$y = a^x$ $y = \log_a x$

Chapter 10

Exponential and Logarithmic Functions

In this chapter we look at two new kinds of functions, the exponential function and the logarithmic function. We begin with the exponential function.

10.1 Exponential Functions

10.1.1 Definition

In an exponential function, the independent variable is in the exponent.

Definition 10.1.1. *An exponential function is a function in the form $f(x) = a^x$ where a is a constant real number and $a > 0$ and $a \neq 1$. The number a is called the base of the function.*

The domain for this function is $\Re$, the set of all real numbers.

So, $f(x) = 2^x$ and $g(x) = (\frac{3}{7})^x$ and $h(x) = (\sqrt{5})^x$ are all examples of exponential functions.

Why do you suppose that we do not allow a to have the value 1? Consider what some of the values would be for $f(x) = 1^x$: $f(2) = 1^2 = 1$, and $f(\frac{1}{2}) = 1^{\frac{1}{2}} = 1$, and $f(0) = 1^0 = 1$. Indeed, for any value of x, $f(x) = 1$. This then is a linear function, the function $f(x) = 1$, so we do not include it in the class of exponential functions.

Why do you suppose that we do not allow a to be a negative number? One obvious reason is that we would have trouble with some x values less than 1. If $a = -4$ and $x = \frac{1}{2}$ we would not be able to evaluate $a^x = (-4)^{\frac{1}{2}} = \sqrt{-4}$. Certainly $\sqrt{-4}$ is undefined in $\Re$. So, we keep our life simple and only allow a to be a positive number, but not equal to 1.

10.1.2 Characteristics and Graphs

A Close Look at $f(x) = 2^x$

Let's look in detail at the particular exponential function $f(x) = 2^x$. We'll start by looking at some specific values of the function:

- Some positive integer values for x: $f(1) = 2^1 = 2$; $f(2) = 2^2 = 4$; $f(3) = 2^3 = 8$; $f(4) = 16$; $f(10) = 1,024$; $f(20) = 1,048,576$; $f(50)$ is a really big number with 16 digits. For $x > 10$ the use of a calculator is forgivable.
- We certainly don't want to forget our y-intercept, where $x = 0$: $f(0) = 2^0 = 1$.
- And now some negative integer values for x: $f(-1) = 2^{-1} = \frac{1}{2}$; $f(-2) = 2^{-2} = \frac{1}{4}$. Indeed we recall that $a^{-p} = \frac{1}{a^p}$ so we can use our work from above to easily get: $f(-3) = \frac{1}{f(3)} = \frac{1}{8}$; $f(-4) = \frac{1}{16}$; $f(-10) = \frac{1}{1,024}$; $f(-20) = \frac{1}{1,048,576}$; and $f(-50)$ is a fraction with a 1 in the numerator and a really big number with 16 digits in the denominator.
- Now let's look at some non-integer values of x: $f(\frac{1}{2}) = 2^{\frac{1}{2}} = \sqrt{2} \approx 1.414$. Similarly, $f(\frac{1}{3}) = 2^{\frac{1}{3}} = \sqrt[3]{2} \approx 1.260$; $f(\frac{1}{4}) = \sqrt[4]{2} \approx 1.189$; $f(\frac{1}{10}) = \sqrt[10]{2} \approx 1.072$. We use a calculator here to get decimal approximations because we will use them in our study below.
- We use the same "reciprocal trick" (and a calculator) to get function values for x's that are negative non-integer values: $f(-\frac{1}{2}) = \frac{1}{\sqrt{2}} \approx 0.707$; $f(-\frac{1}{3}) = \frac{1}{\sqrt[3]{2}} \approx 0.794$; $f(-\frac{1}{4}) \approx 0.841$; $f(-\frac{1}{10}) \approx 0.933$.

We now summarize in a list all of the data we have determined so far.

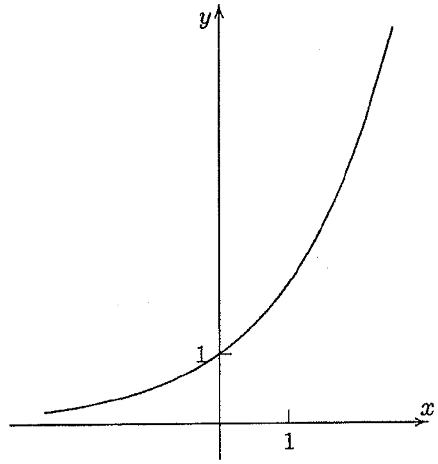
x	$f(x) = 2^x$	x	$f(x) = 2^x$
-20	$\frac{1}{1,048,576}$	$\frac{1}{10}$	1.072
-10	$\frac{1}{1,024}$	$\frac{1}{4}$	1.189
-4	$\frac{1}{16}$	$\frac{1}{3}$	1.260
-3	$\frac{1}{8}$	$\frac{1}{2}$	1.414
-2	$\frac{1}{4}$	1	2
-1	$\frac{1}{2}$	2	4
$-\frac{1}{2}$	0.707	3	8
$-\frac{1}{3}$	0.794	4	16
$-\frac{1}{4}$	0.841	10	1,024
$-\frac{1}{10}$	0.933	20	1,048,576
0	1		

Look carefully at this list of function values. We will make some observations. You should decide if you think that the observation is just peculiar to the specific data chosen, or if it will hold for the whole function $f(x) = 2^x$ or if this observation might hold for all exponential functions.

- First we observe that all of the $f(x)$ values are positive. Is there a value of x that would make 2^x negative? or make 2^x zero? Can you think of a value for a and a value for x so that a^x is negative? or zero? Remember that $a > 0$ and $a \neq 1$.
- Next we observe that the $f(x)$ values are increasing as x increases. When x is the smallest number in the list, $f(x)$ is the smallest number. When x is the largest number, $f(x)$ is the largest number. Do you think this pattern holds for x values not on the list? Is it reasonable that the value of 2^x should increase if x increases? Do you think this holds for a^x?
- We see that $f(0) = 1$. Is this true for all values of a? Is it true that $a^0 = 1$ for all values of a where $a > 0$ and $a \neq 1$?

- Finally, look at the pattern of values as x gets very large or x gets very small. What do you suppose happens to $f(x)$? We have already seen that $f(x) = 2^x$ is increasing, but is it doing so without bound or can we expect its graph to have a horizontal asymptote as $x \to \infty$? What about as $x \to -\infty$? Can you make a general statement about a^x?

We can use (some of) the data above to draw a sketch of the graph of $f(x) = 2^x$ (below). It has been stated earlier that graphing a function is more than just plotting a few points and then "playing dot-to-dot." However, it does turn out this time that the graph for this function is indeed a smooth curve that behaves nicely (it is all connected; doesn't jump around or have "holes" in it).



$y = 2^x$

Irrational Exponents

We have said that the domain for $f(x) = 2^x$ is $\Re$. Yet all of the data points that we plotted were rational x values. We had a variety of values: positive, negative, integer, non-integer. From our study of exponents in Chapter 1 we know how to deal with these. But, how do we deal with x values that are irrational, such as $x = \sqrt{3}$ or $x = \pi$? What does 2^π mean? It does not mean 2 multiplied times itself π times. It does not mean the "πth" root of 2. We can use the graph of $f(x) = 2^x$ to get a value for 2^π. 2^π is simply the y-value on the graph when the x value is π.

You might think that this is a "cheating" way to answer the question, "What do we mean by 2^π?" But, at least it is a reasonable answer. Since the graph of $f(x) = 2^x$ is smooth and increasing, and since $3.1 < \pi < 3.2$, we would want the value of 2^π to be somewhere between the values of $2^{3.1}$ and $2^{3.2}$. We know how to interpret $2^{3.1}$ and $2^{3.2}$ because these exponents are rational. Since $3.1 = \frac{31}{10}$ and $3.2 = \frac{32}{10}$ we understand $2^{3.1} = 2^{\frac{31}{10}} = \sqrt[10]{2^{31}} \approx 8.574$ and $2^{3.2} = 2^{\frac{32}{10}} = \sqrt[10]{2^{32}} \approx 9.198$. Our smooth, connected graph tells us that there IS a value for 2^π and we have figured that it must be between 8.574 and 9.198. We can get closer to the actual value of 2^π by simply choosing rational numbers closer to π than are 3.1 and 3.2. With calculus we have a way of squeezing the interval so closely around π that we say we can know the actual value of 2^π.

So, what is the point of all of this? Certainly from a practical standpoint we will simply use a calculator to get $2^\pi \approx 8.825$. But what we have here is a way to interpret an irrational exponent. The smooth connectedness of the graph of $f(x) = a^x$ gives us a way to understand the values of numbers in the form a^x where the exponent is irrational. We can consider all irrational exponents in the same way that we did here with π.

Exponential Functions with base $a > 1$

Now we will look at some exponential functions besides just $f(x) = 2^x$. Complete the following table and use it to sketch the graphs of $f(x) = 2^x$, $g(x) = 3^x$ and $h(x) = 4^x$ on the same set of axes.

x	$f(x) = 2^x$	$g(x) = 3^x$	$h(x) = 4^x$
-3	$\frac{1}{8}$	$\frac{1}{27}$	$\frac{1}{64}$
-2	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{16}$
-1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
$-\frac{1}{2}$	0.707	$1/\sqrt{3}$	$\frac{1}{2}$
0	1	1	1
$\frac{1}{2}$	1.414	$\sqrt{3}$	2
1	2	3	4
2	4	9	16
3	8	27	64

$2^x > 3^x > 4^x \quad x < 0$

$2^x = 3^x = 4^x, \quad x = 0$

$2^x < 3^x < 4^x, \quad x > 0$

Compare the three graphs:

- When $x < 0$, which function has the highest function values? which has the lowest?
- When $x = 0$, what do you notice?
- When $x > 0$, which function has the highest function values? which has the lowest?

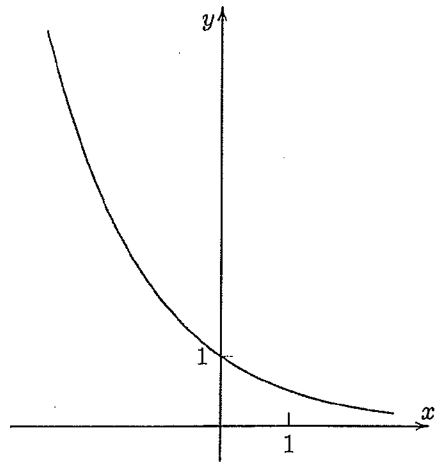
Make a general statement to describe your findings.

Use this statement to sketch in the graph of $k(x) = (\frac{7}{2})^x$ without plotting points.

Exponential Functions with base $0 < a < 1$

You will recall that the definition of exponential function allows for the base a to be any positive number except 1. So far we have dealt only with a values greater than 1. Let's consider the exponential function $f(x) = (\frac{1}{2})^x$. If we make a table of values for this function and look at its graph we see that it is a little different from the ones studied previously, although some of the numbers look quite familiar.

x	$f(x) = (\frac{1}{2})^x$	x	$f(x) = (\frac{1}{2})^x$
-20	$1,048,576$	$\frac{1}{10}$	0.933
-10	$1,024$	$\frac{1}{4}$	0.841
-4	16	$\frac{1}{3}$	0.794
-3	8	$\frac{1}{2}$	0.707
-2	4	1	$\frac{1}{2}$
-1	2	2	$\frac{1}{4}$
$-\frac{1}{2}$	1.414	3	$\frac{1}{8}$
$-\frac{1}{3}$	1.260	4	$\frac{1}{16}$
$-\frac{1}{4}$	1.189	10	$\frac{1}{1,024}$
$-\frac{1}{10}$	1.072	20	$\frac{1}{1,048,576}$
0	1		

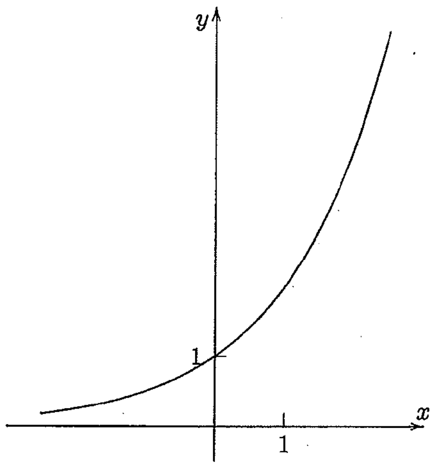


$y = (\frac{1}{2})^x$

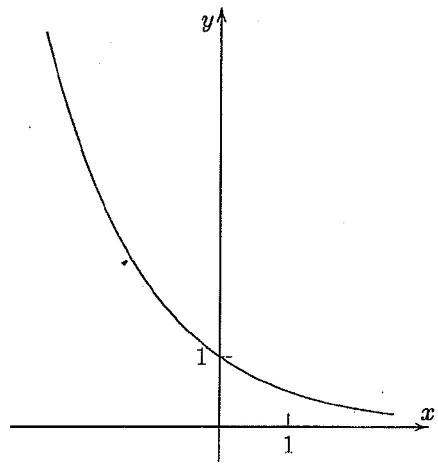
$2^x, \frac{1}{2^x}, 2^{-x}$

The values for the exponential function with base $a = \frac{1}{2}$ and the values for the exponential function with base $a = 2$ are simply reciprocals of each other. We can explain this algebraically: $(\frac{1}{2})^x = \frac{1^x}{2^x} = \frac{1}{2^x}$.

We can take this a step further: $(\frac{1}{2})^x = \frac{1}{2^x} = 2^{-x}$. In Chapter 3 we learned that for any function f, the graph of $f(x)$ and the graph of $f(-x)$ are reflections of each other about the y-axis. Look at the graphs below for $f(x) = 2^x$ and $f(-x) = 2^{-x} = (\frac{1}{2})^x$.



$y = 2^x$



$y = (\frac{1}{2})^x$

What do you suppose the graph of $f(x) = (\frac{1}{3})^x$ looks like? How will it compare to exponential graphs with bases of 3 or $\frac{1}{2}$?. Test your thoughts by sketching (you can do it now without plotting points) the graphs for 2^x and 3^x and then sketching in the graphs for $(\frac{1}{2})^x$ and $(\frac{1}{3})^x$ by using the reflections. Be careful. The only place that any graph will cross another is at the point $(0, 1)$.

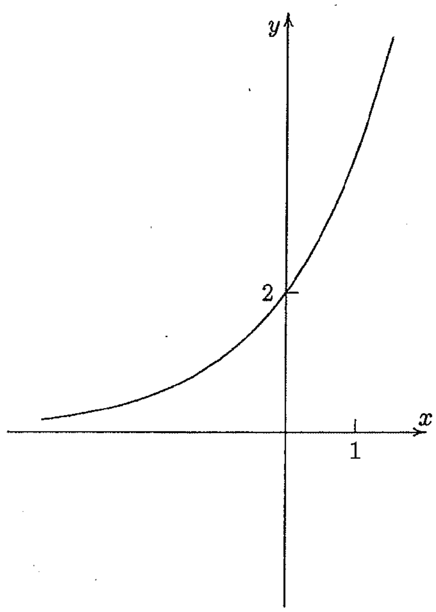
Since we now have some exponential functions whose graphs decrease, you might want to revisit the conclusions you drew concerning exponential functions. We can now state some definite facts about exponential functions.

Important Idea 10.1.1.
Functions in the form $f(x) = a^x$ where $a > 0$ and $a \neq 1$:

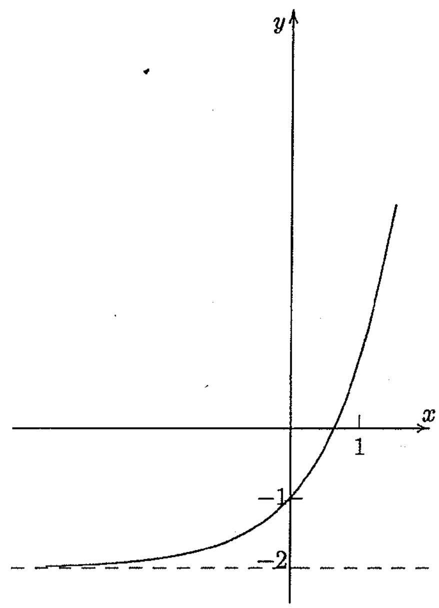
1. *have domain* $\Re$,
2. *have range* $(0, \infty)$ *(the function values are always positive),*
3. *have y-intercept* $(0, 1)$,
4. *and, when* $a > 1$, *we have:*
   (a) f *is increasing,*
   (b) *as* $x \to \infty$, $f \to \infty$, *and*
   (c) *as* $x \to -\infty$, $f \to 0$ *(there is a left horizontal asymptote* $y = 0$*),*
5. *and, when* $0 < a < 1$ *we have:*
   (a) f *is decreasing,*
   (b) *as* $x \to -\infty$, $f \to \infty$, *and*
   (c) *as* $x \to \infty$, $f \to 0$ *(there is a right horizontal asymptote* $y = 0$*).*

Variations

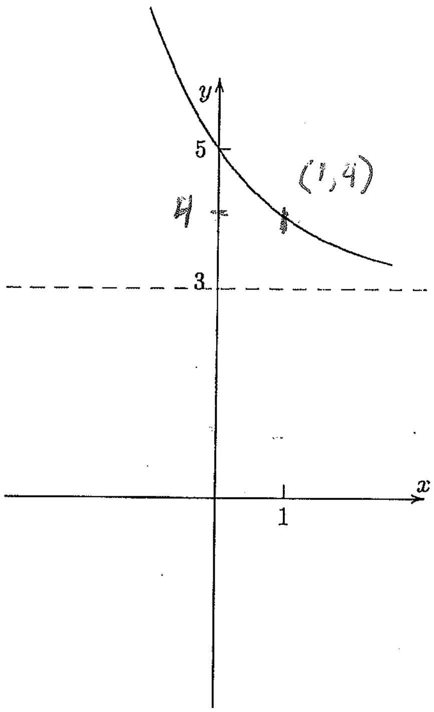
You should become familiar with the graphs of the exponential functions. Adopt them into the "family of functions" studied in chapter 4. These exponential functions can be moved and stretched by applying the same transformations that were used to operate on other functions. Below are some examples.



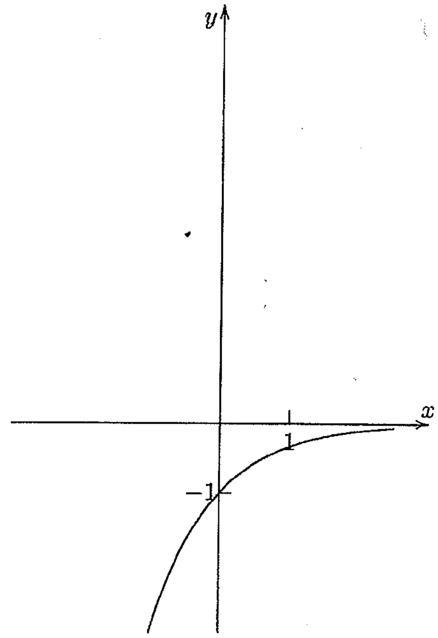
$y = 2^{x+1}$



$y = 3^x - 2$



$y = (\frac{1}{2})^{(x-1)} + 3$



$y = -(\frac{1}{3})^x$

10.1.3 The number 'e'

You are familiar with the number π. While you might not know how the actual value was arrived at you do know that it is the number that gives the ratio of the length of the arc of a semi-circle to the radius of the circle. You know that while π is irrational it can be approximated to any desired level of accuracy. In this course we are usually content with $\pi \approx 3.14$.

We now introduce another helpful irrational number. This number is called 'e.' The geometric development of e is beyond the scope of this course[1] but you will enjoy seeing its development in calculus. This number is mentioned in this course so that when you encounter it in calculus you will be comfortable with it, as you now are with π. The number e has its applications mainly with exponential and logarithmic functions.

The number $e \approx 2.71828$. Just as you think of π as being a little more than 3, you can think of e as being a little less than 3. So, now when you are asked to "pick a number between 1 and 10" you have another alternative: e. While e is a truly wonderful number, it is still a number, bound by all the same algebraic rules as any other constant, so don't let its appearance bother you.

Since e is a real number and $e > 0$ and $e \neq 1$ we can use it as the base for our exponential function.

You know what the graphs of $f(x) = 2^x$ and $f(x) = 3^x$ look like. On the blank page opposite, draw these graphs on the same set of axes. You know that $2 < e < 3$. Use this information to sketch

[1]Author's irrelevant note: I think I've read the phrase "beyond the scope of this course" in every math book I've ever read. I've always wanted to be able to write that myself.