Spin Glasses, Frustration, Clusterability

Selections from
A Mathematical Bibliography of
Signed and Gain Graphs and Allied Areas
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Subject Classification Codes

A code in *lower case* means the topic appears implicitly but not explicitly. A suffix **w** on **SG**, **SD**, **VS** denotes signs used as weights, i.e., treated as the numbers $+1$ and -1, added, and (usually) the sum compared to 0. In a string of codes a colon indicates subcategorization.

Alg Algorithms.
Aut Automorphisms, symmetries, group actions.
B Balance (mathematical), cobalance.
Cl Clusterability.
Cov Covering graphs, double coverings.
D Duality (matroids or matrices).
E Enumeration of types of signed graphs, etc.
Exr Interesting exercises (in an expository work).
Exp Expository.
Fr Frustration index (line index of imbalance) and other measures of imbalance.
Gen Generalization.
GG Gain graphs, voltage graphs, biased graphs; includes Dowling lattices.
I Incidence matrix, Kirchhoff matrix.
K Signed complete graphs.
N Numerical and algebraic invariants of signed graphs, etc.
P All-negative or antibalanced signed graphs; parity-biased graphs.
Phys Applications in physics (partial coverage).
Rand Random signs or gains, signed or gain graphs.
Ref Many references.
SD Signed digraphs: mathematical properties.
SG Signed graphs: mathematical properties.
Str Structure theory.
Sw Switching of signs or gains.
VS Vertex-signed graphs (“marked graphs”); signed vertices and edges.
WG Weighted graphs.

S. Alexander and P. Pincus

1980a Phase transitions of some fully frustrated models. *J. Phys. A: Math. Gen.* 13, No. 1 (1980), 263–273. **(P: Phys)**

Constantin P. Bachas

1984a Computer-intractibility of the frustration model of a spin glass. *J. Physics A* 17 (1984), L709–L712. MR 85j:82043.

The frustration index decision problem on signed (3-dimensional) cubic lattice graphs is NP-complete. [Proof is incomplete; completed and improved by Green (1987a).] [Cf. Barahona (1982a).] **(SG: Fr: Alg)**

R. Balian, J.M. Drouffe, and C. Itzykson

1975a Gauge fields on a lattice. II. Gauge-invariant Ising model. *Phys. Rev. D* 11 (1975), 2098–2103. **(SG: Phys, Sw, B)**

Francisco Barahona

1981a Balancing signed toroidal graphs in polynomial-time. Unpublished manuscript, 1981.

Given a 2-connected Σ whose underlying graph is toroidal, polynomial-time algorithms are given for calculating the frustration index $l(\Sigma)$ and the generating function of switchings Σ^μ by $|E_-(\Sigma^\mu)|$. The technique is to solve a Chinese postman (T-join) problem in the toroidal dual graph, T corresponding to the frustrated face boundaries. Generalizes (1982a). [See (1990a), p. 4, for a partial description.] **(SG: Fr, Alg)**

1982a On the computational complexity of Ising spin glass models. *J. Phys. A: Math. Gen.* 15 (1982), 3241–3253. MR 84c:82022.

The frustration-index problem, that is, minimization of $|E_-(\Sigma^\eta)|$ over all switching functions $\eta : V \to \{\pm 1\}$, for signed planar and toroidal graphs and subgraphs of 3-dimensional grids. Analyzed structurally, in terms of perfect matchings in a modified dual graph, and algorithmically. The last is NP-hard, even when the grid has only 2 levels; the former are polynomial-time solvable even with weighted edges. The last is NP-hard; the former are polynomial-time solvable. Also, the problem of minimizing $|E_-(\Sigma^\eta)| + \sum_v \eta(v)$ for planar grids ("2-dimensional problem with external magnetic field"), which is NP-hard. (This corresponds to adding an extra vertex, negatively adjacent to every vertex.)

(SG: Phys, Fr: D, Alg)

1990a On some applications of the Chinese Postman Problem. In: B. Korte, L. Lovász, H.J. Prömel, and A. Schrijver, eds., *Paths, Flows and VLSI-Layout*, pp. 1–16. Algorithms and Combinatorics, Vol. 9. Springer-Verlag, Berlin, 1990. MR 92b:90139. Zbl. 732.90086.

Section 2: "Spin glasses." **(SG: Phys, Fr: Exp)**

Section 5: "Max cut in graphs not contractible to K_5," pp. 12–13.

(sg: fr: Exp)

Francisco Barahona and Adolfo Casari

1988a On the magnetisation of the ground states in two-dimensional Ising spin glasses. *Comput. Phys. Comm.* 49 (1988), 417–421. MR 89d:82004. Zbl. 814.90132.

(SG: Fr: Alg)

Francisco Barahona, Martin Grötschel, and Ali Ridha Mahjoub

1985a Facets of the bipartite subgrapn polytope. *Math. Operations Res.* 10 (1985), 340–358. MR 87a:05123a. Zbl. 578.05056.

The polytope $P_B(\Gamma)$ is the convex hull in $\mathbb{R}^E$ of incidence vectors of bipartite edge sets. Various types of and techniques for generating facet-defining inequalities, thus partially extending the description of $P_B(\Gamma)$ from the weakly bipartite case (Grötschel and Pulleyblank (1981a)) in which all facets are due to edge and odd-polygon constraints. [Some can be described best via signed graphs; see Poljak and Turzík (1987a).] [A brief expository treatment of the polytope appears in Poljak and Tuza (1995a).] **(sg: p: fr: G)**

Francisco Barahona and Enzo Maccioni

1982a On the exact ground states of three-dimensional Ising spin glasses. *J. Phys. A: Math. Gen.* 15 (1982), L611–L615. MR 83k:82044.

Discusses a 3-dimensional analog of Barahona, Maynard, Rammal, and Uhry (1982a). (Here there may not always be a combinatorial LP optimum; hence LP may not completely solve the problem.) **(SG: Phys, Fr, Alg)**

F. Barahona, R. Maynard, R. Rammal, and J.P. Uhry

1982a Morphology of ground states of two-dimensional frustration model. *J. Phys. A: Math. Gen.* 15 (1982), 673–699. MR 83c:82045.

Sect. 2: "The frustration model as the Chinese postman's problem", describes how to find the frustration index $l(-\Sigma) = \min_\eta |E_-(\Sigma^\eta)|$ (over all switching functions η) of a signed planar graph by solving a Chinese postman (T-join) problem in the planar dual graph, T corresponding to the frustrated face boundaries. [This was solved independently by Katai and Iwai (1978a).] The postman problem is solved by linear programming, in which there always is a combinatorial optimum: see Sect. 3: "Solution of the frustration problem by duality: rigidity". Of particular interest are vertex pairs, esp. edges, for which $\eta(v)\eta(w)$ is the same for every "ground state" (i.e., minimizing η); these are called "rigid". Sect. 5: "Results" (of numerical experiments) has interesting discussion. [Barahona (1981a) generalizes to signed toroidal graphs.]

In the preceding one minimizes $f_0(\eta) = \sum_E \sigma(vw)\eta(v)\eta(w)$. More general problems discussed are (1) allowing positive edge weights (due to variable bond strengths); (2) minimizing $f_0(\eta) + c\sum_V \eta(v)$, with $c \neq 0$ because of an external magnetic field. Then one cannot expect the LP to have a combinatorial optimum. **(SG: Phys, Fr, Fr(Gen), Alg)**

F. Barahona and J.P. Uhry

1981a An application of combinatorial optimization to physics. *Methods Oper. Res.* 40 (1981), 221–224. Zbl. 461.90080. **(SG: Phys, Fr: Exp)**

I. Bieche, R. Maynard, R. Rammal, and J.P. Uhry

1980a On the ground states of the frustration model of a spin glass by a matching method of graph theory. *J. Phys. A: Math. Gen.* 13 (1980), 2553–2576. MR 81g:82037. **(SG: Phys, Fr, Alg)**

A.J. Bray, M.A. Moore, and P. Reed

1978a Vanishing of the Edwards-Anderson order parameter in two- and three-dimensional Ising spin glasses. *J. Phys. C: Solid State Phys.* 11 (1978), 1187–1202.

(**Phys: SG: Fr**)

Debashish Chowdhury

1986a *Spin Glasses and Other Frustrated Systems.* Princeton Univ. Press, Princeton, and World Scientific, Singapore, 1986.

Includes brief survey of how physicists look upon frustration. See esp. Sect. 1.3, "An elementary introduction to frustration", where the signed square lattice graph illustrates balance vs. imbalance; Ch. 20, "Frustration, gauge invariance, defects and SG [spin glasses]", discussing planar duality (see e.g. Barahona (1982a), "gauge theories", where gains are in the orthogonal or unitary group (and switching is called "gauge transformation" by physicists), and functions of interest to physicists; Addendum to Ch. 10, pp. 378–379, mentioning results on when the proportion of negative bonds is fixed and on gauge theories. (**Phys: SG, GG, VS, Fr: Exp, Ref**)

B. Derrida, Y. Pomeau, G. Toulouse, and J. Vannimenus

1979a Fully frustrated simple cubic lattices and the overblocking effect. *J. Physique* 40 (1979), 617–626. (**SG: Phys, Fr**)

1980a Fully frustrated simple cubic lattices and phase transitions. *J. Physique* 41 (1980), 213–221. MR 80m:82020. (**Phys: SG**)

Patrick Doreian and Andrej Mrvar

1996a A partitioning approach to structural balance. *Social Networks* 18 (1996), 149–168.

They propose indices for clusterability that generalize the frustration index. Fix $k \geq 2$ and $\alpha \in [0,1]$. For a partition π of V into k parts, they define $P(\pi) := \alpha n_- + (1-\alpha) n_+$, where $n_+ := |E_+\langle\pi\rangle|$ = number of positive edges between parts and $n_- := |E_- : \pi|$ = number of negative edges within parts. The first proposed measure is $\min P(\pi)$, minimized over k-partitions. [Call this $P_{k,\alpha}$.] A second suggestion is the "negation-minimal index of generalized imbalance [i.e., of clusterability]", the smallest number of edges whose negation (equivalently, deletion) makes Σ clusterable; it $= \min_k 2P_{k,.5}$. [Note that $P(\pi)$ effectively generalizes the Potts Hamiltonian as given by Welsh (1993a). *Question.* Does $P(\pi)$ fit into an interesting generalized Potts model?] [$P(\pi)$ also resembles the Potts Hamiltonian in Fischer and Hertz (1991a) (q.v. for a related research question).]

They employ a local optimization algorithm to evaluate $P_{k,\alpha}$ and find an optimal partition: random descent from partition to neighboring partition, where π and π' are neighbors if they differ by transfer of one vertex or exchange of two vertices between two parts. This was found to work well if repeated many times. [A minimizing partition into at most k parts is equivalent to a ground state of the k-spin Potts model in the form given by Welsh (1993a), but not quite of that in Fischer and Hertz (1991a).]

Terminology: $P(\pi)$ is called the "criterion function" [more explicitly, one might call it the 'clusterability (adjusted by α)' of π]; clusterability is "k-balance" or "generalized balance". The partition's parts are "plus-sets". Signed digraphs are employed in the notation but direction is ignored.

(**SD: sg: B, Cl: Fr(Gen), Alg, PsS**)

1996b Structural balance and partitioning signed graphs. In: A. Ferligoj and A. Kramberger, eds., *Developments in Data Analysis*, pp. 195–208. Metodološki zvezki, Vol. 12. FDV, Ljubljana, Slovenia, 1996.

Similar to (1996a). Some lesser theoretical detail; some new examples. The k-clusterability index $P_{k,\alpha}$ (see (1996a)) is compared for different values of k, seeking the minimum. [But for which value(s) of α is not stated.] Interesting observation: optimal values of k were small. It is said that positive edges between parts are far more acceptable socially than negative edges within parts [thus, in the criterion function α should be rather near 1].

(**SD: sg: B, Cl: Fr(Gen), Alg, PsS**)

K.H. Fischer and J.A. Hertz

1991a *Spin Glasses.* Cambridge Studies in Magnetism: 1. Cambridge Univ. Press, Cambridge, Eng., 1991. MR 93m:82019.

Sect. 2.5: "Frustration", discusses the spin glass Ising model (essentially, signed graphs) in square and cubical lattices, including the "Mattis model" (a switching of all positive signs), as well as a vector analog, the "XY" model (planar spins) and (p. 46) even a general gain-graph model with switching-invariant Hamiltonian. From the point of view of physics (mainly theoretical physics). (**Phys: SG: Fr, Sw: Exp, Ref**)

Sect. 3.7: "The Potts glass". The Hamiltonian (without edge weights) is $H = -\frac{1}{2}\sum \sigma(e_{ij})(k\delta(s_i, s_j) - 1)$. [It is not clear that the authors intend to permit negative edges. If they are allowed, H is rather like Doreian and Mrvar's (1996a) $P(\pi)$. *Question.* Is there a worthwhile generalized signed and weighted Potts model with Hamiltonian that specializes both to this form of H and to P?] [Also cf. Welsh (1993a) on the Ashkin–Teller–Potts model.] (**Phys: sg, cl: Exp**)

M. Grötschel, M. Jünger, and G. Reinelt

1987a Calculating exact ground states of spin glasses: a polyhedral appraoch. In: J.L. van Hemmen and I. Morgenstern, eds., *Heidelberg Colloquium on Glassy Dynamics* (Proc., 1986), pp. 325–353. Lect. Notes in Physics, Vol. 275. Springer-Verlag, Berlin, 1987. MR 88g:82002 (book).

Sect. 2, "The spin glass model": finding the weighted frustration index in a weighted signed graph (Σ, w), or finding a ground state in the corresponding Ising model, is equivalent to the weighted max-cut problem in $(-\Sigma, w)$. This article concerns finding the exact weighted frustration index. Sect. 3, "Complexity", describes previous results on NP-completeness and polynomial-time solvability. Sect. 4, "Exact methods", discusses previous solution methods. Sect. 5, "Polyhedral combinatorics", shows that finding weighted frustration index is a linear program on the cut polytope; also expounds related work. The remainder of the paper concerns a specific cutting-plane method suggested by the polyhedral combinatorics.

(**sg: fr: gen: Alg**)(**Phys**)(**Ref**)

Scott Kirkpatrick

1977a Frustration and ground-state degeneracy in spin glasses. *Phys. Rev. B* 16, No. 10 (1977), 4630–4641. (**Phys: SG, B, Sw**)

Marc Mézard, Giorgio Parisi, and Miguel Angel Virasoro

1987a *Spin Glass Theory and Beyond.* World Scientific Lecture Notes in Physics, Vol. 9. World Scientific, Singapore, 1987. MR 91k:82066.

Focuses on the Sherrington-Kirkpatrick model, i.e., underlying complete graph, emphasising the Parisi-type model (see articles reprinted herein), which posits numerous metastable states, separated by energy barriers of greatly varying heights and subdividing as temperature decreases. Essentially heuristic (as noted in MR): that is, the ideas awaited [and still largely await] mathematical justification.

Many original articles on Ising and vector models (both of which are based on weighted signed graphs) are reprinted herein, though few are of general signed-graphic interest.

[See also, i.a., Toulouse (1977a, etc.), Chowdhury (1986a), Stern (1989a), Fischer and Hertz (1991a), Vincent, Hammann, and Ocio (1992a) for physics, Barahona (1982a, etc.), Grötschel, Jünger, and Reinelt (1987a) for mathematics.] (**Phys, SG: Fr: Exp, Ref**)

Ch. 0, "Introduction", briefly compares, in the obvious way, balance in social psychology with frustration in spin glasses. (**Phys, PsS: SG: B: Exp**)

Pt. I, "Spin glasses", Ch. 2, "The TAP approach": pp. 19–20 describe 1-vertex switching of a weighted signed graph to reduce frustration, not however necessarily producing the frustration index (minimum frustration). (**Phys: SG: Fr, Sw, Alg: Exp**)

Daniel L. Stern

1989a Spin glasses. *Scientific American*, July 1989, 52–59.

Informally describes frustration in spin glasses in terms of randomly ferromagnetic and antiferromagnetic interactions (see Toulouse (1977a)) and gives some history and applications. (**Phys:: sg: b, Rand: Exp**)

Gérard Toulouse

See also B. Derrida and J. Vannimenus.

1977a Theory of the frustration effect in spin glasses: I. *Commun. Phys.* 2 (1977), 115–119.

Introduces the notion of imbalance ("frustration") of a signed graph to account for inherent disorder in an Ising model (here synonymous with a signed graph, usually a lattice graph). (Positive and negative edges are called "ferromagnetic and antiferromagnetic bonds".) Observes that switching the edge signs from all positive (the model of D.D. Mattis, *Phys. Lett.* 56A (1976), 421–?) makes no essential difference. In a planar lattice [or any plane graph] frustration of face boundaries ("plaquettes") can be thought of as curvature, i.e., failure of flatness. Proposes two kinds of asymptotic behavior of frustration as a polygon encloses more plaquettes. The planar-duality approach for finding the states with minimum frustration (i.e., switchings with fewest negative edges); the number of such states is the "ground-state degeneracy" and is important. Ideas are sketched; no proofs. (**SG: Phys, Sw, B**)

1979a Symmetry and topology concepts for spin glasses and other glasses. In: Nonperturbative Aspects in Quantum Field Theory (Proc. Les Houches Winter Adv. Study Inst., 1978). *Physics Rep.* 49, No. 2 (1979), 267–272. MR 82j:82063. (**Phys: SG: Exp**)

Gérard Toulouse and Jean Vannimenus

1977a La frustration: un monde semé de contradictions. *La Recherche* No. 83, Vol. 8 (Nov., 1977), 980–981.

Popular exposition of the elements of frustration in relation to the Ising model [evidently based on Toulouse (1977a)]. Briefly mentions the social psychology application. [See also Stern (1989a).] (**Phys: SG, B: Exp**)

Jean Vannimenus
See also B. Derrida and G. Toulouse.

J. Vannimenus, J.M. Maillard, and L. de Sèze
1979a Ground-state correlations in the two-dimensional Ising frustration model. *J. Phys. C: Solid State Phys.* 12 (1979), 4523–4532. (**Phys: SG: Fr**)

J. Vannimenus and G. Toulouse
1977a Theory of the frustration effect: II. Ising spins on a square lattice. *J. Phys. C: Solid State Phys.* 10 (1977), L537–L541. (**SG: Phys: Fr**)

Jacques Villain
1977a Spin glass with non-random interactions. *J. Phys. C: Solid State Phys.* 10, No. 10 (1977), 1717–1734. (**SG: Phys, Fr, Sw**)

1977b Two-level systems in a spin-glass model: I. General formalism and two-dimensional model. *J. Phys. C: Solid State Phys.* 10 (1977), 4793–4803. (**Phys: SG: Fr**)

1978a Two-level systems in a spin-glass model: II. Three-dimensional model and effect of a magnetic field. *J. Phys. C: Solid State Phys.* 11, No. 4 (1978), 745–752. (**Phys: SG: Fr**)(**GG: Phys, Sw, B**)

E. Vincent, J. Hammann, and M. Ocio
1992a Slow dynamics in spin glasses and other complex systems. In: D.H. Ryan, ed., *Recent Progress in Random Magnets*, pp. 207–236. World Scientific, Singapore, 1992.

Surveys experiments with spin glass materials, especially their aging behavior. Interprets results as tending to support the Parisi-type model (see notes on Mézard, Parisi, and Virasoro (1987a)). (**Phys**)

G.H. Wannier
1950a Antiferromagnetism. The triangular Ising net. *Phys. Rev.* (2) 79 (1950), 357–364. MR 12, 576. Zbl. 38, 419 (e: 038.41904). (**Phys: P: Fr**)