Let R be a ring.

Proposition 1: The intersection of any collection of ideals in R is also an ideal.

Pf: $I_\alpha, \alpha \in S$ are ideals. Consider $J = \bigcap_{\alpha \in S} I_\alpha$. If $a, b \in J$ and $r \in R$ then $a, b \in I_\alpha$ for every α so:

$$a-b \in I_\alpha,\ ra \in I_\alpha,\ ar \in I_\alpha \text{ for every } \alpha \in S$$

and therefore $a-b \in \bigcap_{\alpha \in S} I_\alpha = J$, $ra \in \bigcap_{\alpha \in S} I_\alpha = J$, $ar \in \bigcap_{\alpha \in S} I_\alpha = J$. Since $0 \in I_\alpha$ for all α, we have $0 \in \bigcap I_\alpha = J$. Thus J satisfies all the requirements to be an ideal. ☒

Proposition 2: If I, J are ideals of R then $I+J = \{i+j : i \in I, j \in J\}$ is also an ideal.

Pf: Let $a, b \in I+J$ and $r \in R$. Then $a = i+j$, $b = i_1+j_1$ for some $i, i_1 \in I$ and $j, j_1 \in J$. Then $a-b = (i-i_1)+(j-j_1) \in I+J$, $ra = (ir)+(rj) \in I+J$, $ar = (ir)+(jr) \in I+J$ and $0 = 0+0 \in I+J$. Thus $I+J$ is an ideal. ☒

Proposition 3: If I, J are ideals in R then

$$I \cdot J = \{i_1 j_1 + i_2 j_2 + \dots + i_k j_k : k \in \mathbb{N},\ i_1, \dots, i_k \in I,\ j_1, \dots, j_k \in J\}$$

is an ideal.

Pf: Let $a, b \in I \cdot J$, $r \in R$. Then $a = i_1 j_1 + \dots + i_k j_k$, $b = \tilde{i}_1 \tilde{j}_1 + \dots + \tilde{i}_\ell \tilde{j}_\ell$ for some $i_1, \dots, i_k, \tilde{i}_1, \dots, \tilde{i}_\ell$ in I, $j_1, \dots, j_k, \tilde{j}_1, \dots, \tilde{j}_k$ in J. Then

$$a-b = i_1 j_1 + \dots + i_k j_k + (-\tilde{i}_1)\tilde{j}_1 + \dots + (-\tilde{i}_\ell)\tilde{j}_\ell \in I \cdot J$$

$$ra = (ri_1)j_1 + \dots + (ri_k)j_k \in IJ,\quad ar = i_1(j_1 r) + \dots + i_k(j_k r) \in I \cdot J$$

Also $0 = 0 \cdot 0 \in I \cdot J$ so $I \cdot J$ is an ideal.

Proposition 4: I, J ideals in R then $I \cdot J \subseteq I \cap J$.

Pf: If $a \in I \cdot J$ then $a = i_1 j_1 + i_2 j_2 + \dots + i_k j_k$.
Since $i_1 \in I$, $i_m j_m \in I$ and since $j_m \in J$, $i_m j_m \in J$ so $i_m j_m \in I \cap J$ for $m = 1, 2, \dots, k$. Thus $i_1 j_1 + \dots + i_m j_m \in I \cap J$ i.e. $a \in I \cap J$.
Thus $I \cdot J \subseteq I \cap J$. □

Definition: I, J are coprime if $I + J = R$.

Remark: If R is unital then I, J are coprime if and only if $1 = i + j$ for some $i \in I, j \in J$.

Definition: An ideal I is called maximal if I is proper (i.e. $I \neq R$) and there is no proper ideal J s.t. $I \subseteq J$, $I \neq J$ (i.e. there are no ideals strictly between I and R).
From the correspondence theorem we see:
I is a maximal ideal iff R/I is a non-trivial simple ring.

Suppose now that R is commutative. For any $a \in R$ the set $aR = \{ar : r \in R\}$ is an ideal:
$a \in R$ the set $aR = \{ar : r \in R\}$ is an ideal:
$0 = a \cdot 0$, $ar - as = a(r - s)$, $r_1(ar) = a(r_1 r) = (ar) r_1$ so aR has all the requirements to be an ideal.

If R is also unital then $a = a \cdot 1 \in aR$.

Def: If R is a commutative unital ring then aR is called the principal ideal generated by a.

Thm: R is a simple commutative unital ring if and only if R is a field.

Pf: Let $a \in R$, $a \neq 0$. Then aR is an ideal and $a \in aR$ so $aR \neq \{0\}$. Since R is simple, $R = aR$. Thus $1 = ar$ for some $r \in R$, i.e. a is invertible. We showed that all non-zero elements in R are invertible, so R is a field.

Conversely, if R is a field and I is an ideal in R and $I \neq \{0\}$ then there is $a \in I$, $a \neq 0$. Let $b = a^{-1}$ so for any $r \in R$ we have $r = (rb) \cdot a \in I$ (since $a \in I$) so $R = I$. Thus shows that $\{0\}, R$ are the only ideals in R, i.e. R is simple.

Corollary: R commutative, unital. Then I is maximal ideal in R if and only if R/I is a field.

Fact: In any unital ring every ideal different from R (i.e. proper) is contained in a maximal ideal.

Def: Let R be a commutative unital ring. An ideal I in R is called prime if R/I is an integral domain.

Since every field is an integral domain, we have

Proposition 5: Every maximal ideal is prime.

Pf: I maximal $\Leftrightarrow$ R/I field $\Rightarrow$ R/I integral domain $\Leftrightarrow$ I prime. ☒

Proposition 6: Let R be a commutative, unital ring. Following conditions are equivalent:

(1) I is a prime ideal in R

(2) If $a \cdot b \in I$ for some $a, b \in R$ then either $a \in I$ or $b \in I$.

(3) If J, K are ideals in R and $J \cdot K \subseteq I$ then either $J \subseteq I$ or $K \subseteq I$.

Pf: (1) $\Rightarrow$ (2). Assume I is prime, so R/I is an integral domain. Consider the quotient homo. $\pi: R \to R/I$. Since $ab \in I$ we have $0 = \pi(ab) = \pi(a)\pi(b)$. Since R/I is a domain, either $\pi(a) = 0$ or $\pi(b) = 0$, i.e. either $a \in \ker\pi = I$ or $b \in \ker\pi = I$. So (2) holds.

(2) $\Rightarrow$ (3). Suppose there is $a \in J$, $a \notin I$ and $b \in K$, $b \notin I$. Then $a \cdot b \in J \cdot K \subseteq I$ so $ab \in I$ and $a \notin I$, $b \notin I$. This contradicts (2).

(2) Thus either there is no a s.t. $a \in J$, $a \notin I$, which means $J \subseteq I$ or there is no b s.t. $b \in K$, $b \notin I$ which means $K \subseteq I$.

(3) $\Rightarrow$ (1) Suppose I is not prime. So R/I is not a domain. So there are $a+I \neq 0$, $b+I \neq 0$ in R/I s.t. $(a+I)\cdot(b+I) = 0$ i.e. $ab+I = 0$. This means $a \notin I$, $b \notin I$ and $ab \in I$. Take $J = aR$, $K = bR$. Then $JK = (aR)(bR) = (ab)R \subseteq I$ but $J \not\subseteq I$, $K \not\subseteq I$. This contradicts (3). So I is prime. ☒

Example: Let us look at the ring $\mathbb{Z}$ of integers. Every ideal in $\mathbb{Z}$ is of the form $m\mathbb{Z}$ for some $m \geq 0$.

(1) $m\mathbb{Z} \cap n\mathbb{Z} = \mathrm{lcm}(m,n)\cdot\mathbb{Z}$. This follows from the observation that a number is a multiple of m and multiple of n iff it is a multiple of $\mathrm{lcm}(m,n)$.

(2) $m\mathbb{Z} + n\mathbb{Z} = \gcd(m,n)\mathbb{Z}$. This follows from the fact that $m\mathbb{Z} + n\mathbb{Z} = k\mathbb{Z}$ for some $k \geq 0$. Since $m\mathbb{Z} \subseteq k\mathbb{Z}$, $n\mathbb{Z} \subseteq k\mathbb{Z}$ we see that $k \mid m$ and $k \mid n$ and if $d \mid m$ and $d \mid n$ then d divides every element in $m\mathbb{Z} + n\mathbb{Z}$, so $d \mid k$. Thus k is the largest common divisor of m, n, i.e. $k = \gcd(m,n)$.

(3) $(m\mathbb{Z})\cdot(n\mathbb{Z}) = (m\cdot n)\mathbb{Z}$. In fact $m\mathbb{Z}\cdot n\mathbb{Z}$ is clearly contained in $mn\mathbb{Z}$ and $m\cdot n \in (m\mathbb{Z})\cdot(n\mathbb{Z})$ so $mn\mathbb{Z} \subseteq (m\mathbb{Z})\cdot(n\mathbb{Z})$.

It is true in any commutative unital ring that

$$(aR)\cdot(bR)=(ab)R.$$

(4) $m\mathbb{Z}$ and $n\mathbb{Z}$ are coprime iff $m\mathbb{Z}+n\mathbb{Z}=\mathbb{Z}$ iff $\gcd(m,n)\cdot\mathbb{Z}=\mathbb{Z}$ iff $\gcd(m,n)=1$ iff m,n are relatively prime.

(5) $m\mathbb{Z}$ is a prime ideal iff $m=0$ or m is a prime number.

In fact $\mathbb{Z}/0\cdot\mathbb{Z}\cong\mathbb{Z}$ so $0\cdot\mathbb{Z}=\{0\}$ is a prime ideal.

$\mathbb{Z}/p\mathbb{Z}$ is a field for any prime number p so $p\mathbb{Z}$ is maximal, hence prime.

If $n=ab$, ~~$1<a<n$~~ $1<a<n$ then $a+n\mathbb{Z}$, $b+n\mathbb{Z}$ are non-zero elements in $\mathbb{Z}/n\mathbb{Z}$ and $(a+n\mathbb{Z})\cdot(b+n\mathbb{Z})=0$, i.e. $\mathbb{Z}/n\mathbb{Z}$ is not a domain, so $n\mathbb{Z}$ is not prime.

(6) From our discussion in (5) $m\mathbb{Z}$ is maximal in $\mathbb{Z}$ iff m is a prime number.