GROUP ACTIONS AKA PERMUTATION REPRESENTATIONS
LECTURE 4

Let G be a finite group of order $|G| = p^k m$, where p is a prime, $k > 0$ and $p \nmid m$. Let Syl_p be the set of all Sylow $p-$subgroups of G, i.e. the set of all subgroups of order p^k (we proved that this set is not empty). Let $t_p(G) = |Syl_p(G)|$, i.e. $t_p(G)$ is the numbers of Sylow p subgroups of G. We will write t_p for $t_p(G)$ when there is no danger of confusion which G we have in mind.

The group G acts on $Syl_p(G)$ by conjugation. We are going to examine this action by looking at various subgroups of G acting on $Syl_p(G)$.

Let $P \in Syl_p(G)$. The stabilizer of P is the set of all $g \in G$ such that $gPg^{-1} = P$. We have a name for this subgroup: it is the normalizer $N_G(P)$ of P in G. Clearly $P \subseteq N_G(P)$ and P is a normal subgroup of $N_G(P)$.

We will need the following observation.

Proposition. Suppose that P is a normal Sylow p-subgroup of a group H. If Q is a p-subgroup of H then $Q \subseteq P$.

In order to prove the proposition consider the quotient homomorphism $\pi : H \longrightarrow H/P$. Note that $|H/P| = |H|/|P|$ is not divisible by p. On the other hand, since Q is a p-group, so is the image $\pi(Q)$. Thus $\pi(Q)$ is a p-subgroup of H/P and $p \nmid |H/P|$. This can only happen if $\pi(Q) = \{e\}$, i.e. $Q \subseteq \ker \pi = P$.

Corollary. A finite group G has a normal Sylow p-subgroup if and only if $t_p = 1$.

Indeed, the proposition tells us that if G has a normal Sylow p-subgroup P then $t_p = 1$ (as any other Sylow p-subgroup is contained in P hence equal to P). Conversely, if $t_p = 1$ then $Syl_p(G)$ has one element P and therefore $G = N_G(P)$, i.e. P is normal in G.

We return to our discussion of the action of G on $Syl_p(G)$. Let $P \in Syl_p(G)$. Since P is a normal Sylow p-subgroup of the normalizer $N_G(P)$, the proposition tells us that any p-subgroup of the normalizer $N_G(P)$ is contained in P.

Take any Sylow p-subgroup Q of G and consider its action on $Syl_p(G)$. If $P \in Syl_p(G)$ is a fixed point for Q then $Q \subseteq N_G(P)$ and therefore $Q \subseteq P$ (by the observation in the previous paragraph). Since P and Q have the same order, $P = Q$. Conversely, Q is fixed by conjugation with any element of Q. Thus we see that the action of Q on $Syl_p(G)$ has exactly 1 fixed point, namely Q itself. Since Q is a p-group, Rule 3 from the previous lecture tells us that $t_p = |Syl_p(G)| \equiv 1 \pmod{p}$.

Consider now an orbit $O(P)$ of the action of G on $Syl_p(G)$. G acts on $O(P)$ and so does any subgroup of G. Recall from Rule 1 that $|(O(P)| = |G|/|N_G(P)|$. Since $P < N_G(P)$, we have $p^k||N_G(P)|$ so $|(O(P)||m$. In particular, the size of $O(P)$ is not divisible by p. Let Q be any Sylow p-subgroup of G. Since Q acts on $O(P)$, by Rule 3, Q has a fixed point in $O(P)$ (Q is a p-group action on $O(P)$ and the size of $O(P)$ is not divisible by p). On the other hand, we have seen that Q is the only fixed point of the action of Q on $Syl_p(G)$. This shows that $Q \in O(P)$. In other words, $O(P) = Syl_p(G)$. This means that G acts transitively on $Syl_p(G)$ and $t_p = |O(P)||m$.

Take now any p-subgroup T of G and consider its action on $Syl_p(G)$. We just proved that $p \nmid t_p$. By Rule 3 again, we see that T must have a fixed point. Thus there is a $P \in Syl_p(G)$ such that $T \subseteq N_G(P)$. This implies that $T \subseteq P$. We proved that any p-subgroup of G is contained in a Sylow p-subgroup.

We summarize our investigation in the following fundamental theorem, called **Sylow Theorem**:

Sylow Theorem. Let G be a finite group of order $|G| = p^k m$, where p is a prime, $k > 0$ and $p \nmid m$. Then:

(1) G has at least one subgroup of order p^k, called a Sylow p-subgroup of G.
(2) Any two Sylow p-subgroups of G are conjugate in G.
(3) The number t_p of Sylow $p-$subgroups of G satisfies:

$$t_p \equiv 1 \pmod{p} \quad \text{and} \quad t_p | m.$$

(4) Every $p-$subgroup of G is contained in some Sylow $p-$subgroup.

Sylow Theorem is a fundamental result about finite groups, which is a key tool in deriving various structural properties of finite groups. We will give just one illustration of how to use it by proving the following result.

Theorem. Any finite group of order 4225 is abelian.

In fact, let G be a finite group of order 4225. We have $4225 = 5^2 \cdot 13^2$.

Consider the number t_5 of Sylow 5-subgroups of G. We know that t_5 is a divisor of 13^2 so it is one of $1, 13, 169$. We also know that $t_5 \equiv 1 \pmod{5}$. Thus $t_5 = 1$. This means that G has a normal Sylow 5-subgroup P.

Consider the number t_{13} of Sylow 13-subgroups of G. We know that t_{13} is a divisor of 5^2 so it is one of $1, 5, 25$. We also know that $t_{13} \equiv 1 \pmod{13}$. Thus $t_{13} = 1$. This means that G has a normal Sylow 13-subgroup Q.

We have a normal subgroup P of order 13^2 and a normal subgroup Q or order 5^2. They are both abelian (any group of order p^2, p a prime, is abelian). Note that $P \cap Q = \{e\}$, as the groups have relatively prime orders. By a result we proved earlier, $PQ \cong P \times Q$, so PQ is abelian and has $5^2 \cdot 13^2 = 4225$ elements. Since PQ is a subgroup of G, we have $G = PQ$, so G is abelian.