Homework 6, due on Friday, October 9

Read carefully chapters 14,15, 16 in the textbook. **It is crucial that you read and attempt ALL the exercises at the end of each chapter as they form integral part of the book.**
Solve the following problems.

Problem 1. Let G be a finite group and let H, K be subgroups of G.

a) Suppose that $h, h_1 \in H$ and $k, k_1 \in K$. Prove that $hk = h_1k_1$ if and only if $h_1 = hu$ and $k_1 = u^{-1}k$ for some $u \in H \cap K$.

b) Prove that $|HK| = |H||K|/|H \cap K|$.

Problem 2. Let G be a group and H a subgroup of G.

a) Prove that aHa^{-1} is a subgroup of G for any $a \in G$. Any subgroup of the form aHa^{-1} is called a **conjugate** of H.

b) Prove that $aHa^{-1} = bHb^{-1}$ if and only if $a^{-1}b \in N_G(H)$. Conclude that the number of different conjugates of H is equal to the index $[G : N_G(H)]$.

c) Prove that the intersection of all subgroups conjugate to H is a normal subgroup in G. Show that this subgroup is the largest subgroup of H which is normal in G (this means that any normal in G subgroup of H is contained in the above intersection).

d) Suppose that H has finite index in G. Prove that there is a normal subgroup of G which is contained in H and has finite index in H (use b), c) and Problem 2c) from homework 5).

Problem 3. Let G be a group.

a) Prove that if H is a subgroup of G and K is a normal subgroup of G then the groups KH/K and $H/(H \cap K)$ are isomorphic (hint: consider the restriction of the quotient homomorphism $KH \longrightarrow KH/K$ to the subgroup H of KH and show that it is surjective; then describe its kernel and apply FHT). Consult Problem 2 c) from homework 5.

b) Let K and H be normal subgroups of G such that $K \subseteq H$. Show that H/K is a normal subgroup of G/K. Prove that $(G/K)/(H/K)$ is isomorphic to G/H. Hint: consider the quotient homomorphisms $f : G \longrightarrow G/H$ and $\pi : G \longrightarrow G/K$. Use FHT to get a homomorphism $G/K \longrightarrow G/H$. Note that H/K is the kernel of this homomorphism. Apply FHT again.

Problem 4. Let G be a finite group and let H be a subgroup of G. Let m be the smallest positive integer such that $a^m \in H$. Prove that $< a^m >= H \cap < a >$ and that m divides the order of a. **Hint:** Prove first that $\{k \in \mathbb{Z} : a^k \in H\}$ is a subgroup of $\mathbb{Z}$.

Problem 5. Let G be a group and let f_n be the function $f_n : G \longrightarrow G$, $f_n(g) = g^n$. Prove that if f_3 and f_5 are homomorphisms then G is abelian.

Problem 6. Let $G = \mathbb{Q} \times \mathbb{Z}$ as a set. Here $\mathbb{Q}$ stands for the group of rational numbers and $\mathbb{Z}$ for the integers (both with addition as the group operation). We define a binary operation on the set G as follows:

$$(q, m)(r, n) = (q + 2^m r, m + n)$$

(this construction is a special case of a general construction called a **semidirect product**).

a) Verify that G with this binary operation is a group. What is the inverse of (q, m)?

b) Let $T = \{(q, 0) : q \text{ is an integer}\}$. Prove that T is a subgroup of G.

c) Show that $(0, 1)(q, 0)(0, 1)^{-1} = (2q, 0)$. Conclude that if $t = (0, 1)$ then tTt^{-1} is a proper subgroup of T. What is $t^{-1}Tt$?

d) What is the normalizer of T in G?

Problem 7. Problem G to chapter 15.