Chapter 11

Systems of Equations and Inequalities

11.1 Systems of Equations

General Systems

Given the equation $x^2 + y = 7$ there are many solutions. That is, there are many ordered pairs (x, y) for which this equation is true. Some solutions are $(1, 6)$, $(2, 3)$, $(-5, -18)$ and $(0, 7)$.

Given the equation $y = x + 1$, we can find many solutions to this also. $(1, 2)$, $(2, 3)$ and $(-5, -4)$ are solutions.

In this chapter we are interested in *systems of equations*. A system of equations is a set of two or more equations. The solution to a system of equations is the set of all ordered pairs that are solutions to *all* of the equations in the system. Systems of equations are sometimes called *simultaneous* equations because the equations have to be true simultaneously (at the same time). A system of equations is usually written with a large left-hand brace beside the equations. We can make a system out of the two equations above.

$$\begin{cases} x^2 + y = 7 \\ y = x + 1 \end{cases}$$

The solution to this system is all of the ordered pairs (x, y) that make both of the equations true. We see from our partial lists above that the point $(2, 3)$ is part of the solution to the system. Are there other points?

The second equation in the system tells us that in this system y is the same as $x + 1$. We use this information to rewrite the first equation.

$$x^2 + y = 7$$

$$x^2 + (x + 1) = 7$$

$$x^2 + x - 6 = 0$$

$$(x + 3)(x - 2) = 0$$

$$x = -3 \;\; or \;\; x = 2$$

Since $y = x + 1$ we substitute our x values and get the solutions $(-3, -2)$ and $(2, 3)$.

Example 11.1.1.

Find the solution to the system: $\begin{cases} x + y = 4 \\ x^2 + y^2 - 4x = 0 \end{cases}$

We rewrite the first equation to $x = 4 - y$ and substitute it into the second equation:

$$\begin{aligned} x^2 + y^2 - 4x &= 0 \\ (4-y)^2 + y^2 - 4(4-y) &= 0 \\ 16 - 8y + y^2 + y^2 - 16 + 4y &= 0 \\ 2y^2 - 4y &= 0 \\ 2y(y-2) &= 0 \\ y = 0 \ \text{or} \ y &= 2 \end{aligned}$$

Since $x = 4 - y$ we substitute our y values and get solutions $(4, 0)$ and $(2, 2)$.

In Example 11.1.1 we could have rewritten the first equation as $y = 4 - x$ and substituted this expression for y in the second equation. Indeed, this appears to be the easier method. In theory we could have rewritten the second equation, solving for either x or y, and then substituted into the first equation to find the solution of the system. Do you see why this is unnecessarily awkward?

Comprehension Check 11.1.

Solve the system in Example 11.1.1 by solving the first equation for y instead of x. Check to be sure that you got the same solutions, $(4, 0)$ and $(2, 2)$.

Finding the solution to a system of equations means finding all of the ordered pairs (x, y) that make all of the equations true. The set of ordered pairs that makes any *one* of the equations true can be illustrated on a Cartesian coordinate system as the graph of the equation. So, in theory, we could draw the graphs of all of the equations in the system on the same set of axes and see where they intersect. This would give us all of the points in common, the solution to the system. In fact the instructions to "Find the solution of the system " means the same thing as "Find the intersection of the graphs of the equations" (of the system).

Comprehension Check 11.2.

Carefully graph the equations of the system $\begin{cases} x^2 + y = 7 \\ y = x + 1 \end{cases}$ *and verify that their intersections are $(-3, -2)$ and $(2, 3)$, the solutions obtained earlier.*

While finding the intersections of the graphs does give us the solution to the system, this is not a very practical way to find the solution to a system. To appreciate this point, carefully graph the equations of the following system. They are lines so the graphing is not difficult.

$$\begin{cases} 3x + 5y = 2 \\ 2y - x = 1 \end{cases}$$

What intersection did you find?

Now we will solve this system algebraically. We rewrite the second equation to $x = 2y - 1$ and substitute it into the first equation:

$$\begin{aligned}3(2y-1)+5y&=2\\6y-3+5y&=2\\11y&=5\\y&=\frac{5}{11}\end{aligned}$$

We substitute this value for y into $x = 2y - 1$ and get $x = -\dfrac{1}{11}$. Our solution is $(-\frac{1}{11}, \frac{5}{11})$. Did you get that exact point from your graph? Likely not. If you graphed carefully you probably got a point in the second quadrant, but it is hard to be precise.

In the Example 11.1.1 system that you graphed it was easy to verify that the intersection points matched the solution set. In fact, had you done the graph first, found the intersection points and tested them in the equations (just to make sure) you could have solved your system. The given system of lines was more difficult because the intersection did not have nice integer values. It is hard to "eyeball" numbers like $\frac{5}{11}$ and $-\frac{1}{11}$. But even these numbers are rational. Can you imagine how difficult it would be if you got a solution point that algebraically worked out to be $(\pi, \frac{\sqrt{5}}{7})$? So, although the theory is solid—intersection points of graphs *are* the solutions—in general graphing is not a practical way to solve these problems.

Graphing is not totally useless however. Graphing the equations of a system on the same set of axes can tell us how many solutions there are and can give us a general idea of where to look for them. This technique is very useful in calculus where being close to a solution is the first goal in some algorithmic processes.

Let's consider the system: $\begin{cases} y = \sin(2x) & \textit{where } x \in [0, 2\pi] \\ y = \cos x & \textit{where } x \in [0, 2\pi] \end{cases}$.

Carefully graph each of the equations over the interval $[0, 2\pi]$.

What intersections do you see? If your graph is good, you should see that the curves cross each other at $(\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, 0)$. You can easily algebraically verify that these are solutions to the system. What other intersections are there? You should have one on x interval $(0, \frac{\pi}{2})$ and one on x interval $(\frac{\pi}{2}, \pi)$. You can certainly make some close guesses as to what these points are. It is left as an exercise to find their exact values.

Linear Systems

Suppose we have a system of two linear equations in x and y. How many solutions can this system have? If we think about the graphs of two lines in a plane we can get our answer. The lines might intersect at one point (one solution to the system) or the lines might be parallel and not intersect at all (no solutions to the system) or the lines could be completely co-linear (infinitely many solutions to the system). These are our only choices. We will see what kinds of systems cause these particular results as we solve systems of two linear equations.

Example 11.1.2.

Find the solution to the system: $\begin{cases} 5x - y = -1 \\ -x + y = -5 \end{cases}$

We rewrite the second equation to $y = x - 5$ and substitute it into the first equation:

$$5x - (x - 5) = -1$$
$$4x + 5 = -1$$
$$4x = -6$$
$$x = -\frac{6}{4} = -\frac{3}{2}$$

We substitute this value for x into the second equation:

$$-x + y = -5 \Longrightarrow -(-\frac{3}{2}) + y = -5 \Longrightarrow \frac{3}{2} + y = -\frac{10}{2} \Longrightarrow y = -\frac{13}{2}$$

Once we find a constant value for one variable, it does not matter which system equation is used to find the value of the second variable. We should get the same answer. A good stategy is to use one equation to *find* the value and the other equation to *check* the value. We check our answer for Example 11.1.2 in the other equation:

$$5x - y = 5(-\frac{3}{2}) - (-\frac{13}{2}) = -\frac{15}{2} + \frac{13}{2} = -\frac{2}{2} = -1. \quad \checkmark$$

Two equations are the same if they have exactly the same solution set. For example, the equation $5x - y = -1$ is the same as $5x - y + 3 = 2$. We have simply added 3 to both sides of the equation. These equations are also equal to $3x - y = -1 - 2x$. Here we have subtracted $2x$ from both sides of $5x - y = -1$ [or subtracted $(2x + 3)$ from both sides of $5x - y + 3 = 2$]. If we add (subtract) the same thing to (from) both sides of an equation we don't change the solution set for that equation. We can also write equivalent equations by multiplying or dividing both sides of an equation by the same thing (except zero, of course). $5x - y = -1$ is the same as (has the same solution set as) $10x - 2y = -2$. We use this idea to help us solve systems of equations. Let's return to the system in Example 11.1.2: $\begin{cases} 5x - y = -1 \\ -x + y = -5 \end{cases}$

The solution to a system requires that both equations be true at the same time. We start with the first equation, $5x - y = -1$ and we rewrite it by adding the same thing to both sides. Since we know that in this system $-x + y$ is the same as -5, if we add the $-x + y$ to the left of our original equation and add -5 to the right of our original equation, then we haven't changed the solution set of the original equation: $5x - y = -1 \Longrightarrow (5x - y) + (-x + y) = -1 + -5 \Longrightarrow 4x = -6 \Longrightarrow x = -\frac{3}{2}$. So, the statement $x = -\frac{3}{2}$ is the direct result of assuming the simultaneous truth of the two equations in our system. We can now go back to either of the original statements in the system to get the y-value, $-\frac{13}{2}$. In doing so, we are really doing a series of the same operations on both sides of the equations. Ultimately, the system $\begin{cases} 5x - y = -1 \\ -x + y = -5 \end{cases}$ is equal to the system $\begin{cases} x = -\frac{3}{2} \\ y = -\frac{13}{2} \end{cases}$. Both have the same solution set.

The method just used to solve the system above is called the Method of Elimination. It is usually easier to work with when presented in a vertical orientation: $\begin{array}{rcrcr} 5x & - & y & = & -1 \\ -x & + & y & = & -5 \\ \hline 4x & & & = & -6 \end{array}$

We start with the first equation. Since $-x + y$ is equal to -5 we are adding the same thing to both sides of the first equation. We do our addition vertically and get the resulting equation $4x = -6$.

We then finish the problem as above.

You can probably now see why this is called the Method of Elimination.

We were fortunate in the above problem that our y terms were eliminated by simple addition. Life isn't always this easy. However, we can still use this method if instead of using both system equations as written we use an equivalent equation for one or both of the originals. We do not change the system solution if we replace one (or both) of the equations with equivalent equations. An equivalent equation can be made by multiplying both sides of an original equation by the same non-zero real number.

Suppose we wish to use the Method of Elimination on the system: $\begin{cases} 3x+4y=-8 \\ -x-3y=-4 \end{cases}$.

We can easily see that neither the x terms nor the y terms will be eliminated by simple vertical addition.

We can fix this difficulty by replacing the bottom equation with an equivalent equation gotten by multiplying both sides of the bottom equation by 3. We then add and eliminate the x term.

$$\begin{array}{rcrcr} 3x & + & 4y & = & -8 \\ -3x & - & 9y & = & -12 \\ \hline & - & 5y & = & -20 \end{array}$$

This tells us that $y = 4$. We substitute $y = 4$ into the first original equation to get $3x + 4(4) = -8 \implies 3x = -24 \implies x = -8$. We check the point $(-8, 4)$ in the second system equation, $-(-8) - 3(4) = 8 - 12 = -4$ $\checkmark$ Thus the only solution to the original system of linear equations is the point $(-8, 4)$. This tells us that we have two lines that intersect only once.

We could have chosen to solve the above system by eliminating the y term instead of the x term. In this case we avoid introducing fractions by rewriting BOTH of the original equations. We multiply the top one by 3 and the bottom one by 4. Look at the work below and figure out why we chose those particular values.

$$\begin{cases} 3x+4y=-8 \\ -x-3y=-4 \end{cases} \implies \begin{array}{rcrcr} 9x & + & 12y & = & -24 \\ -4x & - & 12y & = & -16 \\ \hline 5x & & & = & -40 \end{array}$$

This last equation gives us $x = -8$. We can find $y = 4$ and check our answer as per usual.

Example 11.1.3.

Solve the following system three ways. Use the method of Substitution. Use the Method of Elimination twice, once eliminating the x term and once eliminating the y term.

$$\begin{cases} 3x-2y=4 \\ -5x+4y=2 \end{cases}$$

Substitution Method:

$3x - 2y = 4 \implies -2y = 4 - 3x \implies y = -2 + \frac{3}{2}x$.

We substitute this expression for y into the second equation:

$-5x + 4y = 2 \implies -5x + 4(-2 + \frac{3}{2}x) = 2 \implies -5x - 8 + 6x = 2 \implies x = 10$. *We substitute* $x = 10$ *in the first equation to find* y: $3(10) - 2y = 4 \implies 30 - 2y = 4 \implies -2y = -26 \implies y = 13$. *We check the point* $(10, 13)$ *in the second equation:* $-5(10) + 4(13) = -50 + 52 = 2$ $\checkmark$

Method of Elimination—eliminating x:
We multiply the first equation by 5 and the second equation by 3:

$$\begin{cases} 3x-2y=4 \\ -5x+4y=2 \end{cases} \Longrightarrow \begin{array}{rcrcr} 15x & - & 10y & = & 20 \\ -15x & + & 12y & = & 6 \\ \hline & & 2y & = & 26 \end{array}$$

This last line tells us that $y=13$. We substitute into the first equation: $3x-2(13)=4 \Longrightarrow 3x-26=4 \Longrightarrow 3x=30 \Longrightarrow x=10$. It isn't necessary to check this result since we did it above for the Substitution Method.

Method of Elimination—eliminating y:
We only need to multiply one equation. (Why?) We multiply the first equation by 2:

$$\begin{cases} 3x-2y=4 \\ -5x+4y=2 \end{cases} \Longrightarrow \begin{array}{rcrcr} 6x & - & 4y & = & 8 \\ -5x & + & 4y & = & 2 \\ \hline x & & & = & 10 \end{array}$$

We substitute into the first equation: $3(10)-2y=4 \Longrightarrow 30-2y=4 \Longrightarrow -2y=-26 \Longrightarrow y=13.$

Consider the system of equations $\begin{cases} 4x-3y=1 \\ -8x+6y=-2 \end{cases}$

We attempt to solve using the Method of Elimination. If we multiply the top equation by 2 we get:

$$\begin{array}{rcrcr} 8x & - & 6y & = & 2 \\ -8x & + & 6y & = & -2 \\ \hline & & 0 & = & 0 \end{array}$$

We were certainly successful at eliminating things! If we look more closely at the two equations in the system we see that they are equivlent statements. If we multiply the top one by -2 we actually get the bottom one. The two equations are just different algebraic representations for the same line. So the solution sets for both equations are exactly the same. The solution set for the system, then is that same set too, the set of all order pairs that comprise the line. We say that the two equations in the system are co-linear. We will get this result when our attempts to solve the system result in a statement that is always true, such as $0=0$.

Let's make a slight change to the previous system and consider the system $\begin{cases} 4x-3y=1 \\ -8x+6y=3 \end{cases}$

Again we use the Method of Elimination. We multiply the top equation by 2 and get:

$$\begin{array}{rcrcr} 8x & - & 6y & = & 2 \\ -8x & + & 6y & = & 3 \\ \hline & & 0 & = & 5 \end{array}$$

It is no surprise that we again eliminated both of our variables. But our result is different. We ended up with the statement $0=5$. This is never true. This tells us that our two equations cannot both be true at the same time. The lines have no solutions in common. The lines do not intersect. It must be that the lines are parallel. What do we know about parallel lines? They have the same slope, but not the same y-intercept. If we rewrite the two equations into slope-intercept form our system looks like $\begin{cases} y=\frac{4}{3}x-\frac{1}{3} \\ y=\frac{4}{3}x+\frac{1}{2} \end{cases}$

In this form it is clear that both lines have slope $\frac{4}{3}$ but different y-intercepts. When we attempt to solve a system of parallel lines we will arrive at a statement that is always false, such as $0=5$.

Important Idea 11.1.1.

When we algebraically try to solve a system of equations and get a resulting statement that is always true, then the equations in the system are identical. The solution set for the system is the solution set for any one of the equations in the system. In the case of a system of linear equations, *the equations in the system are co-linear.*

When we algebraically try to solve a system of equations and get a resulting statement that is always false, then the equations in the system have no solution. There is no common intersection among the graphs of the equations in the system. In the case of a system of linear equations, the equations in the system represent parallel lines.

11.2 Systems of Inequalities

Graphing Inequalities

Suppose we want to represent the solution set for $y \geq 3$ on a set of Cartesian coordinates. We know how to graph the equation $y = 3$. It is a horizontal line with y-intercept 3. These points should be included in the solution set for $y \geq 3$. What other points are in the set? Certainly $(2, 4)$, $(-7, 9)$, and $(0, \pi)$ are points that satisfy the inequality $y \geq 3$. In fact any point in the plane that lies above the horizontal line $y = 3$ must be included. By similar reasoning, any point below the the line $y = 3$ should not be included. To graph our inequality then we draw the horizontal line $y = 3$ and then shade in the entire region above the line. The shaded region represents the solution set.

Now we want to graph $x < 2$ on a set of Cartesian coordinates. Again, we know how to graph $x = 2$. This is a vertical line with x-intercept 2. This time we do not want to include the points ON the line because we were given a strict inequality which doesn't include x being equal to 2. We do want to include all of the points in the plane that lie to the left of the line $x = 2$ and none of the points that lie to the right. We show this by drawing the boundary $x = 2$ as a dotted line and then shading the region to the left of this line. The idea of using a dotted line to indicate a boundary whose points are not included is similar to the idea of graphing on a number line where we use an open circle to indicate a boundary point that is not included in the desired set.

We can have more interesting graphs than just those bounded by horizontal or vertical lines. Let's show on a graph the solutions to the inequality $y > x + 2$. We know how to draw the line $y = x + 2$. This line is the boundary for our graph. Since we have a strict inequality we draw this line as a dotted line. We have to decide which side of the boundary includes our solution points. Since the inequality says that the y values are GREATER THAN the $x + 2$ values we want to shade above the line. It is always a good idea to choose a test point in the region you believe you want to shade, just to make sure. This is not an insult....it is easy to make a sign mistake with the algebra of inequalities. A point that is definitely above the line is $(0, 100)$. Do we get a true statement when we substitute $x = 0$ and $y = 100$ into the original inequality? Yes. One point that is very often useful to use as a test point is the origin, the point $(0, 0)$. If we substitute the point $(0, 0)$ into the original inequality we get a false statement. This means that the region that includes the test point $(0, 0)$ should NOT be shaded as part of the solution. We can feel pretty good about the accuracy of our graph now.

Example 11.2.1.
Graph the solution set for $x - 2y \leq 6$.

First we rewrite the inequality so that it will be in a format that is easier to graph.

$$x - 2y \leq 6 \Longrightarrow -2y \leq -x + 6 \Longrightarrow y \geq \frac{1}{2}x - 3.$$

Notice that we had to change the inequality sign when we divided both sides by -2 *in the last step.*

We now draw the line $y = \frac{1}{2}x - 3$. *We make it a solid line to indicate that these values are included in the solution set. From the statement* $y \geq \frac{1}{2}x - 3$ *we understand that we want the points whose* y *values are greater than or equal to* $\frac{1}{2}x - 3$, *so we shade above the line. If we test the point* $(0,0)$ *in the original problem we get a false statement. This confirms that the region containing the point* $(0,0)$ *is not the desired region.*

In the last example we had to change the inequality sign as part of our algebra. It is important that when we check our shaded region with a test point that we use the *original* equation for testing. Otherwise we could be making a test in an inequality where we might have made an algebraic error.

Example 11.2.2.
Sketch the graph of $x^2 + y^2 < 4$.

We know that the graph of $x^2 + y^2 = 4$ *is a circle of radius* 2 *with center at the origin. This is our boundary. We draw a dotted line circle because we are given a strict inequality. We want the sum of the squares* $x^2 + y^2$ *to be less than* 4 *so we shade the interior of the circle. Using the test point* $(0,0)$ *confirms this.*

We can use test points to help us determine or verify which region of the plane needs to be shaded. But it would be good to not rely soley on this. THINK about what the inequality is saying and use reason rather than ritual to figure out the shading. In the last example we can see that we are adding two positive numbers, x^2 and y^2. If we want to keep their sum LESS THAN 4, then we can't use the outside of the circle for shading. The outside of the circle contains some very big values for x and y. The inequality that describes shading the outside of the circle is $x^2 + y^2 > 4$.

Example 11.2.3.
Sketch the graph of $y \leq x^2 - 1$.

We can easily sketch the graph of $y = x^2 - 1$. *It is the standard parabola* $y = x^2$ *shifted down one unit. We draw this with a solid line because we do not have a strict inequality. We apply reason: We want in our set the points whose* y *values are LESS THAN OR EQUAL TO those of the parabola. So we shade below the parabola line. We shade everywhere that is not the "interior" of the parabola. If we test* $(0,0)$ *we get a false statement, which helps verify our choice.*

Systems of Inequalities

We know that a system of equations is a pair (or more) of equations and that the solution to a system of equations is the set of all ordered pairs (x, y) that are solutions to *all* of the equations in the system. We have a similar definition for a system of inequalities.

Given the system of inequalities $\begin{cases} y > 3 \\ x \leq 2 \end{cases}$ the solution of the system is the set of all points (x, y) that are in the solution set of both $y > 3$ and $x \leq 2$. We can draw the solution sets of each inequality and then look at their intersection. We saw at the beginning of the section that the graph for $y > 3$ consists of a dotted horizontal line through $(0,3)$ and shading above the line. We saw that the solution for $y \leq 2$ is a solid vertical line through $(2,0)$ with shading to the left of that line. The intersection of these two graphs is a shaded area bounded below by the dotted line $y = 3$ and bounded on the right by the vertical solid line $x = 2$. The intersection of the two boundary lines is the point $(2,3)$. We do not include the point $(2,3)$ in the solution set. (Why?)

The intersections of boundaries for the graph of a system of inequalites are called *vertices.* It is important to specifically state the vertices of the solution so that the boundaries are unambiguous. State whether or not the vertices are part of the solution set. This makes it very clear what is in and what is not in the solution set. There are many applications in economics and optimization theory where the identification of these vertices is very important.

It will be very helpful to your understanding of the next examples if you sketch the graphs as you work through them.

Example 11.2.4.

Find the solution to the system $\begin{cases} x - 2y \leq 6 \\ x > -1 \end{cases}$

We rewrite the first inequality (from Example 11.2.1) into the form $y \geq \frac{1}{2}x - 3$. *We sketch the graph of the line* $y = \frac{1}{2}x - 3$, *making it a solid line. We shade above the line.*
We sketch the graph of $x > -1$: *We draw a dotted vertical line through the point* $(-1,0)$ *and we shade to the right of this dotted line.*
The intersection of these two graphs is the solution set. It is a triangular shaped region bounded on the bottom by the solid line $y = \frac{1}{2}x - 3$ *and bounded on the left by the vertical dotted line* $x = -1$. *The vertex of the region is the intersection of the two boundary lines* $(-1, -\frac{7}{2})$. *The vertex is not included in the solution.*

Example 11.2.5.

Find the solution to the system $\begin{cases} x - 2y \leq 6 \\ x > -1 \\ y \leq 2 \end{cases}$

We notice that the first two inequalities are the same as in Example 11.2.4, so we can begin with the solution graph from that example.
We now add the graph of $y = 2$, *a horizontal line through the point* $(0,2)$. *We make this a solid line and shade below it.*
The intersection of the three solution sets (the solution of the first two now intersected with the solution set of the third) is a triangle. It is bounded on the left by dotted line $x = -1$, *on the top by solid line* $y = 2$ *and on the bottom by solid line* $y = \frac{1}{2}x - 3$. *The vertices are* $(-1,2)$ *(not in the solution set),* $(-1, -\frac{7}{2})$ *(not in the solution set) and* $(10,2)$ *(in the solution set).*

In the last example, the boundaries for the solution formed a triangle. It was clear that we had three vertices. The point $(-1,2)$ was obvious from the graph. The other two points had to be calculated by finding the intersections of the respective pairs of lines. The intersection of lines $x = -1$ and $y = \frac{1}{x} - 3$ is $(-1, -\frac{7}{2})$. The intersection of the lines $y = 2$ and $y = \frac{1}{2}x - 3$ is $(10,2)$. Finding intersections is a matter of solving each pair of equations simultaneously.

Example 11.2.6.

Find the solution to the system $\begin{cases} 3x \geq y + 5 \\ x - 3y < -1 \end{cases}$

We rewrite the two linear inequalities so that they are easier to graph:

$$3x \geq y + 5 \Longrightarrow 3x - 5 \geq y \Longrightarrow y \leq 3x - 5$$

$$x - 3y < -1 \Longrightarrow -3y < -x - 1 \Longrightarrow y > \frac{1}{3}x + \frac{1}{3}$$

*We graph the line $y = 3x - 5$ with a solid line and shade the region below (to the right) of it.
We graph the line $y = \frac{1}{3}x + \frac{1}{3}$ with a dotted line and shade the region above it.
The intersection of these two graphs is the triangular shaped region in the upper right portion of the axes. It is bounded below by the dotted line $y = \frac{1}{3}x + \frac{1}{3}$ and bounded on the left by the solid line $y = 3x - 5$. There is one vertex point, $(2, 1)$ which is not part of the solution set.*

Example 11.2.7.

Find the solution to the system $\begin{cases} y > x^2 - 2 \\ x + y > 0 \end{cases}$

*We rewrite the second equation as $y > -x$.
The graph of the first equation is the parabola $y = x^2$ shifted down two units. We graph this with a dotted line. Since we have y is greater than $x^2 - 2$ we shade the interior of the parabola.
Now we graph the line $y = -x$, using a dotted line. We shade above the line.
The intersection of these two graphs is not the little "banana-shaped" piece enclosed by the two graphs. The intersection is the interior of the parabola that is above the "banana." The solution set is bounded on the left and right by the dotted line parabola and is bounded on the bottom by the dotted line $y = -x$. There are two vertices. To find their coordinates we find where the dotted lines intersect by solving the system* $\begin{cases} y = x^2 - 2 \\ y = -x \end{cases}$ *We can use the substitution method, substituting for y:*

$$y = x^2 - 2 \Longrightarrow -x = x^2 - 2 \Longrightarrow 0 = x^2 + x - 2$$
$$\Longrightarrow 0 = (x + 2)(x - 1) \Longrightarrow x = -2 \text{ or } x = 1$$
$$x = -2 \Rightarrow y = 2 \quad \text{and} \quad x = 1 \Rightarrow y = -1$$

The two vertices are $(-2, 2)$ and $(1, -1)$. Neither is part of the solution set.

Example 11.2.8.

Find the solution to the system $\begin{cases} x^2 + y^2 < 4 \\ y > x \end{cases}$

*The graph of the equation $x^2 + y^2 = 4$ is a circle with center at the origin and radius 2. We draw that with a dotted line. Since $x^2 + y^2$ is less than 4 we shade the interior of the circle.
The graph of $y = x$ is a line through the origin. We include this dotted line on the graph and shade above it.
The intersection of these two graphs is the interior of a semicircle that sits in quadrant II and in half of each of quadrants I and III. There are two vertices where the line $y = x$ intersects the circle $x^2 + y^2 = 4$. To find the coordinates we can formally solve the system* $\begin{cases} x^2 + y^2 = 4 \\ y = x \end{cases}$ *or just recognize that since these coordinates are of the form (x, x), the points must be $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. Neither vertex is part of the solution set.*

Example 11.2.9.

Find the solution to the system $\begin{cases} x \geq y^2 - 1 \\ xy > 0 \end{cases}$

*The graph of $x = y^2 - 1$ is a horizontal parabola with vertex $(-1, 0)$, opening to the right. We graph this with a solid line. We are interested in those points whose x values are greater than $y^2 - 1$ so we shade the interior of the parabola. (If you are unsure what to shade, don't forget that you can pick a test point).
The inequality $xy > 0$ is interesting because at first it looks impossible to deal with algebraically. But, think: we have two values, x and y, whose product is positive. What does that tell us about x and y? They both must have the same sign. This is true for all points in the first and third quadrants.*

This then is the solution set for the inequality $xy > 0$. Shade all of the first and third quadrants. The intersection of the solution sets for these two inequalities has two pieces. One piece is enclosed in the third quadrant, bounded below by the solid line parabola $x = y^2 - 1$ and bounded by the dotted line x and y axes. There are three vertices for this piece. We can readily see from the graph that their coordinates are $(0,0)$, $(-1,0)$ and $(0,-1)$. None are part of the solution set. The second piece of the solution set is entirely in the first quadrant. It is bounded above by the solid line parabola $x = y^2 - 1$, bounded below by the dotted line x-axis and bounded on the left by the dotted line y-axis. The two vertices of this piece are $(0,0)$ and $(0,1)$. Neither is a part of the solution.

Example 11.2.10.

Find the solution to the system $\begin{cases} x + y \leq 3 \\ 2x - y \leq 7 \\ x + 3y > -6 \end{cases}$

We recognize that each of these inequalities is linear and rewrite the system so that it is easier to work with:

$$\begin{cases} y \leq 3 - x \\ y \geq 2x - 7 \\ y > -\frac{1}{3}x - 2 \end{cases}$$

When dealing with a system of more than two inequalties it can be helpful to label each inequality line on the graph. We will call our inequalities T, M, B (top, middle, bottom — referring to their list position in the system). We graph the line $y = 3 - x$ with a solid line, label the line T and shade below it. We graph line $y = 2x - 7$ with a solid line, label it M, and shade above (to the left of) it. We graph line $y = -\frac{1}{3}x - 2$ with a dotted line, label it B and shade above it. What is the intersection of these three graphs? It is not the enclosed triangle.

The solution to the system is the region bounded above by the solid line T, bounded below by the dotted line B and bounded on the right by the solid line M. There are two vertices. We need to find the intersections of lines T and M and of lines M and B. We do not care about the intersection of lines T and B. It is not relevant to the solution.

To get the intersection of lines T and M we solve the system $\begin{cases} y = 3 - x \\ y = 2x - 7 \end{cases}$

We get point $(\frac{10}{3}, -\frac{1}{3})$. This vertex is part of the solution set.

To get the intersection of lines M and B we solve the system $\begin{cases} y = 2x - 7 \\ y = -\frac{1}{3}x - 2 \end{cases}$

We get the vertex point $(\frac{15}{7}, -\frac{19}{7})$. This point is not part of the solution set.

11.3 Exercises

Problems for Section 11.1

Problem 1. Solve the following systems of equations.

(a) $\begin{cases} y = x + 3 \\ y = 9 - x^2 \end{cases}$ (b) $\begin{cases} xy = 4 \\ y = 4x \end{cases}$ (c) $\begin{cases} y = |\sin x| & \text{for } x \in [0, 2\pi] \\ y = |\cos x| & \text{for } x \in [0, 2\pi] \end{cases}$

(d) $\begin{cases} y = x^2 \\ y = -x^2 + 2 \\ y = 2x - 1 \end{cases}$ (e) $\begin{cases} \log x + \log y = 2 \\ \log(2x) - \log y = 1 \end{cases}$ (f) $\begin{cases} y = \sin(2x) & \text{for } x \in [0, 2\pi] \\ y = \cos x & \text{for } x \in [0, 2\pi] \end{cases}$

Problem 2. Graph the equations in the following systems to determine how many solutions there are and to get an estimate of what they are. Then algebraically solve the system to verify your graphing results.

(a) $\begin{cases} x^2+y^2=4 \\ y=-x+2 \end{cases}$ (b) $\begin{cases} x^2-y^2=4 \\ 3x-2y=0 \end{cases}$ (c) $\begin{cases} y=\sin(2x) & \text{for } x\in[0,\pi] \\ y=\sin x & \text{for } x\in[0,\pi] \end{cases}$

Problem 3. Graph the equations in the following systems to determine how many solutions there are. You do not need to find actual solution values.

(a) $\begin{cases} y=\frac{1}{\pi}x \\ y=\lvert\cos x\rvert \end{cases}$ (b) $\begin{cases} y=\lvert x\rvert \\ y=e^x \end{cases}$

Problem 4. Use the method of substitution to solve the following systems of linear equations. Indicate if you found parallel lines or co-linear equations.

(a) $\begin{cases} 3x-y=-3 \\ 3y+19=-5x \end{cases}$ (b) $\begin{cases} 2x-5y=-7 \\ 4y-2x=10 \end{cases}$

Problem 5. Use the method of elimination to solve the following systems of linear equations. Indicate if you found parallel lines or co-linear equations.

(a) $\begin{cases} x+2y=6 \\ 2x-y=7 \end{cases}$ (b) $\begin{cases} 3x-2y=-19 \\ x+4y=-4 \end{cases}$

(c) $\begin{cases} 3x-2y=12 \\ 5x+4=2y \end{cases}$ (d) $\begin{cases} 3x+2y=-5 \\ 4x+3y=1 \end{cases}$

Problem 6. Use any method you wish to solve the following systems of linear equations. Indicate if you found parallel lines or co-linear equations.

(a) $\begin{cases} -8x+y=-2 \\ 4x-3y=1 \end{cases}$ (b) $\begin{cases} 3y=2-x \\ 2x+6y=-3 \end{cases}$ (c) $\begin{cases} y=3x-3 \\ 6x=8+3y \end{cases}$

(d) $\begin{cases} \frac{2}{3}x+\frac{3}{4}y=\frac{5}{6} \\ \frac{4}{3}x-\frac{1}{4}y=\frac{1}{2} \end{cases}$ (e) $\begin{cases} \frac{1}{2}x-y=-3 \\ -x+2y=6 \end{cases}$

Problems for Section 11.2

Problem 1. Graph the following inequalities on Cartesian coordinate axes.

(a) $y>4$ (b) $x\le -\frac{1}{2}$ (c) $y<2x-5$ (d) $3x-y\le 5$

(e) $x\ge y^2$ (f) $\dfrac{x^2}{4}+\dfrac{y^2}{9}\ge 1$ (g) $\lvert x\rvert>3$

Problem 2. Graph the solution to each system of inequalities. Find the coordinates of all vertices of your solution and tell whether they are included in the solution.

(a) $\begin{cases} y < 4 \\ y > x \end{cases}$ (b) $\begin{cases} y > x + 2 \\ y < \frac{1}{2}x + 2 \end{cases}$ (c) $\begin{cases} |y| < 1 \\ |x| < 2 \end{cases}$

(d) $\begin{cases} x + y > 2 \\ x < 3 \\ y < 5 \end{cases}$ (e) $\begin{cases} y < x + 3 \\ -2y < 3x - 6 \\ y > \frac{1}{2}x - 1 \end{cases}$ (f) $\begin{cases} y < -x^2 + 3 \\ y > x^2 - 1 \end{cases}$

(g) $\begin{cases} y < x^2 \\ y \geq -x^2 \end{cases}$ (h) $\begin{cases} y < -x^2 + 4 \\ y \leq 2x + 1 \end{cases}$ (i) $\begin{cases} y \geq x^2 \\ x \geq y^2 \end{cases}$

(j) $\begin{cases} y > x^2 - 1 \\ xy \leq 0 \end{cases}$ (k) $\begin{cases} y < \sin x \\ y > -\sin x \end{cases}$ (l) $\begin{cases} y < e^x \\ y > \ln x \end{cases}$

(m) $\begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{9} < 1 \\ \dfrac{x^2}{9} + \dfrac{y^2}{4} > 1 \end{cases}$

Problem 3. On a set of coordinate axes, draw a circle with center at the origin and radius 3. On the same set of axes, draw a dotted-line circle with center at the origin and radius 1. Shade the donut shaped region between the two graphs. Write a system of inequalities to algebraically describe your sketch.

Problem 4. On a set of coordinate axes, plot the points $(0,3)$, $(6,0)$ and $(-2,-1)$. Connect the points with dotted line segments to form a dotted triangle. Shade the interior of the triangle. Write a system of inequalities that describes your shaded region.

11.4 Answers to Exercises

Answers for Section 11.1 Exercises

Answer to Problem 1.

(a) $(-3,0)$, $(2,5)$ (b) $(1,4)$, $(-1,-4)$ (c) $(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$, $(\frac{3\pi}{4}, \frac{\sqrt{2}}{2})$, $(\frac{5\pi}{4}, \frac{\sqrt{2}}{2})$, $(\frac{7\pi}{4}, \frac{\sqrt{2}}{2})$

(d) $(1,1)$ (e) $(10\sqrt{5}, 2\sqrt{5})$ (f) $(\frac{\pi}{6}, \frac{\sqrt{3}}{2})$, $(\frac{\pi}{2}, 0)$, $(\frac{5\pi}{6}, -\frac{\sqrt{3}}{2})$, $(\frac{3\pi}{2}, 0)$

Answer to Problem 2.
(a) $(0,2)$, $(2,0)$ (b) no solution (c) $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$, $(0,0)$, $(\pi, 0)$

Answer to Problem 3.
(a) three solutions (b) one solution

Answer to Problem 4.
(a) $(-2,-3)$ (b) $(-11,-3)$