13.4 Nonlinear Systems

In this final section, we want to learn how to solve systems of equations that are not necessarily all linear. We call these non-linear systems of equations.

Definition: Non-linear system of equations
A system of equations where one or more equations involved is not a line.

We primarily use the substitution method to solve a non-linear system. However, sometimes elimination will work as well.

Also, just as before, the solution to a non-linear system is all the points of intersection of the graphs of the equations. Therefore, since we now have more that just lines, we can have a variety of numbers of solutions. The graph can intersect once, twice, several times or not at all. Therefore, we should always verify the solution(s) to a system by looking at the graph.

Example 1:

Solve the system.

a. $\begin{aligned} x^2 + y^2 &= 100 \\ y - x &= 2 \end{aligned}$ b. $\begin{aligned} x^2 + y^2 &= 25 \\ y^2 &= x + 5 \end{aligned}$ c. $\begin{aligned} x^2 + 2y^2 &= 12 \\ xy &= 4 \end{aligned}$

Solution:

a. We will solve this system by substitution. So we start by solving the bottom equation for y and then substitute it into the top equation. We get

$$y - x = 2 \Rightarrow y = x + 2$$

$$x^2 + y^2 = 100 \Rightarrow x^2 + (x+2)^2 = 100$$

$$x^2 + x^2 + 4x + 4 = 100$$

$$2x^2 + 4x - 96 = 0$$

$$2(x^2 + 2x - 48) = 0$$

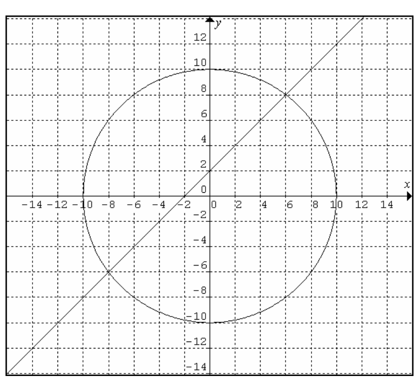
$$2(x+8)(x-6) = 0$$

$$x = -8,\ 6$$

So we have two different x values. This means we should have two points of intersection. Lets now find the y values and verify with a graph.

$$\begin{aligned} y &= x + 2 \\ y &= -8 + 2 \\ y &= -6 \end{aligned} \qquad \begin{aligned} y &= x + 2 \\ y &= 6 + 2 \\ y &= 8 \end{aligned}$$

So our solutions are $(-8, -6)$ and $(6, 8)$. We clearly have a circle and a line, thus we can easily graph them together and get



b. Again we will use substitution to solve. This time notice that the bottom equation is already solved for y^2 and we have a y^2 in the top equation. Thus, that is the substitution we will make. We get

$$\begin{aligned} x^2 + y^2 &= 25 \\ y^2 &= \underbrace{x+5} \end{aligned} \quad \Rightarrow x^2 + (x+5) = 25$$

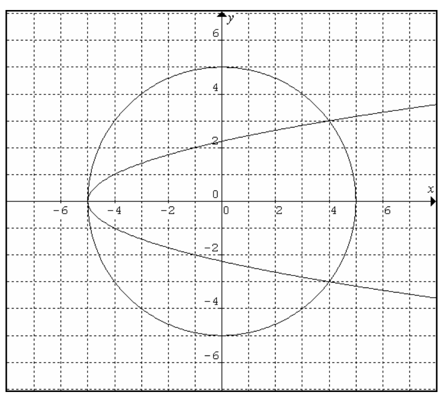
Now we solve for x and then solve for y.

$$\begin{aligned} x^2 + (x+5) &= 25 \\ x^2 + x - 20 &= 0 \\ (x+5)(x-4) &= 0 \\ x &= -5,\ 4 \end{aligned}$$

We substitute these back in to get y. We get

$$\begin{aligned} y^2 &= -5+5 \\ y^2 &= 0 \\ y &= 0 \end{aligned} \qquad\qquad \begin{aligned} y^2 &= 4+5 \\ y^2 &= 9 \\ y &= \pm 3 \end{aligned}$$

So we have three different solutions $(-5,\ 0)$, $(4,\ 3)$ and $(4,\ -3)$. Lets verify with a graph. We have here a parabola and a circle. We get



c. Again, we will use substitution to solve. We need to decide which variable to solve for first. It seems that x or y on the bottom equation would be easiest. So we will solve for x. We get

$$xy = 4 \Rightarrow x = \frac{4}{y}$$

Now we substitute that into the first equation and solve. We have

$$\begin{aligned} x^2 + 2y^2 = 12 \Rightarrow \left(\frac{4}{y}\right)^2 + 2y^2 &= 12 \\ \frac{16}{y^2} + 2y^2 &= 12 \end{aligned}$$

We will have to clear the fractions and solve as we did in chapter 10, that is, using a substitution.

$$y^2\left(\frac{16}{y^2}+2y^2\right)=y^2(12)$$
$$16+2y^4=12y^2$$
$$2y^4-12y^2+16=0$$
Substitute u = y^2
$$2u^2-12u+16=0$$
$$2(u^2-6u+8)=0$$
$$2(u-4)(u-2)=0$$
$$u=4,\ 2$$
Re-substitute u = y^2
$$y^2=4 \qquad y^2=2$$
$$y=\pm 2 \qquad y=\pm\sqrt{2}$$

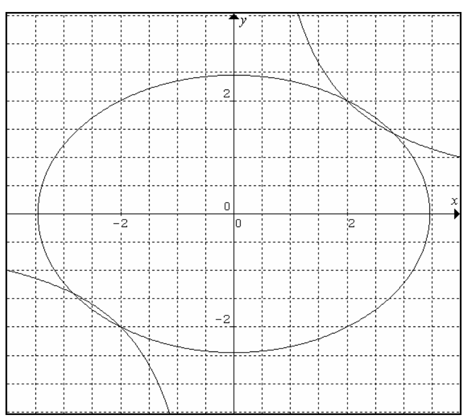
So since we have four different y values we will have to find x for each one. We substitute these in to $x=\dfrac{4}{y}$ to get

$$x=\frac{4}{2} \qquad x=\frac{4}{-2} \qquad x=\frac{4}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} \qquad x=\frac{4}{-\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}$$
$$=2 \qquad =-2 \qquad =\frac{4\sqrt{2}}{2} \qquad =-\frac{4\sqrt{2}}{2}$$
$$=2\sqrt{2} \qquad =-2\sqrt{2}$$

So we have four solutions, $(2,\ 2)$, $(-2,\ -2)$, $(2\sqrt{2},\ \sqrt{2})$ and $(-2\sqrt{2},\ -\sqrt{2})$.

Lets verify with a graph. We have an ellipse and a basic function ($xy=4 \Rightarrow y=\dfrac{4}{x}$).

So we have



Example 2:

Solve the system.

a. $y=3^x$
$y=3^{2x}-2$

b. $y=\log_2(x+1)$
$y=5-\log_2(x-3)$

Solution:

a. This time solving is a little more complicated. There are a variety of direction we could go, however, we are going to start by noticing $3^{2x} = \left(3^x\right)^2$. So our system really is

$$y = 3^x$$
$$y = \left(3^x\right)^2 - 2$$

We can see clearly we will substitute the top equation into the bottom. That is, put y in for 3^x. We get

$$\begin{aligned} y &= (y)^2 - 2 \\ y &= y^2 - 2 \\ y^2 - y - 2 &= 0 \\ (y-2)(y+1) &= 0 \\ y &= 2, -1 \end{aligned}$$

Now we substitute these values back in to get

$$\begin{aligned} 2 &= 3^x \\ x &= \log_3 2 \end{aligned} \qquad\qquad -1 = 3^x$$

However, the second equation is impossible (recall, exponential functions are always positive). Thus, that solution must be omitted. So we have a solution of $(\log_2 3,\ 2)$.

b. Lastly, since these equations are both already solved for y, we can simply set them equal to one another. Then we are left with a logarithmic equation to solve. We get

$$\begin{aligned} \log_2(x+1) &= 5 - \log_2(x-3) \\ \log_2(x+1) + \log_2(x-3) &= 5 \\ \log_2(x+1)(x-3) &= 5 \\ (x+1)(x-3) &= 2^5 \\ x^2 - 2x - 3 &= 32 \\ x^2 - 2x - 35 &= 0 \\ (x-7)(x+5) &= 0 \\ x &= 7, -5 \end{aligned}$$

However, -5 cannot be a solution since is doesn't even check in the equation. Thus we only have x = 7. Now we substitute this back into either original equation to get the y value. We choose the first equation.

$$\begin{aligned} y &= \log_2(7+1) \\ &= \log_2 8 \\ &= 3 \end{aligned}$$

So the solution is $(7,\ 3)$.

13.4 Exercises

Solve the systems.

1. $x^2 + y^2 = 2$, $x + y = 2$

2. $x^2 + y^2 = 25$, $y - x = 1$

3. $25x^2 + 9y^2 = 225$, $5x + 3y = 15$

4. $9x^2 + 4y^2 = 36$, $3x + 2y = 6$

5. $y^2 = x + 3$, $2y = x + 4$

6. $y = x^2$, $3x = y + 2$

7. $x^2 - y^2 = 16$, $x - 2y = 1$

8. $x^2 + 4y^2 = 25$, $x + 2y = 7$

9. $x^2 + y^2 = 18$, $2x + y = 3$

10. $x^2 - y = 3$, $2x - y = 3$

11. $x^2 + y^2 = 20$, $y = x^2$

12. $x^2 - y^2 = 3$, $y = x^2 - 3$

13. $x^2 - x - y = 2$, $4x - 3y = 0$

14. $x^2 - 2x + 2y^2 = 8$, $2x + y = 6$

15. $x^2 + y^2 = 13$, $y = x^2 - 1$

16. $x^2 - y = 5$, $x^2 + y^2 = 25$

17. $x^2 + y^2 = 25$, $2x^2 - 3y^2 = 5$

18. $x^2 + y^2 = 4$, $9x^2 + 16y^2 = 144$

19. $x^2 + y^2 = 13$, $x^2 - y^2 = -16$

20. $x^2 + y^2 = 16$, $y^2 - 2y^2 = 10$

21. $x^2 + y^2 = 20$, $x^2 - y^2 = -12$

22. $x^2 + y^2 = 14$, $x^2 - y^2 = 4$

23. $xy = -\frac{9}{5}$, $3x + 2y = 6$

24. $x + y = -6$, $xy = -7$

25. $y = x^2 - 4$, $x^2 - y^2 = -16$

26. $x^2 + y^2 = 25$, $y^2 = x + 5$

27. $xy = \frac{1}{6}$, $y + x = 5xy$

28. $xy = \frac{1}{12}$, $x + y = 7xy$

29. $x^2 + xy + 2y^2 = 7$, $x - 2y = 5$

30. $x^2 - xy + 3y^2 = 27$, $x - y = 2$

31. $x^2 + y^2 = 5$, $xy = 2$

32. $x^2 + y^2 = 20$, $xy = 8$

33. $3xy + x^2 = 34$, $2xy - 3x^2 = 8$

34. $2xy + 3y^2 = 7$, $3xy - 2y^2 = 4$

35. $\frac{1}{x} + \frac{1}{y} = 5$, $\frac{1}{x} - \frac{1}{y} = -3$

36. $\frac{1}{x} - \frac{1}{y} = 4$, $\frac{1}{x} + \frac{1}{y} = -2$

37. $\frac{2}{x^2}+\frac{5}{y^2}=3$
$\frac{3}{x^2}-\frac{2}{y^2}=1$

38. $\frac{1}{x^2}-\frac{3}{y^2}=14$
$\frac{2}{x^2}+\frac{1}{y^2}=35$

39. $x^4 = y-1$
$y-3x^2+1=0$

40. $x^3-y=0$
$xy-16=0$

41. $y=-\sqrt{x}$
$(x-3)^2+y^2=4$

42. $y=\sqrt{x}$
$(x+2)^2+y^2=1$

43. $y=2^x$
$y=2^{2x}-12$

44. $y=5^x$
$y=5^{2x}-1$

45. $y=e^{4x}$
$y=e^{2x}+6$

46. $y=2e^{2x}$
$y=e^x-1$

47. $y=\log_2(x+4)$
$y=2-\log_2(x+1)$

48. $y=\log_6 x$
$y=-\log_6(x+1)$

49. $y=\log_9(x+1)$
$y=\log_9 x+\frac{1}{2}$

50. $y=\log_{16}(x+3)$
$y=\log_{16}(x-1)+\frac{1}{2}$