Homework 10, due on Friday, November 13

Read carefully chapters 18,19,20 in the textbook. **It is crucial that you read and attempt ALL the exercises at the end of each chapter as they form integral part of the book.**
Solve the following problems.

Problem 1. Let R be a unital ring and let J be an ideal in the ring $M_2(R)$ of all 2×2 matrices with entries in R.

a) Suppose that $a \in R$ is an entry of a matrix belonging to J. Prove that the matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ a & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ 0 & a \end{smallmatrix}\right)$ belong to J.

b) Prove that the set $I = \{a \in R : \ a \text{ is an entry of a matrix in } J\}$ is an ideal of R and $J = M_2(I)$.

c) Let $f : R \longrightarrow S$ be a ring homomorphism. Applying f to every entry of a matrix defines a function $F : M_2(R) \longrightarrow M_2(S)$. Prove that F is a homomorphism. How are the kernels of f and F related?

d) Prove that if K is an ideal in R then $M_2(K)$ is an ideal in $M_2(R)$ and $M_2(R)/M_2(K) \cong M_2(R/K)$. Hint: define a surjective homomorphism from $M_2(R)$ to $M_2(R/K)$ whose kernel is $M_2(K)$.

e) Prove that if R is a field, then the only ideals in $M_2(R)$ are $\{0\}$ and $M_2(R)$, i.e. $M_2(R)$ is a simple ring.

Remark. The problem is true for $n \times n$ matrices for any integer $n \geq 1$.

Problem 2. Let R be a commutative, not necessarily unital, ring.

a) For any $a \in R$ consider the function $l_a : R \longrightarrow R$, $l_a(r) = ar$. Prove that it is a homomorphism of additive groups (note that it usually is NOT a homomorphism of rings). Prove that the kernel and the image of l_a are ideals of R. (Remark: the kernel is often called the **annihilator** of a and the image is **the principal ideal generated by** a).

b) Suppose that R is simple, i.e. $\{0\}$ and R are the only ideals of R. Suppose furthermore, that there are $a, b \in R$ such that $ab \neq 0$. Prove that l_a is a bijection. Conclude that $as = a$ for some $s \in R$.

c) Continue with part b) to show that s is the identity for multiplication, so R is unital. Then prove that R is a field.

d) Suppose that R is simple and $ab = 0$ for every $a, b \in R$. Prove that the additive group of R is cyclic of prime order.

Problem 3. Let R be a ring. Two ideals I, J of R are called **coprime** if $I + J = R$. Suppose that I and J are coprime.

a) Prove that for any $r, t \in R$ there is $s \in R$ such that $s + I = r + I$ and $s + J = t + J$. **Hint:** Write $r = i + j$, $t = i_1 + j_1$ for some $i, i_1 \in I$ and $j, j_1 \in J$ and consider $s = j + i_1$.

b) Let $f_I : R \longrightarrow R/I$ and $f_J : R \longrightarrow R/J$ be the quotient homomorphisms. Define $f : R \longrightarrow (R/I) \times (R/J)$ by $f(r) = (f_I(r), f_J(r))$. Use a) to prove that f is a surjective ring homomorphism. What is $\ker f$?

c) Prove that $R/(I \cap J)$ is isomorphic to $(R/I) \times (R/J)$. Conclude that $\mathbb{Z}/(mn)\mathbb{Z}$ is isomorphic to $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ (as rings) for any relatively prime integers m, n

d) Let R be unital and commutative. Prove that $I \cap J = IJ$. (**Hint:** Write $1 = i + j$ for some $i \in I$, $j \in J$ and use the fact that $x = 1 \cdot x$ for any x.)

e) Suppose R is unital and that I, J, K are ideals of R such that both I, K and J, K are coprime. Prove that IJ and K are coprime. Conclude that $I \cap J$ and K are coprime.

Problem 4. Problem H in chapter 18.

Problem 5. Problem I in chapter 18.

Problem 6. Problem G parts 3, 4, 5 in chapter 19.

Problem 7. Problem E in chapter 20.