Homework 9, due on Friday, November 6

Read carefully chapters 17,18 in the textbook. **It is crucial that you read and attempt ALL the exercises at the end of each chapter as they form integral part of the book.**
Solve the following problems.

Problem 1. Problem E in chapter 17.

Problem 2. a) Solve Problem F in chapter 17.

b) Let G be an abelian group written additively (so the operation is denoted by $+$ and the identity element by 0). Consider the direct product $G \times G$. It is an abelian group and we have the following four homomorphisms:

- $\pi_1 : G \times G \longrightarrow G$, $\pi_1(a, b) = a$.
- $\pi_2 : G \times G \longrightarrow G$, $\pi_2(a, b) = b$.
- $e_1 : G \longrightarrow G \times G$, $e_1(a) = (a, 0)$.
- $e_2 : G \longrightarrow G \times G$, $e_2(a) = (0, a)$.

Suppose we have a group homomorphism $\phi : G \longrightarrow G \times G$. Prove that $f = \pi_1\phi$ and $g = \pi_2\phi$ are endomorphisms of G and $\phi(a) = (f(a), g(a))$ for every $a \in G$.

c) We use the notation from part b). Suppose $\psi : G \times G \longrightarrow G$ is a group homomorphism. Prove that $\alpha = \psi e_1$ and $\beta = \psi e_2$ are endomorphisms of G and $\psi(a, b) = \alpha(a) + \beta(b)$ for every $(a, b) \in G \times G$.

d) Suppose that $\phi : G \longrightarrow G \times G$ is an isomorphism and $\psi : G \times G \longrightarrow G$ is the inverse of ϕ. Using the notation from parts b) and c) prove that

$$\alpha f + \beta g = 1, \quad f\alpha = 1, \quad f\beta = 0, \quad g\alpha = 0, \quad g\beta = 1.$$

Conclude that in the ring $End(G)$ α is left invertible and a right zero divisor.

e) Let G be the set of all sequences of integers, i.e. all functions $f : \mathbb{N} \longrightarrow \mathbb{Z}$. Show that G is an abelian group under the usual addition of functions. Prove that G and $G \times G$ are isomorphic. Hint: split a sequence into two sequences, one using the odd terms and the other using even terms.

Problem 3. Problem G in chapter 17.

Problem 4. Problem H parts 5 and 6 in chapter 17.

Problem 5. a) Solve Problem M parts 1 and 3 in chapter 17.

b) Find two elements a, b in the ring of 2×2 matrices which are nilpotent and such that $a + b$ is not nilpotent.

Problem 6. Problem A parts 1,2,3 in chapter 18.

Problem 7. Problem C parts 8,9 in chapter 18.

Problem 8. Problem F parts 5,7 in chapter 18.