A Bibliography of Sign-Colored Graphs and Colored Matroids and Their Invariants

Compiled by
Thomas Zaslavsky

Department of Mathematical Sciences
Binghamton University
Binghamton, New York, U.S.A. 13902-6000

E-mail: zaslav@math.binghamton.edu

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Preface

A *signed graph* is a graph whose edges are labeled by signs. Several kinds of labelled graph have been called "signed" yet are mathematically very different. I distinguish four types, of which this selection concerns the

- *Sign-colored graphs*, in which the edges are labelled from a two-element set that is acted upon by the sign group: $-$ interchanges labels, $+$ leaves them unchanged. This is the kind of "signed graph" found in knot theory. The natural generalization is to more colors and more general groups—or no group.

I use these standard symbols:

Γ	is a graph (undirected), possibly allowing loops and multiple edges. It is normally finite unless otherwise indicated.
Σ	is a signed graph. Its vertex and edge sets are V and E; its order is $n = \|V\|$. E_+, E_- are the sets of positive and negative edges and Σ_+, Σ_- are the corresponding spanning subgraphs (unsigned).
$[\Sigma]$	is the switching class of Σ.
$A(\ )$	is the adjacency matrix.
Φ	is a gain graph.
Ω	is a biased graph.
$l(\ )$	is the frustration index (= line index of imbalance).
$G(\ )$	is the bias matroid of a signed, gain, or biased graph.
$L(\ ), L_0(\ )$	are the lift and extended lift matroids.

Some standard terminology (much more will be found in the *Glossary* (Zaslavsky 1998c)):

polygon, circle: The graph of a simple closed path, or its edge set.

cycle: In a digraph, a coherently directed polygon, i.e., "dicycle". More generally: in an oriented signed, gain, or biased graph, a matroid circuit (usually, of the bias matroid) oriented to have no source or sink.

Subject Classification Codes

A code in *lower case* means the topic appears implicitly but not explicitly. A suffix **w** on **S**, **SG**, **VS** denotes signs used as weights, i.e., treated as the numbers $+1$ and -1, added, and (usually) the sum compared to 0. A suffix **c** on **SG**, **VS** denotes signs used as colors (often written as the numbers $+1$ and -1), usually permuted by the sign group. In a string of codes a colon precedes subtopics. A code may be refined through being suffixed by a parenthesised code, as **S**(**M**) denoting signed matroids (while **S**: **M** would indicate matroids of signed objects; thus **S**(**M**): **M** means matroids of signed matroids).

A Adjacency matrix, eigenvalues.
Alg Algorithms.
Appl Applications other than (**Chem**), (**Phys**), (**PsS**) (partial coverage).
Aut Automorphisms, symmetries, group actions.
B Balance (mathematical), cobalance.
Cov Covering graphs, double coverings.
D Duality (graphs, matroids, or matrices).
E Enumeration of types of signed graphs, etc.
ECol Edge coloring.
Exp Expository.
Exr Interesting exercises (in an expository work).
Fr Frustration index (line index of imbalance) and other measures of imbalance.
Gen Generalization.
GG Gain graphs, voltage graphs, biased graphs; includes Dowling lattices.
I Incidence matrix, Kirchhoff matrix.
K Signed complete graphs.
Knot Connections with knot theory (sparse coverage if signs are purely notational).
LG Line graphs.
M Matroids, chain-groups, flows.
N Numerical and algebraic invariants of signed graphs, etc.
O Orientations, bidirected graphs.
P All-negative or antibalanced signed graphs; parity-biased graphs.
Phys Applications in physics (partial coverage).
PsS Psychological, sociological, and anthropological applications (partial coverage).
Rand Random signs or gains, signed or gain graphs.
Ref Many references.
S Signed objects other than graphs and hypergraphs: mathematical properties.
SG Signed graphs: mathematical properties.
SH Signed hypergraphs: mathematical properties.
Str Structure theory.
Sw Switching of signs or gains.
T Topology applied to a graph; surface embeddings. (Not applications to topology.)
VS Vertex-signed graphs (“marked graphs”); signed vertices and edges.

C. Bankwitz

1930a Über die Torsionszahlen der alternierenden Knotes. *Math. Ann.* 103 (1930), 145–161.

Introduces the sign-colored graph of a link diagram. [Further work by numerous writers, e.g., S. Kinoshita *et al.* and esp. Kauffman (1989a) and successors.] **(Knot: SGc)**

C.M. Fortuin and P.W. Kasteleyn

1972a On the random cluster model. I. Introduction and relation to other models. *Physica* 57 (1972), 536–564. MR 50 #12107.

Most of the paper recasts classical physical and other models (percolation, ferromagnetic Ising, Potts, graph coloring, linear resistance) in a common form that is generalized in §7, "Random cluster model". The "cluster (generating) polynomial" $Z(\Gamma; p, \kappa)$, where $p \in \mathbb{R}^E$ and $\kappa \in \mathbb{R}$, is a 1-variable specialization of the general parametrized dichromatic polynomial. In the notation of Zaslavsky (1992b) it equals $Q_\Gamma(q, p; \kappa, 1)$, where $q_e = 1 - p_e$. Thus it partially anticipates the general polynomials of Przytycka and Przytycki (1988a), Traldi (1989a), and Zaslavsky (1992b) that were based on Kauffman's (1989a) sign-colored Tutte polynomial. A spanning-tree expansion appears only in the resistance model. A feature [that seems not to have been taken up] is the differentiation relation (7.7) connecting $\partial \ln Z/\partial q_e$ with [I think!] the expectation that the endpoints of e are disconnected in a subgraph. **(sgc: Gen: N, Phys)**

Jay R. Goldman and Louis H. Kauffman

1993a Knots, tangles, and electrical networks. *Adv. Appl. Math.* 14 (1993), 267–306. MR 94m:57013. Zbl. 806.57002. (()**SGc**: **N**, Knot, **Phys**)

François Jaeger

1992a On the Kauffman polynomial of planar matroids. In: Jaroslav Nešetřil and Miroslav Fiedler, eds., *Fourth Czechoslovak Symposium on Combinatorics, Graphs and Complexity* (Prachatice, 1990), pp. 117–127. Ann. Discrete Math., Vol. 51. North-Holland, Amsterdam, 1992. MR 94d:57016. Zbl. 763.05021.

(This is not the colored Tutte polynomial of Kauffman (1989a).) Jaeger shows that the Kauffman polynomial, originally defined for link diagrams and here transformed to an invariant of signed plane graphs, depends only on the edge signs and the polygon matroid. It can also be reformulated to be essentially independent of signs. *Problem.* Define a similar invariant for more general matroids. **(SGc, S(M): N, Knot)**

Louis H. Kauffman

See also J.R. Goldman.

1986a Signed graphs. Abstract 828-57-12, *Abstracts Amer. Math. Soc.* 7, No. 5 (1986), p. 307.

Announcement of (1989a). **(SGc: Knot: N)**

1988a New invariants in the theory of knots. *Amer. Math. Monthly* 95 (1988), 195–242. MR 89d:57005. Zbl. 657.57001.

A leisurely development of Kauffman's combinatorial bracket polynomial of a link diagram and the Jones and other knot polynomials, including the basics of (1989a). **(Knot, SGc: N: Exp)**

†1989a A Tutte polynomial for signed graphs. *Discrete Appl. Math.* 25 (1989), 105–127. MR 91c:05082. Zbl. 698.05026.

The Tutte polynomial, also called "Kauffman's bracket of a signed graph" and equivalent to his bracket of a link diagram, is defined by a sum over spanning trees of terms that depend on the signs and activities of the edges and nonedges of the tree. The point is that the deletion-contraction recurrence over an edge has parameters dependent on the color of the edge; also, the parameters of the two colors are related. The purpose is to develop the bracket of a link diagram combinatorially. §3.2, "Link diagrams": how link diagrams correspond to signed plane graphs. §4, "A polynomial for signed graphs", defines the general sign-colored graph polynomial $Q[\Sigma](A, B, d)$ by deletion-contraction, modified multiplication on components, and evaluation on graphs of loops and isthmi. §5, "A spanning tree expansion for $Q[G]$" [G means Σ], proves $Q[\Sigma]$ exists by producing a spanning-tree expansion, shown independent of the edge ordering by a direct argument. [No dichromatic form of $Q[\Sigma]$ appears in this paper.] §6, "Conclusion", remarks that $Q[\Sigma]$ is invariant under signed-graphic Reidemeister moves II and III. [This significant work, inspired by Thistlethwaite (1988a), led to independent but related generalizations by Przytycka and Przytycki (1988a), Schwärzler and Welsh (1993a), Traldi (1989a), and Zaslavsky (1992b) that were partially anticipated by Fortuin and Kasteleyn (1972a).] (**SGc: N, Knot**)

Shin'ichi Kinoshita

See also T. Yajima.

Shin'ichi Kinoshita and Hidetaka Terasaka

1957a On unions of knots. *Osaka Math J.* 9 (1957), 131–153. MR 20 #4846. Zbl. 080.17001.

Employs the sign-colored graph of a link diagram (Bankwitz 1930a) to form certain combinations of links. (**SGc: Knot**)

L. Lovász, L. Pyber, D.J.A. Welsh, and G.M. Ziegler

1995a Combinatorics in pure mathematics. In: R.L. Graham, M. Grötschel, and L. Lovász, eds., *Handbook of Combinatorics*, Vol. II, Ch. 41, pp. 2039–2082. North-Holland (Elsevier), Amsterdam, and MIT Press, Cambridge, Mass., 1995. MR 97f:00003. Zbl. 851.52017.

§7: "Knots and the Tutte polynomial", considers the signed graph of a knot diagram (pp. 2076–77). (**SGc: Knot**)

Kunio Murasugi

1988a On the signature of a graph. *C.R. Math. Rep. Acad. Sci. Canada* 10 (1988), 107–111. MR 89h:05056.

The signature of a sign-colored graph (see 1989a) is an invariant of the sign-colored graphic matroid. (**SGc: I, m**)

1989a On invariants of graphs with applications to knot theory. *Trans. Amer. Math. Soc.* 314 (1989), 1–49. MR 89k:57016. Zbl. 726.05051.

Studies a dichromatic form, $P_\Sigma(x, y, z)$, of Kauffman's (1989a) Tutte polynomial of a sign-colored graph. The deletion-contraction parameters are $a_\varepsilon = 1$, $b_\varepsilon = x^\varepsilon$ for $\varepsilon = \pm 1$; the initial values are such that $P_\Sigma(x, y, z) = y^{-1} Q_\Sigma(a, b; y, z)$ of Zaslavsky (1992b). The polynomial is shown to be, in effect, an invariant of the sign-colored graphic matroid.

Much unusual graph theory is in here. A special focus is the degrees of the polynomial. First Main Thm. 3.1: Formulas for the maximum and minimum combined degrees of $P_\Sigma(x, y, z)$. §7, "Signature of a graph", studies the signature (σ in the paper, s here) of the Kirchhoff matrix B_Σ obtained by changing the diagonal of $A(\Sigma)$ so the row sums are 0. Prop. 7.2 is a matrix-tree theorem [entirely different from that of Zaslavsky (1982a)]. The Second Main Thm. 8.1 bounds the signature: $|V| - 2\beta_0(\Sigma_-) + 1 \leq s \leq |V| - 2\beta_0(\Sigma_+) + 1$ ($\beta_0 =$ number of components), with equality characterized. The Kirchhoff matrix is further examined later on. §9, "Dual graphs": Differing from most studies, here the dual of a sign-colored plane graph is the planar dual with same edge signs [however, negating all colors is a triviality]. §10, "Periodic graphs": These graphs might be called branched covering graphs of signed gain graphs with finite cyclic gain group. [Thus they generalize the periodic graphs of Collatz (1978a) and others.] §§12–15 concern applications to knot theory. (**SGc: N, I, GG(Cov), D, Knot**)

1991a Invariants of graphs and their applications to knot theory. In: S. Jackowski, B. Oliver, and K. Pawałowski, eds., *Algebraic topology Poznań 1989* (Proc., Poznań, 1989), pp. 83–97. Lecture Notes in Math., Vol. 1474. Springer-Verlag, Berlin, 1991. MR 92m:57015. Zbl. 751.57007.

§§1–3 expound results from (1989a) on the dichromatic polynomial and the signature of a sign-colored graph and knot applications. §5 discusses the signed Seifert graph of a link diagram. (**SGc: N, I, Knot: Exp**)

1993a *Musubime riron to sono ono.* [*Knot Theory and Its Applications.*] 1993.

In Japanese. See (1996a). (**SGc: Knot**)

1996a *Knot Theory and Its Applications.* Birkhäuser, Boston, 1996. MR 97g:57011. Zbl. 864.57001.

Updated translation of (1993a) by Bohdan Kurpita. Pp. 36–37: Construction of signed plane graph from link diagram, and conversely. (**SGc: Knot**)

Kunio Murasugi and Jozef H. Przytycki

1993a *An Index of a Graph with Applications to Knot Theory.* Mem. Amer. Math. Soc., Vol. 106, No. 158. Amer. Math. Soc., Providence, R.I., 1993. MR 94d:57025. Zbl. 792.05047.

Ch. I, "Index of a graph". The "index" is the largest number of "independent" edges, where "independent" has a complicated recursive definition (unrelated to matchings), one of whose requirements is that the edges be "singular" (simple, i.e., nonmultiple links). The positive or negative index of a sign-colored graph is similar except that the independent edges must all be positive or negative. [The general notion is that of the index of a graph-subgraph pair. Edge signs are a way to pick out complementary subgraphs.] Thm. 2.4: Each of these indices is additive on blocks of a bipartite graph. The main interest, because of applications to knot theory, is in bipartite plane graphs. Ch. II, "Link theory": Pp. 26–27 define the sign-colored Seifert graph of an oriented link diagram, to which the graphical index theory is applied. (**SGc: N, D, Knot**)

Teresa M. Przytycka and Józef H. Przytycki

1988a Invariants of chromatic graphs. Tech. Rep. No. 88-22, Univ. of British Columbia, Vancouver, B.C., 1988.

Generalizing concepts from Kauffman (1989a). [See also Traldi (1989a) and Zaslavsky (1992b).] (**SGc: Gen: N, Knot**)

1993a Subexponentially computable truncations of Jones-type polynomials. In: Neil Robertson and Paul Seymour, eds., *Graph Structure Theory* (Proc., Seattle, 1991), pp. 63–108. Contemp. Math., Vol. 147. Amer. Math. Soc., Providence, R.I., 1993. MR 95c:57016. Zbl. 812.57010. (**SGc: N, Knot, Ref**)

Jozef H. Przytycki

See T.M. Przytycka.

W. Schwärzler and D.J.A. Welsh

1993a Knots, matroids and the Ising model. *Math. Proc. Cambridge Philos. Soc.* 13 (1993), 107–139. MR 94c:57019. Zbl. 797.57002.

Tutte and dichromatic polynomials of signed matroids, generalized from Kauffman (1989a); this is the 2-colored case of Zaslavsky's (1992b) strong Tutte functions of colored matroids. [For terminology see Zaslavsky 1992b.] Applications to knot theory.

§2, "A matroid polynomial", is foundational. Prop. 2.1 characterizes strong Tutte functions of signed matroids by two equations connecting their parameters and their values on signed coloops and loops. [If the function is 0 on positive coloops, the proof is incomplete and the functions $=0$ except on $M=\emptyset$ are missed.] Prop. 2.2: The Tutte (basis-expansion) polynomial of a function W of signed matroids is well defined iff W is a strong Tutte function. Eq. (2.8) says $W=$ the rank generating polynomial Q_Σ (here also called W) if certain variables are nonzero; (2.9) shows there are only 3 essential variables since, generically, only the ratio of parameters is essential [an observation that applies to general strong Tutte functions]. Prop. 2.5 computes Q_Σ of a 2-sum.

§3 adapts Q_Σ to Kauffman's and Murasugi's (1989a) signed-graph polynomials and simplifies some of the latter's results (esp. his chromatic degree). §4, "The anisotropic Ising model", concerns the Hamiltonian of a state of a signed graph. The partition function is essentially an evaluation of Q_Σ. §5, "The bracket polynomial", and §6, "The span of the bracket polynomial": Certain substitutions reduce Q_Σ to 1 variable; its properties are examined, esp. in light of knot-theoretic questions. Thm. 6.4 characterizes signed matroids with "full span" (a degree property). §7, "Adequate and semi-adequate link diagrams", generalizes those notions to signed matroids. §8, "Zero span matroids": when does span(bracket) $=0$? Yes if $M=M(\Sigma)$ where Σ reduces by Reidemeister moves to K_1, but the converse is open (and significant if true). (**Sc(M), SGc: N, Knot, Phys**)

Hidetaka Terasaka

See S. Kinoshita.

Morwen B. Thistlethwaite

1988a On the Kauffman polynomial of an adequate link. *Invent. Math.* 93 (1988), 285–296. MR 89g:57009. Zbl. 645.57007.

A 1-variable Tutte-style polynomial Γ_Σ of a sign-colored graph. Fix an edge ordering. For each spanning tree T and edge e, let $\mu_T(e)=-A^{3\tau_T(e)\sigma(e)}$ if e is active with respect to T, $A^{\tau_T(e)\sigma(e)}$ if it is inactive, where $\tau_T(e)=+1$ if $e\in T$, -1 if $e\notin T$. Then $\Gamma_\Sigma(A)=\sum_T\prod_{e\in T}\mu_T(e)$. [In the notation of Zaslavsky (1992a), $\Gamma_\Sigma(A) = Q_\Sigma$ with $a_\varepsilon = A^{-\varepsilon}, b_\varepsilon = A^\varepsilon$ for $\varepsilon = \pm 1$ and $u = v = -(A^2 + A^{-2})$.] §§3 and 4 show Γ_Σ is independent of the ordering. Other sections derive consequences for knot theory. [This marks the invention of a Tutte-style polynomial of a colored, or parametrized or weighted, graph or matroid, developed in Kauffman (1989a) and successors.] (**SGc: Knot: N**)

Lorenzo Traldi

1989a A dichromatic polynomial for weighted graphs and link polynomials. *Proc. Amer. Math. Soc.* 106 (1989), 279–286. MR 90a:57013. Zbl. 713.57003.

Generalizing Kauffman's (1989a) Tutte polynomial of a sign-colored graph, Traldi's "weighted dichromatic polynomial" $Q(\Gamma; t, z)$ is the $Q_\Gamma(1, w; t, z)$ of Zaslavsky (1992b), in which the deletion-contraction parameters $a_e = 1$ and $b_e = w(e)$, the weight of e. Thm. 2 gives the Tutte-style spanning-tree expansion. Thm. 4: Kauffman's Tutte polynomial $Q[\Sigma](A, B, d) = d^{-1} A^{|E_+|} B^{|E_-|} Q_{|\Sigma|}(1, w; d, d)$ for connected Σ, with $w(e) = (AB^{-1})^{\sigma(e)}$. [See Kauffman (1989a) for other generalizations. Traldi gives perhaps too much credit to Fortuin and Kasteleyn (1972a).]

P. 284: Invariance under Reidemeister moves of type II constrains the weighted dichromatic polynomial to, in essence, equal Kauffman's. Thus no generalization is evident in connection with general link diagrams. There is an interesting application to special link diagrams. (**SGc: Gen: N, Knot**)

D.J.A. Welsh [Dominic Welsh]

See also L. Lovász and W. Schwaärzler.

1992a On the number of knots and links. In: G. Halász, L. Lovász, D. Miklós, and T. Szönyi, eds., *Sets, Graphs and Numbers* (Proc., Budapest, 1991), pp. 713–718. Colloq. Math. Soc. János Bolyai, Vol. 60. North-Holland, Amsterdam, 1992. MR 94f:57010. Zbl. 799.57001.

The signed graph of a link diagram is employed to get an upper bound. (**SGc: E**)

1993a *Complexity: Knots, Colourings and Counting.* London Math. Soc. Lecture Note Ser., 186. Cambridge Univ. Press, Cambridge, Eng., 1993. MR 94m:57027. Zbl. 799.68008.

Includes very brief treatments of some appearances of signed graphs.

§2.2, "Tait colourings", defines the signed graph of a link diagram, mentioned again in observation (2.3.1) on alternating links and Prop (5.2.16) on "states models" (from Schwärzler and Welsh (1993a)). §5.6, "Thistlethwaite's nontriviality criterion": the criterion depends on the signed graph.

§2.5, "The braid index and the Seifert index of a link", defines the Seifert graph, a signed graph based on splitting the link diagram. (**SGc, Knot**)

§5.7, "Link invariants and statistical mechanics", defines a relatively simple spin model for signed graphs, with an arbitrary finite number of possible spin values. The partition function is related to link diagrams.

§4.2, "The Ising model", introduces the basic concepts in mathematical terms. §6.4, "The complexity of the Ising model", "Computing ground states of spin systems", pp. 105–107, discusses finding a ground state of the Ising model. This is described as the min-weight cut problem with weights the negatives [this is an error] of the Ising bond interaction values: that is,

the weighted frustration index problem in the negative [erroneous] of the Ising graph. It is the max-cut problem when the Ising graph is balanced (ferromagnetic) [should be antibalanced (antiferromagnetic)]. For external magnetic field, follows Barahona (1982a). **(sg: Fr, Phys)**

§3.6, "Ice models", counts "ice configurations" (certain graph orientations) via poise gains modulo 3, although the counting function is not gain-graphic. **(gg, N, Phys)**

§4.4: "The Ashkin–Teller–Potts model". This treatment of the Potts model has a different Hamiltonian from that of Fischer and Hertz (1991a). [It does not seem that Welsh intends to admit edge signs but if they are allowed then the Hamiltonian (without edge weights) is $-\sum \sigma(e_{ij})(\delta(s_i, s_j) - 1)$. Up to halving and a constant term, this is Doreian and Mrvar's (1996a) clusterability measure $P(\pi)$, with $\alpha = .5$, of the vertex partition induced by the state.] [Also cf. Fischer and Hertz (1991a).] **(cl, Phys)**

1993b The complexity of knots. In: John Gimbel, John W. Kennedy and Louis V. Quintas, eds., *Quo Vadis, Graph Theory?*, pp. 159–171. Ann. Discrete Math., Vol. 55. North-Holland, Amsterdam, 1993. MR 94c:57021. Zbl. 801.68086.

Link diagrams $\leftrightarrow$ dual pairs of sign-colored plane graphs: based on Yajima and Kinoshita (1957a). Unsolved algorithmic problems about knots based on link diagrams; in particular, triviality of diagrams is equivalent to *Problem* 4.2: A polynomial-time algorithm to decide whether the graphical Reidemeister moves can convert a given signed plane graph to one with edges all of one sign. **(SGc: D, Knot: Alg, Exp)**

1993c Knots and braids: some algorithmic questions. In: Neil Robertson and Paul Seymour, eds., *Graph Structure Theory* (Proc., Seattle, 1991), pp. 109–123. Contemp. Math., Vol. 147. Amer. Math. Soc., Providence, R.I., 1993. MR 94g:57014. Zbl. 792.05058.

§1 presents the sign-colored graph of a link diagram and §5, "Reidemeister graphs", describes Schwärtzler and Welsh (1993a). §3 defines the sign-colored Seifert graph. **(SGc. Sc(M): N, Alg, Knot: Exp)**

1997a Knots. In: Lowell W. Beineke and Robin J. Wilson, eds., *Graph Connections: Relationships between Graph Theory and other Areas of Mathematics*, Ch. 12, pp. 176–193. The Clarendon Press, Oxford, 1997. MR 99a:05001 (book). Zbl. 878.57001.

Mostly describes the signed graph of a link diagram and its relation to knot theory, including knot properties deducible directly from the signed graph, the Kauffman bracket and two-variable polynomials, etc. Similar to relevant parts of (1993a). **(SGc: Knot: N: Exp)**

Takeshi Yajima and Shin'ichi Kinoshita

1957a On the graphs of knots. *Osaka Math. J.* 9 (1957), 155–163. MR 20 #4845. Zbl. (e: 080.17002).

Examines the relationship between the two dual sign-colored graphs, Σ and Σ', of a link diagram (Bankwitz 1930a), translating the Reidemeister moves into graph operations and showing that they will convert Σ into Σ'. **(SGc: Knot)**

Thomas Zaslavsky

See also C. Greene, P. Hanlon, and P. Solé.

1992b Strong Tutte functions of matroids and graphs. *Trans. Amer. Math. Soc.* 334 (1992), 317–347. MR 93a:05047. Zbl. 781.05012.

Suppose that a function of matroids with labelled points is defined that is multiplicative on direct sums and satisfies a Tutte-Grothendieck recurrence with coefficients (the "parameters") that depend on the element being deleted and contracted, but not on the particular minor from which it is deleted and contracted: specifically, $F(M) = a_e F(M \setminus e) + b_e F(M/e)$ if e is not a loop or coloop in M. Thm. 2.1 completely characterizes such "strong Tutte functions" for each possible choice of parameters: there is one general type, defined by a rank generating polynomial $R_M(a, b; u, v)$ (the "parametrized rank generating polynomial") involving the parameters $a = (a_e)$, $b = (b_e)$ and the variables u, v, and there are a few special types that exist only for degenerate parameters. All have a Tutte-style basis expansion; indeed, a function has such an expansion iff it is a strong Tutte function (Thms. 7.1, 7.2). The Tutte expansion is a polynomial within each type. If the points are colored and the parameters of a point depend only on the color, one has a multicolored matroid generalization of Kauffman's (1989a) Tutte polynomial of a sign-colored graph. Kauffman's particular choices of parameters are shown to be related to matroid and color duality.

For a graph the "parametrized dichromatic polynomial" $Q_\Gamma = u^{\beta_0(\Gamma)} R_{G(\Gamma)}$, where $G =$ graphic matroid and $\beta_0 =$ number of connected components. A "portable strong Tutte function" of graphs is multiplicative on disjoint unions, satisfies the parametrized Tutte-Grothendieck recurrence, and has value independent of the vertex set. Thm. 10.1: Such a function either equals Q_Γ or is one of two degenerate exceptions. Prop. 11.1: Kauffman's (1989a) polynomial of a sign-colored graph equals $R_{G(|\Sigma|),\sigma}(a, b; d, d)$ for connected Σ, where $a_+ = b_- = B$ and $a_- = b_+ = A$. [Cf. Traldi 1989a).]

[This paper differs from other generalizations of Kauffman's polynomial (Przytycka and Przytycki 1988a, Traldi 1989a, partially anticipated by Fortuin and Kasteleyn 1972a), all of which develop the parametrized dichromatic polynomial of a graph, principally in that it characterizes *all* strong Tutte functions; also in generalizing to matroids and in having little to say about knots. Schwärzler and Welsh (1993a) generalize to signed matroids (and characterize their strong Tutte functions) but not to arbitrary colors.]

(**Sc(M), SGc: Gen: N, D, Knot**)