

Homework 4, due on Friday, September 25

Read carefully chapters 9,10,11,14 in the textbook. **It is crucial that you read and attempt ALL the exercises at the end of each chapter as they form integral part of the book.**

Solve the following problems.

Problem 1. Let G be a group and a, b elements of G which commute, i.e. $ab = ba$. Suppose that a has order k and b has order l . We proved in class that:

1. the order of a^m is equal to $k/\gcd(k, m)$ for any integer m .
2. the order of ab divides $kl/\gcd(k, l)$ and $kl/\gcd(k, l)^2$ divides the order of ab . In particular, if k, l are relatively prime then ab has order kl .

- a) Prove that if d is a positive divisor of k then G has an element of order d .
- b) Prove that G has an element of order $\text{lcm}(k, l) = kl/\gcd(k, l)$ (warning: it does not have to be ab).
- c) Suppose that G is abelian and N is the smallest positive integer such that $g^N = e$ for every $g \in G$. Prove that G has an element of order N . Show a finite non-abelian group for which this is false.
- d) Let $f : G \rightarrow H$ be a homomorphism. Let $a \in G$ be an element of finite order k . Prove that $f(a)$ has also finite order, say m , and m divides k . Prove furthermore that a^n is in the kernel of f if and only if m divides n .

Problem 2. a) Let G be a group. For any element $g \in G$, let $c_g : G \rightarrow G$ be the conjugation by g map defined by $c_g(x) = gxg^{-1}$. Show that c_g is an automorphism of G . (It is called the **inner automorphism** induced by the element g or **the conjugation by g** automorphism.)

- b) Show that the set $\{c_g : g \in G\}$ is a subgroup of $\text{Aut}G$. (This subgroup is called the **inner automorphism group** of G , denoted $\text{Inn}G$.)
- c) Show that the map $g \mapsto c_g$ is a homomorphism from G to $\text{Aut}G$, whose image is the inner automorphism group $\text{Inn}G$ and whose kernel is the center $Z(G)$.

Problem 3. Let G and H be groups.

- a) Prove that the maps $\pi_G : G \times H \rightarrow G$ and $\pi_H : G \times H \rightarrow H$ defined by $\pi_G(g, h) = g$ and $\pi_H(g, h) = h$ are homomorphisms. What are the kernels of these maps?
- b) Suppose that $G = H$ and let $\Delta = \{(a, b) \in G \times G : a = b\}$. Prove that Δ is a subgroup of $G \times G$ which is isomorphic to G .
- c) Let $f_1 : Q \rightarrow G$ and $f_2 : Q \rightarrow H$ be homomorphisms. Define $f : Q \rightarrow G \times H$ by $f(a) = (f_1(a), f_2(a))$. Prove that f is a homomorphism and that $\ker f = \ker f_1 \cap \ker f_2$.

Problem 4. Consider the dihedral group $D_n = \{I, T, \dots, T^{n-1}, S, ST, \dots, ST^{n-1}\}$ of order $2n$. Let $f : D_n \rightarrow A$ be a homomorphism, where A is an abelian group.

- a) Prove that $T^2 \in \ker f$.
- b) Prove that if n is odd then the image of f has either one or two elements. Give example of a homomorphism f with image of order 2.

c) Let n be even. Define a map $g : D_n \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$ by

$$g(S^a T^b) = (a \bmod 2, b \bmod 2).$$

Prove that g is a surjective homomorphism and find its kernel.

d) Prove that if n is even then the image of f is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2$ or the trivial group.

Problem 5. a) Prove that the exponential function $f(x) = e^x$ is an isomorphism from the group of all real numbers with addition to the group of positive real numbers with multiplication.

b) Prove that the group all rational numbers with addition is not isomorphic to the group of positive rational numbers with multiplication.

c) Prove that the group $\mathbb{R}^\times$ of all non-zero real numbers with multiplication is not isomorphic to the group $\mathbb{R}_+^\times$ of all positive real numbers with multiplication. Prove that $\mathbb{R}^\times$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{R}_+^\times$.

d) Prove that every finitely generated subgroup of the group $\mathbb{Q}$ of rational numbers with addition is cyclic but $\mathbb{Q}$ is not cyclic.

Problem 6. Problem H to chapter 9.

Problem 7. Problem E to chapter 11.

Problem 8. Problem G to chapter 14.