

Solutions to Homework 1

Problem 1. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be functions.

- a) Prove that if both f and g are surjective then gf is surjective.
- b) Prove that if gf is injective then f is injective. Show by example that g need not be injective.

Solution. a) Let $c \in C$. Since g is surjective, there is $b \in B$ such that $g(b) = c$. Since f is surjective, there is $a \in A$ such that $f(a) = b$. Thus $(gf)(a) = g(f(a)) = g(b) = c$. We showed that every c is in the range of gf , so gf is surjective.

b) Suppose contrary, that f is not injective. Then there are $a, b \in A$ such that $a \neq b$ and $f(a) = f(b)$. But then $(gf)(a) = g(f(a)) = g(f(b)) = (gf)(b)$, which contradicts the assumption that gf is injective.

To see that g need not be injective consider $f(x) = e^x$ and $g(x) = x^2$ (both as functions from $\mathbb{R}$ to $\mathbb{R}$). Clearly g is not injective, but $gf(x) = e^{2x}$ is injective.

Problem 2. Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be a function defined as follows: $f(m)$ is the last digit in the decimal expansion of m . What is the domain, codomain, range of f ? Find $f(S)$, where S is the set of all even numbers.

- b) What is $f \circ f$?

Solution. The domain of f is the set $\mathbb{Z}$ of all integers and so is the codomain of f . Since a decimal digit is one of the integers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, the range of f is contained in the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. On the other hand, for $i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ we have $f(i) = i$, so each of the digits is a value of f . It follows that the range of f is the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ of the decimal digits. The last digit of an even number is even, so $f(S) = \{0, 2, 4, 6, 8\}$.

- b) Since $f(a) \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for every a and $f(i) = i$ for $i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, we see that $f \circ f(a) = f(a)$ for every a . Thus $f \circ f = f$.

Problem 3. Let $f : A \rightarrow B$ be a function.

- a) Prove that if S, T are subsets of A then $f(S \cup T) = f(S) \cup f(T)$. Show that the analogous statement for intersection is not always true.
- b) Prove that if S, T are subsets of B then $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$, $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$, and $f^{-1}(S \setminus T) = f^{-1}(S) \setminus f^{-1}(T)$.
- c) State and prove a result relating $f^{-1}(S \div T)$ and $f^{-1}(S) \div f^{-1}(T)$.
- d) Prove that if S, T are subsets of A then $f(S) \setminus f(T) \subseteq f(S \setminus T)$. Show by example that the equality does not always hold.

Solution. a) Clearly $f(S) \subseteq f(S \cup T)$ and $f(T) \subseteq f(S \cup T)$. It follows that $f(S) \cup f(T) \subseteq f(S \cup T)$. Conversely, suppose that $b \in f(S \cup T)$. Thus $b = f(a)$ for some $a \in S \cup T$. If $a \in S$ then $b = f(a) \in f(S)$ and if $a \in T$ then $b = f(a) \in f(T)$. In either case, we see that $b \in f(S) \cup f(T)$. This shows that $f(S \cup T) \subseteq f(S) \cup f(T)$. We proved that $f(S) \cup f(T) \subseteq f(S \cup T)$ and $f(S \cup T) \subseteq f(S) \cup f(T)$. This means that $f(S \cup T) = f(S) \cup f(T)$.

To see that the analogous statement for intersections is not always true consider $f(x) = x^2$, $S = \{0, 2\}$, $T = \{0, -2\}$. Then $f(S) = \{0, 4\}$, $f(T) = \{0, 4\}$, and $f(S \cap T) = f(\{0\}) = \{0\}$. Thus

$$f(S) \cap f(T) = \{0, 4\} \neq \{0\} = f(S \cap T).$$

b) Let $x \in f^{-1}(S \cap T)$. This means that $f(x) \in S \cap T$, i.e. $f(x) \in S$ and $f(x) \in T$. Thus $x \in f^{-1}(S)$ and $x \in f^{-1}(T)$, which implies that $x \in f^{-1}(S) \cap f^{-1}(T)$. This proves that $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$.

Conversely, suppose that $x \in f^{-1}(S) \cap f^{-1}(T)$. Then $x \in f^{-1}(S)$ and $x \in f^{-1}(T)$, so $f(x) \in S$ and $f(x) \in T$. Thus $f(x) \in S \cap T$ and therefore $x \in f^{-1}(S \cap T)$. This proves that $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$.

Since we proved that $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$ and $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$, we conclude that $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

The argument for the union follows the same line of reasoning.

Let $x \in f^{-1}(S \cup T)$. This means that $f(x) \in S \cup T$, i.e. $f(x) \in S$ or $f(x) \in T$. Thus $x \in f^{-1}(S)$ or $x \in f^{-1}(T)$, which implies that $x \in f^{-1}(S) \cup f^{-1}(T)$. This proves that $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$.

Conversely, suppose that $x \in f^{-1}(S) \cup f^{-1}(T)$. Then $x \in f^{-1}(S)$ or $x \in f^{-1}(T)$, so $f(x) \in S$ or $f(x) \in T$. Thus $f(x) \in S \cup T$ and therefore $x \in f^{-1}(S \cup T)$. This proves that $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$.

Since we proved that $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$ and $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$, we conclude that $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.

The argument for the difference again follows the same line of reasoning.

Let $x \in f^{-1}(S \setminus T)$. This means that $f(x) \in S \setminus T$, i.e. $f(x) \in S$ and $f(x) \notin T$. Thus $x \in f^{-1}(S)$ and $x \notin f^{-1}(T)$, which implies that $x \in f^{-1}(S) \setminus f^{-1}(T)$. This proves that $f^{-1}(S \setminus T) \subseteq f^{-1}(S) \setminus f^{-1}(T)$.

Conversely, suppose that $x \in f^{-1}(S) \setminus f^{-1}(T)$. Then $x \in f^{-1}(S)$ and $x \notin f^{-1}(T)$, so $f(x) \in S$ and $f(x) \notin T$. Thus $f(x) \in S \setminus T$ and therefore $x \in f^{-1}(S \setminus T)$. This proves that $f^{-1}(S) \setminus f^{-1}(T) \subseteq f^{-1}(S \setminus T)$.

Since we proved that $f^{-1}(S \setminus T) \subseteq f^{-1}(S) \setminus f^{-1}(T)$ and $f^{-1}(S) \setminus f^{-1}(T) \subseteq f^{-1}(S \setminus T)$, we conclude that $f^{-1}(S \setminus T) = f^{-1}(S) \setminus f^{-1}(T)$.

c) We use the equality $S \div T = (S \cup T) \setminus (S \cap T)$ and part b):

$$\begin{aligned} f^{-1}(S \div T) &= f^{-1}((S \cup T) \setminus (S \cap T)) = f^{-1}(S \cup T) \setminus f^{-1}(S \cap T) = \\ &= (f^{-1}(S) \cup f^{-1}(T)) \setminus (f^{-1}(S) \cap f^{-1}(T)) = f^{-1}(S) \div f^{-1}(T). \end{aligned}$$

d) Let $b \in f(S) \setminus f(T)$. Then $b \in f(S)$, so $b = f(a)$ for some $a \in S$. Since $b = f(a) \notin f(T)$, we have $a \notin T$. Thus $a \in S \setminus T$ and therefore $b = f(a) \in f(S \setminus T)$. This proves that $f(S) \setminus f(T) \subseteq f(S \setminus T)$. To see that the equality does not need to hold consider $f(x) = x^2$, $S = \{-2, 2\}$, $T = \{-2\}$. Then $f(S) = f(T) = \{4\}$ so $f(S) \setminus f(T) = \emptyset$. On the other hand $S \setminus T = \{2\}$ and $f(S \setminus T) = \{4\}$.

Problem 4. For a set X we define $P(X)$ as the set of all subsets of X (the book denotes it by P_X). $P(X)$ is called the **power set** of X . The operations $\cup, \cap, \setminus, \div$ are binary operations on $P(X)$. Which of them are associative? Which of them are commutative? Which have the identity element?

Solution. The operations $\cup$ and $\cap$ are clearly associative and commutative. The empty set is the identity element for $\cup$ and the set X is the identity element for $\cap$.

The operation $\setminus$ is not commutative if X is nonempty:

$$X \setminus \emptyset = X \text{ and } \emptyset \setminus X = \emptyset.$$

It is not associative:

$$X \setminus (\emptyset \setminus X) = X \text{ and } (X \setminus \emptyset) \setminus X = \emptyset.$$

The operation $\setminus$ does not have identity element if X is nonempty. Indeed, if A was the identity for $\setminus$ we would have $A \setminus X = X$. But A is a subset of X , so $A \setminus X = \emptyset$ and $X = \emptyset$.

The operation $\div$ is commutative:

$$A \div B = (A \cup B) \setminus (A \cap B) = (B \cup A) \setminus (B \cap A) = B \div A.$$

The empty set is the identity for $\div$: $A \div \emptyset = A$. Finally, the operation $\div$ is associative. To prove this, we need to show that for any three subsets A, B, C of X we have

$$(A \div B) \div C = A \div (B \div C).$$

Note first that an element does not belong to $A \div B$ if and only if either it belongs to both A and B or it belongs to neither A nor B . Thus $C \setminus (A \div B)$ consists of elements which are either in each set A, B, C or are in C but not in A and not in B .

Since an element is in $A \div B$ if and only if it either belongs to A and does not belong to B or it belongs to B and does not belong to A , we see that $(A \div B) \setminus C$ consists of all elements which either belong to A and do not belong to B or C or belong to B and do not belong to A or C .

Putting these observations together, we see that an element belongs to $(A \div B) \div C = (C \setminus (A \div B)) \cup ((A \div B) \setminus C)$ if and only if it either belongs to each of A, B, C or it belongs to exactly one of A, B, C .

Similar discussion shows that an element belongs to $A \div (B \div C)$ if and only if it either belongs to each of A, B, C or it belongs to exactly one of A, B, C . This proves that the sets $(A \div B) \div C$ and $A \div (B \div C)$ are equal.