BIASED GRAPHS. VIII. A CORNUCOPIA OF EXAMPLES (WORKING DRAFT January 24, 2002)

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Abstract. Continuing and applying the work of Part IV, "Geometrical realizations", we develop linear representation theory for the matroids of gain and biased graphs of many types: balanced, signed, and antibalanced graphs; and poise, antidirection, and Hamiltonian bias. We explore cyclic gains for various kinds of bias, since these are crucial for bias-matroid representations over finite fields.

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Introduction

In this paper, a continuation of Part IV [15], we develop a variegated multitude of examples. The main concern in each is whether there are canonical vector representations of the bias and lift matroids, and over which fields (and skew fields). If a canonical representation does not exist one will naturally ask about noncanonical ones, but except for the simplest lower bounds on the order of the field that question is difficult, too much so for us to treat here. A rare exception, perhaps the only one, is Pagano's theorem [5] for signed graphs (see below, Section 2), which is based on deep and difficult work by Whittle [9] on matroids representable over all fields with characteristic other than 2.

We sometimes also discuss the enumeration of regions or faces of real hyperplanar representations. The main work there, according to Corollaries IV.2.3 and IV.4.4, is the evaluation of the chromatic polynomial of the biased graph, for which we refer to Part III and [13].

1. Balanced graphs

For a balanced graph $(\Gamma, \mathcal{C}(\Gamma))$ (see Example I.6.1, Lemma II.2.2, Corollary II.3.3), both bias and lift matroids coincide with the usual polygon matroid $G(\Gamma)$. Therefore, as is well known, they are representable over every field, uniquely so up to projective transformations. Furthermore, they are the only gain graphs that can have binary canonical bias representations.

Proposition 1.1. *If a biased or gain graph has a canonical bias representation over $\mathbb{F}_2$, it is balanced.* □

Suppose $\Phi = (\Gamma, \varphi)$ is a real, multiplicative gain graph—that is, its gain group is $\mathbb{R}^*$—and it is balanced. Since it is balanced, it is (by Lemma I.5.3) a switching of $\{1\}\Gamma$ by a function $\eta : N \to \mathbb{R}^*$. That is, $\varphi(e; v, w) = \eta(v)^{-1} 1 \eta(w)$. A canonical bias vector representation is therefore given by $f(e) = \eta(w)\hat{w} - \eta(v)\hat{v}$ (where e has endpoints v and w). The canonical bias hyperplane representation is by hyperplanes $h(e) = \{x \in F^N : \eta(v)x_v = \eta(w)x_w\}$. The resulting hyperplane arrangement, $\mathcal{H}(\Phi) = \{h(e)\}_{e \in E}$, is a multiplicative scaling of the graphic arrangement $\mathcal{H}(\Gamma)$ (which is identical to $\mathcal{H}(\{1\}\Gamma)$); like the latter, it has $|\chi_\Gamma(-1)|$ regions (since $\chi_\Phi(\lambda) = \chi_\Gamma(\lambda)$ by balance and Section III.3) and its face-number generating polynomial equals $(-1)^{n-c(\Gamma)} w_\Gamma(-x, -1)$ (by Corollary IV.2.3 and Section III.3 again). (When counting regions we always assume Φ is finite.)

If Φ is a real, additive gain graph—that is, its gain group is $\mathbb{R}^+$—and it is balanced, then it is $\{0\}\Gamma$ switched by some function $\eta : N \to \mathbb{R}^+$. The natural lift vector representation is $z(e) = \hat{w} - \hat{v} + [\eta(w) - \eta(v)]b_0$. Dually, the affinographic hyperplane corresponding to e in $\mathbb{A}^N(\mathbb{R})$ has equation $x_w - x_v = \eta(w) - \eta(v)$; so the affinographic arrangement representing $L(\Phi)$ is $\mathcal{A}(\Phi) = \{x_w - x_v = \eta(w) - \eta(v) : e{:}vw \in E\}$. This is an additive scaling of $\mathcal{H}(\Gamma)$ and for that reason has, like $\mathcal{H}(\Phi)$, the same face-number polynomial as does $\mathcal{H}(\Gamma)$. For a representation of $L_0(\Phi)$ we go to $\mathbb{P}^N(\mathbb{R})$ and the arrangement $\mathcal{A}_{\mathbb{P}}(\Phi)$ whose hyperplanes are the ideal hyperplane, representing e_0, and the projective completions of the affine hyperplanes of $\mathcal{A}(\Phi)$. The number of regions is unchanged. (This is obvious by the geometry. Abstractly, it follows from the formulas lying behind the proof of Corollary IV.4.4.)

2. Additively biased (signed) graphs

Additive bias (Example I.6.4) means that each theta graph contains an odd number of balanced circles. Equivalently, it is the bias that arises from a signed graph [11, Theorem 6]: $\Omega = \langle\Sigma\rangle$ for some signed graph Σ. Signed graphs are especially interesting because the gain group provides canonical bias and lift representations over complementary sets of fields. The incidence matrix $\mathbf{I}(\Phi)$ is *totally dyadic*: every nonsingular minor determinant is a signed power of 2 ([12, Lemma 8A.3], or by Equation (IV.2.2)); thus the bias matroid is canonically representable over every field of characteristic not equal to 2 [12, Theorem 8B.1]. On the other hand, the gain group's having order 2 means that the lift matroid is canonically representable over every field of characteristic 2.

In particular, $G(\Sigma)$ is canonically representable by a hyperplane arrangement $\mathcal{H}(\Sigma)$ in $\mathbb{R}^N$. The number of regions formed by this arrangement (in the finite case) equals $|\chi_\Sigma(-1)|$, where $\chi_\Sigma(\lambda)$ is the chromatic polynomial of Σ (see [13] or Section III.4) of which there is an extensive theory in [14] covering many special types of signed graph. The number of faces of $\mathcal{H}(\Sigma)$ of each dimension is given by the Whitney-number polynomial [13, Corollary 4.1′].

The canonical vector bias representations of $\pm K_n$ (the graph with all possible positive and negative links), $\pm K_n^\circ$ (where every vertex gets a negative loop), and $\pm K_n'$ (where every vertex gets a half edge) are the root systems D_n, C_n, and B_n. Their dual hyperplane arrangements and generalizations to sign-symmetric signed graphs, which are signed graphs of the form $\pm\Gamma$ with unbalanced edges added as desired, were studied in [10]. All these are special cases of group expansions (Part V). More general kinds of root subsets are treated, as signed graphs, in [14].

Although the canonical bias representation does not work in characteristic 2 if Ω is unbalanced [12, Theorem 8B.2], there may be noncanonical bias representations, especially binary ones (as we saw in Example IV.2.1) or quaternary ones (representations over $\mathbb{F}_4$). A class of unbalanced signed graphs that have binary representations is discussed in Example 2.1. Exactly which signed graphs have binary or quaternary representations was completely determined by Pagano [5] (assuming sufficient connectivity). He found an interesting new form of quaternary representation he calls an "ω-representation". Suppose there is an orientation of the negative edges of Σ in which each balanced circle has equally many edges in each direction and each unbalanced circle has exactly one more edge in one direction than in the other. (The positive edges are not oriented.) Such an orientation is (adapting Pagano's and Gerards' terminologies) a *negative orientation with discrepancy 1.* Assign gains $\varphi(e; u, v) = 1$ if e is positive and $\varphi(e; u, v) = \omega$ if e is negative and oriented from u to v, where ω is a generator of $\mathbb{F}_4^*$. Thus we have a gain graph Φ with gains in $\mathbb{Z}_3$. Since $\langle\Phi\rangle = \langle\Sigma\rangle$, both Φ and Σ have the same bias matroid. An *ω-representation of* Σ is a canonical representation of $G(\Phi)$ over $\mathbb{F}_4$.

Theorem 2.1 (Pagano [5]). *Assume Σ is 3-connected. For each negative orientation of Σ with discrepancy 1, the corresponding ω-representation of Φ is a vector representation of $G(\Sigma)$ over $\mathbb{F}_4$. If Σ has no negative orientation with discrepancy 1, then $G(\Sigma)$ has no representation in characteristic 2.*

The 3-connected signed graphs that have the required orientations are known from Gerards' indirect method [2]; Pagano determined them independently by a direct analysis.

Obviously, the canonical lift representation of a signed graph is defined only in characteristic 2. It is unknown for which biased graphs $L(\Omega)$ may be representable over other fields, except in the examples of Section 1 and Example 2.1. The answer depends partly on whether Ω may have gains in groups $\mathbb{Z}_p^k$; this is also unknown. We do know from Proposition IV.4.3 that only when such gains exist can $L_0(\Omega)$ be representable in characteristic p.

Example 2.1 (*Signed graphs with no node-disjoint unbalanced circles*). If no connected component of Ω contains two node-disjoint unbalanced circles, then the bias and lift matroids coincide, just as in the balanced case. Although the matroids need not be graphic, if Ω is additively biased they are regular (an example is $-K_5$; see Example 2.2), hence representable over every field by the canonical bias representation of Σ. (For a proof see [16, Corollary 3D].) Lovász used this fact and Seymour's decomposition theorem [6] for regular matroids to describe the structure of Σ [4]; unfortunately this excellent theorem, intended to appear in a large joint paper [3], remains unpublished.

One kind of unbalanced signed graph without node-disjoint unbalanced circles is obtained by taking an all-positive graph $+\Gamma$, adding a negative link e joining two nodes already connected, and contracting the added edge. The bias and lift matroids of this signed graph Σ equal $G(\Gamma)$ since e is a coloop in the matroids of $+\Gamma \cup -e$. Thus $G(\Sigma)$ has a binary representation but it is not a canonical bias representation of Σ.

Example 2.2 (*Parity-biased (antibalanced) graphs*). (Examples I.6.3, III.12.1.) In a *parity-biased* graph the balanced circles are those of even length. This is the bias of an all-negative signed graph. The switchings of such signed graphs are called *antibalanced*. There has been no special research on the vector representations of these graphs, but some particular antibalanced graphs are important.

Example 2.3 (*A noncanonical representation*). For instance, $-K_5$ has a noncanonical binary representation: by [16, Proposition 6A] $G(-K_5)$, which is regular by Example 2.1 for instance, equals Bixby's matroid R_{10}, which is central to the decomposition of regular matroids [6]. This representation cannot be a canonical one of any biased graph, for the biased graph would have to have trivial gains (by Proposition 1.1) so its bias matroid would be graphic, but R_{10} is not a graphic matroid.

3. Poise and modular poise bias

(Example I.6.5.)

4. Antidirection bias

(Example I.6.6.)

5. Seven dwarves: the biased K_4's

(Examples I.7.1–7, II.6.1–7, III.13.7–1.)

6. Hamiltonian bias

Hamiltonian bias (Example III.12.2)—or perhaps better, *Hamiltonian balance*—is any bias in which only Hamiltonian circles are balanced. The Hamiltonian circles to be balanced can be chosen at will since they will always form a linear class. We have *complete Hamiltonian balance* when all the Hamiltonian circles are balanced, while in *Hamiltonian balance over an edge* it is just those Hamiltonian circles that contain a fixed edge that are the balanced ones. Any gains that make only Hamiltonian circles balanced are called *Hamiltonian gains.* Let us write $\mathcal{H}$ for the set of all Hamiltonian circles in Γ and $\mathcal{H}(e)$ for the set of all that contain the edge e.

Hamiltonian bias provides many interesting examples. We look at three. In each we find the nonexistence of some canonical representations and some lower bounds on any possible representation. We prove nonexistence of canonical representations by showing that the bias cannot be obtained from gains; thus we vastly augment the catalog of known biased graphs without gains, which until now consisted of the single graph in Example I.5.8. (But the latter is still the only known minor-minimal nongainable biased graph.)

Example 6.1 (*Wheels*). Label the nodes of W_t, the t-spoke wheel, by z at the hub and $x_1, x_2, \ldots, x_t$ around the rim. Suppose W_t has complete Hamiltonian bias. We want to know what the corresponding gains are so we can describe the canonical representations. By suitable switching we can make the gains on the spokes equal 1. The rim gains $\varphi_i = \varphi(x_i x_{i+1})$ satisfy $\varphi_1\varphi_2\cdots\varphi_{t-1} = 1$ and $\varphi_2\varphi_3\cdots\varphi_t = 1$ due to the Hamiltonian circles $zx_1x_2\cdots x_t z$ and $zx_2\cdots x_t x_1 z$, respectively. It follows that $\varphi_t = \varphi_1$ and by extension

$$\varphi_1 = \varphi_2 = \cdots = \varphi_t = \alpha,$$

say. Furthermore, $\alpha^{t-1} = 1$ in order to make the Hamiltonian circles balanced. Since we want Hamiltonian gains, the order of α cannot be $< t-1$, for that would make a balanced non-Hamiltonian circle. Therefore $\langle\alpha\rangle$ can only be $\mathbb{Z}_{t-1}$.

(By the preceding reasoning, if W_t has gains so that every Hamiltonian circle is balanced, then the essential gain group—the group generated by $\varphi(E)$ if the gains are switched so a spanning tree has all identity gains—is $\mathbb{Z}_{t-1}$ or a homomorphic image. This holds true whether or not any non-Hamiltonian circles are balanced.)

Thus there are canonical bias representations only over fields F for which $F^* \supseteq \mathbb{Z}_{t-1}$. There are canonical lift representations only over fields for which $F^+ \supseteq \mathbb{Z}_{t-1}$. Consequently:

Proposition 6.1. *For $t \geq 3$, $(W_t, \mathcal{H})$ has a canonical bias representation, and $G(W_t^\bullet, \mathcal{H})$ has a representation, over $\mathbb{F}_q$ if and only if $t-1 \mid q-1$, over $\mathbb{R}$ if and only if $t = 3$, and over $\mathbb{C}$ always. $(W_t, \mathcal{H})$ has a canonical lift representation, and $L_0(W_t, \mathcal{H})$ has a representation, in characteristic $p \geq 0$ if and only if $t = p+1$, and then both are representable over the prime field of order p.* □

In interpreting this one should keep in mind that, since W_t has no node-disjoint circles, $G(W_t, \mathcal{B}) = L(W_t, \mathcal{B})$ for any bias. So, $G(W_t, \mathcal{H})$ is representable noncanonically in characteristic $t-1$ if $t-1$ is prime.

The proposition leaves open the possibility of noncanonical bias and lift representations. Although we cannot decide their existence, we can establish lower bounds.

Proposition 6.2. *For $t \geq 4$, there is no bias or lift representation of $(W_t, \mathcal{H})$ over any field of order less than $t-1$.*

Proof. We apply Lemma IV.7.*[L:linesfield]*. We produce an L_{t-1} minor of $G(W_t, \mathcal{H})$ by contracting $x_1x_2, \ldots, x_{t-1}x_t$. This leaves two nodes, z and x (the latter representing all the rim nodes), with an unbalanced loop at x and a t-fold link xz. There is one balanced digon, $\{x_1z, x_tz\}$, from the Hamiltonian circle $zx_1 \cdots x_tz$. Removing one edge from that digon leaves a matroid L_t. □

Whether these bounds may or may not be sharp is uncertain. For $t = 3$ the lower bounds allow all fields for bias and lift representations. We know that $(W_3, \mathcal{H}) = \langle -K_4 \rangle)$ and $G(-K_4) = L(-K_4)$. Proposition 6.1 says that $G(-K_4)$ has a canonical representation over $\mathbb{F}_q$ if q is odd and $L(-K_4)$ has a canonical representation in characteristic 2. Combining these, the matroid has representations over all finite fields. Indeed, we know from [16, Proposition 5A] that $G(-K_4)$ is graphic, a submatroid of $G(K_{3,3})$, so is representable over all fields. For $t = 4, 6$, $t-1$ is prime so $(W_t, \mathcal{H})$ has a canonical lift representation over $\mathbb{F}_{t-1}$, the minimum allowed by Proposition 6.2. For $t = 5$, $t-1$ is a prime power. Whenever this happens, there is a canonical bias representation over $\mathbb{F}_t$. That is one greater than the minimum in Proposition 6.2. Whether there is a representation over $\mathbb{F}_4$, which the lower bound permits, is not known.

Example 6.2 (*Complete graphs*). The complete Hamiltonian balance on K_3 and K_4 is produced by the all-negative signing. (See Examples 2.2 and 5.) With larger complete graphs the situation is completely different.

Lemma 6.3. *On K_n with $n \geq 5$, there is no gain function that yields any Hamiltonian bias whose balanced circles contain all of $\mathcal{H}(e)$.*

Proof. Let us assume K_5 endowed with gains so that every Hamiltonian circle on the edge e_{15} is balanced. (We assume nothing about other circles.) By switching we can assume that the gain on any edge $e_{i,i+1}$ is 1. The Hamiltonian circles on e_{15} give the equations

$$\varphi_{13}\varphi_{24} = \varphi_{24}\varphi_{35} = \varphi_{14}\varphi_{25} = \varphi_{13}\varphi_{42}\varphi_{25} = \varphi_{14}\varphi_{42}\varphi_{35} = 1,$$

where for simplicity we write φ_{ij} instead of $\varphi(e_{ij})$. We deduce that

$$\varphi_{13} = \varphi_{35} = \varphi_{42} = \alpha \qquad \text{and} \qquad \varphi_{52} = \varphi_{41} = \alpha^2,$$

where $\alpha^4 = 1$. Thus we have gains from a cyclic group that could be $\mathbb{Z}_4$, $\mathbb{Z}_2$, or trivial. These gains are not Hamiltonian, since the quadrilateral 1254 has gain 1.

Now let us suppose K_n, where $n > 5$, endowed with similar gains, and let $e = e_{1n}$. Delete all the edges except those of the Hamiltonian circle $v_1v_2 \cdots v_n$ and those amongst the nodes $v_{i+1}, v_{i+2}, \ldots, v_{i+5}$ where $0 \leq i \leq n-5$; call the remaining graph $\bar{K}_5$. It is a subdivision of K_5 with v_{i+j} corresponding to node j in K_5, so the distinguished edge e of K_n lies in the subdivision of edge e_{15} of K_5. Thus by suppressing divalent nodes we have K_5 with gains that make every Hamiltonian circle on e_{15} balanced. The balanced quadrilateral of K_5 whose existence we showed in the first paragraph is the subdivision of a non-Hamiltonian circle in K_n; thus the gains on K_n are not Hamiltonian. □

Incidentally, one can deduce from an extension of the proof of this lemma that the only gains on K_n that could possibly make $\mathcal{H}(e_{1n})$ balanced are the gains $\varphi_{i,i+1} = 1$ for all i and, for $1 \leq i < j \leq n$ with $j - i \geq 2$, $\varphi_{ij} = \alpha^2$ if $j - i$ is odd, $= \alpha$ if $j - i$ is even and i is odd, $= \alpha^{-1}$ if $j - i$ is even and i is even. However, it is not trivial to verify that these gains do make $\mathcal{H}(e_{1n})$ balanced, and I have not tried to decide the question.

Proposition 6.4. *For $n \geq 5$, there is no bias representation of any $(K_n^\bullet, \mathcal{B})$ where $\mathcal{H} \supseteq \mathcal{B} \supseteq \mathcal{H}(e)$. For $n \geq 15$, there is no bias representation of any $(K_n, \mathcal{B})$. For $n \geq 5$, there is no representation of any $L_0(K_n, \mathcal{B})$. For $n \geq 7$, there is no lift representation of any $(K_n, \mathcal{B})$.*

Proof. By the lemma, for $n \geq 5$ a canonical bias representation of $(K_n, \mathcal{B})$ is impossible. Consequently, the first part follows from Proposition IV.2.4 and the nonexistence of a canonical bias representation of $(K_n, \mathcal{B})$.

For the second part, suppose the contrary. Label the nodes in order around a balanced Hamiltonian circle so that $e = e_{1,n}$. By contracting $e_{i,i+1}$ and $e_{i+1,i+2}$ for $i = 1, 4, \ldots, 13$ and $e_{i,i+1}$ for $i = 15, 16, \ldots, n-1$, we get a biased graph that contains a subgraph $\Omega_5 = (K_5^\bullet, \mathcal{B}_5)$ with unbalanced loops at all five nodes and having some form of Hamiltonian bias $\mathcal{B}_5 \supseteq \mathcal{H}_{K_5}(e)$. By contracting the representation, we have (by Theorem IV.2.1(b)) a bias representation of $G(\Omega_5)$. But we have seen that this is impossible.

The third part follows similarly to the first, by Proposition IV.4.3. The fourth part is similar to the second, but it is only necessary to produce one unbalanced loop; then the lift matroid of the contraction equals the complete lift matroid, which is unrepresentable. □

It is not known whether there may exist noncanonical bias representations when $5 \leq n \leq 14$ or lift representations when $n = 5$ or 6, most importantly of $(K_5, \mathcal{B})$ since if that has no noncanonical bias or lift representation, neither does any larger $(K_n, \mathcal{B})$. However, it is certain that the representing field cannot be too small.

Proposition 6.5. *For $n \geq 5$, there is no bias or lift representation of K_n with any Hamiltonian bias $\mathcal{B}$ over any field of order less than 4 if $n = 5$, 8 if $n = 6$, and $\lfloor n^2/4 \rfloor + 1$ if $n > 6$.*

Proof. Again we apply Lemma IV.7.*[L:linesfield]*. We produce L_k minors of $G(K_n, \mathcal{B})$ by dividing the n nodes into two sets, A of $\lfloor n/2 \rfloor$ nodes and B of $\lceil n/2 \rceil$ nodes, then contracting a star on each of these sets. The result when $n > 6$ is a contrabalanced graph of order 2 with a loop at each node and a link of multiplicity $m = \lfloor n/2 \rfloor \lceil n/2 \rceil = \lfloor n^2/4 \rfloor$, whose bias matroid is L_{m+2}. Thus $q \geq m+1$. In case $n \leq 6$ the stars are paths; this may create balanced digons in the contraction. Specifically, if the endpoints of the paths are $a, a' \in A$ and $b, b' \in B$, then $\{ab, a'b'\}$ and $\{ab', a'b\}$ may be balanced. When $n = 5$, furthermore, the contraction has only one loop, at the node resulting from contracting A. Discarding multiple points in the bias matroid, therefore, we have L_5 when $n = 5$ and L_9 when $n = 6$.

To get large L_k minors of $L(K_n, \mathcal{B})$ we carry out the same procedure (with one exception), but since the loops are all in the same atom of the contracted matroid we get L_{m+1} when $n > 6$ and L_5 when $n = 5$. The resulting lower bound $q \geq m$ for $n > 6$ can be raised by 1 because m is never a prime power. For $n = 6$ we vary the construction: take $\#A = 2$ and contract a star on B, so producing an 8-fold link and one loop, whose lift (and bias) matroid is L_9. □

The lower bound for $n = 5$ or 6 can be increased slightly if there are enough unbalanced Hamiltonian circles. For instance, suppose $n = 5$ and $\mathcal{B} \subseteq \mathcal{H}(e)$. Choose $B \supset N(e)$ and take the star in B to have the same endpoints as e. Then in the contraction there are no balanced digons, so the result has bias matroid L_7. Consequently, $G(K_5, \mathcal{B})$ has no representation over $\mathbb{F}_q$ if $q < 6$ (so the smallest possible q is 7). If $n = 6$ and $\mathcal{B} \subseteq \mathcal{H}(e)$, choosing B and the star similarly gives a contraction whose matroid is L_{11}. Thus there is no representation if $q < 10$ (so the smallest possible q is 11).

These lower bounds are purely theoretical. I do not know of any representations of these matroids except for trivial $\mathcal{B}$, such as $\mathcal{B} = \varnothing$, and in particular not for any bias satisfying $\mathcal{H}(e) \subseteq \mathcal{B} \subseteq \mathcal{H}$.

Example 6.3 (*Complete bipartite graphs*). Another Hamiltonian example, broadly similar to K_n but having interesting features of its own, is the complete bipartite graphs $K_{t,t}$ for $t \geq 3$. Here $K_{3,3}$ is an odd man out; the real action begins with $t = 4$. The new feature, as compared with complete graphs, is that the gain group is determined by balance of the Hamiltonian circles on two nonadjacent edges.

We write $\mathcal{H}(e, f)$ for the set of all Hamiltonian circles containing e and f. The vertex set of $K_{t,t}$ is $\{x_1, \ldots, x_t, y_1, \ldots, y_t\}$ with edges $x_i y_j$. We abbreviate $\varphi_{ij} = \varphi(x_i y_j; x_i, y_j)$.

Lemma 6.6. *On $K_{t,t}$ with $t \geq 4$, there is no Hamiltonian gain function whose balanced circle class contains $\mathcal{H}(e, f)$ for nonadjacent edges e and f.*

Sketch of Proof. In general the proof resembles that of Lemma 6.3, but it is more complicated because the base graph $K_{4,4}$ is larger.

We take $e = x_1 y_1$ and $f = x_4 y_4$. By appropriate switching the gains on the octagon $x_1 y_2 x_3 y_4 x_4 y_3 x_2 y_1$ can be taken to be 1. This leaves 6 gains to be determined. Various octagons yield the relations

$$\varphi_{13} = \varphi_{22}, \qquad \varphi_{31} = \varphi_{22}, \qquad \varphi_{24} = \varphi_{33}, \qquad \varphi_{42} = \varphi_{33},$$
$$\varphi_{14} = \varphi_{22}\varphi_{33}, \qquad \varphi_{41} = \varphi_{33}\varphi_{22},$$
$$\varphi_{22}\varphi_{31}\varphi_{14} = 1, \qquad \varphi_{14}\varphi_{42}\varphi_{33} = 1,$$

whence we deduce that $\varphi_{22}\varphi_{33}^3 = 1 = \varphi_{33}\varphi_{22}^3$. One last octagon yields the relation $\varphi_{31}\varphi_{14}\varphi_{42} = \varphi_{33}\varphi_{22}$, from which it follows that $\varphi_{22}^2 = 1$ and thence $\varphi_{33} = \varphi_{22}$. We have shown that the gains are cyclic of order 2, or are trivial. That is, (K_{44}, φ) is balanced or is a signed graph in which the Hamiltonian circle $x_1 y_2 x_3 y_4 x_4 y_3 x_2 y_1$ is all positive, as are the edges $x_1 y_4$ and $x_4 y_1$, while the remaining 6 edges are negative. By switching the signs we get $K_{4,4}$ with all edges positive except e and f; we call this signed graph $K_{4,4}(e, f)$.

In either case, needless to say, there are balanced quadrilaterals, so the gains are non-Hamiltonian.

Now let $t > 4$. By finding suitable subgraphs of $K_{t,t}$ that are isomorphic to $K_{4,4}$, with subdivided e and f, in such a way that $x_1 y_1$ lies in the subdivided e and $x_t y_t$ lies in the subdivided f, we can show that any gains on $K_{t,t}$ that make $\mathcal{H}(x_1 y_1, x_t y_t)$ balanced must be either balanced or switch to $K_{t,t}(x_1 y_1, x_t y_t)$. Again, either way the gains are not Hamiltonian. □

We draw conclusions as in Example 6.2, although we do not push the results as far.

Proposition 6.7. *Let $t \geq 4$, let e, f be nonadjacent edges in $K_{t,t}$, and let $\mathcal{H} \supseteq \mathcal{B} \supseteq \mathcal{H}(e, f)$. There is no projective representation of $G(K_{t,t}^{\bullet}, \mathcal{B})$ or $L_0(K_{t,t}, \mathcal{B})$.* □

Example 6.4 (*The small one: $K_{3,3}$*). $K_{3,3}$ is noteworthy in one respect: $(K_{3,3}, \mathcal{H})$ has gains, and if $K_{3,3}$ is given gains for which $\mathcal{H}(e, f)$ is balanced (where e and f are nonadjacent), then either $\mathcal{H}$ is the class of balanced circles or the whole graph is balanced.

To prove this we take $e = e_{11}$ and $f = e_{22}$. We may take all edges to have identity gains except $e_{13}, e_{31}, 3_{22}$. The hexagons $x_1y_3x_3y_2x_2y_1$ and $x_1y_1x_3y_3x_2y_2$ make $\varphi_{13} = \varphi_{22} = \varphi_{31}$. The hexagon $x_1y_3x_2y_2x_3y_1$ makes the product $\varphi_{22}\varphi_{31}\varphi_{13} = 1$; thus the group is $\mathbb{Z}_3$. One can verify that every hexagon but no quadrilateral is balanced in these gains.

As a consequence we know that the matroid $G(K_{3,3}, \mathcal{H}) = L(K_{3,3}, \mathcal{H})$ (which are equal since there are no node-disjoint circles) has canonical representations over $\mathbb{F}_4$ (a canonical bias representation) and $\mathbb{F}_3$ (a canonical lift representation).

7. Deformations of the braid arrangement

These real affine hyperplane arrangements have recently been extensively studied. They are the affinographic representations $\mathcal{A}(\Phi)$ (see the end of Section IV.4.1) of certain integral additive gain graphs. To describe them let $\vec{K}_n$ denote the complete graph on node set $\{1, 2, \ldots, n\}$ with each edge oriented in the ascending direction. For $A \subseteq \mathbb{R}^+$, $A\vec{K}_n$ is the gain graph having edges of the form (e_{ij}, α) for $i < j$ and $\alpha \in A$. Arrangements $\mathcal{A}(A\vec{K}_n)$, where A is a finite subset of $\mathbb{Z}$, are called "deformations of the braid arrangement". Principal examples are the Shi arrangement, where $A = \{0, 1\}$, the Linial arrangement, where $A = \{1\}$; their "extensions", where $A = \{-l+1, = l+2, \ldots, l-1, l\}$ or $A = \{1, 2, \ldots, l\}$, respectively, where $l = 1, 2, 3, \ldots$; the composed partition arrangement, where $A = \{0, \pm 1\}$, and its extension the l-composed partition arrangement, where $A = \{0, \pm 1, \ldots, \pm l\}$.

(The unextended arrangements are canonical lift representations of instances of poise gain graphs (Example 3): the Linial arrangement represents poise gains on $\vec{K}_n$, while the Shi arrangement corresponds to poise gains on the mixed graph consisting of $\vec{K}_n$ superimposed upon an undirected K_n and the composed partition arrangement corresponds to an undirected K_n superimposed upon the loopless complete digraph.)

Athanasiadis [1] studied several arrangements of this type, epecially to compute their characteristic polynomials—which (as implied by Corollary IV.4.4(a) and Theorem II.5.3) are the balanced chromatic polynomials of the associated gain graphs. Athanasiadis' methods are generally based on gain graph coloring using a large cyclic gain group.

For a more thorough discussion of examples like these, with references, see [1] and [17, Examples 10.5–8].

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