Let $f: G \to H$ be a homomorphism.

① If $K$ is a subgroup of $G$ then $f(K) < H$.

② If $L < H$ then $f^{-1}(L) < G$ and $\ker f \subseteq f^{-1}(L)$

③ If $L \lhd H$ then $f^{-1}(L) \lhd G$

④ If $f$ is surjective and $K \lhd G$ then $f(K) \lhd H$

Proof: We already know ① that image of a group under a homomorphism is a group.

For ② let $a, b \in f^{-1}(L)$. Then $f(a) \in L$ and $f(b) \in L$. Since $L$ is a subgroup, $f(a) \cdot f(b) = f(ab) \in L$ and $f(a)^{-1} = f(a^{-1}) \in L$. Thus $ab \in f^{-1}(L)$ and $a^{-1} \in f^{-1}(L)$. Finally, $f(e) \in L$, so $e \in f^{-1}(L)$.

Suppose now that $L \lhd H$. Let $a \in f^{-1}(L)$ and $g \in G$. Then $f(gag^{-1}) = f(g)f(a)f(g)^{-1} \in L$ since $f(a) \in L$ and $L \lhd H$. Thus $f^{-1}(L)$ is normal in $G$. This proves ③.

To prove ④ let $u \in f(K)$ and $h \in H$. Thus $u = f(a)$ for some $a \in K$. Since $f$ is surjective, $h = f(g)$ for some $g \in G$. Therefore $huh^{-1} = f(g)f(a)f(g)^{-1} = f(gag^{-1}) \in f(K)$ since $gag^{-1} \in K$ (as $K$ is normal in $G$). This proves ④. ∎

We get the following very useful result:

**Theorem (Correspondence Thm):** If $f: G \to H$ is a surjective homomorphism then we have a bijection between subgroups of $H$ and subgroups of $G$ which contain $\ker f$:

$$\{\text{subgroups of } H\} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \{\text{subgroups of } G \text{ which contain } \ker f\}$$

$$\phi(K) = f^{-1}(K) \text{ and } \psi(L) = f(L).$$

In particular, when $\pi: G \to G/N$ is a quotient map we get a bijective correspondence between subgroups of $G/N$ and subgroups of $G$ which contain $N$.

The correspondence takes normal subgroups to normal subgroups.

**Thm:** Let $K, L$ be normal subgroups of $G$ such that $K \cap L = \{e\}$. Then

(1) For any $k \in K$ and $l \in L$ we have $k \cdot l = l \cdot k$

(2) The subgroup $KL = LK$ is isomorphic to $K \times L$

**Pf:** Consider the element $u = klk^{-1}l^{-1}$. Since $L$ is normal we have $klk^{-1} \in L$ so $u = (klk^{-1}) \cdot l^{-1} \in L$. On the other hand, since $K$ is normal we have $lk^{-1}l^{-1} \in K$, so $u = k(lk^{-1}l^{-1}) \in K$. Thus $u \in K \cap L = \{e\}$, i.e. $u = e$. This means that $klk^{-1}l^{-1} = e$, i.e. $kl = lk$.

Now consider the function $f: K \times L \longrightarrow KL$, $f(k,l) = kl$. Clearly $f$ is surjective. Also:

$f((k,l)\cdot(k_1,l_1)) = f((kk_1, ll_1)) = (kk_1)(ll_1) = (kl)(k_1l_1) = f(k,l)f(k_1,l_1)$.

Thus $f$ is a homomorphism. If $f(k,l) = e$ then $kl = e$ so $k = l^{-1}$. Thus $k \in K \cap L = \{e\}$ so $k = e$ and $l = e$.

Thus shows that $\ker f = \{(e,e)\}$, so $f$ is injective.

Therefore $f$ is an isomorphism.

Recall the following corollary of FHT:

if $f: G \to H$ is a surjective homomorphism then

(*) $|G| = |H| \cdot |\ker f|$

(we are assuming that $G$ is a finite group here).

In fact, by FHT, $H$ is isomorphic to $G/\ker f$ and therefore $|H| = |G/\ker f| = [G : \ker f]$. Thus (*) is a consequence of Lagrange's Thm.

We will now illustrate the concepts and results we learned so far in proving the following

Theorem: Let $A$ be a finite abelian group of order $n$ and let $m \mid n$. Then $A$ has a subgroup of order $m$.

Remark: This result is no longer true for non-abelian groups. Consider for example the alternating group $A_5$. We have $|A_5| = 60$. Recall that $A_5$ is a simple group. Thus $A_5$ does not have any subgroup of order 30, as any such subgroup would have index 2, and therefore would be normal. One can show that $A_5$ has no subgroups of order 15 too.