1x.5 Poles and Zeros, and the "s-Plane"

In the frequency domain it's certainly OK to think about a simple *RC* circuit (for example a single-section *RC* lowpass filter) basically as a circuit with a characteristic 3 dB frequency ($f_{3dB} = 1/2\pi RC$), and an asymptotic rolloff (6 dB/octave), with corresponding phase shift (0° for $f \ll f_{3dB}$, going smoothly through $-45°$ at f_{3dB}, and on to $-90°$ for $f \gg f_{3dB}$). This is how we treated it in Chapter 1 of AoE3 (see for example §1.7.8 and Figure 1.104), where we dealt also with the overall scaling of resistor values so that its input impedance did not heavily load the input signal source, and its output impedance was able to drive the load attached to its output. For a more complicated circuit you can write down complex impedances and nodal equations (Kirchhoff's voltage and current laws) to figure out the response; or you can punt and run a SPICE simulation (see Appendix J).

But there's another approach, favored by folks who spend their days designing filters, or worrying about stability of feedback and control circuits.[94] As one of them recently proclaimed, when asked about the utility of the complex frequency plane, "I *live* in the *s*-plane!" Where he lives (with lots of company) is in a plane that is an extension of real frequency (ω, or $2\pi f$) into the complex plane. Just to make things confusing, the real axis corresponds to imaginary frequency: $s = j\omega$ (you can blame this on the Laplace transform). So real frequencies lie along the *y*-axis, and imaginary frequencies (i.e., exponentials) lie along the *x*-axis.[95]

The utility of the *s*-plane comes about because the transfer function (usually written $H(s)$, for the complex ratio of V_{out} to V_{in}) of any linear circuit you can concoct from *R*'s, *L*'s, and *C*'s (or their active circuit analogs) can be written as a fraction that looks like

$$H(s) = a\frac{(s-Z_1)(s-Z_2)\cdots(s-Z_m)}{(s-P_1)(s-P_2)\cdots(s-P_n)} \quad (1x.26)$$

where the *Z*'s ("zeros") are points in the complex plane where the transfer function has value zero (i.e., a null), the *P*'s ("poles") are points where its value is divergent (i.e., infinite), and *a* is an overall multiplicative gain factor. The response of such a circuit is fully described by the location of its poles and zeros (along with the multiplying factor *a*). You'll commonly find engineers talking about the response of some system entirely in terms of the locations of its poles and zeros,[96] evidence of the intuition that can be acquired from such a technique. Let's see how this goes, with a couple of simple examples.

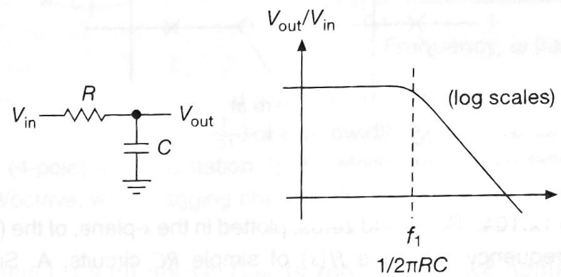

Figure 1x.103. Single-section *RC* lowpass filter circuit, and its frequency response (magnitude of V_{out}/V_{in}) plotted on logarithmic axes.

Example: Single-section *RC* lowpass

Figure 1x.103 shows the familiar single-section *RC* lowpass filter, along with the magnitude of its response V_{out}/V_{in} versus frequency (i.e., $|H(f)|$), plotted on logarithmic axes. As we saw in §1.7.9, it's simply a frequency-dependent voltage divider, whose (complex) response is found by taking ratios of (complex) impedances:

$$\frac{V_{out}}{V_{in}} = \frac{Z_C}{Z_R + Z_C} = \frac{-j/\omega C}{R - j/\omega C} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + sRC} \quad (1x.27)$$

where in the last step we've substituted $s = j\omega$. This has the form of eq'n 1x.26, with unit gain and with a pole lying on the *x*-axis at location $s = -1/RC$ (Fig. 1x.104A). The filter's response would be infinite at a (complex) frequency corresponding to such a pole; but *real* frequencies correspond to points on the *y*-axis, at which the response here is happily finite, and falling with increasing frequency, i.e., with increasing distance from the fearsome pole. We'll see some more intuition presently (look at Fig. 1x.106 if you cannot wait); for now you can deduce, from eq'n 1x.27, that the magnitude of the lowpass filter's response, at any real frequency *f* on the *y*-axis, is inversely proportional to

[94] Topics discussed in Chapters 2, 4, 4x and 6, and Appendix E.

[95] Physicists seem content with the concept of frequency as a complex number. But we've noticed that engineers sometimes exhibit discomfort; they prefer to say that the *s*-plane represents frequency along the *y*-axis, and another quantity, σ, along the *x*-axis. That is, a point in the *s*-plane has coordinates $s = \sigma + j\omega$.

[96] And of their trajectories in the complex plane as various parameters are adjusted; this is known as "root locus."

the distance between that point and the pole, scaled by unit gain at dc. And the corresponding phase shift is just the angle (relative to the x-axis) of the line joining those points.

Exercise 1x.1. Take the challenge: show that we tell the truth, starting with the last form of eq'n 1x.27, and confining your thinking to the s-plane.

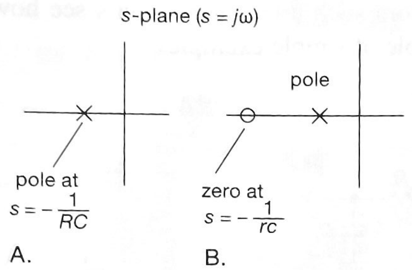

Figure 1x.104. Poles and zeros, plotted in the s-plane, of the (complex) frequency response $H(s)$ of simple RC circuits. A. Single-section lowpass filter (Fig. 1x.103). B. "Pole-zero" lowpass filter with added resistor r (Fig. 1x.105).

More complex circuits will have additional poles (and perhaps zeros, as we'll see next). For example, a filter consisting of a pair of cascaded RC lowpass sections will have a pair of poles on the negative x-axis; its amplitude response is the product (or sum of dB's) of the gain magnitudes for each pole (i.e., inverse of distances from each pole), and its phase shift is the sum of the phase shifts (i.e., angles from each pole). It follows that the ultimate rolloff of a cascade of n lowpass sections goes as $1/f^n$ (i.e., $6n$ dB/octave), with a phase shift of $-90n°$. But you knew that already, right?

Note that a pole on the y-axis corresponds to infinite response at the corresponding real frequency; a stable system has all its poles in the left half plane. And you can guess what happens if there's a pole in the right half plane.[97]

Example: Pole–zero network

When you design amplifiers with negative feedback (the meat of Chapter 4), you often find yourself in a situation where the amplifier itself (without feedback: "open-loop") has a low-frequency pole, and thus open-loop gain dropping at 6 dB/octave with 90° lagging phase shift at higher frequencies. In that case feedback is stable. But there may be a second pole at a higher frequency, causing the open-loop phase shift to approach 180° at a frequency where feedback can produce an oscillation.[98] An elegant solution is the use of a so-called "pole-zero" network, in which an s-plane zero is used to cancel the offending second pole.

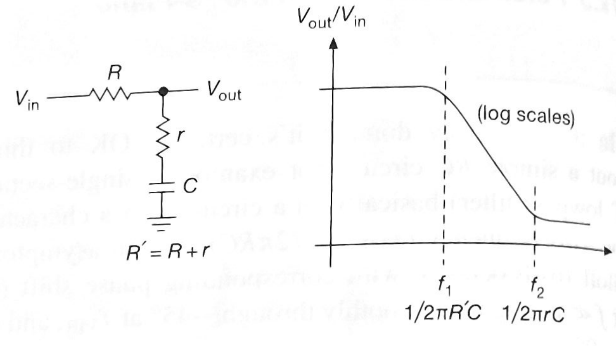

Figure 1x.105. Lowpass filter circuit with added resistor r, and its "pole-zero" frequency response. Its rolloff flattens to an asymptotic attenuation $V_{out}/V_{in} \rightarrow r/(r+R)$ above the rC characteristic frequency $f_2 = 1/2\pi rC$.

The addition of a resistor (Fig. 1x.105) converts the simple RC lowpass into a pole-zero network. The response is easy to figure:

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{R+1/sC}{R+r+1/sC} = \frac{1+srC}{1+s(R+r)C} \qquad (1x.28)$$

where we've sidestepped the j and ω stuff by going directly to the form $Z_C = 1/sC$. For this circuit we'll assume that $r \ll R$, and by inspection we see that the pole has moved to a slightly lower frequency $s = -1/(R+r)C$, and there is now a zero at a higher frequency $s = -1/rC$; these are indicated conventionally by the symbols × and o, as in Figure 1x.104B. The corresponding phase shift (not shown) goes from 0° at low frequencies, to nearly −90° well above f_1, and back down to 0° well above f_2.

Another place you encounter pole–zero response is in the output capacitor of a voltage regulator: the regulator's output resistance forms a pole with the capacitance, and the capacitor's ESR forms a zero. Low-dropout regulators are fussy about their output bypassing, frequently requiring the electrolytic capacitor's ESR to fall in some range of minimum and maximum – see, for example, the discussion in §9.3.7.

Visualizing the response

The transfer function $H(s)$, with its amusing poles and zeros, is defined over the whole complex s-plane. But if you want to know the network's response to signals of (real)

[97] "Do you see what happens, Larry?! Do you see what happens?!"

[98] As we'll see in §4.9.2, the condition is that the phase shift reaches 180° – negative feedback becomes positive feedback – at a frequency where the loop gain is unity.

frequency f, all that matters is the value (amplitude and phase) of $H(s)$ along the s-plane's y-axis. You can visualize a scalar function of two variables as a contour map (a "fishnet"), as in Figure 1x.106 where the magnitude of the transfer function (i.e., $|H(s)|$, the ratio of output amplitude to input amplitude; the circuit's "gain") is shown[99] along slices parallel to the axes. The slice along the imaginary axis reveals the response magnitude, here seen as quite flat out to unit angular frequency (the filter's -3 dB point).

Some folks tell us that they find helpful the image of a rubber sheet, draped over the poles (way up at infinity!), and nailed down to the x–y plane at the zeros and at the far horizons. Hey, whatever works for *them*...

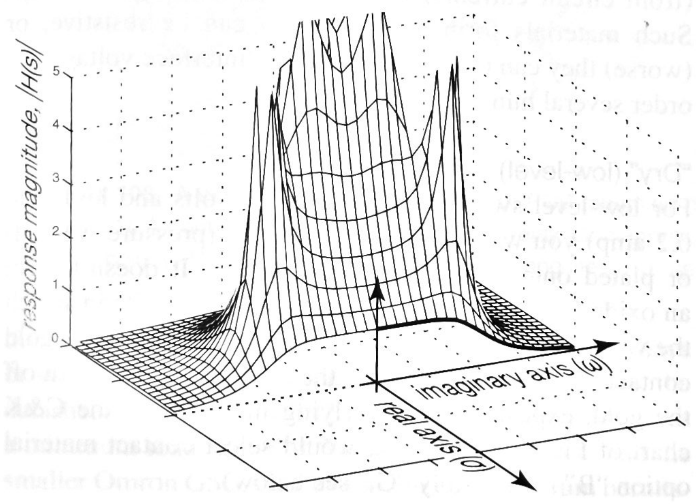

Figure 1x.106. Contour map ("fishnet") of the magnitude of the transfer function V_{out}/V_{in} for a 4-pole Butterworth lowpass filter. The slice along the $j\omega$-axis reveals the response to real frequencies ω. Filters with performance like this cannot be implemented as a passive *RC* circuit; to create off-axis poles you must include inductors, or (equivalently) use an active-filter implementation, see Chapter 6.

A few loose ends

A single-section *RC highpass* filter has a zero at the origin, and a pole at $s = -1/RC$. The zero corresponds to no response (a null) at dc, rising 6 dB/octave (and with a leading $+90°$ phase shift) at increasing frequency. That would continue forever, were it not for the pole that stops the rise at the characteristic 3 dB frequency $f_{3dB} = 1/2\pi RC$; it also cancels the leading phase shift, so the phase shift at

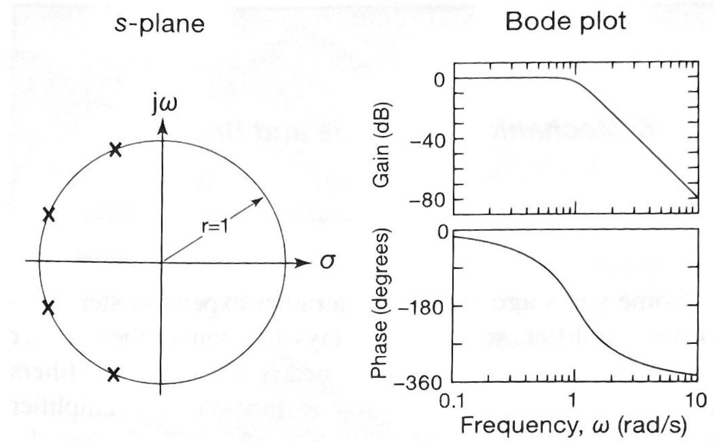

Figure 1x.107. The poles of a Butterworth lowpass filter lie on a circle of radius equal to its 3 dB bandwidth, seen here for a 4th-order (4-pole) implementation. In the stopband its gain falls off at 24 dB/octave, with a lagging phase shift approaching $-360°$.

frequencies well above f_{3dB} is zero. But, once again, you knew that already.

In a real *RLC* circuit there are at least as many poles as zeros. Poles and zeros can be real (as in the preceding examples) or complex ("off-axis"). The latter always occur as conjugate pairs, i.e., as a pole (or zero) pair located symmetrically above and below the s-plane's x-axis, as for example in the 4-pole Butterworth lowpass filter of Figures 1x.106 and 1x.107. For a network with both poles and zeros, the magnitude of the gain at any frequency (represented by a point along the y-axis) is proportional to the product of the distances to each zero, divided by the product of the distances to each pole (replace multiplications and divisions by sums and differences, when figuring gain as dB's). The phase shift is the sum of the angles from the zeros, minus the sum of the angles from the poles.

Exercise 1x.2. Try the recipe for yourself: (a) As you run your finger up the y-axis (frequency) in Figure 1x.104B, draw a sketch of the approximate phase shift of the pole-zero network (Fig. 1x.105) that it models. Does it agree with the statement made in the paragraph that includes eq'n 1x.28? (b) Now puzzle over the fact that, in the limit of high frequency, the ratio of distances to the zero and to the pole approaches unity, which would appear to say that the response goes to unity, a result we know to be wrong. Perhaps eq'n 1x.28 needs to be rearranged into the form of eq'n 1x.26, to bring the overall gain factor in front? Take it from there.

[99] Calculated, with our thanks, by Kent Lundberg, using a combination of his Matlab script and his keen intuition of network behavior.