Figure 1.86. The goal is to charge up the capacitor from a source of dc voltage V_{in}. In the top circuit (Figure 1.86A) we've done it the conventional way, with a series resistor to limit the peak current demanded from the voltage source. OK, it does work – but it has a drawback that can be serious, namely that half the energy is lost as heat in the resistor. By contrast, in the circuit with the inductor (Figure 1.86B) no energy is lost (assuming ideal components); and, as a bonus, the capacitor gets charged to twice the input voltage. The output-voltage waveform is a sinusoidal half-cycle at the resonant frequency $f = 1/2\pi\sqrt{LC}$, a topic we'll see soon (§1.7.14).[34,35]

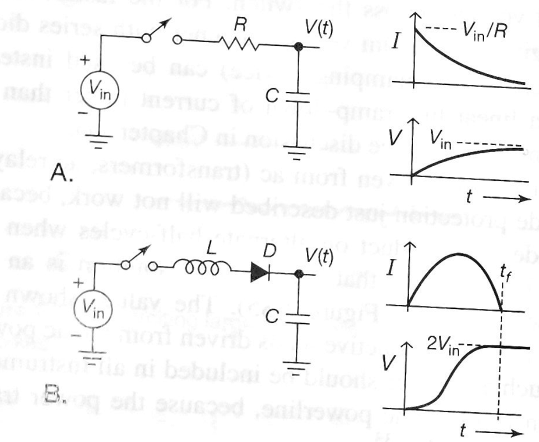

Figure 1.86. Resonant charging is lossless (with ideal components) compared with the 50% efficiency of resistive charging. Charging is complete after t_f, equal to a half-cycle of the resonant frequency. The series diode terminates the cycle, which would otherwise continue to oscillate between 0 and $2V_{in}$.

[34] A mechanical analogy may be helpful here. Imagine dropping packages onto a conveyor belt that is moving at speed v; the packages are accelerated to that speed by friction, with 50% efficiency, finally reaching the belt speed v, at which speed they ride into the sunset. That's resistive charging. Now we try something completely different, namely, we rig up a conveyor belt with little catchers attached by springs to the belt; and alongside it we have a second belt, running at twice the speed ($2v$). Now when we drop a package onto the first conveyor it compresses a spring, then rebounds at $2v$; and it makes a soft landing onto the second conveyor. No energy is lost (ideal springs), and the package rides off into the sunset at $2v$. That's reactive charging.

[35] Resonant charging is used for the high-voltage supply in flashlamps and stroboscopes, with the advantages of (a) full charge between flashes (spaced no closer than t_f), and (b) no current immediately after discharge (see waveforms), thus permitting the flashlamp to "quench" after each flash.

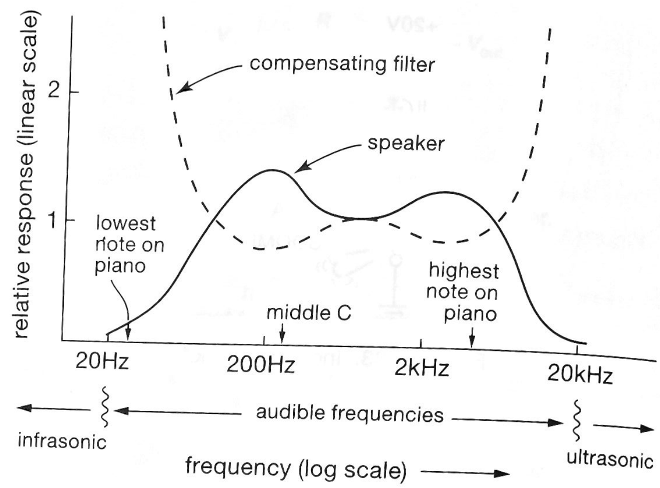

Figure 1.87. Example of frequency analysis: "boom box" loudspeaker equalization. The lowest and highest piano notes, called A0 and C8, are at 27.5 Hz and 4.2 kHz; they are four octaves below A440 and four octaves above middle C, respectively.

1.7 Impedance and reactance

Warning: this section is somewhat mathematical; you may wish to skip over the mathematics, but be sure to pay attention to the results and graphs.

Circuits with capacitors and inductors are more complicated than the resistive circuits we talked about earlier, in that their behavior depends on frequency: a "voltage divider" containing a capacitor or inductor will have a frequency-dependent division ratio. In addition, circuits containing these components (known collectively as *reactive* components) "corrupt" input waveforms such as square waves, as we saw earlier.

However, both capacitors and inductors are *linear* devices, meaning that the amplitude of the output waveform, whatever its shape, increases exactly in proportion to the input waveform's amplitude. This linearity has many consequences, the most important of which is probably the following: *the output of a linear circuit, driven with a sinewave at some frequency f, is itself a sinewave at the same frequency (with, at most, changed amplitude and phase).*

Because of this remarkable property of circuits containing resistors, capacitors, and inductors (and, later, linear amplifiers), it is particularly convenient to analyze any such circuit by asking how the output voltage (amplitude and phase) depends on the input voltage *for sinewave input at a single frequency*, even though this may not be the intended use. A graph of the resulting *frequency response*, in which the ratio of output to input is plotted for each sinewave frequency, is useful for thinking about many kinds of waveforms. As an example, a certain "boom-box" loudspeaker might have the frequency response shown in Figure 1.87, in which the "output" in this case is of course sound pressure, not voltage. It is desirable for a speaker to have a "flat" response, meaning that the graph of sound pressure versus frequency is constant over the band of audible frequencies. In this case the speaker's deficiencies can be corrected by the introduction of a passive filter with the inverse response (as shown) within the amplifiers of the radio.

As we will see, it is possible to generalize Ohm's law, replacing the word "resistance" with "impedance," in order to describe any circuit containing these linear passive devices (resistors, capacitors, and inductors). You could think of the subject of impedance (generalized resistance) as Ohm's law for circuits that include capacitors and inductors.

Some terminology: impedance (**Z**) is the "generalized resistance"; inductors and capacitors, for which the voltage and current are always 90° out of phase, are *reactive*; they have *reactance* (X). Resistors, with voltage and current always in phase, are *resistive*; they have *resistance* (R). In general, in a circuit that combines resistive and reactive components, the voltage and current at some place will have some in-between phase relationship, described by a complex impedance: impedance = resistance + reactance, or $\mathbf{Z} = R + jX$ (more about this later).[36] However, you'll see statements like "the impedance of the capacitor at this frequency is ..." The reason you don't have to use the word "reactance" in such a case is that impedance covers everything. In fact, you frequently use the word "impedance" even when you know it's a resistance you're talking about; you say "the source impedance" or "the output impedance" when you mean the Thévenin equivalent resistance of some source. The same holds for "input impedance."

In all that follows, we will be talking about circuits driven by sinewaves at a single frequency. Analysis of circuits driven by complicated waveforms is more elaborate, involving the methods we used earlier (differential equations) or decomposition of the waveform into sinewaves (Fourier analysis). Fortunately, these methods are seldom necessary.

1.7.1 Frequency analysis of reactive circuits

Let's start by looking at a capacitor driven by a sinewave voltage source $V(t) = V_0 \sin \omega t$ (Figure 1.88). The current is

$$I(t) = C\frac{dV}{dt} = C\omega V_0 \cos \omega t,$$

i.e., a current of amplitude $\omega C V_0$, with its phase leading the input voltage by 90°. If we consider amplitudes only, and disregard phases, the current is

$$I = \frac{V}{1/\omega C}.$$

(Recall that $\omega = 2\pi f$.) It behaves like a frequency-dependent resistance $R = 1/\omega C$, but in addition the current is 90° out of phase with the voltage (Figure 1.89).

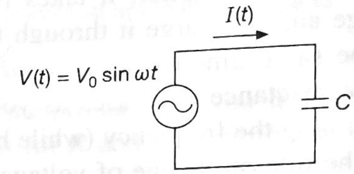

Figure 1.88. A sinusoidal ac voltage drives a capacitor.

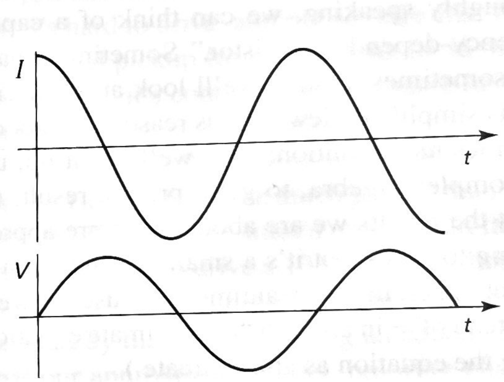

Figure 1.89. Current in a capacitor leads the sinusoidal voltage by 90°.

For example, a 1 μF capacitor put across the 115 volts (rms) 60 Hz powerline draws a current of rms amplitude:

$$I = \frac{115}{1/(2\pi \times 60 \times 10^{-6})} = 43.4\,\text{mA (rms)}.$$

Soon enough we will complicate matters by explicitly worrying about *phase shifts* and the like – and that will get us into some complex algebra that terrifies beginners (often) and mathophobes (always). Before we do that, though, this is a good time to develop intuition about the frequency-dependent behavior of some basic and important circuits that use capacitors, ignoring for the time being the troublesome fact that, when driven with a sinusoidal signal, currents and voltages in a capacitor are not in phase.

As we just saw, the ratio of *magnitudes* of voltage to

[36] But, in a nutshell, the magnitude of **Z** gives the ratio of amplitudes of voltage to current, and the polar angle of **Z** gives the phase angle between current and voltage.

current, in a capacitor driven at a frequency ω, is just

$$\frac{|V|}{|I|} = \frac{1}{\omega C},$$

which we can think of as a sort of "resistance" – the magnitude of the current is proportional to the magnitude of the applied voltage. The official name for this quantity is *reactance*, with the symbol X, thus X_C for the reactance of a capacitor.[37] So, for a capacitor,

$$X_C = \frac{1}{\omega C}. \tag{1.26}$$

This means that a larger capacitance has a smaller reactance. And this makes sense, because, for example, if you double the value of a capacitor, it takes twice as much current to charge and discharge it through the same voltage swing in the same time (recall $I = C dV/dt$). For the same reason the reactance decreases as you increase the frequency – doubling the frequency (while holding V constant) doubles the rate of change of voltage and therefore requires that you double the current, thus half the reactance.

So, roughly speaking, we can think of a capacitor as a "frequency-dependent resistor." Sometimes that's good enough, sometimes it isn't. We'll look at a few circuits in which this simplified view gets us reasonably good results, and provides nice intuition; later we'll fix it up, using the correct complex algebra, to get a precise result. (Keep in mind that the results we are about to get are approximate; we're *lying* to you – but it's a small lie, and anyway we'll tell the truth later. In the meantime we'll use the weird symbol $\asymp$ instead of $=$ in all such "approximate equations," and we'll flag the equation as approximate.)

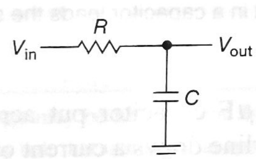

Figure 1.90. Lowpass filter.

A. *RC* lowpass filter (approximate)

The circuit in Figure 1.90 is called a *lowpass filter*, because it passes low frequencies and blocks high frequencies. If you think of it as a frequency-dependent voltage divider, this makes sense: the lower leg of the divider (the capacitor) has a decreasing reactance with increasing frequency, so the ratio of V_{out}/V_{in} decreases accordingly:

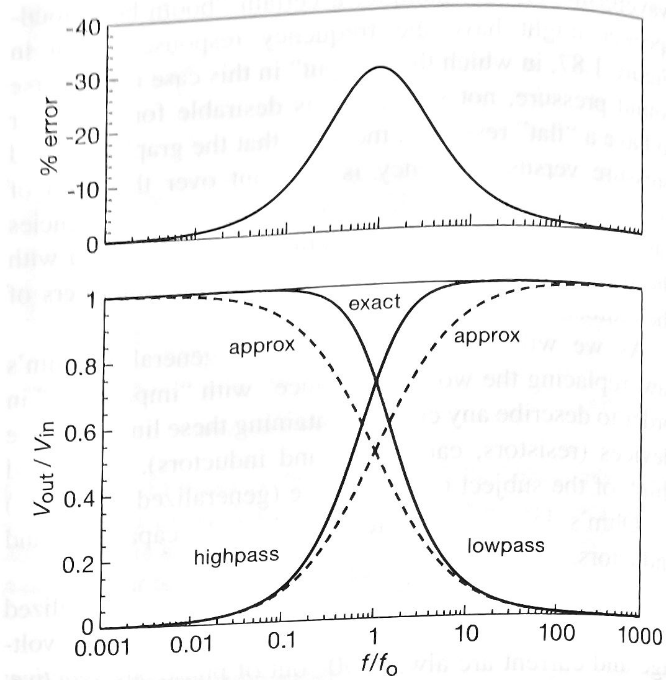

Figure 1.91. Frequency response of single-section *RC* filters, showing the results both of a simple approximation that ignores phase (dashed curve), and the exact result (solid curve). The fractional error (i.e., dashed/solid) is plotted above.

$$\frac{V_{out}}{V_{in}} \asymp \frac{X_C}{R+X_C} = \frac{1/\omega C}{R+1/\omega C} = \frac{1}{1+\omega RC} \quad \text{(approximate!)} \tag{1.27}$$

We've plotted that ratio in Figure 1.91 (and also that of its cousin, the *highpass filter*), along with their exact results that we'll understand soon enough in §1.7.8.

You can see that the circuit passes low frequencies fully (because at low frequencies the capacitor's reactance is very high, so it's like a divider with a smaller resistor atop a larger one) and that it blocks high frequencies. In particular, the crossover from "pass" to "block" (often called the *breakpoint*) occurs at a frequency ω_0 at which the capacitor's reactance ($1/\omega_0 C$) is equal to the resistance R: $\omega_0 = 1/RC$. At frequencies well beyond the crossover (where the product $\omega RC \gg 1$), the output decreases inversely with increasing frequency; that makes sense because the reactance of the capacitor, already much smaller than R, continues dropping as $1/\omega$. It's worth noting that, even with our "ignoration of phase shifts," the equation (and graph) for the ratio of voltages is quite accurate at both low and high frequencies and is only modestly in error around the crossover

[37] Later we'll see *inductors*, which also have a 90° phase shift (though of the opposite sign), and so are likewise characterized by a reactance X_L.

frequency, where the correct ratio is $V_{out}/V_{in} = 1/\sqrt{2} \approx 0.7$, rather than the 0.5 that we got.[38]

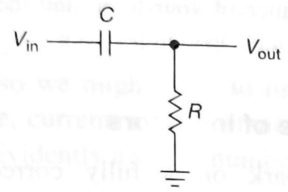

Figure 1.92. Highpass filter.

B. *RC* highpass filter (approximate)

You get the reverse behavior (pass high frequencies, block low) by interchanging *R* and *C*, as in Figure 1.92. Treating it as a frequency-dependent voltage divider, and ignoring phase shifts once again, we get (see Figure 1.91)

$$\frac{V_{out}}{V_{in}} \asymp \frac{R}{R+X_C} = \frac{R}{R+1/\omega C} = \frac{\omega RC}{1+\omega RC} \quad \textit{(approximate!)} \tag{1.28}$$

High frequencies (above the same crossover frequency as before, $\omega \gg \omega_0 = 1/RC$) pass through (because the capacitor's reactance is much smaller than *R*), whereas frequencies well below the crossover are blocked (the capacitor's reactance is much larger than *R*). As before, the equation and graph are accurate at both ends, and only modestly in error at the crossover, where the correct ratio is, again, $V_{out}/V_{in} = 1/\sqrt{2}$.

C. Blocking capacitor

Sometimes you want to let some band of signal frequencies pass through a circuit, but you want to block any steady dc voltage that may be present (we'll see how this can happen when we learn about amplifiers in the next chapter). You can do the job with an *RC* highpass filter if you choose the crossover frequency correctly: a highpass filter always blocks dc, so what you do is choose component values so that the crossover frequency is *below* all frequencies of interest. This is one of the more frequent uses of a capacitor and is known as a dc *blocking capacitor*.

For instance, every stereo audio amplifier has all its inputs capacitively coupled, because it doesn't know what dc level its input signals might be riding on. In such a coupling application you always pick *R* and *C* so that all frequencies of interest (in this case, 20 Hz to 20 kHz) are passed without loss (attenuation). That determines the product *RC*: $RC > 1/\omega_{min}$, where for this case you might choose $f_{min} \approx 5\,\text{Hz}$, and so $RC = 1/\omega_{min} = 1/2\pi f_{min} \approx 30\,\text{ms}$.

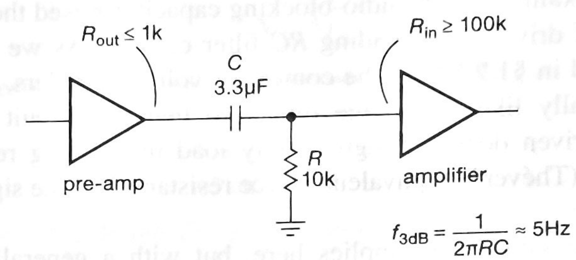

Figure 1.93. "Blocking capacitor": a highpass filter for which all signal frequencies of interest are in the passband.

You've got the product, but you still have to choose individual values for *R* and *C*. You do this by noticing that the input signal sees a load equal to *R* at signal frequencies (where *C*'s reactance is small – it's just a piece of wire there), so you choose *R* to be a reasonable load, i.e., not so small that it's hard to drive, and not so large that the circuit is prone to signal pickup from other circuits in the box. In the audio business it's common to see a value of 10 kΩ, so we might choose that value, for which the corresponding *C* is 3.3 μF (Figure 1.93). The circuit connected to the output should have an input resistance much greater than 10 kΩ, to avoid loading effects on the filter's output; and the driving circuit should be able to drive a 10 kΩ load without significant attenuation (loss of signal amplitude) to prevent circuit loading effects by the filter on the signal source. It's worth noting that our approximate model, ignoring phase shifts, is perfectly adequate for blocking capacitor design; that is because the signal band is fully in the passband, where the effects of phase shifts are negligible.

In this section we've been thinking in the frequency domain (sinewaves of frequency *f*). But it's useful to think in the time domain, where, for example, you might use a blocking capacitor to couple pulses, or square waves. In such situations you encounter waveform *distortion*, in the form of "droop" and overshoot (rather than the simple amplitude attenuation and phase shift you get with sinusoidal waves). Thinking in the time domain, the criterion you use to avoid waveform distortion in a pulse of duration *T* is that the time constant $\tau = RC \gg T$. The resulting droop is approximately T/τ (followed by a comparable overshoot at the next transition).

You often need to know the reactance of a capacitor at a given frequency (e.g., for design of filters). Figure 1.100 in §1.7.8 provides a very useful graph covering large ranges

[38] Of course, it fails to predict anything about phase shifts in this circuit. As we'll see later, the output signal's phase lags the input by 90° at high frequencies, going smoothly from 0° at low frequencies, with a 45° lag at ω_0 (see Figure 1.104 in §1.7.9).

of capacitance and frequency, giving the value of $X_C = 1/2\pi fC$.

D. Driving and loading *RC* filters

This example of an audio-blocking capacitor raised the issue of driving and loading *RC* filter circuits. As we discussed in §1.2.5A, in the context of voltage dividers, you generally like to arrange things so that the circuit being driven does not significantly load the driving resistance (Thévenin equivalent source resistance) of the signal source.

The same logic applies here, but with a generalized kind of resistance that includes the reactance of capacitors (and inductors), known as *impedance*. So a signal source's impedance should generally be small compared with the impedance of the thing being driven.[39] We'll have a precise way of talking about impedance shortly, but it's correct to say that, apart from phase shifts, the impedance of a capacitor is equal to its reactance.

What we want to know, then, are the input and output impedances of the two simple *RC* filters (lowpass and highpass). This sounds complicated, because there are four impedances, and they all vary with frequency. However, if you ask the question the right way, the answer is simple, and the same in all cases!

First, assume that in each case the right thing is being done to the other end of the filter: when we ask the input impedance, we assume the output drives a high impedance (compared with its own); and when we ask the output impedance, we assume the input is driven by a signal source of low internal (Thévenin) impedance. Second, we dispose of the variation of impedances with frequency by asking only for the *worst-case* value; that is, we care what only the *maximum* output impedance of a filter circuit may be (because that is the worst for driving a load), and we care about only the *minimum* input impedance (because that is the hardest to drive).

Now the answer is astonishingly simple: in all cases the worst-case impedance is just *R*.

Exercise 1.23. Show that the preceding statement is correct.

So, for example, if you want to hang an *RC* lowpass filter onto the output of an amplifier whose output resistance is 100 Ω, start with $R = 1\text{k}$, then choose *C* for the breakpoint you want. Be sure that whatever loads the output has an input impedance of at least 10k. You can't go wrong.

Exercise 1.24. Design a two-stage "bandpass" *RC* filter, in which the first stage is highpass with a breakpoint of 100 Hz, and the second stage is lowpass with a breakpoint of 10 kHz. Assume the input signal source has an impedance of 100Ω. What is the worst-case output impedance of your filter, and therefore what is the minimum recommended load impedance?

1.7.2 Reactance of inductors

Before we embark on a fully correct treatment of impedance, replete with complex exponentials and the like, let's use our approximation tricks to figure out the reactance of an inductor.

It goes as before: we imagine an inductor *L* driven by a sinusoidal voltage source of angular frequency ω such that a current $I(t) = I_0 \sin \omega t$ is flowing.[40] Then the voltage across the inductor is

$$V(t) = L\frac{dI(t)}{dt} = L\omega I_0 \cos \omega t.$$

And so the ratio of *magnitudes* of voltage to current – the resistance-like quantity called *reactance* – is just

$$\frac{|V|}{|I|} = \frac{L\omega I_0}{I_0} = \omega L.$$

So. for an inductor,

$$X_L = \omega L. \tag{1.29}$$

Inductors, like capacitors, have a frequency-dependent reactance; however, here the reactance *increases* with increasing frequency (the opposite of capacitors, where it *decreases* with increasing frequency). So, in the simplest view, a series inductor can be used to pass dc and low frequencies (where its reactance is small) while blocking high frequencies (where its reactance is high). You often see inductors used this way, particularly in circuits that operate at radio frequencies; in that application they're sometimes called *chokes*.

1.7.3 Voltages and currents as complex numbers

At this point it is necessary to get into some complex algebra; you may wish to skip over the math in some of the following sections, taking note of the results as we derive them. A knowledge of the detailed mathematics is not necessary for understanding the remainder of the book. Very little mathematics will be used in later chapters. The section ahead is easily the most difficult for the reader with little mathematical preparation. *Don't be discouraged!*

As we've just seen, there can be phase shifts between

---

[39] With two important exceptions, namely, transmission lines and current sources.

[40] We take the easy path here by specifying the *current*, rather than the voltage; we are rewarded with a simple derivative (rather than a simple integral!).

the voltage and current in an ac circuit being driven by a sinewave at some frequency. Nevertheless, as long as the circuit contains only *linear* elements (resistors, capacitors, inductors), the magnitudes of the currents everywhere in the circuit are still proportional to the magnitude of the driving voltage, so we might hope to find some generalization of voltage, current, and resistance in order to rescue Ohm's law. Evidently a single number won't suffice to specify the current, say, at some point in the circuit, because we must somehow have information about both the magnitude and phase shift.

Although we can imagine specifying the magnitudes and phase shifts of voltages and currents at any point in the circuit by writing them out explicitly, e.g., $V(t) = 23.7\sin(377t + 0.38)$, it turns out that we can meet our requirements more simply by using the algebra of complex numbers to *represent* voltages and currents. Then we can simply add or subtract the complex number representations, rather than laboriously having to add or subtract the actual sinusoidal functions of time themselves. Because the actual voltages and currents are real quantities that vary with time, we must develop a rule for converting from actual quantities to their representations, and vice versa. Recalling once again that we are talking about a single sinewave frequency, ω, we agree to use the following rules.

1. Voltages and currents are *represented* by the complex quantities $\mathbf{V}$ and $\mathbf{I}$. The voltage $V_0\cos(\omega t + \phi)$ is to be represented by the complex number $V_0e^{j\phi}$. Recall that $e^{j\phi} = \cos\phi + j\sin\phi$, where $j = \sqrt{-1}$.
2. We obtain *actual* voltages and currents by multiplying their complex number representations by $e^{j\omega t}$ and then taking the real part: $V(t) = \mathcal{R}e(\mathbf{V}e^{j\omega t})$, $I(t) = \mathcal{R}e(\mathbf{I}e^{j\omega t})$.

In other words,

circuit voltage versus time		complex number representation
$V_0\cos(\omega t + \phi)$	$\longrightarrow$ $\longleftarrow$ multiply by $e^{j\omega t}$ and take real part	$V_0e^{j\phi} = a + jb$

(In electronics, the symbol j is used instead of i in the exponential in order to avoid confusion with the symbol i meaning small-signal current.) Thus, in the general case, the actual voltages and currents are given by

$$V(t) = \mathcal{R}e(\mathbf{V}e^{j\omega t})$$
$$= \mathcal{R}e(\mathbf{V})\cos\omega t - \mathcal{I}m(\mathbf{V})\sin\omega t$$

$$I(t) = \mathcal{R}e(\mathbf{I}e^{j\omega t})$$
$$= \mathcal{R}e(\mathbf{I})\cos\omega t - \mathcal{I}m(\mathbf{I})\sin\omega t.$$

For example, a voltage whose complex representation is

$$\mathbf{V} = 5j$$

corresponds to a (real) voltage versus time of

$$V(t) = \mathcal{R}e[5j\cos\omega t + 5j(j)\sin\omega t]$$
$$= -5\sin\omega t \text{ volts.}$$

1.7.4 Reactance of capacitors and inductors

With this convention we can apply complex Ohm's law correctly to circuits containing capacitors and inductors, just as for resistors, once we know the reactance of a capacitor or inductor. Let's find out what these are. We begin with a simple (co)sinusoidal voltage $V_0\cos\omega t$ applied across a capacitor:

$$V(t) = \mathcal{R}e(V_0e^{j\omega t}).$$

Then, using $I = C(dV(t)/dt)$, we obtain

$$I(t) = -V_0C\omega\sin\omega t = \mathcal{R}e\left(\frac{V_0e^{j\omega t}}{-j/\omega C}\right),$$
$$= \mathcal{R}e\left(\frac{V_0e^{j\omega t}}{\mathbf{Z}_C}\right)$$

i.e., for a capacitor

$$\mathbf{Z}_C = -j/\omega C \quad (= -jX_C);$$

$\mathbf{Z}_C$ is the impedance of a capacitor at frequency ω; it is equal in magnitude to the reactance $X_C = 1/\omega C$ that we found earlier, but with a factor of $-j$ that accounts for the 90° leading phase shift of current versus voltage. As an example, a 1 μF capacitor has an impedance of $-2653j\,\Omega$ at 60 Hz, and $-0.16j\,\Omega$ at 1 MHz. The corresponding reactances are 2653 Ω and 0.16 Ω.[41] Its reactance (and also its impedance) at dc is infinite.

If we did a similar analysis for an inductor, we would find

$$\mathbf{Z}_L = j\omega L \quad (= jX_L).$$

A circuit containing only capacitors and inductors always has a purely imaginary impedance, meaning that the voltage and current are always 90° out of phase – it is purely reactive. When the circuit contains resistors, there is also

[41] Note the convention that the reactance X_C is a real number (the 90° phase shift is implicit in the term "reactance"), but the corresponding impedance is purely imaginary: $\mathbf{Z} = R - jX$.

a real part to the impedance. The term "reactance" in that case means the imaginary part only.

1.7.5 Ohm's law generalized

With these conventions for representing voltages and currents, Ohm's law takes a simple form. It reads simply

$$\mathbf{I} = \mathbf{V}/\mathbf{Z},$$
$$\mathbf{V} = \mathbf{IZ},$$

where the voltage represented by $\mathbf{V}$ is applied across a circuit of impedance $\mathbf{Z}$, giving a current represented by $\mathbf{I}$. The complex impedance of devices in series or parallel obeys the same rules as resistance:

$$\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \cdots \quad \text{(series)}, \tag{1.30}$$

$$\mathbf{Z} = \frac{1}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \cdots} \quad \text{(parallel)}. \tag{1.31}$$

Finally, for completeness we summarize here the formulas for the impedance of resistors, capacitors, and inductors:

$$\begin{aligned} \mathbf{Z}_\mathrm{R} &= R && \text{(resistor)}, \\ \mathbf{Z}_\mathrm{C} &= -j/\omega C = 1/j\omega C && \text{(capacitor)}, \\ \mathbf{Z}_\mathrm{L} &= j\omega L && \text{(inductor)}. \end{aligned} \tag{1.32}$$

With these rules we can analyze many ac circuits by the same general methods we used in handling dc circuits, i.e., application of the series and parallel formulas and Ohm's law. Our results for circuits such as voltage dividers will look nearly the same as before. For multiply-connected networks we may have to use Kirchhoff's laws, just as with dc circuits, in this case using the complex representations for V and I: the sum of the (complex) voltage drops around a closed loop is zero, and the sum of the (complex) currents into a point is zero. The latter rule implies, as with dc circuits, that the (complex) current in a series circuit is the same everywhere.

Exercise 1.25. Use the preceding rules for the impedance of devices in parallel and in series to derive the formulas (1.17) and (1.18) for the capacitance of two capacitors (a) in parallel and (b) in series. *Hint*: in each case, let the individual capacitors have capacitances C_1 and C_2. Write down the impedance of the parallel or series combination; then equate it to the impedance of a capacitor with capacitance C. Then find C.

Let's try out these techniques on the simplest circuit imaginable, an ac voltage applied across a capacitor, which we looked at earlier, in §1.7.1. Then, after a brief look at power in reactive circuits (to finish laying the groundwork), we'll analyze (correctly, this time) the simple but extremely important and useful RC lowpass and highpass filter circuits.

Imagine putting a 1 μF capacitor across a 115 volts (rms) 60 Hz powerline. What current flows? Using complex Ohm's law, we have

$$\mathbf{Z} = -j/\omega C.$$

Therefore, the current is given by

$$\mathbf{I} = \mathbf{V}/\mathbf{Z}.$$

The phase of the voltage is arbitrary, so let us choose $\mathbf{V} = A$, i.e., $V(t) = A\cos\omega t$, where the amplitude $A = 115\sqrt{2} \approx 163$ volts. Then

$$\mathbf{I} = j\omega CA \approx -0.061\sin\omega t.$$

The resulting current has an amplitude of 61 mA (43 mA rms) and leads the voltage by 90°. This agrees with our previous calculation. More simply, we could have noticed that the impedance of the capacitor is negative imaginary, so whatever the absolute phase of V, the phase of I_{cap} must lead by 90°. And in general the phase angle between current and voltage, for any two-terminal RLC circuit, is equal to the angle of the (complex) impedance of that circuit.

Note that if we wanted to know just the magnitude of the current, and didn't care what the relative phase was, we could have avoided doing any complex algebra: if

$$\mathbf{A} = \mathbf{B}/\mathbf{C},$$

then

$$A = B/C,$$

where A, B, and C are the magnitudes of the respective complex numbers; this holds for multiplication, also (see Exercise 1.26). Thus, in this case,

$$I = V/Z = \omega CV.$$

This trick, which we used earlier (because we didn't know any better), is often useful.

Surprisingly, there is no power dissipated by the capacitor in this example. Such activity won't increase your electric bill; you'll see why in the next section. Then we will go on to look at circuits containing resistors and capacitors with our complex Ohm's law.

Exercise 1.26. Show that, if $\mathbf{A} = \mathbf{BC}$, then $A = BC$, where A, B, and C are magnitudes. *Hint*: represent each complex number in polar form, i.e., $\mathbf{A} = Ae^{i\theta}$.

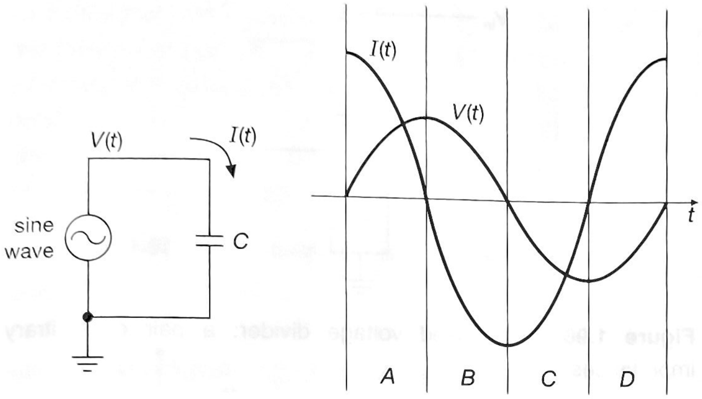

Figure 1.94. The power delivered to a capacitor is zero over a full sinusoidal cycle, owing to the 90° phase shift between voltage and current.

1.7.6 Power in reactive circuits

The instantaneous power delivered to any circuit element is always given by the product $P = VI$. However, in reactive circuits where V and I are not simply proportional, you can't just multiply their amplitudes together. Funny things can happen; for instance, the sign of the product can reverse over one cycle of the ac signal. Figure 1.94 shows an example. During time intervals A and C, power is being delivered to the capacitor (albeit at a variable rate), causing it to charge up; its stored energy is increasing (power is the rate of change of energy). During intervals B and D, the power delivered to the capacitor is negative; it is discharging. The average power over a whole cycle for this example is in fact exactly zero, a statement that is always true for any purely reactive circuit element (inductors, capacitors, or any combination thereof). If you know your trigonometric integrals, the next exercise will show you how to prove this.

Exercise 1.27. Optional exercise: prove that a circuit whose current is 90° out of phase with the driving voltage consumes no power, averaged over an entire cycle.

How do we find the average power consumed by an arbitrary circuit? In general, we can imagine adding up little pieces of VI product, then dividing by the elapsed time. In other words,

$$P = \frac{1}{T}\int_0^T V(t)I(t)\,dt, \tag{1.33}$$

where T is the time for one complete cycle. Luckily, that's almost never necessary. Instead, it is easy to show that the average power is given by

$$P = \mathcal{R}e(\mathbf{VI}^*) = \mathcal{R}e(\mathbf{V}^*\mathbf{I}), \tag{1.34}$$

where $\mathbf{V}$ and $\mathbf{I}$ are complex rms amplitudes (and an asterisk means *complex conjugate* – see the math review, Appendix A, if this is unfamiliar).

Let's take an example. Consider the preceding circuit, with a 1 volt (rms) sinewave driving a capacitor. We'll do everything with rms amplitudes, for simplicity. We have

$$\mathbf{V} = 1,$$

$$\mathbf{I} = \frac{\mathbf{V}}{-j/\omega C} = j\omega C,$$

$$P = \mathcal{R}e(\mathbf{VI}^*) = \mathcal{R}e(-j\omega C) = 0.$$

That is, the average power is zero, as stated earlier.

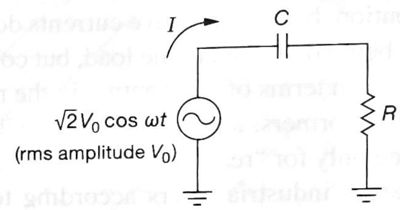

Figure 1.95. Power and power factor in a series *RC* circuit.

As another example, consider the circuit shown in Figure 1.95. Our calculations go like this:

$$\mathbf{Z} = R - \frac{j}{\omega C},$$

$$\mathbf{V} = V_0,$$

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{V_0}{R - (j/\omega C)} = \frac{V_0\,[R + (j/\omega C)]}{R^2 + (1/\omega^2 C^2)},$$

$$P = \mathcal{R}e(\mathbf{VI}^*) = \frac{V_0^2 R}{R^2 + (1/\omega^2 C^2)}.$$

(In the third line we multiplied numerator and denominator by the complex conjugate of the denominator in order to make the denominator real.) The calculated power[42] is less than the product of the magnitudes of $\mathbf{V}$ and $\mathbf{I}$. In fact, their ratio is called the *power factor*:

$$|\mathbf{V}||\mathbf{I}| = \frac{V_0^2}{[R^2 + (1/\omega^2 C^2)]^{1/2}},$$

$$\text{power factor} = \frac{\text{power}}{|\mathbf{V}||\mathbf{I}|} = \frac{R}{[R^2 + (1/\omega^2 C^2)]^{1/2}}$$

[42] It's always a good idea to check limiting values: here we see that $P \to V^2/R$ for large C; and for small C the magnitude of the current $|I| \to V_0/X_C$, or $V_0\omega C$, thus $P \to I^2R = V_0^2\omega^2C^2R$, in agreement at both limits.

in this case. The power factor is the cosine of the phase angle between the voltage and the current, and it ranges from 0 (purely reactive circuit) to 1 (purely resistive). A power factor of less than 1 indicates some component of reactive current.[43] It's worth noting that the power factor goes to unity, and the dissipated power goes to V^2/R, in the limit of large capacitance (or of high frequency), where the reactance of the capacitor becomes much less than R.

Exercise 1.28. Show that all the average power delivered to the preceding circuit winds up in the resistor. Do this by computing the value of V_R^2/R. What is that power, in watts, for a series circuit of a 1 μF capacitor and a 1.0k resistor placed across the 115 volt (rms), 60 Hz powerline?

Power factor is a serious matter in large-scale electrical power distribution, because reactive currents don't result in useful power being delivered to the load, but cost the power company plenty in terms of I^2R heating in the resistance of generators, transformers, and wiring. Although residential users are billed only for "real" power [$\mathscr{R}e(\mathbf{VI}^*)$], the power company charges industrial users according to the power factor. This explains the capacitor yards that you see behind large factories, built to cancel the inductive reactance of industrial machinery (i.e., motors).

Exercise 1.29. Show that adding a series capacitor of value $C = 1/\omega^2 L$ makes the power factor equal 1.0 in a series RL circuit. Now do the same thing, but with the word "series" changed to "parallel."

1.7.7 Voltage dividers generalized

Our original voltage divider (Figure 1.6) consisted of a pair of resistors in series to ground, input at the top and output at the junction. The generalization of that simple resistive divider is a similar circuit in which either or both resistors are replaced with a capacitor or inductor (or a more complicated network made from R, L, and C), as in Figure 1.96. In general, the division ratio $V_{\text{out}}/V_{\text{in}}$ of such a divider is not constant, but depends on frequency (as we have already seen, in our approximate treatment of the lowpass and highpass filters in §1.7.1). The analysis is straightforward:

$$\mathbf{I} = \frac{\mathbf{V}_{\text{in}}}{\mathbf{Z}_{\text{total}}},$$

$$\mathbf{Z}_{\text{total}} = \mathbf{Z}_1 + \mathbf{Z}_2$$

[43] Or, for nonlinear circuits, it indicates that the current waveform is not proportional to the voltage waveform. More on this in §9.7.1.

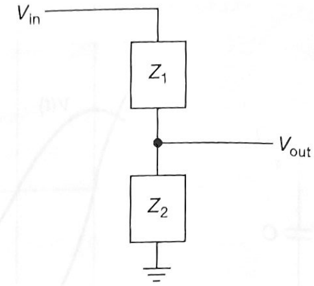

Figure 1.96. Generalized voltage divider: a pair of arbitrary impedances.

$$\mathbf{V}_{\text{out}} = \mathbf{I}\mathbf{Z}_2 = \mathbf{V}_{\text{in}} \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}.$$

Rather than worrying about this result in general, let's look at some simple (but very important) examples, beginning with the RC highpass and lowpass filters we approximated earlier.

1.7.8 *RC* highpass filters

We've seen that by combining resistors with capacitors it is possible to make frequency-dependent voltage dividers, owing to the frequency dependence of a capacitor's impedance $\mathbf{Z}_C = -j/\omega C$. Such circuits can have the desirable property of passing signal frequencies of interest while rejecting undesired signal frequencies. In this subsection and the next we revisit the simple lowpass and highpass RC filters, correcting the approximate analysis of §1.7.1; though simple, these circuits are important and widely used. Chapter 6 and Appendix E describe filters of greater sophistication.

Referring back to the classic RC highpass filter (Figure 1.92), we see that the complex Ohm's law (or the complex voltage-divider equation) gives

$$\mathbf{V}_{\text{out}} = \mathbf{V}_{\text{in}} \frac{R}{R - j/\omega C} = \mathbf{V}_{\text{in}} \frac{R(R + j/\omega C)}{R^2 + (1/\omega^2 C^2)}.$$

(For the last step, multiply top and bottom by the complex conjugate of the denominator.) Most often we don't care about the phase of V_{out}, just its amplitude:

$$V_{\text{out}} = (\mathbf{V}_{\text{out}} \mathbf{V}^*_{\text{out}})^{1/2}$$

$$= \frac{R}{[R^2 + (1/\omega^2 C^2)]^{1/2}} V_{\text{in}}.$$

Note the analogy to a resistive divider, where

$$V_{\text{out}} = \frac{R_2}{R_1 + R_2} V_{\text{in}}.$$

Here the impedance of the series RC combination

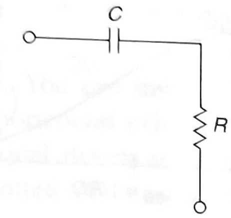

Figure 1.97. Input impedance of unloaded highpass filter.

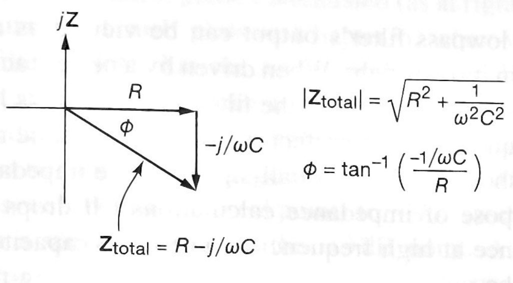

Figure 1.98. Impedance of series *RC*.

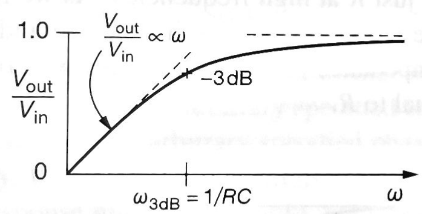

Figure 1.99. Frequency response of highpass filter. The corresponding phase shift goes smoothly from +90° (at $\omega = 0$), through +45° (at $\omega_{3\text{dB}}$), to 0° (at $\omega = \infty$), analogous to the lowpass filter's phase shift (Figure 1.104).

(Figure 1.97) is as shown in Figure 1.98. So the "response" of this circuit, ignoring phase shifts by taking magnitudes of the complex amplitudes, is given by

$$V_{\text{out}} = \frac{R}{[R^2 + (1/\omega^2 C^2)]^{1/2}} V_{\text{in}}$$
$$= \frac{2\pi f RC}{[1 + (2\pi f RC)^2]^{1/2}} V_{\text{in}} \qquad (1.35)$$

and looks as shown in Figure 1.99 (and earlier in Figure 1.91).

Note that we could have gotten this result immediately by taking the ratio of the *magnitudes* of impedances, as in Exercise 1.26 and the example immediately preceding it; the numerator is the magnitude of the impedance of the lower leg of the divider (*R*), and the denominator is the magnitude of the impedance of the series combination of *R* and *C*.

As we noted earlier, the output is approximately equal to the input at high frequencies (how high? $\omega \gtrsim 1/RC$) and goes to zero at low frequencies. The highpass filter is

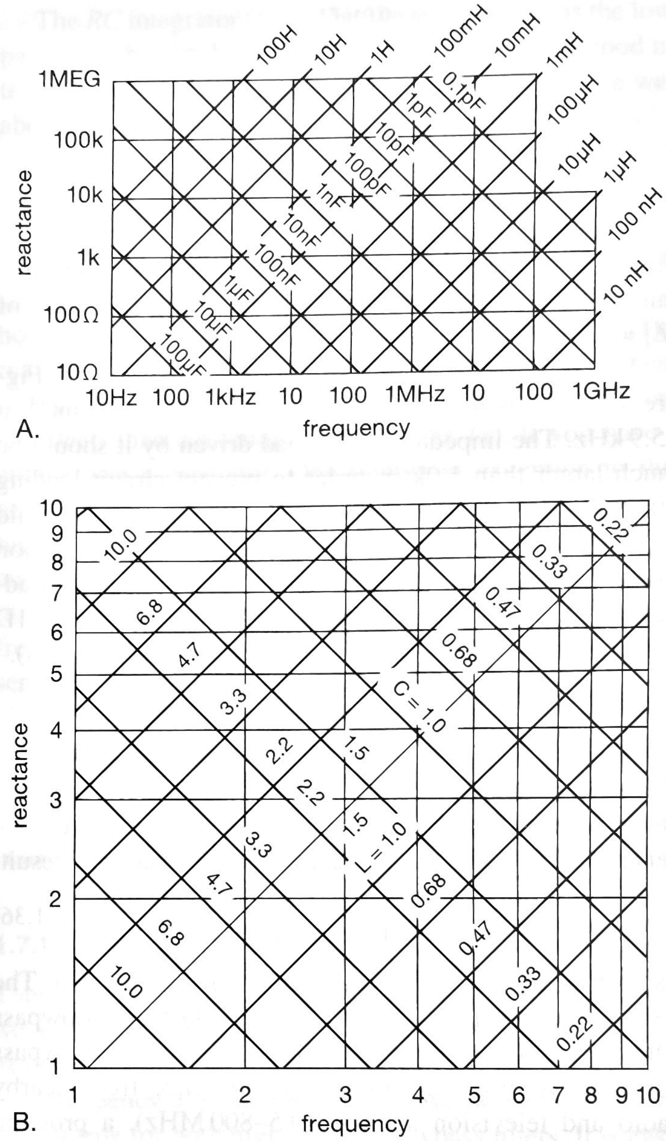

Figure 1.100. A: Reactance of inductors and capacitors versus frequency; all decades are identical, except for scale. B: A single decade from part A expanded, with standard 20% component values (EIA "E6") shown.

very common; for instance, the input to the oscilloscope can be switched to "ac coupling." That's just an *RC* highpass filter with the bend at about 10 Hz (you would use ac coupling if you wanted to look at a small signal riding on a large dc voltage). Engineers like to refer to the −3 dB "breakpoint" of a filter (or of any circuit that behaves like a filter). In the case of the simple *RC* high-pass filter, the −3 dB breakpoint is given by

$$f_{3\text{dB}} = 1/2\pi RC.$$

You often need to know the impedance of a capacitor at a given frequency (e.g., for the design of filters).

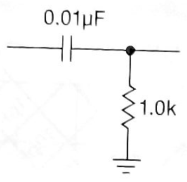

Figure 1.101. Highpass filter example.

Figure 1.100 provides a very useful graph covering large ranges of capacitance and frequency, giving the value of $|\mathbf{Z}| = 1/2\pi fC$.

As an example, consider the filter shown in Figure 1.101. It is a highpass filter with the 3 dB point[44] at 15.9 kHz. The impedance of a load driven by it should be much larger than 1.0k in order to prevent circuit loading effects on the filter's output, and the driving source should be able to drive a 1.0k load without significant attenuation (loss of signal amplitude) in order to prevent circuit loading effects by the filter on the signal source (recall §1.7.1D for worst-case source and load impedances of *RC* filters).

1.7.9 *RC* lowpass filters

Revisiting the lowpass filter, in which you get the opposite frequency behavior by interchanging *R* and *C* (Figure 1.90, repeated here as Figure 1.102), we find the accurate result

$$V_{out} = \frac{1}{(1+\omega^2 R^2 C^2)^{1/2}} V_{in} \qquad (1.36)$$

as seen in Figure 1.103 (and earlier in Figure 1.91). The 3 dB point is again at a frequency[45] $f = 1/2\pi RC$. Lowpass filters are quite handy in real life. For instance, a lowpass filter can be used to eliminate interference from nearby radio and television stations (0.5–800 MHz), a problem that plagues audio amplifiers and other sensitive electronic equipment.

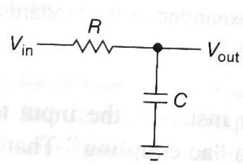

Figure 1.102. Lowpass filter.

Exercise 1.30. Show that the preceding expression for the response of an *RC* lowpass filter is correct.

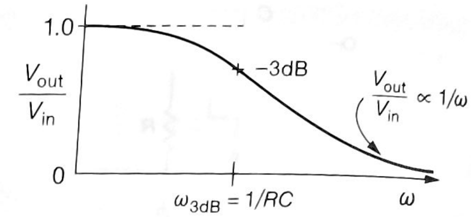

Figure 1.103. Frequency response of lowpass filter.

The lowpass filter's output can be viewed as a signal source in its own right. When driven by a perfect ac voltage (zero source impedance), the filter's output looks like *R* at low frequencies (the perfect signal source can be replaced with a short, i.e., by its small-signal source impedance, for the purpose of impedance calculations). It drops to zero impedance at high frequencies, where the capacitor dominates the output impedance. The signal driving the filter sees a load of *R* plus the load resistance at low frequencies, dropping to just *R* at high frequencies. As we remarked in §1.7.1D, the worst-case source impedance and the worst-case load impedance of an *RC* filter (lowpass or highpass) are both equal to *R*.

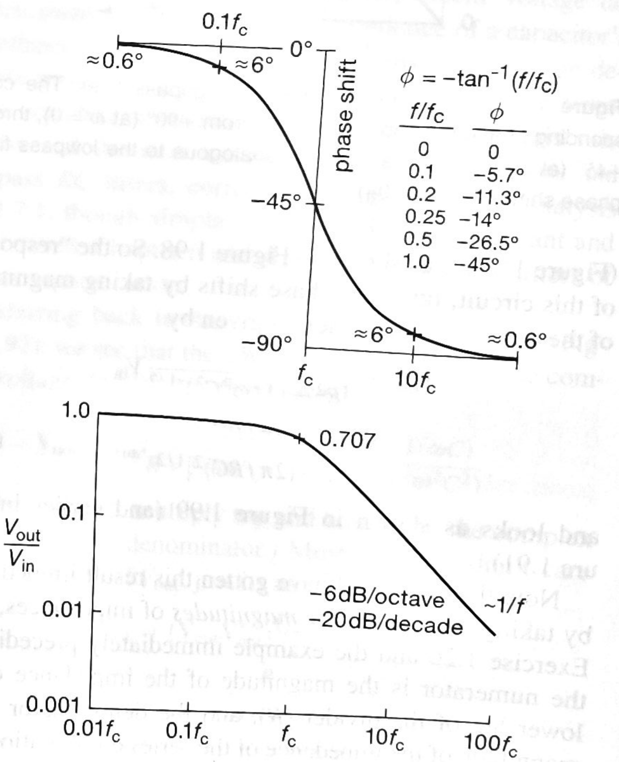

Figure 1.104. Frequency response (phase and amplitude) of lowpass filter plotted on logarithmic axes. Note that the phase shift is −45° at the −3 dB point and is within 6° of its asymptotic value for a decade of frequency change.

[44] One often omits the minus sign when referring to the −3 dB point.

[45] As mentioned in §1.7.1A, we often like to define the breakpoint frequency $\omega_0 = 1/RC$, and work with frequency ratios ω/ω_0. Then a useful form for the denominator in eq'n 1.36 is $\sqrt{1+(\omega/\omega_0)^2}$. The same applies to eq'n 1.35, where the numerator becomes ω/ω_0.

In Figure 1.104, we've plotted the same lowpass filter response with logarithmic axes, which is a more common way that it's done. You can think of the vertical axis as decibels, and the horizontal axis as octaves (or decades). On such a plot, equal distances correspond to equal ratios. We've also plotted the phase shift, using a linear vertical axis (degrees) and the same logarithmic frequency axis. This sort of plot is good for seeing the detailed response even when it is greatly attenuated (as at right); we'll see a number of such plots in Chapter 6, when we treat active filters. Note that the filter curve plotted here becomes a straight line at large attenuations, with a slope of −20 dB/decade (engineers prefer to say "−6 dB/octave"). Note also that the phase shift goes smoothly from 0° (at frequencies well below the breakpoint) to −90° (well above it), with a value of −45° at the −3 dB point. A rule of thumb for single-section *RC* filters is that the phase shift is $\approx 6°$ from its asymptotic value at $0.1f_{3\,\mathrm{dB}}$ and at $10f_{3\,\mathrm{dB}}$.

Exercise 1.31. Prove the last assertion.

An interesting question is the following: is it possible to make a filter with some arbitrary specified amplitude response and some other arbitrary specified phase response? Surprisingly, the answer is no: the demands of causality (i.e., that response must follow cause, not precede it) force a relationship between phase and amplitude response of realizable analog filters (known officially as the Kramers–Kronig relation).

1.7.10 *RC* differentiators and integrators in the frequency domain

The *RC* differentiator that we saw in §1.4.3 is exactly the same circuit as the highpass filter in this section. In fact, it can be considered as either, depending on whether you're thinking of waveforms in the time domain or response in the frequency domain. We can restate the earlier time-domain condition for its proper operation ($dV_{\mathrm{out}} \ll dV_{\mathrm{in}}$) in terms of the frequency response: for the output to be small compared with the input, the signal frequency (or frequencies) must be well below the 3 dB point. This is easy to check: suppose we have the input signal $V_{\mathrm{in}} = \sin \omega t$. Then, using the equation we obtained earlier for the differentiator output, we have

$$V_{\mathrm{out}} = RC\frac{d}{dt}\sin \omega t = \omega RC \cos \omega t,$$

and so $dV_{\mathrm{out}} \ll dV_{\mathrm{in}}$ if $\omega RC \ll 1$, i.e., $RC \ll 1/\omega$. If the input signal contains a range of frequencies, this must hold for the highest frequencies present in the input.

The *RC* integrator (§1.4.4) is the same circuit as the lowpass filter; by similar reasoning, the criterion for a good integrator is that the lowest signal frequencies must be well above the 3 dB point.

1.7.11 Inductors versus capacitors

Instead of capacitors, inductors can be combined with resistors to make lowpass (or highpass) filters. In practice, however, you rarely see *RL* lowpass or highpass filters. The reason is that inductors tend to be more bulky and expensive and perform less well (i.e., they depart further from the ideal) than capacitors (see Chapter *1x*). If you have a choice, use a capacitor. One important exception to this general statement is the use of ferrite beads and chokes in high-frequency circuits. You just string a few beads here and there in the circuit; they make the wire interconnections slightly inductive, raising the impedance at very high frequencies and preventing oscillations, without the added series resistance you would get with an *RC* filter. An *RF choke* is an inductor, usually a few turns of wire wound on a ferrite core, used for the same purpose in RF circuits. Note, however, that inductors are *essential* components in (a) *LC* tuned circuits (§1.7.14), and (b) switch-mode power converters (§9.6.4).

1.7.12 Phasor diagrams

There's a nice graphical method that can be helpful when we are trying to understand reactive circuits. Let's take an example, namely, the fact that an *RC* filter attenuates 3 dB at a frequency $f = 1/2\pi RC$, which we derived in §1.7.8. This is true for both highpass and lowpass filters. It is easy to get a bit confused here, because at that frequency the reactance of the capacitor equals the resistance of the resistor; so you might at first expect 6 dB attenuation (a factor of 1/2 in voltage). That is what you would get, for example, if you were to replace the capacitor with a resistor of the same impedance magnitude. The confusion arises because the capacitor is reactive, but the matter is clarified by a phasor diagram (Figure 1.105). The axes are the real (resistive) and imaginary (reactive) components of the impedance. In a series circuit like this, the axes also represent the (complex) voltage, because the current is the same everywhere. So for this circuit (think of it as an *RC* voltage divider) the input voltage (applied across the series *RC* pair) is proportional to the length of the hypotenuse, and the output voltage (across *R* only) is proportional to the length of the *R* leg of the triangle. The diagram represents the situation at the frequency where the capacitor's reactance equals *R*, i.e., $f = 1/2\pi RC$, and shows that the ratio of output voltage to input voltage is $1/\sqrt{2}$, i.e., -3 dB.

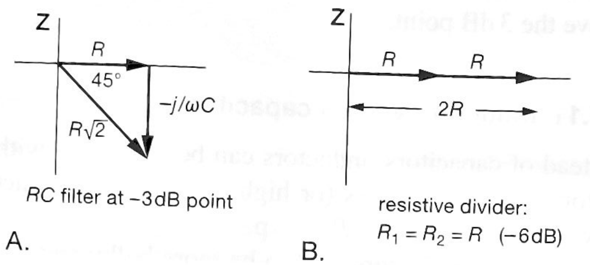

Figure 1.105. Phasor diagram for lowpass filter at 3 dB point.

The angle between the vectors gives the phase shift from input to output. At the 3 dB point, for instance, the output amplitude equals the input amplitude divided by the square root of 2, and it leads by 45° in phase. This graphical method makes it easy to read off amplitude and phase relationships in *RLC* circuits. You can use it, for example, to get the response of the highpass filter that we previously derived algebraically.

Exercise 1.32. Use a phasor diagram to derive the response of an *RC* high-pass filter: $V_{out} = V_{in}R/\sqrt{R^2 + (1/\omega^2 C^2)}$.

Exercise 1.33. At what frequency does an *RC* lowpass filter attenuate by 6 dB (output voltage equal to half the input voltage)? What is the phase shift at that frequency?

Exercise 1.34. Use a phasor diagram to obtain the lowpass filter response previously derived algebraically.

In the next chapter (§2.2.8) we'll see a nice example of phasor diagrams in connection with a constant-amplitude phase-shifting circuit.

1.7.13 "Poles" and decibels per octave

Look again at the response of the *RC* lowpass filter (Figures 1.103 and 1.104). Far to the right of the "knee" the output amplitude is dropping proportional to $1/f$. In one octave (as in music, one octave is twice the frequency) the output amplitude will drop to half, or -6 dB; so a simple *RC* filter has a 6 dB/octave falloff. You can make filters with several *RC* sections; then you get 12 dB/octave (two *RC* sections), 18 dB/octave (three sections), and so on. This is the usual way of describing how a filter behaves beyond the cutoff. Another popular way is to say a "three-pole filter," for instance, meaning a filter with three *RC* sections (or one that behaves like one). (The word "pole" derives from a method of analysis that is beyond the scope of this book and that involves complex transfer functions in the complex frequency plane, known by engineers as the "*s*-plane." This is discussed in the advanced volume, in Chapter *1x*.)

A caution on multistage filters: you can't simply cascade several identical filter sections in order to get a frequency response that is the concatenation of the individual responses. The reason is that each stage will load the previous one significantly (since they're identical), changing the overall response. Remember that the response function we derived for the simple *RC* filters was based on a zero-impedance driving source and an infinite-impedance load. One solution is to make each successive filter section have much higher impedance than the preceding one. A better solution involves active circuits like transistor or operational amplifier (op-amp) interstage "buffers," or active filters. These subjects will be treated in Chapters 2–4, 6, and 13.

1.7.14 Resonant circuits

When capacitors are combined with inductors or are used in special circuits called active filters, it is possible to make circuits that have very sharp frequency characteristics (e.g., a large peak in the response at a particular frequency), compared with the gradual characteristics of the *RC* filters we've seen so far. These circuits find applications in various audio and RF devices. Let's now take a quick look at *LC* circuits (there will be more on them, and active filters, in Chapter 6 and in Appendix E).

A. Parallel and series *LC* circuits

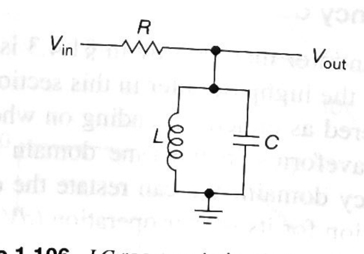

Figure 1.106. *LC* resonant circuit: bandpass filter.

First, consider the circuit shown in Figure 1.106. The impedance of the *LC* combination at frequency f is just

$$\frac{1}{Z_{LC}} = \frac{1}{Z_L} + \frac{1}{Z_C} = \frac{1}{j\omega L} - \frac{\omega C}{j}$$

$$= j\left(\omega C - \frac{1}{\omega L}\right),$$

i.e.,

$$Z_{LC} = \frac{j}{(1/\omega L) - \omega C}.$$