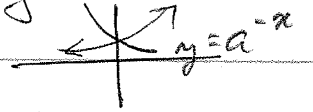
Lecture notes
10/29
10/30

Definition + Properties of logarithms (exponents)

First, review

$y = a^x$ exponential fcn



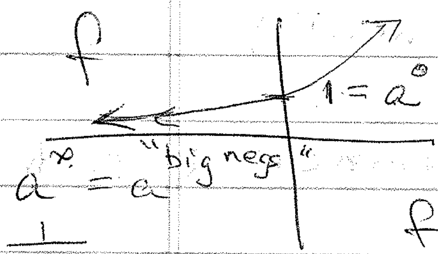
where $0 < a < 1$ & $a > 1$

It turns out that we don't need to consider too hard $0 < a < 1$ b/c if $a \in \mathbb{Z}^+$ then $\frac{1}{a}$ is between $0 + 1$. But $\left(\frac{1}{a}\right)^x = \cancel{a} \frac{1^x}{a^x}$

But $\frac{1^x}{a^x} = \frac{1}{a^x} = a^{-x}$

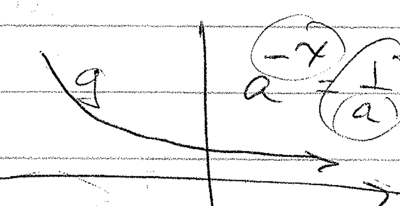
~~So $y = a^x$ is~~

So we get the full picture of exp. fcn behavior by looking at



$\cancel{y}$ $f(x) = \boxed{a^x}$

$g(x) = a^{-x}$



f reflects over y-axis: $a^x \rightarrow a^{-x}$

f + g are 1-1

Dom f, g $= \mathbb{R}$

Ran f, g : $y \cancel{\geq} 0$ $(y > 0)$

Conclusion: We only concern ourselves with bases $a > 1$ since $a^{-x} = \left(\frac{1}{a}\right)^x$

Bases we consider are like

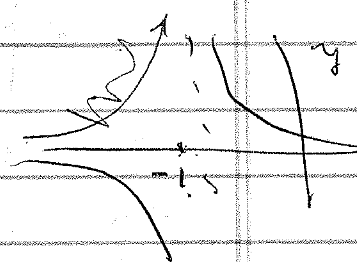
2, 3, 10, $\frac{1}{2}$, e (irrat.)

let a^{-x} do this job

Science -

Calculus

$$e = \boxed{\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n}$$



$y = \frac{1}{x+1}$

$\lim_{x \to -1^+}$

$e \approx 2.718$

$n \to \infty$

$n \in \mathbb{Z}$

$$\left(1 + \frac{1}{1}\right)^1 = 2$$

$$\left(1 + \frac{1}{2}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

$$\left(1 + \frac{1}{3}\right)^3 = \left(\frac{4}{3}\right)^3 = \frac{64}{27}$$

$\vdots$

natural base $\equiv e \approx 2\frac{3}{4}$

natural logarithm (exponent)

JZ asked how to deal with things like 7^e.

$\Leftarrow y = 7^e \equiv$ a number x such that $\log_e x = 7$

We leave 7^e as is, an irrational number that lies on the graph of $y = 7^x$ between 7^2 and 7^3.

Now, logarithmic function $y = \log_a x$.

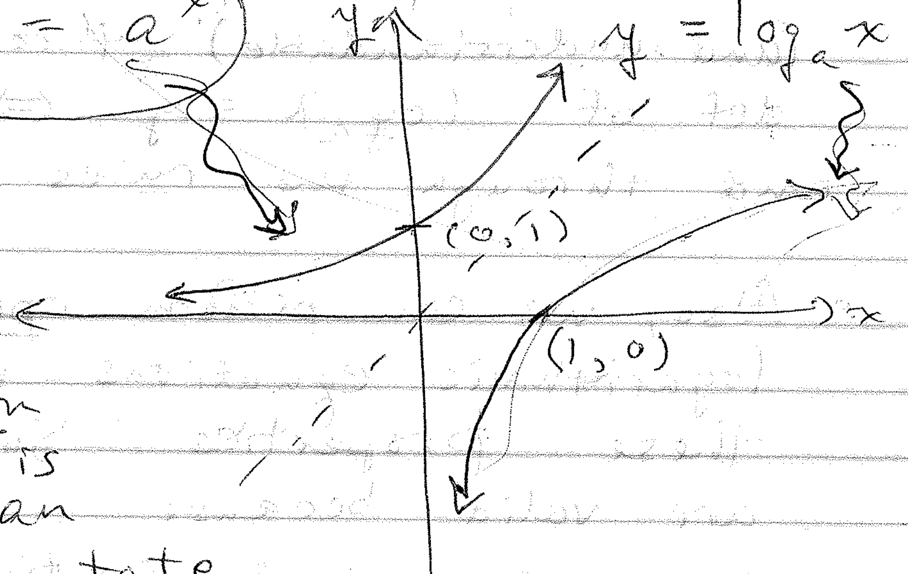
~~Recall~~ The **inverse** of $f(x) = a^x$
is $\log_a x = f^{-1}(x)$

$y = a^x$

$y = \log_a x$

$\frac{1}{3}$ $\quad \frac{1}{3}^x$

$= 3^{-x}$



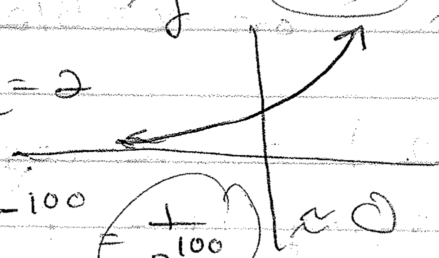
Caution
$y = x$ is
not an
asymptote

~~If~~ Asymptotes of exp + log fcns

$y = a^x$ as $x \to \infty$
$a^x \to \infty$

$a = 2$

as $x \to -\infty$
$a^x \to 0$

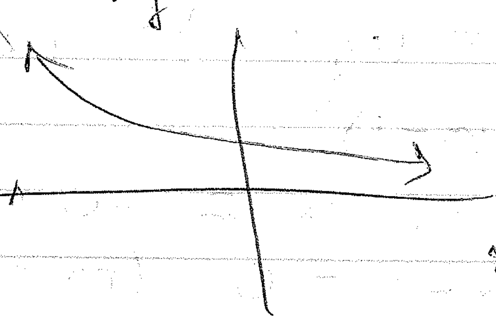


$2^{-100} = \frac{1}{2^{100}} \approx 0$

$\Rightarrow$ x-axis is the
left asymptote
of $y = a^x$

$y = a^{-x}$ as $x \to \infty$
$a^{-x} \to 0$

as $x \to -\infty$
$a^{-x} \to \infty$



x-axis is the
right asymptote

Properties of logs — p. 340

- These 7 properties are verifiable (and understandable) through the def of $\log_a x = y \iff x = a^y$ and through the rules of exponents.

- Also, we can perform operations on logarithmic equations by employing these properties. Such operations are valid because $y = \log x$ is a one-one and continuous function

For example: $2^{x+1} = 2^{3x-4}$
is solved simply as $x + 1 = 3x - 4$
or $x = 5/2$

Likewise, $\log(x+2) = 2\log(x-1)$ *
is solved by employing the property 7.

Prop. #7: $\log_a m^p = p \log_a m$

So $2\log(x-1) = \log(x-1)^2$ (right side *)
and $\log(x+2) = \log(x-1)^2$ (left side *)
$\Longrightarrow x+2 = (x-1)^2$
so $x^2 - 2x + 1 - x - 2 = 0$
$x^2 - 3x - 1 = 0$ (no soln!)

p. 340 Proving the properties of logs

#1. - 3. we understand + verify using the def of logarithm.

#4 is a little trickier; it says $\boxed{a^{\log_a m} = m}$ Really?

Proof Consider that $y = \log_a m$ means $\boxed{a^y = m}$ by def.

Then $a^y = a^{\log_a m}$ by substitution & log is 1-1 and continuous (rules of)

But by def $a^y = m$

so $a^{\log_a m} = m$ (Try memorizing this for the mental strength it builds :))

Prop #5 $\log_a mn = \log_a m + \log_a n$

"Labels"

Begin with letting $\boxed{u = \log_a m}$ and $\boxed{v = \log_a n}$

By def. of log, $\boxed{a^u = m}$ and $\boxed{a^v = n}$

We note that $\boxed{a^u \cdot a^v = a^{u+v} = m \cdot n}$.

Now, noting $\log_a a^{u+v} = u + v$ (prop #3)

Substituting on both sides our givens:

$\log_a m \cdot n = \log_a m + \log_a n$ //

#6 $\log_a \frac{m}{n} = \log_a m - \log_a n$

Begin with letting $\boxed{u = \log_a m}$ and $\boxed{v = \log_a n.}$ Again, by def, $\boxed{a^u = m}$, $\boxed{a^v = n}$.

We note that $\boxed{\frac{m}{n} = \frac{a^u}{a^v} = a^{u-v}}$

and $\log_a a^{u-v} = u - v$ (prop #3)

Substituting our givens:

$$\log_a \frac{m}{n} = \log_a m - \log_a n \quad //$$

Final property: Changing bases

Use $x = y \iff \log_a x = \log_b y$

to go from $y = \log_a x$ to $y = \dfrac{\log_b x}{\log_b a}$

$$y = \log_a x$$

$$a^y = x$$

$$\log_b a^y = \log_b x$$

$$y \log_b a = \log_b x$$

$$y = \log_b x / \log_b a \quad //$$