Chapter 3

Functions

Use combination of the 2 defs:
A fcn. consists of 2 sets X + Y + a rule (call it f) that assigns to each element $x \in X$ a unique element $y \in Y$. X is the domain of f (dom f)
Y is the range of f (ran f)

"One reason many students struggle so much in calculus is that they don't really understand functions."[1]

3.1 Function Definition and Notation

Definition of Function

There are two common perspectives on the rigorous definition of *function*. Both are valid.

Definition 3.1.1.

Definition 1: *A function consists of two sets, X and T, and a rule that assigns to each element x in the set X a unique element y in the set T. The set X is called the domain of the function. The set T is called the target space of the function. The range of the function is the set Y that consists of only those elements y in set T that are assigned to domain elements by the rule.*

In this definition we have the possibility that the range of the function, set Y, is a proper subset of the target set T. For this course we will not complicate matters by dealing with target sets. We will concern ourselves with domain X and range Y. Our work with functions can be accomplished by using Definition 2 below.

Definition 2: *A function is a set of ordered pairs (x, y) such that for each element x there is a unique element y. The set of all first coordinates (x's) is called the domain of the function. The set of all second coordinates (y's) is called the range of the function.*

Can you see that Definition 2 is equivalent to Definition 1 if $Y = T$?

A few points must be made that are critical to the understanding of just what a function is:

- An essential part of a function is the domain set. Without this set, there is no function.
- Each element in the domain is assigned to a unique (only one) range element.
- Two or more elements in the domain can be assigned to the same element in the range.

[1] Michael Putney, long-time successful mathematics teacher, Vestal Sr. High School. Author's note: When I first heard this from Mr. Putney, I thought it a rather simplistic analysis of the situation. After teaching first semester calculus over a dozen times, I am convinced that he is right.

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Suppose a set X consists of the possible scores on a ten-point quiz. $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Further, suppose that there is a rule that assigns letter grades based on the quiz score:

Good ex.

Score	$0-4$	5	$6,7$	8	$9,10$
Grade	F	D	C	B	A

This situation describes a function. The domain is the set of scores X and the range is the set of letter grades, $Y = \{A, B, C, D, F\}$. The table describes the assignment of the elements of X. Each quiz score (domain element) is assigned to exactly one letter grade (range element). It is OK that more than one domain element is assigned to the same range element (two domain elements are assigned letter grades C and A and four domain elements are assigned to range element F).

Instead of using a table and list we could view this function as a set of ordered pairs: { (0,F), (1,F), (2,F), (3,F), (4,F), (5,D), (6,C), (7,C), (8,D), (9,A), (10,A)}. Note that for every first element, there is a unique second element. It is important to know that these are *ordered* pairs. The first coordinates must be domain elements and the second coordinates must be range elements.

Suppose a set X consists of all real numbers, so $X = \Re$. Further suppose that there is a rule that takes each element of X and squares it.

This situation describes a function. The domain is $\Re$ and the range Y is the set of non-negative real numbers (why?). Each number x in $\Re$ is assigned a unique number x^2 in Y. It is OK that more than one domain element is assigned to the same range element (for example, both 2 and -2 are assigned to the range value 4). We would not want to individually list all of the ordered pairs in this function but we could express them several ways: $\{(x, y), x \in \Re : (x, y) = (x, x^2)\}$ or $\{(x, y), x \in \Re : y = x^2\}$ or simply $\{(x, x^2), x \in \Re\}$.

Suppose we had the following set of ordered pairs:
$\{(0, 1), (5, 4), (8, 6), (2, 5), (5, 6)\}$.

This is not a function. We have a domain set $X = \{0, 5, 8, 2, 6\}$. We have a range set $Y = \{1, 4, 6, 5\}$. But the domain element 5 does not have a unique assignment. It is assigned to both 4 and 6.

Function Notation

Definition 3.1.2.

The independent variable of a function is the letter (or other symbol) that is used to represent members of the domain set.

The dependent variable of a function is the letter (or other symbol) that is used to represent the members of the range set.

We will use the expression "is a function of". A translation of this is "depends on." We would say that letter grades are "a function of" quiz scores. This simply means that the letter grades "depend on" the quiz scores. We use this language to help us know what the independent variable is. It is the object of the word "of" in the phrase "a function of." In many cases it is obvious from the context of the situation as to which is the independent variable and which is the dependent variable. Sometimes it is not. The matter is made very clear when using standard function notation.

Important Idea 3.1.1.

The notation "$f(x)$" is used to denote the dependent value of a function whose name is f and whose independent variable is x.

Sometimes students find function notation confusing. It is helpful to think that the expression "$f(x)$" refers to a range value, a y value if you prefer. y is the same thing as $f(x)$. When we write $y = f(x)$ we are just giving the name y to the dependent variable. This can be useful when we think of y as the second coordinate of ordered pairs (x, y), but it is really redundant. We can just as well write our ordered pairs as $(x, f(x))$.

Students are usually introduced to y as a dependent variable early in their algebra careers and become comfortable with it. If you find function notation confusing then you can just mentally substitute y whenever you see $f(x)$. $f(x)$ *is* y. The advantage to the function notation is that you get the specific information telling you that x is the independent variable. You also then have the possibility of working with more than one function at the same time and not getting them confused. You call one f and the other g and everything is clear. With the y notation, you just call everything y.

Suppose we call the function used for quiz scores and grades g. Then we can think $g(score) = grade$. Specifically, $g(4) = F$ and $g(8) = B$. The "=" means "is." So, the value of g (that is, $g(x)$) is (=) a letter grade. In the statement $g(4) = F$, we get all of the information we need: "g" refers to the score/grade table, "4" refers to an element in the domain and "F" refers to the range value that is assigned to 4 by g.

The squaring function described above can be written as $f(x) = x^2$. We arbitrarily name the function f. The independent variable is x. The dependent variable $f(x)$ is equal to the square of the independent variable. Some values of the function are $f(3) = 3^2 = 9$, $f(-6) = (-6)^2 = 36$, $f(10) = 100$, $f(0) = 0$.

Natural Domain of a Function

It was stated earlier that the domain is an essential part of a function. Sometimes we specifically state the domain, as in the examples above with the grades and with the squaring. Sometimes the domain is not specifically stated but implied by what makes sense for the specific application. For example, if we had a function (h) that calculated taxi fare (F), dependent on the time duration(t) of the ride, we would write $h(t) = F$. By the situation we assume that t cannot be a negative number. The implied domain of h would then be $[0, \infty)$.

When our function involves an algebraic expression, like $f(x) = x^2$, but does not have a specific domain stated we assume that the domain is all values of $\Re$ for which the function would make sense. This is called the natural domain of the function. In the case of $f(x) = x^2$ the natural domain would be $\Re$ because it is possible to square any real number.

When looking for the natural domain of a function, it is often easier to look for values that cannot belong to the domain and exclude them from $\Re$, rather than to begin to list all of the values that can belong to the domain. At this point, the values that we must exclude from a domain are those which would give zeros in a denominator or which would give negative values under even roots.

We will use the notation D_f to represent the domain of f and R_f to represent the range of f.

Example 3.1.1.

Find the natural domain of $f(x) = \dfrac{1}{x^2 - x - 6}$ *and evaluate* $f(2), f(0),$ *and* $f(3)$.

p. 66

$$f(x) = \frac{1}{x^2 - x - 6} = \frac{1}{(x-3)(x+2)}.$$

alternate preferred

Since we cannot h~~ave zero values in a denominator~~, x cannot have the value 3 or the value -2. Thus the domain, D_f is $(-\infty,-2)\cup(-2,3)\cup(3,\infty)$.
$f(2) = \dfrac{1}{(2-3)(2+2)} = \frac{1}{-4}$. $f(0) = \dfrac{1}{-6}$ *$f(3)$ is undefined because 3 is not in D_f.*

Example 3.1.2.
Find the domain for $g(x) = \sqrt{4-x^2}$

We know that we cannot have a negative value under a square root. The values of x that would make $(4-x^2)<0$ are those where $x^2>4$. This would be all values of x greater than 2 or less than -2. It is alright to have the square root of zero. So, $D_g = [-2,\ 2]$.

Comprehension Check 3.1.

1. *Which of the following describe a function? If a function, state the domain.*
   (a) *The set of points* $\{(1,2),(3,4),(5,6),(7,1)\}$ ✓
   (b) *The set of points* $\{(1,1),(2,2),(3,3),(4,4)\}$ ✓
   (c) *The set of points* $\{(1,4),(1,3),(4,5),(7,5)\}$ X
   (d) $\{(-1,1),(1,1),(-2,2),(2,2),\ldots\}$
2. *S is a set of 10 students. D is a set of two dormitories. f is a table that matches students and dormitoies. Which describes a function: $f(s)=d$? $f(d)=s$? neither? both?*
3. *For the function $f(x) = \dfrac{3x-6}{x+1}$, find $f(10)$, $f(0)$, $f(-4)$. What is the domain of f?*

$y = |x|$

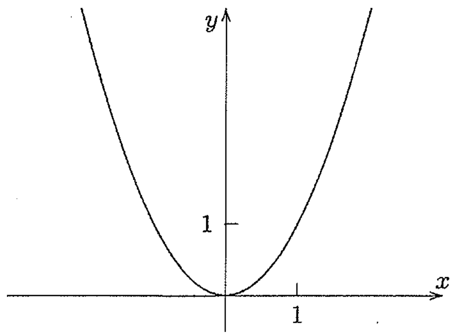
The Vertical Line Test

We have said that a function is a set of ordered pairs. We are used to drawing pictures (graphs) of sets of ordered pairs on coordinate axes. The horizontal axis generally belongs to the independent variable and the vertical axis belongs to the dependent variable. As seen in Comprehension Check 3.1 not all sets of ordered pairs represent functions. The function definition requires that for each domain element, x, there is a *unique* range element, y. What does this mean for our graph? It means that the graph of a function cannot have two different points on it that both have the same x coordinate.

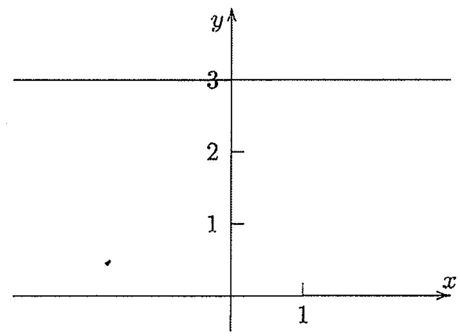
Important Idea 3.1.2.
The Vertical Line Test: No vertical lines drawn through the graph of a function can intersect the graph more than once. If there can be drawn any *vertical line that intersects the graph at more than one point, then the graph does not represent a function.*

All points on a vertical line have the same first coordinate. Since a function cannot contain two different points with the same first coordinate, there can be at most one graph point per vertical line.

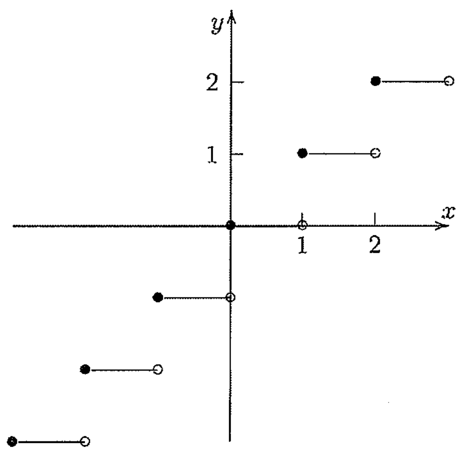
The following graphs represent functions:



$y = x^2$

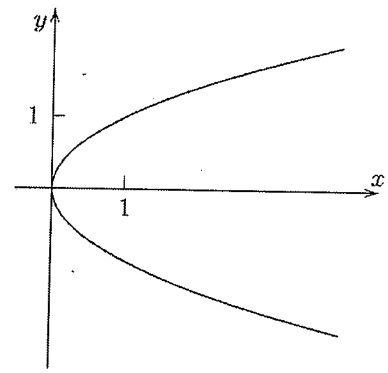


$y = 3$

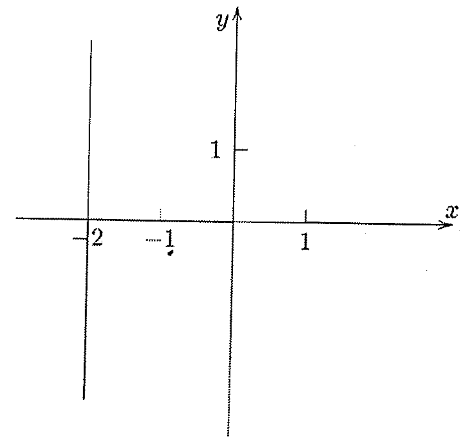


The greatest integer function

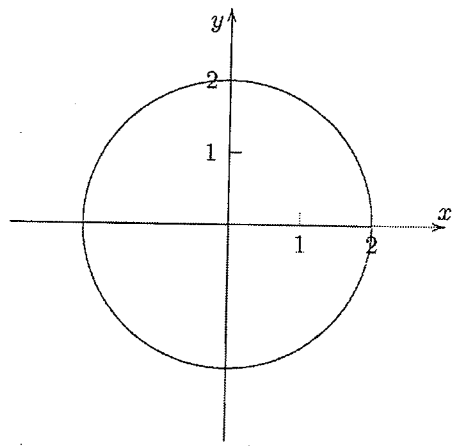
The following graphs do not represent functions:



$x = y^2$



$x = -2$



$x^2 + y^2 = 4$

Piecewise-Defined Functions

A "piecewise-defined function" is a function whose domain is divided into disjoint sets. Consider the function f below whose domain is $\mathbb{Z}$, the set of integers.

$$f(x) = \begin{cases} x & \textit{if} \quad \textit{x is an even integer} \\ -x & \textit{if} \quad \textit{x is an odd integer} \end{cases}$$

For this function, if the domain value x is even, the y value is equal to the x value. If the domain value x is odd, the y value is the same as the x value except that the sign is switched. $f(2) = 2$, $f(3) = -3$, $f(0) = 0$, $f(-5) = -(-5) = 5$.

For a piecewise-defined function we take our domain element, check to see which piece of the partitioned domain it belongs to, and then use only the rule associated with that piece of the domain to calculate the y value.

Example 3.1.3.

Given: $g(x) = \begin{cases} 1-x & if \quad x<0 \\ 2x-3 & if \quad x \geq 0 \end{cases}$

Find: D_g, $g(-2)$, $g(0)$, $g(\frac{1}{2})$, $g(5)$

The domain for a piecewise-defined function is very easy to find. We just look at all of the domain values listed. For g *we have* $x<0$ *and* $x \geq 0$ *so* $D_g = \Re$.

To find $g(-2)$ *we see that* -2 *is in the part of the domain* $x<0$ *so we use the corresponding rule* $(1-x)$ *to evaluate* $x=-2$. *Thus,* $g(-2) = 1-(-2) = 3$. *Each of the other* x *values* $(0, \frac{1}{2}, 5)$ *belong to the piece of the domain described by* $x \geq 0$ *so we use the rule* $(2x-3)$ *to evaluate them. Thus,* $g(0) = -3$, $g(\frac{1}{2}) = -2$ *and* $g(5) = 7$.

Important Idea 3.1.3.

A piecewise-defined function is a function. *The domain is divided into disjoint sets so there is no ambiguity as to how to find the* y *value for any given* x. *There is be only one correct* y *value for each* x.

A piecewise-defined function is one function. It is not two functions or three functions, but one *function whose domain just happens to be expressed in pieces.*

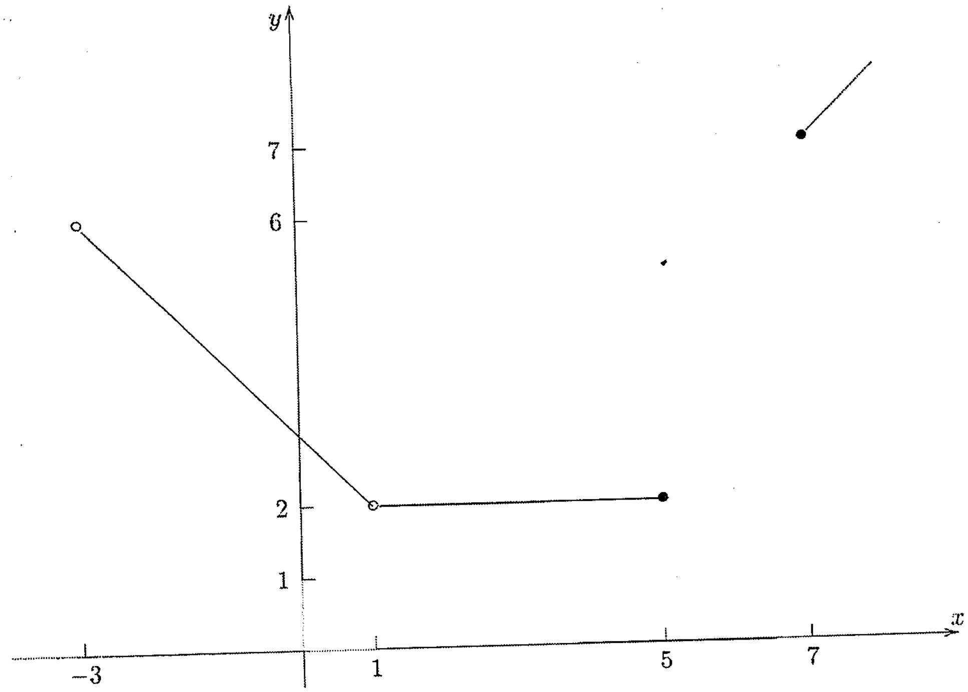
Example 3.1.4.

Given: $h(x) = \begin{cases} 3-x & if \quad -3<x<1 \\ 2 & if \quad 1<x \leq 5 \\ x & if \quad x \geq 7 \end{cases}$

Find: D_h, $h(-2)$, $h(0)$, $h(1)$, $h(3)$, $h(41)$

$D_h = (-3,1) \cup (1,5] \cup [7,\infty)$. $h(-2) = 5$, $h(0) = 3$,
$h(1)$*is undefined,* $h(3) = 2$, $h(4) = 2$, $h(41) = 41$

There will be times when we want to graph a piecewise-defined function. This is not a problem. After all, a graph is just an illustration of the set of ordered pairs that make up the function. So to graph a piecewise defined function just graph each of the pieces over that part of the x-axis that corresponds to its domain piece. Be sure to only graph points that are part of the function. Resist the temptation to simply plot a few points and "play dot-to-dot." Below is the graph of h from Example 3.1.4.



h

3.2 Function Operations

Definitions and Simple Operations

Definition 3.2.1.
Two functions are equal *if they are both comprised of exactly the same set of ordered pairs.*

So, for two functions to be equal they must both have the same domain *and* the same rule that assigns range values to the domain elements.

Example 3.2.1.
The functions $f(x) = \sqrt{x^2}$ *and* $g(x) = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$ *are equal.*

Both functions have domain $\Re$. *Both functions' rules evaluating the domain elements are the same (verify this by testing a few values until you are convinced that this is true).*

Example 3.2.2.
The functions $f(x) = x + 3$ *and* $g(x) = \dfrac{x^2 - 9}{x - 3}$ *are not equal.*

$D_f = \Re$ *but* D_g *does not contain the element* $x = 3$. *Since their domains are not equal, the functions cannot be the same.*

In example 3.2.2 it is tempting to argue that $\frac{x^2-9}{x-3} = \frac{(x+3)(x-3)}{(x-3)} = (x+3)$ and so $f = g$. We need to be very careful here. When reducing the fraction we are really dividing the numerator and denominator by the expression $(x-3)$. We can only do this with the stipulation that $x \neq 3$, otherwise we would be dividing by zero. It *is* true that for all values of x *except* $x = 3$ the ordered pairs of f are the same as the ordered pairs of g. But since function f contains the point $(3, 6)$ and function g does not, then $f \neq g$. It is interesting to note here that the graphs of f and g are indentical except that g has an open circle at the point $(3, 6)$ to indicate that this point is missing from the function g.

Since we now know what it means for functions to be equivalent, it is reasonable to ask about adding functions or multiplying them or squaring them. We have definitions for these operations. Since functions are defined by both their domains and their rule that assigns range values to their domain elements, we must consider both the domain and the rule when we think about algebraic operations on functions.

Definition 3.2.2.
Given function f with domain D_f and function g with domain D_g we define the following:

1. $(f+g)(x) = f(x) + g(x)$ *where* $D_{f+g} = D_f \cap D_g$
2. $(f-g)(x) = f(x) - g(x)$ *where* $D_{f-g} = D_f \cap D_g$
3. $(f \cdot g)(x) = f(x) \cdot g(x)$ *where* $D_{f \cdot g} = D_f \cap D_g$
4. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ *where* $D_{\frac{f}{g}} = D_f \cap D_g$ *and* $g(x) \neq 0$

It should make sense that for any specific value of x it would be meaningless to add $f(x)$ and $g(x)$ if x was not in the domain for one of those functions. So the domain for the function $(f+g)$ can only contain values that are in both the domain of f and the domain of g. This is precisely the set $D_f \cap D_g$. This concept holds true for $(f-g)$, $f \cdot g$ and $\frac{f}{g}$ as well. We have an additional domain requirement when we deal with $\frac{f}{g}$. We must eliminate from the domain all x values for which $g(x) = 0$. Why?

Example 3.2.3.
Given $f(x) = x^2 + 3$ and $g(x) = x - 7$, the following are true:

- $(f+g)(x) = f(x) + g(x) = (x^2+3) + (x-7) = x^2 + x - 4.$
- $(f+g)(3) = 8$
- $(f \cdot g)(1) = -24$
- *The domain for* $\left(\frac{f}{g}\right)(x) = (-\infty, 7) \cup (7, \infty)$.
- *The domain for* $(f-g)$ *is* $\Re$.
- $f^2(x) = (f \cdot f)(x) = (x^2+3)(x^2+3) = [f(x)]^2$

Composition of Functions

Suppose we have the functions $f(x) = x^2+3$ and $g(x) = x-7$ as in the previous example. Function f takes its input, squares it and then adds three to it. Function g simply takes its input and subtracts 7. Both functions have domain $\Re$. When we choose a number, say 5, from $\Re$ we calculate $f(5)$ by squaring 5 and then adding 3. We simplify the arithmetic and so say $f(5) = 28$. We can calculate $g(5)$ by taking 5 and subtracting 7. We say $g(5) = -2$.

Suppose that instead of taking a constant and using it as the input to f or g we used an algebraic expression, like "$2x$". Function f would take $2x$, square it and then add 3. $f(2x) = (2x)^2 + 3 = 4x^2 + 3$. Function g would simply take the $2x$ and subtract 7. $g(2x) = (2x) - 7 = 2x - 7$. By using an algebraic expression as input to functions we have created new functions.

Suppose instead of $2x$ we wish to use the function g as input to the function f. The notation for this would be $f(g(x))$. We evaluate this: $f(g(x)) = f(x-7) = (x-7)^2 + 3 = x^2 - 14x + 49 + 3 = x^2 - 14x + 52$.

What would $g(f(x))$ be? Here we use f as input to g. $g(f(x)) = g(x^2+3) = (x^2+3)-7 = x^2-4$.

This process of using one function as input to another function is called *composition of functions*. We have a special notation to indicate a composition.

Important Idea 3.2.1.
The composition of the functions f and g is written $(f \circ g)$ and defined: $(f \circ g)(x) = f(g(x))$. The domain for function $(f \circ g)$ is $\{x : x \in D_g \text{ and } g(x) \in D_f\}$.

Order is important here. $(f \circ g)(x) = f(g(x))$, while $(g \circ f)(x) = g(f(x))$. We saw using $f(x) = x^2 + 3$ and $g(x) = x - 7$ that $(f \circ g)(x) = x^2 - 14x + 52$, but $(g \circ f)(x) = x^2 - 4$. These are NOT the same.

The domain of the new function $(f \circ g)$ is important. It is not simply the intersection of the domains of f and g. To understand the reasoning behind the strange-looking domain, look at what the function $(f \circ g)$ does with its input x. $(f \circ g)(x) = f(g(x))$. First the x is input to the function g. For this to make sense we must require that $x \in D_g$. Then the result that we get from this first operation, $g(x)$, is input to the function f. For this to make sense we have to be sure that the input to f is in the domain of f. Hence we require $g(x) \in D_f$. This leads to our statement that the domain for function $(f \circ g)$ is $\{x : x \in D_g \text{ and } g(x) \in D_f\}$.

Example 3.2.4.
Given $f(x) = \frac{1}{x}$ and $g(x) = x - 3$, find $(f \circ g)$, and its domain.

$(f \circ g)(x) = f(g(x)) = f(x-3) = \frac{1}{x-3}$.

$D_g = \Re$. $D_f = \{x : x \neq 0\}$. *So $D_{(f \circ g)}$ requires that $x \in \Re$ and that $g(x) \neq 0$. This last expression says, $(x-3) \neq 0$, or $x \neq 3$. Combining these requirements we get that $D_{(f \circ g)} = \{x : x \neq 3\}$.*

If we look at the expression for $(f \circ g)$ in the last example, and we look at the domain, it would be reasonable to ask, "Why do we have to go through the steps to find the domain when we can just figure it out from the $\frac{1}{x-3}$ expression for $(f \circ g)$?" Let's look at another example to help answer that question.

Example 3.2.5.
Given $f(x) = x^2 - 2$ *and* $g(x) = \sqrt{4 - x^2}$, *find* $(f \circ g)$, *and its domain.*

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{4 - x^2}) = (\sqrt{4 - x^2})^2 - 2 = (4 - x^2) - 2 = 2 - x^2.$$

$D_g = \{x : -2 \leq x \leq 2\}$. $D_f = \Re$. *So,* $D_{(f \circ g)}$ *requires that* $x \in [-2, 2]$ *and that* $\sqrt{4 - x^2} \in \Re$. *Combining these requirements, we get that* $D_{(f \circ g)} = [-2, 2]$.

If we attempted to figure the domain of $(f \circ g)$ from the statement $(f \circ g)(x) = 2 - x^2$ we would determine that the domain is $\Re$. This would be incorrect. If we look at the expression for $(f \circ g)$ prior to simplification we get a correct picture. $(f \circ g)(x) = (\sqrt{4 - x^2})^2 - 2$ makes it clear that our domain can only contain x values for which $(4 - x^2)$ is positive. The order of operation is critical. The square root operation on the variable occurs before the second squaring operation eliminates the square root.

In fact, we do not have to write all of the steps used for figuring the domain of $(f \circ g)$. We *can* look at the expression that we get when we actually do the composition. But we must look at that expression prior to any simplification. As in the previous example, simplification can hide the true domain of the function.

Important Idea 3.2.2.
A function is defined by both its domain and its rule that assigns range values to domain elements. So, when creating a function $(f \circ g)$ *from two other functions* f *and* g *it is essential that we calculate the domain of* $(f \circ g)$. *This domain should be specifically stated if it is different from the natural domain of the algebraic expression of* $(f \circ g)$.

Example 3.2.5 asked for both the expression of $(f \circ g)$ and its domain. If it had only asked for "the function $(f \circ g)$", we would have had to supply the domain anyway because it is different from the natural domain of the function's algebraic expression. For Example 3.2.5, the function $(f \circ g)(x)$ must be written with its domain: $(f \circ g)(x) = 2 - x^2$ where $x \in [-2,\ 2]$.

Example 3.2.6.
Given: $f(x) = \frac{1}{x-3}$, $g(x) = x + 5$, $h(x) = 3 - \frac{1}{x}$.
Find: $(f \circ g)$, $(h \circ f)$, $(h \circ f \circ g)$ *and* $(g \circ g)$.

$$(f \circ g)(x) = f(g(x)) = f(x + 5) = \frac{1}{(x+5)-3} = \frac{1}{x+2}.$$

$$(h \circ f)(x) = h(f(x)) = h(\tfrac{1}{x-3}) = 3 - \frac{1}{\frac{1}{x-3}} = 3 - (x - 3) = -x + 6.$$

$$D_{(h \circ f)} = \{x : x \neq 3\}.$$

$$(h \circ f \circ g) = h(f(g(x))) = h(\tfrac{1}{x+2}) = 3 - \frac{1}{\frac{1}{x+2}} = 3 - (x + 2) = -x + 1.$$

$$D_{(h \circ f \circ g)} = \{x : x \neq -2\}.$$

$$(g \circ g) = g(g(x)) = g(x + 5) = (x + 5) + 5 = x + 10.$$

Comprehension Check 3.2.
This is a quick check-up on concepts and notation.
For $f(x) = 4 - 3x$, *find the following:*

$f(2)$	$f(-3)$	$f(2)+f(-3)$	$f(2+-3)$
$f(2x)$	$2f(x)$	$f(x^2)$	$(f(x))^2$
$x^2f(x)$	$f(f(x))$	$f(x+2)$	$f(-x)$
$-f(x)$	$-f(-x)$	$f(\frac{1}{x})$	$\frac{1}{f(x)}$

The Difference Quotient

In calculus you will encounter an expression called the *difference quotient.* It isn't necessary for this course that you understand why it is important or that you memorize it. But, it gives us good practice for working with compositions of functions so we include it here. The difference quotient is the expression:

$$\frac{f(x+h)-f(x)}{h}$$

Example 3.2.7.

Express the difference quotient for the function $f(x)=x^2$. Simplify.

$$\frac{f(x+h)-f(x)}{h}=\frac{(x+h)^2-x^2}{h}=\frac{x^2+2xh+h^2-x^2}{h}$$
$$=\frac{2xh+h^2}{h}=\frac{h(2x+h)}{h}=2x+h$$

Express the difference quotient for the function $g(x)=3x-7$. Simplify.

$$\frac{g(x+h)-g(x)}{h}=\frac{[3(x+h)-7]-[3x-7]}{h}=\frac{3x+3h-7-3x+7}{h}=\frac{3h}{h}=3$$

Recognizing Composition of Functions

It is useful to be able to recognize a function that is constructed by the composition of two, or more, other functions. Generally, when a function has more than one operation acting on an independent variable, we can rewrite the function as a composition of other functions.

Example 3.2.8.

Given function h, find functions f and g such that $(f\circ g)=h$.

$h(x)=\sqrt{x^3-1}$. *We choose $f(x)=\sqrt{x}$ and $g(x)=x^3-1$.*

Then $f(g(x))=f(x^3-1)=\sqrt{x^3-1}=h(x)$.

Our selections for f and g are not unique. We could have chosen $f(x)=\sqrt{x-1}$ and $g(x)=x^3$ (check this out). One thing we do need to be mindful of for all of our composition choices is the order of operation. We saw that in general $(f\circ g)$ is not the same as $(g\circ f)$. Always test your selections of f and g to make sure that your composition works.

Example 3.2.9.

Given function h, find functions f and g such that $(f\circ g)=h$.

$h(x)=\dfrac{2}{x+5}+x+5$. *We choose $f(x)=\dfrac{2}{x}+x$ and $g(x)=x+5$.*

Then $f(g(x)) = \dfrac{2}{(x+5)} + (x+5) = h(x)$

3.3 Function Characteristics

When a realtor is advertising a house he will use descriptive phrases for the property such as "house suited to its natural setting" (which is interpreted to mean "located in swamp, questionable plumbing"). Likely the realtor will indicate the basic architectural style: modern, colonial, ranch, split level, Cape Cod. However there are some basic facts that go into every house listing: number of bedrooms, number of bathrooms, total square footage, type of garage, type of heat, annual property tax assesment. These characteristics apply to any house, regardless of the style or location. With this information we can better compare the choices on the market.

Functions are similar (except that we are not trying to sell them). There are basic "styles" of functions. We have separate chapters in this text for polynomial functions, rational functions, trigonometric functions and logarithmic functions. When we study each of these types of functions we will look at some basic characteristics which enable us to be able to compare them, thus help us to better understand them. In this section we look at what three of these characteristics are.

Function Intercepts

One (two?) interesting feature of a function is its itercepts. Where does the graph of the function cross the x-axis and the y-axis?

The y-intercept of a function is the point where the graph of the function crosses the y-axis. A function does not have to have a y-intercept, but it cannot have more than one. Why? Every point on the y-axis has first coordinate $x = 0$. By definition of *function* there can be only one y value for any given x value. It is easy to find the y-intercept for a function. Since the x coordinate must be 0 we simply substitute the value 0 in for x and the result is the y value.

Example 3.3.1.
Find the y-intercept for $f(x) = 3x^2 - 2x + 6$ *and for* $g(x) = \dfrac{6x+2}{x}$.

$f(0) = 6$, *so the y-intercept is* 6.

$g(0)$ *is undefined (zero in denominator) so g does not have a y-intercept.*

The y-intercept is a point; it has two coordinates. We should say that the y-intercept is $(0, 6)$ but as long as we use the words "y-intercept" we can unambiguously mean the point $(0, 6)$ by simply saying "The y-intercept is 6."

An x-intercept of a function is a point where the graph of the function crosses the x-axis. It is where the y value of the function is 0. A function can have any number of x intercepts. We find them by finding the values of x for which y is equal to zero.

Example 3.3.2.
Find the x-intercept(s) for $f(x) = x^3 - x$.

$f(x) = x^3 - x = x(x^2 - 1) = x(x+1)(x-1)$. $f(x) = 0$ *for* $x = 0, 1, -1$, *so these are the x-intercepts.*

The function "rules" for inverse functions are such that one reverses the effect of the other. This is why their compositions always return the original input: $f[f^{-1}(x)] = x$ and $f^{-1}[f(x)] = x$. We think of the squaring operation and the taking of a square root as each operation reversing the other. We would like to think of them as inverses, but strictly they are not. There is the domain problem. However, we can recognize the domain problem and pose another function to be the inverse of $g(x) = \sqrt{x}$. We suggest that $g^{-1}(x) = x^2$ for $x \geq 0$. If we restrict the domain of the squaring function, then we do have an inverse function for the square root function. The problem we had before with $g[f(x)] = g(x^2) = \sqrt{(x^2)} = |x|$ goes away if we restrict $x \geq 0$. We will look at these particular functions in more detail in the Family of Simple Functions chapter.

3.5 Exercises

Problems for Section 3.1

Problem 1. Which of the following are functions? If not a function, tell why not. Assume x would represent the independent variable.

(a) $y = x^2$ (b) $x^2 = y - 2x$ (c) $\{(2,-1),(4,3),(1,2),(0,3)\}$

(d) $y = \begin{cases} -x & if \quad x < 0 \\ 2 & if \quad x > 0 \end{cases}$ (e) $y^2 = x$ (f) $\{(4,2),(-3,1),(2,4),(-3,-2)\}$ X

$x \to \pm\sqrt{y}$ eg $\{(1,1),(-1$

Problem 2. Given f where $D_f = \{-1,0,1,3,7\}$ and $f(x) = x^2 + 3$, find R_f, the range of f.

Problem 3. For each of the following functions, find the domain and range. Then evaluate the function at $-1, 0, 1$ and 2 if in the domain.

(a) $f(x) = \dfrac{1}{x-2}$ (b) $f(x) = \{(2,1),(4,3),(1,2),(0,-3)\}$

(c) $f(x) = \sqrt{2-x} + 1$ (d) $A(r) = \pi r^2$ (area of a circle as a function of its radius)

Problem 4. For each of the following functions, find the domain:

(a) $f(x) = x^2 - x + 2$ (b) $y = \dfrac{x-5}{3x^2+12x}$ (c) $g(x) = \dfrac{1}{x} + \dfrac{4}{x+3}$ (d) $f(x) = \sqrt{\dfrac{x^2+1}{-x}}$

(e) $h(x) = \sqrt[4]{x+5}$ (f) $s(x) = \sqrt[3]{3x+7}$ (g) $y = \dfrac{x+2}{x^2-4}$

Problem 5. Given $f(x) = (3+x)^2$, show that $f(2) + f(3) \neq f(5)$.

Problem 6. For each of the following functions find the domain and evaluate the function for $x = -2, -\frac{1}{2}, 0, 1$ and $\frac{9}{4}$ if in the domain. Also, for functions g and h, graph the function and give its range.

(a) $f(x) = \begin{cases} \sqrt{1-x^2} & if \quad -1 \leq x \leq 1 \\ \frac{1}{x} & if \quad x > 1 \end{cases}$ (b) $g(x) = \begin{cases} -3 & if \quad x < 0 \\ x & if \quad x > 0 \end{cases}$

(c) $h(x) = \begin{cases} -1 & if \quad x \text{ is an integer} \\ 1 & if \quad x \text{ is not an integer} \end{cases}$

Problem 7. The cost of renting a car for a week is \$250 plus 25 cents per mile driven. Write a function that describes the cost of week's rental as a function of the number of miles driven. Identify your independedent variable, your dependent variable and the function's domain and range.

Problem 8. Write an equation that expresses the area of a square in terms of its perimeter.

Problem 9. The length of a rectangle is three inches more than its width. Write an equation that expresses the area of the rectangle as a function of its width. Then write an equation that expresses the area of the rectangle as a function of its length.

Problems for Section 3.2

Problem 1. Given $f(x) = \frac{1}{x}$ and $g(x) = x$. Does $(f \circ f) = g$? Justify your answer.

Problem 2. Show that the domains for $f(x) = \sqrt{\dfrac{x}{x-2}}$ and $g(x) = \dfrac{\sqrt{x}}{\sqrt{x-2}}$ are not the same. This means that $f \neq g$.

Problem 3. Given functions $f(x) = \dfrac{3}{x+1}$ and $g(x) = \dfrac{x+2}{x-1}$, find:

(a) $(f+g)$ and the domain of $(f+g)$ (b) $(f-g)$ and the domain of $(f-g)$

(c) $(f \cdot g)$ and the domain of $(f \cdot g)$ (d) $\dfrac{f}{g}$ and the domain of $\dfrac{f}{g}$

(e) $(f+g)(2)$ and $(f \cdot g)(-2)$

Problem 4. Given functions f and g below, find $(f+g)(0)$, $(f+g)(5)$ and function $(f+g)(x)$

$$f(x) = \begin{cases} x+3 & if \quad x \leq 0 \\ 2x-5 & if \quad x > 0 \end{cases} \qquad g(x) = \begin{cases} 3x-1 & if \quad x < -1 \\ 4-x & if \quad x \geq -1 \end{cases}$$

Problem 5. Given $f(x) = 1 - 2x^2$ and $g(x) = x + 1$, find $(f \circ g)$, $(g \circ f)$, $(f \circ f)$, $(g \circ g)$.

Problem 6. For each of the following pairs f and g, find $(f \circ g)$ and $(g \circ f)$ and their domains.

(a) $f(x) = \sqrt{x} \quad g(x) = x^2$ (b) $f(x) = x + 2 \quad g(x) = \dfrac{1}{x-3}$

Problem 7. For functions f and g below, find $(f \circ g)(4)$, $(f \circ g)(-4)$ and $(f \circ g)(x)$.

$$f(x) = \begin{cases} 2x & if \quad x < 0 \\ x^2 & if \quad x > 3 \end{cases} \qquad g(x) = 1 - x$$

Problem 8. Find $(f \circ g)$ if $f = \{(-1,2),(0,7),(2,5)\}$ and $g = \{(-1,1),(3,0),(4,-1)\}$.

Problem 9. Evaluate the Difference Quotient (see page 75) for the following functions:

(a) $f(x) = \dfrac{1}{x}$ (b) $f(x) = x^2 + x$