BIASED GRAPHS. V. GROUP AND BIASED EXPANSIONS (WORKING DRAFT August 8, 2004)

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Abstract. We study group and biased expansions of graphs: uniqueness of gains given either the bias or a matroid; the characteristics of subexpansions; the automorphism group; the intimate connection with quasigroups. We develop linear representation theory for matroids of expansions, generalizing Dowling's geometries. Examples include complex analogs of root systems and their Poincaré polynomials, as well as a few affine, projective, and hypercube geometries.

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INTRODUCTION TO EXPANSIONS

In this fifth part of the series "Biased graphs" we delved into one of the most important familes of gain and biased graphs: those obtained by expanding the edge set of an ordinary graph in a regular way, either in a fashion based on a gain group or abstractly mimicking the combinatorics of a group-based expansion. In Parts I–IV [15] we introduced biased graphs and gain graphs and their bias and lift matroids (whose definitions are recalled in Section 1, so it is not necessary that the reader be familiar with the previous parts of the series); here we apply these concepts to group and biased expansion graphs. These expansions and their matroids include structures that are equivalent to Dowling's lattices, isotopy and conjugacy classes of n-ary quasigroups, and 3-nets (or 3-webs) and have applications to the associative law (see [17]) and projective geometry (see Part VI).

The *group expansion* of an ordinary graph Δ by a group $\mathfrak{G}$, in brief the *$\mathfrak{G}$-expansion* of Δ, is the graph, denoted by $\mathfrak{G}\Delta$, whose node set is $N(\Delta)$ and whose edge set is $\mathfrak{G} \times E(\Delta)$, the endpoints of an edge ge (this is shorthand for (g, e)) being the same as those of e. For the sake of notation we orient Δ in an arbitrary way. The *gain* of ge is g if taken along the orientation of e and g^{-1} if taken against the orientation. (For complete details see Section 1.) The *projection* $p : E(\mathfrak{G}\Delta) \to E(\Delta)$ maps ge to e. The *full $\mathfrak{G}$-expansion* $\mathfrak{G}\Delta^\bullet$ is $\mathfrak{G}\Delta$ with an unbalanced edge (a half edge or unbalanced loop) attached to each node. Group expansions were introduced in Examples I.6.7 and III.3.6.

A *biased expansion* of Δ is a combinatorial generalization of a group expansion. It is a biased graph Ω (a graph with "balanced circles", definition in Section 1.1) together with a projection mapping $p : \|\Omega\| \to \Delta$ that is surjective, is the identity on nodes, and has the properties that no balanced digon projects to a single edge and that, for each circle $C = e_1 e_2 \cdots e_l$ in Δ and each $\tilde{e}_1 \in p^{-1}(e_1), \ldots, \tilde{e}_{l-1} \in p^{-1}(e_{l-1})$, there is a unique $\tilde{e}_l \in p^{-1}(e_l)$ for which $\tilde{e}_1 \tilde{e}_2 \cdots \tilde{e}_l$ is balanced. A full biased expansion is similar to a full group expansion.

The original reason for my interest in group and biased expansion graphs was that the bias matroids of full group expansions of complete graphs, $G(\mathfrak{G}K_n^\bullet)$, are the well-known Dowling geometries of a group, introduced in [5] (preceded in [4] by the projectively represented version for the multiplicative groups $\mathbb{F}_q^*$). The bias matroids of group expansions themselves, $G(\mathfrak{G}K_n)$, are what have been called *jointless* Dowling geometries [9, Section 8.5]. In previous work, these and other matroids based on $\mathfrak{G}K_n$ were studied in Example III.5.7, and expansions by the sign group $\{\pm\} \cong \mathbb{Z}_2$ were treated at length in [13]. Here we generalize in two ways: to group expansions that have arbitrary groups and base graphs, as above, and to a combinatorial generalization without groups.

Besides matroids there are other reasons to be interested in expansions. Perhaps the strongest is the connection with n-ary quasigroups, which were introduced by Belousov and Sandik [1]. Biased expansions of a triangle can be regarded (as Dowling knew implicitly [5, pp. 78–79]) as graph-theoretic realizations of quasigroups and Latin squares (Section 1.5) and as cryptomorphisms of 3-nets (see Section VI.*[net]*). Thus biased expansions may be regarded as, *inter alia*, graphical generalizations of quasigroups. We develop this point of view in [17], the result being a fairly complete theory of which biased expansions are group expansions.

The purpose of this article is, first, to develop basic properties of group and biased expansion graphs, and second, to apply the theory of biased graphs and their matroids and linear representations from Parts I–IV to group and biased expansions. To these ends we especially study isomorphisms and automorphisms of expansions and their matroids. Then we look at a variety of special examples. One is maximal biased expansions, whose matroids are in a way the closest generalization of Dowling geometries. (These are a focus of [17], but without matroid theory; here we mention only the characteristic polynomial of their bias matroids.) Another is a family of complex generalizations of root systems that correspond to certain group expansions and partial group expansions of arbitrary graphs. We calculate, in terms of the chromatic polynomial of the base graph, the Poincaré polynomial of the complement of the complex hyperplane arrangement that is the canonical bias hyperplanar representation of the gain graph. The main task, according to Corollaries IV.2.3 and IV.4.5, is to evaluate the chromatic polynomial of the biased graph, for which we depend on Part III.

A very special kind of example is constituted by those few biased expansions whose matroids are equal to affine or projective geometries or, in one lonely case, the matroid of vertices of a cube.

The various sections are largely autonomous. First are those (Sections 2–6) that concern the basic combinatorial properties of group and biased expansions and their matroids, such as isomorphisms and automorphisms (Sections 3, 4, 5, and, perhaps despite a first impression, 2) and substructures (Section 6). Section 7 treats the most important invariant of the matroids of the maximal biased expansions that were characterized in [17]. Sections 8–11 focus on representations and examples of biased and especially group expansions in linear, projective, and affine spaces.

This article is in part a sequel to Parts I, II, and IV, in which the notions of gain and biased graphs, their bias and lift matroids, and their geometrical representations were defined. Especially, in the latter sections we want to exemplify the representation theory of Part IV. To avoid extensive repetition, in those sections we assume that the reader is familiar with some necessary definitions from Part IV, but we recall all basic definitions in the next section.

1. Preliminaries and fundamentals

Here we collect a few old and new definitions and some elementary but not uninteresting observations about expansions.

1.1. **Basic concepts.**

Edges in a graph are of four types: *links* (two distinct endpoints), *loops* (two coincident endpoints), *half edges* (one endpoint), and *loose edges* (no endpoints). These are explained more fully in Part I; here we mostly need only the first two kinds, which are called *ordinary edges*. A graph all of whose edges are ordinary is an *ordinary graph*; one whose edges are all links is a *link graph*; a link graph without parallel edges is *simple*. A *circle*, also called a polygon, cycle, or circuit, is the graph or edge set of a simple closed path. The sum (symmetric difference) of sets is written $S+T$. This applies in particular to circles, regarded as edge sets, and especially circles whose union is a theta graph. A graph is *inseparable*, or *2-connected*, or a *block*, if any two edges lie in a common circle. A *block* of a graph is a maximal inseparable subgraph.

Suppose Δ is an ordinary graph and γ is a cardinal number (which may be infinite): by $\gamma\Delta$ we mean Δ with every edge replaced by γ copies of itself. (Thus the underlying graph of a regular biased expansion $\gamma \cdot \Delta$ is $\gamma\Delta$; note the importance of the dot in the notation.)

A *biased graph* $\Omega = (\|\Omega\|, \mathcal{B})$ consists of an underlying graph $\|\Omega\|$ together with a class $\mathcal{B}$ of circles, called *balanced circles*, such that no theta subgraph contains exactly two balanced circles. The *balance-closure* of $S \subseteq E(\Omega)$ is

$$\mathrm{bcl}(S) = S \cup \big\{e \in E : e \in C \subseteq S \cup \{e\} \text{ for some balanced circle } C\big\}.$$

The biased graph consisting of a graph Γ in which every circle is balanced is denoted $\langle\Gamma\rangle$.

A *gain graph* $\Phi = (\|\Phi\|, \varphi)$ with *gain group* $\mathfrak{G}$ is a graph $\|\Phi\|$ together with an orientable *gain function* $\varphi : E(\Phi) \to \mathfrak{G}$; that is, φ is defined on oriented edges and, letting e^{-1} denote e with the opposite orientation, $\varphi(e^{-1}) = \varphi(e)^{-1}$. A group expansion, obviously, is a gain graph. A circle in Φ is called *balanced* if the product of the gains of its edges is 1, the group identity; thus Φ produces a biased graph $\langle\Phi\rangle$. Not all biased graphs are *gainable* (obtained from gain graphs); see Example I.5.8, Section 1.5, or [17]. *Switching* a gain graph Φ by a *switching function* $\eta : N \to \mathfrak{G}$ means replacing the gain function φ by $\varphi^\eta(e; v, w) = \eta(v)^{-1}\varphi(e; v, w)\eta(w)$; the switched gain graph is Φ^η. Two gain graphs are *switching equivalent* if one is a switching of the other. Since the balanced circle class of a gain graph is unaltered by switching, the various matroids of the gain graph are also unchanged. Switching has no effect if η is constant and its value lies in the center $Z(\mathfrak{G})$. The switching (equivalence) class of Φ is denoted by $[\Phi]$.

A gain or biased graph is *full* if it has an unbalanced edge (a half edge or unbalanced loop) attached to each node. If we have a biased graph Ω and we attach an unbalanced edge at every node, we write the resulting graph as $\Omega^\bullet$ and call it *filled* (or full). If we attach unbalanced edges only to the members of $X \subseteq N$, we write the result as $\Omega^{(X)}$ and call it *partially filled*. If we attach unbalanced edges to k arbitrary nodes, we write $\Omega^{(k)}$. This includes the cases in which $X = N$ or $\varnothing$.

The *bias matroid* $G(\Omega)$ is the matroid with ground set $E(\Omega)$ whose circuits are the balanced circles and the theta graphs and handcuffs that contain no balanced circle. The *lift matroid* $L(\Omega)$ is the matroid on the same ground set whose circuits are the balanced circles and the theta graphs and bracelets that contain no balanced circle. The *complete* (or *extended*) *lift matroid* $L_0(\Omega)$ is the lift matroid with an *extra point* e_0 that can be regarded, for the purpose of defining L_0, as an unbalanced edge attached to any node. For a more complete presentation of the matroids see Part II.

A *long line* in a matroid is a line that contains at least four atoms.

1.2. **Basics of expansions.**

Now that we have defined biased graphs, we can state a precise definition of biased expansions. Ω is a *biased expansion* of Δ, written $\Omega \downarrow_p \Delta$ or simply $\Omega \downarrow \Delta$, if Δ is an ordinary graph, $p : \|\Omega\| \to \Delta$ is a homomorphism that is surjective and is the identity on nodes, and

(i) for $e \in E(\Delta)$, $p^{-1}(e)$ contains no balanced digon;

(ii) (the *Circle Lifting Property*) for each circle $C = e_1e_2 \cdots e_l$ in Δ and each $\tilde{e}_1 \in p^{-1}(e_1)$, $\ldots$, $\tilde{e}_{l-1} \in p^{-1}(e_{l-1})$, there is a unique $\tilde{e}_l \in p^{-1}(e_l)$ for which $\tilde{e}_1\tilde{e}_2 \cdots \tilde{e}_l$ is balanced;

(iii) Ω is a biased graph.

We call Ω a *regular* or γ*-fold* biased expansion if each $p^{-1}(e)$ has the same cardinality γ; then γ is the *multiplicity* of the expansion. We introduced biased expansions in the particular case of regular biased expansions in Example III.3.8, but regular expansions are not really so special, because a biased expansion of a block must be regular. A biased expansion is *trivial* if it is regular with multiplicity 1.

Normally we can confine our attention to simple base graphs. It is clear that a biased expansion of an ordinary graph Δ is the union of arbitrary biased expansions of the blocks of Δ. Biased expansion of a loop is uninteresting. Suppose e and f are parallel links in Δ: then in a biased expansion Ω of Δ, there is a bijection between $p^{-1}(e)$ and $p^{-1}(f)$ such that, if $\tilde{e}$ and $\tilde{f}$ correspond, then $\tilde{e}\tilde{f}$ is balanced and, for any set $\tilde{P} \subseteq E(\Omega)$ that contains neither $\tilde{e}$ nor $\tilde{f}$, $\tilde{P} \cup \{\tilde{e}\}$ is a balanced circle if and only if $\tilde{P} \cup \{\tilde{f}\}$ is. Thus, Ω is completely determined by $\Omega \setminus p^{-1}(f)$. Moreover, any gains φ for Ω are completely determined by the gains on $\Omega \setminus p^{-1}(f)$ by the equation $\varphi(\tilde{f}) = \varphi(\tilde{e})$.

Regularity makes it convenient to restrict to expansions of blocks (inseparable graphs). Often we can do so, because the definition of a biased expansion applies only within blocks; that is, $\Omega \downarrow_p \Delta$ if and only if, for each block Δ' of Δ, $p^{-1}(\Delta')$ is a biased expansion of Δ'. This leads us to observe that, if $\Omega \downarrow \Delta$ and Δ' is any subgraph of Δ, then $p^{-1}(\Delta')$ is a biased expansion of Δ'. One can say something similar about contraction; it respects biased expansion but in a somewhat complicated way for which we refer the reader to [17, Section 2.3].

One of the basic properties of expansions is the existence of balanced copies of Δ.

Lemma 1.1. *Let Ω be a biased expansion of an ordinary graph Δ. For any balanced set $\tilde{S}$ in Ω, there is a balanced subgraph $\tilde{\Delta} \subseteq \Omega$ such that $p : \tilde{\Delta} \to \Delta$ is an isomorphism and $\tilde{S} \subseteq E(\tilde{\Delta})$.*

Proof. Take a maximal forest F in S and extend it to a maximal forest F_0 of Ω. Then $S \subseteq \operatorname{bcl} F \subseteq \operatorname{bcl} F_0$ by definition of balance-closure. Take $E(\tilde{\Delta}) = \operatorname{bcl}(F_0)$. Then $\tilde{\Delta}$ is balanced by Proposition I.3.1, and $p\big|_{\tilde{\Delta}}$ is surjective and one-to-one by the definition of biased expansion. $\square$

A second basic property concerns switching in a group expansion.

Lemma 1.2. *Given any balanced edge set $\tilde{S} \subseteq E(\mathfrak{G}\Delta)$, we may switch gains to φ^η such that $(\varphi^\eta)^{-1}(1) \supseteq \tilde{S}$.*

Proof. Immediate from Lemma I.5.3. □

1.3. **More about the definition.**

There are several equivalent variations on the definition of a biased expansion; we mention a few. Clearly, (i) may be restricted to edges that are isthmi of Δ. Or, if (i) is retained and we adopt the property

(o) for a loop $e \in E(\Delta)$, $p^{-1}(e)$ contains at most one balance loop,

uniqueness may be omitted in (ii), due to (iii).

A *lift* of $S \subseteq E(\Delta)$ is any $\tilde{S} \subseteq E(\Omega)$ such that $p\big|_{\tilde{S}}$ is a bijection onto S. Consider the property

(ii′) every balanced subset $\tilde{S} \subseteq E(\Omega)$ extends uniquely to a balanced lift of $\operatorname{clos}_\Delta p(\tilde{S})$.

(The closure is in the usual polygon or cycle matroid of Δ.)

Proposition 1.3. *Assuming (i) and (iii), property (ii′) is equivalent to (ii).*

Proof. That (ii′) implies (ii) is obvious.

Assume (ii) and let $\tilde{S}$ be balanced in Ω. By (i), and (ii) for loops, $\tilde{S}$ is a lift of $S = p(\tilde{S})$. $\tilde{S}$ extends uniquely by (ii), because $\operatorname{clos}_\Delta S = S \cup \{e \in E(\Delta) :$ there is a circle $C \subseteq S \cup e\}$ and the extension of $\tilde{S}$ exists by Lemma 1.1 applied to $p^{-1}(\operatorname{clos}_\Delta S)$. □

Again, (i) may be restricted to isthmi if we have (ii′). Or, if (i) is retained and we adopt (o), uniqueness may be omitted in (ii′). We may go further; consider the property

(ii″) every balanced subset $\tilde{S} \subseteq E(\Omega)$ extends to a balanced lift of $E(\Delta)$

(which has no uniqueness clause).

Proposition 1.4. *Assuming (i) and (iii), the combination of (ii″) and (o) is equivalent to (ii).*

Proof. Clearly, (ii″) implies (ii′) without uniqueness. To derive the former from the latter, extend $\tilde{S}$ so it contains a maximal forest of Ω but no new circles, then apply (ii′) without uniqueness; this proves (ii″). □

In (ii′) or (ii″) we may restrict the arbitrary balanced set $\tilde{S}$ to be a forest, or a maximal forest.

1.4. **Recognizing expansions.**

Here is another fundamental question. How can we recognize biased expansions amongst all biased graphs? We say Ω is a *simply biased link graph* if every edge is a link and there are no balanced digons. The heart of the question is to recognize which of this kind of graph are biased expansions.

Proposition 1.5 (Recognition). *A simply biased link graph Ω is a biased expansion graph if and only if, for every unbalanced circle $\tilde{C}$ in Ω of length at least 3 and every $\tilde{e} \in \tilde{C}$, there is an edge $\tilde{e}^*$ parallel to $\tilde{e}$ such that $(\tilde{C} \setminus \tilde{e}) \cup \tilde{e}^*$ is balanced.*

Proof. Necessity is obvious.

For sufficiency, let Δ be the simplification of $\|\Omega\|$ and $p : \Omega \to \Delta$ the obvious mapping. Take a circle C in Δ and $e \in C$. Lift $C \setminus e$ to a path $\tilde{P}$ and lift e to any $\tilde{e} \in p^{-1}(e)$. If $\tilde{P} \cup \tilde{e}$ is unbalanced, by hypothesis there exists $\tilde{e}^*$ making $\tilde{P} \cup \tilde{e}^*$ balanced. Thus (ii) is satisfied except for uniqueness, but as (i) is also satisfied, we are done. □

1.5. **Quasigroups and Latin squares.**

For K_3 there is a whole algebraico-combinatorial theory of biased expansions that does not generalize to larger complete graphs. The reason is that biased expansions of a triangle are cryptomorphic to equivalence classes of quasigroups (and their multiplication tables, which are Latin squares). Thus we have a whole class of biased expansions (of small order) that are not group expansions.

A *quasigroup* is a set with a binary operation such that the equation $xy = z$ has a unique solution for any one variable given the values of the other two. Let $\mathfrak{Q}$ be a quasigroup and construct $\langle \mathfrak{Q} K_3 \rangle$, the *quasigroup expansion* of K_3 by $\mathfrak{Q}$, as follows: The nodes are v_1, v_2, v_3. There is an edge ae_{ij} for every $a \in \mathfrak{Q}$ and $ij = 12, 23, 13$. The balanced circles are the triangles $\{ae_{12}, be_{23}, ce_{13}\}$ such that $ab = c$ in $\mathfrak{Q}$.

Proposition 1.6. *The quasigroup expansion $\langle \mathfrak{Q} K_3 \rangle$ is a biased expansion of K_3. Conversely, every biased expansion of K_3 has the form $\langle \mathfrak{Q} K_3 \rangle$ for a quasigroup $\mathfrak{Q}$.*

Furthermore, two quasigroup expansions $\langle \mathfrak{Q}_1 K_3 \rangle$ and $\langle \mathfrak{Q}_2 K_3 \rangle$ are isomorphic if and only if $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are equivalent by a combination of isotopy and conjugation.

Proof. We explain the converse in the first part. To construct $\mathfrak{Q}$, we define it to be a set in one-to-one correspondence with each $p^{-1}(e_{ij})$, where $E(K_3) = \{e_{12}, e_{23}, e_{13}\}$. The multiplication is defined by $ab = c$ if $\{ae_{12}, be_{23}, ce_{13}\}$ is balanced. The quasigroup property, that in the equation $ab = c$ any two of a, b, c determine the third uniquely, follows from the fact that in a biased expansion of K_3 the balanced circles are a linear class and there are no balanced digons.

Isotopy means changing the three bijections by which $\mathfrak{Q}$ corresponds to the edge classes. *Conjugacy* means interchanging the roles of the edges e_{ij}, or equivalently, applying an automorphism of K_3. (See [3].) It is clear that biased expansions of K_3 correspond to equivalence classes of quasigroups under isotopy and conjugacy. □

Dowling mentions the lattices $\operatorname{Lat} G(\mathfrak{Q} K_3^\bullet)$ of full quasigroup expansions in [5, pp. 78–79].

The quasigroup expansion is a biased graph, not a gain graph: the labels are not gains because there is no notion of inversion in a quasigroup. That means we have two ways to expand by a group $\mathfrak{Q} = \mathfrak{G}$: the quasigroup expansion $\langle \mathfrak{Q} K_3 \rangle$ (written with brackets) and the gain graph $\mathfrak{G} K_3$. Fortunately, the definitions are compatible. If we treat the edge labels in $\langle \mathfrak{Q} K_3 \rangle$, with node set $\{v_1, v_2, v_3\}$, as gains assigned to edges oriented from the low to the high end, then $\langle \mathfrak{Q} K_3 \rangle = \langle \mathfrak{G} K_3 \rangle$.

Another way to look at quasigroups, and biased expansions of K_3, is in terms of Latin squares. Suppose we have a $\gamma \cdot K_3$. Order $p^{-1}(e_{12})$ and $p^{-1}(e_{23})$ arbitrarily. Form a $\gamma \times \gamma$ square whose (k, l)th square has entry $L(k, l) = \tilde{e}_{13}$ where $\tilde{e}_{13}$ is the edge that makes a balanced triangle with the k^{th} edge in $p^{-1}(e_{12})$ and the l^{th} edge in $p^{-1}(e_{23})$. Then L is a Latin square. The construction is reversible, of course, and it is clearly a cryptomorphism of quasigroups whose elements are arbitrarily linearly ordered. It is known which biased expansions of K_3 are group expansions because this is really a reinterpretation of the question of when a Latin square is isotopic to the multiplication table of a group. That is answered by the quadrangle criterion of Latin squares [3, Theorem 1.2.1(2)], which we translate into the language of expansions in [17, Section 1.2].

The viewpoint of quasigroups, or Latin squares, generalizes to multary operations through larger circles; see [17, Proposition 1.1]. But one may naturally wonder about higher analogs of quasigroups based on complete graphs instead of circles. They are very restricted, according to a result of Kahn and Kung.

Proposition 1.7 ([7, Section 7, pp. 490–492])**.** *A biased expansion of a complete graph of order at least* 4 *is a group expansion.*

The reason is that K_4, as the base of a biased expansion, encodes the associative law. This is the theme of [17].

1.6. **Biased graphs vs. biased expansions.**

It would be pleasant to be able to say that every biased graph is a minor of some biased expansion graph. If so, one could say that biased expansions encompass all biased graphs. However, there is a counterexample.

First, we need a lemma. We assume familiarity with contraction of a biased graph, which is defined in Part I. A *minor* of Ω is any subgraph of a contraction, including any subgraph of Ω itself.

Lemma 1.8. *Let Ω be a biased expansion of a link graph Δ. Any contraction of Ω consists of a biased expansion Ω_0 of a minor Δ_0 of Δ, possibly together with loops and loose and half edges. The edge fibers of Ω_0 are the sets $p^{-1}(e)$ for $e \in E(\Delta_0)$; thus if Ω is regular, Ω_0 is regular with the same multiplicity.*

Proof. Consider Ω/S where $S \subseteq E(\Omega)$. Let $\tilde{T}$ be a maximal forest in S, projecting to T, and let $S' = S \setminus \tilde{T}$. Further, let Ω' be $\Omega/\tilde{T}$ with loops removed, so only links remain. Ω' projects onto $\Delta' = \Delta/\operatorname{clos}_\Delta T$ through a mapping p' induced by p. Indeed, $E(\Omega') = p^{-1}(E(\Delta'))$ because any edge of Ω that does not project into $E(\Delta')$ either belongs to $\tilde{T}$ or forms a circle with some edges of $\tilde{T}$ and therefore is a loop in the contraction.

Thus, Ω/S is a contraction of $\Omega/\tilde{T}$ by a set S' of loops. Contracting balanced loops merely deletes them. Contracting unbalanced loops deletes their supporting nodes and converts any uncontracted edges incident with those nodes into half or loose edges. Letting Ω_0 be Ω/S with loops and half and loose edges deleted, and letting Δ_0 be Δ' with those nodes deleted that support unbalanced loops of $\Omega/\tilde{T}$, we see that p' induces $p_0 : E(\Omega_0) \to \Delta_0$ and $p_0^{-1}(e) = p^{-1}(e)$ for every edge $e \in E(\Delta_0)$. As $\Omega_0|p^{-1}(e) = \Omega|p^{-1}(e)$, property (i) is satisfied. We verify (ii$'$). Take a circle C_0 in Δ_0. It is the contraction of some circle C in Δ by the edge set $C \cap T$. Choose $e \in C_0$ and lift $C_0 \setminus e$ arbitrarily to $\tilde{P}_0$ in Ω_0. This lift can be extended to a lift $\tilde{P}$ of $C \setminus e$ in Ω and completed to a balanced lift of C by choosing suitable $\tilde{e} \in p^{-1}(e)$. Then $\tilde{P}_0 \cap \tilde{e}$ is balanced in Ω_0; thus (ii) holds in Ω_0. □

Example 1.1. Let Ω_0 be the ungainable biased graph of Example I.5.8. Ω_0 has underlying graph $2C_4$ with nonadjacent node pairs v_1v_3 and v_2v_4. Let $\Omega_1 = (2C_4 \cup \{e_{13}, e_{24}\}, \mathcal{B}(\Omega_0))$; that is, we add to Ω_0 two diagonal edges, to make the underlying graph complete, but no new balanced circles. Then Ω_1 is not a minor of any biased expansion.

Proof. Ω_1 is not gainable because Ω_0 is not. Suppose Ω_1 contained in a biased expansion Ω. We may assume Ω to have order 4. Since it expands K_4, it must have gains by Proposition 1.7. This is a contradiction; therefore Ω cannot exist.

Now suppose Ω is a biased expansion of which Ω_1 is a minor. That means $\Omega_1 = (\Omega/S) \setminus T$ for edge sets S and T. By Lemma 1.8 Ω/S consists of a biased expansion, call it Ω_2, together with nonlink edges. Therefore, $\Omega_2 \supseteq \Omega_1$; but as Ω_2 is a biased expansion of K_4, that is impossible. □

2. Uniqueness of gains

In a group expansion the gains are essentially unique as long as Δ is sufficiently well connected—this is Theorem 2.1(b), and for the special case $\Delta = K_n$ it follows from Dowling's Theorem 8 [5]. Thus the canonical representations are projectively unique up to embedding of $\mathfrak{G}$ in F^* or F^+. What is more, by Theorem IV.7.1, if $\mathfrak{G}$ is nontrivial there are only canonical representations.

Theorem 2.1. *Let $\mathfrak{G}$ and $\mathfrak{H}$ be groups and let Δ be a 2-connected link graph of order at least 3.*

(a) *Suppose Ω, a biased expansion of Δ, has gains in $\mathfrak{H}$; that is, there is a gain graph $\Psi = (\|\Omega\|, \psi, \mathfrak{H})$ such that $\langle\Psi\rangle = \Omega$. Then $\mathfrak{H}$ has a subgroup $\mathfrak{G}'$ such that Ψ switches to $\langle\mathfrak{G}'\Delta\rangle$, whence $\Omega \cong \langle\mathfrak{G}'\Delta\rangle$.*

(b) *Suppose $\langle\mathfrak{G}\Delta\rangle$ has gains in $\mathfrak{H}$; that is, there is a gain graph $\Psi = (\|\mathfrak{G}\Delta\|, \psi, \mathfrak{H})$ such that $\langle\Psi\rangle = \langle\mathfrak{G}\Delta\rangle$. Then $\mathfrak{H}$ has a subgroup $\mathfrak{G}'$ such that $\mathfrak{G}' \cong \mathfrak{G}$ and Ψ switches to $\mathfrak{G}'\Delta$.*

(c) *If $\langle\mathfrak{G}\Delta\rangle \cong \langle\mathfrak{H}\Delta\rangle$, then $\mathfrak{G} \cong \mathfrak{H}$. More precisely, suppose $F : \|\mathfrak{G}\Delta\| \to \|\mathfrak{H}\Delta\|$ is a graph isomorphism by which $\langle\mathfrak{G}\Delta\rangle \cong \langle\mathfrak{H}\Delta\rangle$. Then there is a group isomorphism $f : \mathfrak{G} \to \mathfrak{H}$ such that (F, f) is a switching isomorphism of $\mathfrak{G}\Delta$ with $\mathfrak{H}\Delta$.*

The theorem says that a biased graph, if it has gains, is a group expansion $\langle\mathfrak{G}\Delta\rangle$ and has gains only in the one group $\mathfrak{G}$, aside from the elementary modifications of enlarging the gain group and switching.

The second half of (c) needs some explanation. Suppose we have biased graphs Ω and Υ and a graph isomorphism $F : \|\Omega\| \to \|\Upsilon\|$. We say Ω and Υ are *isomorphic by F* if F carries balanced circles to balanced circles and unbalanced circles to unbalanced ones. Suppose now we have two gain graphs, Φ with gain group $\mathfrak{G}$ and Ψ with gain group $\mathfrak{H}$, and graph and group isomorphisms $F : \|\Omega\| \to \|\Upsilon\|$ and $f : \mathfrak{G} \to \mathfrak{H}$. We say (F, f) is a *switching isomorphism* of Φ with Ψ if, after suitably switching Φ and Ψ, we have $\psi(F(e)) = f(\varphi(e))$ for all $e \in E_*(\Phi)$. It is clear that the switching can be done entirely in one gain graph, either Φ or Ψ.

Proof of (a). We may assume by switching that some balanced lift of Δ has gain 1 in Ψ. (See Lemma 1.1.) Then we have a family of mappings $\psi_{e,ij} : p^{-1}(e) \to \mathfrak{H}$, one for each oriented edge $e{:}v_iv_j$ of Δ, by restricting the gain function of Ψ; that is, $\psi_{e,ij}(ge) = g$. By the definition, $\psi_{e,ji}(\tilde{e}) = \psi_{e,ij}(\tilde{e})^{-1}$.

Now we show that all $\psi_{e,ij}$ have the same range $\mathfrak{G}_{e,ij} = \{g \in \mathfrak{H} : \exists ge \in p^{-1}(e)\}$. Let $C = e_1e_2\cdots e_l$ be a circle of length $l \geq 3$ in Δ, the endpoints of e_i being v_{i-1} and v_i with $v_0 = v_l$. A lift $\tilde{C} = (g_1e_1)(g_2e_2)\cdots(g_le_l)$ is balanced in Ψ $\iff$ $g_1g_2\cdots g_l = 1$. Choose gains $g_i = 1$ for all $i \neq 1, 2, j$. Because $\langle\Psi\rangle$ is a biased expansion of Δ, for any choice of $g_1e_1, g_2e_2 \in E(\Psi)$ there exists an edge g_je_j making $\tilde{C}$ balanced. Then $g_1g_2 = g_j^{-1}$. For now let us take $g_2 = 1$; it follows that $\mathfrak{G}_{e_1,01} \subseteq \mathfrak{G}^{-1}_{e_j,j-1,j}$. By a similar argument $\mathfrak{G}_{e_j,j-1,j} \subseteq \mathfrak{G}^{-1}_{e_1,01}$ and indeed $\mathfrak{G}_{e_i,i-1,i} = \mathfrak{G}^{-1}_{e_j,j-1,j}$ for all distinct i, j. Thence, $\mathfrak{G}_{e_1,01} = \mathfrak{G}_{e_2,21} = \mathfrak{G}_{e_3,23} = \mathfrak{G}_{e_1,10}$. In other words, all $\mathfrak{G}_{e_i,i,i-1}$ are equal and self-inverse, for $e_i \in C$. Call their common value $\mathfrak{G}'$. By 2-connection of Δ, the same holds for all edges in Δ.

From the equation $g_1g_2 = g_j^{-1}$ we see that $\mathfrak{G}' \cdot \mathfrak{G}' \subseteq \mathfrak{G}'$. Thus, $\mathfrak{G}'$ is a subgroup of $\mathfrak{H}$ and Ψ is the same as $\mathfrak{G}'\Delta$ except for the technicality, which we purposely overlooked in stating the conclusion, that the gain function of Ψ has codomain $\mathfrak{H}$ while that of $\mathfrak{G}'\Delta$ has codomain $\mathfrak{G}'$. $\square$

Proof of (b). The spirit of the proof resembles that of (a). First switch Ψ so that every edge with gain 1 in $\mathfrak{G}\Delta$ also has gain 1 in Ψ. This is possible by Lemma 1.1 because $\langle\Psi\rangle = \langle\mathfrak{G}\Delta\rangle$.

We define a family of mappings $f_{e,ij} : \mathfrak{G} \to \mathfrak{H}$, one for each oriented edge $e{:}v_iv_j$ of Δ, by the formula

$$f_{e,ij}(\varphi_{ij}(ge)) = \psi_{ij}(ge). \tag{2.1}$$

Here φ_{ij} means the gain of ge as oriented from v_i to v_j; *a priori*, $f_{e,ij}$ need not be the same as $f_{e,ji}$. (In $\psi_{ij}(ge)$, remember that ge is the name of an edge in $\|\Psi\| = \|\mathfrak{G}\Delta\|$; its gain in Ψ need not have anything to do with g.) Due to our foresighted switching, $f_{e,ij}(1) = 1$.

The relationship between $f_{e,ij}$ and $f_{e,ji}$ can be calculated.

$$f_{e,ji}(\varphi_{ji}(ge)) = \psi_{ji}(ge) = \psi_{ij}(ge)^{-1} = f_{e,ij}(\varphi_{ij}(ge))^{-1} = f_{e,ij}(g)^{-1}$$

and

$$f_{e,ji}(\varphi_{ji}(ge)) = f_{e,ji}(\varphi_{ij}(ge)^{-1}) = f_{e,ji}(g^{-1}),$$

so that

$$f_{e,ji}(g) = f_{e,ij}(g^{-1})^{-1}. \tag{2.2}$$

Now we show that all $f_{e,ij}$ are the same. Choose C as in the proof of (a). Consider a corresponding circle in $\mathfrak{G}\Delta$: $\tilde{C} = (g_1e_1, g_2e_2, \ldots, g_le_l)$. We compare balance of $\tilde{C}$ in $\mathfrak{G}\Delta$ and Ψ:

$$\begin{aligned} g_1g_2\cdots g_l = 1 &\iff \tilde{C} \text{ is balanced in } \mathfrak{G}\Delta \iff \tilde{C} \text{ is balanced in } \Psi \\ &\iff f_{e_1,01}(g_1)f_{e_2,12}(g_2)\cdots f_{e_l,l-1,l}(g_l) = 1. \end{aligned}$$

Now choose $g_i = 1$ except for g_1, g_2, and g_j where $2 < j \leq l$. Thus

$$g_1g_2g_j = 1 \iff f_{e_1,01}(g_1)f_{e_2,12}(g_2)f_{e_j,j-1,j}(g_j) = 1.$$

Using Equation (2.2) we can rewrite this as

$$f_{e_j,j,j-1}(g_1g_2) = f_{e_1,01}(g_1)f_{e_2,12}(g_2). \tag{2.3}$$

Setting $g_1 = 1$ and $g_2 = 1$ in turn shows that $f_{e_1,01} = f_{e_j,j,j-1} = f_{e_2,12}$. By symmetry all $f_{e_i,i-1,i}$ are equal and are equal to $f_{e_i,i,i-1}$. Furthermore, substitution in (2.3) shows that this function, call it f, is a homomorphism. Because it is an injection, f is a monomorphism. Finally, by 2-connectedness any two nonparallel edges of Δ belong to a circle of length 3 or more; so $f_{e,ij} = f$ for all edges of Δ. Therefore we have a uniquely determined monomorphism $f : \mathfrak{G} \to \mathfrak{H}$. Taking $\mathfrak{G}' = f(\mathfrak{G})$, we see that Ψ (after switching) $= \mathfrak{G}'\Delta$. $\square$

Proof of (c). We can assume that $\langle\mathfrak{G}\Delta\rangle = \langle\mathfrak{H}\Delta\rangle$. Thus by part (b), there is a monomorphism $f : \mathfrak{G} \to \mathfrak{H}$; but in this case it is a surjection, whence an isomorphism. It is clear that f is the required isomorphism. $\square$

In (b), the deduction from (2.3) that f is a well defined homomorphism is similar to [5, Lemma at (5.2)]. We are essentially giving a biased-graphic version of the proof that a quasigroup that is isotopic to a group is not isotopic to any other group; this is a standard result in the theory of quasigroups and Latin squares (see for instance [3, Section 1.3]).

Corollary 2.2. *Let Δ be a 2-connected link graph of order at least 3. Suppose φ_1 and φ_2 are two $\mathfrak{G}$-gain functions for a biased expansion $\langle\mathfrak{G}\Delta\rangle$. Then φ_2 is obtained from φ_1 by switching and applying an automorphism of $\mathfrak{G}$.*

Proof. Apply Theorem 2.1(c) with $\mathfrak{G} = \mathfrak{H}$ and $F =$ the identity mapping. $\square$

Corollary 2.3. *If Δ is a 2-connected link graph of order at least 3 and $\langle \mathfrak{H}\Delta \rangle \supseteq \langle \mathfrak{G}\Delta \rangle$, then $\mathfrak{G}$ is isomorphic to a subgroup of $\mathfrak{H}$.*

Proof. Apply Theorem 2.1(b). $\square$

The assumption of 2-connectedness is essential in Theorem 2.1(a, b). We demonstrate this with counterexamples.

Example 2.1. Consider a simple graph $\Delta = \Delta_1 \cup_r \Delta_2$, the union of two inseparable graphs of order at least 3 that intersect only in the node r. Suppose $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are nonisomorphic subgroups of $\mathfrak{H}$ of the same order. We take $\Psi = \mathfrak{G}_1\Delta_1 \cup_r \mathfrak{G}_2\Delta_2$ in Theorem 2.1(a). Then $\langle \Psi \rangle$ is a regular biased expansion of Δ. Nevertheless it is not possible to switch Ψ into the form $\mathfrak{G}'\Delta$; for if it were, then by Theorem 2.1(c) applied to $\mathfrak{G}_1\Delta_1$ and $\mathfrak{G}_2\Delta_2$, $fG' \cong \mathfrak{G}_1$ and $fG' \cong \mathfrak{G}_2$, which is impossible.

Example 2.2. Consider a simple graph $\Delta = \Delta_1 \cup_r \Delta_2$, the union of two inseparable graphs of order at least 3 that intersect only in the node r. Suppose $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are different but isomorphic subgroups of $\mathfrak{H}$. We take $\mathfrak{G} = \mathfrak{G}_1$ in Theorem 2.1(b) and $\Psi = \mathfrak{G}_1\Delta_1 \cup_r \mathfrak{G}_2\Delta_2$. It is not possible to switch Ψ into the form $\mathfrak{G}'\Delta$.

To see this, consider a simple closed walk C_i, from r to r, in Δ_i. Let $\tilde{C}_i$ denote any simple closed walk in $\mathfrak{G}_i\Delta_i$ that projects to C_i. By choosing $\tilde{C}_i$ suitably we can make $\varphi(\tilde{C}_i)$ be any element of $\mathfrak{G}_i$. Switching by $\eta : N \to \mathfrak{G}$ gives $\varphi^\eta(\tilde{C}_i) = \varphi(\tilde{C}_i)^{\eta(r)}$, the conjugate of $\varphi(\tilde{C}_i)$ by $\eta(r)$. If we switch so that Φ^η has the form $\mathfrak{H}_1\Delta_1 \cup_r \mathfrak{H}_2\Delta_2$ for groups $\mathfrak{H}_1$ and $\mathfrak{H}_2$, then

$$\mathfrak{H}_i = \{\varphi^\eta(\tilde{C}_i) : \tilde{C}_i \text{ projects to } C_i\} = \{\varphi(\tilde{C}_i)^{\eta(r)} : \tilde{C}_i \text{ projects to } C_i\} = \mathfrak{G}_i^{\eta(r)}.$$

Therefore, by switching we can never make $\mathfrak{H}_1 = \mathfrak{H}_2$.

3. Automorphisms of group expansions

The automorphisms of matroids of partially filled biased expansions are, for the most part, precisely the automorphisms of the expansions themselves. In the case of group expansions the latter have a simple and natural description; that is the topic of this section. (The automorphism groups of nongroup biased expansions are still mysterious.)

Bonin found the automorphism groups of the partially filled Dowling geometries $G(\mathfrak{G}K_n^{(k)})$ employing a geometric presentation [2]. Here we extend his work to matroids of arbitrary group expansions, developing the theory in terms of biased graphs, which we find easier to work with than geometrical representations (and, indeed, that is the main virtue of biased graph theory). Since we write all actions exponentially, our notation differs from Bonin's and in particular our group $\operatorname{Aut}\langle\mathfrak{G}K_n\rangle$ is antiisomorphic to his. (It is the reversed group defined in [11, Section 8??].)

Consider $\langle\mathfrak{G}\Delta\rangle$, where Δ is a link graph. We want to know the automorphism group, $\operatorname{Aut}\langle\mathfrak{G}\Delta\rangle$. An essential part of that group is the fibered automorphism group, $\operatorname{Aut}_p\langle\mathfrak{G}\Delta\rangle$, which consists of the automorphisms that act only within fibers; that is, they correspond to the identity automorphism of the base graph. More generally we shall consider a base graph together with a partition π of the node set, written Δ^π. The question is to describe the group $\operatorname{Aut}\langle\mathfrak{G}\Delta^\pi\rangle$ consisting of those automorphisms of $\langle\mathfrak{G}\Delta\rangle$ of which every node orbit is contained in a block of the partition; that is, the orbit partition of N refines π. The reason for the partition is to handle partly filled expansions; this will be explained at Theorem 3.3.

For $g \in \mathfrak{G}$, define η_g to be the constant function $\eta_g \equiv g$ and α_g to be conjugation by g, i.e., $h^{\alpha_g} = g^{-1}hg$.

Each of the groups $\operatorname{Aut}\Delta^\pi$ (the automorphism group of Δ restricted to those automorphisms that respect π), $\operatorname{Aut}\mathfrak{G}$, and $\mathfrak{G}^N$ (the group of switching functions) acts on $\langle\mathfrak{G}\Delta^\pi\rangle$. The actions of $f \in \operatorname{Aut}\Delta^\pi$, $\alpha \in \operatorname{Aut}\mathfrak{G}$, and $\eta \in \mathfrak{G}^N$ on $\langle\mathfrak{G}\Delta^\pi\rangle$ are by

$$(ge)^{\tilde f} = gf(e), \qquad (ge)^{\tilde\alpha} = g^\alpha e, \qquad (ge)^{\tilde\eta} = \eta(v)^{-1}g\eta(w)e,$$

where e is oriented $e{:}vw$ and $\tilde\theta$ denotes the action of θ. Thus the product set

$$\mathfrak{A} = \operatorname{Aut}\Delta^\pi \times \operatorname{Aut}\mathfrak{G} \times \mathfrak{G}^N$$

acts on $\langle\mathfrak{G}\Delta^\pi\rangle$; however, not as the product group. We shall determine a group structure on $\mathfrak{A}$ of which $\operatorname{Aut}\langle\mathfrak{G}\Delta^\pi\rangle$ is a homomorphic image.

First we point out that the actions of $\operatorname{Aut}\Delta^\pi$ and $\operatorname{Aut}\mathfrak{G}$ are faithful but that of $\mathfrak{G}^N$ has kernel $\hat Z(\mathfrak{G})$, the center of $\mathfrak{G}$ embedded diagonally in $\mathfrak{G}^N$.

Next we define a natural action of $\operatorname{Aut}\mathfrak{G}$ on $\mathfrak{G}^N$, taking η to η^α where $\eta^\alpha(v) = (\eta(v))^\alpha$. This action defines a semidirect product $\operatorname{Aut}\mathfrak{G} \ltimes \mathfrak{G}^N$. One readily calculates that

$$(ge)^{\tilde\eta\tilde\alpha} = (ge)^{\tilde\alpha\widetilde{\eta^\alpha}}, \tag{3.1}$$

so the action on $\langle\mathfrak{G}\Delta\rangle$ defines a mapping $\operatorname{Aut}\mathfrak{G} \ltimes \mathfrak{G}^N \to \operatorname{Aut}_p\langle\mathfrak{G}\Delta\rangle$. Note that for a fibered automorphism π can be ignored.

Lemma 3.1. *Let Δ be an inseparable link graph of order at least 3. The fibered automorphism group of a group expansion $\langle\mathfrak{G}\Delta\rangle$ is*

$$\operatorname{Aut}_p\langle\mathfrak{G}\Delta\rangle = [\operatorname{Aut}\mathfrak{G} \ltimes \mathfrak{G}^N]/\tilde{\mathfrak{G}},$$

where $\tilde{\mathfrak{G}} = \{(\alpha_h^{-1}, \eta_h) : h \in \mathfrak{G}\} \cong \mathfrak{G}/Z(\mathfrak{G})$. (In particular, $\tilde\alpha_h = \tilde\eta_h$ for $h \in \mathfrak{G}$.)

Proof. By Corollary 2.2, for any fiber-preserving automorphism $F : \langle\mathfrak{G}\Delta\rangle \to \langle\mathfrak{G}\Delta\rangle$ there exist $\alpha \in \operatorname{Aut}\mathfrak{G}$ and $\eta \in \mathfrak{G}^N$ such that

$$\varphi(F(\tilde{e})) = \varphi^{\eta\alpha}(\tilde{e}),$$

or equivalently,

$$F(ge) = [\eta(v)^{-1} g \eta(w)]^{\alpha} e = (ge)^{\tilde{\eta}\tilde{\alpha}}.$$

This is the identity function if and only if $\eta = \eta_h$ (by setting $g = 1$) and $\alpha = \alpha_{h^{-1}}$. Thus the kernel of the action is $\tilde{\mathfrak{G}}$. Note that the kernel of the action of $\mathfrak{G}^N$ itself is absorbed into $\tilde{\mathfrak{G}}$. □

If Δ had an isthmus e, then $\operatorname{Aut}_p\langle\mathfrak{G}\Delta\rangle$ would contain arbitrary permutations of $p^{-1}(e)$, regardless of the gain group. The hypotheses eliminate this and similar possibilities that would invalidate the lemma.

The next lemma and the theorem must be restricted to simple base graphs because, if Δ has parallel links e_1, e_2, then an automorphism of $\langle\mathfrak{G}\Delta^\pi\rangle$ might exchange ge_1 with ge_2 for some but not all group elements g.

Lemma 3.2. *Let Δ be a simple graph in which every block has order at least 3. The automorphism group of a group expansion $\langle\mathfrak{G}\Delta^\pi\rangle$ is*

$$\operatorname{Aut}\langle\mathfrak{G}\Delta^\pi\rangle = \operatorname{Aut}\Delta^\pi \ltimes \operatorname{Aut}_p\langle\mathfrak{G}\Delta\rangle.$$

Proof. We show first that $\operatorname{Aut}\Delta^\pi$ and $\operatorname{Aut}_p\langle\mathfrak{G}\Delta\rangle$ generate the whole automorphism group. Suppose we have an automorphism F of $\langle\mathfrak{G}\Delta^\pi\rangle$. It is clear that it induces a unique automorphism f of Δ^π. Applying f^{-1} to $\langle\mathfrak{G}\Delta^\pi\rangle$, we see that $F \circ f^{-1} \in \operatorname{Aut}_p\langle\mathfrak{G}\Delta^\pi\rangle$.

If $F = F'$, then $f = f'$ so $F \circ f^{-1} = F' \circ (f')^{-1}$. This shows that, as a set, $\operatorname{Aut}\langle\mathfrak{G}\Delta^\pi\rangle$ equals the product of $\operatorname{Aut}\Delta^\pi$ and $\operatorname{Aut}_p\langle\mathfrak{G}\Delta\rangle$. The semidirect product structure can be defined in terms of the actions; that is, for $f \in \operatorname{Aut}\Delta^\pi$ and $\tilde{\theta} \in \operatorname{Aut}_p\langle\mathfrak{G}\Delta\rangle$ we define $\tilde{\theta}^f$ to be $\tilde{f}^{-1} \circ \tilde{\theta} \circ \tilde{f}$, which clearly does preserve fibers. □

Now, $\operatorname{Aut}\Delta^\pi$ acts faithfully on $\operatorname{Aut}\mathfrak{G} \ltimes \mathfrak{G}^N$ by taking $\alpha\eta$ to $\alpha\eta^f$, where $\eta^f(v) = \eta(f^{-1}(v))$, i.e., $\eta^f = f^{-1} \circ \eta$. This action defines a semidirect product

$$\mathfrak{A} = \operatorname{Aut}\Delta^\pi \ltimes [\operatorname{Aut}\mathfrak{G} \ltimes \mathfrak{G}^N]. \tag{3.2}$$

(This is the group structure we want on the set $\mathfrak{A}$.) Since

$$(ge)^{\tilde{\alpha}\tilde{\eta}\tilde{f}} = \eta(v)^{-1} g^{\alpha} \eta(w) f(e)$$

and

$$(ge)^{\tilde{f}\tilde{\alpha}\widetilde{\eta^f}} = \big[g^{\alpha} f(e)\big]^{\widetilde{\eta^f}} = \eta\big(f^{-1}(f(v))\big)^{-1} g^{\alpha} \eta\big(f^{-1}(f(w))\big) f(e) = (ge)^{\tilde{\alpha}\tilde{\eta}\tilde{f}},$$

one sees that the actions lead to a mapping

$$\operatorname{Aut}\Delta^\pi \ltimes [\operatorname{Aut}\mathfrak{G} \ltimes \mathfrak{G}^N] \to \operatorname{Aut}\langle\mathfrak{G}\Delta^\pi\rangle.$$

Theorem 3.3. *Let Δ be a simple graph in which every block has order at least 3 and let π be a partition of N. Then*

$$\operatorname{Aut}\langle\mathfrak{G}\Delta^\pi\rangle = \operatorname{Aut}\Delta^\pi \ltimes \operatorname{Aut}_p\langle\mathfrak{G}\Delta\rangle = \mathfrak{A}/\tilde{\mathfrak{G}}.$$

The stabilizer of $1\Delta^\pi$ is

$$\operatorname{Aut}\Delta^\pi \times \big[(\operatorname{Aut}\mathfrak{G} \ltimes \tilde{\mathfrak{G}})/\tilde{\mathfrak{G}}\big] = \operatorname{Aut}\Delta^\pi \times \operatorname{Aut}\mathfrak{G}.$$

and $1\Delta^\pi$ is fixed by $\operatorname{Aut}\mathfrak{G}$ only.

Proof. The first equality follows by combining the lemmas. For the second, it is easy to see that the kernel of the action of $\mathfrak{A}$ on $\langle\mathfrak{G}\Delta^\pi\rangle$ is $\tilde{\mathfrak{G}}$. It follows by standard group theory that $\mathfrak{A}/\tilde{\mathfrak{G}} = \operatorname{Aut}\Delta^\pi \ltimes \big[(\operatorname{Aut}\mathfrak{G} \ltimes \mathfrak{G}^N)/\tilde{\mathfrak{G}}\big]$; then we apply Lemma 3.1.

For (f, α, η) to stabilize $1\Delta^\pi$, η must be η_h for some $h \in \mathfrak{G}$, and that is sufficient for stabilization. Hence the stabilizer is

$$\operatorname{Aut}\Delta^\pi \times (\operatorname{Aut}\mathfrak{G} \ltimes \tilde{\mathfrak{G}})/\tilde{\mathfrak{G}}.$$

In the second factor, $\operatorname{Aut}\mathfrak{G} \ltimes \tilde{\mathfrak{G}} = \{\alpha_h^{-1}\eta_h : h \in \mathfrak{G}\}$ and $\tilde{\mathfrak{G}} = \{\alpha_h^{-1}\eta_h : h \in \mathfrak{G}\}$; therefore the factor is essentially $\operatorname{Aut}\mathfrak{G}$. □

The reason for the partition π is that we want to know the automorphism group of a partially filled expansion $\langle\mathfrak{G}\Delta^{(X)}\rangle$, where there is one half edge at each node in $X \subseteq N$. In that case the partition $\pi = \{X, X^c\}$. As the logic is the same for any partition, there is no reason to avoid generality.

Lemma 3.1 tells us that the entire action of $\operatorname{Aut}_p\langle\mathfrak{G}\Delta\rangle$ on a fiber $p^{-1}(e)$ is equivalent to moving a fixed edge $\tilde{e}$ to an arbitrary $\tilde{e}^*$ and then permuting $p^{-1}(e) \setminus \tilde{e}^*$ by the action of $\operatorname{Aut}\mathfrak{G}$. (We see this by switching first and then applying a group automorphism.)

Corollary 3.4. *Assuming Δ is a block of order at least 3, $\operatorname{Aut}_p\langle\mathfrak{G}\Delta\rangle$ acts as the symmetric group on a fiber $p^{-1}(e)$ if and only if $\mathfrak{G} = \mathbb{Z}_\gamma$ for $\gamma \leq 3$ or $\mathfrak{G} = V_4$.*

Here V_4 is the Klein four-group. The corollary is immediate from the easy lemma:

Lemma 3.5. *The only groups for which $\operatorname{Aut}\mathfrak{G}$ is the symmetric group of $\mathfrak{G} \setminus 1$ are V_4 and $\mathbb{Z}_\gamma$ for $\gamma \leq 3$.*

A tantalizing final question:

Problem 3.6. Is there a way of distinguishing group from nongroup expansions by their automorphism groups?

4. Uniqueness of graphs

The bias matroids of full biased expansions, and the complete lift matroids of nontrivial biased expansions, in most cases determine the biased expansions themselves. This is easy to see for expansions of K_n (and the case $\Delta = K_n$ with the bias matroid is implicit in [5], as we mentioned at the end of Section IV.2.1), but not quite so obvious in general—indeed, there are exceptions due to loops or monovalent nodes in Δ or other reasons.

Proposition 4.1. *Let Δ be a link graph in which every node has at least two neighbors, and let Ω be a nontrivial regular biased expansion of Δ. If Υ is a biased graph without isolated nodes such that $G(\Upsilon) \cong G(\Omega^\bullet)$, then $\Upsilon \cong \Omega^\bullet$ (if we treat unbalanced loops as half edges) by the same edge bijection as in the matroid isomorphism.*

If $G(\Upsilon) \cong G(\Omega)$ and Ω has multiplicity at least 4, then $\Upsilon \cong \Omega$ by the same bijection as in the matroid isomorphism.

Proof. We assume all unbalanced edges in Ω and Υ are half edges. A half edge of $\Omega^\bullet$ is the same as an edge belonging to at least two long lines in $G(\Omega^\bullet)$. In Υ, every edge that belongs to at least two long lines is a half edge. These half edges identify an injection $f_0 : N(\Omega) \to N(\Upsilon)$. For a link in Ω, its endpoints in Υ determined by which long line it belongs to. Therefore f_0 is surjective and $\|\Omega^\bullet\| \cong \|\Upsilon\|$ by the matroid isomorphism. A balanced circle is identified by having $\operatorname{rk} C < \#C$; thus $\Omega^\bullet \cong \Upsilon$.

PROOF OF SECOND PARAGRAPH. □

Corollary 4.2. *Let $\mathfrak{G}$ and $\mathfrak{H}$ be groups and Δ a link graph. If $G(\mathfrak{G}\Delta^\bullet) \cong G(\mathfrak{H}\Delta^\bullet)$, then $\langle \mathfrak{G}\Delta \rangle \cong \langle \mathfrak{H}\Delta \rangle$.*

Proof. We may assume $\mathfrak{G}$ and $\mathfrak{H}$ are nontrivial. If Δ is a forest, the conclusion is obvious. If not, apply Proposition 4.1. □

Proposition 4.3. *Let Δ be a link graph in which every node has at least two neighbors and let Ω be a nontrivial regular biased expansion of Δ. If Υ is a biased graph without isolated nodes such that $L_0(\Upsilon) \cong L_0(\Omega)$, then $\Upsilon \cong \Omega$ up to Whitney 2-isomorphism operations. The matroid isomorphism itself is a 2-isomorphism of Υ with Ω, except possibly when $\Delta = K_3$ and $\Omega = 2 \cdot K_3 = \langle \pm K_3 \rangle$.*

The same conclusions follow if $L(\Psi) \cong L(\Omega)$ and Ω has multiplicity at least 4.

Proof. If Ω has multiplicity 3 or more, e_0 is the unique point of $L_0(\Omega)$ that belongs to every long line. Thus, the isomorphism $f : L_0(\Omega) \to L_0(\Upsilon)$ preserves e_0. Therefore $\|\Omega\|$ and $\|\Upsilon\|$ are 2-isomorphic. It follows (as in the previous proposition) that Ω and Υ are 2-isomorphic (as defined in Section IV.5.1).

If Ω has multiplicity 2, let us suppose that $\Delta \neq K_3$, so $n \geq 4$. Since Δ is neither K_3 nor a star, every edge e in Δ has a nonadjacent edge f. Then any $\tilde{e} \in p^{-1}(e)$ and $\tilde{f} \in p^{-1}(f)$ do not lie in a 3-point line together. The only point in $L_0(\Omega)$ that belongs to a 3-point line with every other point is e_0. Thus we have identified e_0 uniquely. The remainder of the proof is as in the case of higher multiplicity.

Suppose $\Omega = \langle \pm K_3 \rangle$. Then $L_0(\Omega) = F_7$, the Fano plane, and no other biased graph has this property [16, Proposition 3A]. Therefore $\Upsilon \cong \Omega$, but the transitivity of the automorphism group of F_7 implies that $f(e_0)$ may not equal e_0, so the biased-graph isomorphism need not be obtained from the matroid isomorphism f even in combination with Whitney 2-isomorphism operations. □

Corollary 4.4. *Let $\mathfrak{G}$ and $\mathfrak{H}$ be groups and Δ a simple graph. If $L_0(\mathfrak{G}\Delta) \cong L_0(\mathfrak{H}\Delta)$, then $\langle\mathfrak{G}\Delta\rangle \cong \langle\mathfrak{H}\Delta\rangle$.*

Proof. For nontriviality we may assume that $\#\mathfrak{G} > 1$ and Δ is a block with $n \geq 3$. Then Proposition 4.3 applies. □

Example 4.1. A reason for requiring multiplicity at least 4 in the case of $G(\Omega)$ in Proposition 4.1 and that of $L(\Omega)$ in Proposition 4.3 is that $G(\mathbb{Z}_3 K_3) \cong \mathbb{A}^2(\mathbb{F}_3)$ (Example 10.3), so it has automorphisms that are not automorphisms of $\mathbb{Z}_3 K_3$. However, it may be that the lift case is still valid in multiplicity 3 with one exception.

Example 4.2. Another reason is that $G(\pm K_3) = G(K_4)$ (Example IV.2.4). Thus there has to be an exception to the conclusions of both propositions in multiplicity 2.

Problem 4.5. Extend Propositions 4.1 and 4.3 to $G(\Omega)$ and $L(\Omega)$ in multiplicity 3 and, if possible, 2. Multiplicity 2 may be especially troublesome because of Example 4.2. However, it may be that the only exceptions are those based on K_3.

A corollary of Theorem 2.1 and our results on matroids is a generalization of Dowling's Theorem 8 [5], which says that $G(\mathfrak{G}K_n^\bullet) \cong G(\mathfrak{H}K_n^\bullet) \iff \mathfrak{G} \cong \mathfrak{H}$.

Corollary 4.6. *Let $\mathfrak{G}$ and $\mathfrak{H}$ be groups and Δ a simple graph, not a forest. Then*

$$G(\mathfrak{G}\Delta^\bullet) \cong G(\mathfrak{H}\Delta^\bullet) \iff \mathfrak{G} \cong \mathfrak{H} \iff L_0(\mathfrak{G}\Delta) \cong L_0(\mathfrak{H}\Delta).$$

Proof. Combine Theorem 2.1(b) and Corollaries 4.2 and 4.4. □

5. MATROID AUTOMORPHISMS

The automorphism groups of the partially filled Dowling geometries $G(\mathfrak{G}K_n^{(k)})$ were worked out by Bonin [2] (with one error; see Theorem 5.1 below). His results can be extended, not without extra difficulty, to all matroids of group expansions; that is what we do here.

Theorem 5.1. $\operatorname{Aut} G(\Omega^{(k)}) = \operatorname{Aut} \Omega^{(k)}$ *(the extras being all one kind of edge) and* $\operatorname{Aut} L(\Omega) = \operatorname{Aut} L_0(\Omega) = \operatorname{Aut} \Omega$*, with EXCEPTIONS.*

6. Balance-closed subgraphs

An edge set S of Ω is *balance-closed* if it equals its balance-closure $\operatorname{bcl} S$. Balance-closed subsets of a group or biased expansion have a very simple description. Recall that in a graph Δ there is a closure operator $\operatorname{clos}_\Delta$ on the edges defined by the matroid $G(\Delta)$.

Proposition 6.1. *A subgraph of a biased expansion of Δ is balance-closed if and only if it is a biased expansion of a closed subgraph of Δ.*

Proof. We may assume Δ is a block. Let Ω be a biased expansion of Δ with projection $p : \Omega \to \Delta$ and let $\Omega' \subseteq \Omega$.

Suppose Ω' is a biased expansion of a closed subgraph $\Delta' \subseteq \Delta$. If $\tilde{C}$ is a balanced circle in Ω such that $\tilde{C} \setminus \tilde{e} \subseteq E(\Omega')$ for some $\tilde{e} \in \tilde{C}$, then $p(\tilde{e}) \in \operatorname{clos}_\Delta p(\tilde{C} \setminus \tilde{e}) \subseteq \operatorname{clos}_\Delta E(\Delta') = E(\Delta')$. Therefore there must be an edge $\tilde{e}'$ in Ω' for which $(\tilde{C} \setminus \tilde{e}) \cup \tilde{e}'$ is a balanced circle. But $\tilde{e}$ is the only such edge in Ω. Hence, $\tilde{C} \subseteq \Omega'$. We conclude that Ω' is balanced-closed.

Contrariwise suppose Ω' is a balance-closed subgraph of Ω. By reasoning like that above, it is clear that Ω' is a biased expansion of $\Delta' = p(\Omega')$. To prove Δ' is closed in Δ, let C be a circle in Δ such that $C \setminus e \subseteq E(\Delta')$ for some $e \in C$. Lift $C \setminus e$ to $\tilde{P}$ in Ω'. There is an edge $\tilde{e}$ of Ω for which $p(\tilde{e}) = e$ and $\tilde{P} \cup \tilde{e}$ is a balanced circle. By balance-closure, $\tilde{e}$ is in Ω'. Therefore, $e \in p(\Omega')$; so Δ' is closed in Δ. $\square$

Proposition 6.2. *Let $\Phi \subseteq \mathfrak{G}\Delta$, where Δ is a link graph and $||\Phi||$ is 2-connected. Then Φ is balance-closed if and only if it is a switching of a subgroup expansion $\mathfrak{H}\Delta'$ where $\mathfrak{H} \leq \mathfrak{G}$ and Δ' is a closed block subgraph of Δ.*

Proof. Suppose Φ switches to $\mathfrak{H}\Delta'$. Then $\langle \Phi \rangle$ is a biased expansion of Δ', hence is balance-closed in $\mathfrak{G}\Delta$ by Proposition 6.1.

Conversely, if Φ is balance-closed, then $\langle \Phi \rangle$ is a biased expansion of a closed subgraph $\Delta' \subseteq \Delta$, a block by the assumption on Φ. Switch $\mathfrak{G}\Delta$ so that Φ has a spanning tree with identity gains. Then $\Phi \supseteq \{1\}\Delta'$ due to balance-closure.

Let $p\big|_\Phi : \Phi \to \Delta'$ be the projection: we wish to prove that $\mathfrak{H}_e = \varphi(p\big|_\Phi^{-1}(e))$ is the same for every $e \in E(\Delta')$ and is a group. Note that $\mathfrak{H}_{e^{-1}} = \mathfrak{H}_e^{-1}$, so $\mathfrak{H}_e$ may depend on the orientation of e. Take any two edges $e, f \in E(\Delta')$ and a circle $C \ni e, f$, which exists because Δ' is a block. Orient e and f similarly in C and write $ePfQ$, where P and Q are the paths joining e and f in C. Lift C to a balanced circle $\tilde{C} = \tilde{e}\tilde{P}\tilde{f}\tilde{Q} \subseteq \Phi$ in which $\tilde{P} \cup \tilde{Q} \subseteq \{1\}\Delta' \subseteq \Phi$. Since $\tilde{C}$ is balanced, $1 = \varphi(\tilde{C}) = \varphi(\tilde{e})\varphi(\tilde{f})$; that is, $\tilde{e}$ and $\tilde{f}$ have inverse gains. Therefore $\mathfrak{H}_e = \mathfrak{H}_f^{-1}$. This conclusion holds for any two edges in Δ'. Now choose $C = e_1e_2e_3\cdots$ in Δ' and orient e_1, e_2, e_3 similarly along C. Thus $\mathfrak{H}_{e_1} = \mathfrak{H}_{e_2}^{-1} = \mathfrak{H}_{e_3} = \mathfrak{H}_{e_1}^{-1}$: that is, $\mathfrak{H}_e$ is independent of how e is oriented, so we may safely write $\mathfrak{H}_e = \mathfrak{H}$ for all oriented edges in Δ'. Finally, let $a, b \in \mathfrak{H}$ and choose a balanced $\tilde{C} = \tilde{e}_1\tilde{e}_2\tilde{e}_3\cdots$ so that $\varphi(\tilde{e}_1) = a$, $\varphi(\tilde{e}_2) = b$, $\varphi(\tilde{e}_i) = 1$ for $i \geq 4$; then $ab = \varphi(\tilde{e}_3)^{-1} \in \mathfrak{H}$. Thus $\mathfrak{H}$ is a subgroup of $\mathfrak{G}$ and Φ (after switching) equals $\mathfrak{H}\Delta'$. $\square$

7. Maximal biased expansions

Now we first come to matroid theory, with the computation of the chromatic and characteristic polynomials of maximal biased expansions and their matroids. Given a biased expansion Ω of Δ and nonadjacent nodes v_i, v_j, suppose there exists a biased expansion Ω' of $\Delta \cup e_{ij}$ such that $(p')^{-1}(\Delta) = \Omega$ and $p'\big|_{\Omega} = p$. Then we say Ω *extends* to the edge e_{ij}. If Δ is simple and Ω does not extend to any edge not in Δ, then Ω is a *maximal biased expansion.* This is the case, for instance, when Δ is complete. Because $\langle \mathfrak{G} K_n \rangle$ is the only maximal biased expansion that is gain-biased and $G(\mathfrak{G} K_n^\bullet)$ is the Dowling matroid of $\mathfrak{G}$ of rank n (see Example III.5.7), I like to think of the matroid $G(\Omega^\bullet)$ of a maximal biased expansion as a natural close generalization of a Dowling matroid.

Although a group expansion is maximal only if the base graph is complete, there do exist maximal biased expansions of incomplete graphs. We show in [17] that a regular expansion of finite order is maximal only if Δ is the result of iterated edge amalgamation of complete graphs and of circles whose length is at least 3. (*Edge amalgamation* means identifying one edge in each of two disjoint graphs.) Also, if the multiplicity of expansion is no more than 3, then the expansion is a group expansion so Δ must be complete. This makes it easy to write a formula for the characteristic polynomial of a generalized Dowling geometry. Recall that a circle of length l has chromatic polynomial

$$\chi_{C_l}(\lambda) = (\lambda - 1)\big[(\lambda-1)^{l-1} - (-1)^{l-1}\big].$$

Theorem 7.1. *Suppose that $\Omega = \gamma \cdot \Delta$ is a maximal γ-fold biased expansion graph of order $n \geq 3$ and that Δ is obtained by a series of edge amalgamations from c_l circle graphs of orders $l \geq 3$ and k_m complete graphs of orders $m \geq 3$. Then the characteristic polynomial of the full bias matroid $G(\Omega^\bullet)$ is*

$$p_{G(\Omega^\bullet)}(\lambda) = (\lambda-1)(\lambda-1-\gamma) \cdot \prod_{l=3}^{\infty} \left[\frac{(\lambda-1-\gamma)^{l-1} - (-\gamma)^{l-1}}{\lambda - 1} \right]^{c_l}$$
$$\cdot \prod_{m=3}^{\infty} \big[(\lambda-1-2\gamma)(\lambda-1-3\gamma)\cdots(\lambda-1-[m-1]\gamma)\big]^{k_m}.$$

The characteristic polynomials of the lift and complete lift matroids $L(\Omega)$ and $L_0(\Omega)$ are

$$p_{L_0(\Omega)}(\lambda) = (\lambda-1)(\lambda-\gamma) \cdot \prod_{l=3}^{\infty} \left[\frac{(\lambda-\gamma)^{l-1} - (-\gamma)^{l-1}}{\lambda} \right]^{c_l}$$
$$\cdot \prod_{m=3}^{\infty} \big[(\lambda-2\gamma)(\lambda-3\gamma)\cdots(\lambda-[m-1]\gamma)\big]^{k_m},$$
$$p_{L(\Omega)}(\lambda) = p_{L_0(\Omega)}(\lambda) + (\lambda-1) \cdot \prod_{l=3}^{\infty} \left[\frac{(\lambda-1)^{l-1} - (-1)^{l-1}}{\lambda} \right]^{c_l}$$
$$\cdot \prod_{m=3}^{\infty} \big[(\lambda-2)(\lambda-3)\cdots(\lambda-[m-1])\big]^{k_m},$$

except that $p_{L(\Omega)}(\lambda) = \lambda^{-1}\chi_\Delta(\lambda)$ when $\gamma = 1$.

Proof. We know that $p_{G(\Omega^\bullet)}(\lambda) = \chi^{\mathrm{b}}_\Omega(\lambda-1)$ (Theorem III.5.1) and $\chi^{\mathrm{b}}_\Omega(y) = \gamma^n \chi_\Delta(y/\gamma)$ (Examples III.3.6–8). Thus the problem reduces to computing the chromatic polynomial of an edge amalgamation $\Gamma_1 \cup_e \Gamma_2$, where e denotes the amalgamated edge. This is well known to be $\chi_{\Gamma_1}(y)\chi_{\Gamma_2}(y)/y(y-1)$, so

$$\chi_\Delta(z) = z(z-1)\cdot \prod_{l=3}^{\infty}\left[\frac{(z-1)^{l-1}-(-1)^{l-1}}{z}\right]^{c_l} \cdot \prod_{m=3}^{\infty}\big[(z-2)(z-3)\cdots(z-[m-1])\big]^{k_m}.$$

Since Ω is connected, we know from Theorem III.5.2 that

$$p_{L_0(\Omega)}(\lambda) = \lambda^{-1}(\lambda-1)\chi^{\mathrm{b}}_\Omega(\lambda)$$

and

$$p_{L(\Omega)}(\lambda) = \lambda^{-1}\big[(\lambda-1)\chi^{\mathrm{b}}_\Omega(\lambda)+\chi_\Delta(\lambda)\big].$$

The desired formulas follow easily. The exceptional case is due to the fact that, $1\cdot\Delta$ being balanced, $L(1\cdot\Delta)$ has rank $n-1$. All other matroids have rank n. $\square$

8. LINEAR REPRESENTATION

The basic fact about vector representation of *group* expansions is a classification. We speak of *bias* and *lift* representations depending on which matroid, bias or lift, we are representing. Canonical bias and lift representations and other terminology are defined in Sections IV.2.2 and IV.4.1.

Proposition 8.1. *Let Δ be a 2-connected simple graph of order $n \geq 3$. Any linear representation of $G(\mathfrak{G}\Delta)$ or $L(\mathfrak{G}\Delta)$ is a canonical bias or lift representation of $\mathfrak{G}\Delta$. The representation over a skew field F is projectively unique up to gain-group embedding, except that when $n = 3$, $G(\mathfrak{G}\Delta) = L(\mathfrak{G}\Delta)$, $\mathfrak{G}$ is nontrivial, and $\mathfrak{G}$ embeds in both F^+ and F^*, then the canonical bias and lift representations are projectively inequivalent even up to gain-group embedding.*

Proof. If $\mathfrak{G}$ is trivial, the matroid is graphic, so it has projectively unique representation. Otherwise, Theorem IV.7.1 applies. The inequivalence of bias and lift representations is explained in Section IV.7. If $n \geq 4$, Δ contains two node-disjoint edges, and thence $\mathfrak{G}\Delta$ contains node-disjoint digons; thus $G \neq L$. □

We conclude that, if $n \geq 4$, then $G(\mathfrak{G}\Delta)$ or $L(\mathfrak{G}\Delta)$ has as many inequivalent representations over F as there are embeddings of $\mathfrak{G}$ into F^* or F^+ that are inequivalent under automorphisms of F. If $\Delta = K_3$, then $G = L$ and there are as many inequivalent representations as there are inequivalent embeddings of fG into F^* and F^+. The proofs are by Theorem IV.7.1 and the ensuing remarks and Propositions IV.2.9 and IV.4.9.

For separable graphs we have a partial version of Proposition 8.1.

Corollary 8.2. *Let Δ be a separable simple graph of which each block has order at least 3.*

(a) Suppose Δ has two node-disjoint circles or $\mathfrak{G}$ embeds only in F^* or only in F^+. The bias representation of $\mathfrak{G}\Delta$ is canonical and projectively unique if $\mathfrak{G}$ embeds uniquely into F^*, but otherwise there is a noncanonical bias representation. The lift representation of $\mathfrak{G}\Delta$ is canonical and projectively unique if $\mathfrak{G}$ embeds uniquely into F^+, but otherwise it need not be canonical.

(b) Suppose Δ has no two node-disjoint circles, so that $G(\mathfrak{G}\Delta) = L(\mathfrak{G}\Delta)$, and $\mathfrak{G}$ embeds in both F^* and F^+. Every representation of $G(\mathfrak{G}\Delta)$ is a canonical bias or lift representation if $\mathfrak{G}$ embeds uniquely in F^* and uniquely in F^+, but not otherwise.

Proof. We analyze the behavior of representations without any assumptions on embeddability of $\mathfrak{G}$.

Let us consider first the lift representation of a block $\mathfrak{G}\Delta_i$. Since it is canonical (with respect to some embedding $\mathfrak{G} \hookrightarrow F^+$), it extends (uniquely) to a representation of $L_0(\mathfrak{G}\Delta_i)$. Let $\hat{E}_i$ be the set representing $\mathfrak{G}\Delta_i$. $\hat{E}_i$ spans a subspace t_i that contains a point $\hat{e}_0$ representing the extra point of $L_0(\mathfrak{G}\Delta_i)$. The representation of $L_0(\mathfrak{G}\Delta_i)$ is projectively unique (up to gain-group embedding in F^+) and the subspaces t_i are projectively unique since they are independent except for their common point $\hat{e}_0$, because $L_0(\mathfrak{G}\Delta)$ is the parallel connection of the $L_0(\mathfrak{G}\Delta_i)$ (Theorem IV.5.4.). Therefore the representation of $L_0(\mathfrak{G}\Delta)$ is unique up to the choice of embedding of $\mathfrak{G}$. However, this choice can be made *independently on each block.* Hence, if there is no more than one block and there is more than one embedding of $\mathfrak{G}$ in F^+, the representation of $L(\mathfrak{G}\Delta)$ is not unique up to gain-group embedding for $\mathfrak{G}\Delta$ as a whole.

A similar argument applies to bias representations. The representation of a block of $\mathfrak{G}\Delta_i$ is canonical and therefore extends to $\mathfrak{G}\Delta_i$. Blocks that share a node v have representations that overlap at the representing vector $\hat{v}$. The representation of $\mathfrak{G}\Delta$ is canonical if a consistent choice of embedding $\mathfrak{G} \hookrightarrow F^*$ is made for each block, but not otherwise. $\square$

The principal representation theorem for *biased* expansions is:

Proposition 8.3. *Let Ω be a biased expansion of a 2-connected link graph Δ. Any representation of $G(\Omega)$ or $L(\Omega)$ is a canonical bias or lift representation. If Ω has a canonical bias representation over a skew field F, then Ω has the form $\langle\mathfrak{G}\Delta\rangle$ where $\mathfrak{G} \leq F^*$. If Ω has a canonical lift representation over F, then Ω has the form $\langle\mathfrak{G}\Delta\rangle$ where $\mathfrak{G} \leq F^+$.*

Proof. It follows from Theorem IV.7.1 that any bias or lift representation is canonical with respect to gains on $\|\Omega\|$ from F^* or F^+, and from Theorem 2.1 that $\Omega = \langle\mathfrak{G}\Delta\rangle$ where $\mathfrak{G} \leq F^*$ for a canonical bias representation and $\mathfrak{G} \leq F^+$ for a canonical lift representation. $\square$

Thus a nongroup biased expansion can have a bias or lift representation only in a nondesarguesian projective geometry. If its order is greater than 3 (and it is 2-connected), it has no projective representations at all.

An important corollary of Theorem 2.1 characterizes representability and unique representability of a group-expansion matroid, in the former aspect generalizing Dowling's Theorems 9–10 [5], which concern $G(\mathfrak{G}K_n^\bullet)$.

Corollary 8.4. *Let Δ be a simple graph that is not a forest, let $X \subseteq N$, and let $\mathfrak{G}$ be a group.*

(a) $G(\mathfrak{G}\Delta^{(X)})$ is representable over a skew field F if and only if $\mathfrak{G}$ is isomorphic to a subgroup of F^*, and $L(\mathfrak{G}\Delta)$ is representable over F if and only if $\mathfrak{G}$ is isomorphic to a subgroup of F^+, except when $\Delta = K_3$ (ignoring isolated nodes) and $X = \varnothing$. In the latter case, where $G(\mathfrak{G}K_3) = L(\mathfrak{G}K_3)$, this matroid is representable over F if and only if $\mathfrak{G}$ embeds in F^* or F^+.

(b) The bias representations of $\mathfrak{G}\Delta^{(X)}$ and lift representations of $\mathfrak{G}\Delta$ are projectively unique up to gain-group embedding, except when $\Delta = K_3$ (ignoring isolated nodes) and $X = \varnothing$. In the latter case there are two projectively inequivalent representations of $G(\mathfrak{G}K_3)$ up to gain-group embedding.

(c) If $\mathfrak{G}$ embeds uniquely in F^* or F^+ (up to automorphisms of F), then the representations of $G(\mathfrak{G}\Delta^{(X)})$ or $L(\mathfrak{G}\Delta)$, respectively, are projectively unique, unless $\Delta = K_3$ (ignoring isolated nodes) and $X = \varnothing$. In the latter case there is unique representation if $\mathfrak{G}$ embeds uniquely in F^* but not at all in F^+, or the reverse; while $G(\mathfrak{G}K_3)$ has exactly two inequivalent representations over F if $\mathfrak{G}$ embeds uniquely in both F^* and F^+ (which requires that F be infinite).

Proof. Combine Theorems IV.7.1 and 2.1. $\square$

In part (b) there is a hidden special case due to the equation $G(\mathfrak{G}K_3) = L(\mathfrak{G}K_3)$. Here we may have both canonical bias representations from embeddings $\mathfrak{G} \hookrightarrow F^*$ and canonical lift representations from embeddings $\mathfrak{G} \hookrightarrow F^+$. Part (b) means that every representation of $G(\mathfrak{G}K_3)$ is projectively equivalent to a canonical bias or lift representation (but not both at once, as that is clearly impossible) derived from an embedding of $\mathfrak{G}$ into F^* or F^+, respectively.

An amusing application of Corollary 8.4 is the following.

Corollary 8.5. *Let Δ be a simple graph without isthmi or isolated nodes and not K_3. Let its blocks be Δ_i for $i \in I$, an index set, and let $\mathfrak{G}_i$, $i \in I$, be groups. Let $\Phi = \bigcup_{i\in I} \mathfrak{G}_i \Delta_i$. Then $G(\Phi)$ is representable over F if and only if every $\mathfrak{G}_i$ is embeddable in F^*, and $L(\Phi)$ is representable over F if and only if every $\mathfrak{G}_i$ is embeddable in F^+.* □

If we take all $\mathfrak{G}_i$ of the same order but not all isomorphic we have an example of a biased expansion that behaves much like a group expansion although it is not one.

9. Root systems and complex analogs

The classical root systems A_{n-1}, D_n, and B_n or C_n are canonical representations of $G(\{1\}K_n)$, $G(\pm K_n)$, and $G(\pm K_n^\bullet)$. (See below, and for more detail [13].) As complex analogs we define sets obtained from a complex primitive γ-th root of unity where $\gamma \geq 2$. Let ζ be such a root and in $\mathbb{C}^n$ let

$$\begin{aligned} D_{n,\varepsilon}(\gamma) &= \{\zeta^k b_i + \varepsilon \zeta^l b_j : 1 \leq i < j \leq n;\ 0 \leq k,l < \gamma\}, \\ B_{n,\varepsilon}(\gamma) &= D_{n,\varepsilon}(\gamma) \cup \{\zeta^k b_i : i \in [n];\ 0 \leq k < \gamma\}, \\ C_{n,\varepsilon}(\gamma) &= \{\zeta^k b_i + \varepsilon \zeta^l b_j : 1 \leq i \leq j \leq n;\ 0 \leq k,l < \gamma\} \setminus \{0\} \\ &= D_{n,\varepsilon}(\gamma) \cup \{(\zeta^k + \varepsilon \zeta^l) b_i : i \in [n];\ 0 \leq k,l < \gamma\} \setminus \{0\}, \end{aligned}$$

where $\varepsilon = \pm 1$ and $\{b_1, \ldots, b_n\}$ is the standard basis of $\mathbb{C}^n$. The classical root systems D_n, B_n, and C_n are the special cases with $\gamma = 2$ and $\varepsilon = -$, taken as sets in $\mathbb{R}^n$.

If γ is even, the value of ε in these definitions is immaterial.

We also define intermediate systems $B_{n,m,\varepsilon}(\gamma)$ and $C_{n,m,\varepsilon}(\gamma)$ in which the definition of $B_{n,\varepsilon}$ or $C_{n,\varepsilon}$ is modified by taking $i \in [m]$, where $0 \leq m \leq n$.

Provided that we take only one vector from each complex line it generates, $D_{n,-}(\gamma)$ is a canonical bias representation of $\mathbb{Z}_\gamma K_n$. Furthermore, with the same proviso, $B_{n,m,-}(\gamma)$ and $C_{n,m,-}(\gamma)$ are canonical bias representations of $\mathbb{Z}_\gamma K_n^{(m)}$, where the unbalanced edges are half edges for $B_{n,-}$ and unbalanced loops for $C_{n,-}$. Accordingly, the matroids of $B_{n,-}(\gamma)$ and $C_{n,-}(\gamma)$, after simplification, are the Dowling geometries of cyclic groups, $G(\mathbb{Z}_\gamma K_n^\bullet)$; that of $D_{n,-}(\gamma)$ is the jointless Dowling geometry $G(\mathbb{Z}_\gamma K_n)$. When $\gamma = 2$ we have the classical root systems D_n, B_n, and C_n and intermediate systems, only in complex space rather than real space.

If γ is odd, $D_{n,+}(\gamma)$ is more complicated. Let θ be a square root of ζ, thus a primitive 2γ-th root of unity. Then $b_j + \zeta^k b_i = b_j - \theta^{2k+\gamma} b_i$, so

$$D_{n,+}(\gamma) = \{\zeta^l (b_j - \theta^{2k+1} b_i) : 1 \leq i < j \leq n;\ 0 \leq k,l < \gamma\}.$$

The matroid is $G(\{1,3,\ldots,2\gamma-1\}K_n)$ with gain group $\mathbb{Z}_{2\gamma}$. As before, $M(B_{n,+}(\gamma)) = M(C_{n,+}(\gamma)) = G(\{1,3,\ldots,2\gamma-1\}K_n^\bullet)$.

The linear dual of $B_{n,m,\varepsilon}(\gamma)$ and $C_{n,m,\varepsilon}(\gamma)$ is

$$B^*_{n,m,\varepsilon}(\gamma) = \begin{cases} \mathcal{H}(\mathbb{Z}_\gamma K_n^{(m)}) & \text{if } \varepsilon = - \text{ or } \gamma \text{ is even;} \\ \mathcal{H}(\{1,3,\ldots,2\gamma-1\}K_n^{(m)}) & \text{if } \varepsilon = + \text{ and } \gamma \text{ is odd.} \end{cases}$$

An arrangement of the first type consists of all hyperplanes in $\mathbb{C}^n$ of the forms

$$\begin{aligned} x_j &= \xi^k x_i \ \text{ for } \ 0 \leq k < \gamma, \quad i < j, \\ x_i &= 0 \text{ for } \ 1 \leq i \leq m; \end{aligned}$$

one of the second type consists of all hyperplanes of the forms

$$\begin{aligned} x_j &= -\xi^k x_i \ \text{ for } \ 0 \leq k < \gamma, \quad i < j, \\ x_i &= 0 \text{ for } \ 1 \leq i \leq m. \end{aligned}$$

Ehrenborg and Readdy in [6] have a notion of "Dowling transform" of a hyperplane arrangement that, when applied to the complete-graph arrangement (or "braid arrangement") $A^*_{n-1} = \mathcal{H}(K_n)$, or that arrangement with m coordinate hyperplanes added, gives the linear dual $D^*_{n,m,-}$. Their theorem proves (in a different way) our formula for the characteristic polynomial of $D^*_{n,n,-} = B^*_n$ as well as much more. The "Dowling transform", though expressed geometrically, is equivalent to expanding arbitrary $\mathbb{R}^*_{>0}$-gain graphs by a cyclic group. Koban has generalized this to expansions of gain graphs and biased graphs by arbitrary groups and even to biased expansions [8].

The case of odd γ with $\varepsilon = +$ does not correspond to a group expansion, but it seems so natural a generalization of root systems that we treat it here anyway.

Proposition 9.1. *The complement in $\mathbb{C}^n$ of the complex hyperplane arrangement $B^*_{n,m,\varepsilon}(\gamma)$, where $0 \le m \le n$ and $\gamma \ge 3$, has Poincaré polynomial*

$$\big(1+y\big)\big(1+y[1+\gamma]\big)\cdots\big(1+y[1+(n-2)\gamma]\big)\big(1+y[(n-1)(\gamma-1)+m]\big)$$

if $\varepsilon = -$ or γ is even, and

$$\sum_{l=0}^{n}\big(1+y\big)\big(1+y[1+2\gamma]\big)\cdots\big(1+y[1+(l-1)2\gamma]\big)\big\{S(n,l)\gamma^{n-l}+(n-m)S(n-1,l)\gamma^{n-l-1}\big\}$$

if $\varepsilon = +$ and γ is odd.

Proof. This is the special case $\Delta = K_n$ of Proposition 9.2. For a direct proof apply Corollary IV.2.3(b) together with Example III.6.7 with $\Phi = \mathbb{Z}_\gamma K_n^{(m)}$ for the first formula, Example 9.1 for the second formula. □

The first part ($\varepsilon = -$) was proved in [6] by a different method.

Example 9.1 (*Complete odd gain graphs with even cyclic gains*). We want the chromatic polynomial of the gain graph $\Phi = \{1,3,\dots,2\gamma-1\}K_n^{(m)}$ with gain group $\mathbb{Z}_{2\gamma}$, where $\gamma \ge 1$. The first step is to compute the balanced chromatic polynomial by coloring with color set $C^*_k = [k] \times \mathbb{Z}_{2\gamma}$ (Section III.4). Suppose we use exactly l numerical colors $i \in [k]$ and the node set with color i is X_i. Within each color class X_i we have to assign group colors $\alpha \in \mathbb{Z}_{2\gamma}$ so that adjacent nodes have difference not in $\{1,3,\dots,2\gamma-1\}$. Hence all group colors are odd, or all are even, and they are otherwise arbitrary. There are $2\gamma^{\#X_i}$ ways to do this. There are $S(n,l)$ ways to choose l color classes and $(k)_l$ ways to assign them l numerical colors. (Here $(x)_l$ is the falling factorial $x(x-1)\cdots(x-l+1)$.) Consequently, with the substitution $\lambda = 2\gamma k$, we have

$$\chi^{\mathrm{b}}_\Phi(\lambda) = \sum_{l=0}^{n} S(n,l)2^l\gamma^n\left(\frac{\lambda}{2\gamma}\right)_l.$$

The actual chromatic polynomial is obtained by Theorem III.6.1:

$$\chi_\Phi(\lambda) = \sum_{l=0}^{n}(2\gamma)^l\left\{S(n,l)\gamma^{n-l}+(n-m)S(n-1,l)\gamma^{n-l-1}\right\}\left(\frac{\lambda-1}{2\gamma}\right)_l.$$

We can generalize the complex root-system analogs by introducing a base graph Δ on the node set $[n]$. Define

$$D_{n,\varepsilon}(\gamma,\Delta) = \{\xi^k b_i + \varepsilon\xi^l b_j : i<j \text{ and } ij \in E(\Delta);\ 0 \le k,l < \gamma\}$$

and for $X \subseteq N$,

$$BC_{n,\varepsilon}(\gamma,\Delta,X) = D_{n,\varepsilon}(\gamma,\Delta) \cup \{\text{at least one vector } cb_i,\ c \ne 0,\ \text{for each } i \in X\}.$$

Then

$$M\big(BC_{n,\varepsilon}(\gamma,\Delta,X)\big) = G(\mathbb{Z}_\gamma \Delta^{(X)})$$

if $\varepsilon = -$ or γ is even, but for odd γ,

$$M\big(BC_{n,+}(\gamma,\Delta,X)\big) = G(\{1,3,\dots,2\gamma-1\}\Delta^{(X)})$$

with gain group $\mathbb{Z}_{2\gamma}$.

The complex hyperplane arrangement dual to $BC_{n,\varepsilon}(\gamma,\Delta,X)$ is

$$BC^*_{n,\varepsilon}(\gamma,\Delta,X) = \begin{cases} \mathcal{H}(\mathbb{Z}_\gamma \Delta^{(X)}) & \text{if } \varepsilon = - \text{ or } \gamma \text{ is even,} \\ \mathcal{H}(\{1,3,\dots,2\gamma-1\}\Delta^{(X)}) & \text{if } \varepsilon = + \text{ and } \gamma \text{ is odd.} \end{cases}$$

A node set W is *stable* in Δ if $E(\Delta){:}W = \varnothing$. The lattice of partitions of W is written Π_W. The subgraph induced by $\pi \in \Pi_W$ is $\Delta{:}\pi = \bigcup_{B\in\pi}(\Delta{:}B)$.

Proposition 9.2. *Let Δ be a simple graph on node set $N = [n]$, $X \subseteq [n]$, and $\gamma \geq 2$. The complement in $\mathbb{C}^n$ of the hyperplane arrangement $BC^*_{n,\varepsilon}(\gamma,\Delta,X)$ has Poincaré polynomial*

$$\sum_{\substack{X\subseteq W\subseteq N \\ W^c \text{ stable in } \Delta}} (-y)^{\#W^c}(-\gamma y)^{\#W}\chi_{\Delta:W}\left(-\frac{\gamma y+1}{\gamma y}\right)$$

if $\varepsilon = -$ or γ is even, and

$$\sum_{\substack{X\subseteq W\subseteq N \\ W^c \text{ stable in } \Delta}} \sum_{\pi\in\Pi_W} \frac{(-y)^n\gamma^{\#W}2^{c(\Delta:\pi)}}{(-2\gamma y)^{\#\pi}}\big(1+y\big)\big(1+y[1+2\gamma]\big)\cdots\big(1+y[1+(\#\pi-1)2\gamma]\big)$$

if $\varepsilon = +$ and γ is odd.

Proof. By Corollary IV.2.3(b), we need the chromatic polynomial $\chi_\Phi(\lambda)$ where $\Phi = \mathbb{Z}_\gamma\Delta^{(X)}$ in the first part and $\{1,3,\dots,2\gamma-1\}\Delta^{(X)}$ with gain group $\mathbb{Z}_{2\gamma}$ in the second part. The former is from Example III.6.6(a), the latter from Example 9.2. □

Example 9.2 (*Oddly expanded graphs with even cyclic gains*). We want the chromatic polynomial of $\Phi = \{1,3,\dots,2\gamma-1\}\Delta^{(X)}$ with gain group $\mathbb{Z}_{2\gamma}$, where $N(\Delta) = [n]$ and $\gamma \geq 1$. (The case $\gamma = 1$ gives the all-negative signed graph $-\Delta^{(X)}$, previously studied in [14, Section 5].) The analysis is similar to that in Example 9.1 except that a color class need not be connected, so a separate choice of odd or even group colors must be made for each component of $\Delta{:}X_i$. This accounts for the factor $2^{c(\Delta:B)}$ in the formula

$$\chi^{\mathrm{b}}_\Phi(\lambda) = \sum_{\pi\in\Pi_n} \gamma^n 2^{c(\Delta:\pi)}\left(\frac{\lambda}{2\gamma}\right)_{\#\pi},$$

from which by Theorem III.6.1 we obtain

$$\chi_\Phi(\lambda) = \sum_{\substack{X\subseteq W\subseteq N \\ W^c \text{ stable in } \Delta}} \sum_{\pi\in\Pi_W} \gamma^{\#W}2^{c(\Delta:\pi)}\left(\frac{\lambda-1}{2\gamma}\right)_{\#\pi}.$$

10. AFFINE AND PROJECTIVE GEOMETRIES

A curious fact is that there is a handful of group-expansion matroids that are affine and projective geometries. We describe the examples first, then give the proof.

Example 10.1 (*Binary affine plane*). The affine plane $\mathbb{A}^2(\mathbb{F}_2)$ has matroid C_4, the 4-point circuit. The group expansions with this matroid are $G(P_2^{(2)})$, where P_2 is a path of length 2 and the two unbalanced edges are placed at the endpoints of the path, $G(\langle C_4\rangle) = L(\langle C_4\rangle)$, and $G(\pm P_2) = L(\pm P_2)$.

Example 10.2 (*Fano plane*). The Fano plane $\mathbb{P}^2(\mathbb{F}_2)$ equals $L_0(\pm K_3)$ (which we can also think of as $L(\pm K_3^{(1)})$). By [16, Proposition 3A], no other bias or lift matroid of a biased graph is this plane.

Example 10.3 (*Ternary affine plane*). It is known that the matroid $G(\mathbb{Z}_3 K_3) = L(\mathbb{Z}_3 K_3)$, the rank-3 jointless Dowling geometry of $\mathbb{Z}_3$, is the affine plane $\mathbb{A}^2(\mathbb{F}_3)$ (see [9, Section 8.5]). Indeed, we can be more precise: that plane is the canonical lift representation of $\mathbb{Z}_3 K_3$.

Example 10.4 (*Binary affine space*). The lift matroid $L(\pm C_4)$ equals $\mathbb{A}^3(\mathbb{F}_2)$. (I thank Steven Tedford for help with this example.)

The proofs of these examples are contained in the following result.

Proposition 10.1. *The only ways in which a partially filled group expansion $\Phi = \mathfrak{G}\Delta^{(k)}$ can have bias or lift matroid equal to the matroid of an affine or projective geometry of dimension at least 2 are those in Examples 10.1–10.4. The only other examples in which $G(\Phi)$ or $L(\Phi)$ contains a spanning affine geometry of dimension at least 2 and is contained in the corresponding projective space are $L_0(\pm K_4) = M(\mathbb{P}^3(\mathbb{F}_2)\setminus$ two points on a line through $e_0)$ and $L_0(\pm(K_4 \setminus \text{edge}) = M(\mathbb{P}^3(\mathbb{F}_2)\setminus$ four points on two lines through $e_0)$.*

Proof. We assume that $M = G(\Phi)$ or $L(\Phi)$ contains a spanning affine geometry $M(\mathbb{A}^d(F))$. (Remember that L_0 is essentially a special case of L obtained by adding a half edge e_0 to the biased graph.) The case of trivial gain group is elementary because M is graphic. We may therefore assume $\#\mathfrak{G} > 1$. This implies that $d = n-1$ and, by Theorem IV.7, that Φ has only canonical bias and lift representations.

The fact that $M \subseteq \mathbb{P}^{n-1}(F)$ means Φ is representable over F. Since the representation is canonical, $\mathfrak{G} \leq F^*$ or F^+, respectively. But there must be equality because M contains $\mathbb{A}^{n-1}(F)$. Thus there are two cases: $\Phi = F^*\Delta^{(k)}$ and we have a canonical bias representation in $\mathbb{P}^{n-1}(F)$, or $\Phi = F^+\Delta^{(k)}$ and we have a canonical lift representation in $\mathbb{P}^{n-1}(F)$.

In the bias case, the canonical representation omits all points of the form $(1, a_1, \ldots, a_{n-1}) \in \mathbb{P}^{n-1}(F)$ with $a_1, \ldots, a_{n-1} \in F^*$, yet it contains the complement of a hyperplane. That can only happen if $\#F^* = 1$. But then $\mathfrak{G}$ is trivial, contradicting the hypothesis. Thus, this case cannot exist.

In the lift case, the representation, being canonical, extends to $L_0(\Phi)$, which is covered by the $\binom{n}{2}$ lines through e_0. Since $\mathbb{A}^{n-1}(F)$ is covered by no fewer than q^{n-2} lines (where $q = \#F$), we have $q^{n-2} \leq \binom{n}{2}$. The only solutions for which $n \geq 3$ are: $n = 3$ with $q \leq 3$; $n = 4$ with $q = 2$; and $n = 5$ with $q = 2$. We take up these four cases individually.

When $n = q = 3$ the field is $\mathbb{F}_3$ and the gain group is $\mathbb{Z}_3$. Since $\mathbb{A}^2(\mathbb{F}_3)$ is covered by no fewer than 3 lines, the gain graph is $\mathbb{Z}_3 K_3$. The representation of $L_0(\mathbb{Z}_3 K_3)$ consists of three lines through the one point e_0 in $\mathbb{P}^2(\mathbb{F}_3)$, so its complement is two points on one line l through e_0. That is, $L(\mathbb{Z}_3 K_3)$ is represented by $\mathbb{P}^2(\mathbb{F}_3) \setminus l$, which is $\mathbb{A}^2(\mathbb{F}_3)$. This is Example 10.3.

When $q = 2$, the field is $\mathbb{F}_2$ and the gain graph is a signed graph. In dimension $n - 1 = 2$, $L_0(\pm\Delta)$ is covered by 2 or 3 lines. In the former case $\Delta = P_2$ and in the latter $\Delta = K_3$. These two cases give Examples 10.1 and 10.2.

In dimension $n - 1 = 3$, we know $L_0(\pm K_4)$ represents in $\mathbb{P}^3(\mathbb{F}_2)$ and uses all but 2 points. These points generate a line l that contains e_0, because $L_0(\pm K_4)$ is the union of 6 lines through e_0. Deleting two well-chosen parallel pairs of edges from $\pm K_4$ leaves $\pm\Delta$ with $L(\pm\Delta) \cong M(\mathbb{A}^3(\mathbb{F}_2))$. The deleted pairs cannot be adjacent; for if they were, Δ would be separable so $L(\pm\Delta)$ would be a 2-sum (by Theorem IV.5.4); but $M(\mathbb{A}^3(\mathbb{F}_2))$ is not a 2-sum. Therefore $\Delta = C_4$ and $L(\pm C_4) \cong M(\mathbb{A}^3(\mathbb{F}_2))$. This is the only way to get a submatroid of $L_0(\pm K_4)$ that is an affine 3-space.

In dimension $n-1 = 4$, we know $L_0(\pm K_5)$ is represented by a set in $\mathbb{P}^4(\mathbb{F}_2)$, call it again E_0. Suppose a subset $A \subseteq E_0$ has matroid $M(\mathbb{A}^4(\mathbb{F}_2))$. Then $M(A \cup e_0)/e_0$ (in the affine space) $\cong L_0((\pm K_5) \mid A)/e_0$, but the former matroid is not graphic while the latter is contained in $G(2K_5)$. Thus $M(\mathbb{A}^4(\mathbb{F}_2))$ cannot be contained in $L_0(\pm K_5)$. ☐

The extension of Proposition 10.1 to biased expansions is:

Corollary 10.2. *The only ways in which a partially filled regular biased expansion Ω can have bias or lift matroid that contains a spanning affine geometry of dimension at least 2 are for Ω to equal $\langle\Phi\rangle$ where Φ is as in Proposition 10.1.*

Proof. Following the proof of Proposition 10.1 for Ω instead of Φ, and assuming nontriviality, we see that representability of the matroid, being canonical, requires Ω to have the form $\langle\mathfrak{G}\Delta^{(k)}\rangle$. Then Proposition 10.1 applies. The only possible exception is when the projective and affine geometries are nondesarguesian planes; but then the affine plane, having a biased-graphic matroid of rank 3, must be the union of at most 3 lines. Hence its order is at most 3 so it is coordinatizable. ☐

Example 10.5 (*Non-Fano plane*). The non-Fano plane F_7^- equals $G(\pm K_3^{(1)})$. For the proof, apply Example 10.2. By treating e_0 as a half edge at v_1 and relaxing the circuit hyperplane $\{e_0, e_{23}\}$ in $L(\pm K_3^{(v_1)})$ we get $G(\pm K_3^{(v_1)})$. Correspondingly, relaxing a circuit hyperplane in F_7 gives F_7^-.

It can be shown that $\langle\pm K_3^{(1)}\rangle$ is the only biased graph for which $G(\Omega) = F_7^-$. (I thank Lori Koban for help with this example.)

11. Hypercubes

The *verticial hypercube* Q^d is the set $\{0,1\}^d$, or equivalently $\{1,-1\}^d$, in $\mathbb{A}^d(\mathbb{R})$. Another way to describe $M(Q^d)$ is as the simplified matroid of linear dependence of $\{1,-1\}^{d+1} \subseteq \mathbb{R}^{d+1}$.

Example 11.1 (*Verticial cube*). The matroid $M(Q^3)$ equals $G(\pm C_4)$. The proof depends on Example 10.4. We know that $M(Q^3)$ is obtained from $M(\mathbb{A}^3(\mathbb{F}_2))$ by relaxing two complementary circuit-hyperplanes (see [10, matroid R_8, pp. 508–509]) and that $G(\pm C_4)$ is obtained from $L(\pm C_4)$ in the same way since the lift hyperplanes $\{\pm e_{12}, \pm e_{34}\}$ and $\{\pm e_{23}, \pm e_{14}\}$ are independent in G. (I thank Steven Tedford for help with this example.)

Thus, $M(Q^3)$ is a submatroid of the linear dependence matroid of the root system D_4. Specifically, it equals $M(x(\pm C_4))$. We may ask whether Q^3, divided by $\sqrt{2}$ to make lengths correspond, is metrically equivalent to $x(\pm C_4)$; or rather, because of central symmetry, we should compare $Q = \{1,-1\}^4/\sqrt{2}$ to $\pm x(\pm C_4) = \{\pm b_i \pm b_{i+1} : i = 1,2,3,4\}$ (where b_5 means b_1). These sets are in fact isometrically equivalent; the proof is by comparison of the angles made by the various vectors. Therefore Q^3 is, in essence, metrically contained in D_4.

Example 11.2 (*Verticial hypercubes*). The matroid of the verticial square Q^2 equals C^4, and is therefore biased-graphic by Example 10.1. But what about higher cubes? In fact, there is no $d > 3$ for which $M(Q^d)$ is a matroid of a partially filled group expansion.

For the proof, suppose $M(Q^d)$ with $d > 3$ is a matroid of a partially filled group expansion. Since $M(Q^3)$ is nongraphic (as its one-point contraction is F_7^-, by Examples 11.1 and 10.5), the gain group is nontrivial. Because in Q^d every line has 2 points, the gain group is $\mathbb{Z}_2$; so the gain graph is $\pm\Delta^{(X)}$. For the same reason, and because $M(Q^d)$ is connected, $X = \varnothing$. Since $M(Q^d)$ is representable over $\mathbb{R}$ and Δ has order $n > 3$, by Theorem IV.7.1 the matroid is $G(\pm\Delta)$. Again by the size of lines, Δ is bipartite. Finally, we know that $n = \operatorname{rk} G(\pm\Delta) = \operatorname{rk} M(Q^d)) = d + 1$. These facts are incompatible with the obvious requirement that $\#Q^d \leq \#E(\pm\Delta)$.

Example 11.3 (*Complex hypercubes*). We might consider a complex generalization. Let ζ be a complex primitive k-th root of unity for $k \geq 3$. Let $Q^d(k) = \{(\zeta^{k_1}, \dots, \zeta^{k_d}) : k_1, \dots, k_d \in [k]\} \subseteq \mathbb{A}^d(\mathbb{C})$. Is there any $d \geq 2$ for which the matroid of $Q^d(k)$ equals that of a partially filled group expansion? There is not.

The gain group cannot be larger than k; that follows from the existence of a k-point line in Q^d when $k > 3$, such as $\{(\zeta^j, \dots, \zeta^j) : j \in [k]\}$, and from the nonexistence of a long line when $k = 3$. Then counting points and other simple arguments as in the previous example show that the only possibility is that $M(Q^2(3))$ might equal $G(\mathbb{Z}K_3)$. But $G(\mathbb{Z}_3 K_3) = \mathbb{A}^2(\mathbb{F}_3)$, which cannot be embedded in characteristic 0.

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