

Homework 5, due on Friday, October 2

Read carefully chapters 12,13,14,15 in the textbook. **It is crucial that you read and attempt ALL the exercises at the end of each chapter as they form integral part of the book.**

Solve the following problems.

Problem 1. Let G be a group and let H, K be subgroups of G .

a) Suppose that $h_1, h_2 \in H$. Prove that $h_1(H \cap K) = h_2(H \cap K)$ iff $h_1K = h_2K$. Conclude that $[H : H \cap K] \leq [G : K]$.

b) Let L be a subgroup of H . Prove that $[G : L]$ is finite iff both $[G : H]$ and $[H : L]$ are finite and then $[G : L] = [G : H][H : L]$. Hint: Note that H is partitioned into $[H : L]$ left cosets of L in H and conclude that each left coset of H in G is also partitioned into $[H : L]$ left cosets of L in G .

c) Suppose that $[G : H]$ and $[G : K]$ are finite. Prove that $[G : H \cap K] \leq [G : H][G : K]$ (so, in particular, $[G : H \cap K]$ is finite).

Problem 2. Let G be a group.

a) Show that intersection of any collection $\{N_i : i \in I\}$ of normal subgroups of G is a normal subgroup of G .

b) Show that if N is a subgroup of G such that $[G : N] = 2$ then N is normal in G .

c) Prove that if H is a subgroup of G and N is a normal subgroup of G then

1. $H \cap N$ is a normal subgroup of H .
2. $HN = NH$
3. HN is a subgroup of G
4. if H is also normal in G then HN is normal in G .

d) Show that the subgroup Δ of $G \times G$ in Problem 3 b) from Homework 4 is normal if and only if G is abelian.

e) Show that the subgroup $\text{Inn}G$ of $\text{Aut}G$ is normal (see problem 2 from homework 4).

f) Let G be the group from Problem 3 of Homework 2. Let $H = \{T_{a,b} \in G : a \text{ is a rational number}\}$. Prove that H is a normal subgroup of G .

Problem 3. Let G be a group and let H be a subgroup of G . Define the **normalizer** of H in G as $N_G(H) = \{a \in G : aH = Ha\}$.

a) Prove that $a \in N_G(H)$ iff $c_a(H) = H$, where c_a is the conjugation by a .

b) Prove that $N_G(H)$ is a subgroup of G and that $H \subseteq N_G(H)$ and the center $Z(G) \subseteq N_G(H)$.

c) Prove that H is a normal subgroup of $N_G(H)$. Conclude that $H \triangleleft G$ iff $N_G(H) = G$.

d) Prove that $N_G(H)$ is the union of all those left cosets of H in G which are also right cosets of H in G .

e) Let $G = D_4$, $H = \{I, S\}$. Find $N_G(H)$.

Problem 4. Prove that if A, B are normal subgroups of a group G then $G/(A \cap B)$ is isomorphic to a subgroup of $(G/A) \times (G/B)$. Hint: Use Problem 3 c) from homework 4.

b) Let G be a group and let M, N be normal subgroups of G . Suppose that G/M and G/N are abelian. Prove that $G/(M \cap N)$ is also abelian.

Problem 5. Let G be a group such that $G/Z(G)$ is cyclic (here $Z(G)$ is the center of G). Prove that G is abelian.

Problem 6. Problem H to chapter 13.

Problem 7. Problem I to chapter 13.

Problem 8. Problem D to chapter 15.