Solutions to Homework 5

Problem 1. Let G be a group and let H, K be subgroups of G.

a) Suppose that $h_1, h_2 \in H$. Prove that $h_1(H \cap K) = h_2(H \cap K)$ iff $h_1 K = h_2 K$. Conclude that $[H : H \cap K] \leq [G : K]$.

b) Let L be a subgroup of H. Prove that $[G : L]$ is finite iff both $[G : H]$ and $[H : L]$ are finite and then $[G : L] = [G : H][H : L]$. Hint: Note that H is partitioned into $[H : L]$ left cosets of L in H and conclude that each left coset of H in G is also partitioned into $[H : L]$ left cosets of L in G.

c) Suppose that $[G : H]$ and $[G : K]$ are finite. Prove that $[G : H \cap K] \leq [G : H][G : K]$ (so, in particular, $[G : H \cap K]$ is finite).

Solution: a) Note that $h_1(H \cap K) = h_2(H \cap K)$ iff $h_2^{-1} h_1 \in H \cap K$ iff $h_2^{-1} h_1 \in K$ (since we know that $h_2^{-1} h_1 \in H$), iff $h_1 K = h_2 K$. It follows that the map $\Phi : H/(H \cap K) \longrightarrow G/K$ given by $\Phi(h(H \cap K)) = hK$ is well defined and injective. Thus $[H : H \cap K] \leq [G : K]$.

Remark. When we define the function Φ above we take a coset in $H/(H \cap K)$ and write it as $h(H \cap K)$. There are many choices of h which give the same coset, so to make sure that Φ is well defined we need to check that its definition does not depend on which h we use, i.e. if $h_1(H \cap K) = h(H \cap K)$ we need to make sure that $hK = h_1 K$. This is exactly what we did in the first part of the solution.

b) Suppose that G/L is finite. Note that if $aL = bL$ then $b^{-1}a \in L$. Since $L \subseteq H$, we have $b^{-1}a \in H$, so $aH = bH$. It follows that the map $G/L \longrightarrow G/H$ given by $aL \mapsto aH$ is well defined and clearly surjective. Since G/L is finite, so is G/H. Observe that H/L is a subset of G/L consisting of those cosets of L which are of the form hL for some $h \in H$. Thus H/L is finite.

Suppose now that both G/H and H/L are finite. Let $g_1 H, ..., g_s H$ be all the different cosets of H in G, $s = [G : H]$. Similarly, let $t_1 L, ..., t_t L$ be all the different cosets of L in H, $t = [H : L]$. We claim that the cosets $g_i t_j L$ are pairwise distinct and give all cosets of L in G. In fact, if $g_i t_j L = g_u t_v L$ then $t_v^{-1} g_u^{-1} g_i t_j \in L$. Since $L \subseteq H$ and $t_j, t_v \in H$, we get $g_u^{-1} g_i \in H$, so $g_i H = g_u H$ and therefore $i = u$. Thus $t_v^{-1} g_u^{-1} g_i t_j = t_v^{-1} t_j \in L$, so $t_j L = t_v L$ and therefore $j = v$. This proves that the cosets $g_i t_j L$, $i = 1, 2, ..., s; j = 1, 2, ..., t$ are pairwise distinct. If $g \in G$ then $g \in g_i H$ for some i, so $g = g_i h$ for some $h \in H$. Furthermore, $h \in t_j L$ for some j, so $h^{-1} t_j \in L$. We see that $g^{-1}(g_i t_j) = h^{-1} g_i^{-1} g_i t_j = h^{-1} t_j \in L$, i.e. $gL = (g_i t_j)L$. This proves that each coset of L in G is equal to one of $(g_i t_j)L$. It follows that G/L has st elements, i.e. $[G : L] = [G : H][H : L]$.

c) By a) we have $[H : H \cap K]$ is finite and $[H : H \cap K] \leq [G : K]$. Taking $L = H \cap K$ in b) we see that

$$[G : H \cap K] = [G : H][H : H \cap K] \leq [G : H][G : K].$$

Problem 2. Let G be a group.

a) Show that intersection of any collection $\{N_i : i \in I\}$ of normal subgroups of G is a normal subgroup of G.

b) Show that if N is a subgroup of G such that $[G : N] = 2$ then N is normal in G.

c) Prove that if H is a subgroup of G and N is a normal subgroup of G then

1. $H \cap N$ is a normal subgroup of H.

2. $HN = NH$

3. HN is a subgroup of G

4. if H is also normal in G then HN is normal in G.

d) Show that the subgroup Δ of $G \times G$ in Problem 3 b) from Homework 4 is normal if and only if G is abelian.

e) Show that the subgroup Inn$G$ of Aut$G$ is normal (see problem 2 from homework 4).

f) Let G be the group from Problem 3 of Homework 2. Let $H = \{T_{a,b} \in G : a \text{ is a rational number}\}$. Prove that H is a normal subgroup of G.

Solution. a) Recall that a subgroup N of G is normal if for any $g \in G$ and any $n \in N$ we have $gng^{-1} \in N$. Let N be the intersection of the subgroups N_i, $i \in I$, and let $n \in N$ and $g \in G$. Since n is in the intersection, it is in every N_i. Since N_i is normal, $gng^{-1} \in N_i$. Thus gng^{-1} is in each N_i, $i \in I$ and therefore gng^{-1} is in the intersection N. This proves that N is normal.

b) Since $[G : N] = 2$, the sets G/N and $N\backslash G$ have two elements. Let $G/N = \{N, aN\}$. Since the cosets partition G, we see that G is the union of disjoint sets N and aN. Thus $aN = G - N$ (the complement of N in G).

Similarly, if $N\backslash G = \{N, Nb\}$ then $Nb = G - N = aN$. Thus every left coset of N in G is also a right coset, so N is normal in G.

c) Let $n \in H \cap N$ and $h \in H$. Since N is normal and $n \in N$, we have $hnh^{-1} \in N$. Since H is a subgroup and h, n, h^{-1} are all in H, we have $hnh^{-1} \in H$. It follows that $hnh^{-1} \in H \cap N$. This proves 1. (that $H \cap N$ is normal in H).

We will show now that $HN \subseteq NH$. Take any element $hn \in HN$, where $h \in H$ and $n \in N$. We have $hn = (hnh^{-1})h$. Since $hnh^{-1} \in N$ (we use here the fact that N is normal) and $h \in H$ we see that $hn \in NH$. In the same way we show that $NH \subseteq HN$. Thus $HN = NH$. This proves 2.

Note that from 2. we have $(HN)(HN) = (HN)(NH) = H(NN)H = HNH = HHN = HN$ (we use the fact that multiplication of subsets is associative and that $KK = K$ for any subgroup K). This shows that HN is closed under multiplication. A different argument: if hn, h_1n_1 are in HN then

$$(hn)(h_1n_1) = (hh_1)(h_1^{-1}nh_1n_1) \in HN$$

since $hh_1 \in H$ and $h_1^{-1}nh_1 \in N$. Since e is in both N and H, $e = ee \in HN$. Finally, if $hn \in HN$ then $(hn)^{-1} = n^{-1}h^{-1} \in NH$. Since $NH = HN$, we see that $(hn)^{-1} \in HN$. This proves that HN is a subgroup.

If both H and N are normal then for any $g \in G$ we have $gH = Hg$ and $gN = Ng$ so

$$g(HN) = (gN)H = (Ng)H = N(gH) = N(Hg) = (NH)g.$$

This means that HN is a normal subgroup of G.

d) If G is abelian then $G \times G$ is abelian and therefore every subgroup of $G \times G$ is normal. In particular, $\Delta \lhd G \times G$.

Suppose that G is not abelian. Thus there are two elements g, h in G such that $gh \neq hg$, i.e. $hgh^{-1} \neq g$. Note that $(g, g) \in \Delta$ and $(h, e) \in G \times G$ but $(h, e)(g, g)(h, e)^{-1} = (hgh^{-1}, g) \notin \Delta$. Thus Δ is not normal in $G \times G$.

e) Let $\phi \in \mathrm{Aut}G$ and $c_a \in \mathrm{Inn}G$. We need to prove that $\phi c_a \phi^{-1} \in \mathrm{Inn}G$. Note that

$$(\phi c_a \phi^{-1})(x) = \phi(a\phi^{-1}(x)a^{-1}) = \phi(a)\phi(\phi^{-1}(x))\phi(a^{-1}) = \phi(a)x\phi(a)^{-1} = c_{\phi(a)}(x)$$

so $\phi c_a \phi^{-1} = c_{\phi(a)} \in \mathrm{Inn}G$. This shows that $\mathrm{Inn}G \lhd \mathrm{Aut}G$.

f) First note that H is indeed a subgroup. In fact, $e = T_{1,0} \in H$. Recall that $T_{a,b}T_{c,d} = T_{ac,ad+b}$. If $T_{a,b}, T_{c,d} \in H$ then $a, c \in \mathbb{Q}$ so ac is rational and therefore $T_{a,b}T_{c,d} \in H$. Also, $T_{a,b}^{-1} = T_{a^{-1},-a^{-1}b}$ so $T_{a,b}^{-1} \in H$ (since a^{-1} is rational). This shows that H is a subgroup. Now if $T_{a,b} \in G$ and $T_{c,d} \in H$ then

$$T_{a,b}T_{c,d}T_{a,b}^{-1} = T_{ac,ad+b}T_{a^{-1},-a^{-1}b} = T_{aca^{-1},ac(-a^{-1}b)+ad+b} = T_{c,-cb+ad+b} \in H$$

since c is rational. Thus H is a normal subgroup.

Problem 3. Let G be a group and let H be a subgroup of G. Define the **normalizer** of H in G as $N_G(H) = \{a \in G : aH = Ha\}$.

a) Prove that $a \in N_G(H)$ iff $c_a(H) = H$, where c_a is the conjugation by a.

b) Prove that $N_G(H)$ is a subgroup of G and that $H \subseteq N_G(H)$ and the center $Z(G) \subseteq N_G(H)$.

c) Prove that H is a normal subgroup of $N_G(H)$. Conclude that $H \lhd G$ iff $N_G(H) = G$.

d) Prove that $N_G(H)$ is the union of all those left cosets of H in G which are also right cosets of H in G.

e) Let $G = D_4$, $H = \{I, S\}$. Find $N_G(H)$.

Solution. a) Suppose that $a \in N_G(H)$. Then $aH = Ha$ so $aHa^{-1} = (Ha)a^{-1} = Haa^{-1} = H$. Thus for any $h \in H$ we have $c_a(h) = aha^{-1} \in aHa^{-1} = H$. Thus $c_a(H) \subseteq H$. Since $aHa^{-1} = H$, given $h \in H$ there is $h_1 \in H$ such that $h = ah_1a^{-1} = c_a(h)$. Thus $H \subseteq c_a(H)$. This shows that $H = c_a(H)$.

Suppose now that $c_a(H) = H$. Note that for any $h \in H$ we have $ah = (aha^{-1})a = c_a(h)a \in Ha$. Thus $aH \subseteq Ha$. Also, any $h \in H$ is of the form $c_a(h_1)$ for some $h_1 \in H$ so $ha = c_a(h_1)a = ah_1a^{-1}a = ah_1 \in aH$, which proves that $Ha \subseteq aH$. Thus $aH = Ha$.

b) Suppose that $a, b \in N_G(H)$. Then $c_a(H) = H = c_b(H)$ by a). Thus $c_{ab}(H) = c_a(c_b(H)) = c_a(H) = H$ and therefore $ab \in N_G(H)$ by a) again. Since $c_a(H) = H$ and c_a is a bijection, we have $c_a^{-1}(H) = H$. But $c_a^{-1} = c_{a^{-1}}$, so $a^{-1} \in N_G(H)$ by a). Finally, if $a \in H$ then $aH = H = Ha$, so $a \in N_G(H)$. Thus $H \subseteq N_G(H)$.

Recall now that $a \in Z(G)$ iff c_a is the identity map. But the identity map maps H onto itself, so $Z(G) \subseteq N_G(H)$ by a).

c) H is a subgroup of $N_G(H)$ by b). By definition of $N_G(H)$, $aH = Ha$ for all $a \in N_G(H)$. Thus $H \lhd N_G(H)$. It follows that if $N_G(H) = G$ then $H \lhd G$. Conversely, if $H \lhd G$ then $aH = Ha$ for every $a \in G$ so $N_G(H) = G$.

d) Let T be the union of all the left cosets of H which are also right cosets. If aH is also a right coset then $aH = Hb$ for some b. If c is any element in $aH = Hb$ then $aH = cH$ and $Hb = Hc$. Thus $cH = Hc$ and $c \in N_G(H)$. This proves that $N_G(H)$ contains all left cosets of H which are also right cosets, i.e. $T \subseteq N_G(H)$. Conversely, if $a \in N_G(H)$ then $aH = Ha$, so $a \in aH \subseteq T$. This shows that $N_G(H) \subseteq T$. Consequently, we see that $N_G(H) = T$.

Problem 4. a) Prove that if A, B are normal subgroups of a group G then $G/(A \cap B)$ is isomorphic to a subgroup of $(G/A) \times (G/B)$. Hint: Use Problem 3 c) from homework 4.

b) Let G be a group and let M, N be normal subgroups of G. Suppose that G/M and G/N are abelian. Prove that $G/(M \cap N)$ is also abelian.

Solution. a) Consider the quotient homomorphisms $\pi_1 : G \longrightarrow G/A$ and $\pi_2 : G \longrightarrow G/B$. By problem 3 c) this induces a homomorphism $f : G \longrightarrow (G/A) \times (G/B)$ and $\ker f = \ker \pi_1 \cap \ker \pi_2 = A \cap B$. Let $\pi : G \longrightarrow G/(A \cap B)$ be the quotient homomorphism. Then π is surjective and $\ker \pi = A \cap B = \ker f$. By the Fundamental Homomorphism Theorem, there is unique $\overline{f} : G/(A \cap B) \longrightarrow (G/A) \times (G/B)$ such that $f = \overline{f}\pi$. Since $\ker \pi = \ker f$, $\ker \overline{f} = \{e\}$ so $\overline{f}$ is injective.

b) By part a), $G/(M \cap N)$ is isomorphic to a subgroup of $G/M \times G/N$. Since G/M and G/N are abelian, the group $G/M \times G/N$ is also abelian and so is every subgroup of $G/M \times G/N$.

Here is a different argument, without using part a). Consider two elements $x, y \in G/(M \cap N)$. Thus $x = a(M \cap N)$, $y = b(M \cap N)$ for some $a, b \in G$. We want to prove that $xy = yx$, i.e. that $(ab)(M \cap N) = (ba)(M \cap N)$. This holds iff $(ba)^{-1}(ba) \in M \cap N$.

Since G/M is abelian, we have $(aM)(bM) = (bM)(aM)$, i.e. $(ab)M = (ba)M$. Thus $(ba)^{-1}(ab) \in M$. The same argument applied to G/N shows that $(ba)^{-1}(ab) \in N$. Thus $(ba)^{-1}(ba) \in M \cap N$, which is exactly what is needed to show that $xy = yx$. This proves that $G/(M \cap N)$ is abelian.

Problem 5. Let G be a group such that $G/Z(G)$ is cyclic (here $Z(G)$ is the center of G). Prove that G is abelian.

Solution: The assumption that $G/Z(G)$ is cyclic means that there is $x \in G/Z(G)$ such that every element of $G/Z(G)$ is a power of x. We can write $x = gZ(G)$ for some $g \in G$. If $a \in G$ then $aZ(G) = x^k = g^k Z(G)$ for some integer k. This means that $a = g^k z$ for some $z \in Z(G)$. Similarly, if b is another element of G then $b = g^l s$ for some integer l and some $s \in Z(G)$. Since z and s commute with all elements in G we have

$$ab = (g^k z)(g^l s) = g^k g^l zs = g^{k+l} zs \quad \text{and} \quad ba = (g^l s)(g^k z) = g^l g^k sz = g^{k+l} zs.$$

Thus $ab = ba$, which proves that G is abelian.

Problem 6. Problem H to chapter 13.

Solution: 1. If every non-trivial element of G has order 2 then G is abelian. Let $a \neq b$ be two non-trivial elements of G. Then $H = \{e, a, b, ab\}$ is a subgroup of G of order 4. Take $c \in G$ which is not in H. Then the elements c, ac, bc, abc are distinct and none of them is in H. Thus $e, a, b, ab, c, ac, bc, abc$ are 8 different elements in G, so $G = \{e, a, b, ab, c, ac, bc, abc\}$. Now the function $f : G \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ defined by $f(a) =$

$(1,0,0), f(b) = (0,1,0), f(c) = (0,0,1)$, $f(ab) = (1,1,0)$, $f(ac) = (1,0,1)$, $f(bc) = (0,1,1)$, $f(a,b,c) = (1,1,1)$ is an isomorphism.

2) Note that ba is not in H, so it must be in Hb. If $ba = a^2b$ then $bab = a^2$ (since $b = b^{-1}$). Then $e = a^4 = babbab = ba^2b = bba = a$, which is not possible. Since $ba \neq b$, we have either $ba = ab$ or $ba = a^3b$

3) If $ba = ab$ then G is abelian. Then H and $N = \{e, b\}$ are normal subgroups of G and have trivial intersection. Thus $G = HN$ is isomorphic to $H \times N$. Since $H \cong Z_4$ and $N \cong Z_2$, we have $G \cong Z_4 \times Z_2$.

4) Suppose that $ba = a^3b = a^{-1}b$. Recall that $D_4 = \{I, T, T^2, T^3, S, TS, T^2S, T^3S\}$. Sending T to a,S to b, and in general S^iT^j to b^ia^j is easily seen to be an isomorhism of groups.

5) If b, ab, a^2b, a^3b all have order 4 then b^2 has order 2 hence is not in Hb. Thus b^2 is in H and the only element in H of order 2 is a^2. Thus $b^2 = a^2$.

6) If $ab = ba$ then a and b commute so $(ab)^2 = a^2b^2 = a^4 = e$ so ab would have order 2 contrary to our assumption.

If $ba = a^2b$ then since $a^2 = b^2$ we have $ba = b^3$ and $a = b^2$, which is not possible as b^2 has order 2 and a has order 4. Thus we have $ba = a^3b$.

7) Writing $a = i$, $b = j$, $ab = k$, $a^2 = -I$, $e = I$ we see that $i^2 = j^2 = k^2 = -I$ and the multiplication table for G is the same as for the quaternion group Q_8 from Problem 6 in homework 2. Thus $G \cong Q_8$.

Problem 7. Problem I to chapter 13.

Solution. 1. Clearly $a \sim a$ as $a = eae^{-1}$. If $a \sim b$ then $a = xbx^{-1}$ for some x and therefore $b = x^{-1}ax$, so $b \sim a$. Finally, if $a \sim b$ and $b \sim c$ then $a = xbx^{-1}$ and $b = ycy^{-1}$ for some x, y, so $a = x(ycy^{-1})x^{-1} = (xy)c(xy)^{-1}$ and therefore $a \sim c$. Thus $\sim$ is an equivalence relation.

2) This was proved in part a) of Quiz 2.

3) We have $x^{-1}ax = y^{-1}ay$ if and only if

$$a = xy^{-1}ayx^{-1} = (xy^{-1})a(xy^{-1})^{-1}$$

which is equivalent to $(xy^{-1})a = a(xy^{-1})$, i.e. $xy^{-1} \in C(a)$.

4) By 3), $x^{-1}ax = y^{-1}ay$ if and only if $xy^{-1} \in C(a)$, which is equivalent to $C(a)x = C(a)y$.

5) Part 4) confirms that the function which sends an element $x^{-1}ax$ of the conjugacy class of a to the coset $C(a)x$ is well defined (i.e. does not depend on which x is used) and injective. It is clearly surjective. Thus we have a bijection between the conjugacy class of a and the right cosets of $C(a)$ in G.

6) Since $[G : C(a)]$ is the number of right cosets of $C(a)$ in G, the conjugacy class of a has $[G : C(a)]$ elements by 5).

Problem 8. Problem D to chapter 15.

Solution. Let $\pi : G \longrightarrow G/H$ be the quotient homomorphism.

1) Let $a \in G$. Then $\pi(a)$ has finite order, say m. Thus $e = \pi(a)^m = \pi(a^m)$, so $a^m \in \ker \pi = H$. Since every element of H has finite order, there is k such that $e = (a^m)^k = a^{mk}$. Thus a has finite order.

2) Assume G is abelian. Let $a \in G$. There is $z \in G/H$ such that $\pi(a) = z^2$. Since π is surjective, $z = \pi(b)$ for some $b \in G$. Thus $\pi(ab^{-2}) = e$ so $ab^{-2} \in \ker \pi = H$. By our assumption, $ab^{-2} = t^2$ so $a = t^2b^2 = (tb)^2$ (here we use the assumption that G is abelian).

3) This uses the same idea as part 1. Let $a \in G$. Then $\pi(a)$ has order p^m for some m. Thus $e = \pi(a)^{p^m} = \pi(a^{p^m})$, so $a^{p^m} \in \ker \pi = H$. Since H is a p-group, there is k such that $e = (a^{p^m})^{p^k} = a^{p^{m+k}}$. Thus the order of a divides p^{m+k} and therefore it is a power of p.

4) Assume H and G/H are finitely generated. Recall that the subgroup $< S >$ generated by a subset S of a group G is the intersection of all subgroups of G which contain S. In particular, if a subgroup of G contains S then it contains $< S >$.

Suppose that G/H is generated by a finite subset $S = \{z_1, \ldots, z_k\}$. Since π is surjective, $z_i = \pi(a_i)$ for some $a_i \in G$. Let $T = \{b_1, \ldots, b_m\}$ generate H. Take $R = \{a_1, \ldots, a_k\} \cup T$ and let $K =< R >$. Since $T \subseteq R \subseteq < R >= K$, we have $H =< T > \subseteq K$. The subgroup $\pi(K)$ of G/H contains the set S, so $G/H =< S > \subseteq \pi(K) \subseteq G/H$. We see that $\pi(K) = G/H$. Since $H \subseteq K$ and $\pi(K) = G/H$, we have $G = \pi^{-1}(G/H) = K$ by the correspondence theorem. Thus G is generated by the finite set R.