotopy pullback. Now we can simplify it up to weak equivalence to the map

$$X_i^H\longrightarrow \mathrm{Map}_*(EO(i),X_i)^H\times^h_{\mathrm{Map}_*(EO(i),Y_i)^H}Y_i^H.$$

But $\mathrm{Map}_*(EO(i), X_i)^H$ is a valid model for the homotopy orbits X^{hH}, so this map is a weak equivalence iff the square (A.1.31) is a homotopy pullback. □

Proposition A.1.32. *There is a **flat level model structure** on* $\mathbf{Sp}^O$ *in which*

- *the cofibrations are the flat cofibrations,*
- *the weak equivalences are the maps that are weak equivalences* $X_i \to Y_i$ *at each spectrum level, and*
- *the fibrations are the maps satisfying the conditions in Lemma A.1.31.*

This is cofibrantly generated by the maps I and J given in Definition A.1.29.

Proof. Most of the points in Proposition 5.2.6 are now easy to check. The hard part was Lemma A.1.31, which allows us to see that a map is an acyclic J-fibration iff each map $X_i^H \to Y_i^H$ is an acyclic fibration. Then it is comparatively easy to see this is the same thing as an I-fibration. □

For the stable model structure, all we have to do is add the second set of maps from Definition 5.6.3, so that now we have the larger set of generating acyclic cofibrations

$$\begin{aligned} J = &\left\{ \mathscr{G}_i((O(i)/H)\times D^k\times\{0\})_+ \longrightarrow \mathscr{G}_i((O(i)/H)\times D^k\times I)_+ \;:\; k,i\geq 0,\ H\leq O(i) \right\} \\ &\cup \left\{ \mathscr{G}_i[EO(i)\times(O(i)/H)\longrightarrow CEO(i)\times(O(i)/H)]_+ \square \left(S_+^{k-1}\longrightarrow D_+^k\right) \;:\; k,i\geq 0,\ H\leq O(i) \right\} \\ &\cup \left\{ k_{i,j} \square \left(S_+^{d-1}\longrightarrow D_+^d\right) \;:\; i,j,d\geq 0 \right\}. \end{aligned}$$

This really is exactly the same maps $k_{i,j}$ as in Definition 5.6.3, defined using free spectra, not semifree spectra! It turns out that semifree spectra are unnecessary for this last step.

Theorem A.1.33. *There is a **flat stable model structure** on* $\mathbf{Sp}^O$ *in which*

- *the cofibrations are the flat cofibrations,*
- *the weak equivalences are the usual stable equivalences, and*
- *the fibrations are the maps satisfying the conditions in both Lemma A.1.31 and Lemma 5.6.6. Spelling this out, each map* $X_i^H \to Y_i^H$ *is a Serre fibration and each of the following squares is a homotopy pullback:*

$$\begin{array}{ccc} X_i^H & \longrightarrow & X_i^{hH} \\ \downarrow{\scriptstyle p_i^H} & & \downarrow{\scriptstyle p_i^{hH}} \\ Y_i^H & \longrightarrow & Y_i^{hH} \end{array} \qquad \begin{array}{ccc} X_i & \longrightarrow & \Omega^j X_{i+j} \\ \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle \Omega^j p_{i+j}} \\ Y_i & \longrightarrow & \Omega^j Y_{i+j}. \end{array}$$

This is cofibrantly generated by the maps I from Definition A.1.29 and the maps J given just above.

Proof. Again, most of the points in Proposition 5.2.6 are easy to check, or were checked already when we proved Theorem 6.1.37. The interesting part is that, by a combination of the proofs of Proposition A.1.32 and Theorem 6.1.37, a map is a J-fibration iff each map $X_i^H \to Y_i^H$ is a fibration and the above two squares are both homotopy pullbacks. Since homotopy fixed points commute with loopspaces and homotopy pullbacks, this gives for each subgroup $H \leq O(i)$ the homotopy pullback square

$$\begin{array}{ccc} X_i^{hH} & \longrightarrow & \Omega^j(X_{i+j}^{hH}) \\ \downarrow & & \downarrow \\ Y_i^{hH} & \longrightarrow & \Omega^j(Y_{i+j}^{hH}) \end{array}$$

and then by pasting homotopy pulbacks together we conclude that the square

$$\begin{array}{ccc} X_i^{H} & \longrightarrow & \Omega^j(X_{i+j}^{H}) \\ \downarrow & & \downarrow \\ Y_i^{H} & \longrightarrow & \Omega^j(Y_{i+j}^{H}) \end{array}$$

is a homotopy pullback as well. Therefore in a J-fibration $X \to Y$, if we shift by i and then take H-fixed points, the resulting map of orthogonal spectra $(\mathrm{sh}^i X)^H \to (\mathrm{sh}^i Y)^H$ is a stable fibration as well. Now it is straightforward to see that the J-fibration $X \to Y$ is additionally a stable equivalence iff it is a level equivalence, iff each $X^H \to Y^H$ is a level equivalence, iff $X \to Y$ is an I-fibration. □

We can also vary this by only using generating cofibrations and acyclic cofibrations in positive degrees:

$$\begin{aligned} I^+ &= \left\{ \mathscr{G}_i((O(i)/H) \times S^{k-1})_+ \longrightarrow \mathscr{G}_i((O(i)/H) \times D^k)_+ \,:\, k \geq 0,\ i > 0,\ H \leq O(i) \right\} \\ J^+ &= \left\{ \mathscr{G}_i((O(i)/H) \times D^k \times \{0\})_+ \longrightarrow \mathscr{G}_i((O(i)/H) \times D^k \times I)_+ \,:\, k \geq 0,\ i > 0,\ H \leq O(i) \right\} \\ &\quad \cup \left\{ \mathscr{G}_i[EO(i) \times (O(i)/H) \longrightarrow CEO(i) \times (O(i)/H)]_+ \,\square\, \left(S_+^{k-1} \longrightarrow D_+^k\right) \,:\, k \geq 0,\ i > 0,\ H \leq O(i) \right\} \\ &\quad \cup \left\{ k_{i,j} \,\square\, \left(S_+^{d-1} \longrightarrow D_+^d\right) \,:\, i > 0,\ j, d \geq 0 \right\}. \end{aligned}$$

Theorem A.1.34. *There is a **flat positive model structure** on $\mathbf{Sp}^O$ in which*

- *the cofibrations are the flat positive cofibrations, generated by the set I^+ just above,*
- *the weak equivalences are the usual stable equivalences, and*

- *the fibrations are the maps in which each $X_i^H \to Y_i^H$ is a Serre fibration and each of the following squares is a homotopy pullback, but only for n and i positive:*

$$\begin{array}{ccc} X_i^H & \longrightarrow & X_i^{hH} \\ \downarrow{\scriptstyle p_i^H} & & \downarrow{\scriptstyle p_i^{hH}} \\ Y_i^H & \longrightarrow & Y_i^{hH} \end{array} \qquad \begin{array}{ccc} X_i & \longrightarrow & \Omega^j X_{i+j} \\ \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle \Omega^j p_{i+j}} \\ Y_i & \longrightarrow & \Omega^j Y_{i+j}. \end{array}$$

There is also a similar model structure on $\mathbf{Sp}^\Sigma_{\mathbf{Top}}$ and $\mathbf{Sp}^\Sigma_{\mathbf{sSet}}$, where we replace every instance of $O(i)$ with Σ_i and define the semifree spectrum $\mathscr{G}_i$ using the sphere category $\mathbf{S}^\Sigma$ instead of $\mathbf{S}^O$. See [Shi04, BDS22] for more details. There is also a "global" model structure on orthogonal spectra that uses the flat cofibrations but whose weak equivalences measure a kind of equivariant homotopy group for every compact Lie group G, see [Sch18, Thm 4.3.18].

We conclude this subsection by checking that these model structures all interact in the correct way with the smash product of diagram spectra.

Proposition A.1.35. *The smash product $\wedge$ is a Quillen tensor with respect to all of the above model structures.*

Proof. We do the proof in orthogonal spectra. The proof follows the same pattern as Proposition 6.3.4. The key fact is that the smash product of semifree spectra is semifree:

$$\mathscr{G}_i A \wedge \mathscr{G}_j B \cong \mathscr{G}_{i+j}(O(i+j)_+ \wedge_{O(i)\times O(j)} (A \wedge B)), \tag{A.1.36}$$

which follows immediately from checking that the right adjoints all agree.

Now if we take a pushout-product of two maps in I, we get a semifree spectrum on an unbased space of the form

$$O(i+j) \times_{O(i)\times O(j)} (O(i)/H \times O(j)/K) \times (S^{k-1} \to D^k) \square (S^{\ell-1} \to D^\ell)$$

$$\cong O(i+j)/(H \times K) \times (S^{k+\ell-1} \to D^{k+\ell}),$$

and indeed this is one of the (generating) cofibrations. If instead one map is in I and the other is in one of the first to sets in J, then the pushout-product is an acyclic cofibration using the same method, and using Lemma A.1.16 to show that the map on the inside of the semifree spectrum is an acyclic Bredon cofibration.

In the final case where we take a map $A \to X$ in I and a map $(B \to Y) = k_{i,j} \square \left(S^{d-1}_+ \longrightarrow D^d_+\right)$, it appears we might have to do more work to rerrange the pushout-product of a semifree spectrum with $k_{i,j}$. But there is a cheap way out. We already know the pushout-product is a flat cofibration. To show it is a weak equivalence, by the method used in the proof of

Lemma 5.6.5, we only have to show that the maps $A \otimes B \to A \otimes Y$ and $X \otimes B \to X \otimes Y$ are weak equivalences. But this follows from Exercise 6.6.12, because the spectra B and Y are cofibrant in the usual stable model structure, so a tensor product with them has the homotopy type of the left-derived tensor product. □

Remark A.1.37. The motivation for the term "flat" is that $X \wedge (-)$ preserves all stable equivalences when X is flat cofibrant, not just cofibrant, generalizing Exercise 6.6.12. See [Sch18, Thm 3.5.10].

Remark A.1.38. As far as the author is aware, the flat model structure and its applications first appeared in [Shi04]. The equivariant verison for genuine G-spectra appeared in the thesis of Stolz, which was eventually incorporated into the paper [BDS22]. The book [Sch18] also contains a wealth of information on this and related model structures.

A.1.5 Pushout-powers of flat cofibrations

Now that we have the positive flat model structure, we can finish the proof of the hypotheses of Proposition A.1.25, and use this to prove that the symmetric pushout-power $f^{\square n}/\Sigma_n$ preserves all cofibrations, and all acyclic cofibrations between cofibrant objects, in the positive flat model structure. This finishes the proof in Theorem 6.5.33 that commutative ring spectra have a model structure. (Phew!)

Lemma A.1.39. *The symmetric pushout-power $f \mapsto f^{\square n}/\Sigma_n$ preserves flat cofibrations.*

Proof. By Proposition A.1.25 it suffices to check this when f is a generating flat cofibration. So let $f : \mathscr{G}_i((O(i)/H) \times S^{k-1})_+ \longrightarrow \mathscr{G}_i((O(i)/H) \times D^k)_+$ for some $i, k \geq 0$. Using the n-fold version of the isomorphism (A.1.36), we see that for any $O(i)$-space A we have

$$(\mathscr{G}_i A)^{\wedge n} \cong \mathscr{G}_{ni}(O(ni)_+ \wedge_{O(i)\times\cdots\times O(i)} A^{\wedge n}).$$

From Proposition 6.2.6, each element $\sigma \in \Sigma_n$ acts by its usual left action on $A^{\wedge n}$, together with right multiplication of the element of $O(ni)$ by σ^{-1}, regarded as a block permutation that shifts the n different blocks of i coordinates around. (The reader can verify that this action respects the relation that moves elements of $O(i)^{\times n}$ back and forth between $O(ni)$ and $A^{\wedge n}$.)

Taking A to be of the form $((O(i)/H) \times B)_+$, this gives

$$(\mathscr{G}_i((O(i)/H) \times B)_+)^{\wedge n} \cong \mathscr{G}_{ni}(O(ni)/(H \times \cdots \times H) \times B^{\times n})_+.$$

Now feeding the map $S^{k-1} \to D^k$ into the B slot and using that $\mathscr{G}_{ni}(O(ni)/(H\times\cdots\times H)\times(-))$ preserves colimits, we get a semifree spectrum on the pushout-product

$$\mathscr{G}_{ni}(O(ni)/(H \times \cdots \times H) \times (S^{k-1} \to D^k)^{\square n})_+. \tag{A.1.40}$$

By Lemma A.1.20 the map $(S^{k-1} \to D^k)^{\square n}$ is a Bredon Σ_n-cofibration. So to prove our claim we can replace it with a single Bredon Σ_n-cell:

$$\mathscr{G}_{ni}(O(ni)/(H \times \cdots \times H) \times \Sigma_n/K \times (S^{\ell-1} \to D^\ell))_+. \tag{A.1.41}$$

The space $O(ni)/(H^{\times n}) \times \Sigma_n/K$ is a smooth manifold with smooth action by the compact Lie group $(O(ni) \times \Sigma_n)$, with $O(ni)$ acting by multiplication on the left on $O(ni)$, and Σ_n acting both on the left on Σ_n and also on the right on $O(ni)$. Therefore by Lemma A.1.3 it is a $(O(ni) \times \Sigma_n)$-CW complex. Quotienting by the Σ_n-action is an example of induction from Lemma A.1.11 along the homomorphism $O(ni) \times \Sigma_n \to O(ni)$, so after applying $/\Sigma_n$ we get an $O(ni)$-CW complex. Therefore (A.1.41) is a semifree spectrum on a relative $O(ni)$-CW complex, which by Lemma A.1.30 gives a flat cofibration. □

Lemma A.1.42. *On positive flat cofibrant spectra, the symmetric tensor power $(-)^{\wedge n}/\Sigma_n$ is level equivalent to the homotopy orbits $(-)^{\wedge n}_{h\Sigma_n}$, and therefore preserves all stable equivalences.*

In other words, the collapse map

$$(E\Sigma_n)_+ \wedge_{\Sigma_n} X^{\wedge n} \longrightarrow *_+ \wedge_{\Sigma_n} X^{\wedge n} = X^{\wedge n}/\Sigma_n.$$

is a level equivalence of spectra, whenever X is positive flat cofibrant. This result is useful for more than just the model structure on CAlg(**Sp**); it's also used to prove the rectification theorem for E_∞-ring spectra (**??**).

Proof. It suffices to show that the smash power $X^{\wedge n}$ is at every spectrum level a Borel Σ_n-cofibrant based space, in other words a retract of a based free Σ_n-cell complex. Because that implies that the orbits and homotopy orbits are equivalent.

By Lemma A.1.21 applied to the map $* \to X$, we see that $X^{\wedge n}$ is equivariantly a cell complex built out of cells of the form $\Sigma_n \wedge_{\Sigma_{k_1} \times \cdots \times \Sigma_{k_m}} (f_1^{\square k_1} \square \cdots \square f_m^{\square k_m})$. So it suffices to show that each of these morphisms individually is a Borel Σ_n-cofibration at each spectrum level. By Lemma A.1.11, it suffices to prove that $f_1^{\square k_1} \square \cdots \square f_m^{\square k_m}$ is a Borel $(\Sigma_{k_1} \times \cdots \times \Sigma_{k_m})$-cofibration at each spectrum level.

Now we just write out the product $f_1^{\square k_1} \square \cdots \square f_m^{\square k_m}$ explicitly:

$$\begin{aligned}
f_j &= \mathscr{G}_{i_j}((O(i_j)/H_j) \times (S^{d_j-1} \to D^{d_j}))_+ \\
f_j^{\square k_j} &= \mathscr{G}_{k_j i_j}((O(k_j i_j)/(H_j^{\times k_j}) \times (S^{d_j-1} \to D^{d_j})^{\square k_j})_+ \\
\square_{j=1}^m f_j^{\square k_j} &= \mathscr{G}_{\sum_j k_j i_j}((O(\textstyle\sum_j k_j i_j)/(\prod_j H_j^{\times k_j}) \times \square_{j=1}^m (S^{d_j-1} \to D^{d_j})^{\square k_j})_+ \\
(\square_{j=1}^m f_j^{\square k_j})_\ell &= \mathbf{S}^O(\textstyle\sum_j k_j i_j, \ell) \wedge_{O(\sum_j k_j i_j)} ((O(\sum_j k_j i_j)/(\prod_j H_j^{\times k_j}) \times \square_{j=1}^m (S^{d_j-1} \to D^{d_j})^{\square k_j})_+ \\
&\cong \mathbf{S}^O(\textstyle\sum_j k_j i_j, \ell) \wedge_{O(\sum_j k_j i_j)} ((O(\sum_j k_j i_j)/(\prod_j H_j^{\times k_j}))_+ \wedge (\square_{j=1}^m (S^{d_j-1} \to D^{d_j})^{\square k_j})_+
\end{aligned}$$

For each j, the map $(S^{d_j-1} \to D^{d_j})^{\square k_j}$ is a Bredon Σ_{k_j}-cofibration by Lemma A.1.20, so their pushout-product $\square_{j=1}^m (S^{d_j-1} \to D^{d_j})^{\square k_j}$ is a Bredon $(\Sigma_{k_1} \times \cdots \times \Sigma_{k_m})$-cofibration by Lemma A.1.15.

Next we check that $O(\sum_j k_j i_j)/(\prod_j H_j^{\times k_j})$ is a free $(\Sigma_{k_1} \times \cdots \times \Sigma_{k_m})$-CW complex. It is certainly a smooth manifold with a smooth action by this group, so it is an equivariant CW complex by Lemma A.1.3. To check that the action is free, it suffices to pass to the larger quotient $O(\sum_j k_j i_j)/(\prod_j O(i_j)^{\times k_j})$. We identify this manifold with the set of all ordered n-tuples of mutually orthogonal subspaces of $O(\sum_j k_j i_j)$ in which the first k_1 subspaces are i_1-dimensional, the next k_2 subspaces are i_2-dimensional, and so on. Along this identification, the action of the group $(\Sigma_{k_1} \times \cdots \times \Sigma_{k_m})$ permutes the subspaces, and so it is clearly free. Therefore $O(\sum_j k_j i_j)/(\prod_j H_j^{\times k_j})$ is a free $(\Sigma_{k_1} \times \cdots \times \Sigma_{k_m})$-CW complex.

Now let $p = \sum_{j=1}^m k_j i_j$. As in the proof of Lemma A.1.30, the operation $\mathbf{S}^O(p,\ell) \wedge_{O(p)} (-)$ sends this to a based free $(\Sigma_{k_1} \times \cdots \times \Sigma_{k_m})$-CW complex, by an induction on each free $O(p)$-cell of $\mathbf{S}^O(p,\ell)$. Therefore the entire expression

$$\mathbf{S}^O(\textstyle\sum_j k_j i_j, \ell) \wedge_{O(\sum_j k_j i_j)} ((O(\sum_j k_j i_j)/(\prod_j H_j^{\times k_j}))_+$$

is a based free $(\Sigma_{k_1} \times \cdots \times \Sigma_{k_m})$-CW complex. By Lemma A.1.15, the smash product of this with the Bredon $(\Sigma_{k_1} \times \cdots \times \Sigma_{k_m})$-cofibration $\square_{j=1}^m (S^{d_j-1} \to D^{d_j})^{\square k_j}$ gives a Borel $(\Sigma_{k_1} \times \cdots \times \Sigma_{k_m})$-cofibration, so $f_1^{\square k_1} \square \cdots \square f_m^{\square k_m}$ is a Borel $(\Sigma_{k_1} \times \cdots \times \Sigma_{k_m})$-cofibration at each spectrum level. This finishes the proof. □

Remark A.1.43. The symmetric tensor power $(-)^{\wedge n}/\Sigma_n$ does not preserve weak equivalences if we drop the "positive" condition on the cofibrant objects. Indeed, using Lemma A.1.42, we can show that it takes the stable equivalence $F_1 S^1 \to F_0 S^0$ to a map of spectra that is equivalent to $\Sigma^\infty_+ B\Sigma_n \to \Sigma^\infty_+ * = \mathbb{S}$, which is not a stable equivalence.

Now by the formal argument in Proposition A.1.25, we conclude:

Corollary A.1.44. *$f^{\square n}/\Sigma_n$ preserves all flat cofibrations, and flat acyclic cofibrations between positive flat cofibrant spectra.*

Corollary A.1.45. *$f^{\square n}/\Sigma_n$ sends all cofibrations to flat cofibrations, and sends all acyclic cofibrations between positive cofibrant spectra to acyclic flat cofibrations.*

Since the flat cofibrations are also h-cofibrations of spectra in the sense of Definition 6.1.27, this is enough to finish the proof Theorem 6.5.33, that commutative ring spectra $\mathrm{CAlg}(\mathbf{Sp})$ have a model structure right-induced from the positive model structure on spectra. Alternatively, we could use flat cofibrations throughout:

Theorem A.1.46. *If we give $\mathbf{Sp}^O$, $\mathbf{Sp}^\Sigma_{\mathbf{Top}}$, or $\mathbf{Sp}^\Sigma_{\mathbf{sSet}}$ the positive flat model structure, then the categories of modules $\mathrm{Mod}(R)$, algebras $\mathrm{Alg}(\mathbf{Sp})$, and commutative algebras $\mathrm{CAlg}(\mathbf{Sp})$ all*

have right-induced model structures. In these model structures, every cofibrant algebra or cofibrant commutative algebra is also flat cofibrant as a spectrum.

Remark A.1.47. Our proof that CAlg(**Sp**) has a model structure right-induced from the positive model structure on **Sp** follows the pattern of [Shi04, Whi17] and [BGH22, 5.8]. An older alternative approach is given in [MMSS01], which avoids the flat cofibrations in a roundabout way.

A.2 More about homotopy colimits and limits

In this section we prove some of the facts about homotopy colimits whose proofs are a little too long or technical for the main text. This includes Quillen's lemma and a more general statement of descent for homotopy colimits, the commutation of geometric realization with certain homotopy pullbacks, and various improvements to the assumptions about when the Bousfield-Kan homotopy colimit and homotopy limit have the correct homotopy type.

A.2.1 Horizontal descent and the proof of Quillen's lemma

The basic ingredient we need for this proof is the fact that we can glue fibrations together. There are two ways to glue them, "horizontally" and "vertically:"

- Given Hurewicz fibrations with equivalent homotopy fibers

$$\begin{array}{ccccc} F_1 & \xleftarrow{\sim} & F_0 & \xrightarrow{\sim} & F_2 \\ \downarrow & & \downarrow & & \downarrow \\ E_1 & \longleftarrow & E_0 & \longrightarrow & E_2 \\ \downarrow & & \downarrow & & \downarrow \\ B_1 & \longleftarrow & B_0 & \longrightarrow & B_2, \end{array}$$

  the map of homotopy pushouts $E_1 \cup^h_{E_0} E_2 \to B_1 \cup^h_{B_0} B_2$ is also a quasifibration (and so up to equivalence it is also a Hurewicz fibration with the fiber equivalent to the F_i).

- Given Hurewicz fibrations with the same base space

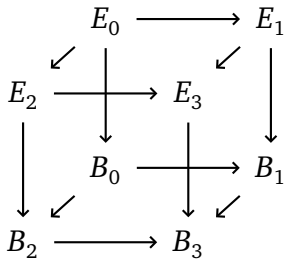
$$\begin{array}{ccccc} F_1 & \longleftarrow & F_0 & \longrightarrow & F_2 \\ \downarrow & & \downarrow & & \downarrow \\ E_1 & \longleftarrow & E_0 & \longrightarrow & E_2 \\ \downarrow & & \downarrow & & \downarrow \\ B & = & B & = & B, \end{array}$$

  the homotopy pushout $E_1 \cup^h_{E_0} E_2 \to B$ is a Hurewicz fibration whose fiber is $F_1 \cup^h_{F_0} F_2$.

More homotopically, if we take a homotopy pushout of three fiber sequences, and we want the pushout to also be a fiber sequence, it is enough to assume that the fibers are equivalent (horizontal pasting), or the base spaces are all equivalent (vertical pasting).

Let's give a proof of the horizontal version of this where the fibers are all equivalent. Since the claim only involves calculating the homotopy fiber of the map $E_1 \cup^h_{E_0} E_2 \to B_1 \cup^h_{B_0} B_2$, it is enough to show that the pullback of this map to any of the B_i is equivalent to E_i again. So the claim is reduced to the following theorem.

Theorem A.2.1 (Horizontal descent for homotopy pushouts)**.** *In the following cube of spaces, if the top and bottom faces are homotopy pushouts and the left and back faces (the ones adjacent to $E_0 \to B_0$) are homotopy pullbacks, then the front and right faces (the ones adjacent to $E_3 \to B_3$) are also homotopy pullbacks.*



This theorem occurs many times in the literature, for instance famously in [Seg74, Lem 1.7], but the proof is usually not easy, see e.g. [Fuc71] and [DL59]. A slick modern approach using minimal Kan fibrations is provided by Lurie in [Lur09, 6.1.3.12], see also [Lur25, Theorem 011U], and a more "cells and lifts" style proof appears in [Hat02, 4K.1]. We give another proof here, in a style similar to Hatcher's but a bit more geometric.

Proof. Let us first consider what it means for a commuting square

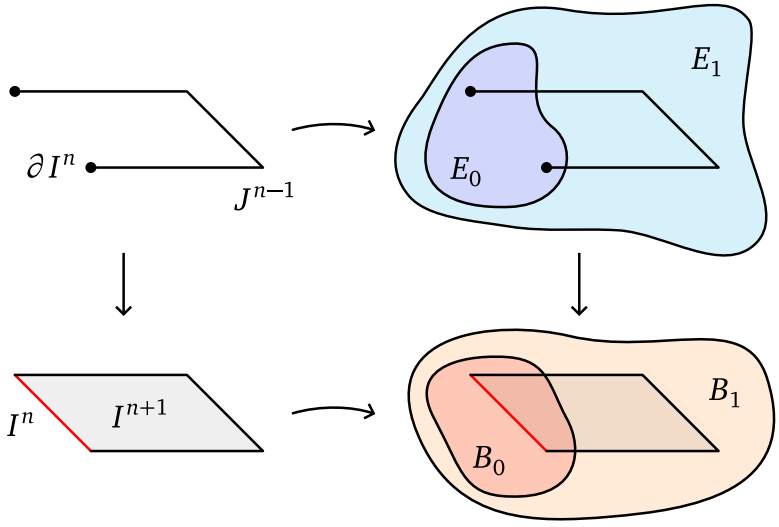
$$\begin{array}{ccc} E_0 & \longrightarrow & E_1 \\ \downarrow & & \downarrow \\ B_0 & \longrightarrow & B_1, \end{array}$$

to be a homotopy pullback. This is true iff $E_0 \to B_0 \times^h_{B_1} E_1$ is a weak equivalence, or equivalently, any map of pairs $(D^n, S^{n-1}) \to (B_0 \times^h_{B_1} E_1, E_0)$ can be deformed to one in which the disc lifts to E_0 (and if desired this homotopy can always be done rel S^{n-1}).

First we check that this is equivalent to the claim that for any commuting diagram of pairs as shown, the horizontal maps may be modified by a homotopy so that a lift of pairs exists. (Recall that $J^n \subseteq I^{n+1}$ is the union of all but one of the faces, and I^n is the remaining face.)

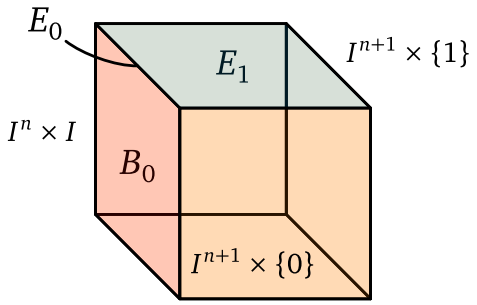
$$\begin{array}{ccc} (J^n, \partial I^n) & \longrightarrow & (E_1, E_0) \\ \downarrow & \nearrow & \downarrow \\ (I^{n+1}, I^n) & \longrightarrow & (B_1, B_0). \end{array} \tag{A.2.2}$$

To perform this rearrangement, we pick a homeomorphism $(D^n \times I) \cup_{(S^{n-1}\times I)} S^{n-1} \cong I^{n+1}$ that takes $D^n \times \{0\}$ to the left-hand face I^n and $D^n \times \{1\}$ to the union of the remaining faces J^n. This allows us to turn each map $D^n \to B_0 \times^h_{B_1} E_1$ that on S^{n-1} lifts to E_0, into a map $I^{n+1} \to B_1$ with I^n landing in B_0, J^n lifting to E_1, and on their intersection ∂I^n a further lift to E_0, giving the commuting square in (A.2.2), or the square illustrated below.



A lift of $D^n \to B_0 \times^h_{B_1} E_1$ to a map $D^n \to E^0$ produces a particular kind of lift in this square, namely a lift in which $I^n \times I \to E_1$ is constant in the I-coordinate and is therefore determined by a single map $I^n \to E_0$. However, any lift in the square (A.2.2) can be deformed to such a "constant" lift simply by shrinking the cube I^{n+1} along itself to the left-hand face I^n. So these two conditions are equivalent.

Next we check that if the maps $E_i \to B_i$ are fibrations, then (A.2.2) always has a *strict* lift, without needing to change the horizontal maps up to homotopy. We know that it has a lift up to homotopy, and this produces a map of the cube $I^{n+1} \times I$ to B_1 that has lifts along the left and top faces as shown.



But since $E_0 \to B_0$ is a fibration, we can lift the left-hand face $I^n \times I$ to E_0, and then since $E_1 \to B_1$ is a fibration we can lift the rest of $I^{n+1} \times I$ to E_1. Restricting to $I^{n+1} \times \{0\}$, therefore the original map of pairs had a lift to (E_1, E_0), as claimed.

Now that we understand this condition better, we replace I^{n+1} by an $(n+1)$-simplex Δ^{n+1}, and in the place of I^n and J^n we take one face Δ^n and the union of the remaining faces, which is a horn Λ^n. This horn intersects the last face Δ^n along the boundary $\partial\Delta^n$.

In our given cube of spaces, let us replace all of the maps $E_i \to B_i$ by fibrations for $i = 0, 1, 2$. Then form B_3 and E_3 as the corresponding mapping cylinders, and let $\widetilde{B}_3$ and $\widetilde{E}_3$ be the corresponding mapping "dumbbells:"

$$
\begin{aligned}
B_3 &= B_1 \cup_{(B_0 \times \{0\})} (B_0 \times [0,1]) \cup_{(B_0 \times \{1\})} B_2 \\
E_3 &= E_1 \cup_{(E_0 \times \{0\})} (E_0 \times [0,1]) \cup_{(E_0 \times \{1\})} E_2 \\
\widetilde{B}_3 &= (B_1 \times [0,1/3]) \cup_{(B_0 \times \{1/3\})} (B_0 \times [1/3,2/3]) \cup_{(B_0 \times \{2/3\})} (B_2 \times [2/3,1]) \\
\widetilde{E}_3 &= (E_1 \times [0,1/3]) \cup_{(E_0 \times \{1/3\})} (E_0 \times [1/3,2/3]) \cup_{(E_0 \times \{2/3\})} (E_2 \times [2/3,1])
\end{aligned}
$$

Note that in the category of arrows, $\widetilde{E}_3 \to \widetilde{B}_3$ is homotopy equivalent to $E_3 \to B_3$, the homotopy being one that in the interval direction stretches $[1/3, 2/3]$ to the entire interval and compresses $[0, 1/3]$ and $[2/3, 1]$ to the two endpoints.

Our goal is to prove that one of the squares involving $E_3 \to B_3$ is a homotopy pullback. By the above considerations, it suffices to find a lift in each diagram of pairs of the following form.

$$
\begin{array}{ccccc}
(\Lambda^n, \partial\Delta^n) & \longrightarrow & (E_3, E_2) & \longrightarrow & (\widetilde{E}_3, E_2) \\
\downarrow & \nearrow & \downarrow & & \downarrow \\
(\Delta^{n+1}, \Delta^n) & \longrightarrow & (B_3, B_2) & \longrightarrow & (\widetilde{B}_3, B_2).
\end{array}
\tag{A.2.3}
$$

Throughout we let $p\colon \Delta^{n+1} \to [0,1]$ be the composition of the map $\Delta^{n+1} \to B_3$ with the projection to the cylinder coordinate $B_3 \to I$.

We are allowed to modify the horizontal maps on the left-hand side of (A.2.3) up to homotopy first, and we will do so. We first perform iterated barycentric subdivision on

Δ^{n+1} until the simplices are small enough that their images under p have length smaller than $1/4$. Let $h\colon I \to I$ be the homotopy such that h_0 is the identity, h_1 is the map that stretches $[1/4, 3/4]$ to the entire interval and compresses $[0, 1/4]$ and $[3/4, 1]$ to the two endpoints, and h_t is the linear interpolation between these two. Then for each $\sigma \subseteq \Delta^{n+1}$ we do a straight-line homotopy of maps $\sigma \to I$ so that the vertices of σ go to their images under the compression map h_1, and on the rest of σ we get the linear interpolation. Note that when any point of σ touches 0 or 1, the whole simplex is contained in $[0, 1/4)$ or $(3/4, 1]$, respectively, and in this case the simplex is just deformed to the constant map at 0 or 1, respectively.

We then extend this homotopy of maps $\Delta^{n+1} \to I$ to a homotopy of maps $\Delta^{n+1} \to B_3$ by noting that any point with image in $(0, 1)$ either stays in that set or goes straight to 0 or 1, and so on the B_0 coordinate we can take the identity or apply the canonical map to B_1 or B_2. And if a point has image in $\{0, 1\}$ then its image is stationary throughout the homotopy, so there is no need to modify the coordinate in B_1 or B_2. Furthermore, on the subspace Δ^n this homotopy keeps us in B_2, and on Λ^n the chosen lift to E_3 can also be deformed in a compatible way, using the identity map of E_0 or the canonical maps to E_1 and E_2.

After this modification, the map $p\colon \Delta^{n+1} \to I$ is linear on each sub-simplex, and so the preimages of 0 and 1 give two subcomplexes $K_1, K_2 \subseteq \Delta^{n+1}$, with the last face Δ^n contained in K_2. Let N_1 and N_2 be the closed regular neighborhoods of K_1 and K_2, respectively. They are disjoint since after the above modification each simplex still has length less than $1/2$ in I, and so under the new definition of p we have $p(N_1) \subseteq [0, 1/2)$ and $p(N_2) \subseteq (1/2, 1]$. Let N_0 be the closure of their complement, and let $C_1 = N_0 \cap N_1$ and $C_2 = N_0 \cap N_2$. So we have decomposed Δ^{n+1} as the union $N_1 \cup_{C_1} N_0 \cup_{C_2} N_2$.

We now modify the map p a little further, by taking each vertex in C_1 and moving its image in $[0, 1]$ to the point $1/3$, and similarly moving each vertex in C_2 to $2/3$. Then we move all remaining vertices in N_0 into the interval $(1/3, 2/3)$. Again we linearly interpolate to get a map on the entire simplex, and in the B_0 or E_0 direction we simply take the identity map. After this modification we have the equalities

$$K_1 = p^{-1}(0), \quad N_1 = p^{-1}([0, 1/3]), \quad C_1 = p^{-1}(1/3), \quad N_0 = p^{-1}([1/3, 2/3]),$$

$$C_2 = p^{-1}(2/3), \quad N_2 = p^{-1}([2/3, 1]), \quad K_2 = p^{-1}(1).$$

Now we construct the desired lift $(\Delta^{n+1}, \Delta^n) \to (\widetilde{E}_3, E_2)$ in the diagram (A.2.3), one simplex at a time. Since we took iterated barycentric subdivision, it is possible to factor $\Lambda^n \to \Delta^{n+1}$ into a sequence of extensions of the form $\Lambda^k \to \Delta^{k+1}$ for $k \leq n$. For each such

extension we construct a lift of the form

$$\begin{array}{ccc} \Lambda^k & \longrightarrow & \widetilde{E}_3 \\ \downarrow & \nearrow & \downarrow \\ \Delta^{k+1} & \longrightarrow & \widetilde{B}_3. \end{array}$$

This will automatically send Δ^n to the subspace E_2.

We come now to the main difficulty in this proof: $\widetilde{E}_3 \to \widetilde{B}_3$ is not a fibration! However, we have carefully arranged things so that the desired lift will still exist. If $\Delta^{k+1} \subseteq N_0$, then we only have to find a lift

$$\begin{array}{ccc} \Lambda^k & \longrightarrow & E_0 \times [1/3, 2/3] \\ \downarrow & \nearrow & \downarrow \\ \Delta^{k+1} & \longrightarrow & B_0 \times [1/3, 2/3], \end{array}$$

and this exists because $E_0 \to B_0$ is a fibration. If $\Delta^{k+1} \subseteq N_1$, then it is a join of some vertices in K_1 and some vertices in C_1. If the missing face Δ^k has at least one vertex in K_1, then we only have to find a lift

$$\begin{array}{ccc} \Lambda^k & \longrightarrow & E_1 \times [0, 1/3] \\ \downarrow & \nearrow & \downarrow \\ \Delta^{k+1} & \longrightarrow & B_1 \times [0, 1/3], \end{array}$$

and this exists because $E_1 \to B_1$ is a fibration. If the missing face Δ^k has all vertices in C_1, then the opposite vertex is still in K_1, and we have a more complicated lifting diagram of pairs

$$\begin{array}{ccc} (\Lambda^k, \partial\Delta^k) & \longrightarrow & (E_1 \times [0, 1/3], E_0 \times \{1/3\}) \\ \downarrow & \nearrow & \downarrow \\ (\Delta^{k+1}, \Delta^k) & \longrightarrow & (B_1 \times [0, 1/3], B_0 \times \{1/3\}), \end{array}$$

but the lift exists by our proof that (A.2.2) has a strict lift if the maps $E_0 \to B_0$ and $E_1 \to B_1$ are fibrations. When $\Delta^{k+1} \subseteq N_2$ a similar argument holds. Note that as we construct these extensions, the simplices in C_1 and C_2 are always lifted to E_0 and not just to E_1 or E_2, which allows the remaining simplices in N_0 to be filled without any issues. This finishes the proof. □

To state the next two results, consider the category **Q** whose objects are maps of spaces

$E \to B$, and whose morphisms are homotopy pullback squares:

$$\begin{array}{ccc} F & \xrightarrow{\sim} & F' \\ \downarrow & & \downarrow \\ E & \longrightarrow & E' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B'. \end{array}$$

Lemma A.2.4. *The category* **Q** *has all coproducts, homotopy pushouts, and homotopy sequential colimits. In particular, for any pushout diagram in* **Q** *of the form*

$$\begin{array}{ccccc} F_1 & \xleftarrow{\sim} & F_0 & \xrightarrow{\sim} & F_2 \\ \downarrow & & \downarrow & & \downarrow \\ E_1 & \longleftarrow & E_0 & \longrightarrow & E_2 \\ \downarrow & & \downarrow & & \downarrow \\ B_1 & \longleftarrow & B_0 & \longrightarrow & B_1, \end{array}$$

the homotopy pushouts of the rows define another object of **Q** *with the same fiber, and for any sequential colimit diagram in* **Q** *of the form*

$$\begin{array}{ccccccc} F_0 & \xrightarrow{\sim} & F_1 & \xrightarrow{\sim} & F_2 & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ B_0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots \end{array}$$

the sequential homotopy colimits of the rows define another object of **Q** *with the same fiber.*

Proof. The first claim about coproducts is true because the homotopy fiber of a map of coproducts is measured on each component separately. The second claim about pushouts follows immediately from Theorem A.2.1. For the third claim we make the maps into closed inclusions and then note that the formation of the homotopy fiber commutes up to equivalence with colimits along closed inclusions, as in Exercise 1.7.31. □

Quillen's lemma (Theorem 7.4.33) now follows as a special case of this general theorem about homotopy colimits:

Theorem A.2.5 (Horizontal descent for homotopy colimits)**.** *Suppose* **I** *is a well-based topological category and* $X \to Y$ *is an* **I***-diagram in* **Q***. In other words for each morphism* $i \to j$ *in* **I***, the square*

$$\begin{array}{ccc} X(i) & \longrightarrow & X(j) \\ \downarrow & & \downarrow \\ Y(i) & \longrightarrow & Y(j) \end{array}$$

is a homotopy pullback (i.e. Cartesian). Then for any object i *the induced square*

$$\begin{array}{ccc} X(i) & \longrightarrow & B(X, \mathbf{I}, *) \\ \downarrow & & \downarrow \\ Y(i) & \longrightarrow & B(Y, \mathbf{I}, *) \end{array}$$

is also a homotopy pullback (Cartesian).

We prove this simultaneously with its variant for geometric realization:

Theorem A.2.6 (Horizontal descent for geometric realization)**.** *Suppose* $X_\bullet \to Y_\bullet$ *is a map of Reedy cofibrant simplicial spaces such that for every* $[m] \to [n]$ *in* $\mathbf{\Delta}$*, the square*

$$\begin{array}{ccc} X_n & \longrightarrow & X_m \\ \downarrow & & \downarrow \\ Y_n & \longrightarrow & Y_m \end{array}$$

is a homotopy pullback (Cartesian). Then for any $n \geq 0$ *the induced square*

$$\begin{array}{ccc} X_n \simeq \Delta^n \times X_n & \longrightarrow & |X_\bullet| \\ \downarrow & & \downarrow \\ Y_n \simeq \Delta^n \times Y_n & \longrightarrow & |Y_\bullet| \end{array}$$

is also a homotopy pullback (Cartesian).

In all of these cases, it is most intuitive to think of the Cartesian condition as saying that all of the maps $X(i) \to Y(i)$ have the "same homotopy fiber," and the map of homotopy colimits or realizations has this same homotopy fiber as well. If the maps $X(i) \to Y(i)$ are quasifibrations then we can phrase this in terms of the actual fiber: the corresponding map of bar constructions $B(X, \mathbf{I}, *) \to B(Y, \mathbf{I}, *)$ or of realizations $|X_\bullet| \to |Y_\bullet|$ is also a quasifibration.

Proof of Theorem A.2.5 and Theorem A.2.6. We first prove Theorem A.2.5 when $\mathbf{I}$ is discrete. We show that $B(X,\mathbf{I},*) \to B(Y,\mathbf{I},*)$ has the same homotopy fiber as the maps $X(i) \to Y(i)$, one skeleton at a time. Since $\mathbf{I}$ is discrete, the conditions for Lemma 8.2.36 are satisfied, and so we get the pushout square (8.2.37), which in this case becomes

$$\begin{array}{ccc} \partial\Delta^n \times \coprod_{c_0\to c_1\to\cdots\to c_n} X(c_0) & \longrightarrow & \Delta^n \times \coprod_{c_0\to c_1\to\cdots\to c_n} X(c_0) \\ \downarrow & & \downarrow \\ \mathrm{Sk}^{n-1}B(X,\mathbf{I},*) & \longrightarrow & \mathrm{Sk}^n B(X,\mathbf{I},*). \end{array}$$

The map of diagrams $X \to Y$ induces, on each corner of this square, a map $E \to B$ and therefore an object of $\mathbf{Q}$. The top horizontal map of the above diagram induces a map in $\mathbf{Q}$ by inspection. So if we assume that the map $\mathrm{Sk}^{n-1}B(X,\mathbf{I},*) \to \mathrm{Sk}^{n-1}B(Y,\mathbf{I},*)$ has the correct homotopy fiber, then the left vertical of the above diagram induces a map in $\mathbf{Q}$, and so the remaining two maps induce a map in $\mathbf{Q}$ by Lemma A.2.4, showing that the map $\mathrm{Sk}^n B(X,\mathbf{I},*) \to \mathrm{Sk}^n B(Y,\mathbf{I},*)$ also has the correct homotopy fiber. Therefore by induction this is true for all values of n. The map from each of these to the colimit over n is also a map in $\mathbf{Q}$ by Lemma A.2.4, and so $B(X,\mathbf{I},*) \to B(Y,\mathbf{I},*)$ has the correct homotoy fiber, which finishes the proof.

Theorem A.2.6 follows immediately since by Proposition 8.6.24 the realization is a special case of the homotopy colimit. (We could have argued this directly, but it takes a little longer to discuss what happens to the latching objects. This is one place where it is very convenient to go back and forth between hocolims and realizations.)

Now to prove Theorem A.2.5 when $\mathbf{I}$ is a topological category, since $\mathbf{I}$ is well-based the homotopy colimit can be described as the realization of a Reedy cofibrant simplicial space $N_\bullet(X,\mathbf{I},*)$, so by Theorem A.2.6 it suffices to show that every face and degeneracy map in the two-sided nerve induces a homotopy pullback square:

$$\begin{array}{ccc} \coprod_{c_0,c_1,\dots,c_n} X(c_0)\times\mathbf{I}(c_0,c_1)\times\cdots\times\mathbf{I}(c_{n-1},c_n) & \longrightarrow & \coprod_{d_0,d_1,\dots,d_m} X(d_0)\times\mathbf{I}(d_0,d_1)\times\cdots\times\mathbf{I}(d_{m-1},d_m) \\ \downarrow & & \downarrow \\ \coprod_{c_0,c_1,\dots,c_n} Y(c_0)\times\mathbf{I}(c_0,c_1)\times\cdots\times\mathbf{I}(c_{n-1},c_n) & \longrightarrow & \coprod_{d_0,d_1,\dots,d_m} Y(d_0)\times\mathbf{I}(d_0,d_1)\times\cdots\times\mathbf{I}(d_{m-1},d_m). \end{array}$$

The homotopy pullback becomes the disjoint union over all $c_0,\dots,c_n$ of the pullbacks of

$$\begin{array}{ccc} & & X(d_0)\times\mathbf{I}(d_0,d_1)\times\cdots\times\mathbf{I}(d_{m-1},d_m) \\ & & \downarrow \\ Y(c_0)\times\mathbf{I}(c_0,c_1)\times\cdots\times\mathbf{I}(c_{n-1},c_n) & \longrightarrow & Y(d_0)\times\mathbf{I}(d_0,d_1)\times\cdots\times\mathbf{I}(d_{m-1},d_m), \end{array}$$

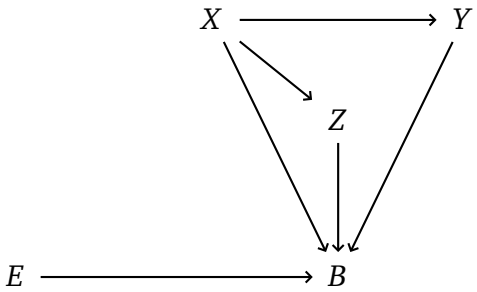
each of which gives $X(c_0)\times\mathbf{I}(c_0,c_1)\times\cdots\times\mathbf{I}(c_{n-1},c_n)$, and so the proof is complete. □

Remark A.2.7. Descent is not supposed to hold in every model category; it is supposed to be a special property that holds for spaces, and for other categories like **sSet** that act like spaces. In stable settings such as spectra, it is possible to prove a version of descent using the fact that fiber sequences are the same thing as cofiber sequences, but this is a more trivial result and is not as important as the result for spaces.

A.2.2 Vertical descent and the homological form of Quillen's lemma

In this section we give a quick but self-contained proof of the vertical form of descent for homotopy colimits. As before, the only difficult case is for homotopy pushouts:

Theorem A.2.8 (Vertical descent for homotopy pushouts)**.** *For any diagram of spaces of the form*



the homotopy pushout of the sub-diagram $Y \leftarrow X \rightarrow Z$ commutes with the homotopy pullback $(-) \times^h_B E$. In other words, if we arrange for $X \rightarrow Y$ to be a cofibration and $E \rightarrow B$ to be a fibration, then the following square is a homotopy pushout:

$$\begin{array}{ccc} X \times_B E & \longrightarrow & Y \times_B E \\ \downarrow & & \downarrow \\ Z \times_B E & \longrightarrow & (Y \cup_X Z) \times_B E. \end{array}$$

In other words, "homotopy pushouts are preserved by homotopy pullbacks along a fixed map."

Proof. There are a couple of ways to prove this; for instance we could argue that the functor $(-) \times^h_B E$ is equivalent to a left Quillen functor, and therefore it preserves homotopy colimits. For a direct proof that uses some fiberwise homotopy theory, without loss of generality the maps from X, Y, and Z to B are all Hurewicz fibrations, and we just have to show that the pushout $Y \cup_X Z \rightarrow B$ is also a Hurewicz fibration. (This form of the result is originally due to Clapp, see [Cla81, Prop 1.3].) Because then by the same argument the homotopy pushout of $Y \times_B E$, $X \times_B E$, and $Z \times_B E$ is a fibration over E, and the map from this fibration to $(Y \cup_X Z) \times_B E$ is a map of fibrations over E that is an equivalence on the fibers, so the map itself is an equivalence.

To prove this smaller pushout-fibration lemma, we first prove it in the case that $X \to Y$ has the *fiberwise* homotopy extension property, where the retraction $Y \times I \to (X \times I) \cup_{(X \times \{0\})} (Y \times \{0\})$ can be chosen to commute with the projection map back to B. Choose a path-lifting functions $L_X : X \times_B B^I \to X^I$, and extend this to a path-lifting function on Y by choosing a lift in the below square.

$$\begin{array}{ccc} Y \times_B B^I \times \{0\} \cup_{X \times_B B^I \times \{0\}} X \times_B B^I \times I & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow \\ Y \times_B B^I \times I & \longrightarrow & B, \end{array}$$

This is possible by the assumption that $X \to Y$ had the fiberwise homotopy extension property. This lift creates a path-lifting function $L_Y : Y \times_B B^I \to Y^I$ that agrees with L_X when restricted to X. Finally, choose a third, unrelated path-lifting function $L_Z : Z \times_B B^I \to Z^I$. To glue these together to a path-lifting function for $Y \cup_X Z$, we use L_Z on Z, and a combination of L_Y and L_Z on $Y \setminus X$.

The retraction that we defined in $Y \times I$ creates an assignment of each point $y \in Y$ to a path $h(t, y)$, such that the points close to X are sent to a path that goes into X and then remains constant in X for some time (more time as we get closer to X). (This is the standard formula for turning a cofibration into an NDR-pair, see [May99].) So we define the path lifting on $Y \setminus X$ as follows: first apply L_Y to get a path $\sigma : I \to Y$. Then we take the path $h(t, \sigma(t))$, in other words the diagonal in the square $I^2 \to Y$ defined by $h(t, \sigma(s))$. If we are far from X, we take this path as the definition of our path-lifting function on $y \in Y$. When we are close to X then at some point t the path $h(t, \sigma(t))$ lands in X, and then stays fixed. When this happens, we switch over and lift the remainder of our path in B using L_Z instead. (The point at which we switch over can be given an explicit formula from the NDR-pair structure.) This recipe actually makes sense for all of Y, and on the subspace X it returns L_Z, so our choices of path-lifting now glue together to give a single path-lifting function on $Y \cup_X Z$.

Finally, if $X \to Y$ only has the homotopy extension property, but X and Y are both Hurewicz fibrations over B, then $(X \times \{0\}) \to (X \times I)$ clearly has the fiberwise homotopy extension property, so by the result we just proved, $(X \times I) \cup_{X \times \{0\}} (Y \times \{0\})$ is a Hurewicz fibration over B. It follows that there exists a lift

$$\begin{array}{ccc} (X \times I) \cup_{X \times \{0\}} (Y \times \{0\}) & = & (X \times I) \cup_{X \times \{0\}} (Y \times \{0\}) \\ \downarrow{\scriptstyle i} & \nearrow & \downarrow \\ Y \times I & \longrightarrow & B, \end{array}$$

and this lift shows that $X \to Y$ in fact had the fiberwise homotopy extension property all along! □

Next, we use this to prove a homological version of horizontal descent from the previous section, and use this to prove Proposition 9.1.34, which is a key ingredient in the group completion theorem. The key idea is fiberwise suspension.

Definition A.2.9. For any map of topological spaces $p : E \to B$, the (unreduced) fiberwise suspension $S_B E$ is the homotopy pushout of the diagram $B \leftarrow E \to B$ in which both maps are p. This has a map to B, and the fiber is the unreduced suspension of the fiber, SF.

The following is an immediate consequence of the proof of Theorem A.2.8.

Lemma A.2.10. *If $E \to B$ is a Hurewicz fibration then $S_B E \to B$ is a Hurewicz fibration as well. More generally, the homotopy fiber of $S_B E \to B$ is the unreduced suspension of the homotopy fiber of $E \to B$.*

Now consider the category $\mathbf{Q}_{H\mathbb{Z}}$ whose objects are maps of spaces $E \to B$, and whose morphisms are weakly homologically Cartesian squares, i.e. squares in which the induced map of fibers $F \to F'$ induces an isomorphism on homology with $\mathbb{Z}$ coefficients. (By the universal coefficient theorem, they also induce an isomorphism on homology with coefficients in any abelian group.)

Lemma A.2.11. *The category $\mathbf{Q}_{H\mathbb{Z}}$ has all homotopy pushouts and homotopy sequential colimits, in the same sense as Lemma A.2.4 above.*

Proof. Consider the functor $\mathbf{Q}_{H\mathbb{Z}} \to \mathbf{Q}$ that takes two-fold fiberwise suspension (Definition A.2.9). This preserves all weak equivalences, and on the homotopy fibers has the effect of suspending twice. A map of spaces $F \to F'$ induces an isomorphism on homology iff $S^2 F \to S^2 F'$ is a weak equivalence, so this is well-defined. Now we observe that fiberwise suspension commutes with the kinds of homotopy pushouts and sequential colimits we are considering, so the colimit after suspending twice lands in $\mathbf{Q}$ iff the original colimit landed in $\mathbf{Q}_{H\mathbb{Z}}$. Therefore the claim follows from Lemma A.2.4. □

Theorem A.2.12 (Homological descent for homotopy colimits)**.** *Suppose $\mathbf{I}$ is a well-based topological category and $X \to Y$ is an $\mathbf{I}$-diagram in $\mathbf{Q}_{H\mathbb{Z}}$. In other words for each morphism $i \to j$ in $\mathbf{I}$, the square*

$$\begin{array}{ccc} X(i) & \longrightarrow & X(j) \\ \downarrow & & \downarrow \\ Y(i) & \longrightarrow & Y(j) \end{array}$$

induces an isomorphism on the homology of the vertical fibers. Then for any object i the induced square

$$\begin{array}{ccc} X(i) & \longrightarrow & B(X, \mathbf{I}, *) \\ \downarrow & & \downarrow \\ Y(i) & \longrightarrow & B(Y, \mathbf{I}, *) \end{array}$$

also induces an isomorphism on the homology of the vertical fibers.

The proof is identical to Theorem A.2.5. Proposition 9.1.34 now follows as a special case.

Remark A.2.13. In the place of the two-fold fiberwise suspension, we could have taken any spectrum A and applied a more sophisticated operation $\Omega^\infty_B(A\overline{\wedge}(-))$ that transforms each fiber F into a space $\Omega^\infty(A\wedge F_+)$ that captures the A-homology of F (see [Mal23] for details). The same argument then shows that homotopy colimits have descent for extraordinary homology; this gives a proof of the version of the group completion theorem from [Nik17].

A.2.3 Simplifications of the categorical bar construction

In Section 7.4.2 we gave a simplification of the categorical bar construction that is very useful for actually writing out examples. In this section we prove that it is correct, and adapt it to any simplicial model category.

Suppose $\mathbf{I}$ is a discrete category and the non-identity morphisms are closed under composition. This means that $\mathbf{V} = \mathbf{sSet}$ or $\mathbf{Top}$, so let's pick one of those cases, and say that Y is the diagram in $\mathbf{V}$, and X is the diagram in the larger category $\mathbf{C}$. We'll write all of the tensors as $\otimes$ to make the expressions cleaner.

Then we claim that the bar construction is generated by the coproduct

$$\coprod_{n\geq 0}\left(\coprod_{c_0\to c_1\to\cdots\to c_n}\Delta[n]\otimes X(c_0)\otimes Y(c_n)\right),$$

where $c_0\to\cdots\to c_n$ runs over all strings of composable, non-identity morphisms, modulo the face relations. To be more precise, there is a coequalizer diagram

$$\begin{array}{c}\displaystyle\coprod_{n\geq 0}\coprod_{0\leq i\leq n}\left(\coprod_{c_0\to c_1\to\cdots\to c_n}\Delta[n-1]\otimes X(c_0)\otimes Y(c_n)\right)\\ \downarrow\downarrow\\ \displaystyle\coprod_{n\geq 0}\left(\coprod_{c_0\to c_1\to\cdots\to c_n}\Delta[n]\otimes X(c_0)\otimes Y(c_n)\right)\longrightarrow B(X,\mathbf{I},Y),\end{array}\tag{A.2.14}$$

in which the two parallel maps apply either the ith coface $d^i\colon\Delta[n-1]\to\Delta[n]$ or the ith face map to $\coprod_{c_0\to c_1\to\cdots\to c_n}X(c_0)\otimes Y(c_n)$, as in Example 7.4.10. The map to $B(X,\mathbf{I},Y)$ arises because $B(X,\mathbf{I},Y)$ is defined using more terms and more relations than the ones given.

Proposition A.2.15. *The diagram* (A.2.14) *is in fact a coequalizer. In other words,* $B(X,\mathbf{I},Y)$ *is given by this smaller set of generators and relations.*

Proof. Since $\mathbf{I}$ is discrete, the simplicial object $B^{\otimes}_{\bullet}(X,\mathbf{I},Y)$ is at each level the coproduct over all strings of morphisms $c_0 \to c_1 \to \cdots \to c_n$ of the terms $X(c_0)\otimes Y(c_n)$. The latching map includes those summands in which at least one of the morphisms is an identity map. So it is the inclusion of a summand, and the complementary summand is the coproduct over the non-identity maps. Therefore Lemma 8.2.36 gives pushout squares of the form

$$\begin{array}{ccc} \coprod_{c_0\to c_1\to\cdots\to c_n} \partial\Delta[n]\otimes X(c_0)\otimes Y(c_n) & \longrightarrow & \coprod_{c_0\to c_1\to\cdots\to c_n} \Delta[n]\otimes X(c_0)\otimes Y(c_n) \\ \downarrow & & \downarrow \\ \mathrm{Sk}^{n-1}B(X,\mathbf{I},Y) & \longrightarrow & \mathrm{Sk}^{n}B(X,\mathbf{I},Y) \end{array} \tag{A.2.16}$$

where now we are taking coproducts only over strings of non-identity morphisms $c_0 \to \cdots \to c_n$. Taking the colimit over n, this rearranges into the statement that (A.2.14) is a coequalizer. □

Next, as in Section 7.5.4, we get a simplification of $F(X,\mathbf{I},Y)$ when $\mathbf{I}$ is discrete and the non-identity morphisms are closed under composition. So $\mathbf{V} = \mathbf{sSet}$ or $\mathbf{Top}$. Recall that X is the diagram in $\mathbf{V}$, and Y is the diagram in the larger category $\mathbf{C}$. Then the cobar construction is given by the product

$$\prod_{n\geq 0}\left(\prod_{c_0\to c_1\to\cdots\to c_n} Y(c_n)^{\Delta[n]\otimes X(c_0)}\right),$$

where $c_0 \to \cdots \to c_n$ runs over all strings of composable, non-identity morphisms, and then we restrict to the subobject that respects the face relations. To be more precise, there is an equalizer diagram

$$\begin{array}{ccc} F(X,\mathbf{I},Y) & \longrightarrow & \prod_{n\geq 0}\left(\prod_{c_0\to c_1\to\cdots\to c_n} Y(c_n)^{\Delta[n]\otimes X(c_0)}\right) \\ & & \downarrow\downarrow \\ & & \prod_{n\geq 0}\prod_{0\leq i\leq n}\left(\prod_{c_0\to c_1\to\cdots\to c_n} Y(c_n)^{\Delta[n-1]\otimes X(c_0)}\right) \end{array} \tag{A.2.17}$$

in which the two parallel maps apply either the ith coface $d^i\colon \Delta[n-1]\to\Delta[n]$ or the ith coface map to $\prod_{c_0\to c_1\to\cdots\to c_n} X(c_0)\otimes Y(c_n)$, as in Example 7.5.13. The map to $B(X,\mathbf{I},Y)$ arises because $B(X,\mathbf{I},Y)$ is defined using more terms and more relations than the ones given.

Proposition A.2.18. *The diagram* (A.2.17) *is in fact an equalizer.*

Proof. The proof is the same as in Proposition A.2.15. In particular, we use Lemma 8.4.17 to get for each n a pullback square

$$\begin{array}{ccc} \mathrm{coSk}^n \mathrm{Tot}(X^\bullet) & \longrightarrow & \mathrm{coSk}^{n-1}\mathrm{Tot}(X^\bullet) \\ \downarrow & & \downarrow \\ \prod_{c_0\to c_1\to\cdots\to c_n} Y(c_n)^{\Delta[n]\otimes X(c_0)} & \longrightarrow & \prod_{c_0\to c_1\to\cdots\to c_n} Y(c_n)^{\partial\Delta[n]\otimes X(c_0)} \end{array} \tag{A.2.19}$$

in which the product is only over strings of non-identity morphisms $c_0 \to \cdots \to c_n$. □

A.2.4 Improving the Bousfield-Kan homotopy colimit

In Section 8.3 we proved that the bar construction defines the homotopy colimit in any cofibrantly generated simplicial model category. To be more specific, Theorem 8.3.19 says that $B(QX, \mathbf{I}, *)$ is the homotopy colimit of $X : \mathbf{I} \to \mathbf{C}$, when $\mathbf{I}$ is a discrete or **sSet**-enriched indexing category, and $\mathbf{C}$ is any cofibrantly generated simplicial model category.

In this section we extend this theorem to a few other settings. First, by Corollary 8.3.21, it's possible to let $\mathbf{I}$ take values in topological spaces, spectra, or any other simplicial symmetric monoidal category $\mathbf{V}$ that acts on $\mathbf{C}$. But then $\mathbf{I}$ needs to be pointwise cofibrant.

However, if $\mathbf{I}$ isn't pointwise cofibrant, we can still replace it by a pointwise cofibrant category $Q\mathbf{I}$, by the following theorem.

Suppose $\mathbf{V}$ is any of the categories from Theorem 6.5.20, or the versions of these categories based on simplicial sets: $\mathbf{V} = \mathbf{sSet}$, $\mathbf{sSet}_*$, or $\mathbf{Sp}^\Sigma_{\mathbf{sSet}}$.

Theorem A.2.20 (Right-induced model structure for $\mathbf{V}$-categories)**.** *There is a model structure on the category of (small)* $\mathbf{V}$*-enriched categories* $\mathbf{I}$ *with fixed object set* S*, such that*

- *the weak equivalences are the pointwise equivalences, and*
- *the fibrations are the pointwise fibrations, which give fibrations on* $\mathbf{I}(a, b)$ *separarately for all* $a, b \in S$*.*

Every cofibration in this model structure is a pointwise cofibration.

The proof is almost identical to the proof of the model structure for algebras from Section 6.5.3, except that we replace the notion of an "algebra in $\mathbf{V}$" by the notion of a "category enriched in $\mathbf{V}$," which is a many-objects version of an algebra. See Exercise

8.7.41. (This theorem appears in the literature in [SS03a, Prop 6.3], and earlier in [DK80, Prop 7.2] in the special case that $\mathbf{V} = \mathbf{sSet}$.)

Notice that the initial object in $\mathbf{V}$-enriched categories is the unit category $\mathbf{U}$ with

$$\begin{cases} \mathbf{U}(a,b) = I & \text{if } a = b, \\ \mathbf{U}(a,b) = \emptyset & \text{if } a \neq b. \end{cases}$$

So, a category $\mathbf{I}$ that is cofibrant in the above model structure is always pointwise cofibrant, since $\mathbf{U} \to \mathbf{I}$ is a pointwise cofibration. Therefore, for any $\mathbf{V}$-category $\mathbf{I}$ we get a cofibrant replacement $q : Q\mathbf{I} \to \mathbf{I}$, such that $Q\mathbf{I}$ is pointwise cofibrant and q is a pointwise equivalence. If we assume in addition that the category $\mathbf{V}^{\mathbf{I}}$ of $\mathbf{I}$-diagrams in $\mathbf{V}$ also has the projective model structure (which is again true in all of our examples), we get the following as a consequence of Theorem 8.3.19.

Corollary A.2.21. *If $Q\mathbf{I} \to \mathbf{I}$ denotes any such pointwise-cofibrant replacement of $\mathbf{I}$, the category $\mathbf{C}^{\mathbf{I}}$ has a cofibrant replacement functor Q, and the unit $I \in \mathbf{V}$ is cofibrant, then the (enriched) homotopy colimit of any diagram X over $\mathbf{I}$ exists, and is given by*

$$\left(\operatorname*{hocolim}_{c \in \mathbf{I}} X(c)\right) \simeq B(QX, Q\mathbf{I}, \mathrm{const}(I)).$$

So any time we have a diagram of spectra X, and the diagram is indexed by a *topological* category $\mathbf{I}$, we can form its homotopy colimit. We just have to take a cofibrant replacement of the spectra $X(c)$ and the mapping spaces $\mathbf{I}(a,b)$, and then form the bar construction.

Another way to improve Corollary 8.3.21 is to choose a broader class of cofibrations in $\mathbf{V}$ and $\mathbf{C}$ that satisfy the assumptions above Theorem 8.2.27, using both $\otimes$ and $\boxtimes$. For instance:

Example A.2.22.

- When $\mathbf{V} = \mathbf{C} = \mathbf{sSet}$, the cofibrancy conditions we gave in Corollary 8.3.21 are always true, so there is no way to improve them. The homotopy colimit is $B(X, \mathbf{I}, *)$. The same is true if $\mathbf{C}$ or both $\mathbf{V}$ and $\mathbf{C}$ are $\mathbf{sSet}_*$.
- When $\mathbf{V} = \mathbf{C} = \mathbf{Top}$, we can use unbased Hurewicz cofibrations everywhere. Then every diagram X or Y is already pointwise cofibrant, and for $\mathbf{I}$ to be pointwise cofibrant, it's enough for each space $\mathbf{I}(c,c)$ to be well-based.

  This recovers our earlier theorem that the bar construction in $\mathbf{Top}$ is homotopy invariant (Theorem 7.4.17), and that if $\mathbf{I}$ is discrete or well-based, then $B(X, \mathbf{I}, *)$ is the homotopy colimit of X (Corollary 7.4.24).

We give two more examples where the assumptions can be improved.

Theorem A.2.23. *In* $\mathbf{V} = \mathbf{C} = \mathbf{Top}_*$ *the bar construction* $B^\wedge(X, \mathbf{I}, Y)$ *preserves equivalences, and therefore the homotopy colimit of* X *is given by* $B^\wedge(X, \mathbf{I}, *)$*, when the following maps are all unbased h-cofibrations.*

$$* \longrightarrow X(c), \qquad S^0 \longrightarrow \mathbf{I}(c,c), \qquad * \longrightarrow \mathbf{I}(a,b), \qquad * \longrightarrow Y(c) \tag{A.2.24}$$

This recovers the statement that $B(X, \mathbf{I}, *)$ is the homotopy colimit so long as X and $\mathbf{I}$ are well-based. We've been seeing versions of this claim ever since Definition 1.5.12, but this version works for any category $\mathbf{I}$, not just coproducts, pushouts, and sequential colimits.

Proof. We take the variant of the assumptions above Theorem 8.2.27 that uses both $\otimes$ and $\boxtimes$. It is easy to verify these for the unbased h-cofibrations, and therefore the realization preserves equivalences of cofibrant objects when the latching maps are Reedy unbased h-cofibrations. Under the given assumptions on X, $\mathbf{I}$, and Y, the proof of Theorem 8.3.12 shows that the latching maps of $B^\wedge(X, \mathbf{I}, Y)$ are Reedy unbased h-cofibrations, so the bar construction preserves weak equivalences. □

Theorem A.2.25. *In* $\mathbf{V} = \mathbf{Top}$ *and* $\mathbf{C} = \mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}}$, $\mathbf{Sp}^{\Sigma}_{\mathbf{Top}}$, *or* $\mathbf{Sp}^{O}_{\mathbf{Top}}$, *the bar construction* $B(X, \mathbf{I}, Y)$ *preserves equivalences when* X *is any diagram of spectra, and* $\mathbf{I}$ *and* Y *are pointwise Quillen cofibrant in unbased spaces. Therefore the homotopy colimit of an* $\mathbf{I}$*-diagram of spectra* X *is given by* $B(X, \mathbf{I}, *) = B^\wedge(X, \mathbf{I}_+, S^0)$ *when* $\mathbf{I}$ *is pointwise Quillen cofibrant and* X *is arbitrary.*

Proof. Again, it is enough to check that the Quillen cofibrations in $\mathbf{V} = \mathbf{Top}$ and the h-cofibrations in $\mathbf{C} = \mathbf{Sp}$ satisfy the assumptions above Theorem 8.2.27. This is done in Exercise 8.7.42. □

Remark A.2.26. We could also apply Theorem A.2.23 to every level of the spectra, and conclude that $B(X, \mathbf{I}, *)$ is the homotopy colimit when the maps of (A.2.24) are Hurewicz cofibrations (at each spectrum level for $X(c)$).

The third and final improvement of Corollary 8.3.21 is that $\mathbf{C}$ doesn't even have to be a simplicial model category, it can be an arbitrary model category.

Theorem A.2.27. *If* $\mathbf{C}$ *is any model category and* $\mathbf{I}$ *is discrete, then homotopy colimits along* $\mathbf{I}$ *always exist.*

However, the proof still uses the idea of a simplicial model category. It turns out that in any model category, you can make a janky version of a tensor with **sSet**: for each X, factor the map $X \amalg X \to X$ into a cofibration and weak equivalence, and then *define* $\Delta[1] \otimes X$ to be the new space constructed by this factorization. Then you take similar factorizations to construct $\Delta[2] \otimes X$ and all of the higher simplices as well. These factorizations give a "framing" of $\mathbf{C}$, which is close enough to a tensor with **sSet** that you can still define the bar construction $B(X, \mathbf{I}, *)$ and the above arguments go through. See [DHKS04] and [Hov99] for more details.

A.2.5 Improving the Bousfield-Kan homotopy limit

In this section we describe the corresponding improvements of the Bousfield-Kan model for the homotopy limit. As before, whenever the model structure on **V**-categories **I** from Theorem A.2.20 and the projective model structure for **I**-diagrams exist, we get the following more general formula for the homotopy limit. (We still assume that the unit $I \in \mathbf{V}$ is cofibrant.)

Corollary A.2.28. *The (enriched) homotopy limit of any diagram Y over* **I** *exists, and is given by*

$$\left(\operatorname*{holim}_{c\in\mathbf{I}} Y(c)\right) \simeq F(\mathrm{const}(I), Q\mathbf{I}, RY).$$

In particular, any time we have a diagram of spectra Y, and the diagram is indexed by a *topological* category **I**, we can form its homotopy colimit. We just have to take a fibrant replacement of the spectra $Y(c)$ and the cofibrant replacement of the mapping spaces $\mathbf{I}(a,b)$, before taking the cobar construction.

One might also hope to choose a broader class of fibrant objects for which the cobar construction preserves equivalences. This isn't actually possible in spectra, as far as the author is aware, but it is possible if we restrict to finite cell categories **I**.

Definition A.2.29. Let $\mathbf{V} = \mathbf{Top}$. We say that **I** is a **finite cell category** if each mapping space $\mathbf{I}(a,b)$ is a finite cell complex and the realization $B\mathbf{I} = |B(*,\mathbf{I},*)|$ is also a finite cell complex. We say that the diagram X on **I** is a **pointwise finite cell** if each of the spaces $X(a)$ is a finite cell complex. For $\mathbf{V} = \mathbf{sSet}$, we say that **I** and X are finite cell if their geometric realizations are finite cell.

Theorem A.2.30. *When* **I** *is finite cell and X is pointwise finite cell, if* **C** *is any category of diagram spectra, then the cobar construction $F(X,\mathbf{I},Y)$ preserves equivalences, and therefore the homotopy limit of Y is given by $F(*,\mathbf{I},Y)$, when Y is level fibrant.*

Recall that "level fibrant" means that the levels Y_n are fibrant spaces. When "space" means topological space, this is always true. When "space" means simplicial set, this means that Y_n is a Kan complex. So for the categories of spectra based on topological spaces, in other words $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}}$, $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}}$, or $\mathbf{Sp}^{\mathbb{N}}_{\mathbf{Top}}$, the cobar construction $F(*,\mathbf{I},Y)$ is always the homotopy limit if **I** is finite cell and Y is *any* diagram of spectra.

Proof. It suffices to see that all pointwise equivalences of such diagrams induce stable equivalences on the cobar construction. A pointwise equivalence of diagrams induces a stable equivalence on each level of the cosimplicial spectrum $F^\bullet(X,\mathbf{I},Y)$, using the fact that **I** has only finitely many objects, and that finite products and cotensors $F(K,-)$ for K a finite cell complex always preserve stable equivalences of spectra. The proof

of Theorem 8.5.12 shows that $F^\bullet(X, \mathbf{I}, Y)$ is Reedy *level* fibrant: its matching maps are levelwise fibrations of spectra.

To show this gives an equivalence on the totalization, we follow the proof of Theorem 8.4.11 using the *level* fibrations of spectra instead of the stable fibrations. We have the same axioms as in Lemma 8.4.12, except for the limit axiom. (The co-gluing lemma is subtle: a pullback square along a level fibration is a homotopy pullback square in spectra, which gets a long exact sequence on stable homotopy groups. Therefore a stable equivalence of pullback diagrams gives a stable equivalence on the pullbacks.)

We are therefore able to prove that each equivalence of Reedy level fibrant cosimplicial spectra induces an equivalence on $\mathrm{coSk}^n\mathrm{Tot}$ for each n, though not on the limit. However, since $\mathbf{I}$ is a finite cell category, the latching maps of $N_\bullet(*, \mathbf{I}, *)$ are isomorphisms for sufficiently large n. Therefore, for $n \gg 0$ the matching maps for $F^\bullet(X, \mathbf{I}, Y)$ are also isomorphisms. It follows from the pullback square (8.4.16) that the maps $\mathrm{coSk}^n\mathrm{Tot} \to \mathrm{coSk}^{n-1}\mathrm{Tot}$ are isomorphisms for sufficiently large n. Therefore an equivalence on the nth coskeleton for every finite n implies an equivalence on the totalization, finishing the proof. □

Finally, as in Theorem A.2.27, it isn't necessary for $\mathbf{C}$ to be simplicial. See [DHKS04] and [Hov99] for more details.

Theorem A.2.31. *If* $\mathbf{C}$ *is any model category and* $\mathbf{I}$ *is discrete, then homotopy limits along* $\mathbf{I}$ *always exist.*

A.3 Monads and model categories of algebras

In this section we go deeper into some of the technical proofs from Chapter 9. The key idea we need to organize these proofs is the concept of a **monad**. It can come off as very abstract, so to motivate it, let's being by giving an explicit construction of colimits in the category of $\mathcal{O}$-algebras, when $\mathcal{O}$ is any operad. Then we'll show how to organize and streamline the construction using monads.

A.3.1 Free $\mathcal{O}$-algebras and colimits of $\mathcal{O}$-algebras

Suppose $\mathbf{O} \in \mathrm{Op}(\mathbf{C})$ is an operad in any symmetric monoidal category $\mathbf{C}$. Suppose that $\mathbf{C}$ has all colimits and that the tensor $\otimes$ preserves colimits in each slot.

Definition A.3.1. For any object $X \in \mathrm{ob}\,\mathbf{C}$, we define the **free $\mathcal{O}$-algebra** on X as

$$T_{\mathcal{O}}(X) := \coprod_{n\geq 0} \mathcal{O}(n) \otimes_{\Sigma_n} X^{\otimes n} = \mathcal{O}(0) \amalg (\mathcal{O}(1) \otimes X) \amalg (\mathcal{O}(2) \otimes_{\Sigma_2} (X \otimes X)) \amalg \cdots .$$

The tensor over Σ_n means that we take the tensor product and then quotient by the relation $(f\sigma, x_1, \dots, x_n) = (f, x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$. (We can quotient by relations like this in any category with all colimits; see the discussion in the proof of Lemma 6.5.13.)

The $\mathcal{O}$-algebra structure is defined on $T_{\mathcal{O}}(X)$ by the operad composition maps. For instance, one piece of this action is

$$\mathcal{O}(n)\otimes\bigotimes_{i=1}^{n}(\mathcal{O}(k_i)\otimes X^{\otimes k_i}) \cong \Big(\mathcal{O}(n)\otimes\bigotimes_{i=1}^{n}\mathcal{O}(k_i)\Big)\otimes X^{\otimes(\sum_i k_i)} \xrightarrow{\circ} \mathcal{O}\Big(\sum_i k_i\Big)\otimes X^{\otimes(\sum_i k_i)}. \quad \text{(A.3.2)}$$

The required identity, associativity, and symmetry axioms for this action follow from the same axioms for the operad $\mathcal{O}$. We let $F_{\mathcal{O}}(X)$ denote $T_{\mathcal{O}}(X)$ with this $\mathcal{O}$-algebra structure. In other words, $T_{\mathcal{O}}(X)$ is the underlying object in $\mathbf{C}$ of the $\mathcal{O}$-algebra $F_{\mathcal{O}}(X)$.

Lemma A.3.3. *The functor $F_{\mathcal{O}}$ is the left adjoint of the forgetful functor $U_{\mathcal{O}}$.*

Proof. Let $X \in \mathbf{C}$ and $Y \in \mathrm{Alg}_{\mathcal{O}}(\mathbf{C})$. We form the bijection

$$f : \coprod_{n\geq 0} \mathcal{O}(n)\otimes_{\Sigma_n} X^{\otimes n} \to Y \qquad \longleftrightarrow \qquad g : X \to U_{\mathcal{O}}(Y)$$

by restricting each map f to the $n = 1$ term to get g, and extending each map g to the remaining terms by applying $g^{\otimes n}$ and then applying the $\mathcal{O}$-action on Y. A straightforward diagram-chase shows that every map f must be of this form, so these are inverses. □

Lemma A.3.4. *If $\mathbf{C}$ has all colimits and limits, and $\otimes$ preserves colimits in each slot, then the category $\mathrm{Alg}_{\mathcal{O}}(\mathbf{C})$ has all colimits and limits as well.*

Proof. We have already proven this in the special case that $\mathcal{O} = \mathrm{Assoc}$ and Comm in Lemma 6.5.13 and Lemma 6.5.24, respectively. For a general operad, the limit of a diagram of $\mathcal{O}$-algebras X_i is constructed in essentially the same way: we take the limit in as a diagram in $\mathbf{C}$, and then give the limit an $\mathcal{O}$-action by filling in the diagram

$$\begin{array}{ccc} \mathbf{O}(n)\otimes\Big(\lim_{(\mathbf{C})} X_i\Big)^{\otimes n} & \dashrightarrow & \lim_{(\mathbf{C})} X_i \\ \downarrow & & \downarrow \\ \mathcal{O}(n)\otimes X_i^{\otimes n} & \longrightarrow & X_i \end{array}$$

using the universal property of the limit. The fact that this is the limit in $\mathcal{O}$-algebras follows the same pattern as Exercise 6.6.30.

As before, colimits are more complicated. Given a diagram of $\mathcal{O}$-algebras $\mathscr{I} \to \mathrm{Alg}_{\mathcal{O}}(\mathbf{C})$, we take all possible tensor products

$$\coprod_{n\geq 0}\ \coprod_{i_1,\dots,i_n\in \mathrm{ob}\,\mathbf{I}} \mathcal{O}(n)\otimes X_{i_1}\otimes\cdots\otimes X_{i_n}, \quad \text{(A.3.5)}$$

and then impose the following variant of the relations from Lemma 6.5.24:

- For each arrow $f : i \to j$ in $\mathbf{I}$, each expression involving X_i is identified with the corresponding expression with X_j in its place, along the map f.
- For every expression $\mathscr{O}(n) \otimes X_{i_1} \otimes \cdots \otimes X_{i_n}$ and every $\sigma \in \Sigma_n$, the following two images are identified:

$$\begin{array}{ccc} & \nearrow & \mathscr{O}(n) \otimes X_{i_1} \otimes \cdots \otimes X_{i_n} \\ \mathscr{O}(n) \otimes X_{i_1} \otimes \cdots \otimes X_{i_n} & & \\ & \searrow & \mathscr{O}(n) \otimes X_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes X_{i_{\sigma^{-1}(n)}}. \end{array}$$

  The first map applies σ to $\mathscr{O}(n)$, and the second permutes the Xs using σ.
- Every pair of expressions of the following form is identified:

$$\begin{array}{ccc} \mathscr{O}(n) \otimes (\mathscr{O}(k_1) \otimes X_{i_1} \otimes \cdots \otimes X_{i_1}) \otimes \cdots \otimes (\mathscr{O}(k_n) \otimes X_{i_n} \otimes \cdots \otimes X_{i_n}) & & \\ \downarrow & \searrow & \\ & & \mathscr{O}(n) \otimes X_{i_1} \otimes \cdots \otimes X_{i_n} \\ \mathscr{O}\big(\textstyle\sum_i k_i\big) \otimes (X_{i_1} \otimes \cdots \otimes X_{i_1}) \otimes \cdots \otimes (X_{i_n} \otimes \cdots \otimes X_{i_n}). & & \end{array}$$

  The left-hand map applies the operad composition $\circ$, while the right-hand map applies the $\mathscr{O}$-action on X. Informally, this means that composite operations in the spaces $\mathscr{O}(-)$ can be factored and then the smaller functions can act on the terms X_i. In the special case that $n = 1$, this gives an identification

$$\mathscr{O}(k) \otimes X_i^{\otimes k} \cong I \otimes \mathscr{O}(k) \otimes X_i^{\otimes k} \xrightarrow{\text{id}\otimes\text{act}} \mathscr{O}(1) \otimes X_i. \tag{A.3.6}$$

We call the resulting quotient $\operatorname*{colim}_{\mathrm{Alg}_{\mathscr{O}}(\mathbf{C})} X_i$. This is an $\mathscr{O}$-algebra by essentially the same construction that we did for $F_{\mathscr{O}}$, and the map from each X_i into this colimit is an $\mathscr{O}$-algebra homomorphism using (A.3.6). Any collection of compatible $\mathscr{O}$-algebra homomorphisms $f_i : X_i \to Y$ induce a map from this colimit to Y by applying f_i to all the X_i terms and then applying the $\mathscr{O}$-action of Y. This is well-defined and a map of $\mathscr{O}$-algebras, using the algebra axioms for Y. And this is the only way to extend the maps $\mathscr{O}(1) \otimes X_i \to Y$ to a map on the entire colimit as a map of $\mathscr{O}$-algebras, so our colimit $\mathscr{O}$-algebra satisfies the universal property of the colimit in $\mathrm{Alg}_{\mathscr{O}}(\mathbf{C})$. □

A.3.2 Monads

The above construction of the colimit in $\mathrm{Alg}_{\mathscr{O}}(\mathbf{C})$ can be simplified, and also generalized, using the following language.

Definition A.3.7. A **monad**, or **triple**, on a category $\mathbf{C}$ is a functor $T : \mathbf{C} \to \mathbf{C}$ with natural transformations $\mu : T \circ T \longrightarrow T$ and $\eta : \mathrm{id}_{\mathbf{C}} \longrightarrow T$ making the following diagrams commute:

$$\begin{array}{ccc} T\circ T\circ T & \xrightarrow{\mu\circ 1} & T\circ T \\ \downarrow{\scriptstyle 1\circ\mu} & & \downarrow{\scriptstyle \mu} \\ T\circ T & \xrightarrow{\mu} & T \end{array} \qquad \begin{array}{ccccc} T & \xrightarrow{1\circ\eta} & T\circ T & \xleftarrow{\eta\circ 1} & T \\ & \searrow^{1} & \downarrow{\scriptstyle \mu} & \swarrow_{1} & \\ & & T & & \end{array}$$

An **algebra** over T is an object $X \in \mathbf{C}$ with an action map $\alpha\colon T(X) \to X$ such that the analogous diagrams commute:

$$\begin{array}{ccc} T\circ T(X) & \xrightarrow{\mu\circ 1} & T(X) \\ \downarrow{\scriptstyle 1\circ\alpha} & & \downarrow{\scriptstyle \alpha} \\ T(X) & \xrightarrow{\alpha} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\eta} & T(X) \\ & \searrow^{1} & \downarrow{\scriptstyle \alpha} \\ & & X \end{array}$$

A map of algebras is a map in $\mathbf{C}$ commuting with the T-action. We let $\mathrm{Alg}_T(\mathbf{C})$ denote the category of T-algebras in $\mathbf{C}$.

Example A.3.8. If $\mathbf{C}$ is a symmetric monoidal category with all colimits, such that $\otimes$ preserves colimits in each variable, then for any operad $\mathcal{O}$ in $\mathbf{C}$, the functor $T_{\mathcal{O}} = U_{\mathcal{O}}F_{\mathcal{O}} : \mathbf{C} \to \mathbf{C}$, in other words the free $\mathcal{O}$-algebra regarded just an object of $\mathbf{C}$, is an example of a monad. The composition $T_{\mathcal{O}} \circ T_{\mathcal{O}} \to T_{\mathcal{O}}$ is exactly the map (A.3.2) that comes from composition in the operad $\mathcal{O}$, and the transformation $\mathrm{id}_{\mathbf{C}} \to T_{\mathcal{O}}$ is the inclusion of X into the term $\mathcal{O}(1) \otimes X$ along the identity element of $\mathcal{O}(1)$.

The following is an exercise in diagram-chasing.

Lemma A.3.9. *An algebra over the monad $T_{\mathcal{O}}$ is the same thing as an algebra over the operad $\mathcal{O}$. In other words,* $\mathrm{Alg}_{T_{\mathcal{O}}}(\mathbf{C}) \cong \mathrm{Alg}_{\mathcal{O}}(\mathbf{C})$.

So we have arrived at a more concise and formal definition of $\mathrm{Alg}_{\mathcal{O}}(\mathbf{C})$.

Example A.3.10. Writing this out for the associative operad for spaces, the monad $T_{\mathrm{Assoc}} = T_{\mathrm{Alg}}$ is the "free associative monoid" functor

$$T_{\mathrm{Alg}}(X) = (*) \amalg (X) \amalg (X \times X) \amalg (X \times X \times X) \amalg \cdots$$

It is not hard to see that a map $T_{\mathrm{Alg}}(X) \to X$ gives X an n-fold multiplication for every n. With a little more work one can see that the diagrams in Definition A.3.7 commute iff this corresponds to a single multiplication that is associative and unital.

Example A.3.11. If (L, R) is any adjunction of categories, then the composite RL is a monad, with η the unit of the adjunction $\mathrm{id} \to RL$, and μ given by the counit

$$\mathrm{id} \circ \epsilon \circ \mathrm{id} : RL \longrightarrow RLRL.$$

Therefore $\Omega^n\Sigma^n$ is an example of a monad, one that doesn't come from an operad, see Example A.3.17 below.

Example A.3.12. As a special case of the previous example, if $\alpha\colon \mathbf{I} \to \mathbf{J}$ is any functor of indexing categories, then the left Kan extension and restriction form an adjunction $(\alpha_!, \alpha^*)$ between $\mathbf{Top}^{\mathbf{I}}$ and $\mathbf{Top}^{\mathbf{J}}$. Therefore $T_\alpha = \alpha^*\alpha_!$ is a monad on $\mathbf{Top}^{\mathbf{I}}$.

For instance, if $\mathbf{I} = \{\bullet, \bullet\}$ and $\mathbf{J} = \{\bullet \to \bullet\}$, then the monad $T_{\mathbf{I}\to\mathbf{J}}$ takes the diagram $\{A, B\}$ to the diagram $\{A \to (A \amalg B)\}$. Algebras over this monad are essentially the thing as $\mathbf{J}$-diagrams!

If T is a monad on any category $\mathbf{C}$, then there is a forgetful functor $U_T\colon \mathrm{Alg}_T(\mathbf{C}) \to \mathbf{C}$, and a free functor $F_T\colon \mathbf{C} \to \mathrm{Alg}_T(\mathbf{C})$ that sends each object X to the object $F_T(X) := T(X)$, with T-action arising from the composition in T.

Lemma A.3.13. *The functor F_T is the left adjoint of the forgetful functor U_T.*

Proof. Let $X \in \mathbf{C}$ and $Y \in \mathrm{Alg}_T(\mathbf{C})$. We form the bijection

$$f : T(X) \to Y \qquad \leftrightarrow \qquad g : X \to U_T(Y)$$

by composing each map f with the unit $\mathrm{id}_{\mathbf{C}} \to T$, to get g, and taking each map g to the composite of $T(g)$ and the algebra structure map $T(Y) \to Y$. A straightforward diagram-chase shows that these are inverses. □

If the monad came from an adjunction between $\mathbf{C}$ and another category $\mathbf{D}$, one might wonder whether this new adjunction between $\mathbf{C}$ and $\mathrm{Alg}_{RL}(\mathbf{C})$ has anything to do with the original one. It turns out that sometimes it is equivalent, and sometimes it is not.

There is always a canonical map $\mathbf{D} \to \mathrm{Alg}_{RL}(\mathbf{C})$, given by sending each object Y to RY with the action $(RL)RY = R(RL)Y \to R(\mathrm{id}_{\mathbf{D}})Y = RY$. We say that (L, R) is a **monadic adjunction** when this gives an equivalence of categories $\mathbf{D} \simeq \mathrm{Alg}_{RL}(\mathbf{C})$. Intuitively, monadic adjunctions are free-forgetful adjunctions in which the objects of the larger category are objects of the smaller category with some kind of additional "algebraic" structure.

Example A.3.14. The adjunction between $\mathbf{C}$ and $\mathrm{Alg}_T(\mathbf{C})$ is always monadic. The adjunction (Σ, Ω) is not monadic, because Ω does not form an equivalence between based spaces and $\Omega\Sigma$-algebras of based spaces. The adjunction $(\alpha_!, \alpha^*)$ is monadic precisely when α is essentially surjective.

Now we will use this language to simplify the construction of the colimit from Lemma A.3.4. We do have to make a mild assumption on the monad T:

Definition A.3.15. A functor $T : \mathbf{C} \to \mathbf{D}$ **preserves reflexive coequalizers** if for any coequalizer diagram in $\mathbf{C}$

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{h} Z,$$

if there is a map $j : Y \to X$ such that $fj = \mathrm{id}_Y = gj$, then the diagram

$$T(X) \underset{T(g)}{\overset{T(f)}{\rightrightarrows}} T(Y) \xrightarrow{T(h)} T(Z)$$

is also a coequalizer.

Example A.3.16. The n-fold product of spaces $X \rightsquigarrow X^{\times n}$ is a functor that preserves reflexive coequalizers. The same is true for the nth tensor power $X \rightsquigarrow X^{\otimes n}$ in any symmetric monoidal category $\mathbf{C}$ in which the tensor preserves colimits in each slot. See Exercise 9.6.32. Therefore if T comes from an operad in a symmetric monoidal category with this mild condition, then T preserves reflexive coequalizers.

Example A.3.17. The monad $\Omega\Sigma$ does not preserve reflexive coequalizers. Consider the two maps $I \amalg \{*\} \rightrightarrows I$ that are the identity on I, and that send $*$ to 0 or 1, respectively. The coequalizer is $I/\{0,1\} \cong S^1$. One can check however that the resulting map $\Omega\Sigma I \to \Omega\Sigma S^1$ is not surjective, so there is no way that it can be a coequalizer of any pair of maps.

Lemma A.3.18. *If* $\mathbf{C}$ *is any category with all colimits and limits, and* T *is a monad on* $\mathbf{C}$ *that preserves reflexive coequalizers, then the category* $\mathrm{Alg}_T(\mathbf{C})$ *has all colimits and limits as well.*

Proof. As in Lemma A.3.4, the limit of a diagram of T-algebras X_i is constructed by taking the limit in as a diagram in $\mathbf{C}$, and then giving the limit a T-action by filling in the diagram

$$\begin{array}{ccc} T\left(\lim\limits_{(\mathbf{C})} X_i\right) & \dashrightarrow & \lim\limits_{(\mathbf{C})} X_i \\ \downarrow & & \downarrow \\ T(X_i) & \longrightarrow & X_i \end{array}$$

using the universal property of the limit. The fact that this is the limit in T-algebras follows the same pattern as Exercise 6.6.30.

The colimit of a diagram of T-algebras X_i is constructed as the coequalizer

$$T\left(\operatorname*{colim}_{(\mathbf{C})} T(X_i)\right) \rightrightarrows T\left(\operatorname*{colim}_{(\mathbf{C})} X_i\right) \longrightarrow \operatorname*{colim}_{\mathrm{Alg}_T(\mathbf{C})} X_i, \tag{A.3.19}$$

where one of the two maps applies the actions $T(X_i) \to X_i$, while the other applies the canonical interchange $(\operatorname{colim} T(X_i)) \to T(\operatorname{colim} X_i)$ and then the multiplication of T. This

coequalizer is reflexive, so when we apply T it is still a coequalizer. So we may define the T-algebra structure by the map of coequalizer diagrams

$$\begin{array}{ccccc} TT\left(\operatorname*{colim}_{(\mathbf{C})} T(X_i)\right) & \rightrightarrows & TT\left(\operatorname*{colim}_{(\mathbf{C})} X_i\right) & \longrightarrow & T\left(\operatorname*{colim}_{\mathrm{Alg}_T(\mathbf{C})} X_i\right) \\ \downarrow{\scriptstyle \mu\circ\mathrm{id}} & & \downarrow{\scriptstyle \mu\circ\mathrm{id}} & & \vdots \\ T\left(\operatorname*{colim}_{(\mathbf{C})} T(X_i)\right) & \rightrightarrows & T\left(\operatorname*{colim}_{(\mathbf{C})} X_i\right) & \longrightarrow & \operatorname*{colim}_{\mathrm{Alg}_T(\mathbf{C})} X_i. \end{array}$$

This is a T-algebra structure using the monad axioms for T. (It might even be better to write each of the leftmost copies of T in (A.3.19) as F_T, to indicate that T acts by multiplication into the leftmost copy of T.)

Given a single T-algebra Y and compatible maps of T-algebras X_i, we form the map $T(\operatorname{colim} X_i) \to T(Y) \to Y$ by taking the colimit map in $\mathbf{C}$ and then using the T-algebra structure on Y. This respects the relation and so passes to a map $\operatorname*{colim}_{\mathrm{Alg}_T(\mathbf{C})} X_i \to Y$, which we check is a map of T-algebras, and this is the unique T-algebra map extending the maps $T(X_i)$ by the universal property of the coequalizer. Therefore we have the colimit of X in the category of T-algebras in $\mathbf{C}$. □

Now we can prove that algebras over any monad in **Top** have a model structure.

Proposition A.3.20. *For any monad T on* **Top***, if T is topologically enriched and preserves reflexive coequalizers, then the category* $\mathrm{Alg}_T(\mathbf{Top})$ *has a right-induced model structure.*

As before, this means that a map of T-algebras $X \to X'$ is a weak equivalence or fibration iff it is a weak homotopy equivalence or Serre fibration, respectively, after forgetting the T-action. (The cofibrations in the model structure are then generated by the maps $F_T(S^{n-1}) \to F_T(D^n)$.)

Proof. We already proved this when T is the associative or commutative monad, in Proposition 9.1.3. The proof for a general monad T is identical, except that we use (A.3.19) to guarantee that colimits and limits exist in $\mathrm{Alg}_T(\mathbf{Top})$. □

Remark A.3.21. This proof works equally well when **Top** is replaced by $\mathbf{Top}^{\mathbf{I}}$ with its projective model structure, as we use in the proof of Proposition A.3.37 below.

Corollary A.3.22. *For any operad $\mathcal{O} \in \mathrm{Op}(\mathbf{Top})$, the category* $\mathrm{Alg}_{\mathcal{O}}(\mathbf{Top})$ *has a right-induced model structure.*

Now, you might expect that this proof that $\mathrm{Alg}_T(\mathbf{C})$ applies in any model category $\mathbf{C}$ with any monad T satisfying mild assumptions. Unfortunately, to make this work, one of the

assumptions you have to make are not so mild. The mild one is that you need to ensure that the small-object argument will run. The more severe one is that every $F_T(J)$-cell complex needs to be a weak equivalence.

In **Top**, we used a hack to prove this, namely that $D^n \times I$ deformation retracts onto $D^n \times \{0\}$, and therefore $F_T(D^n \times I)$ deformation retracts onto $F_T(D^n \times \{0\})$. That trick only works in **Top**, $\mathbf{Top}_*$, or diagrams in these two categories. For other symmetric monoidal model categories such as $\mathbf{Sp}^\Sigma$ or $\mathbf{Sp}^O$, there is no general theory of how to do this for every monad, so we will have to be content with a model structure for algebras over an operad instead.

Theorem A.3.23. *Suppose* $\mathbf{C} = \mathbf{Top}$, $\mathbf{Top}_*$, $\mathbf{sSet}$, $\mathbf{sSet}_*$, $\mathbf{Sp}^\Sigma_+$, $\mathbf{Sp}^O_+$, $\mathbf{Sp}^\Sigma_{\mathbf{sSet},+}$, *or* $\mathbf{Ch}_{\geq 0}(k)$ *or* $\mathbf{Ch}(k)$ *in which* $\mathbb{Q} \subseteq k$, *and let* $\mathscr{O}$ *be any operad in* $\mathbf{C}$. *Then* $\mathrm{Alg}_{\mathscr{O}}(\mathbf{C})$ *has a right-induced model structure coming from* $\mathbf{C}$.

We give a brief summary of the proof, the full version of which can be found in [BGH22, 5.8.1]. The essential difficulty is to show that $T(J)$-cell complexes are weak equivalences. For that, we need to describe pushouts in T-algebras of the form

$$\begin{array}{ccc} T(K) & \xrightarrow{T(f)} & T(L) \\ \downarrow{\scriptstyle \tilde{g}} & & \downarrow \\ X & \longrightarrow & P, \end{array}$$

where K and L are objects in the underlying category $\mathbf{C}$ and X is a T-algebra, just as we did when $T = T_{\mathrm{Assoc}}$ in Lemma 6.5.16 and $T = T_{\mathrm{Comm}}$ in Lemma 6.5.29. As you might guess from those two lemmas, there should be a general formula that expresses $X \to P$ as the composition of pushout of maps that are obtained from $f^{\square n}$ somehow.

Lemma A.3.24. *The map* $X \to P$, *considered as a map in the underlying category* $\mathbf{C}$, *is a countable composition of pushouts of maps of the form*

$$\mathscr{U}^{\mathscr{O}}_X(n) \otimes_{\Sigma_n} (f^{\square n}).$$

Here $\mathscr{U}^{\mathscr{O}}_X$ is the **enveloping operad**, whose algebras are the same thing as $\mathscr{O}$-algebras X' with an $\mathscr{O}$-algebra map $X \to X'$. Concretely, $\mathscr{U}^{\mathscr{O}}_X(n)$ is the object of $\mathbf{C}$ given by the coequalizer

$$\coprod_{\ell \geq 0} \mathscr{O}(\ell + n) \otimes_{\Sigma_\ell} (T_{\mathscr{O}} X)^{\otimes \ell} \rightrightarrows \coprod_{\ell \geq 0} \mathscr{O}(\ell + n) \otimes_{\Sigma_\ell} X^{\otimes \ell} \longrightarrow \mathscr{U}^{\mathscr{O}}_X(n). \qquad \text{(A.3.25)}$$

See [BGH22, Const 5.6.4 and Eq. (5.6.3)]. At any rate, the point is that if $f^{\square n}$ is an acyclic cofibration between cofibrant objects, and tensoring with the space $\mathscr{U}^{\mathscr{O}}_X(n)$ and then dividing out by Σ_n has the homotopy type of the homotopy orbits by Σ_n, then we

can prove that this makes $X \to P$ into a weak equivalence, with enough cofibrancy so that if we compose several of these maps we still get a weak equivalence. Of course, for this to work in spectra we need to use the positive cofibrations, so that $f^{\square n}$ is an acyclic flat positive cofibration as in Corollary A.1.45. This concludes our summary of the proof.

See also [BGH22, 5.6], [Spi01, Prop 8], and [PS18, Prop 5.7] for various versions of the construction, along with [PS18, Section 5] for a nice survey of further results about when the right-induced model structure on $\mathrm{Alg}_{\mathcal{O}}(\mathbf{C})$ exists.

A.3.3 Change of monad

Now that we can work with T-algebras up to weak equivalence for a fixed monad T, we turn our attention to changing the monad. Our goal is to transform algebras over a given operad $\mathcal{O}$ into algebras over any equivalent operad $\mathcal{O}'$. So for example, if $\mathcal{O}$ is equivalent to Assoc, we would like to show that $\mathcal{O}$-algebras can be turned Assoc-algebras!

To begin, suppose $\phi : T \to T'$ is any map of monads. That is, a natural transformation that commutes with the multiplication and unit maps. For instance, it might come from a map of operads $\mathcal{O} \to \mathcal{O}'$, by applying the map of operads to each of the terms $\mathcal{O}(n)$ that appears to turn it into $\mathcal{O}'(n)$

Every T'-algebra X becomes an T-algebra $\phi^* X$ in a canonical way: we compose the action $T'(X) \to X$ with the map of monads $T(X) \to T'(X)$ to get a new action $T(X) \to X$. We call this the **restriction of monad** along ϕ, by analogy with Definition 6.5.8.

Proposition A.3.26. *Let $\phi : T \to T'$ be any map of monads on the same category $\mathbf{C}$. If $\mathbf{C}$ has all colimits and T' preserves reflexive coequalizers, then the restriction of monad functor*

$$\phi^* \mathrm{Alg}_{T'}(\mathbf{C}) \to \mathrm{Alg}_T(\mathbf{C})$$

*has a left adjoint called **extension of monad** $\phi_!(X) = T' \otimes_T X$.*

If furthermore $\mathbf{C}$ is a cofibrantly generated model category, and the categories of algebras have the right-induced model structure, then $(\phi_!, \phi^*)$ is a Quillen adjunction. If in addition $T(X) \to T'(X)$ is an equivalence on every cofibrant object X, then $(\phi_!, \phi^*)$ is a Quillen equivalence.

When the map of monads ϕ comes from a map of operads $\mathcal{O} \to \mathcal{O}'$, we call these functors **restriction of operad** ϕ^* and **extension of operad** $\phi_!$.

Proof. We first prove that $\phi_!$ exists. Following our intuition for what the expression "$T' \otimes_T X$" should mean, we take the coequalizer

$$T'T(X) \rightrightarrows T'(X) \longrightarrow \phi_! X,$$

where one of the two maps is the T-action on X, and the other map applies ϕ to the inner copy of T and then applies the multiplication on T'. Since the coequalizer is reflexive, applying T' on the outside gives another coequalizer shown along the top row, and then we can use the multiplication of T' to define a map to the bottom row as shown:

$$\begin{array}{ccccc} T'T'T(X) & \rightrightarrows & T'T'(X) & \longrightarrow & T'\phi_!X \\ \downarrow{\scriptstyle \mu\circ\mathrm{id}} & & \downarrow{\scriptstyle \mu\circ\mathrm{id}} & & \vdots \\ T'T(X) & \rightrightarrows & T'(X) & \longrightarrow & \phi_!X \end{array}$$

We then check this is a T'-action, which only requires the the monad axioms for T'. (Imagine how difficult it would be to describe this if we were still using the explicit formulas in terms of an operad $\mathcal{O}$! This is why it is important to think in terms of monads for these proofs.)

It also follows that the map $T'(X) \to \phi_!X$ is a T'-algebra map, and a map of T'-algebras $\phi_!X \to Y$ is determined by the resulting map $T'(X) \to Y$. Using this, it is a straightforward diagram-chase to show that $\phi_!$ is indeed the left adjoint of ϕ^*.

It is now easy to see that $(\phi_!, \phi^*)$ is a Quillen adjunction when the categories of algebras have the right-induced model structure. Since ϕ^* doesn't change the algebra, it preserves weak equivalences and fibrations of T'-algebras, and we are done.

For the final claim, let X be a T-algebra that is a $F_T(I)$-cell complex, where I is the set of generating cofibrations in $\mathbf{C}$. Let $K \to L$ denote a generic map in the set I. Working inductively up the skeleta, we see that the counit of the adjunction $X \to \phi^*\phi_!X$ induces a map of cell complexes in $\mathbf{C}$ that is an equivalence on each level. This uses the fact that the maps $T(K \to L)$ and $T'(K \to L)$ are cofibrations of T-algebras, so at each stage of the cell complex, we are forming a pushout of cofibrant objects along a cofibration, which in any model category is always a left-derived pushout (Exercise 5.8.12), and similarly a sequential colimit along cofibrations always preserves equivalences. Therefore the weak equivalences $T(K) \to T'(K)$ and $T(L) \to T'(L)$ and the weak equivalence between the previous stage of the cell complexes gives a weak equivalence on the next stage.

Now it follows that for any T'-algebra Y, a map $X \to \phi^*Y$ is a weak equivalence iff $\phi_!X \to Y$ is a weak equivalence. (This is another example of the argument from Exercise 6.6.16.) This finishes the proof. □

Example A.3.27. The left-derived extension of operad functor gives us a way to take algebras over the little intervals operad to strictly associative algebras, $\mathrm{Alg}_{\mathscr{C}_1}(\mathbf{Top}) \to \mathrm{Alg}(\mathbf{Top})$. This is a Quillen equivalence by Proposition A.3.26, so we have finished the proof of the rectification theorem (Corollary 9.3.26), that every A_∞-space is equivalent as an A_∞ space to one that has a strictly associative and unital multiplication!

Example A.3.28. For every topological category $\mathbf{I}$ with object set S, $\mathbf{I}$-diagrams in spaces are the same thing as algebras over a monad on $\mathbf{Top}^{\times S}$ that sends every tuple of spaces

$(X_s)_{s\in S}$ to the tuple $(\coprod_{t\in S} X_s \times \mathbf{I}(s,t))_{s\in S}$. The above theorem recovers the restriction and left Kan extension adjunction from Proposition 5.4.17, and the proof in Exercise 5.8.16 that this is a Quillen equivalence when $\mathbf{I} \to \mathbf{J}$ is a pointwise equivalence.

The change-of-monads theorem implies that we can pass algebras back and forth between different operads without losing information, *if* their associated monads are equivalent on cofibrant spaces. It is *not* enough for the operads themselves to be equivalent!

Definition A.3.29. We say that a map of operads $\mathcal{O} \to \mathcal{O}'$ is a **monadic equivalence** if the induced map of monads $T_{\mathcal{O}}(X) \to T_{\mathcal{O}'}(X)$ is an equivalence for every cell complex X.

By definition, any two A_∞ operads are monadically equivalent to Assoc. And any two E_∞ operads are equivalent by taking their product. So we have finished the proof of Theorem 9.3.25.

A.3.4 The monadic bar construction

It is sometimes helpful to describe the homotopy type of the left-derived extension of monad functor, $\mathbb{L}\phi_!$, using the bar construction.

Definition A.3.30. For any map of monads $\phi : T \to T'$ and T-algebra X, the **monadic bar construction** $B(T', T, X)$ is the realization of the simplicial space $N_\bullet(T', T, X)$ such that

$$N_n(T', T, X) = T' \circ T^{\circ n}(X),$$

with face maps arising from the multiplication on T, the multiplication $T' \circ T \to T'$ coming from ϕ, and the T-action on X, just as in the bar construction Definition 7.3.7. The degeneracy maps come from the unit maps of T.

We will analyze this in the special case that $\mathbf{C} = \mathbf{Top}$, or more generally $\mathbf{Top}^{\mathbf{I}}$, under the assumption that

Assumption A.3.31.

- The monad T' preserves reflexive coequalizers,
- T' is topologically enriched, so that we have assembly maps

  $$\Delta^n \times T'(X) \to T'(\Delta^n \times X),$$

  and furthermore
- T' commutes with realization, in the sense that for any simplicial object $X_\bullet$ the map of geometric realizations induced by the assembly maps of T'

  $$|T'X_\bullet| \longrightarrow T'|X_\bullet|$$

  is a homeomorphism.

Example A.3.32. These assumptions are satisfied for $T_{\mathcal{O}}$ for any operad $\mathcal{O}$. They are also satisfied for the monad $\alpha^*\alpha_!$ when α is any map of topological categories $\mathbf{I} \to \mathbf{J}$. In both cases, this follows from the fact that finite products and colimits commute with geometric realization.

These assumptions guarantee that the realization of a simplicial object in T'-algebras is again a T'-algebra, under the action

$$T'|X_\bullet| \xleftarrow{\cong} |T'X_\bullet| \longrightarrow |X_\bullet|,$$

where the first map is the commutation of T' with realization, and the second map is the action of T' on each level.

Lemma A.3.33. *For any such monad T', the realization functor $\mathrm{Alg}_{T'}(\mathbf{Top}^{\mathbf{I}})^{\Delta^{\mathrm{op}}} \to \mathrm{Alg}_{T'}(\mathbf{Top}^{\mathbf{I}})$ has a left-derived functor, given by replacing the simplicial T'-algebra one in which the underlying simplicial space is Reedy h-cofibrant.*

Proof. We just have to show that each simplicial T'-algebra can be replaced by an equivalent one that is Reedy cofibrant, in a functorial way. One way to do this is to use the Reedy model structure inherited from the model structure on $\mathrm{Alg}_{T'}(\mathbf{Top})$ constructed in Proposition A.3.20. □

For the purposes of the following lemma, let us say that a map of monads $T \to T'$ is a weak equivalence if $T(X) \to T'(X)$ is a weak equivalence for all X. And when we take the monadic bar construction, we implicitly left-derive the realization so that it is a T'-algebra and the realization functor preserves weak equivalences.

Lemma A.3.34 (Homotopy invariance of the monadic bar construction)**.** *Under these assumptions, the monadic bar construction $B(T', T, X)$ preserves weak equivalences in each slot.*

Proof. The assumptions we made guarantee that the expression at each level $T' \circ T^{\circ n}(X)$ preserves all weak equivalences, so we are done. □

Lemma A.3.35 (Bar lemma)**.** *There is a homotopy equivalence of T-algebras*

$$B(T, T, X) \xrightarrow{\sim} X.$$

Proof. The same proof as in Lemma 7.3.11 shows that there is such a homotopy equivalence if we take the realization of $N_\bullet(T, T, X)|$ and don't left-derive it. So we need to check that this has the same homotopy type as the left-derived realization. The simplest way to see that is to map the simplicial object forward to $N_\bullet(|TS_\bullet(-)|, |TS_\bullet(-)|, X)$, which is a Reedy cofibrant object in simplicial $|TS_\bullet(-)|$-algebras. The map of simplicial objects is

an equivalence on the realization and on each simplicial level, and the target object is Reedy cofibrant, which confirms that the source object has realization equivalent to the left-derived realization. □

Corollary A.3.36. *The bar construction $B(T', T, X)$ is a formula for the left-derived extension of monad functor $\mathbb{L}\phi_! \colon \mathrm{Alg}_T \to \mathrm{Alg}_{T'}$.*

As one application, we can now be more explicit about how every A_∞-algebra X is replaced by a weakly equivalent associatve algebra. If $\mathscr{O}$ is the A_∞ operad that acts on X, then the replacement is given by the zig-zag

$$X \xleftarrow{\sim} B(\mathscr{O}, \mathscr{O}, X) \xrightarrow{\sim} B(\mathrm{Assoc}, \mathscr{O}, X),$$

where both of the maps are $\mathscr{O}$-algebra homomorphism, but the $\mathscr{O}$-algebra structure on the right-hand space arises by restriction of operad from an Assoc-algebra structure. In other words, it is a rectification of X to a space with a strictly associative multiplication.

As another application, we use the monadic bar construction to prove Proposition 9.4.38. Suppose $\mathscr{O}$ is an operad in which the balanced product $\mathscr{O}(n) \times_{\Sigma_n} X^{\times n}$ is always equivalent to the corresponding homotopy orbits $\mathscr{O}(n) \times^h_{\Sigma_n} X^{\times n}$. Let $\mathscr{O}^\otimes$ be its corresponding category of operators from Example 9.4.36.

Proposition A.3.37. *Under these assumptions, there is a weak equivalence between the identity functor on $\mathrm{Alg}_{\mathscr{O}^\otimes}(\mathbf{Top})$ and a functor that sends every $\mathscr{O}^\otimes$-algebra to one that is in standard form.*

Proof. We follow the proof from [MMO25, 5.27], adapted to the unbased version of Π. Let $\mathbb{D}$ be the monad on Π-spaces such that $\mathbb{D}$-algebras are $\mathscr{O}^\otimes$-diagrams. Writing out the definition, we have a coequalizer

$$\coprod_{\ell,m\geq 0} X_\ell \times \Pi(\ell_+, \mathbf{m}_+) \times \mathscr{O}^\otimes(\mathbf{m}_+, \mathbf{n}_+) \rightrightarrows \coprod_{m\geq 0} X_m \times \mathscr{O}^\otimes(\mathbf{m}_+, \mathbf{n}_+) \longrightarrow (\mathbb{D}X)_n,$$

which simplifies to

$$(\mathbb{D}X)_n \cong \coprod_{m\geq 0} \coprod_{\alpha \colon \mathbf{m}\to\mathbf{n}} \prod_{j=1}^{n} \mathscr{O}(\alpha^{-1}(j)) \times_{\Sigma_{\alpha^{-1}(j)}} X_{\alpha^{-1}(j)},$$

because the projection relations from Π have the effect of canceling out the points of $\mathbf{m}_+ \to \mathbf{n}_+$ that go to the basepoint, leaving only a map of unbased sets $\mathbf{m} \to \mathbf{n}$, and then the permutations in Π mod out by the symmetric group actions $\Sigma_{\alpha^{-1}(j)}$.

On the other hand, we have a monad on $\mathbf{Top}^\Pi$ given by $RT_\mathscr{O}L$, where the monad multiplication applies the unit of the adjunction (L, R) and then the multiplication on $T_\mathscr{O}$ to give

$$RT_\mathscr{O}LRT_\mathscr{O}L \longrightarrow RT_\mathscr{O}T_\mathscr{O}L \longrightarrow RT_\mathscr{O}L.$$

We form a map of monads $\phi : \mathbb{D} \to RT_{\mathscr{O}}L$ by applying the Segal maps to each term:

$$\coprod_{\alpha\colon \mathbf{m}\to\mathbf{n}} \prod_{j=1}^{n} \mathscr{O}(\alpha^{-1}(j)) \times_{\Sigma_{\alpha^{-1}(j)}} X_{\alpha^{-1}(j)} \to \coprod_{\alpha\colon \mathbf{m}\to\mathbf{n}} \prod_{j=1}^{n} \mathscr{O}(\alpha^{-1}(j)) \times_{\Sigma_{\alpha^{-1}(j)}} X_1^{\times\alpha^{-1}(j)}$$
$$\cong \prod_{j=1}^{n} \coprod_{m_j\geq 0} \mathscr{O}(m_j) \times_{\Sigma_{m_j}} X_1^{\times m_j}.$$

Therefore the left-derived extension of monad functor $\mathbb{L}\phi_!$ takes each $\mathbb{D}$-algebra to a $RT_{\mathscr{O}}L$-algebra. Note that any $RT_{\mathscr{O}}L$-algebra is a $\mathbb{D}$-algebra by neglect of structure. It is also in standard form, because we may pull out the R from the formula for $\phi_!$. So it remains to show that this new $\mathbb{D}$-algebra is equivalent to the old one, when X has the Segal condition.

This is where the monadic bar construction comes to the rescue. We describe our map from the old $\mathbb{D}$-algebra to the new one, up to equivalence, as a map of bar constructions

$$B(\mathbb{D},\mathbb{D},X) \longrightarrow B(RT_{\mathscr{O}}L,\mathbb{D},X).$$

We just have to check that this is an equivalence at each simplicial level separately. At level zero it is the map $\mathbb{D}(X) \to RT_{\mathscr{O}}L(X)$, which is an equivalence when X has the Segal condition, by inspection. At higher simplicial levels, we notice that $\mathbb{D}X$ always has the Segal condition no matter what, and so $\mathbb{D}\circ\mathbb{D}^{\circ n}(X) \to RT_{\mathscr{O}}L\circ\mathbb{D}^{\circ n}(X)$ is always a weak equivalence when $n \geq 1$. This finishes the proof that the new $\mathbb{D}$-algebra is equivalent to the old one when X has the Segal condition. □

The same proof shows a corresponding result for the A_∞ version of categories of operators defined above Theorem 9.4.40. The only difference in the proof is that there are no symmetric group actions, and we only allow functions $\mathbf{m}_+ \to \mathbf{n}_+$ that come from the subcategory $\mathbf{\Delta}^{\mathrm{op}}$.

For more technical properties of monads and the monadic bar construction see e.g. [May72, Har10, GKRW23].

A.3.5 Sketch proof of the May recognition principle

We have proven the Segal recognition principle already in Theorem 9.4.15 and Theorem 9.4.19. It is beyond our scope to give a full treatment of the May recognition principle (Theorem 9.3.28 and Theorem 9.3.29), but since we have developed many of the tools for the proof, we will at least describe how these tools are used to prove it.

The idea is to focus on the map of operads $\phi : \mathscr{C}_0 \to \mathscr{C}_n$. Recall that an $\mathscr{C}_0$-algebra is a based space, so we think of the restriction of operad operation ϕ^* as taking each $\mathscr{C}_n$-space to the underlying based space, where the basepoint is the "identity element for the

multiplication," in other words the action of the unique point $* \in \mathscr{C}_n(0)$. When we take the extension of operad operation $\phi_! = T_{\mathscr{C}_0 \to \mathscr{C}_n}$, we don't quite get the free $\mathscr{C}_n$-algebra on an unbased space

$$T_{\mathscr{C}_n}(X) = \coprod_{k \geq 0} \mathscr{C}_n(k) \times_{\Sigma_k} X^{\times k},$$

but rather a relative version of this where the basepoint of X is factored in:

$$\phi_!(X) = T_{\mathscr{C}_0 \to \mathscr{C}_n}(X) = \left(\coprod_{k \geq 0} \mathscr{C}_n(k) \times_{\Sigma_k} X^{\times k} \right) / \sim,$$

The relation says that when one of the coordinates of X^k is the basepoint, we plug in the unique point $* \in \mathscr{C}_n(0)$ into the corresponding slot in $\mathscr{C}_n(k)$, giving a point in $\mathscr{C}_n(k-1)$. In symbols, this means that

$$\begin{aligned} &(f(-,\dots,-,-,-,\dots,-), x_1, \dots, x_{i-1}, *, x_{i+1}, \dots, x_k) \\ \sim\ &(f(-,\dots,-,*,-,\dots,-), x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k). \end{aligned}$$

For instance, if we do this with the associative operad, we get the space

$$T_{\mathscr{C}_0 \to \mathrm{Assoc}}(X) = \left(\coprod_{k \geq 0} \mathscr{C}_n(k) \times_{\Sigma_k} X^{\times k} \right) / (x_1, \dots, x_{i-1}, *, x_{i+1}, \dots, x_k) \sim (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k).$$

This is the free topological monoid on a space *that already has a chosen identity element*. You might call it the "semi-free monoid" instead, to distinguish it from the free monoid construction that we usually consider. This is also called the the **James construction** on X.

Lemma A.3.38 (Cohen–May–Segal approximation)**.** *There is an equivalence of monads* $\Omega^n \Sigma^n \simeq T_{\mathscr{C}_n, \mathscr{C}_0}$.

The proof of this is an induction on n. In the base case of $n = 0$ it is trivial, while in the case of $n = 1$, if we use the associative operad in the statement, it says that the James construction on X is a model for $\Omega\Sigma X$.

The monadic bar construction then allows us to replace each $\mathscr{C}_n$ algebra X by the weakly equivalent $\mathscr{C}_n$-algebras

$$X \xleftarrow{\sim} B(T_{\mathscr{C}_0 \to \mathscr{C}_n}, T_{\mathscr{C}_0 \to \mathscr{C}_n}, X) \xrightarrow{\sim} B(\Omega^n \Sigma^n, T_{\mathscr{C}_0 \to \mathscr{C}_n}, X).$$

We then define $B^n X := B(\Sigma^n, T_{\mathscr{C}_0 \to \mathscr{C}_n}, X)$, in other words the bar construction where we only apply Σ^n at the end. We have to check that this makes sense, and that we get well-defined maps $\Sigma^n T_{\mathscr{C}_0 \to \mathscr{C}_n} \to T_{\mathscr{C}_0 \to \mathscr{C}_n}$ that are associative and unital in the appropriate

sense. In other words, Σ^n is a "left module" over the monad $T_{\mathscr{C}_0 \to \mathscr{C}_n}$. Note that $B^n X$ is an $(n-1)$-connected based space since realization preserves connectivity.

Then we give a map in the homotopy category of $\mathscr{C}_n$-algebras $X \to \Omega^n B^n X$ by showing that there is a commutation that pulls the n-fold loop space out of the bar construction, $B(\Omega^n \Sigma^n, T_{\mathscr{C}_0 \to \mathscr{C}_n}, X) \to \Omega^n B(\Sigma^n, T_{\mathscr{C}_0 \to \mathscr{C}_n}, X)$. This map is not an equivalence in general, but one shows that it is an equivalence when X is grouplike using Corollary 8.6.35, and this gives the adjunction of homotopy categories with the desired properties.

Finally, if X is an $\mathscr{C}_\infty$ algebra, then for each $n \geq 0$ it is an $\mathscr{C}_n$-algebra by restriction. One checks that the deloopings $B^n X$ fit together into a positive Ω-spectrum, and this defines the delooping spectrum $B^\infty X$. The full details of the argument are given in [May72].

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