

Homework 2, due on Friday, September 4

Read carefully chapters 3,4,5 in the textbook. **It is crucial that you read and attempt the exercises at the end of each chapter as they form integral part of the book.** Solve the following problems.

Problem 1. Recall that $P(X)$ denotes the set of all subsets of X (the book denotes it by P_X). Prove that $P(X)$ with the binary operation $\div$ (symmetric difference) is a group (see problem C to chapter 3 in the textbook).

Problem 2. a) Let G be a group with the binary operation $\cdot$, A any set and $f : A \rightarrow G$ a bijection. Define a binary operation $*$ on A as follows:

$$a * b = f^{-1}(f(a) \cdot f(b)).$$

Prove that A with $*$ as binary operation is a group. What is the identity of this group?

b) Apply part a) to the following cases. In each case find an explicit formula for $a * b$ (compare it to problem A in chapter 3 in the textbook).

1. $G = \mathbb{R}$ with addition, $A = \mathbb{R}$, $f(x) = x + k$ (k a fixed real number)
2. $G = \mathbb{R} \setminus \{0\}$ with multiplication, $A = \mathbb{R} \setminus \{0\}$, $f(x) = x/2$.
3. $G = \mathbb{R} \setminus \{0\}$ with multiplication, $A = \mathbb{R} \setminus \{-1\}$, $f(x) = x + 1$.
4. $G = \mathbb{R}$ with addition, $A = (-1, 1)$, $f(x) = \operatorname{arctanh}(x)$ is the inverse function to the hyperbolic tangent $\tanh(x) = (e^x - e^{-x})/(e^x + e^{-x})$. Hint: Prove that

$$\tanh(x + y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}.$$

Problem 3. Let a, b be real numbers, $a \neq 0$. Define the function $T_{a,b} : \mathbb{R} \rightarrow \mathbb{R}$ by $T_{a,b}(x) = ax + b$. Let G be the set of all such functions, i.e.

$$G = \{T_{a,b} : a, b \in \mathbb{R}, a \neq 0\}.$$

a) Prove that G with composition of functions as the binary operation is a group.

b) Suppose that $z \neq 0$. Show that $T_{a,b}$ and $T_{1,z}$ commute iff $a = 1$.

Problem 4. Let G be a group such that $g^2 = e$ for every $g \in G$. Prove that G is abelian.

Problem 5. Let G be the set of all bijections $f : \mathbb{Z} \rightarrow \mathbb{Z}$ which preserve distance, i.e. such that $|f(i) - f(j)| = |i - j|$ for all integers i, j .

a) Show that G is a subgroup of $S_{\mathbb{Z}}$. It is called the **infinite dihedral group** and it is often denoted by D_{∞} .

b) The group G contains elements T, S such that $T(a) = a + 1$ and $S(a) = -a$ for all integers a . Prove that $S * T = T^{-1} * S$. Show that the subgroup $\langle T \rangle$ is infinite. What is $\langle S \rangle$?

c) Show that if $F \in G$ and $F(0) = 0$ then either $F = 1$ (the identity) or $F = S$.

d) Show that every element of G is of the form T^i or ST^i for a unique integer i .

e) Suppose that $T^5 S^7 T^3 = S^a T^b$. Find a and b .

Problem 6. In the group $GL_2(\mathbb{C})$ of all invertible 2×2 matrices with entries in complex numbers consider the matrices $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $i = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$, $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $k = ij = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$. Let Q_8 be the set $\{I, -I, i, -i, j, -j, k, -k\}$.

a) Show that Q_8 is a subgroup of $GL_2(\mathbb{C})$. Write the table of multiplication in Q_8 . Q_8 is called the **quaternion group**.

b) List all subgroups of Q_8 .