GROUP ACTIONS AKA PERMUTATION REPRESENTATIONS, LECTURE 1

We start right away by introducing the concept of a group action on a set. Leter we will discuss a broader context for this notion.

Definition 1. *A (left)* **action** *of a group G on a set X is a function $* : G \times X \longrightarrow X$ which to any element $g \in G$ and any $s \in X$ assigns an element $g * s$ of X and has the following two properties:*
(a) $f * (g * s) = (fg) * s$ for any $f, g \in G$ and $s \in X$;
(b) $e * s = s$ for any $s \in X$, where e is the identity element of G.

Note that (a) and (b) mean that the action $*$ is compatible with the group structure of G.

Cnsider now an action $*$ of a group G on a set X. There are two very useful notions associated to any such action.

First we introduce the notion of an **orbit** of an element $s \in X$ under the action of G. In plane words, the orbit $O(s)$ of s consists of all elements of X which can be obtained by acting by some element of G on s, i.e. we have

Definition 2. *The* **orbit** *$O(s)$ of s under the action of G is the set*

$$O(s) = \{g * s : g \in G\}.$$

Note that $s \in O(s)$ for every $s \in X$. The main property of orbits is contained in the following

Lemma 1. *If $s, t \in S$ then either $O(s) = O(t)$ or $O(s) \cap O(t) = \emptyset$.*

Proof: First note that if $v \in O(s)$ then $v = f * s$ for some $f \in G$. Thus, for any $g \in G$ we have $g * v = g * (f * s) = (gf) * s \in O(s)$. This shows that $O(v)$ is a subset of $O(s)$. On the other hand, we have $f^{-1} * v = f^{-1} * (f * s) = (f^{-1}f) * s = e * s = s$, so $s = f^{-1} * v \in O(v)$. As above, this implies that $O(s) \subseteq O(v)$, so we have in fact

$O(s) = O(v)$. In other words, the orbit of any element belonging to a given orbit O coincides with O.

Suppose now that the orbits of s and t are not disjoint, so there is $v \in O(s) \cap O(t)$. Then we have $O(s) = O(v) = O(t)$ by the above discussion. $\square$

The lemma says that **the orbits partition the set X into pairwise disjoint subsets.**

The second notion we want to introduce is the notion of a **stabilizer** $St(s)$ of any $s \in X$. The definition is very simple:

Definition 3. *The* **stabilizer** *of $s \in X$ is the subset $St(s) = \{g \in G : g * s = s\}$ of G. Another very common notation for the stabilizer of s is G_s (we will use both notations).*

In plain words, the stabilizer of s consists of all those elements of G which act trivially on s (i.e. which fix s). The main fact about a stabilizer is that it is a subgroup of G.

Lemma 2. *(1) For any $s \in X$, the stabilizer $St(s)$ is a subgroup of G.*

(2) $St(g * s) = gSt(s)g^{-1}$ for any $g \in G$ and $s \in X$.

Proof: Clearly $e \in St(s)$. If $f, g \in St(s)$ then $f * s = s = g * s$ so $(fg) * s = f*(g*s) = f*s = s$, i.e. $fg \in St(s)$. Also, $g^{-1}*s = g^{-1}*(g*s) = (g^{-1}g)*s = e*s = s$ so $g^{-1} \in St(s)$. This proves (1).

In order to establish (2) note that $f \in St(g*s)$ iff $f*(g*s) = g*s$ iff $(fg)*s = g*s$ iff $g^{-1} * ((fg) * s) = g^{-1} * (g * s) = s$ i.e. iff $(g^{-1}fg) * s = s$ which is equivalent to $g^{-1}fg \in St(s)$ i.e. $f \in gSt(s)g^{-1}$. This proves (2). $\square$

The connection between orbits ans stabilizers is captured in the following very useful result.

Orbit-Stabilizer Theorem. *The number of elements in the orbit $O(s)$ is equal to the index $[G : St(s)]$ of the stabilizer $St(s)$ in G: $|O(s)| = [G : St(s)]$. In particular, if G is finite, then $|O(s)| = |G|/|St(s)|$, so $|O(s)|$ divides $|G|$.*

Proof: Our proof is based on the following observation:

$$g * s = h * s \text{ if and only if } gSt(s) = hSt(s).$$

In fact, $g * s = h * s$ iff $s = (g^{-1}h) * s$ iff $g^{-1}h \in St(s)$ iff $gSt(s) = hSt(s)$.
Using this observation we see that the formula $F(gSt(s)) = g * s$ is well defined and gives a function $F : G/St(s) \longrightarrow O(s)$ which is injective. Clerly $g * s = F(gSt(s))$, so F is surjective. Thus F is a bijection from the set of all left cosets of $St(s)$ onto the orbit $O(s)$. Thus the sets $G/St(s)$ and $O(s)$ have the same number of elements: $[G : St(s)] = |O(s)|$.
If G is finite then Lagrage's Theorem tells us that $|G| = |St(s)|[G : St(s)]$, so $|O(s)| = |G|/|St(s)|$. $\square$

Remark. It follows that the number $|St(t)|$ is the same for any element $t \in O(s)$ (since $O(t) = O(s)$). This however should not be surprising at all, since we proved in Lemma 2 that the groups $St(t)$ and $St(s)$ are conjugate in G, so in particular they have the same number of elements.

We end this note by a discussion of how group actions fit into representation theory. If G is an abstract group, it is often a very fruitful idea to investigate representations of G on objects of a suitable category. By a **representation** we mean here simply a group homomorphism from G to the group $\mathrm{Aut}X$ of all automorphisms of the object X.

For example, we could take a vector space V and consider a representation of G on V. Such representations, i.e. homomorphisms from G to $GL(V)$ are called **linear representations** (here $GL(V)$ is the group of all linear automorphisms of V; if $V = \mathbb{R}^n$ then $GL(V)$ can be identifies with the group $GL_n(\mathbb{R})$ of all invertible $n \times n$ matrices). They play a fundamental role in group theory and many other parts of mathematics.

Another important example form representations of groups on groups, i.e. homomorphisms from G to $\mathrm{Aut}H$ for some group H. Such representations are related to a fundamental construction in group theory, called a **semidirect product**.

Two representations $f_i : G \longrightarrow \mathrm{Aut}X_i$, $i = 1, 2$, are called **equivalent** if there is an isomorphism $\alpha : X_1 \longrightarrow X_2$ such that $\alpha f_1(g) = f_2(g)\alpha$ for evry $g \in G$ (note that α induces a group homomorphism from $\mathrm{Aut}X_1$ to $\mathrm{Aut}X_2$ by $u \mapsto \alpha u \alpha^{-1}$). More generally, we define a morphism between the representations f_1 and f_2 to be any morphism $\alpha : X_1 \longrightarrow X_2$ such that $\alpha f_1(g) = f_2(g)\alpha$ for all $g \in G$

We say that a representation f is **faithful** if $\ker f$ is trivial, i.e. if f is an injection.

It turns out that group actions are essentailly the same as **permutation representations**, i.e. representations on sets. More precisesly, a permutation representation of G on a set X is simply a group homomorphism from G to the group S_X of all permutations of X.

We will now explain how we identify actions and permutation representations.

Suppose first that we have a homomorphism $\phi : G \longrightarrow S_X$. So for $g \in G$ the element $\phi(g)$ is a bijection of X. Define $*$ by $g * s = \phi(g)(s)$. We need to check that conditions (a) and (b) are satisfied, which is a very simple consequence of the fact that ϕ is a homomorphism:

(a) We have $(fg) * s = \phi(fg)(s) = (\phi(f)\phi(g))(s) = \phi(f)(\phi(g)(s)) = \phi(f)(g * s) = f * (g * s)$.

(b) $e * s = \phi(e)(s) = id(s) = s$ for any $s \in X$.

The action we just defined is the action associated to a permutation representation ϕ.

Conversely, suppose we have an action $*$ of G on X. We need to construct a homomorphism $\phi : G \longrightarrow S_X$ corresponding to this action. Note that each $g \in G$ gives rise to a function $L_g : X \longrightarrow X$ defined by $L_g(s) = g * s$. Note that $L_g L_{g^{-1}}(s) = L_g(g^{-1} * s) = g * (g^{-1} * s) = (gg^{-1}) * s = e * s = s$, so $L_g L_{g^{-1}} = id$. Similarly, $L_{g^{-1}} L_g = id$, which shows that L_g is a bijection of X (since it has an inverse function $L_{g^{-1}}$). Thus we get a function $\phi : G \longrightarrow S_X$ defined by $\phi(g) = L_g$. It remains to verify that ϕ is a homomorphism, which is a quite simple task:

$\phi(fg)(s) = L_{fg}(s) = (fg) * s = f * (g * s) = L_f(L_g(s)) = \phi(f)(\phi(g)(s))$ for any $s \in X$ so $\phi(fg) = \phi(f)\phi(g)$.

The reader will easily verify that the constructions of the action from a permutation representation and the permutation representation from an action are inverse to each other and this allows us to identify actions of G on X and homomorphisms from G to S_X.