Homework 8, due on Friday, October 30

Read carefully chapters 14,15, 16 in the textbook and the notes for the lectures on the course web page. **It is crucial that you read and attempt ALL the exercises at the end of each chapter as they form integral part of the book.**
Solve the following problems.

Problem 1. Let G be a group and H a subgroup of G. Let $X = G/H$ be the set of all left cosets of H in G. Define $* : G \times X \longrightarrow X$ by $g * (aH) = (ga)H$.

a) Prove that $*$ is an action of G on X. Show that this action is transitive and that $St(aH) = aHa^{-1}$ for every $a \in G$.

b) According to the discussion in the lecture on October 9 (see the notes) this action corresponds to a homomorphism $\phi : G \longrightarrow S_X$. Prove that kernel of this homomorphism is the largest normal subgroup of G which is contained in H (i.e. that any normal subgroup of G which is contained in H is also contained in this kernel).

c) Suppose that G is finite and there is a prime divisor p of $|G|$ such that $1 < [G : H] < p$. Prove that G is not simple. Hint: Prove that the homomorphism ϕ from part b) is not injective.

d) Suppose that G is finite and that $[G : H]$ is equal to the smallest prime divisor p of $|G|$. Prove that H is normal in G. Hint: Consider the homomorphism ϕ from part b) and prove that $\ker \phi = H$.

Problem 2. a) Find all Sylow $2-$subgroups of S_4. Hint: The group S_4 can be considered as the group of all permutations of the vertices $1, 2, 3, 4$ of a square (numbered clockwise). It contains as a subgroup the dihedral group D_8 of order 8 (which consists of those permutations which are isometries; thus the rotation clockwise by 90 degrees is $R = (1, 2, 3, 4)$ and the reflection in the diagonal joining 1 and 3 is $S = (2, 4)$).

b) What is the number of Sylow $3-$subgroups of S_4?

Problem 3. Let G be a finite group and let P be a Sylow $p-$subgroup of G (for some prime p).

a) Suppose that $f : G \longrightarrow H$ is a surjective homomorphism. Prove that $f(P)$ is a Sylow $p-$subgroup of H.

b) Let N be a normal subgroup of G. Prove that $P \cap N$ is a Sylow $p-$subgroup of N.

Problem 4. Let G be a finite group of order pq, where $p < q$ are primes.

a) Show that G has a normal Sylow $q-$subgroup.

b) Suppose that $p \nmid (q - 1)$. Prove that G has a normal Sylow $p-$subgroup.

c) Suppose that $p \nmid (q - 1)$. Let P be the Sylow $p-$subgroup of G and let Q be the Sylow $q-$subgroup of G. Prove that $G \cong P \times Q$ conclude that G is cyclic.

Problem 5. This problem sketches a different proof of existence of Sylow $p-$subgroups. Let p be a prime. Let G be a finite group and suppose that every group of order smaller than $|G|$ has a Sylow $p-$subgroup (so this proof goes by induction on $|G|$). If $p \nmid |G|$, there is nothing to prove, so we assume that $p||G|$. We use the action of G on itself by

conjugation. Recall that the stabilizer of an element $a \in G$ is simply its centralizer $C(a)$ and the orbit of a is the conjugacy classes of a. In particular, the fixed points of this action are the elements of the center $Z(G)$.

a) Prove that if $p \nmid |Z(G)|$ then there is a non-central element a whose conjugacy class has size not divisible by p. Then justify the following claims:

- $C(a)$ is a proper subgroup of G so it has a Sylow $p-$subgroup P by the inductive assumption;
- p does not divide the index $[G : C(a)]$;
- P is a Sylow p-subgroup of G.

b) Suppose that $p||Z(G)|$. Since $Z(G)$ is abelian, it has an element g of order p (we do not want to use Caychy's theorem here, only the special case for abelian groups since then we can derive Cauchy's theorem from the existence of Sylow subgroups). Show that $Q =< g >$ is a normal subgroup of G of order p. Consider the quotient homomorphism $f : G \longrightarrow G/Q$. Since $|G/Q| < |G|$, G/Q has a Sylow $p-$subgroup P. Prove that $f^{-1}(P)$ is a Sylow $p-$subgroup of G.