

Homework 3, due on Monday, September 14

Read carefully chapters 7,8,9,10,11 in the textbook. **It is crucial that you read and attempt the exercises at the end of each chapter as they form integral part of the book.**

Solve the following problems.

Problem 1. Problem E to chapter 4.

Problem 2. Problem G to chapter 4.

Problem 3. a) Find the product

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 2 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 2 & 5 \end{pmatrix}$$

b) Find the cycle decomposition and order of

1. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 4 & 2 & 7 & 6 & 9 & 8 & 5 \end{pmatrix};$
2. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix};$
3. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 5 & 6 & 7 & 4 \end{pmatrix}.$

c) Express the following permutations as products of disjoint cycles and find their orders

1. $(1, 2, 3, 4, 5)(1, 2, 3, 4, 6)(1, 2, 3, 4, 7);$
2. $(1, 2, 3)(3, 4, 5, 6)(1, 2, 3)^{-1};$
3. $(1, 2, 3, 4, 5, 6)^3.$

Express each of these permutations as a product of transpositions.

d) Find a permutation $\rho \in S_n$ such that $\rho\tau\rho^{-1} = \sigma$ or prove that no such ρ exists, where

1. $\tau = (1, 3, 4), \sigma = (2, 3, 5), n = 5;$
2. $\tau = (1, 4)(2, 3, 5), \sigma = (2, 4, 1)(5, 3), n = 5;$
3. $\tau = (1, 3, 5, 7)(2, 4, 6), \sigma = (1, 6)(4, 5, 7, 3, 2).$

Problem 4. a) Let σ, τ be two transpositions in $S_n, n \geq 3$. Show that $\sigma\tau$ can be expressed as a product of 3-cycles (not necessarily disjoint).

b) Prove that every element of $A_n, n \geq 3$ is a product of 3-cycles.

Problem 5. a) Which of the following permutations are even?

1. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 5 & 1 & 3 & 7 & 8 & 9 & 6 \end{pmatrix};$
2. $(1, 2, 3, 4, 5, 6)(7, 8, 9);$

3. $(1, 2)(1, 2, 3)(4, 5)(5, 6, 8)(1, 7, 9)$.

b) Prove that a k -cycle is even iff k is odd.

c) In the even permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 2 & & & 7 & 8 & 9 & 6 \end{pmatrix}$$

two entries are missing. Find the missing entries.

Problem 6. Let H be a subgroup of S_n generated by $(1, 2)$ and $(1, 2, 3, \dots, n)$. Prove that $H = S_n$. Hint: Show that H contains all transpositions.

Problem 7. Let G be a group. Define the **center** of G as the subset $Z(G)$ of all elements which commute with every element of G , i.e.

$$Z(G) = \{g \in G : ag = ga \text{ for every } a \in G\}.$$

a) Prove that $Z(G)$ is a subgroup of G .

b) Find $Z(D_3)$, $Z(D_4)$, $Z(Q_8)$ and $Z(D_\infty)$.

c) What is $Z(D_n)$?

d) What is $Z(S_n)$?