

## Quizzes for Math 401

**QUIZ 1.** a) Define a group.

b) Is the set  $\{a, b, c\}$  with the binary operation  $*$  defined by the table below a group? Explain your answer.

| $*$ | $a$ | $b$ | $c$ |
|-----|-----|-----|-----|
| $a$ | $a$ | $b$ | $c$ |
| $b$ | $b$ | $b$ | $c$ |
| $c$ | $c$ | $c$ | $b$ |

**Solution.** a) A group is a set  $G$  with a binary operation  $*$  such that

- $a * (b * c) = (a * b) * c$  for any  $a, b, c \in G$ , i.e.  $*$  is associative.
- There is  $e \in G$  such that  $a * e = a = e * a$  for every  $a \in G$ , i.e. the operation  $*$  has an identity.
- For every  $a \in G$  there is  $b \in G$  such that  $a * b = e = b * a$ , i.e. every element in  $G$  has an inverse.

b) This is not a group. Note that we have  $b * a = b * b$ . If this was a group, we could use cancellation to conclude  $a = b$ , which is false.

**Quiz 2.** a) Let  $G$  be a group and  $a \in G$  any element. Define a subset  $C(a)$  of  $G$  as follows:

$$C(a) = \{g \in G : ag = ga\},$$

i.e.  $C(a)$  is the set of all elements in  $G$  which commute with  $a$ . Prove that  $C(a)$  is a subgroup of  $G$ .

b) Let  $S$  be a subset of a group  $G$ . How is the subgroup generated by  $S$  defined?

**Solution.** a) Clearly the identity  $e$  commutes with  $a$ :  $ae = a = ea$ . Thus  $e \in C(a)$ . Let  $g, h$  belong to the set  $C(a)$ . We need to show that  $gh$  and  $g^{-1}$  are also in  $C(a)$ . We know that  $ag = ga$  and  $ah = ha$ . Thus

$$a(gh) = (ag)h = (ga)h = g(ah) = g(ha) = (gh)a$$

so  $gh \in C(a)$ . Also, from  $ag = ga$  we get

$$g^{-1}(ag) = g^{-1}(ga) = (g^{-1}g)a = ea = a$$

so  $(g^{-1}a)g = a$  and therefore  $g^{-1}a = ag^{-1}$ . Thus  $g^{-1} \in C(a)$ . Thus  $C(a)$  is indeed a subgroup of  $G$ . It is called **the centralizer** of  $a$  in  $G$ .

b) If  $S = \emptyset$  is the empty set then the subgroup  $\langle \emptyset \rangle$  generated by  $S$  is defined as  $\langle \emptyset \rangle = \{e\}$ . If  $S$  is nonempty, then the subgroup  $\langle S \rangle$  generated by  $S$  consists of all elements in  $G$  which can be expressed as a product of finitely many elements in  $S$  and their inverses (repetitions allowed). Thus an element  $g$  is in  $\langle S \rangle$  if and only if it can be expressed as

$$g = s_1 s_2 \dots s_k$$

for some  $k \geq 1$ , where for every  $i$  either  $s_i$  is in  $S$  or  $s_i^{-1}$  is in  $S$ .

Alternatively,  $\langle S \rangle$  is the intersection of all subgroups of  $G$  which contain  $S$ .

**Quiz 3.** Let  $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 7 & 6 & 9 & 1 & 3 & 4 & 2 \end{pmatrix}$ .

a) Write  $\sigma$  as a product of disjoint cycles.

b) What is the order of  $\sigma$ ? Explain your answer.

c) Is  $\sigma$  even or odd? Explain your answer.

d) Write  $\sigma^{-1}$  and  $\sigma^2$  as a  $2 \times 9$  matrix:

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ ? & ? & ? & ? & ? & ? & ? & ? & ? \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ ? & ? & ? & ? & ? & ? & ? & ? & ? \end{pmatrix}.$$

**Solution:** a) We have  $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 7 & 6 & 9 & 1 & 3 & 4 & 2 \end{pmatrix} = (1, 8, 4, 6)(2, 5, 9)(3, 7)$ .

b) The order of a permutation is the least common multiple of the lengths of all cycles appearing in the disjoint cycle decomposition of the permutation. Thus the order of  $\sigma$  is  $\text{lcm}(4, 3, 2) = 12$ .

c) A permutation is even (odd) if and only if its decomposition into a product of disjoint cycles contains an even (odd) number of cycles of even length. Thus  $\sigma$  is even, as it has 2 cycles of even length.

d)

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 9 & 7 & 8 & 2 & 4 & 3 & 1 & 5 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 3 & 1 & 2 & 8 & 7 & 6 & 5 \end{pmatrix}.$$

**Quiz 4.** a) Let  $G$  be a group and  $g \in G$  an element of order 16. What is the order of  $g^{12}$ ? Explain your answer.

b)  $G$  is a group and the function  $f : G \rightarrow G$  defined by  $f(g) = g^2$  is a homomorphism.

1. Explain what it means that  $f$  is a homomorphism.

2. Prove that  $G$  is commutative.

**Solution.** a) We proved in class that if the order of a group element  $g$  is  $k$  then for every integer  $m$  the order of  $g^m$  is  $k/\text{gcd}(k, m)$ . Thus the order of  $g^{12}$  is  $16/\text{gcd}(16, 12) = 4$ .

b) 1. When we say that  $f(g) = g^2$  is a homomorphism we mean that for any  $a, b \in G$  we have  $(ab)^2 = f(ab) = f(a)f(b) = a^2b^2$ .

2. From 1. we know that  $a^2b^2 = (ab)^2 = abab$ . Cancelling  $a$  on the left and  $b$  on the right we get  $ab = ba$ . Since  $a, b$  are arbitrary elements of  $G$ , we see that  $G$  is commutative.