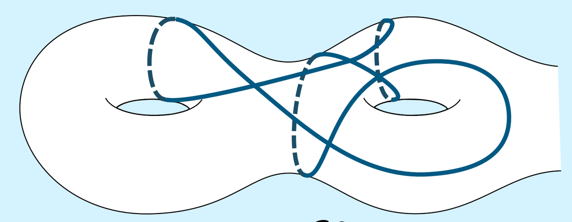
Non-Simple Geodesics on Surfaces

Counting Theorems

1 Previous Results

S - surface
$\mathcal{G}^c$ - set of closed geodesics
$l(\gamma)$ - geodesic length
$i(\gamma,\gamma)$ - self-intersection number



$\gamma \in \mathcal{G}^c$

$$\mathcal{G}^c(L) = \{\gamma \in \mathcal{G}^c \mid l(\gamma) \le L\}$$

Theorem 1 (Margulis) *Let S be a closed, negatively curved surface. Then*

$$\#\mathcal{G}^c(L) = \frac{e^{\delta L}}{\delta L}$$

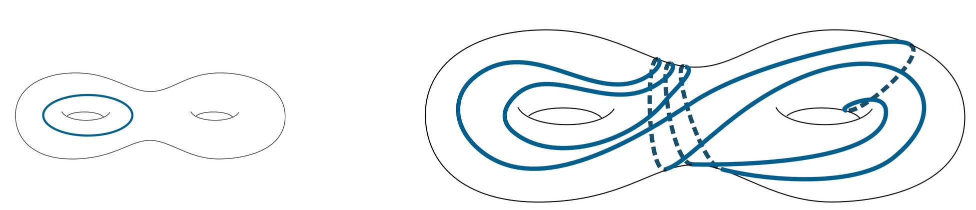
$A \sim B$ if $\lim_{L\to\infty} \frac{A}{B} = 1$

where δ is the topological entropy of the geodesic flow.

NB: If S is hyperbolic, then $\delta = 1$.

$$\mathcal{G}^c(L,K) = \{\gamma \in \mathcal{G}^c \mid l(\gamma) \le L, i(\gamma,\gamma) \le K\}$$

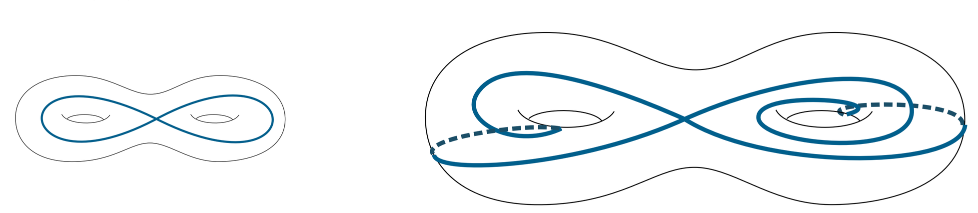
Question 1 *Suppose $K = K(L)$ is a function of L. What can be said about the asymptotic growth of $\#\mathcal{G}^c(L,K)$?*



Theorem 2 (Mirzakhani) *Let S be a hyperbolic genus g surface with n punctures. Then*

$$\#\mathcal{G}^c(L,0) \sim c(S)L^{6g-6+2n}$$

where $c(S)$ is a constant depending only on the geometry of S.



Theorem 3 (Rivin) *Let S be a hyperbolic genus g surface with n punctures. Then*

$$\#\mathcal{G}^c(L,1) \sim c'(S)L^{6g-6+2n}$$

where $c'(S)$ is a constant depending only on the geometry of S.

Question 2 *For fixed L and K, what are the best bounds for $\#\mathcal{G}^c(L,K)$?*

NB: Trivial upper and lower bounds are $0 \le \#\mathcal{G}^c(L,K) \le \#\mathcal{G}^c(L)$.

Consequence of Athreya-Bufetov-Eskin-Mirzakhani + Basmajian + others:

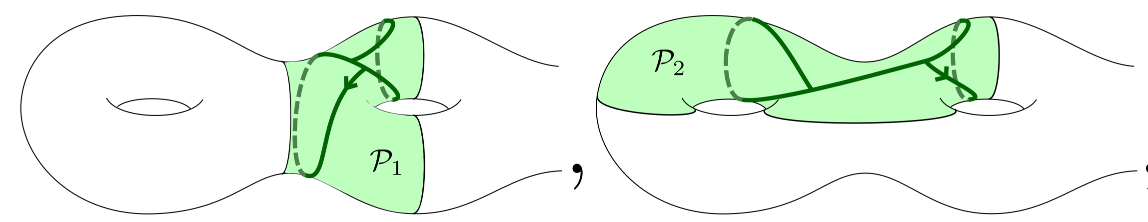
Proposition 1 *On a hyperbolic genus g surface with n punctures,*

$$\#\mathcal{G}^c(L,K) \asymp f(K)L^{6g-+2n}$$

$A \asymp B$ if $\frac{1}{c}B \le A \le cB$

where $f(K)$ is the number of Mod_g orbits in $\mathcal{G}^c(L,K)$.

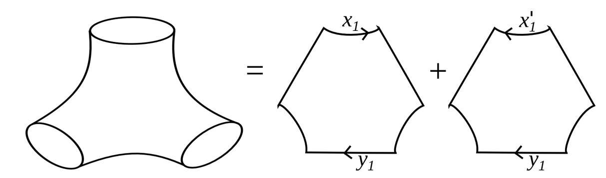
1 Combinatorial Model on Arbitrary Surfaces



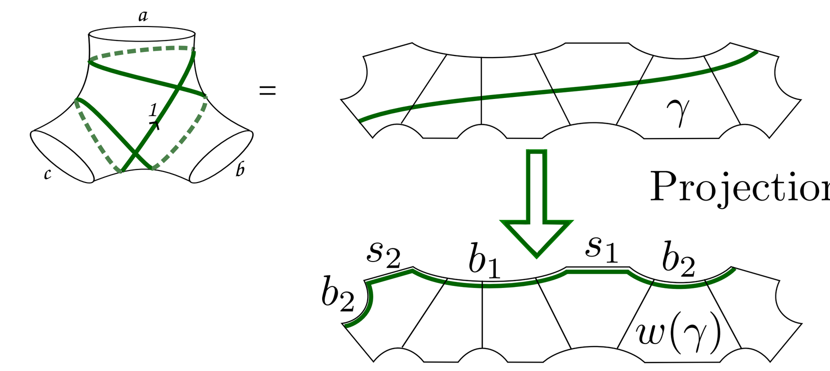
$\gamma \to \begin{matrix}\mathcal{P}_1, \mathcal{P}_2, \ldots \\ \gamma_1, \gamma_2, \ldots\end{matrix}$

Reduce to pairs of pants.

2 Model on Pairs of Pants



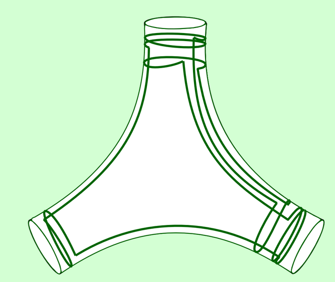
Pants as union of two hexagons.



Get word in hexagon edges.

$\gamma \to w(\gamma) = b_1 s_1 \ldots b_n s_n$,
b_i on $\partial\mathcal{P}$, s_i - seam of $\mathcal{P}$.

- $\gamma \to w(\gamma)$ is 1-1
- $l(\gamma) \asymp |w(\gamma)|$



- Relationship between $i(\gamma,\gamma)$ and $w(\gamma)$:

$$n^2, \sum_{i=1}^n |b_i| \lesssim i(\gamma,\gamma) \lesssim \sum_{i=1}^n i|b_i|$$

2 For Pairs of Pants

$\mathcal{P}$- hyperbolic pair of pants with geodesic boundary

Theorem (S) *On a pair of pants $\mathcal{P}$,*

$$e^{c\sqrt{K}} \le \#\mathcal{G}^c(L,K) \le \min\{e^{c\sqrt{K}\log\frac{L}{K}}, e^{c'\sqrt{K}\log c'\sqrt{K}}\}$$

L, K compete

where c depends on the geometry of $\mathcal{P}$, and $c \to 0$ as the lengths of $\partial\mathcal{P}$ go to infinity, and c' is a universal constant.

Corollary (S) *If $K = K(L)$ is s.t. $K = o(L^2)$, then*

$$\#\mathcal{G}^c(L,K) = o(\#\mathcal{G}^c(L))$$

$$K \approx L^2$$

Theorem 4 (Lalley) *Let S be a closed hyperbolic surface. Choose $\gamma_L \in \mathcal{G}^c(L)$ at random for each $L \in \mathbb{N}$. Then*

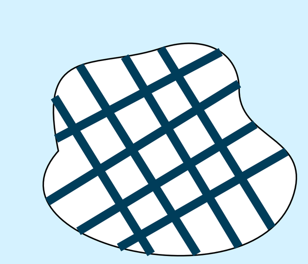
$$\lim_{L\to\infty} \frac{i(\gamma_L,\gamma_L)}{L^2} = \kappa$$

for almost any choice of sequence $\{\gamma_L\}$, where κ depends only on the geometry of S.

Theorem 5 (Basmajian) *Let S be a hyperbolic surface. Then*

$$i(\gamma,\gamma) \le \lambda l(\gamma)^2$$

for any $\gamma \in \mathcal{G}^c$, where λ depends only on the geometry of S.



γ in nbhd of S

3 For an Arbitary Surface

Conjecture *On an arbitrary surface S,*

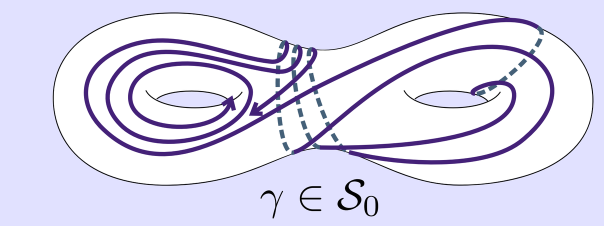
$$\#\mathcal{G}^c(L,K) \le \min\{e^{cL}, e^{K^{3/4}\log q(K,L)}\}$$

where $q(K,L)$ is a rational function in K and L and c is a constant depending only on the geometry of S.

Birman-Series Type Theorem

1 The Original Theorem

$\mathcal{G}$ - set of complete geodesics
$\mathcal{S}_K = \{\gamma \in \mathcal{G} \mid i(\gamma,\gamma) \le K\}$
$\mathcal{T}_K$ - points on some geodesic in S_K

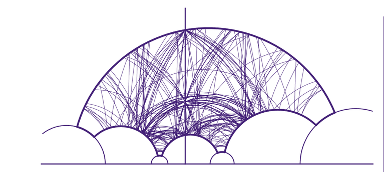


$\gamma \in \mathcal{S}_0$

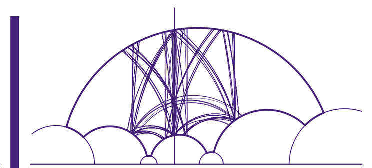
NB: Since most complete geodesics have infinitely many self-intersections, the geodesics in S_K should be thought of as almost simple.

Theorem 1 (Birman-Series) *Let S be a hyperbolic surface. Then T_K is nowhere dense and has Hausdorff dimension 1.*

Once-punctured torus:



Points on arbitrary closed geodesics. (Dense)



S_0 (Credit: Buser, Parlier)

few intersections
$\Downarrow$
"sparse" set

Question 1 *For what families $\mathcal{G}' \subset \mathcal{G}$ of complete geodesics with infinitely many self-intersections do the conclusions of Birman-Series hold?*

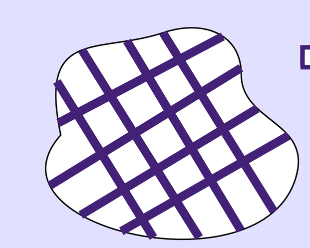
- On, eg, closed S, points on $\gamma \in \mathcal{G}$ dense.
- If $\gamma \in \mathcal{G}$ parameterized by arc-length and

$$\gamma_l = \gamma|_{[-\frac{l}{2},\frac{l}{2}]}$$

then

$$i(\gamma_l,\gamma_l) \sim cl^2$$

almost always.



γ_l in nbhd of S

Look at $\gamma \in \mathcal{G}$ s.t.

$$i(\gamma_l,\gamma_l) \lesssim \epsilon l^2$$

2 For Pairs of Pants

$$\mathcal{G}_\epsilon = \{\gamma \in \mathcal{G} \mid \limsup_{l\to\infty} \frac{i(\gamma_l,\gamma_l)}{l^2} < \epsilon\}$$

$\mathcal{F}_\epsilon$ - set of points on some $\gamma \in \mathcal{G}_\epsilon$

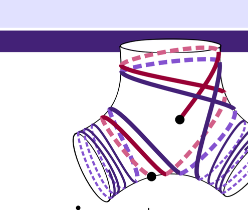
Theorem 2 (S) *On $\mathcal{P}$, $\mathcal{F}_\epsilon$ has Hausdorff dimension $\mu(\epsilon)$ where $\lim_{\epsilon\to 0}\mu(\epsilon) =$ 1. In particular, F_0 has Hausdorff dimension 1.*

But, $\mathcal{F}_\epsilon$ is not nowhere dense. In fact, it can have positive Lebesgue measure.

$\mathcal{F}$ - set of points on some $\gamma \in \mathcal{G}$

Proposition 3 (S) *If $\mathcal{F}_\epsilon^c$ denotes the closure of $\mathcal{F}_\epsilon$ in $\mathcal{P}$, then*

$$\mathcal{F}_\epsilon^c = \mathcal{F}$$



Approximate — by —

More regularity: $\mathcal{G}_\epsilon(L) = \{\gamma \in \mathcal{G}_\epsilon \mid i(\gamma_l,\gamma_l) < 5\epsilon l^2, \forall l \ge L\}$

$\mathcal{F}_\epsilon(L)$ - set of points on some $\gamma \in \mathcal{G}_\epsilon(L)$.

Theorem 4 (S) *There is an ϵ_0 s.t. $\forall \epsilon < \epsilon_0$, $\mathcal{F}_\epsilon(L)$ is nowhere dense for all L.*

3 For an Arbitary Surface

Conjecture 1 *On an arbitrary surface, the Birman-Series theorem holds when*

$$i(\gamma_l,\gamma_l) = o(l^{3/4})$$